63. $r_1 = \dfrac{ed}{1 + \sin\theta}$ and $r_2 = \dfrac{ed}{1 - \sin\theta}$

Points of intersection: $(ed, 0)$, (ed, π)

$$r_1\text{: } \frac{dy}{dx} = \frac{\left(\dfrac{ed}{1+\sin\theta}\right)(\cos\theta) + \left(\dfrac{-ed\cos\theta}{(1+\sin\theta)^2}\right)(\sin\theta)}{\left(\dfrac{-ed}{1+\sin\theta}\right)(\sin\theta) + \left(\dfrac{-ed\cos\theta}{(1+\sin\theta)^2}\right)(\cos\theta)}$$

At $(ed, 0)$, $\dfrac{dy}{dx} = -1$. At (ed, π), $\dfrac{dy}{dx} = 1$.

$$r_2\text{: } \frac{dy}{dx} = \frac{\left(\dfrac{ed}{1-\sin\theta}\right)(\cos\theta) + \left(\dfrac{ed\cos\theta}{(1-\sin\theta)^2}\right)(\sin\theta)}{\left(\dfrac{-ed}{1-\sin\theta}\right)(\sin\theta) + \left(\dfrac{ed\cos\theta}{(1-\sin\theta)^2}\right)(\cos\theta)}$$

At $(ed, 0)$, $\dfrac{dy}{dx} = 1$. At (ed, π), $\dfrac{dy}{dx} = -1$.

Therefore, at $(ed, 0)$ we have $m_1 m_2 = (-1)(1) = -1$, and at (ed, π) we have $m_1 m_2 = 1(-1) = -1$. The curves intersect at right angles.

Review Exercises for Chapter 9

1. Matches (d) - ellipse

3. Matches (a) - parabola

5.

$$16x^2 + 16y^2 - 16x + 24y - 3 = 0$$

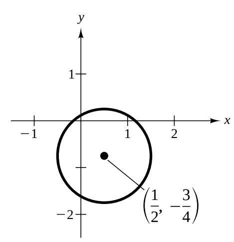
$$\left(x^2 - x + \frac{1}{4}\right) + \left(y^2 + \frac{3}{2}y + \frac{9}{16}\right) = \frac{3}{16} + \frac{1}{4} + \frac{9}{16}$$

$$\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{3}{4}\right)^2 = 1$$

Circle

Center: $\left(\frac{1}{2}, -\frac{3}{4}\right)$

Radius: 1



7.

$$3x^2 - 2y^2 + 24x + 12y + 24 = 0$$

$$3(x^2 + 8x + 16) - 2(y^2 - 6y + 9) = -24 + 48 - 18$$

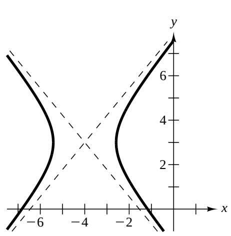
$$\frac{(x+4)^2}{2} - \frac{(y-3)^2}{3} = 1$$

Hyperbola

Center: $(-4, 3)$

Vertices: $\left(-4 \pm \sqrt{2}, 3\right)$

Asymptotes: $y = 3 \pm \sqrt{\dfrac{3}{2}}(x + 4)$



9.

$$3x^2 + 2y^2 - 12x + 12y + 29 = 0$$

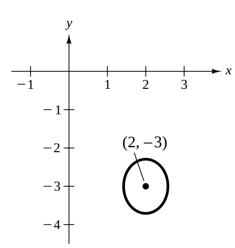
$$3(x^2 - 4x + 4) + 2(y^2 + 6y + 9) = -29 + 12 + 18$$

$$\frac{(x-2)^2}{1/3} + \frac{(y+3)^2}{1/2} = 1$$

Ellipse

Center: $(2, -3)$

Vertices: $\left(2, -3 \pm \dfrac{\sqrt{2}}{2}\right)$



11. Vertex: $(0, 2)$

Directrix: $x = -3$

Parabola opens to the right

$p = 3$

$(y - 2)^2 = 4(3)(x - 0)$

$y^2 - 4y - 12x + 4 = 0$

13. Vertices: $(-3, 0), (7, 0)$

Foci: $(0, 0), (4, 0)$

Horizontal major axis

Center: $(2, 0)$

$a = 5, c = 2, b = \sqrt{21}$

$$\frac{(x-2)^2}{25} + \frac{y^2}{21} = 1$$

15. Vertices: $(\pm 4, 0)$

Foci: $(\pm 6, 0)$

Center: $(0, 0)$

Horizontal transverse axis

$a = 4, c = 6, b = \sqrt{36 - 16} = 2\sqrt{5}$

$$\frac{x^2}{16} - \frac{y^2}{20} = 1$$

17. $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1, a = 3, b = 2, c = \sqrt{5}, e = \dfrac{\sqrt{5}}{3}$

By Example 5 of Section 9.1,

$$C = 12\int_0^{\pi/2} \sqrt{1 - \left(\frac{5}{9}\right)\sin^2\theta}\, d\theta \approx 15.87.$$

19. $y = x - 2$ has a slope of 1. The perpendicular slope is -1.

$$y = x^2 - 2x + 2$$

$$\frac{dy}{dx} = 2x - 2 = -1 \text{ when } x = \frac{1}{2} \text{ and } y = \frac{5}{4}.$$

Perpendicular line: $\quad y - \dfrac{5}{4} = -1\left(x - \dfrac{1}{2}\right)$

$$4x + 4y - 7 = 0$$

21. (a) $V = (\pi ab)(\text{Length}) = 12\pi(16) = 192\pi \text{ ft}^3$

(b) $$F = 2(62.4)\int_{-3}^{3}(3 - y)\frac{4}{3}\sqrt{9 - y^2}\, dy = \frac{8}{3}(62.4)\left[3\int_{-3}^{3}\sqrt{9 - y^2}\, dy - \int_{-3}^{3} y\sqrt{9 - y^2}\, dy\right]$$

$$= \frac{8}{3}(62.4)\left[\frac{3}{2}\left(y\sqrt{9 - y^2} + 9\arcsin\frac{y}{3}\right) + \frac{1}{3}(9 - y^2)^{3/2}\right]_{-3}^{3}$$

$$= \frac{8}{3}(62.4)\left[\frac{3}{2}\left(\frac{9\pi}{2}\right) - \frac{3}{2}\left(-\frac{9\pi}{2}\right)\right] = \frac{8}{3}(62.4)\left(\frac{27\pi}{2}\right) \approx 7057.274$$

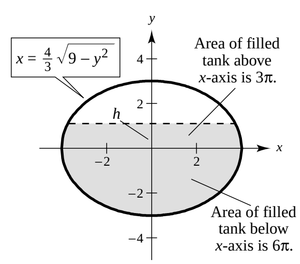
(c) You want $\frac{3}{4}$ of the total area of 12π covered. Find h so that

$$2\int_0^h \frac{4}{3}\sqrt{9 - y^2}\, dy = 3\pi$$

$$\int_0^h \sqrt{9 - y^2}\, dy = \frac{9\pi}{8}$$

$$\frac{1}{2}\left[y\sqrt{9 - y^2} + 9\arcsin\left(\frac{y}{3}\right)\right]_0^h = \frac{9\pi}{8}$$

$$h\sqrt{9 - h^2} + 9\arcsin\left(\frac{h}{3}\right) = \frac{9\pi}{4}.$$



By Newton's Method, $h \approx 1.212$. Therefore, the total height of the water is $1.212 + 3 = 4.212$ ft.

(d) Area of ends $= 2(12\pi) = 24\pi$

Area of sides $=$ (Perimeter)(Length)

$$= 16\int_0^{\pi/2}\left(\sqrt{1 - \left(\frac{7}{16}\right)\sin^2\theta}\right) d\theta(16) \quad \text{[from Example 5 of Section 9.1]}$$

$$\approx 256\left(\frac{\pi/2}{12}\right)\left[\sqrt{1 - \left(\frac{7}{16}\right)\sin^2(0)} + 4\sqrt{1 - \left(\frac{7}{16}\right)\sin^2\left(\frac{\pi}{8}\right)} + 2\sqrt{1 - \left(\frac{7}{16}\right)\sin^2\left(\frac{\pi}{4}\right)}\right.$$
$$\left. + 4\sqrt{1 - \left(\frac{7}{16}\right)\sin^2\left(\frac{3\pi}{8}\right)} + \sqrt{1 - \left(\frac{7}{16}\right)\sin^2\left(\frac{\pi}{2}\right)}\right] \approx 353.65$$

Total area $= 24\pi + 353.65 \approx 429.05$

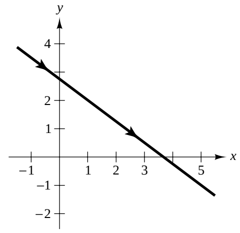
23. $x = 1 + 4t, y = 2 - 3t$

$t = \dfrac{x-1}{4} \Rightarrow y = 2 - 3\left(\dfrac{x-1}{4}\right)$

$y = -\dfrac{3}{4}x + \dfrac{11}{4}$

$4y + 3x - 11 = 0$

Line

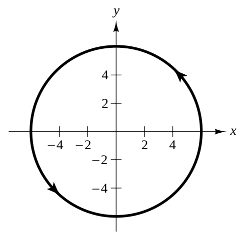


25. $x = 6\cos\theta, y = 6\sin\theta$

$\left(\dfrac{x}{6}\right)^2 + \left(\dfrac{y}{6}\right)^2 = 1$

$x^2 + y^2 = 36$

Circle

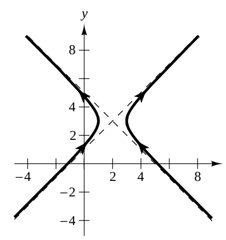


27. $x = 2 + \sec\theta, y = 3 + \tan\theta$

$(x-2)^2 = \sec^2\theta = 1 + \tan^2\theta = 1 + (y-3)^2$

$(x-2)^2 - (y-3)^2 = 1$

Hyperbola



29. $x = 3 + (3 - (-2))t = 3 + 5t$

$y = 2 + (2 - 6)t = 2 - 4t$

(other answers possible)

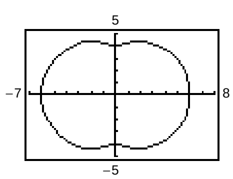
31. $\dfrac{(x+3)^2}{16} + \dfrac{(y-4)^2}{9} = 1$

Let $\dfrac{(x+3)^2}{16} = \cos^2\theta$ and $\dfrac{(y-4)^2}{9} = \sin^2\theta$.

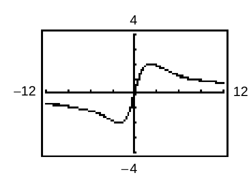
Then $x = -3 + 4\cos\theta$ and $y = 4 + 3\sin\theta$.

33. $x = \cos 3\theta + 5\cos\theta$

$y = \sin 3\theta + 5\sin\theta$



35. (a) $x = 2\cot\theta, y = 4\sin\theta\cos\theta, 0 < \theta < \pi$



(b) $(4 + x^2)y = (4 + 4\cot^2\theta)4\sin\theta\cos\theta$

$= 16\csc^2\theta \cdot \sin\theta \cdot \cos\theta$

$= 16\dfrac{\cos\theta}{\sin\theta}$

$= 8(2\cot\theta)$

$= 8x$

37. $x = 1 + 4t$

$y = 2 - 3t$

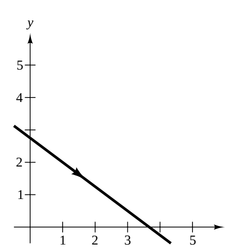
(a) $\dfrac{dy}{dx} = -\dfrac{3}{4}$

No horizontal tangents

(b) $t = \dfrac{x-1}{4}$

$y = 2 - \dfrac{3}{4}(x-1) = \dfrac{-3x+11}{4}$

(c)



39. $x = \dfrac{1}{t}$

$y = 2t + 3$

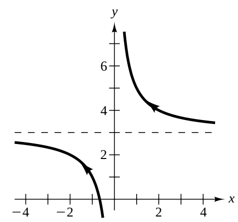
(a) $\dfrac{dy}{dx} = \dfrac{2}{-1/t^2} = -2t^2$

No horizontal tangents $(t \neq 0)$

(b) $t = \dfrac{1}{x}$

$y = \dfrac{2}{x} + 3$

(c)



41. $x = \dfrac{1}{2t + 1}$

$y = \dfrac{1}{t^2 - 2t}$

(a) $\dfrac{dy}{dx} = \dfrac{\dfrac{-(2t-2)}{(t^2-2t)^2}}{\dfrac{-2}{(2t+1)^2}} = \dfrac{(t-1)(2t+1)^2}{t^2(t-2)^2} = 0$ when $t = 1$.

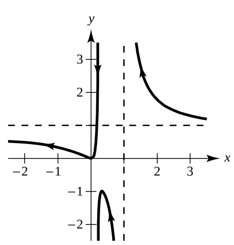
Point of horizontal tangency: $\left(\frac{1}{3}, -1\right)$

(b) $2t + 1 = \dfrac{1}{x} \implies t = \dfrac{1}{2}\left(\dfrac{1}{x} - 1\right)$

$$y = \frac{1}{\frac{1}{2}\left(\frac{1-x}{x}\right)\left[\frac{1}{2}\left(\frac{1-x}{x}\right) - 2\right]}$$

$$= \frac{4x^2}{(1-x)^2 - 4x(1-x)} = \frac{4x^2}{(5x-1)(x-1)}$$

(c)



43. $x = 3 + 2\cos\theta$

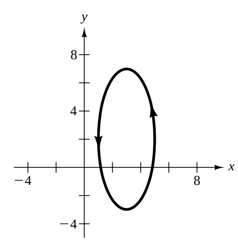
$y = 2 + 5\sin\theta$

(a) $\dfrac{dy}{dx} = \dfrac{5\cos\theta}{-2\sin\theta} = -2.5\cot\theta = 0$ when $\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$.

Points of horizontal tangency: $(3, 7)$, $(3, -3)$

(b) $\dfrac{(x-3)^2}{4} + \dfrac{(y-2)^2}{25} = 1$

(c)



45. $x = \cos^3\theta$

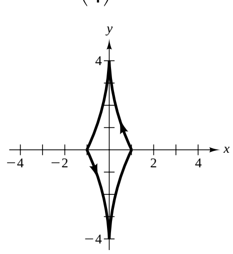
$y = 4\sin^3\theta$

(a) $\dfrac{dy}{dx} = \dfrac{12\sin^2\theta\cos\theta}{3\cos^2\theta(-\sin\theta)} = \dfrac{-4\sin\theta}{\cos\theta} = -4\tan\theta = 0$ when $\theta = 0, \pi$.

But, $\dfrac{dy}{dt} = \dfrac{dx}{dt} = 0$ at $\theta = 0, \pi$. Hence no points of horizontal tangency.

(b) $x^{2/3} + \left(\dfrac{y}{4}\right)^{2/3} = 1$

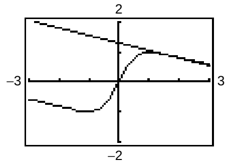
(c)



47. $x = \cot\theta$

$y = \sin 2\theta = 2\sin\theta\cos\theta$

(a), (c)



(b) At $\theta = \dfrac{\pi}{6}, \dfrac{dx}{d\theta} = -4, \dfrac{dy}{d\theta} = 1$, and $\dfrac{dy}{dx} = -\dfrac{1}{4}$

49. $x = r(\cos\theta + \theta\sin\theta)$

$y = r(\sin\theta - \theta\cos\theta)$

$$\frac{dx}{d\theta} = r\theta\cos\theta$$

$$\frac{dy}{d\theta} = r\theta\sin\theta$$

$$s = r\int_0^{\pi}\sqrt{\theta^2\cos^2\theta + \theta^2\sin^2\theta}\,d\theta$$

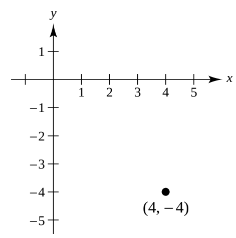
$$= r\int_0^{\pi}\theta\,d\theta = \frac{r}{2}\Big[\theta^2\Big]_0^{\pi} = \frac{1}{2}\pi^2 r$$

51. $(x, y) = (4, -4)$

$r = \sqrt{4^2 + (-4)^2} = 4\sqrt{2}$

$\theta = 7\dfrac{\pi}{4}$

$(r, \theta) = \left(4\sqrt{2}, \dfrac{7\pi}{4}\right), \left(-4\sqrt{2}, \dfrac{3\pi}{4}\right)$



53.

$$r = 3\cos\theta$$
$$r^2 = 3r\cos\theta$$
$$x^2 + y^2 = 3x$$
$$x^2 + y^2 - 3x = 0$$

55.

$$r = -2(1 + \cos\theta)$$
$$r^2 = -2r(1 + \cos\theta)$$
$$x^2 + y^2 = -2\left(\pm\sqrt{x^2 + y^2}\right) - 2x$$
$$(x^2 + y^2 + 2x)^2 = 4(x^2 + y^2)$$

57.

$$r^2 = \cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$r^4 = r^2\cos^2\theta - r^2\sin^2\theta$$
$$(x^2 + y^2)^2 = x^2 - y^2$$

59.

$$r = 4\cos 2\theta\sec\theta$$
$$= 4(2\cos^2\theta - 1)\left(\frac{1}{\cos\theta}\right)$$
$$r\cos\theta = 8\cos^2\theta - 4$$
$$x = 8\left(\frac{x^2}{x^2 + y^2}\right) - 4$$
$$x^3 + xy^2 = 4x^2 - 4y^2$$
$$y^2 = x^2\left(\frac{4 - x}{4 + x}\right)$$

61. $(x^2 + y^2)^2 = ax^2y$

$$r^4 = a(r^2\cos^2\theta)(r\sin\theta)$$
$$r = a\cos^2\theta\sin\theta$$

63. $x^2 + y^2 = a^2\left(\arctan\dfrac{y}{x}\right)^2$

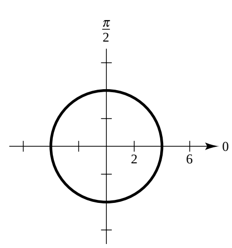
$$r^2 = a^2\theta^2$$

65. $r = 4$

Circle of radius 4

Centered at the pole

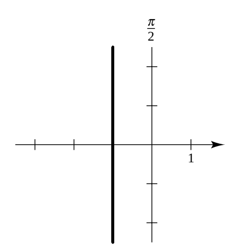
Symmetric to polar axis, $\theta = \pi/2$, and pole



67. $r = -\sec\theta = \dfrac{-1}{\cos\theta}$

$r\cos\theta = -1, x = -1$

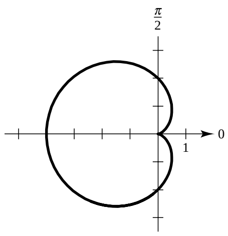
Vertical line



69. $r = -2(1 + \cos\theta)$

Cardioid

Symmetric to polar axis

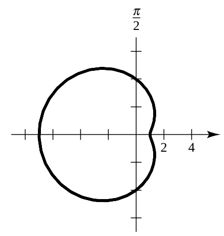


θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	-4	-3	-2	-1	0

71. $r = 4 - 3\cos\theta$

Limaçon

Symmetric to polar axis



θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	1	$\frac{5}{2}$	4	$\frac{11}{2}$	7

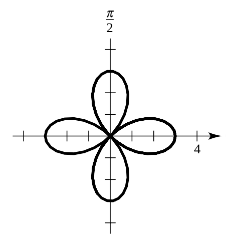
73. $r = -3\cos(2\theta)$

Rose curve with four petals

Symmetric to polar axis, $\theta = \dfrac{\pi}{2}$, and pole

Relative extrema: $(-3, 0), \left(3, \dfrac{\pi}{2}\right), (-3, \pi), \left(3, \dfrac{3\pi}{2}\right)$

Tangents at the pole: $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$



75. $r^2 = 4\sin^2(2\theta)$

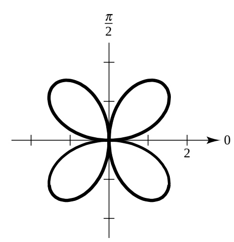
$r = \pm 2\sin(2\theta)$

Rose curve with four petals

Symmetric to the polar axis, $\theta = \dfrac{\pi}{2}$, and pole

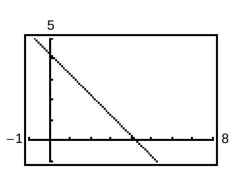
Relative extrema: $\left(\pm 2, \dfrac{\pi}{4}\right), \left(\pm 2, \dfrac{3\pi}{4}\right)$

Tangents at the pole: $\theta = 0, \dfrac{\pi}{2}$



77. $r = \dfrac{3}{\cos[\theta - (\pi/4)]}$

Graph of $r = 3\sec\theta$ rotated through an angle of $\pi/4$



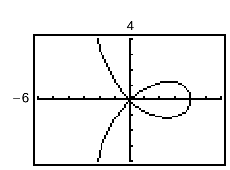
79. $r = 4\cos 2\theta \sec\theta$

Strophoid

Symmetric to the polar axis

$r \Rightarrow -\infty$ as $\theta \Rightarrow \dfrac{\pi^-}{2}$

$r \Rightarrow -\infty$ as $\theta \Rightarrow \dfrac{-\pi^+}{2}$



81. $r = 1 - 2\cos\theta$

(a) The graph has polar symmetry and the tangents at the pole are

$$\theta = \frac{\pi}{3}, -\frac{\pi}{3}.$$

(b) $\dfrac{dy}{dx} = \dfrac{2\sin^2\theta + (1 - 2\cos\theta)\cos\theta}{2\sin\theta\cos\theta - (1 - 2\cos\theta)\sin\theta}$

Horizontal tangents: $-4\cos^2\theta + \cos\theta + 2 = 0, \cos\theta = \dfrac{-1 \pm \sqrt{1+32}}{-8} = \dfrac{1 \pm \sqrt{33}}{8}$

When $\cos\theta = \dfrac{1 \pm \sqrt{33}}{8}, r = 1 - 2\left(\dfrac{1 + \sqrt{33}}{8}\right) = \dfrac{3 \mp \sqrt{33}}{4}$,

$$\left[\frac{3 - \sqrt{33}}{4}, \arccos\left(\frac{1 + \sqrt{33}}{8}\right)\right] \approx (-0.686, 0.568)$$

$$\left[\frac{3 - \sqrt{33}}{4}, -\arccos\left(\frac{1 + \sqrt{33}}{8}\right)\right] \approx (-0.686, -0.568)$$

$$\left[\frac{3 + \sqrt{33}}{4}, \arccos\left(\frac{1 - \sqrt{33}}{8}\right)\right] \approx (2.186, 2.206)$$

$$\left[\frac{3 + \sqrt{33}}{4}, -\arccos\left(\frac{1 - \sqrt{33}}{8}\right)\right] \approx (2.186, -2.206).$$

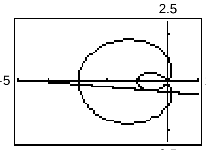
Vertical tangents:

$$\sin\theta(4\cos\theta - 1) = 0, \sin\theta = 0, \cos\theta = \frac{1}{4},$$

$$\theta = 0, \pi, \theta = \pm\arccos\left(\frac{1}{4}\right), (-1, 0), (3, \pi)$$

$$\left(\frac{1}{2}, \pm\arccos\frac{1}{4}\right) \approx (0.5, \pm 1.318)$$

(c)



83. Circle: $r = 3\sin\theta$

$$\frac{dy}{dx} = \frac{3\cos\theta\sin\theta + 3\sin\theta\cos\theta}{3\cos\theta\cos\theta - 3\sin\theta\sin\theta} = \frac{\sin 2\theta}{\cos^2\theta - \sin^2\theta} = \tan 2\theta \text{ at } \theta = \frac{\pi}{6}, \frac{dy}{dx} = \sqrt{3}$$

Limaçon: $r = 4 - 5\sin\theta$

$$\frac{dy}{dx} = \frac{-5\cos\theta\sin\theta + (4 - 5\sin\theta)\cos\theta}{-5\cos\theta\cos\theta - (4 - 5\sin\theta)\sin\theta} \text{ at } \theta = \frac{\pi}{6}, \frac{dy}{dx} = \frac{\sqrt{3}}{9}$$

Let α be the angle between the curves:

$$\tan\alpha = \frac{\sqrt{3} - (\sqrt{3}/9)}{1 + (1/3)} = \frac{2\sqrt{3}}{3}.$$

Therefore, $\alpha = \arctan\left(\dfrac{2\sqrt{3}}{3}\right) \approx 49.1°$.

85. $r = 1 + \cos\theta, r = 1 - \cos\theta$

The points $(1, \pi/2)$ and $(1, 3\pi/2)$ are the two points of intersection (other than the pole). The slope of the graph of $r = 1 + \cos\theta$ is

$$m_1 = \frac{dy}{dx} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \frac{-\sin^2\theta + \cos\theta(1 + \cos\theta)}{-\sin\theta\cos\theta - \sin\theta(1 + \cos\theta)}.$$

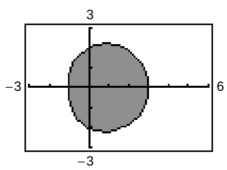
At $(1, \pi/2)$, $m_1 = -1/-1 = 1$ and at $(1, 3\pi/2)$, $m_1 = -1/1 = -1$. The slope of the graph of $r = 1 - \cos\theta$ is

$$m_2 = \frac{dy}{dx} = \frac{\sin^2\theta + \cos\theta(1 - \cos\theta)}{\sin\theta\cos\theta - \sin\theta(1 - \cos\theta)}.$$

At $(1, \pi/2)$, $m_2 = 1/-1 = -1$ and at $(1, 3\pi/2)$, $m_2 = 1/1 = 1$. In both cases, $m_1 = -1/m_2$ and we conclude that the graphs are orthogonal at $(1, \pi/2)$ and $(1, 3\pi/2)$.

87. $r = 2 + \cos\theta$

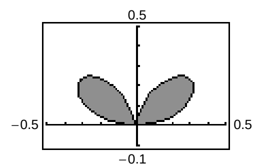
$$A = 2\left[\frac{1}{2}\int_0^{\pi} (2 + \cos\theta)^2\, d\theta\right] \approx 14.14 \quad \left(\frac{9\pi}{2}\right)$$



89. $r = \sin\theta \cdot \cos^2\theta$

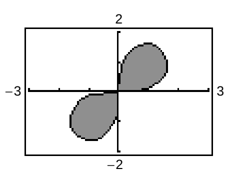
$$A = 2\left[\frac{1}{2}\int_0^{\pi/2} (\sin\theta\cos^2\theta)^2\, d\theta\right]$$

$$\approx 0.10 \left(\frac{\pi}{32}\right)$$



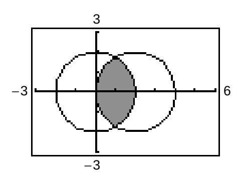
91. $r^2 = 4\sin 2\theta$

$$A = 2\left[\frac{1}{2}\int_0^{\pi/2} 4\sin 2\theta\, d\theta\right] = 4$$



93. $r = 4\cos\theta, r = 2$

$$A = 2\left[\frac{1}{2}\int_0^{\pi/3} 4\, d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2} (4\cos\theta)^2\, d\theta\right] \approx 4.91$$

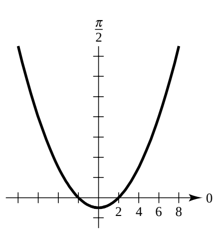


95. $s = 2\int_0^{\pi} \sqrt{a^2(1 - \cos\theta)^2 + a^2\sin^2\theta}\, d\theta$

$$= 2\sqrt{2}\,a\int_0^{\pi} \sqrt{1 - \cos\theta}\, d\theta = 2\sqrt{2}\,a\int_0^{\pi} \frac{\sin\theta}{\sqrt{1 + \cos\theta}}\, d\theta = \left[-4\sqrt{2}\,a(1 + \cos\theta)^{1/2}\right]_0^{\pi} = 8a$$

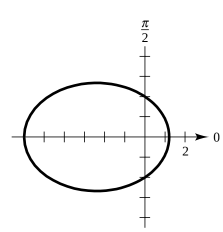
97. $r = \dfrac{2}{1 - \sin\theta}, e = 1$

Parabola



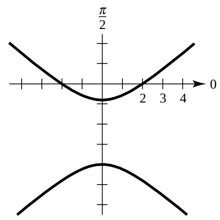
99. $r = \dfrac{6}{3 + 2\cos\theta} = \dfrac{2}{1 + (2/3)\cos\theta}, e = \dfrac{2}{3}$

Ellipse



101. $r = \dfrac{4}{2 - 3\sin\theta} = \dfrac{2}{1 - (3/2)\sin\theta}, e = \dfrac{3}{2}$

Hyperbola



103. Circle

Center: $\left(5, \dfrac{\pi}{2}\right) = (0, 5)$ in rectangular coordinates

Solution point: $(0, 0)$

$$x^2 + (y - 5)^5 = 25$$
$$x^2 + y^2 - 10y = 0$$
$$r^2 - 10r\sin\theta = 0$$
$$r = 10\sin\theta$$

105. Parabola

Vertex: $(2, \pi)$

Focus: $(0, 0)$

$e = 1, d = 4$

$$r = \frac{4}{1 - \cos\theta}$$

107. Ellipse

Vertices: $(5, 0), (1, \pi)$

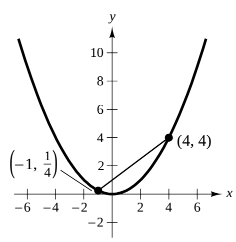
Focus: $(0, 0)$

$a = 3, c = 2, e = \dfrac{2}{3}, d = \dfrac{5}{2}$

$$r = \frac{\left(\frac{2}{3}\right)\left(\frac{5}{2}\right)}{1 - \left(\frac{2}{3}\right)\cos\theta} = \frac{5}{3 - 2\cos\theta}$$

Problem Solving for Chapter 9

1. (a)



(b) $x^2 = 4y$

$2x = 4y'$

$y' = \dfrac{1}{2}x$

$y - 4 = 2(x - 4) \implies y = 2x - 4$ Tangent line at $(4, 4)$

$y - \dfrac{1}{4} = -\dfrac{1}{2}(x + 1) \implies y = -\dfrac{1}{2}x - \dfrac{1}{4}$ Tangent line at $\left(-1, \dfrac{1}{4}\right)$

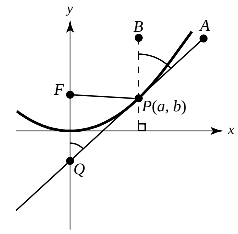
Tangent lines have slopes of 2 and $-1/2 \implies$ perpendicular.

(c) Intersection:

$$2x - 4 = -\frac{1}{2}x - \frac{1}{4}$$
$$8x - 16 = -2x - 1$$
$$10x = 15$$
$$x = \frac{3}{2} \implies \left(\frac{3}{2}, -1\right)$$

Point of intersection, $(3/2, -1)$, is on directrix $y = -1$.

3. Consider $x^2 = 4py$ with focus $(0, p)$.



Let $p(a, b)$ be point on parabola.

$zx = 4py' \implies y' = \dfrac{x}{2p}$

$y - b = \dfrac{a}{2p}(x - a)$ Tangent line

For $x = 0$, $y = b + \dfrac{a}{2p}(-a) = b - \dfrac{a^2}{2p} = b - \dfrac{4pb}{2p} = -b$.

Thus, $Q = (0, -b)$.

$\triangle FQP$ is isosceles because

$$|FQ| = p + b$$

$$\begin{aligned}|FP| = \sqrt{(a-0)^2 + (b-p)^2} &= \sqrt{a^2 + b^2 - 2bp + p^2}\\ &= \sqrt{4pb + b^2 - 2bp + p^2}\\ &= \sqrt{(b+p)^2}\\ &= b + p.\end{aligned}$$

Thus, $\measuredangle FQP = \measuredangle BPA = \measuredangle FPQ$.

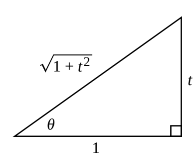
5. (a) In $\triangle OCB$, $\cos\theta = \dfrac{2a}{OB} \implies OB = 2a \cdot \sec\theta$.

In $\triangle OAC$, $\cos\theta = \dfrac{OA}{2a} \implies OA = 2a \cdot \cos\theta$.

$$\begin{aligned}r = OP = AB = OB - OA &= 2a(\sec\theta - \cos\theta)\\ &= 2a\left(\frac{1}{\cos\theta} - \cos\theta\right)\\ &= 2a \cdot \frac{\sin^2\theta}{\cos\theta}\\ &= 2a \cdot \tan\theta\sin\theta\end{aligned}$$

(b) $x = r\cos\theta = (2a\tan\theta\sin\theta)\cos\theta = 2a\sin^2\theta$

$y = r\sin\theta = (2a\tan\theta\sin\theta)\sin\theta = 2a\tan\theta \cdot \sin^2\theta,\ -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$



Let $t = \tan\theta$, $-\infty < t < \infty$.

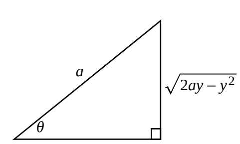
Then $\sin^2\theta = \dfrac{t^2}{1+t^2}$ and $x = 2a\dfrac{t^2}{1+t^2}$, $y = 2a\dfrac{t^3}{1+t^2}$.

(c)

$$\begin{aligned}r &= 2a\tan\theta\sin\theta\\ r\cos\theta &= 2a\sin^2\theta\\ r^3\cos\theta &= 2a\,r^2\sin^2\theta\\ (x^2+y^2)x &= 2ay^2\\ y^2 &= \frac{x^3}{(2a - x)}\end{aligned}$$

7. $y = a(1 - \cos\theta) \implies \cos\theta = \dfrac{a - y}{a}$

$\theta = \arccos\left(\dfrac{a-y}{a}\right)$



$$\begin{aligned}x &= a(\theta - \sin\theta)\\ &= a\left(\arccos\left(\frac{a-y}{a}\right) - \sin\left(\arccos\left(\frac{a-y}{a}\right)\right)\right)\\ &= a\left(\arccos\left(\frac{a-y}{a}\right) - \frac{\sqrt{2ay - y^2}}{a}\right)\end{aligned}$$

$x = a \cdot \arccos\left(\dfrac{a-y}{a}\right) - \sqrt{2ay - y^2},\ 0 \le y \le 2a$

9. For $t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$

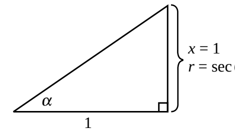
$$y = \frac{2}{\pi}, \frac{-2}{3\pi}, \frac{2}{5\pi}, \frac{-2}{7\pi}, \ldots$$

Hence, the curve has length greater that

$$S = \frac{2}{\pi} + \frac{2}{3\pi} + \frac{2}{5\pi} + \frac{2}{7\pi} + \cdots$$
$$= \frac{2}{\pi}\left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots\right)$$
$$> \frac{2}{\pi}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots\right)$$
$$= \infty.$$

11. (a) Area $= \int_0^\alpha \frac{1}{2}r^2\,d\theta$

$$= \frac{1}{2}\int_0^\alpha \sec^2\theta\,d\theta$$



(b) $\tan\alpha = \frac{h}{1} \Rightarrow$ Area $= \frac{1}{2}(1)\tan\alpha$

$$\Rightarrow \tan\alpha = \int_0^\alpha \sec^2\theta\,d\theta$$

(c) Differentiating, $\frac{d}{d\alpha}(\tan\alpha) = \sec^2\alpha$.

13. If a dog is located at (r, θ), then its neighbor is at $\left(r, \theta + \frac{\pi}{2}\right)$:

$$(x, y) = (r\cos\theta, r\sin\theta) \text{ and } (x, y) = (-r\sin\theta, r\cos\theta).$$

The slope joining these points is

$$\frac{r\cos\theta - r\sin\theta}{-r\sin\theta - r\cos\theta} = \frac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta} = \text{ slope of tangent line at } (r, \theta).$$

$$\frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta}$$

$$\Rightarrow \frac{dr}{d\theta} = -r$$
$$\frac{dr}{r} = -d\theta$$
$$\ln r = -\theta + C_1$$
$$r = e^{-\theta + C_1}$$
$$r = Ce^{-\theta}$$

$$r\left(\frac{\pi}{4}\right) = \frac{d}{\sqrt{2}} \Rightarrow r = Ce^{-\pi/4} = \frac{d}{\sqrt{2}} \Rightarrow C = \frac{d}{\sqrt{2}}e^{\pi/4}$$

Finally, $r = \frac{d}{\sqrt{2}}e^{((\pi/4) - \theta)}$.

15. (a) The first plane makes an angle of 70° with the positive x-axis, and is 150 miles from P:

$$x_1 = \cos 70°(150 - 375t)$$
$$y_1 = \sin 70°(150 - 375t)$$

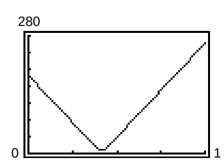
Similarly for the second plane,

$$x_2 = \cos 135°(190 - 450t)$$
$$= \cos 45°(-190 + 450t)$$
$$y_2 = \sin 135°(190 - 450t)$$
$$= \sin 45°(190 - 450t)$$

(b) $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

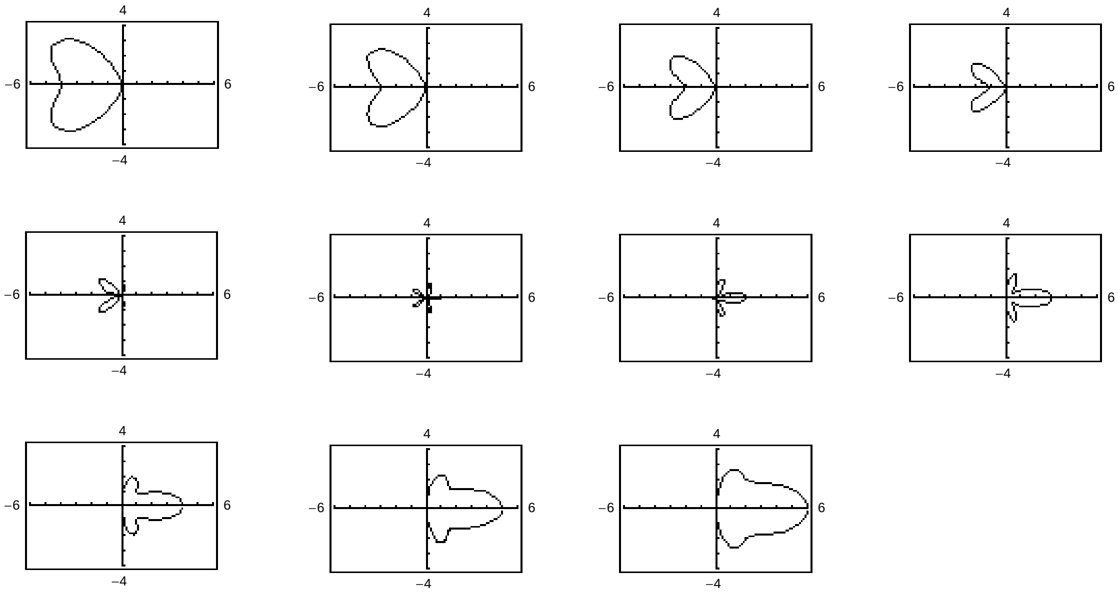
$$= [[\cos 45(-190 + 450t) - \cos 70(150 - 375t)]^2 + [\sin 45(190 - 450t) - \sin 70(150 - 375t)]^2]^{1/2}$$

(c)



The minimum distance is 7.59 miles when $t = 0.4145$.

17.



$n = 1, 2, 3, 4, 5$ produce "bells"; $n = -1, -2, -3, -4, -5$ produce "hearts".