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ADVANCED COMBINATORICS

The Art of Finite and Infinite Expansions

REVISED AND ENLARGED EDITION

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TABLE OF CONTENTS

INTRODUCTION

Notwithstanding its title, the reader will not find in this book a systematic account of this huge subject. Certain classical aspects have been passed by, and the true title ought to be "Various questions of elementary combinatorial analysis". For instance, we only touch upon the subject of graphs and configurations, but there exists a very extensive and good literature on this subject. For this we refer the reader to the bibliography at the end of the volume.

The true beginnings of combinatorial analysis (also called combinatory analysis) coincide with the beginnings of probability theory in the 17th century. For about two centuries it vanished as an autonomous subject. But the advance of statistics, with an ever-increasing demand for configurations as well as the advent and development of computers, have, beyond doubt, contributed to reinstating this subject after such a long period of negligence.

For a long time the aim of combinatorial analysis was to count the different ways of arranging objects under given circumstances. Hence, many of the traditional problems of analysis or geometry which are concerned at a certain moment with finite structures, have a combinatorial character. Today, combinatorial analysis is also relevant to problems of existence, estimation and structuration, like all other parts of mathematics, but exclusively for *finite* sets.

My idea is here to take the uninitiated reader along a path strewn with particular problems, and I can very well imagine that this journey may jolt a student who is used to easy generalizations, especially when only some of the questions I treat can be extended at all, and difficult or unsolved extensions at that, too. Meanwhile, the treatise remains firmly elementary and almost no mathematics of advanced college level will be necessary.

At the end of each chapter I provide statements in the form of *exercises* that serve as *supplementary material*, and I have indicated with a star those that seem most difficult. In this respect, I have attempted to write down these 219 questions with their *answers*, so they can be consulted as a kind of *compendium*.

The first items I should quote and recommend from the bibliography are the three great classical treatises of Netto, MacMahon and Riordan. The bibliographical references, all between brackets, indicate the author's name and the year of publication. Thus, [Abel, 1826] refers, in the *bibliography of articles*, to the paper by Abel, published in 1826. Books are indicated by a star. So, for instance, [*Riordan, 1968] refers, in the *bibliography of books*, to the book by Riordan, published in 1968. Suffixes a, b, c, distinguish, for the same author, different articles that appeared in the same year.

Each chapter is virtually independent of the others, except of the first; but the use of the index will make it easy to consult each part of the book separately.

I have taken the opportunity in this English edition to correct some printing errors and to improve certain points, taking into account the suggestions which several readers kindly communicated to me and to whom I feel indebted and most grateful.

SYMBOLS AND ABBREVIATIONS

$\mathfrak{A}_k(N)$	set of k-arrangements of N
$\mathbf{B}_{n,k}$	partial Bell polynomials
$\mathbf{C}$	set of complex numbers
$\mathbf{E}(X)$	expectation of random variable X
GF	generating function
N	denotes, throughout the book, a finite set with n elements, $\lvert N\rvert = n$
$\mathbf{N}$	set of integers $\geqslant 0$
$\mathbf{P}(A)$	probability of event A
$\mathfrak{P}(N)$	set of subsets of N
$\mathfrak{P}'(N)$	set of nonempty subsets of N
$\mathfrak{P}_k(N)$	set of subsets of N containing k elements
$A+B$	$=A \cup B$, understanding that $A \cap B = \emptyset$
$\mathbf{R}$	set of real numbers
RV	random variable
$\mathbf{Z}$	set of all integers $\gtrless 0$
Δ	difference operator
■	indicates beginning and end of the proof of a theorem
$:=$	equals by definition
$[n]$	the set $\{1, 2, 3, \ldots, n\}$ of the first n positive integers
$n!$	n factorial = the product $1.2.3.\ldots.n$
$(x)_k$	$=x(x-1)\ldots(x-k+1)$
$\langle x\rangle_k$	$=x(x+1)\ldots(x+k-1)$
$[x]$	the greatest integer less than or equal to x
$\lVert x\rVert$	the nearest integer to x
$\binom{n}{k}$	binomial coefficient $=(n)_k/k!$
$s(n,k)$	Stirling number of the first kind
$S(n,k)$	Stirling number of the second kind
$\lvert N\rvert$	number of elements of set N
$\underset{\cdot}{k}$	bound variable, with dot underneath
$\complement A$, $\bar{A}$	complement of subset A
$\complement_n f$	coefficient of t^n in the formal series f
$\{x \mid \mathscr{P}\}$	set of all x with property $\mathscr{P}$
N^M	set of maps of M into N

VOCABULARY OF COMBINATORIAL ANALYSIS

In this chapter we define the language we will use and we introduce those elementary concepts which will be referred to throughout the book. As much as possible, the chosen notations will not be new; we will use only those that actually occur in publications. We will not be afraid to use two different symbols for the same thing, as one may be preferable to the other, depending on circumstances; thus, for example, $\bar{A}$ and $\complement A$ both denote the complement of A, $A\cap B$ and AB stand for the intersection of A and B, etc. For the rest, it seems desirable to avoid taking positions and to obtain the flexibility which is necessary to be able to read different authors.

1.1. SUBSETS OF A SET, OPERATIONS

In the following we suppose the reader to be familiar with the rudiments of set theory, in the naive sense, as they are taught in any introductory mathematics course. This section just defines the notations.

N, **Z**, **R**, **C** denote the set of the non-negative integers including zero, the set of all integers $\gtrless 0$, the set of the real numbers and the set of the complex numbers, respectively.

We will sometimes use the following *logical abbreviations*: $\exists$ (=there exists at least one), $\forall$ (=for all), $\Rightarrow$ (=implies), $\Leftarrow$ (=if), $\Leftrightarrow$ (=if and only if).

When a set Ω and one of its elements ω is given, we write "$\omega\in\Omega$" and we say "ω is element of Ω" or also "ω belongs to Ω" or "ω in Ω". Let H be the subset of elements ω of Ω that have a certain property $\mathscr{P}$, $H\subset\Omega$, then we denote this by:

$$[1a] \qquad H := \{\omega \mid \omega\in\Omega, \mathscr{P}\},$$

and we say this as follows: "H equals *by definition* the set of elements ω of Ω satisfying $\mathscr{P}$". When the list of elements $a, b, c, \ldots, l$ that constitute together N, is known, then we also write:

$$N := \{a, b, c, \ldots, l\}.$$

If N is a finite set, $|N|$ denotes the *number of its elements*. Hence $|N| = \operatorname{card} N = \textit{cardinal}$ of N, also denoted by $\bar{N}$.

$\mathfrak{P}(N)$ is the set of all subsets of N, *including the empty set*; $\mathfrak{P}^*(N)$ denotes the set of all *nonempty* subsets, or *combinations*, or *blocks*, of N; hence, when A is a subset of N, we will denote this by $A \subset N$ or by $A \in \mathfrak{P}(N)$, as we like. For A, B subsets of N, $A, B \subset N$ we recall that

$$A \cap B := \{x \mid x \in A, x \in B\},$$
$$A \cup B := \{x \mid x \in A \quad \text{or} \quad x \in B\},$$

(the or is not exclusive) which are the *intersection* and the *union* of A and B. It sometimes will happen somewhere that we write AB instead of $A \cap B$, for reasons of economy. (See, for example, Chapter IV.) For each *family* $\mathscr{F}$ of subsets of N, $\mathscr{F} := (A_i)_{i \in I}$, we denote:

$$\bigcap_{i \in I} A_i := \{x \mid \forall i \in I, x \in A_i\}, \qquad \bigcup_{i \in I} A_i := \{x \mid \exists i \in I, x \in A_i\}.$$

The (set theoretic) *difference* of two subsets A and B of N is defined by:

[1b] $\quad A \setminus B := \{x \mid x \in A, x \notin B\}.$

The *complement* of A $(\subset N)$ is the subset $N \setminus A$ of N, also denoted by $\bar{A}$, or $\complement A$, or $\complement_N A$. The operation which assigns to A the set $\bar{A}$ is called *complementation*. Clearly:

[1c] $\quad A \setminus B = A \cap \bar{B}.$

$\mathfrak{P}(N)$ is made into a *Boolean algebra* by the operations $\cup$, $\cap$ and $\complement$. Such a structure consists of a certain set M (here $= \mathfrak{P}(N)$) with two operations $\vee$ and $\wedge$ (here $\vee = \cup$, $\wedge = \cap$), and a map of M into itself: $a \to \bar{a}$ (here $A \to \bar{A} = \complement A$) such that *for all* $a, b, c, \ldots \in M$, *we have*:

[1d] (I) $\quad (a \vee b) \vee c = a \vee (b \vee c)$,

(II) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ (*associativity of* $\vee$ *and* $\wedge$).

(III) $\quad a \vee b = b \vee a$,

(IV) $a \wedge b = b \wedge a$ (*commutativity of* $\vee$ *and* $\wedge$).

(V) *There exists a (unique) neutral element denoted by* 0, *for* $\vee$: $a \vee 0 = 0 \vee a = a$

(VI) *There exists a (unique) neutral element denoted by* 1, *for* $\wedge$: $a \wedge 1 = 1 \wedge a = a$.

(VII) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ (*distributivity of* $\wedge$ *with respect to* $\vee$).

(VIII) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ (*distributivity of* $\vee$ *with respect to* $\wedge$).

(IX) *Each* $a \in M$ *has a complement denoted by* $\bar{a}$ *such that* $a \wedge \bar{a} = 0$, $a \vee \bar{a} = 1$.

The most important interrelations between the operations $\bigcup$, $\bigcap$, $\complement$ are the following:

DeMorgan formulas. *Let* $(A_i)_{i \in I}$ *and* $(B_\kappa)_{\kappa \in K}$ *be two families of* N, $A_i \subset N$, $B_\kappa \subset N$, $i \in I$, $\kappa \in K$. *Then*:

[1e] $\quad \complement\left(\bigcup_{i \in I} A_i\right) = \bigcap_{i \in I} (\complement A_i)$

[1f] $\quad \complement\left(\bigcap_{i \in I} A_i\right) = \bigcup_{i \in I} (\complement A_i)$

[1g] $\quad \left(\bigcup_{i \in I} A_i\right) \cap \left(\bigcup_{\kappa \in K} B_\kappa\right) = \bigcup_{(i, \kappa) \in I \times K} (A_i \cap B_\kappa)$

[1h] $\quad \left(\bigcap_{i \in I} A_i\right) \cup \left(\bigcap_{\kappa \in K} B_\kappa\right) = \bigcap_{(i, \kappa) \in I \times K} (A_i \cup B_\kappa)$.

A *system* $\mathscr{S}$ of N is a nonempty (unordered) set of blocks of N, without repetition $(\Leftrightarrow \mathscr{S} \in \mathfrak{P}^*(\mathfrak{P}^*(N)))$; a k-*system* is a system consisting of k blocks.

1.2. PRODUCT SETS

Let be given m finite sets N_i, $1 \leqslant i \leqslant m$, and recall that the *product set* $\prod_{i=1}^m N_i$ or *Cartesian product* of the N_i is the set of the *m-tuples* $(y) := (y_1, y_2, \ldots, y_m)$, where $y_i \in N_i$ for all $i = 1, 2, \ldots, m$. The product set is also denoted by $N_1 \times N_2 \times \cdots \times N_m$ or by $N_1 N_2 N_3 \ldots N_m$ if there is no danger for confusion. We call y_i the *projection* of (y) on N_i, denoted by $pr_i(y)$.

If $N_1 = N_2 = \cdots = N_m = N$, the product set is also denoted by N^m; the *diagonal* Δ of N^m is hence the set of the m-tuples such that $y_1 = y_2 = \cdots = y_m$.

Theorem. *The number of elements of the product set of a finite number* *of finite sets satisfies:*

[2a] $$\Big|\prod_{i=1}^{n} N_i\Big| = \prod_{i=1}^{n} |N_i| = |N_1|\cdot|N_2|\dots|N_n|.$$

■ In fact, the number of n-tuples $(y_1, y_2, \dots, y_n)$ is equal to the product of the number of possible choices for y_1 in N_1, which is $|N_1|$, by the number of possible choices of y_2 in N_2, which is $|N_2|$, etc., by the number of possible choices of y_n in N_n, which is $|N_n|$, because these choices can be done *independently* from each other. ■

Example. What is the number $d(n)$ of factors of n, with prime decomposition $n = p_1^{\nu_1}p_2^{\nu_2}\dots p_k^{\nu_k}$? To choose any factor $p_1^{\delta_1}p_2^{\delta_2}\dots p_k^{\delta_k}$ of n is the same as to choose the sequence $(\delta_1, \delta_2, \dots, \delta_k)$ of exponents such that $\delta_i \in A_i := \{0, 1, 2, \dots, \nu_i\}$, $i = 1, 2, \dots, k$. Then, $d(n) = |A_1 \times A_2 \times \dots \times A_k| = |A_1|\cdot|A_2|\dots|A_k| = (\nu_1+1)(\nu_2+1)\dots(\nu_k+1)$.

1.3. MAPS

Let $\mathfrak{F}(M, N)$ or N^M be the set of the mappings f of M into N: to each $x \in M$, f associates a $y \in N$, the *image* of x by f, denoted by $y = f(x)$. We write often $f: M \to N$ instead of $f \in \mathfrak{F}(M, N)$. As M and N are finite, $m = |M|$, $n = |N|$, we can number the elements of M, so let $M = \{x_1, x_2, \dots, x_m\}$. It is clear that giving f is equivalent to giving a list of m elements of N, say $(y_1, y_2, \dots, y_m)$, written in a certain order and with *repetitions allowed*. By giving the list we mean then that y_i is the image of x_i, $1 \le i \le m$: $y_i = f(x_i)$. In other words, giving f is equivalent to giving an m-tuple $\in N^m$, also called an *m-selection*. In this way we find the justification for the notation N^M for $\mathfrak{F}(M, N)$. Taking [2a] into account, we also have proved the following.

THEOREM A. *The number of maps of M into N is given by*

[3a] $$|\mathfrak{F}(M, N)| = |N^M| = |N|^{|M|}.$$

For each subset $A \subset M$, we denote:

[3b] $$f(A) := \{f(x) \mid x \in A\}.$$

In this way a map is defined from $\mathfrak{P}(M)$ into $\mathfrak{P}(N)$, which is called the *extension of f to the set of subsets of M*. This is also denoted by f.

For all $y \in N$, the subset of M

[3c] $$f^{-1}(y) := \{x \mid f(x) = y\},$$

which may be empty, is called the *pre-image* or *inverse image* of y by f.

THEOREM B. *The number of subsets of M, the empty set included, is given by:*

[3d] $$|\mathfrak{P}(M)| = 2^{|M|}.$$

■ Let N be the set with two elements 0 and 1. We identify a subset $A \subset M$ with the mapping f from M into N defined by: $f(x) = 1$ for $x \in A$, and $f(x) = 0$ otherwise (f is often called the *characteristic function*). In this way we have established a one-to-one correspondence between the sets $\mathfrak{P}(M)$ and N^M. Hence, by [3a], $\mathfrak{P}(M)$ has the same number of elements as N^M, which is $|N|^{|M|} = 2^{|M|}$.

For computing $u_m = |\mathfrak{P}(M)|$, we can also remark that there are just as many subsets of M that do not contain a given point x as there are subsets containing it, namely u_{m-1} in both cases. Hence $u_m = u_{m-1} + u_{m-1} = 2u_{m-1}$, which combined with $u_0 = 1$ gives $u_m = 2^m$ indeed. ■

We recall that $f \in N^M$ is called *injective* (or is said to be an *injection*) if the images of two different elements are different: $x_1 \ne x_2 \Rightarrow f(x_1) \ne f(x_2)$; f is called *surjective* (or is said to be a *surjection*) if every element of N is image of some element in M: $\forall y \in N, \exists x \in M, f(x) = y$; finally f is called *bijective* (or is said to be a *bijection*) if f is surjective as well as injective; in the last case the *inverse* or *reciprocal* of f, denoted by f^{-1}, is defined by $y = f^{-1}(x)$, if and only if $x = f(y)$, where $x \in M$, $y \in N$.

To *count* a certain finite set E, in other words, to determine its size, consists in principle of constructing a *bijection* of E onto another set F, whose number of elements is known already, then $|E| = |F|$.

EXAMPLE. Let E be the set of all subsets of N with *even* size, and F the set of the others (with *odd* size). We can choose $x \in N$ and build a bijection f of E into F as follows: $f(A) = A \cup \{x\}$ or $A \setminus \{x\}$ according to $x \notin A$ or $x \in A$. Thus, $|E| = |F| = (\frac{1}{2})|\mathfrak{P}(N)| = 2^{n-1}$. (See also p. 13.)

1.4. ARRANGEMENTS, PERMUTATIONS

From now we denote for each integer $k \ge 1$:

[4a] $$[k] := \{1, 2, \dots, k\} = \text{the set of the first } k \text{ integers} \ge 1.$$

DEFINITION A. *A k-arrangement α of a set N, $1 \leqslant k \leqslant n = |N|$, is an injective map α from $[k]$ into N* (formerly called 'variation'). *We shall denote the set of k-arrangements of N by $\mathfrak{A}_k(N)$.*

Giving such an α is hence equivalent to giving first a subset of k elements of N:

$$B = \alpha([k]) = \{\alpha(1), \alpha(2), \ldots, \alpha(k)\},$$

and secondly a *numbering* from 1 to k of the elements of B, so finally, a totally ordered subset of k elements of N, which will often be called a k-arrangement of N too (not quite correct, but quite convenient).

We introduce now the following notations:

[4b] $$n! := \prod_{i=1}^{n} i = 1\cdot 2\cdot 3 \cdot \ldots \cdot n, \quad \text{if} \quad n \geqslant 1; \quad 0! = 1.$$

[4c] $$(n)_k := \prod_{i=1}^{k} (n-i+1) = \frac{n!}{(n-k)!} = n(n-1)\ldots(n-k+1), \quad \text{if} \quad k \geqslant 1; \ (n)_0 = 1.$$

[4d] $$\langle n\rangle_k := \prod_{i=1}^{k} (n+i-1) = \frac{(n+k-1)!}{(n-1)!} = n(n+1)\ldots(n+k-1), \quad \text{if} \quad k \geqslant 1; \ \langle n\rangle_0 = 1.$$

$n!$ is called *n factorial*; $(n)_k$ is sometimes called *falling factorial n* (of order k), and $\langle n\rangle_k$ is sometimes called *rising factorial n* (of order k), or also the *Pochhammer* symbol. So, $(n)_n = \langle 1\rangle_n = n!$, $\langle n\rangle_k = (n+k-1)_k$, $(n)_k = \langle n-k+1\rangle_k$, etc. These notations are not yet fixed well. The use of $(n)_k$ in the sense indicated, is inspired by formula [5a] (p. 8) that associates the symbols $(n)_k$ and $\binom{n}{k}$ with each other in a symmetrical way, both using parentheses. The symbol $\langle n\rangle_k$ that we introduce here for lack of any better is not standard, and is often written $(n)_k$ in texts on hypergeometric series. For the reader familiar with the Γ function:

[4e] $$n! = \Gamma(n+1), \qquad (n)_k = \Gamma(n+1)/\Gamma(n-k+1), \qquad \langle n\rangle_k = \Gamma(n+k)/\Gamma(n).$$

Besides, for complex z (and k integer $\geqslant 0$), $(z)_k$ and $\langle z\rangle_k$ still make sense:

[4f] $$(z)_k := z(z-1)\ldots(z-k+1), \quad (z)_0 = 1$$

[4g] $$\langle z\rangle_k := z(z+1)\ldots(z+k-1), \quad \langle z\rangle_0 = 1,$$

and hence they can be considered as polynomials of degree k in the indeterminate z.

THEOREM A. *The number of k-arrangements of N, $1 \leqslant k \leqslant n = |N|$, equals:*

[4h] $$|\mathfrak{A}_k(N)| = (n)_k = n(n-1)\ldots(n-k+1).$$

■ There are evidently n choices possible for the image $\alpha(1)$ of 1 ($\in [k]$); after the choice of $\alpha(1)$ is made, there are left only $(n-1)$ possibilities for $\alpha(2)$, because α is injective, so $\alpha(2) \neq \alpha(1)$; similarly, there are left for $\alpha(3)$ only $(n-2)$ possible choices, because $\alpha(3) \neq \alpha(2)$ and $\alpha(3) \neq \alpha(1)$, etc.; finally, for $\alpha(k)$ there are just $(n-k+1)$ possible choices left. The number of α is hence equal to the product of all these numbers of choices. This is equal to $n(n-1)(n-2)\ldots(n-k+1)$. ■

Note. If $k > n$, then $(n)_k = 0$, and [4h] is still valid.

DEFINITION B. *A permutation of a set N is a bijective map of N onto itself. We denote the set of permutations of N by $\mathfrak{S}(N)$.*

THEOREM B. *The number of permutations of N, $|N| = n \geqslant 1$, equals n!*

■ One can argue as in the proof of Theorem A above. One may also observe that there is a bijection between $\mathfrak{S}(N)$ and $\mathfrak{A}_n(N)$. ■

5. COMBINATIONS (WITHOUT REPETITIONS) OR BLOCKS

DEFINITION A. *A k-combination, or k-block, of a finite set N is a nonempty subset of k elements of N: $B \subset N$, $1 \leqslant k = |B| \leqslant n = |N|$. If one does not know in advance whether $k \geqslant 1$, one says rather k-subset of N ($k \geqslant 0$). We denote the set of k-subsets of N by $\mathfrak{P}_k(N)$.*

A k-block is also called a *combination of k to k of the n elements of N*. *Pair* and *2-block* are synonymous; similarly, *triple* or *triad* and *3-block*, etc.

Next we show three other ways to specify a k-subset of N, $|N| = n$.

THEOREM A. *There exists a bijection between $\mathfrak{P}_k(N)$ and the set of functions $\varphi: N \to \{0, 1\}$, for which the sum of the values equals k: $\sum_{x \in N} \varphi(x) = k$.*

THEOREM B. *There exists a bijection between $\mathfrak{P}_k(N)$ and the set of solu-*

tions of the equation $x_1+x_2+\cdots+x_n=k$, *for which all* x_i *equal* 0 *or* 1.

THEOREM C. *Giving a* $B\in\mathfrak{P}_k(N)$ *is equivalent to giving a distribution of k indistinguishable balls in n distinct boxes, each box containing at most one ball.*

■ For Theorem A, it is sufficient to define for each $B\in\mathfrak{P}_k(N)$ the *characteristic function* $\varphi=\varphi_B$ by $\varphi(y)=1$ if $y\in B$ and $=0$ otherwise. For Theorem B we number the elements of N from 1 to n, $N=\{y_1,y_2,\ldots,y_n\}$; for each $B\in\mathfrak{P}_k(N)$ we define $x_i=x_i(B)$ by $x_i=1$ if $y_i\in B$ and $=0$ otherwise. Finally, for Theorem C each box is associated with a point $y\in N$; to every $B\in\mathfrak{P}_k(N)$ we associate the following distribution: the box associated with y contains a ball if $y\in B$ and no ball if $y\notin B$. ■

THEOREM D. *The number of k-subsets of* N, $0\leqslant k\leqslant n=|N|$, *denoted by* $\binom{n}{k}$, *equals:*

$$[5a]\qquad \binom{n}{k}:=|\mathfrak{P}_k(N)|\overset{(*)}{=}\frac{(n)_k}{k!}=\frac{n(n-1)\ldots(n-k+1)}{k!}$$

$$=\frac{n!}{k!\,(n-k)!}=\binom{n}{n-k}.$$

We will adopt the notation $\binom{n}{k}$, used almost in this form by Euler and fixed by Raabe, with the exclusion of all other notations, as this notation is used in the great majority of the present literature, and its use is even so still increasing. This symbol has all the qualities of a good notation: economical (no new letters introduced), expressive (it is very close in appearance to the explicit value $\frac{(n)_k}{k!}$), typical (no risk of being confused with others), and beautiful. In certain cases, one might prefer (a,b) instead of $\binom{a+b}{a}$ (see pp. 27 and 28), so that $(a,b)=(b,a)$ is perfectly symmetric in a and b. We recall anyway the 'French' notation C_n^k and the 'English' notation ${}^nC_k$.

■ We prove equality (*); the others are immediate consequences. If $k=0$, $(n)_k/0!=1$ [4b, 1] (p. 6), and $|\mathfrak{P}_k(N)|=1$ because $\mathfrak{P}_0(N)$ contains only the empty subset of N. Let us suppose $k\geqslant 1$. With every arrangement $\alpha\in\mathfrak{A}_k(N)$, we associate $B=f(\alpha)=\{\alpha(1),\alpha(2),\ldots,\alpha(k)\}\in\mathfrak{P}_k(N)$ (p. 7). f is a map from $\mathfrak{A}_k(N)$ into $\mathfrak{P}_k(N)$ such that for all $B\in\mathfrak{P}_k(N)$ we have:

$$[5b]\qquad |f^{-1}(B)|=k!,$$

since there are $k!$ possible numberings of B (= the number of k-arrangements of B). Now the set of preimages $f^{-1}(B)$, which are mutually disjoint, covers $\mathfrak{A}_k(N)$ entirely as B runs through $\mathfrak{P}_k(N)$. Hence, the number of elements of $\mathfrak{A}_k(N)$ equals the sum of all $|f^{-1}(B)|$, where $B\in\mathfrak{P}_k(N)$, which is [5c (*)]. Hence, by [4b] (p. 7) for equality (**), and by [5b] for (***)

$$[5c]\qquad (n)_k\overset{(**)}{=}|\mathfrak{A}_k(N)|\overset{(*)}{=}\sum_B|f^{-1}(B)|\overset{(***)}{=}k!\,|\mathfrak{P}_k(N)|;$$

hence $|\mathfrak{P}_k(N)|=(n)_k/k!$. ■

The argument we just have used is sometimes called the 'shepherd's principle': for counting the number of sheep in a flock, just count the legs and divide by 4.

DEFINITION B. *The integers* $\binom{n}{k}$ *are called* binomial coefficients.

We will see the justification of this name on p. 12.

DEFINITION C. *The double sequence* $\binom{n}{k}$ *which is defined by* [5a] *for* $(n,k)\in\mathbb{N}^2$ *(and equal to 0 for* $k>n$*) will be defined from now on also for* $(x,y)\in\mathbb{C}^2$ *in the following way:*

$$[5d]\qquad \binom{x}{y}:=\begin{cases}\dfrac{(x)_y}{y!} & \text{if } x\in\mathbb{C},\ y\in\mathbb{N}\\ 0 & \text{if } x\in\mathbb{C},\ y\notin\mathbb{N}\end{cases}$$

where $(x)_k:=x(x-1)\ldots(x-k+1)$ for any $k\in\mathbb{N}$, $(x)_0=1$.
We will constantly use this convention in the sequel.

Theorem E. *The binomial coefficients satisfy the following recurrence relations:*

$$[5e]\qquad \binom{n}{k}=\binom{n-1}{k-1}+\binom{n-1}{k};\quad k,n\geq 1.$$

$$[5f]\qquad \binom{n}{k}=\frac{n}{k}\binom{n-1}{k-1}=\frac{n-k+1}{k}\binom{n}{k-1};\quad k,n\geq 1.$$

$$[5g]\qquad \binom{n+1}{k+1}=\binom{k}{k}+\binom{k+1}{k}+\cdots+\binom{n}{k}=\sum_{l=k}^{n}\binom{l}{k};\quad k,n\geq 0.$$

$$[5h]\qquad \binom{n-1}{k}=\binom{n}{k}-\binom{n}{k-1}+\cdots+(-1)^k\binom{n}{0}=\sum_{l=0}^{k}(-1)^{k-l}\binom{n}{l}.$$

If n is replaced by a real or complex number x (x can also play the role of an indeterminate variable), then [5e, f, h] still hold, and we have instead of [5g], for each integer k≥0:

$$[5g']\qquad \binom{x+1}{k+1}=\binom{x}{k}+\binom{x-1}{k}+\cdots+\binom{x-n}{k}+\binom{x-n}{k+1}.$$

■ [5e, f] can be verified by substituting the values [5a, b] for the binomial coefficients [5a']; hence [5g'] follows by applying [5e] to each of the terms of the sum $\sum_i\binom{x+1-i}{k+1}$ followed by the evident simplification. For [5h], an analogous method works (a generalization is found at the end of Exercise 30, p. 169).

As an example, we will also give combinatorial proofs of [5e, f, g].

For [5e], let us choose a point $x\in N$, $|N|=n$, and let $\mathscr{S}$ and $\mathscr{T}$ respectively be the system of k-blocks of N that contain or not contain respectively the point x. Clearly, $\mathscr{S}\cap\mathscr{T}=\emptyset$, so:

$$[5i]\qquad |\mathfrak{P}_k(N)|=|\mathscr{S}|+|\mathscr{T}|.$$

Now every $B\in\mathscr{S}$ corresponds to exactly one $B'\in\mathfrak{P}_{k-1}(N\setminus\{x\})$, namely $B\setminus\{x\}$, hence:

$$[5j]\qquad |\mathscr{S}|=|\mathfrak{P}_{k-1}(N\setminus\{x\})|=\binom{n-1}{k-1}.$$

Also, $\mathscr{T}=\mathfrak{P}_k(N\setminus\{x\})$, hence:

$$[5k]\qquad |\mathscr{T}|=\binom{n-1}{k}.$$

Finally, [5i, j, k] imply [5e].

For [5f], let us take the interpretation of $\binom{n}{k}$ as the number of distributions of balls in boxes (Theorem C, p. 3). We form all the $\binom{n}{k}$ distributions successively. Then we need in total $k\binom{n}{k}$ balls. The n boxes play a symmetric role, so every box receives $(1/n)\,k\binom{n}{k}$ times a ball. Now, every distribution that gives a ball to a given box, corresponds to exactly one distribution of $(k-1)$ balls in the remaining $(n-1)$ boxes. These are $\binom{n-1}{k-1}$ in number, so as result we find that $(k/n)\binom{n}{k}=\binom{n-1}{k-1}$.

For [5g], we number the elements of N, $N=\{x_1,x_2,\ldots,x_n\}$. We put for $l=1,2,\ldots$

$$\mathscr{S}_l=\{B\mid B\in\mathfrak{P}_{k+1}(N);\ x_1,x_2,\ldots,x_{l-1}\notin B;\ x_l\in B\}.$$

Evidently, each $B\in\mathfrak{P}_{k+1}(N)$ belongs to exactly one $\mathscr{S}_l$, $l\in[n]$. So

$$[5l]\qquad \binom{n}{k}=|\mathscr{S}_1|+|\mathscr{S}_2|+\cdots.$$

Now, every $B\in\mathscr{S}_l$ corresponds to exactly one:

$$C:=B\setminus\{x_l\}\in\mathfrak{P}_k(N\setminus\{x_1,x_2,\ldots,x_l\}).$$

Hence:

$$[5m]\qquad |\mathscr{S}_l|=|\mathfrak{P}_k(N\setminus\{x_1,x_2,\ldots,x_l\})|=\binom{n-l}{k},$$

and we see that [5m, l] imply [5g]. ■

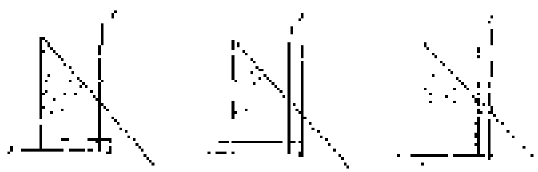
Pascal triangle (or *arithmetical triangle*) is the name for the infinite table, which is obtained by placing each number $\binom{n}{k}$ at the intersection of the n-th row and the k-th column, $k,n\geq 0$ (Figure 1). The numerical computation of the first values can be quickly done, by using [5e] and the initial values $\binom{0}{k}=0$ except for $\binom{0}{0}=1$.

Each recurrence relation [5e, f, g] can be advantageously visualized by a *diagram* (Figures 2a, b, c); in every Pascal triangle represented by the

n \ k	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	1	1	0	0	0	0	0
2	1	2	1	0	0	0	0
3	1	3	3	1	0	0	0
4	1	4	6	4	1	0	0
5	1	5	10	10	5	1	0
...	...						

Fig. 1.

shaded area, the heavy dots represent the pairs (n, k) such that the corresponding $\binom{n}{k}$ are related by a linear recurrence relation (that is, with



(a) (b) (c)

Fig. 2.

coefficients that are possibly not constant with respect to n and k, as for example [5c]). Diagrams 2a, b, c are said to be of the second, first and $(n-k+1)$-st order, respectively, as their associated recurrence relations. A table of binomial coefficients is presented on p. 306.

1.6. BINOMIAL IDENTITY

THEOREM A. (Newton binomial formula, or binomial identity). *If x and y are commuting elements $(xy=yx)$ of a ring, then we have for each integer $n \geq 0$:*

[6a] $$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots.$$

Note. If the ring does not have an identity, we must interpret $x^0 y^n$ and $x^n y^0$ as y^n and x^n, respectively. We can also consider [6a] as an identity between polynomials of the *indeterminates* x and y.

■ Let us examine the coefficients $c_{k,l}$ of the expansion of:

[6b] $$(x+y)^n = P_1 P_2 \cdots P_n = \sum_{k,l} c_{k,l} x^k y^l, \quad P_i = x+y, \quad i \in [n].$$

The term $x^k y^l$ is obtained by choosing k of the n factors P_i, $i \in [n]$, in the sense that one multiplies the terms x of these factors by the terms y of the remaining $(n-k)$ factors. So $l = n-k$. Hence the coefficient $c_{k,l} = c_{k,n-k}$ equals the number of different choices of the k factors P_i among the n, hence equals $\binom{n}{k}$ (Theorem D, p. 8). ■

For instance, if $x = y = 1$, then we have $\sum_k \binom{n}{k} = 2^n$ and thus we find again the result of p. 3: the total number of subsets of N equals 2^n. If $x = -1$, $y = 1$ we obtain $\sum_k (-1)^k \binom{n}{k} = 0$, in other words: in N there are just as many 'even' as 'odd' subsets (see also p. 5).

Now we evaluate the n-th power of the difference operator.

THEOREM B. *Let Δ be the difference operator, which assigns to every function $f \in A^{\mathbb{R}}$, defined on the real numbers, and with values in a ring A, the function $g = \Delta f$, which is defined by $g(x) = f(x+1) - f(x)$, $x \in \mathbb{R}$. For each integer $n \geq 2$, we define $\Delta^n f := \Delta(\Delta^{n-1} f)$, and we denote $\Delta^n f(x)$ instead of $(\Delta^n f)(x)$. Then we have:*

[6c] $$\Delta^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+k), \quad n = 0, 1, 2, \ldots$$

■ Let E be the *translation* operator defined by $Ef(x) := f(x+1)$ and I the identity operator, $If = f$. Clearly, $\Delta = E - I$. Now E and I commute in the ring of operators acting on $A^{\mathbb{R}}$. Hence, defining $E^k = E(E^{k-1}) = E(E(E^{k-2})) = \cdots$, we have, by [6a]:

$$\Delta^n = (E-I)^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} E^k$$

(since $I^{n-k} = I$), from which [6c] follows, as $E^k f(x) = f(x+k)$. ■

In the case of a sequence u_n, $n\in\mathbb{N}$, [6d] implies:

[6d'] $$\Delta^r u_n = \sum_{k=0}^{r} (-1)^{r-k}\binom{r}{k} u_{n+k},$$

where $\Delta u_n = u_{n+1}-u_n$, $\Delta^2 u_n = \Delta(\Delta u_n) = u_{n+2}-2u_{n+1}+u_n$, etc. $\Delta u_n\geqslant 0$, for all n, means that u_n is *increasing*, $\Delta^2 u_n\geqslant 0$ for all n, means that u_n is *convex*.

If Δ operates on one of the variables of a function of several variables, one can *place a dot over* the variable concerned to indicate this. So we write

[6e] $$\Delta f(\dot{u}, v) := f(u+1, v) - f(u, v),$$
$$\Delta f(u, \dot{v}) := f(u, v+1) - f(u, v).$$

Examples. (1) $\Delta^k 0^n$ means the value of $\Delta^k x^n$ in the point $x=0$, and then [6d] gives

[6f] $$\Delta^k 0^n = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n,$$

which are, up to a coefficient $k!$, the *Stirling numbers of the second kind* (cf. p. 204).

(2) $$\Delta^n \frac{1}{(x)_k} = (-1)^n \frac{(k)_n}{(x+n)_{k+n}} \quad \text{(by induction)}.$$

We cite also the following interesting arithmetical property of binomial coefficients:

THEOREM C. *For each prime number p, we have:*

[6g] $$\binom{p}{k} \equiv 0 \pmod p, \quad \text{except} \quad \binom{p}{0}=\binom{p}{p}=1.$$

In other words:

[6g'] $$(1+x)^p \equiv 1+x^p \pmod p,$$

which means that these two polynomials have the same coefficients in $\mathbb{Z}/p\mathbb{Z}$. (Exercise 17, p. 78 gives many other arithmetical properties of the binomial coefficients.)

■ As $\binom{p}{k} = \frac{(p)_k}{k!}$ is an integer, $k!$ divides $(p)_k = p(p-1)\cdots$, and it is relatively prime with respect to p if $1\leqslant k\leqslant p-1$; hence it divides $(p-1)_{k-1}$ according to the lemma of Gauss. Thus $(p-1)_{k-1}/k!$ is an integer $\lambda\geqslant 0$, hence $\binom{p}{k} = p\lambda \equiv 0 \pmod p$. ■

1.7. COMBINATIONS WITH REPETITIONS

DEFINITION. *A p-combination with repetition (or unordered p-selection, or p-CR) of a finite set N, is a list of p elements, all taken from N, repetitions allowed, but the order in the list not taken into account. We denote the set of p-CR of N by* $\mathfrak{C}_p(N)$.

For example, $[a,a,c,b]$ and $[c,b,a,a]$ are identical 4-CR of $\{a,b,c\}$. Each p-block of N can be considered as a p-CR of N.

THEOREM A. *There exists a bijection between* $\mathfrak{C}_p(N)$ *and the set of functions* $\phi: N\to\mathbb{N}$ *for which the sum of the values p equals* $\sum_{y\in N}\phi(y)$.

THEOREM B. *There is a bijection between* $\mathfrak{C}_p(N)$ *and the set of integer solutions, consisting of integers* $\geqslant 0$, *of the equation:*

[7a] $$x_1+x_2+\cdots+x_n = p.$$

Each solution of [7a] is also called *composition of p into n summands* (see Exercise 25, p. 123).

THEOREM C. *Each* $C\in\mathfrak{C}_p(N)$ *corresponds to exactly one distribution of p indistinguishable balls into n distinct boxes.*

The reasoning is the same as for Theorems A, B, and C, on p. 7.

■ For Theorem A, define for each $C\in\mathfrak{C}_p(N)$ the function $\phi=\phi_C\in\mathbb{N}^N$ by $\phi(y)$:= the number of times that y appears in C. For Theorem B, $N=\{x_1, x_2, \ldots, x_n\}$ and $x_i := \phi_C(x_i)$. For Theorem C, identify each point $y\in N$ with a box. ■

THEOREM D. *The number of p-CR of a finite set N,* $|N|=n$, $p\geqslant 0$, *equals* *the binomial coefficient 'with repetition'* $\left(\binom{n}{p}\right)$, *defined by:*

[7b] $$|\Omega_p(N)| = \left(\binom{n}{p}\right) = \frac{\langle n\rangle_p}{p!} = \binom{n+p-1}{p} = (n, p-1).$$

■ We give two proofs of this theorem.

(I) We partition the set of solutions of [7a] (we denote the number of these solutions by $T(n,p)$) into two kinds. First, the solutions such that $x_n=0$; there are evidently $T(n-1,p)$ of them. Next, the solutions for which $x_n\geq 1$; if for these we put $x_n' := x_n-1$ (≥ 0), these solutions correspond each to exactly one solution of $x_1+x_2+\cdots+x_n'=p-1$, of which there are $T(n,p-1)$. Finally,

[7c] $$T(n,p) = T(n-1,p)+T(n,p-1).$$

To this relation we still must add the following *initial conditions*, which follow from [7a]:

[7d] $$T(n,0)=1, \quad T(1,p)=1.$$

Now the double sequence $T(n,p)$ is completely determined. As a matter of fact, the sequence $U(n,p) := \binom{n+p-1}{p}$ evidently satisfies the recurrence relation [7c] as well as the 'boundary conditions' [7d]. Hence $T(n,p)=U(n,p)$.

(II) We represent the n boxes $y_1, y_2, \ldots, y_n$ of T ... side by side. We number the separations between the boxes by $z_1, \ldots, z_{n-1}$, going from left to right (Figure 3). Let now $N = \{y_1, y_2, \ldots, y_n\}$ be the set of these boxes and let $Z := [n+p-1] = \{1, 2, \ldots, n+p-1\}$. Now we define the map f from $\Omega_p(N)$ into $\mathfrak{P}_{n-1}(Z)$ as follows: with every distribution of balls associated with $T\in\Omega_p(N)$ we associate the $(n-1)$ block

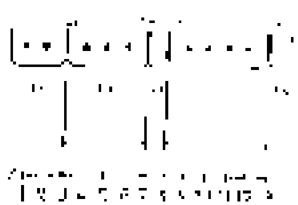


Fig. 3. $n=5$, $p=9$.

$f(T) \subset Z$:

$$f(T) = \{a_1+1,\ a_1+a_2+2,\ \ldots,\ a_1+\cdots+a_{n-1}+n-1\},$$

where $a_i = a_T(y_i)$ stands for the number of repetitions of y_i in the p-combination T. Clearly, f is bijective, hence the result. ■

The binomial coefficient 'with repetition' $\left(\binom{n}{p}\right)$, also called a *figurate number*, can evidently be expressed as a function of the 'symmetrical binomial coefficient' $(a,b) := \binom{a+b}{b}$ of p. 8. We obtain:

$$\left(\binom{n}{p}\right) = (n-1, p) = (p, n-1).$$

Example. We compute the number of monomials in the most general polynomial P of degree k and n indeterminates $x_1, x_2, \ldots, x_n$, ... In the second case, we observe that the list of exponents $\nu_1, \ldots, \nu_n$ of a monomial $x_1^{\nu_1}\cdots x_n^{\nu_n}$ corresponds to a solution with nonnegative integers ν_i of the inequality $\nu_1+\cdots+\nu_n\le k$, which is equivalent to the solution in nonnegative integers of $\nu_1+\cdots+\nu_n+\nu_{n+1}=k$, $\nu_{n+1}\in\mathbb{N}$, $n\in[n+1]$. By [7b] the number is consequently equal to $\binom{(n+1)+k-1}{k} = \binom{n+k}{k}$, which is a result one would ... (see also Exercise 18, p. 77).

Some properties of the binomial coefficients 'with repetition'. First, we have, by [8a] p. 8, ... we obtain:

$$\left(\binom{x}{k}\right) = \binom{x+k-1}{k} = \frac{\langle x\rangle_k}{k!} = \frac{x(x+1)\cdots(x+k-1)}{k!}.$$

Then we have properties for $\left(\binom{x}{k}\right)$ very analogous to those of Theorem B, p. 12, for example:

[7e] $$\left(\binom{x}{k}\right) = \left(\binom{x}{k-1}\right) + \left(\binom{x-1}{k}\right)$$

[7f] $$\left(\binom{x}{k}\right) = \frac{x}{k}\left(\binom{x+1}{k-1}\right) = \frac{x+k-1}{k}\left(\binom{x}{k-1}\right)$$

[7g] $$\left(\binom{n+1}{k}\right) = \sum_{j=0}^{k}\left(\binom{n}{j}\right)$$

The proofs, all very easy, are left to the reader.

Abelian words. One can also give a more abstract definition of the concept of combination with repetitions, which is important to know. Let $\mathfrak{X}$ be a nonempty set, the *alphabet*; we denote the set of finite sequences f of elements of $\mathfrak{X}$ (also called *letters*) by $\mathfrak{X}^*$. We denote $f=(x_{i_k})_{k\in[r]}=(x_{i_1}, x_{i_2}, \ldots, x_{i_r})$, where r is a variable integer $\geqslant 1$. Such a sequence f is also called an *r-arrangement with repetition of* $\mathfrak{X}$. Hence, when $\mathfrak{X}^{[r]}$ has the meaning given on p. 4, and when we make the convention to let the empty set $\emptyset$, denoted by 1, also belong to $\mathfrak{X}^*$, then we have:

$$\mathfrak{X}^* := \{1\} \cup \Big(\bigcup_{r\geqslant 1} \mathfrak{X}^{[r]}\Big).$$

The sequence $f=(x_{i_1}, x_{i_2}, \ldots, x_{i_r})$ will be identified with the *monomial* or *word* $x_{i_1}x_{i_2}\ldots x_{i_r}$. In this form, the integer r is called the *degree* of the monomial or the *length* of the word f. By definition, the length of 1 is 0. In the case $\mathfrak{X}$ is finite, $\mathfrak{X}:=\{x_1, x_2, \ldots, x_n\}$, we can denote by a_i the number of times that the letter x_i occurs in the word f, $a_i\geqslant 0$, $i\in[n]$. In that case we often say that f has the *specification* $(a_1, a_2, a_3, \ldots, a_n)$. For example, for $\mathfrak{X}:=\{x, y\}$, $\mathfrak{X}^{[3]}$ consists of the following 8 words: xxx, xxy, xyx, yxx, xyy, yxy, yyx, yyy. These can also be written: x^3, x^2y, xyx, yx^2, xy^2, yxy, y^2x, y^3. The specifications are then (3, 0), (2, 1), (2, 1), (2, 1), (1, 2), (1, 2), (1, 2), (0, 3), respectively.

The set $\mathfrak{X}^*$ is equipped with an associative composition law, the *product by juxtaposition*, which associates with two words $f=x_{i_1}x_{i_2}\ldots x_{i_r}$ and $g=x_{j_1}x_{j_2}\ldots x_{j_s}$ the *product word* $fg=x_{k_1}x_{k_2}\ldots x_{k_{r+s}}$, where $x_{k_t}=x_{i_t}$ if $t\leqslant r$, and $x_{k_t}=x_{j_{t-r}}$ if $t>r$. One also says that fg is the *concatenation* of f and g. This composition law is associative, and has the empty word 1 as unit element. In this way $\mathfrak{X}^*$ becomes a monoid (that is to say a set with an associative multiplication, and a unit element), which is called the *free monoid generated by* $\mathfrak{X}$. Furthermore, when we denote the set of words of length r by $\mathfrak{X}^{[r]}$, we identify $\mathfrak{X}^{[1]}$ with $\mathfrak{X}$, so $\mathfrak{X}\subset\mathfrak{X}^*$.

We introduce an equivalence relation on $\mathfrak{X}^*$, by defining two words f and g to be equivalent if and only if they consist of the same letters, *up to order, but with the same number of repetitions*. The equivalence class that contains f, is called *the abelian class of f*, or also the *abelian word f*. There is a one-to-one correspondence between the abelian classes and the maps ψ from $\mathfrak{X}$ into $\mathbb{N}$ that are everywhere zero except for a finite number of points. In fact, if we index the set Y of the $y\in\mathfrak{X}$ where $\psi(y)\neq 0$,

in such a way that $Y=\{y_1, y_2, \ldots, y_l\}$, then we can bijectively associate with ψ the abelian class of the word:

$$y_1^{\psi(y_1)}y_2^{\psi(y_2)}\ldots y_l^{\psi(y_l)} := \underbrace{y_1y_1\ldots}_{\psi(y_1)\text{ times}}\;\underbrace{y_2y_2\ldots}_{\psi(y_2)\text{ times}}\;\ldots\;\underbrace{y_ly_l\ldots}_{\psi(y_l)\text{ times}}.$$

If $\mathfrak{X}$ is finite, $\mathfrak{X}=N$, it is clear that an abelian word is just a combination with repetitions, of N (Definition, p. 15).

The set of abelian words $\tilde{\mathfrak{X}}^*$ can also be made into a monoid, when we consider it as a part of $\mathbb{N}^{\mathfrak{X}}$; this last set is equipped with the usual addition of functions ψ. In this way we define the *free abelian monoid generated by* $\mathfrak{X}$.

1.8. Subsets of $[n]$, random walk

Let N be a finite totally ordered set (Definition D, p. 59), with n elements, which we identify with $[n]:=\{1, 2, \ldots, n\}$. We are going to give several interpretations to the specification of a subset $P\subset[n]$, of cardinal $p\,(=|P|)$. We introduce moreover:

$$q:=|\bar{P}|=|\complement P|=n-p.$$

(I) To give a $P\subset[n]$ is equivalent to giving an integer-valued sequence $x(t)$, defined by:

$$[8a] \qquad x(t)-x(t-1)=\begin{cases}+1 & \text{if } t\in P\\ -1 & \text{if } t\notin P\end{cases};\quad t\in[n],\quad x(0):=0.$$

One can represent $x(t)$ by a broken line, which is straight between the points with coordinates $(t, x(t))$. Thus, Figure 4 represents the $x(t)$ associated with the block:

$$[8b] \qquad P=\{3, 5, 6, 7, 8, 10, 11, 12\}\subset[12].$$

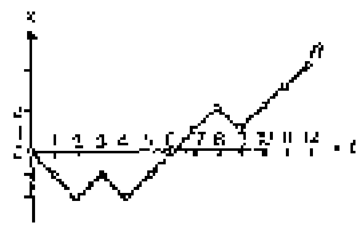


Fig. 4.

Evidently, $p+q=n$ and $x(n)=(x(n)-x(n-1))+\cdots+(x(2)-x(1))+x(1)=p-q$; hence:

[8c] $\quad p=\frac{1}{2}(n+x(n)),\qquad q=\frac{1}{2}(n-x(n)).$

This way of determining $P\subset[n]$ suggests a process, if we imagine that t represents successive instants $1, 2, \ldots, n$.

(2) Giving $P\subset[n]$ is also equivalent to giving the results of a game of heads or tails, played with n throws of a coin, if we agree that

$$x(t)-x(t-1)=1 \Leftrightarrow \text{the } t\text{th throw is } tails \quad (t\in[n]).$$

The numbers p, q of [8c] are then the numbers of *tails* and *heads* obtained in the course of the game, respectively. Because of this interpretation, the sequence $x(t)$ is often called *random walk*; it translates the (stochastic) movements by jumps of ± 1 of a moving point on the x-axis, whose motion occurs only at the times $t=1, 2, \ldots, n$ (a kind of Brownian movement on a line).

Giving $P=[n]$ is also equivalent to giving the successive results of drawing balls from a vase, which contains p black and q white balls, and agreeing that $x(t)-x(t-1)=1 \Leftrightarrow$ the tth ball drawn is black ($t\in[n]$).

(3) One often prefers in combinatorial analysis to represent $P\subset[n]$ by a polygonal line $\mathscr{P}$ which joins the origin $(0,0)$ with the point B with coordinates (p,q) such that the horizontal sides, having lengths one and also called *horizontal steps*, correspond to the points of p, and the vertical sides correspond to the points of the complement of p. Thus, Figure 5 represents the subset p defined by [8b]. Such a polygonal line may be called '*minimal path*' joining O to B (of length $n=p+q$). (In fact, there does not exist a shorter path of length less than n, which joins O to B.

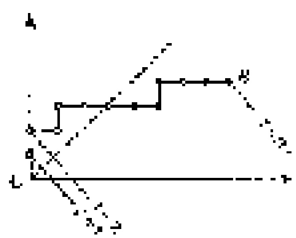


Fig. 5.

consisting of unit length straight sections bounded by points with integer coordinates.)

(4) Finally, giving $P\subset[n]$ is also equivalent to giving a *word* f with two letters a and b, of length n, where the letter a occurs p times, and the letter b occurs q times, $p=|P|$ (see p. 18). Thus, the word representing P of [8b] is *bbabaaabaaa*.

Now we treat two examples of enumerations in $[n]$.

THEOREM A. ([Gergonne, 1812], [Muir, 1901]). *Let $f_l(n,p)$ be the number of p-blocks $P\subset[n]$ with the following property: between two arbitrary points of P are at least $l\,(\geqslant 0)$ points of $[n]$ which do not belong to P. Then:*

[8d] $\quad f_l(n,p)=\binom{n-(p-1)l}{p}.$

■ Let P be $\{i_1, i_2, \ldots, i_p\}$, $1\leqslant i_1<i_2<\cdots<i_p\leqslant n$, and $y_k:=i_k-i_{k-1}-1$, $y_1:=i_1-1$, $y_{p+1}:=n-i_p$. Giving P is equivalent to giving a solution with integers y_i of:

[8e] $\quad y_1+y_2+\cdots+y_p+y_{p+1}=n-p$
$\qquad y_k\geqslant l$ if $2\leqslant k\leqslant p$, $\quad y_1$ and $y_{p+1}\geqslant 0.$

We put $z_k:=y_k-l$ if $2\leqslant k\leqslant p$, and $z_1:=y_1$, $z_{p+1}:=y_{p+1}$. Then $z_i\geqslant 0$, for every $i\in[p+1]$ and [8e] is equivalent to:

[8f] $\quad z_1+z_2+\cdots+z_p+z_{p+1}=n-p-(p-1)l,$

which has $\binom{n-(p-1)l}{p}$ solutions, by Theorems B and D, of pp. 15. ■

Observe that $l=-1$ recovers [7b] p. 16...!

(For other problems concerning the blocks of $[n]$, the reader is referred to [*David, Barton, 1962], pp. 85–101, [Abramson, 1964, 1965], [Abramson, Moser, 1966, 1969], [Church, Gould, 1967], [Kaplansky, 1943, 1945], [(René) Lagrange, 1963], [Mood, 1940].)

THEOREM B (*of André*). *Let p and q be integers, such that $1\leqslant p<q$, $p+q=n$. The number of minimal paths joining O with the point $M(p,q)$ (in the sense of (3) on p. 20) that do not have any point in common with the line $x=y$, except the point O, is* $\frac{q-p}{q+p}\binom{n}{p}$. *In other words, if there is a ballot, for*

which candidates $\mathscr{P}$ and $\mathscr{Q}$ receive p and q votes respectively (so $\mathscr{Q}$ is elected), then the probability that candidate $\mathscr{Q}$ has constantly the majority during the counting of the votes is equal to $(q-p)/(q+p)$.

This is the famous *ballot problem*, formulated by [Bertrand, 1887]; we give the elegant solution of [André, 1887]. Désiré André, born Lyon, 1840, died Paris, 1917, devoted most of his scientific activity to combinatorial analysis. A list and a summary of his principal works are found in [*André, 1910]. See also Exercises 11 and 13 pp. 258 and 260.

■ We first formulate the *principle of reflection*, which essentially is due to André. Let be given a line D parallel to the line $x=y$, and two points A, B lying on the same side of D (for instance above, as in Figure 6). The number of *minimal* paths (the adjective minimal will be omitted in the sequel) joining A with B that intersect or touch D, is equal to the number

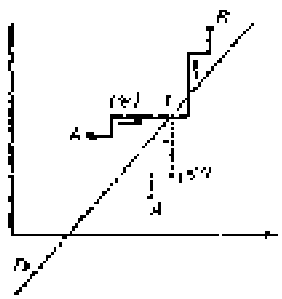


Fig. 6.

of paths joining B with the point A' which lies symmetric to A with respect to D. Indeed, when I stands for the first point that $\mathscr{C}$ has in common with D, going from A to B, we can let the path $\mathscr{C}=(A, I, B)$ correspond to the path $\mathscr{C}'=(A', I, B)$, which is just the same as $\mathscr{C}$ between I and B, but with the part $A'I$ just equal to the image by reflection with respect to D of the part AI of $\mathscr{C}$.

Now let $\mathbf{C}(A, B)$ be the set of paths joining $A(x_A, y_A)$ with $B(x_B, y_B)$, $0 \leqslant x_A \leqslant x_B$, $0 \leqslant y_A \leqslant y_B$. Clearly, the number of paths joining A with B equals:

$$[8a] \qquad \mathbf{C}(A, B) = \binom{x_B+y_B-x_A-y_A}{x_B-x_A},$$

because giving a path is equivalent to choosing a set of (x_B-x_A) horizontal segments among $(x_B+y_B-x_A-y_A)$ places (the duration of the walk).

Let us call a *suitable* path one that satisfies the hypotheses of Theorem B. The number of suitable paths, which is the number of paths joining $W(0,1)$ with $B(p,q)$ without intersecting the line $x=y$, is hence, by the principle of reflection equal to $|\mathbf{C}(W, B)|-|\mathbf{C}(V, B)|$ (Figure 7), which means, by [8a], equal to $\binom{p+q-1}{p}-\binom{p+q-1}{p-1}$, hence the result, after simplifications.

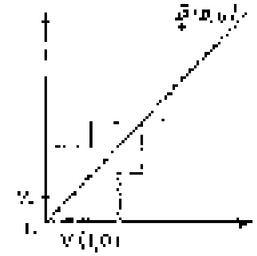


Fig. 7.

The probabilistic interpretation supposes that every path $\in \mathbf{C}(O, B)$ is equally probable, so that the probability we look for is the quotient of the number of suitable paths (which we found already), and the total number of paths joining O with B, which is $|\mathbf{C}(O, B)|=\binom{p+q}{p}$; we find that the probability is $(q-p)/(q+p)$, as announced. Every step represents a vote, the horizontal ones being for $\mathscr{P}$ and the vertical ones for $\mathscr{Q}$. ■ For other problems related to the problem of the ballot, see [Carlitz, Riordan, 1964], [*Feller, 1968, I], p. 67–97, [Goodman, Narayana, 1967], [Guilbaud, Rosenstiehl, 1960], [Kreweras, 1965, 1966a], [Narayana, 1965, 1967], [Riordan, 1964], [Sen, 1964], [*Spitzer, 1964], and especially [*Takács, 1967]. The reader should also solve Exercises 21–22 on pp. 81–82.

1.9. Subsets of $\mathbf{Z}/n\mathbf{Z}$

Let N be a finite set of n points placed on a circle with equal distances between two adjoining points. We identify this set with the set of residue classes modulo n, denoted by $[n]$:

[9a] $N = [\bar n] = \mathbf{Z}/n\mathbf{Z} = \{\bar 0, \bar 1, \bar 2, \ldots, \overline{n-1}\}$.

Figure 8 represents the block $P = \{\bar 2, \bar 3, \bar 5, \bar 6, \bar 7, \bar 8, \overline{11}, \overline{12}, \overline{13}\} \in [\overline{16}]$, with which we can associate the *circular word bbaabaaaabbaaabb*, where the i-th term equals a or b according to whether $\bar i \in P$ or $\notin P$, $0 \le i \le n-1$.

We show now an example of enumeration in $[\bar n]$.

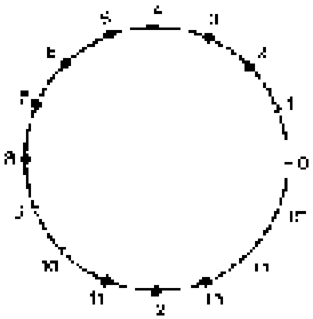


FIG. 8.

THEOREM ([Kaplansky, 1943]). *Let $g_l(n,p)$ be the number of p-blocks $P \subset [\bar n]$ with the following property: between any two points u and v of P (that means on each of the two open arcs uv of the circle on which we think $[\bar n]$ situated) there are at least $l\ (\ge 0)$ points of $[\bar n]$ that do not belong to P. Then:*

[9b] $$g_l(n,p) = \frac{n}{n-pl}\binom{n-pl}{p}.$$

■ When $\mathscr{A}$ stands for the set of the $P \subset [\bar n]$ that satisfy the condition mentioned in the theorem, $[\bar l] := \{\bar 0, \bar 1, \ldots, \overline{l-1}\}$, then we let:

$$\mathscr{A}_i := \{P \mid P \in \mathscr{A},\ P \cap [\bar l] = \{\bar i\}\}, \quad i = 0, 1, 2, \ldots, l-1,$$
$$\mathscr{A}^* := \{P \mid P \in \mathscr{A},\ P \cap [\bar l] = \emptyset\}.$$

$\mathscr{A}^*$ and the $\mathscr{A}_i$ evidently partition $\mathscr{A}$ into $l+1$ disjoint subsets. Hence:

[9c] $$g_l(n,p) = |\mathscr{A}| = |\mathscr{A}^*| + \sum_{i=0}^{l-1} |\mathscr{A}_i|.$$

Now, choosing $P \in \mathscr{A}_i$ is equivalent to choosing on the *straight* interval $[\overline{i+l+1}, \overline{i+l+2}, \ldots, \overline{i+n-l-1}]$ the $(p-1)$-block $P' := P \setminus \{\bar i\}$ with $n-2l-1$

elements. Hence, by Theorem A (p. 24), we have:

[9d] $$|\mathscr{A}_i| = f_l(n-2l-1,\ p-1), \quad 0 \le i \le l-1.$$

Similarly, choosing $P \in \mathscr{A}^*$ is equivalent to choosing it on the *straight* interval $[\bar l, \overline{l+1}, \ldots, \overline{n-1}]$ with $n-l$ elements. Hence:

[9e] $$|\mathscr{A}^*| = f_l(n-l,\ p).$$

Finally, [9c, d, e] imply, by [8d] (p. 24) for the equality (*), and with simplifications for (**):

$$g_l(n,p) = l f_l(n-2l-1,\ p-1) + f_l(n-l,\ p) =$$
$$\overset{(*)}{=} l\binom{n-pl-1}{p-1} + \binom{n-pl}{p} \overset{(**)}{=} \frac{n}{n-pl}\binom{n-pl}{p}. \quad ■$$

It would be interesting to give a combinatorial significance of $g_l(n,p)$ for $l<0$. Also see Exercise 10, p. 173.

1.10. DIVISIONS AND PARTITIONS OF A SET; MULTINOMIAL IDENTITY

DEFINITION A. *Let $\mathscr{M}$ be a finite (ordered) sequence of subsets, distinct or not, empty or not, of a set N:*

$$\mathscr{M} = (A_1, A_2, \ldots, A_m), \quad A_i \subset N, \quad i \in [m], \quad m \ge 1.$$

We say that $\mathscr{M}$ is a division of N (confusion with partition (Definition C, p. 30), should be avoided), or m-division if we want to specify of how many subsets it consists, if the union of the A_i, $i \in [m]$, is N, and if these A_i are mutually disjoint. We denote:

[10a] $$N = A_1 + A_2 + \cdots + A_m \quad \text{or} \quad N = \sum_{i=1}^{m} A_i$$

*(notation of [*Neveu, 1964], p. 3) as one wishes.*

For example, with $N = \{a, b, c, d, e\}$, $A_1 := \emptyset$, $A_2 := \{b, d\}$, $A_3 := \emptyset$, $A_4 := \{a, c, e\}$, the ordered set $\mathscr{M} := (A_1, A_2, A_3, A_4)$ is a 4-division of N. For each division, the nonempty subsets are evidently different and mutually disjoint, and between their cardinalities the following relation exists:

[10b] $$|N| = \sum_{i=1}^{m} |A_i| = |A_1| + |A_2| + \cdots + |A_m|.$$

Many identities are only the consequences of [10b]: one counts a set in two different ways, which gives a combinatorial proof of the identity which is to be examined.

Examples. (1) Let E be the set of nonempty subsets of $A := \{1, 2, \ldots, n+1\}$, and let us call $E_j\,(\subset E)$ the set of subsets of A for which the greatest element is $j\,(\geq 1)$. Evidently, $E = \sum_{j=1}^{n+1} E_j$. Now, $|E| = 2^{n+1} - 1$ and $|E_j| = 2^{j-1}$ (the number of subsets of $\{1, 2, \ldots, j-1\}$). Then, by using [10b] we obtain: $2^{n+1} - 1 = 1 + 2 + 2^2 + \cdots + 2^n$. More generally, for any integers $x, y, n \geq 1$, we could prove by a strictly combinatorial argument the well-known identity:

$$x^{n+1} - y^{n+1} = (x-y)(x^n + x^{n-1}y + x^{n-2}y^2 + \cdots + y^n).$$

(2) Let $Z = X + Y$ be a division of the set Z, $x := |X| \geq 0$, $y := |Y| \geq 0$. We denote E for the set of all $A \subset Z$ such that $|A| = n$ $(E = \mathfrak{P}_n(Z))$, and E_k for the set of all $B \in E$ such that $|B \cap X| = k$. Clearly, $E = \sum_{k=0}^{n} E_k$. Now, from $|E| = \binom{x+y}{n}$ and $|E_k| = \binom{x}{k}\binom{y}{n-k}$ follows the Vandermonde convolution (see p. 44):

$$\binom{x+y}{n} = \sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k}.$$

(3) With $Z = X + Y$ once again, let E be the set of functions f from $[n]$ into Z, and let E_k consist of all f such that $|f^{-1}(X)| = k$. We have $E = \sum_{k=0}^{n} E_k$, $|E| = (x+y)^n$, $|E_k| = \binom{n}{k} x^k y^{n-k}$. Therefore

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

(4) By considering E, the set of functions f from $\{x, y, z\}$ into $[n+1] = \{1, 2, 3, \ldots, n+1\}$ such that $f(x) < f(z)$, $f(y) < f(z)$, and the following subsets: (i) $E_k := \{f \mid f(z) = k+1\}$, (ii) $A := \{f \mid f(x) = f(y)\}$, (iii) $B := \{f \mid f(x) < f(y)\}$, (iv) $C := \{f \mid f(x) > f(y)\}$, we find $E = \sum_{k=1}^{n} E_k = A + B + C$, i.e., with [10b]: $|E| = \sum_{k=1}^{n} k^2 = \binom{n+1}{2} + \binom{n+1}{3} + \binom{n+1}{3} = \frac{1}{6} n(n+1)(2n+1)$. (See also p. 155 and Exercise 4, p. 220.)

THEOREM A. *Let $(a_1, a_2, \ldots, a_m)$ be a sequence of m integers ≥ 0 such that:*

$$a_1 + a_2 + \cdots + a_m = n, \qquad m \geq 1, \quad n \geq 0.$$

Then the number of divisions $\mathscr{A} = (A_1, A_2, \ldots, A_m)$ of N, $|N| = n$, such that $|A_i| = a_i$, $i \in [m]$, also called $(a_1, a_2, \ldots, a_m)$-divisions, is equal to (note that $0! = 1$):

[10c] $$\frac{n!}{a_1!\,a_2!\cdots a_m!} \quad \textit{and can be denoted by} \quad \binom{n}{a_1, a_2, \ldots, a_m}$$

or, even better, by:

[10c'] $$(a_1, a_2, \ldots, a_m).$$

Until recently one said that $\mathscr{A}$ was a *permutation with repetition* of a_1 elements of N, a_2 elements of N, etc. Notation [10c'] which we introduce here and whose virtues we wish to recommend now, is not standard yet, but seems to become more and more in use. Anyway, it has the qualities of a good notation (cf. p. 8) and it is hard to imagine a simpler one. Moreover, it has the advantage over [10c] of being *coherent* with the classical notation of the *binomial* coefficients. In fact, if we use [10c] for the case of binomial coefficients, we get the notation $\binom{n}{k, n-k}$ for $\binom{n}{k}$, which is undesirable. On the contrary, it seems good to extend the usual notation for the binomial coefficients in the case of $\binom{x}{k}$, with x real or complex variable, by the following notation:

[10c''] $$\binom{x}{k_1, k_2, \ldots, k_m} := \frac{(x)_{k_1+k_2+\cdots+k_m}}{k_1!\,k_2!\cdots k_m!} = \frac{x(x-1)(x-2)\cdots(x-k_1-k_2-\cdots-k_m+1)}{k_1!\,k_2!\cdots k_m!},$$

because in this case, for $a_1 + a_2 + \cdots + a_m = n$, we have in our notations:

$$(a_1, a_2, \ldots, a_m) = \binom{n}{a_1, a_2, \ldots, a_m} = \binom{n}{a_2, a_3, \ldots, a_m}, \text{ etc.},$$

which harmonizes perfectly with the binomial and multinomial notations. (This last notation can be found in the *Repertorium* by [*Pascal, 1910], I, p. 51.)

■ As $\mathscr{A}$ is ordered, giving $\mathscr{A}$ means first giving A_1, then A_2, then A_3, then A_4, etc. Now the number of possible choices for $A_1 \subset N$, $|N| = n$, $|A_1| = a_1$, equals $\binom{n}{a_1}$, by [5a] (p. 8). Such a choice being made, the number of possible choices for $A_2 \subset N \setminus A_1$, $|N \setminus A_1| = n - a_1$, $|A_2| = a_2$, is $\binom{n-a_1}{a_2}$, etc. The required number (of the possible $\mathscr{A}$) hence is equal to:

$$\binom{n}{a_1}\binom{n-a_1}{a_2}\cdots\binom{n-a_1-\cdots-a_{k-1}}{a_k},$$

which is equal to [10c] after simplification. ■

The notation [10a] suggests us to write $U - V$ instead of $U \setminus V$, as in [1h] (p. 2), if $V \subset U$. In other words, for three subsets U, V, W of N:

[10d] $\quad W = U - V \Leftrightarrow U = V + W \Leftrightarrow W = U \setminus V \quad \text{and} \quad V \subset U.$

The following notation also originates from [10a]:

[10e] $\quad A_1 + A_2 + \cdots + A_l = N \Leftrightarrow A_1 \cup \cdots \cup A_l = N \quad \text{and} \quad A_i \cap A_j = \varnothing, \quad 1 \leqslant i < j \leqslant l.$

THEOREM B (multinomial identity). *If $x_1, x_2, \ldots, x_m$ are commuting elements of a ring ($\Leftrightarrow x_i x_j = x_j x_i$, $1 \leqslant i < j \leqslant m$), then we have for all integers $n \geqslant 0$:*

[10f] $$\Big(\sum_{i=1}^{m} x_i\Big)^n = (x_1 + x_2 + \cdots + x_m)^n = \sum (a_1, a_2, \ldots, a_m)\, x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m},$$

the last summation takes place over all m-tuples $(a_1, a_2, \ldots, a_m)$ of positive or zero integers $a_i \geqslant 0$ such that $a_1 + a_2 + \cdots + a_m = n$.

Because of this,

DEFINITION B. *The numbers:*

$$(a_1, a_2, \ldots, a_m) = \frac{(a_1 + a_2 + \cdots + a_m)!}{a_1!\, a_2! \cdots a_m!} = \frac{n!}{a_1!\, a_2! \cdots a_m!}$$

are called multinomial coefficients.

For n, m fixed, the number of multinomial coefficients equals the number of solutions of $a_1 + \cdots + a_m = n$, which is $\binom{n+m-1}{n}$, by Theorems B and D (p. 15). A table of the multinomial coefficients can be found on p. 309.

■ We argue as in the proof of Theorem A (p. 12). Let:

[10g] $$(x_1 + x_2 + \cdots + x_m)^n = P_1 P_2 \cdots P_n = \sum c_{a_1, a_2, \ldots, a_m} x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m},$$

with $P_i = x_1 + x_2 + \cdots + x_m$, the summation taking place over all systems of integers $(a_1, a_2, \ldots, a_m)$ that occur as exponents of the terms on the right-hand side of [10g]. Obtaining $x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m}$ in the expansion of the product $P_1 P_2 \cdots$ is equivalent to giving a *division* of the set $\{P_1, P_2, \ldots, P_n\}$ into subsets $A_1, A_2, \ldots, A_m$ such that $|A_i| = a_i$, $i \in [m]$. This we do with the understanding that this division corresponds to multiplying the 'x_1' of the a_1 factors $P_i \in A_1$ by the 'x_2' of the a_2 factors $P_i \in A_2$, etc. (If $a_i = 0$, then one just multiplies by 1). Hence, on one hand:

[10h] $\quad a_1 + a_2 + \cdots + a_m = n, \qquad a_i \in \mathbb{N}, \quad i \in [m];$

on the other hand, the number of terms $x_1^{a_1} x_2^{a_2} \cdots$, where the a_i are fixed such that [10h] holds, is equal to $(a_1, a_2, \ldots, a_m)$, by [10c]. ■

Thus, $(x_1 + x_2 + \cdots + x_m)^2 = \sum_{i=1}^{m} x_i^2 + 2 \sum_{1 \leqslant i < j \leqslant m} x_i x_j$, because the solutions of $a_1 + \cdots + a_m = 2$ are of the form: (I) $a_i = 2$, $a_j = 0$ if $j \neq i$, in which case [10c] $= 1$; (II) $a_i = a_j = 1$ if $i \neq j$, $a_k = 0$ if $k \neq i, j$, in which case [10c] $= 2$. In the same manner, $(x_1 + x_2 + \cdots)^3 = \sum x_i^3 + 3 \sum x_i x_j^2 + 6 \sum x_i x_j x_k$, $(x_1 + x_2 + \cdots)^4 = \sum x_i^4 + 4 \sum x_i x_j^3 + 6 \sum x_i^2 x_j^2 + 12 \sum x_i x_j x_k^2 + 24 \sum x_i x_j x_k x_l$. Moreover, the number of $\sum$'s in the expansion of $(x_1 + x_2 + \cdots)^n$ is exactly $p(n)$, the number of partitions of n, p. 94. (See also Exercise 28, p. 125, and Exercise 9, p. 158.) Multinomial coefficients enjoy congruence properties, analogous to [6c, g] p. 14, the proof being very similar.

THEOREM C. *For any prime number p and $a_1 + a_2 + a_3 + \cdots = p$, we have*

$$(a_1, a_2, a_3, \ldots) \equiv 0 \pmod{p},$$

except $(p, 0, 0, \ldots) = (0, p, 0, \ldots) = \cdots = 1$.

In other words, for variables $x_1, x_2, \ldots, x_m$:

$$(x_1 + x_2 + \cdots + x_m)^p = x_1^p + x_2^p + \cdots + x_m^p.$$

Definition C. *A non-ordered (finite) set $\mathscr{P}$ of p blocks of N (= p-system of N, cf. p. 3), $\mathscr{P}\in\mathfrak{P}^p(N)$, is called a* partition *of N, or* p-partition *if one wants to specify the number of its blocks, if the union of all blocks of $\mathscr{P}$ equals N, and if these blocks are mutually disjoint.*

Hence in a *partition*, as opposed to a *division* (1) no 'subset' is empty; (2) the 'subsets' are not labelled.

Similar to [10a], we denote for such a partition, in order to express the fact that $B, B'\in\mathscr{P} \Rightarrow B\cap B'=\emptyset$:

$$N=\sum_{B\in\mathscr{P}} B, \qquad \forall B\in\mathscr{P}, \quad |B|\geq 1.$$

Evidently there is a bijection between the set of equivalence relations of N and the set of partitions of N; we just associate with every equivalence relation E the partition whose blocks are the equivalence classes of E.

Theorem D. *Let f be a map of M into N, $f\in N^M$. The set of the nonempty pre-images $f^{-1}(y)$, $y\in N$ (p. 5) constitutes a partition of M, which is called the* partition induced by f *on M.*

This is evident. It follows in particular, for each $f\in N^M$ that:

[10f] $$|M|=\sum_{y\in N}|f^{-1}(y)|.$$

1.11. Bound Variables

It is well known that a finite sum of n terms $x_1, x_2, \dots, x_n$, real numbers, or, more generally, in a ring, is denoted by $x_1+x_2+\cdots+x_n$ (such a way of writing, of course, does not mean at all that $n\geq 3$), or even better:

[11a] $$\sum_{i=1}^{n} x_i.$$

We generalize this notation. Let m be an integer ≥ 1, and f a real-valued function (or, more generally, with values in a ring) defined for all *points* (= m-tuples) $c=(c_1, c_2, \dots, c_m)$ of a product set:

[11b] $$E:=E_1\times E_2\times\cdots\times E_m.$$

(Frequently we will have $E_1=E_2=\cdots=E_m=\mathbf{N}$.) If f is only defined on $\Omega(\subset E)$, it will be extended to the whole of E by 0, in most cases. Let us consider a finite set $F\subset E$. The expression S, denoted in any of the following four ways:

[11c] $$(S=)\sum_{c\in F} f(c)=\textstyle\sum_{c\in F} f(c) = \displaystyle\sum_{(c_1,c_2,\dots,c_m)\in F} f(c_1,c_2,\dots,c_m) = \textstyle\sum_{(c_1,c_2,\dots,c_m)\in F} f(c_1,c_2,\dots,c_m),$$

equals *by definition* the finite sum of the values of f in each point c of F, which is called the *summation set*. If $F\cap E=\emptyset$, we give S the value 0.

[11d] Empty sum convention: $\sum_{c\in\emptyset} f(c)=0$.

Sometimes we qualify S by saying that it is a *multiple sum of order* m. For $m=1, 2, 3, \dots$, one says usually *simple, double or triple* sum.

It is clear that the value [11c] of S is completely determined by F and f. Thus, S *does not depend* on $c=(c_1, c_2, \dots, c_m)$, even though c occurs in formula [11c]. For this reason, the letters c or $(c_1, c_2, \dots, c_m)$ are called *bound variables* of the summation (*dummy* or *dead* are also used synonymously for bound). It is useful to note the analogy with the notation $I=\int_a^b f(x)\,dx$ of the integral, in which x is also a bound (real) variable, while I only depends on a, b and f.

Usually, the summation set F is defined by a certain number of conditions or restrictions, $\mathscr{C}_1, \mathscr{C}_2, \dots, \mathscr{C}_r$ on the $c_1, c_2, \dots, c_m$; these conditions will just be translated by saying that the point c belongs to the subsets $F_1, F_2, \dots, F_r$. We will therefore write any of the following:

[11e] $$(S=)\sum_{\mathscr{C}_1,\mathscr{C}_2,\dots,\mathscr{C}_r} f(c)=\textstyle\sum_{\mathscr{C}_1,\mathscr{C}_2,\dots,\mathscr{C}_r} f(c) = \displaystyle\sum_{c\in F_1\cap F_2\cap\cdots\cap F_r} f(c)=\textstyle\sum_{c\in F_1\cap F_2\cap\cdots\cap F_r} f(c).$$

For example, [11f] is equivalent with [11a]:

[11f] $$\sum_{1\leq i\leq n} x_i \quad\text{or}\quad \textstyle\sum_{1\leq i\leq n} x_i.$$

If the expression for the $\mathscr{C}_i$ is not very simple, it is better to avoid writing it underneath or on the side of the summation sign $\sum$, and following it. In that case one uses a phrase like "the summation takes place over all c such that ...".

Quite often one needs some letters different from $c_1, c_2, \dots$, say $d_1, d_2, \dots$

in the detailed description of the conditions $\mathscr{C}_i$. It is important to distinguish these from the bound variables, especially in the case that we wish to use notation [1e]. Therefore we introduce the

[1g] DOT CONVENTION: *every letter with a dot underneath stands for a bound variable.*

Of course, we do not have to dot every bound variable: in [1f], for example, there is but one possible interpretation. We must try to limit the dots to the cases where there is possible danger of confusion or ambiguity (examples follow). Furthermore, each variable needs only to be pointed *once*, and not every time it appears in the conditions $\mathscr{C}_1, \mathscr{C}_2, \ldots$. In general, however, we are not at all embarrassed by excesses, as far as this is concerned. The use of dots under the bound variables is imposed upon us by our total and absolute rejection of the notation by repeated $\sum$ signs (which is still commonly used), for any multiple sum of order m (Theorem B below).

Before demonstrating the preceding by examples, we still put the

[1h] NONNEGATIVE INTEGER CONVENTION: *in the sequel of this book each bound variable will represent an integer ≥ 0 unless stated otherwise.*

Now we give the following results:

THEOREM A *(associativity). For all partitions $\varpi = (\Gamma_1, \Gamma_2, \ldots, \Gamma_q)$ of Γ, $\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_q$, we have:*

[1i] $$S := \sum_{\gamma \in \Gamma} f(\gamma) = \sum_{1 \leq j \leq q} \Big(\sum_{\gamma \in \Gamma_j} f(\gamma) \Big)$$

THEOREM B *(analogue of the Fubini theorem for multiple integrals):*

[1j] $$\sum_{(e_1, e_2) \in E_1 \times E_2} f(e_1, e_2) = \sum_{e_1 \in E_1} \Big(\sum_{e_2 \in E_2} f(e_1, e_2) \Big) = \sum_{e_2 \in E_2} \Big(\sum_{e_1 \in E_1} f(e_1, e_2) \Big)$$

[1k] $$\sum_{(e_1, e_2, e_3) \in E_1 \times E_2 \times E_3} f(e_1, e_2, e_3) = \sum_{e_1 \in E_1} \Big(\sum_{(e_2, e_3) \in E_2 \times E_3} f(e_1, e_2, e_3) \Big) = \sum_{e_1 \in E_1} \Big(\sum_{e_2 \in E_2} \Big(\sum_{e_3 \in E_3} f(e_1, e_2, e_3) \Big) \Big) = \text{etc.}$$

(For the number of possible '[illegible] formulas' see Exercise 20 on p. 228.)

Examples. (I) *To calculate, for $n \geq 0$ integer, the double sum:*

$$S := \sum_{e_1 + e_2 = n} e_1 e_2 .$$

We get, if we reduce it to a simple sum:

$$S = \sum_{0 \leq e_1 \leq n} e_1 (n - e_1) = n \sum_{0 \leq e_1 \leq n} e_1 - \sum_{0 \leq e_1 \leq n} e_1^2$$

$$= n \frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6} = \frac{n(n^2 - 1)}{6} .$$

(See also Exercise 28 on p. 85 for a generalization.)

(II) *To calculate, for a and b complex, $a, b, ab \neq 1$ and n an integer ≥ 0, the double sum:*

$$S := \sum_{0 \leq j \leq k \leq n} a^j b^k .$$

We can do this as follows, where we use Theorem B for the equality (*):

$$S \overset{(*)}{=} \sum_{0 \leq j \leq n} \Big(a^j \sum_{j \leq k \leq n} b^k \Big) = \sum_{0 \leq j \leq n} a^j \frac{b^{n+1} - b^j}{b - 1}$$

$$= \frac{b^{n+1}}{b - 1} \sum_{0 \leq j \leq n} a^j - \frac{1}{b - 1} \sum_{0 \leq j \leq n} (ab)^j$$

$$= \frac{b^{n+1}(a^{n+1} - 1)}{(b - 1)(a - 1)} - \frac{(ab)^{n+1} - 1}{(b - 1)(ab - 1)} .$$

We could also have started with $S = \sum_{0 \leq k \leq n} (b^k \sum_{0 \leq j \leq k} a^j)$.

(III) *For any finite set N, $|N| = n$, to calculate the double sum:*

$$S := \sum_{A, B \subset N} |A \cap B| .$$

(The summation is taken over all pairs of subsets $(A, B) \in \mathfrak{P}(N) \times \mathfrak{P}(N)$.) By Theorem B, we get for S:

$$\sum_{A \subset N} \Big(\sum_{B \subset N} |A \cap B| \Big) = \sum_{A \subset N} \Big(\sum_{0 \leq i \leq |A|} \Big(\sum_{|A \cap B| = i} |A \cap B| \Big) \Big).$$

Now it is easy to see, that the number of subsets $B (\subset N)$ such that

$|A\cap B| = i$, where A is fixed, equals $\binom{|A|}{i}\cdot 2^{n-|A|}$, which is the number of i-subsets of A times the number of subsets of $N - A$. Hence, as $\sum_{i=0}^{a} i\binom{a}{i} = a2^{a-1}$ (which results from taking the derivative of the polynomial $(1+x)^a = \sum_{i=0}^{a}\binom{a}{i}x^i$, and substituting 1 for x), which we use for equality (*), we get for S:

$$\sum_{A\subset N}\Big(2^{n-|A|}\sum_{0\le i\le |A|} i\binom{|A|}{i}\Big) \overset{(*)}{=} \sum_{A\subset N} 2^{n-|A|}\cdot|A|\cdot 2^{|A|-1} = 2^{n-1}\sum_{A\subset N}|A| = 2^{n-1}\cdot n2^{n-1} = n4^{n-1}.$$

More symmetrically, we could have said also:

$$S = \sum_{K\subset N}\Big(\sum_{A\cap B=K}|A\cap B|\Big) = \sum_{K\subset N} 3^{n-|K|}|K| = \sum_{0\le k\le n}\binom{n}{k}3^{n-k}k = n4^{n-1}.$$

(Furthermore, $\sum|A_1\cap\cdots\cap A_k| = n2^{k(n-1)}$, where $A_1, A_2,\ldots,A_k\subset N$.)

In certain cases, we can immediately lower the order of a summation by applying Theorems A and B.

Theorem C. *If* $f(c_1,c_2,\ldots,c_m) = f_1(c_1,c_2,\ldots,c_k)\cdot f_2(c_{k+1},\ldots,c_m)$, $0<k<m$, *then:*

[11l]
$$\sum_{(c_1,c_2,\ldots,c_m)\in\mathfrak{C}_1\times\mathfrak{C}_2\times\cdots\times\mathfrak{C}_m} f(c_1,c_2,\ldots,c_m) = \Big(\sum_{(c_1,\ldots,c_k)\in\mathfrak{C}_1\times\cdots\times\mathfrak{C}_k} f_1(c_1,\ldots,c_k)\Big)\times\Big(\sum_{(c_{k+1},\ldots,c_m)\in\mathfrak{C}_{k+1}\times\cdots\times\mathfrak{C}_m} f_2(c_{k+1},\ldots,c_m)\Big).$$

Particularly:

[11m]
$$\sum_{(c_1,\ldots,c_m)\in\mathfrak{C}_1\times\cdots\times\mathfrak{C}_m} g_1(c_1)\cdots g_m(c_m) = \Big(\sum_{c_1\in\mathfrak{C}_1} g_1(c_1)\Big)\Big(\sum_{c_2\in\mathfrak{C}_2} g_2(c_2)\Big)\cdots\Big(\sum_{c_m\in\mathfrak{C}_m} g_m(c_m)\Big).$$

It will be noticed that this theorem bears some analogy to the theorem on double integrals: if $\Delta=[a,b]\times[c,d]$ then $\iint_\Delta f(x)g(y)\,dx\,dy = (\int_a^b f(x)\,dx)(\int_c^d g(y)\,dy)$.

Clearly, everything that has been said in this section about the notation of finite sums, can be repeated, with the necessary changes, for any expression in which addition is replaced by an internal associative and commutative composition law on the range of f. Thus, we denote:

$\prod_{1\le k\le n} x_k$ for the product $x_1x_2\cdots x_n$;

$\bigcup_{1\le k\le n} A_k$ for the union $A_1\cup A_2\cup\cdots\cup A_n$;

$\bigcap_{1\le k\le n} A_k$ for the intersection $A_1\cap A_2\cap\cdots\cap A_n$.

Conventions [11g, h] still hold for $\prod$, $\bigcup$, $\bigcap$, but [11d] (p. 31) is replaced by [11n, o, p]:

[11n] EMPTY PRODUCT CONVENTION: $\prod_{c\in\emptyset} f(c) := 1$.

[11o] EMPTY UNION CONVENTION: $\bigcup_{c\in\emptyset} A(c) := \emptyset$, where $A(c)\subset N$.

[11p] EMPTY INTERSECTION CONVENTION: $\bigcap_{c\in\emptyset} A(c) := N$, where $A(c)\subset N$.

Example. *Compute, for n integer* ≥ 1, *the double product:*

$$P := \prod_{0\le k+j\le n} a^k b^j.$$

We can work this out as follows, using [5g] on p. 10 for (*):

$$P = \prod_{0\le l\le n}\Big(\prod_{k+j=l} a^k b^j\Big) = \prod_{0\le l\le n}\Big(\prod_{0\le k\le l} a^k b^{l-k}\Big) = \prod_{0\le l\le n}\big(a^{l(l+1)/2}\cdot b^{l(l+1)/2}\big) = \prod_{0\le l\le n}(ab)^{\binom{l+1}{2}} = (ab)^{\sum_{0\le l\le n}\binom{l+1}{2}} \overset{(*)}{=} (ab)^{\binom{n+2}{3}}.$$

More generally, it can be found without difficulty that the l-th order product $\prod a_1^{p_1}a_2^{p_2}\cdots a_l^{p_l}$, where $p_1+p_2+\cdots+p_l\le n$, has the value $(a_1a_2\cdots a_l)^q$ with $q=\binom{l+n}{l+1}$.

1.12. FORMAL SERIES

(I) *General remarks*

The concept of formal power series is a generalization of polynomial. We think the best is to sketch here the outlines of the theory, following Bourbaki ([*Bourbaki, *Algèbre*, chap. 4, 5, 1959], p. 52 (9); see also [*Dubreil (P. and M.-L.), 1964], p. 124–31, [*Lang, 1965], p. 146, [*Zariski, Samuel, II, 1960], p. 129); we will refer to this author for proofs and more details.

In this section, *each small Greek letter represents a finite sequence of k integers* ≥ 0, where k is an integer ≥ 1, which is given once and for all. Such a sequence is sometimes also called a *multi index*. Thus, if we write $\bar{k} = \mathbf{N}^{[k]}$, in which $[k] := \{1, 2, \ldots, k\}$, then $\alpha \in \bar{k}$ means that $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, where $\alpha_i \in \mathbf{N}$.

We may denote:

[12a] $\alpha! := \alpha_1!\alpha_2!\ldots\alpha_k!,$

$|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_k,$

[12a'] $c_\alpha := c_{\alpha_1, \alpha_2, \ldots, \alpha_k}, \quad t^\alpha := t_1^{\alpha_1} t_2^{\alpha_2} \ldots t_k^{\alpha_k}.$

We will consider the case of formal series in k variables over a field C (often $C = \mathbf{R}$ or $\mathbf{C}$).

DEFINITION A. *A formal power series f in k indeterminates (or variables) $t_1, t_2, \ldots, t_k$ over C is a formal expression of the following type:*

[12b]
$$f = f(t) = f(t_1, t_2, \ldots, t_k) = \sum_{\alpha \in \bar{k}} a_\alpha t^\alpha = \sum_{\alpha_1, \alpha_2, \ldots, \alpha_k \geq 0} a_{\alpha_1, \alpha_2, \ldots, \alpha_k} t_1^{\alpha_1} t_2^{\alpha_2} \ldots t_k^{\alpha_k},$$

where $a_\alpha = a_{\alpha_1, \alpha_2, \ldots, \alpha_k}$, the coefficients of f, form a multiple series of order k with values in C. Each expression $a_\mu t^\mu = a_{\mu_1, \mu_2, \ldots, \mu_k} t_1^{\mu_1} \ldots t_k^{\mu_k}$ is called a monomial of f. As the $\mu_1, \mu_2, \ldots, \mu_k$ are bound variables, they can have a dot underneath. We denote $\mathbf{C}[[t_1, t_2, \ldots, t_k]]$, or even better $\mathbf{C}_k[[t]]$, which is called the set of formal series f.

f is a *polynomial* if all coefficients except a finite number of them equal zero, which is usually formulated by saying "*almost all a_α are zero*". In

simple cases we sometimes avoid to write [12b] by using an *ellipsis mark*, three consecutive periods, especially if there is only one indeterminate. For example:

$$f = 1 + t + t^2 + \cdots := \sum_{n \geq 0} t^n \in \mathbf{R}_1[[t]] = \mathbf{R}[[t]].$$

Every power series in several variables, which is convergent in a certain polydisc, can be interpreted as a formal series. Conversely, with every formal series in several indeterminates can be associated with a power series that perhaps converges in the point 0 only. The following expansions:

[12c] $$\exp t := \sum_{n \geq 0} \frac{t^n}{n!}$$

[12d] $$\log(1 + t) := \sum_{n \geq 1} (-1)^{n-1} \frac{t^n}{n}$$

[12e] $$(1 + t)^x := \sum_{n \geq 0} \binom{x}{n} t^n = \sum_{n \geq 0} (x)_n \frac{t^n}{n!} \qquad (x \in \mathbf{C})$$

[12e'] $$(1 - t)^{-x} := \sum_{n \geq 0} \binom{-x}{n} (-1)^n t^n = \sum_{n \geq 0} \langle x \rangle_n \frac{t^n}{n!} = \sum_{n \geq 0} \binom{x}{n} t^n$$

can be as well considered as functions in their radius of convergence as well as certain formal series, which are called respectively, *formal exponential series, formal logarithm, formal binomial series* (in the 1st and 2nd form). Moreover, for [12e'] we have also, if x is an *integer* ≥ 1: $(1-t)^{-x} = \sum_{n \geq 0} \binom{n + x - 1}{x - 1} t^n$. Furthermore, the series [12e, e'] can also be interpreted as series in two indeterminates t and x.

From now on, in the sequel of this book, each power series must be considered as a formal series, unless explicitly stated otherwise.

As in the case of polynomials, $\mathbf{C}_k[[t]]$ becomes an integral domain, if we provide it with addition and multiplication as follows: for every $f = \sum a_\alpha t^\alpha$ and $g = \sum b_\alpha t^\alpha$ where $\alpha \in \bar{k}$:

[12f] $$f + g := \sum_{\mu \in \bar{k}} c_\mu t^\mu, \quad \text{where} \quad c_\mu := a_\mu + b_\mu$$

[12g] $$fg := \sum_{\mu \in \bar{k}} d_\mu t^\mu,$$

where

$$c_\nu = c_{\nu_1,\ldots,\nu_k} = \sum_{\mu+\lambda=\nu} a_\mu b_\lambda = \sum a_{\mu_1,\ldots,\mu_k} b_{\lambda_1,\ldots,\lambda_k},$$

the last summation taken over all sequences of integers ≥ 0, $(\mu_1,\ldots,\mu_k,\lambda_1,\ldots,\lambda_k)$ such that $\mu_1+\lambda_1=\nu_1,\ldots,\mu_k+\lambda_k=\nu_k$ (hence we have $(\nu_1+1)\ldots(\nu_k+1)$ terms in the last summation).

The *homogeneous part of f of degree m* is the formal polynomial:

[12h] $$f_{(m)} := \sum_{|\nu|=m} a_\nu t^\nu = \sum_{\nu_1+\cdots+\nu_k=m} a_{\nu_1,\ldots,\nu_k} t_1^{\nu_1}\ldots t_k^{\nu_k}.$$

The *constant term* of f is $a_0=f_{(0)}$, also denoted by $f(0)$. The *order* of f (which we suppose different from the series 0, all whose coefficients equal zero), is the smallest integer $n\geq 0$, such that $f_{(n)}\neq 0$. For example, $\omega(t_1t_2+(t_1t_2)^2+\cdots)=2$. Clearly, $\omega(fg)=\omega(f)+\omega(g)$. The series 1 is the series all whose terms are zero except the constant term, which equals 1. For example, by [12e, e'], we have formally:

$$(1+t)^x(1+t)^{-x}=1,$$

which results from the same property for the associated convergent expansions.

(II) *Summable families of formal series*

Let $(f_i)_{i\in L}$ be a family of formal series of $C[[t]]$ (often $L=N$ or N^k).

DEFINITION B. *A family $(f_i)_{i\in L}$ is called summable, if for each sequence $\nu\in N^k$, the coefficient $a_{i,\nu}$ of t^ν in f_i equals 0 for* almost all $i\in L$ *(except a finite number, see p. 26). The sum $g=\sum_{\nu} b_\nu t^\nu$ of this family is then defined by:*

[12i] $b_\nu :=$ *the coefficient of t^ν in the finite sum $\sum f_i$, where $i\in L$, and $\omega(f_i)\leq|\nu|$.*

We denote $g=\sum_{i\in L} f_i$.

For $L=N$, (f_i) is evidently summable if and only if the order $\omega(f_i)$ tends to infinity, when i tends to infinity.

We give two examples. (1) The family

$$f_{\mu_1,\mu_2} := \sum_{(\mu_1,\mu_2)\in N^2} t_1^{(\mu_1+1)} t_2^{2(\mu_2+1)} = \sum_{\mu_1\geq 0} t_1^{(\mu_1+1)} \sum_{\mu_2\geq 0} t_2^{2(\mu_2+1)} = t_1 t_2^2 (1-t_1)^{-1}(1-t_2^2)^{-1}$$

is summable, $(\mu_1,\mu_2)\in N^2$. If in the definition of f_{μ_1,μ_2} the exponents (μ_1+1) and $2(\mu_2+1)$ are replaced by μ_1 and $2\mu_2$, then the family is not summable anymore. (2) The family $f_{(n)}$ of homogeneous parts of f, [12h], is summable, and $f=\sum_{n\geq 0} f_{(n)}$. Moreover, we have the 'Cauchy product' form for the series h, which is the product of f and g:

[12j] $$h=fg=\sum h_{(n)}, \quad h_{(n)}=\sum_{0\leq l\leq n} f_{(l)}g_{(n-l)}.$$

THEOREM A (*associativity*). *Let be given a summable family of formal series $(f_i)_{i\in L}$ with sum g, and $(L_j)_{j\in J}$ a division (p. 25), possibly infinite, of L: $L=\sum_{j\in J} L_j$. Then every subfamily $(f_i)_{i\in L_j}$ is summable with sum $g_j := \sum_{i\in L_j} f_i$, and we have $g(=\sum_{i\in L} f_i)=\sum_{j\in J} g_j$.*

THEOREM B (*products*). *Let $(f_i)_{i\in L}$ and $(g_m)_{m\in M}$ be two summable families. Then the family $(f_i g_m)_{(i,m)\in L\times M}$ is summable, and we have $\sum_{(i,m)\in L\times M} f_i g_m = (\sum_{i\in L} f_i)(\sum_{m\in M} g_m)$.*

The generalization to a finite product is evident.

(III) *Multipliable families of formal series*

DEFINITION C. *A family of formal series $(f_i)_{i\in L}$ is called* multipliable *if for almost all (p. 26) $i\in L$, firstly the constant term of f_i equals 1, secondly the coefficient $a_{i,\nu}$ of t^ν in f_i equals 0, for each sequence $\nu\in N^k$ such that $|\nu|\geq 1$. The* product *$g=\sum_{\nu} b_\nu t^\nu$ of this family is then defined by:*

[12k] $b_\nu :=$ *the coefficient of t^ν in the finite product $\prod f_i$, where $i\in L$, and $\omega(f_i-f_i(0))\leq|\nu|$.*

We denote $g=\prod_{i\in L} f_i$.

For $L=N$, (f_i) is multipliable if the order $\omega(f_i-f_i(0))$ tends to infinity, when l tends to infinity. For example, $f_l := (1+t_1 t_2^l)$ is multipliable. Every *finite* family is evidently multipliable, and we get back definition [12g] for the product. Explicitly, for *one single* variable t and *one sequence* (f_i) of formal series, $i=1,2,\dots$, $f_i := \sum_{n\ge 0} a_{i,n} t^n$, we have, if we write out the bound variables n_i completely in (*):

$$[12l]\quad \prod_{i\ge1} f_i \overset{(*)}{=} \prod_{i\ge1}\left(\sum_{n_i\ge0} a_{i,n_i} t^{n_i}\right) = \sum_{n_1,n_2,\dots\ge0} a_{1,n_1}a_{2,n_2}\cdots t^{n_1+n_2+\cdots} = \sum_{n\ge0} t^n\left(\sum_{n_1+n_2+\cdots=n} a_{1,n_1}a_{2,n_2}\cdots\right),$$

where the last summation makes sense, because it contains only a finite number of terms (cf. Definition C). (On this subject, see also p. 130.)

(IV) *Substitution (also called composition) of formal series*

THEOREM C. *Let $(g_i)_{i\in[p]}$ be p formal series $\in C_q[[t]]$ without constant terms: $\omega(g_i)\ge1$. We can 'substitute' g_i for t_i, $i\in[p]$, into every formal series $f=\sum_{\mu\in\mathbb{N}^p} a_\mu t^\mu\in C_p[[t]]$. In this way we obtain a new formal series, called the* **composition** *of f and g, and denoted $f(g_1,g_2,\dots,g_p)$ or $f\circ g$, which belongs again to $C_q[[t]]$. By definition, $f\circ g$ equals the sum of the summable family $a_{\mu_1,\dots,\mu_p}(g_1)^{\mu_1}\cdots(g_p)^{\mu_p}$, where $\mu=(\mu_1,\mu_2,\dots,\mu_p)\in\mathbb{N}^p$.*

For example, using [12c, d], it can be verified that:

$$\log(\exp t)=t,\qquad \exp\{\log(1+t)\}=1+t.$$

Now we want to find the formal expansion of $h:=(1+t_1+t_2+\cdots+t_q)^x\in\mathbb{R}_q[[t]]$. Applying Theorem C, with $f:=(1+u)^x\in\mathbb{R}_1[[u]]$, $g:=t_1+t_2+\cdots+t_q\in\mathbb{R}_q[[t]]$, we get by using [12e] (p. 37) for equality (*) and [10f] (p. 28) for (**):

$$h\overset{(*)}{=}\sum_{n\ge0}\binom{x}{n}g^n=\sum_{n\ge0}\binom{x}{n}(t_1+\cdots+t_q)^n\overset{(**)}{=}\sum_{n\ge0}\left\{\binom{x}{n}\sum_{\nu_1+\cdots+\nu_q=n}\frac{n!}{\nu_1!\cdots\nu_q!}t_1^{\nu_1}\cdots t_q^{\nu_q}\right\},$$

which gives after simplifications:

$$[12m]\quad (1+t_1+t_2+\cdots+t_q)^x=\sum_{\nu_1,\dots,\nu_q\ge0}(x)_{\nu_1+\nu_2+\cdots+\nu_q}\frac{t_1^{\nu_1}t_2^{\nu_2}\cdots t_q^{\nu_q}}{\nu_1!\,\nu_2!\cdots\nu_q!}=\sum_{\nu\in\mathbb{N}^q}(x)_{|\nu|}\frac{t^\nu}{\nu!}=\sum_{\nu_1,\dots,\nu_q\ge0}\binom{x}{\nu_1,\nu_2,\dots,\nu_q}t_1^{\nu_1}t_2^{\nu_2}\cdots t_q^{\nu_q},$$

with the notation [10e'] (p. 27).

Similarly, we obtain:

$$[12m']\quad (1-t_1-t_2-\cdots-t_q)^{-x}=\sum_{\nu_1,\dots,\nu_q\ge0}\langle x\rangle_{\nu_1+\nu_2+\cdots+\nu_q}\frac{t_1^{\nu_1}t_2^{\nu_2}\cdots t_q^{\nu_q}}{\nu_1!\,\nu_2!\cdots\nu_q!}=\sum_{\nu\in\mathbb{N}^q}\langle x\rangle_{|\nu|}\frac{t^\nu}{\nu!}=\sum_{\nu_1,\dots,\nu_q\ge0}\left\langle\begin{matrix}x\\ \nu_1,\nu_2,\dots,\nu_q\end{matrix}\right\rangle t_1^{\nu_1}t_2^{\nu_2}\cdots t_q^{\nu_q},$$

using an evident extension of the notations [7b] (p. 16) and [10e'] (p. 27).

We can also establish, using multinomial coefficients $(\nu):=(\nu_1,\nu_2,\dots,\nu_q)$ of [10e'] (p. 27), the corresponding expansions for log:

$$\log(1+t_1+t_2+\cdots+t_q)=\sum_{\nu_1+\nu_2+\cdots+\nu_q\ge1}(-1)^{\nu_1+\cdots+\nu_q-1}\times\frac{(\nu_1,\nu_2,\dots,\nu_q)}{\nu_1+\nu_2+\cdots+\nu_q}t_1^{\nu_1}t_2^{\nu_2}\cdots t_q^{\nu_q}=\sum_{|\nu|\ge1}(-1)^{|\nu|-1}\frac{(\nu)}{|\nu|}t^\nu,$$

$$-\log(1-t_1-t_2-\cdots-t_q)=\sum_{|\nu|\ge1}\frac{(\nu)}{|\nu|}t^\nu.$$

(V) *Transformations of formal series*

With every formal series $f=\sum_{n\ge0}a_nt^n$ in one **indeterminate** t, we can associate the *formal derivative*, denoted by:

$$[12n]\quad Df=\frac{df(t)}{dt}=\sum_{n\ge0}na_nt^{n-1}=\sum_{n\ge0}(n+1)a_{n+1}t^n.$$

and also the *formal primitive*:

$$[12o]\qquad Pf=\int_0^t f(x)\,dx=\sum_{n\geq 0} a_n\frac{t^{n+1}}{n+1}.$$

All the usual properties hold: $DPf=f$, $D(fg)=(Df).g+f.(Dg)$, etc.

The iterates of these operators can easily be found. For the derivation we have:

$$D^k f=\sum_{n\geq k}(n)_k a_n t^{n-k}=\sum_{m\geq 0}\langle m+1\rangle_k a_{m+k} t^m,$$

and for the primitivation we have:

$$P^k f=\sum_{n\geq 0} a_n\frac{t^{n+k}}{\langle n+1\rangle_k}=$$

$$=\sum_{m\geq k} a_{m-k}\frac{t^m}{(m)_k}=\int_0^t\frac{(t-x)^{k-1}}{(k-1)!}f(x)\,dx.$$

These concepts can be generalized without difficulties to more indeterminates. For example, for $f=\sum_{\nu\in N^k} a_\nu t^\nu$ and $\alpha\in N^k$, we define:

$$[12p]\qquad D^\alpha f:=\frac{\partial^{\alpha_1+\alpha_2+\cdots+\alpha_k}}{\partial t_1^{\alpha_1}\partial t_2^{\alpha_2}\cdots\partial t_k^{\alpha_k}}f(t_1,t_2,\ldots,t_k)$$

$$:=\sum_{\nu_1,\ldots,\nu_k\geq 0}(\nu_1)_{\alpha_1}\cdots(\nu_k)_{\alpha_k}a_{\nu_1,\ldots,\nu_k}t_1^{\nu_1-\alpha_1}\cdots t_k^{\nu_k-\alpha_k}.$$

We mention here also the transformation that associates to every double series $f(x,y)=\sum_{m,n\geq 0}a_{m,n}x^m y^n$ its *diagonal series* $\varphi(t)=\sum_{n\geq 0}a_{n,n}t^n$. When $f(x,y)$ converges, we have ([Hautus, Klarner, 1971]):

$$[12q]\qquad \varphi(t)=\frac{1}{2\pi i}\int_{|z|=\varepsilon} f\left(z,\frac{t}{z}\right)\frac{dz}{z},$$

where ε and $|t|$ are sufficiently small, so that $f(x,y)$ is regular for $|x|<\varepsilon$ and $|y|<|t|/\varepsilon$. In general, it is tantamount to saying that the circle $|z|=\varepsilon$ contains all the poles of $f(z,t/z)$ that tend to 0 when t tends to 0. For instance, for $f(x,y)=\sum_{m,n}(m,n)x^m y^n=(1-x-y)^{-1}$, where the $(m,n)=\binom{m+n}{m}$ are the binomial coefficients in the symmetrical notation (p. 8),

the diagonal $\varphi(t)=\sum_{n\geq 0}\binom{2n}{n}t^n$ equals the residue of $(1-z-t/z)^{-1}z^{-1}$ in the point $z=(1-(1-4t)^{1/2})/2$, in other words $(1-4t)^{-1/2}$. This result is of course well-known (see Exercise 22 (1), p. 81).

(VI) *Formal Laurent series*

These series are written analogously to the preceding [12b] (p. 36), but here the indices and the exponents $\nu_1,\nu_2,\ldots,\nu_k$ can take all integer values $\gtrless 0$, with the condition that the coefficients $a_{\nu_1,\ldots,\nu_k}$ that contain at least one index <0, are *almost all zero*. For example:

(1) With one single indeterminate t: $(t^2+t^3+\cdots)^{-1}=(t^2(1-t)^{-1})^{-1}=t^{-2}-t^{-1}$.

(2) With two indeterminates t_1 and t_2: $\sum t_1^{\nu_1}t_2^{\nu_2}$, $\nu_1\leq\nu_2\leq 2\nu_1+10$, where the integers ν_1,ν_2 can be negative as well as positive or zero.

All the preceding operations, summable families, derivation, etc., can be easily done for such series.

(VII) *Formal series in 'noncommutative' indeterminates* ([Schützenberger, 1961])

Let $\mathfrak{X}^*$ stand for the free monoid generated by $\mathfrak{X}$ (see p. 18) and let $f(\mu)=a_\mu$ be a map from $\mathfrak{X}^*$ into a certain ring A (μ is a *word* over $\mathfrak{X}$). If we write f as a formal series: $f=\sum_{\mu\in\mathfrak{X}^*}a_\mu\mu$, then the set $A^{\mathfrak{X}^*}$ of these maps f becomes an algebra called the *monoid algebra* $\mathfrak{X}^*$ if, for $g=\sum_{\mu\in\mathfrak{X}^*}b_\mu\mu$, we put $f+g:=\sum_{\mu\in\mathfrak{X}^*}(a_\mu+b_\mu)\mu$ and $fg:=\sum_{\mu\in\mathfrak{X}^*}c_\mu\mu$, where $c_\mu=\sum a_\kappa b_\lambda$, the finite summation being taken over all pairs (κ,λ) of words such that $\kappa\lambda=\mu$, in the sense of the juxtaposition product of p. 18. If $\mathfrak{X}$ is finite and if one considers the Abelian words of $\mathfrak{X}$, then the ordinary formal series studied above are found back again.

1.13. GENERATING FUNCTIONS (abbreviated GF)

(I) *Simple sequences*

DEFINITION. *Let be given a real or complex sequence (in this book actually often consisting of positive integers with a combinatorial meaning), then we call* ordinary GF, exponential GF, *and more generally,* GF *according* *to* Ω_n *of the sequence* a_n, *the following three formal series* Φ, Ψ *and* Φ_Ω, *respectively, where* Ω_n *is a fixed given sequence:*

[13a] $$\Phi(t):=\sum_{n\ge 0} a_n t^n,\quad \Psi(t):=\sum_{n\ge 0} a_n\frac{t^n}{n!},\quad \Phi_\Omega(t):=\sum_{n\ge 0}\Omega_n a_n t^n.$$

The most interesting case is that where (at least) one of the entire series **[13a]** has a positive nonzero radius of convergence R, and converges for $|t|<R$ to a composition of elementary known functions; in this case the properties of these functions can be used to give new information about the a_n. (For a detailed study of the relation between a_n and their *GF*, the reader is referred to any work on difference calculus; for example [*Jordan (Ch.), 1947] or [*Milne Thomson, 1933].)

Example A. $a_n:=\binom{x}{n}$, where $x\in\mathbb{R}$ or $\mathbb{C}$. Then $\Phi(t)=\sum_{n\ge 0}\binom{x}{n}t^n=\sum_{n\ge 0}(x)_n t^n/n!=(1+t)^x$, which converges for $|t|<1$ (if $x\in\mathbb{C}$ one chooses the value of $\Phi(t)$ that equals 1 for $t=0$). If we compare the coefficients of $t^n/n!$ in the first and the last member of equalities **[13b]**:

[13b] $$\sum_{n\ge 0}(x+y)_n\frac{t^n}{n!}=(1+t)^{x+y}=(1+t)^x(1+t)^y=\left(\sum_{k\ge 0}(x)_k\frac{t^k}{k!}\right)\left(\sum_{l\ge 0}(y)_l\frac{t^l}{l!}\right),$$

we obtain the *Vandermonde* convolution, in two forms:

[13c] $$(x+y)_n=\sum_{0\le k\le n}\binom{n}{k}(x)_k(y)_{n-k},$$

[13c'] $$\binom{x+y}{n}=\sum_{0\le k\le n}\binom{x}{k}\binom{y}{n-k}$$

(see also p. 26). Similarly, one shows, using $\sum_{n\ge 0}\langle x\rangle_n(t^n/n!)=(1-t)^{-x}$:

[13d] $$\langle x+y\rangle_n=\sum_{0\le k\le n}\binom{n}{k}\langle x\rangle_k\langle y\rangle_{n-k},$$

[13d'] $$\left\langle {x+y \atop n}\right\rangle=\sum_{0\le k\le n}\left\langle {x \atop k}\right\rangle\left\langle {y \atop n-k}\right\rangle.$$

Example B. Fibonacci numbers. These are integers F_n defined by:

[13e] $$F_n=F_{n-1}+F_{n-2},\quad n\ge 2;\qquad F_0=F_1=1.$$

We want to find the ordinary GF, $\Phi=\sum_{n\ge 0}F_nt^n$:

$$\Phi=1+t+\sum_{n\ge 2}(F_{n-1}+F_{n-2})t^n=1+t\Phi+t^2\Phi.$$

Comparing the first and the last member of these equalities we obtain:

[13f] $$\Phi=\sum_{n\ge 0}F_nt^n=\frac{1}{1-t-t^2}.$$

If we decompose this rational function into partial fractions, putting the roots of $1-t-t^2=0$ equal to $-\alpha,-\beta$, we get:

[13g] $$\Phi=\frac{1}{\beta-\alpha}\left(\frac{\beta}{1-\beta t}-\frac{\alpha}{1-\alpha t}\right)=\frac{1}{\sqrt5}\left(\sum_{n\ge 0}\beta^{n+1}t^n-\sum_{n\ge 0}\alpha^{n+1}t^n\right).$$

Hence, identifying the coefficients of t^n in [13f, g]:

[13h] $$F_n=\frac{\beta^{n+1}-\alpha^{n+1}}{\sqrt5},$$

where

$$\alpha:=\frac{1-\sqrt5}{2},\qquad \beta:=\frac{1+\sqrt5}{2}.$$

(One can take also as initial conditions $F_0=0$, $F_1=1$ [*Hardy, Wright, 1960], p. 148, in which case $\Phi=t(1-t-t^2)^{-1}$ and $F_n=(\beta^n-\alpha^n)/\sqrt5$.) Here we find the golden ratio, $\beta=1.61803\ldots$ of the Renaissance architects.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
F_n	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987

n	16	17	18	19	20	21	22	23	24	25
F_n	1597	2584	4181	6765	10946	17711	28657	46368	75025	121393

Moreover, if we let $\lfloor x\rceil$ denote the integer closest to x (x not supposed to be half-integral), then [13h] shows easily that $F_n=\lfloor\beta^{n+1}/\sqrt5\rceil$.

The Fibonacci numbers have a simple combinatorial meaning: F_{n+1} is the number of subsets of $[n]=\{1,2,\ldots,n\}$ such that *no two elements are adjacent* (Subsets with 0 or 1 element are convenient). In fact, according to [5d] (p. 21), the number F_{n+1} of such subsets equals $\sum_p \binom{n+1-p}{p}$. Hence, it follows that $F_{n+2}=\sum\binom{n-p}{p-1}+\sum\binom{n-p}{p}=F_{n+1}+F_n$ (by [5c], p. 10) and $F_0=F_1=1$. Thus, the sequences F_n and F_n coincide, because they satisfy the same defining recurrence relation. (See also Exercise 13, p. 76, and Exercise 31, p. 86.) It can also be shown that the number G_n of subsets of $[\bar{n}]$ (p. 24) such that any two points are not adjacent, equals F_n+F_{n-2} (subset $\emptyset$ is convenient), in other words $G_n=\alpha^n+\beta^n$, $G_n=G_{n-1}+G_{n-2}$ and $\sum_{n\geq 1}G_n t^n=(t+2t^2)(1-t-t^2)^{-1}$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
G_n	0	1	3	4	7	11	18	29	47	76	123	199	322

More generally, defining $(1-t-t^{l+1})^{-1}=:\sum_{n\geq 0}F(n,l)\,t^n$, it can be proved that $F(n+l,l)$ is the number of subsets $B\subset[n]$ such that any two elements of B are always separated by at least $l(\geq 0)$ elements of $\complement B$. For subsets $B\subset[\bar{n}]$ with the same property, the number is $G(n,l)$ where $(t+(l+1)t^{l+1})(1-t-t^{l+1})^{-1}=\sum_{n\geq 1}G(n,l)\,t^n$.

(II) *Multiple sequences*

The concept of GF can be immediately generalized to multiple sequences. We explain the case of double sequences. The three most used GF are the following formal series:

$$\Phi(t,u):=\sum_{n,k\geq 0}a_{n,k}t^n u^k,\qquad \Psi(t,u):=\sum_{n,k\geq 0}a_{n,k}\frac{t^n}{n!}\frac{u^k}{k!},$$
$$\Theta(t,u):=\sum_{n,k\geq 0}a_{n,k}\frac{t^n}{n!}u^k,$$

the last one, Θ, being especially used in the case of a *triangular* sequence ($\Leftrightarrow a_{n,k}=0$, if not $0\leq k\leq n$). We now investigate the double sequence of binomial coefficients, $a_{n,k}:=\binom{n}{k}$, as an example:

$$\Phi(t,u)=\sum_{n,k\geq 0}\binom{n}{k}t^n u^k=\sum_{n\geq 0}t^n\Big(\sum_{0\leq k\leq n}\binom{n}{k}u^k\Big)$$
$$=\sum_{n\geq 0}t^n(1+u)^n=[1-t(1+u)]^{-1},$$

which converges if $|t(1+u)|<1$.

$$\Theta(t,u)=\sum_{n,k\geq 0}\binom{n}{k}\frac{t^n}{n!}u^k=\sum_{n\geq 0}\frac{t^n}{n!}(1+u)^n=\exp\{t(1+u)\}.$$
$$\Psi(t,u)=\sum_{n,k\geq 0}\binom{n}{k}\frac{t^n u^k}{n!\,k!}=\sum_{0\leq k\leq n}\frac{(tu)^k}{(k!)^2}\cdot\frac{t^{n-k}}{(n-k)!}$$
$$=\sum_{k\geq 0}\frac{(ut)^k}{(k!)^2}\sum_{m\geq 0}\frac{t^m}{m!}=(\exp t)\cdot I_0(2\sqrt{ut}),$$

where $I_0(z):=\sum_{k\geq 0}(z/2)^{2k}(k!)^{-2}$ is the modified Bessel function of order 0; because this function is complicated, $\Psi(t,u)$ is not considered very interesting.

(III) *General remarks on generating functions*

We return to the case of a simple sequence a_n.

(1) If the power series $f(z)=\sum_{n\geq 0}a_n z^n$ converges for a complex z ($\Leftrightarrow f(z)$ is an entire function), then the Cauchy integral theorem gives:

$$[13i]\qquad a_n=\frac{1}{2\pi i}\oint f(z)\,z^{-n-1}\,dz,$$

where the integral is taken over a simple curve enclosing the origin, and oriented counterclockwise. Usually, when $f(z)$ is 'elementary', [13i] can very well be used for estimating a_n for great n by the Laplace method or the saddlepoint method (see, for instance, [*De Bruijn, 1961]). In the case that the radius of convergence of $f(z)$ is finite, a Darboux type method can be used (see p. 277).

(2) Of course one can associate with the sequence still others than those of [13a], for example:

$$[13j]\qquad \Omega(t)=\sum_{n\geq 0}a_n\frac{n!}{(t)_{n+1}},$$
$$[13k]\qquad \Lambda(t)=\sum_{n\geq 1}a_n\frac{t^n}{1-t^n},$$
$$[13l]\qquad N(x):=\sum_{n\geq 0}a_n\frac{(x)_n}{n!},$$

which are called respectively '*factorial GF*' (mostly studied by [*Nielsen, 1924]), '*Lambert GF*' (see Exercise 16, p. 161), and '*Newton GF*' (see Exercise 6, p. 221).

(3) Among the several GF defined in [13a, b, c] are all kinds of relations that allow us to pass from one to the other. We cite for example: $\Phi(1/z)=z\int_0^\infty e^{-zt}\Psi(t)\,dt$ (called the Laplace-Carson transform of Ψ), $\Omega(z)=\int_0^1 t^{z-1}\Phi(1-t)\,dt$.

1.14. LIST OF THE PRINCIPAL GENERATING FUNCTIONS

(I) ***Bernoulli and Euler numbers and polynomials***

Bernoulli numbers B_n, Euler numbers E_n, Bernoulli polynomials $B_n(x)$ and Euler polynomials $E_n(x)$ are defined by:

[14a] $$\frac{t}{e^t-1} := \sum_{n\geq 0} B_n\frac{t^n}{n!},\qquad \frac{te^{xt}}{e^t-1} := \sum_{n\geq 0} B_n(x)\frac{t^n}{n!}$$

[14b] $$\frac{2e^t}{e^{2t}+1}=\frac{1}{\operatorname{ch} t}=\sum_{n\geq 0} E_n\frac{t^n}{n!},\qquad \frac{2e^{xt}}{e^t+1}=\sum_{n\geq 0} E_n(x)\frac{t^n}{n!}.$$

(Many generalizations have been suggested). Bernoulli numbers, denoted by b_n in Bourbaki, are sometimes also defined by:

$$t(e^t-1)^{-1}=1-\tfrac{1}{2}t+\sum_{k=1}^{\infty}(-1)^{k-1}\mathcal{B}_k t^{2k}/(2k)!$$

Each $\mathcal{B}_k$ is then >0, and equals $(-1)^{k+1}B_{2k}$ as a function of our Bernoulli numbers.

Their most important properties are:

[14c] $B_n = B_n(0)$, $\quad E_n = 2^n E_n(\tfrac{1}{2})$

[14d] $B_{2h+1} = E_{2h-1} = 0$, for $h = 1, 2, 3, \dots$

[14e] $B_n'(x) = nB_{n-1}(x)$, $\quad E_n'(x) = nE_{n-1}(x)$

[14f] $B_n(x+1) - B_n(x) = nx^{n-1}$,
$E_n(x+1) + E_n(x) = 2x^n$

[14g] $$B_n(x) = \sum_k \binom{n}{k} B_k x^{n-k},$$
$$E_n(x) = \sum_k \binom{n}{k}\frac{E_k}{2^k}\left(x-\frac{1}{2}\right)^{n-k}.$$

[14h] $B_n(1-x) = (-1)^n B_n(x)$, $\quad E_n(1-x) = (-1)^n E_n(x)$

For instance, [14d] follows from the fact that the functions $t(e^t-1)^{-1}-B_0-B_1t$ and $(\operatorname{ch} t)^{-1}$ are even; [14e] follows from the fact that, for $\Phi := te^{xt}(e^t-1)^{-1}$, we have $\partial\Phi/\partial x = t\Phi$, etc. (For a table of B_n and E_n, see [*Abramowitz, Stegun, 1964], p. 810, for $n\leq 60$, and [Knuth, Buckholtz, 1967] for $n\leq 250$ and $n\leq 120$. Applications are found in Exercises 36 and 37, pp. 88 and 89.) The first values of B_n and E_n are:

n	0	1	2	4	6	8	10	12
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{30}$	$\frac{1}{42}$	$-\frac{1}{30}$	$\frac{5}{66}$	$-\frac{691}{2730}$
E_n	1	0	-1	5	-61	1385	-50521	2702765

n	14	16	18	20
B_n	$\frac{7}{6}$	$-\frac{3617}{510}$	$\frac{43867}{798}$	$-\frac{174611}{330}$
E_n	-199360981	19391512145	-2404879675441	370371188237525

(For more information about this subject, see, for instance, [*Campbell, 1966], [*Jordan, 1947], [*Nielsen, 1906].)

We may also define *Genocchi numbers* G_n by:

$$\frac{2t}{e^t+1}=t\left(1-\operatorname{th}\frac{t}{2}\right)=\sum_{n\geq 1} G_n\frac{t^n}{n!}.$$

Then we have $G_3=G_5=G_7=\dots=0$ and $G_{2n}=2(1-2^{2n})B_{2n}=2nE_{2n-1}(0)$, which shows their close relationship with the Bernoulli numbers (used in Exercise 36, p. 89 for 'computing' B_n).

n	1	2	4	6	8	10	12	14	16	18	20
G_n	1	-1	1	-3	17	-155	2073	-38227	929569	-28820619	1109652905

(II) ***Some sequences of 'orthogonal' polynomials***

(The most complete study is made by [*Szegö, 1957].)

We list the GF:

[14i] *The Chebyshev polynomials of the first kind* $T_n(x)$:

$$\frac{1-tx}{1-2tx+t^2} := \sum_{n\geq 0} T_n(x)\,t^n.$$

[14j] The *Chebishev polynomials of the second kind* $U_n(x)$:

$$\frac{1}{1-2tx+t^2} := \sum_{n\geq 0} U_n(x)\,t^n.$$

After some manipulations this implies:

[14k] $\cos n\varphi = T_n(\cos\varphi)$, $\quad \dfrac{\sin(n+1)\varphi}{\sin\varphi} = U_n(\cos\varphi)$.

[14l] The *Legendre* polynomials $P_n(x)$:

$$\frac{1}{\sqrt{1-2tx+t^2}} := \sum_{n\geq 0} P_n(x)\,t^n.$$

[14m] The *Gegenbauer* polynomials $C_n^{(\lambda)}(x)$:

$$(1-2tx+t^2)^{-\lambda} := \sum_{n\geq 0} C_n^{(\lambda)}(x)\,t^n,$$

where $\lambda\in\mathbb{C}$; hence $C_n^{(1/2)}=P_n$, $C_n^{(1)}=U_n$. (These are also called *ultraspherical* polynomials. See Exercise 32, p. 87.)

[14n] The *Hermite* polynomials $H_n(x)$:

$$\exp(-t^2+2tx) := \sum_{n\geq 0} H_n(x)\frac{t^n}{n!}.$$

[14o] The *Laguerre* polynomials $L_n^{(\alpha)}(x)$:

$$(1-t)^{-1-\alpha}\exp\frac{tx}{t-1} := \sum_{n\geq 0} L_n^{(\alpha)}(x)\,t^n \quad (\alpha\in\mathbb{C}).$$

(III) *Stirling numbers*

The *Stirling numbers of the first kind* $s(n,k)$ and *of the second kind* $S(n,k)$ can be defined by the following double GF:

[14p] $$(1+t)^u := 1+\sum_{1\leq k\leq n} s(n,k)\frac{t^n}{n!}u^k$$

[14q] $$\exp\{u(e^t-1)\} := 1+\sum_{1\leq k\leq n} S(n,k)\frac{t^n}{n!}u^k.$$

Because these numbers are very important in combinatorial analysis, we will make a special study of them in Chapter V.

The double GF in the definition can be avoided, if we observe that:

$$(1+t)^u = \exp\{u\log(1+t)\} = \sum_{k\geq 0} u^k\frac{\log^k(1+t)}{k!} \Rightarrow$$

[14r] $$\frac{\log^k(1+t)}{k!} := \sum_{n\geq k} s(n,k)\frac{t^n}{n!}$$

$$\exp\{u(e^t-1)\} = \sum_{k\geq 0} u^k\frac{(e^t-1)^k}{k!} \Rightarrow$$

[14s] $$\frac{(e^t-1)^k}{k!} := \sum_{n\geq k} S(n,k)\frac{t^n}{n!}.$$

(IV) *Eulerian numbers*

The Eulerian numbers $A(n,k)$ (not to be confused with Euler numbers E_n, p. 48) are generated as follows:

[14t] $$\mathfrak{A}(t,u) := \frac{1-u}{e^{t(u-1)}-u} := 1+\sum_{1\leq k\leq n} A(n,k)\frac{t^n}{n!}u^{k-1}.$$

It is easily verified that:

$$(u-u^2)\frac{\partial\mathfrak{A}}{\partial u} + (tu-1)\frac{\partial\mathfrak{A}}{\partial t} + \mathfrak{A} = 0,$$

from which follows, if we put the coefficient of $u^{k-1}t^n/n!$ in this partial differential equation equal to 0, the following recurrence relation:

[14u] $$A(n+1,k) = (n-k+2)A(n,k-1)+kA(n,k), \quad n\geq 0,\ k\geq 2,$$

with initial conditions: $A(n,1)=1$ for $n\geq 0$ and $A(0,k)=0$ if $k\geq 2$. Another GF, denoted by $\mathfrak{A}_*$, is sometimes easier to handle:

[14v] $$\mathfrak{A}_*(t,u) := \mathfrak{A}\left(tu,\frac{1}{u}\right) = 1+u\{\mathfrak{A}(t,u)-1\} =$$
$$= 1+\sum_{1\leq k\leq n} A(n,k)\frac{t^n}{n!}u^k = \frac{1-u}{1-ue^{t(1-u)}}.$$

(A combinatorial interpretation and a table of $A(n, k)$ is given on p. 243.)

1.15. BRACKETING PROBLEMS

We will treat in some detail these famous examples of the use of GF.

(I) *Catalan problem*

Consider a product P of n numbers $X_1, X_2, \ldots, X_n$ *in this order*, $P = X_1 X_2 \ldots X_n$. We want to determine the number of different ways of putting brackets in this product, each way corresponding to a computation of the product by successive multiplications of precisely two numbers each time ([Catalan, 1838]). Thus, $a_3 = 2$, $a_4 = 5$, $a_2 = 1$, according to the following list of bracketings:

$$[15a]\quad (X_1X_2)(X_3X_4),\ ((X_1X_2)X_3)X_4,\ (X_1(X_2X_3))X_4,\ X_1((X_2X_3)X_4),\ X_1(X_2(X_3X_4)).$$

One could also suppose that the sequence, or *word*, $S = X_1, X_2, \ldots, X_n$ is taken from a set with a multiplicative way or composition law, which is neither associative nor commutative; then a_n is the number of correct ways of putting brackets, also called *well-bracketed words*, in S. One can also reason from a single element $X \in E$, and observe that a_n is the number of ways we can interpret a product all whose n factors equal X in E. For $n = 4$ we get then for the list in [15a] the following:

$$[15b]\quad X^2.X^2,\ (X^2.X)X,\ (X.X^2)X,\ X(X^2.X),\ X(X.X^2).$$

Notations [15a, b] become quickly clumsy and difficult to handle, but we observe that any nonassociative product also can be represented by a *bifurcating tree*. Figure 9 (corresponding to $n=4$) shows what we mean. The *height* of the tree is the number of levels above the *root* R (it

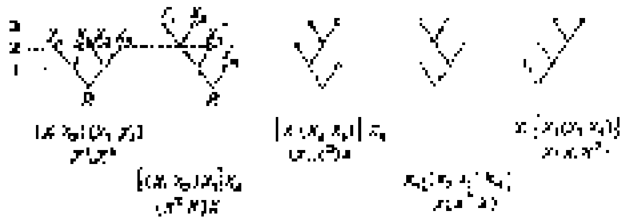


Fig. 9.

is 2 for the first tree, and 3 for the four others). There are $n-2$ nodes, or bifurcations different from R.

We try to find a recurrence relation between the a_n. The last multiplication, which ends the product of all factors $X_1, X_2, \ldots, X_n$ in this order, operates on a product of the first k letters and a product of the last $n-k$ letters, for some k such that $1 \leqslant k \leqslant n-1$. The first k letters can be bracketed in a_k different ways, and the $(n-k)$ last ones can be bracketed in different ways. Thus we get, collecting all possibilities as k ranges over $[n-1]$,

$$[15c]\quad a_n = \sum_{1\leqslant k\leqslant n-1} a_k a_{n-k},\qquad n \geqslant 2.$$

We put:

$$[15d]\quad a_0 := 0,\qquad a_1 := 1.$$

Let now $\mathfrak{A}(t)$ be the GF of the a_n. Then we get, using [15c] for equality (*) and [15d] for (**) and Theorem B of p. 39 for (***):

$$\begin{aligned}\mathfrak{A} = \mathfrak{A}(t) &:= \sum_{n\geqslant 0} a_n t^n = t + \sum_{n\geqslant 2} a_n t^n\\ &\overset{(*)}{=} t + \sum_{n\geqslant 2} t^n \Big(\sum_{1\leqslant k\leqslant n-1} a_k a_{n-k}\Big)\\ &\overset{(**)}{=} t + \sum_{h,k\geqslant 0} a_h a_k t^{h+k}\\ &\overset{(***)}{=} t + \Big(\sum_{h\geqslant 0} a_h t^h\Big)\Big(\sum_{k\geqslant 0} a_k t^k\Big) = t + \mathfrak{A}^2\\ &\Rightarrow \mathfrak{A}^2 - \mathfrak{A} + t = 0,\quad \mathfrak{A}(0) = 0\\ &\overset{(****)}{\Rightarrow} \mathfrak{A}(t) = \tfrac{1}{2}\big(1 - \sqrt{1-4t}\big).\end{aligned}$$

In the implication (****), we have considered $\mathfrak{A}$ as a *function* of t, hence as solution of the preceding quadratic equation. The expansion of the root with [12c] (p. 37) gives us then the required value of a_n, which is often called the *Catalan number*:

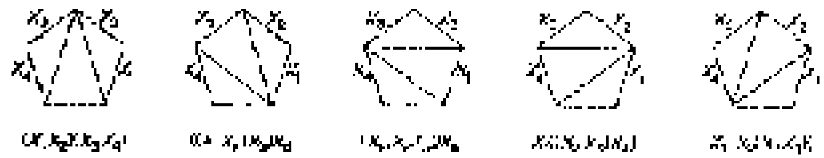
$$[15e]\quad a_n = \frac{1}{n}\binom{2n-2}{n-1}.$$

We list the first few values of a_n:

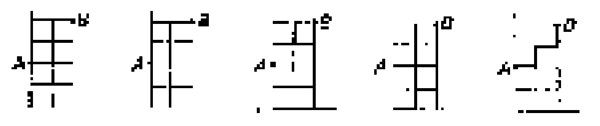
n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
a_n	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	2674440	9694845	35357670

Let us finally mention two other representations of Catalan bracketings.

(1) *Triangulations of a convex polygon* (see also Exercise 8, p. 74). The following example clearly explains the rule:



(2) *Majority paths* (from André, p. 22). Every path joins $A(0, 2)$ to $B(n-2, n)$ with the following convention: any opening bracket (signifies a vertical step and any letter different from X_{n-1} and X_n a horizontal step.



Using Theorem B (p. 21), with $p=n-2$, $q=n$, we easily obtain [15e].

(II) *Wedderburn-Etherington commutative bracketing problem*

([Wedderburn, 1922], [Etherington, 1937], [Harary, Prins, 1959]. For another aspect of this problem, see [Melzak, 1968].)

We suppose E this time to be *commutative*, and we call the number of interpretations of X^n in the sense of [15h] (p. 52) now b_n. Thus $b_2=1$ and $b_3=1$, because $X^2.X=X.X^2$, $b_4=2$, because $(X^2.X)X=(X.X^2)X=X(X.X^2)=X(X^2.X)$. If one prefers, one can also consider b_n as the number of *binary trees*, two trees being considered identical if and only if one can be transformed into the other by reflections with respect to the vertical axes through the nodes. Thus, Figure 10 shows that $b_5=3$:

Fig. 10.

We obtain again a recurrence relation, this time again by inspecting the last multiplication performed, but now it depends on whether n is odd or even:

$$b_{2p-1}=b_1b_{2p-2}+b_2b_{2p-3}+\cdots+b_{p-1}b_p;\quad p\geqslant 2,$$

$$b_{2p}=b_1b_{2p-1}+b_2b_{2p-2}+\cdots+b_{p-1}b_{p+1}+\binom{b_p+1}{2};\quad p\geqslant 1.$$

This can also be written, when we put $b_0:=0$, $b_1:=1$, $b_2:=1$, $b_x=0$ for $x\notin\mathbb{N}$, as follows:

$$b_n=\sum_{\substack{1\leqslant i<j\leqslant n\\ i+j=n}} b_ib_j+\tfrac{1}{2}b_{n/2}+\tfrac{1}{2}(b_{n/2})^2,\quad n\geqslant 2,$$

$$\mathfrak{B}(t):=\sum_{n\geqslant 0} b_nt^n=t+\underbrace{\sum_{n\geqslant 2}t^n\Big(\sum_{\substack{0\leqslant i<j\leqslant n\\ i+j=n}} b_ib_j\Big)}_{(1)}+\tfrac{1}{2}\sum_{n\geqslant 2} b_{n/2}t^n+\tfrac{1}{2}\sum_{n\geqslant 2}(b_{n/2})^2t^n.$$

Now:

$$(1)=\sum_{j>i\geqslant 0} b_ib_jt^{i+j}=\tfrac{1}{2}\Big(\sum_{i,j\geqslant 0} b_ib_jt^{i+j}-\sum_{i\geqslant 0} b_i^2t^{2i}\Big)=\tfrac{1}{2}\Big(\mathfrak{B}^2(t)-\sum_{i\geqslant 0} b_i^2t^{2i}\Big).$$

Hence,

$$\mathfrak{B}(t)=t+\tfrac{1}{2}\mathfrak{B}^2(t)+\tfrac{1}{2}\mathfrak{B}(t^2).$$

This is a functional equation, which can be simplified by putting $\mathscr{W}(t):=1-\mathfrak{B}(t)=1-\sum_{n\geqslant 1}b_nt^n$; then we get:

[15i] $\quad\mathscr{W}(t^2)=2t+\mathscr{W}^2(t).$

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
b_n	1	[illegible]	1	2	3	6	11	23	46	98	207	451	983	2179	4850	10905	24631

n	18	19	20	21	22	23	24	25	26
b_n	56011	127912	293547	676157	1563372	3626149	8436379	19680277	46026618

For a method giving an asymptotic equivalent, see [Otter, 1948]; after a computation due to Bender, $b_n\sim 0.31877662\ldots(2.48325354\ldots)^n n^{-3/2}$.

(II) *Generalized bracketing problem of Schröder* ([Schröder, 1870])

We return to the *noncommutative* case, and we compute the number c_n of bracketings of $X_1, X_2, \ldots, X_n$ where we allow this time in each bracket an arbitrary number of *adjacent* factors. For example, for $n=4$, we must extend the list of [15a] by the following of Figure 11: (thus $c_4=11$)

Fig. 11.

For a recurrence relation we consider again the last multiplication: this time there are not just two factors to be multiplied, but $l(\geq 2)$, of which l_1 factors consist of one letter, l_2 of two letters, etc. Hence:

[15g] $$l_1+l_2+\cdots+l_{n-1}+l_n=l,\qquad l_1+2l_2+\cdots+(n-1)l_{n-1}+nl_n=n,$$

with $l_n=0$, because $l\geq 2$. Now, there are $l!/(l_1!\,l_2!\ldots l_n!)$ different ways to *arrange* these l factors of the last operation, because the choice of a particular sequence of these l factors just means giving a $(l_1, l_2, \ldots, l_n)$-division of $[l]$ (cf. p. 27). Hence:

$$c_n=\sum \frac{l!}{l_1!\,l_2!\ldots l_n!}c_1^{l_1}c_2^{l_2}\ldots c_n^{l_n},\quad n\geq 2,\quad c_0:=0,\quad c_1:=1,$$

where the summation takes place over the $l_1, l_2, \ldots$ such that [15g] *and* $l\geq 2$ ($\Rightarrow n\geq 2$). Thus:

$$\mathfrak{C}:=\sum_{n\geq 0}c_nt^n=t+\sum_{n\geq 2}c_nt^n$$
$$=t+\sum_{l_1+l_2+\cdots\geq 2}\frac{(l_1+l_2+\cdots)!}{l_1!\,l_2!\ldots}(c_1t)^{l_1}(c_2t^2)^{l_2}\ldots$$
$$=t+\sum_{l\geq 2}\left\{\sum_{l_1+l_2+\cdots=l}\frac{l!}{l_1!\,l_2!\ldots}(c_1t)^{l_1}(c_2t^2)^{l_2}\ldots\right\}$$
$$=t+\sum_{l\geq 2}(c_1t+c_2t^2+\cdots)^l=t+\sum_{l\geq 2}\mathfrak{C}^l=t+\frac{\mathfrak{C}^2}{1-\mathfrak{C}}$$
$$\Rightarrow 2\mathfrak{C}^2-(1+t)\mathfrak{C}+t=0,\quad \mathfrak{C}(0)=0.$$

Hence, when we consider $\mathfrak{C}(t)$ as a function of t, we get:

$$\mathfrak{C}(t)=\tfrac{1}{4}(1+t-\sqrt{1-6t+t^2}).$$

If we expand the root $(1+u)^{1/2}$, $u:=-6t+t^2$, and rearrange the t^n, we get by using [12m] (p. 41):

[15h] $$c_n=\sum_{0\leq k\leq (n/2)}(-1)^k\frac{1\cdot 3\cdot 5\ldots(2n-2k-3)}{k!\,(n-2k)!}3^{n-2k}2^{-k-2}.$$

In fact, the c_n can be computed more quickly if we have a linear recurrence relation for them. Such a recurrence relation always exists for the Taylor coefficients of any algebraic function ([Comtet, 1964]), the coefficients being polynomials in n. In the case of $\mathfrak{C}(t)$, which is clearly algebraic, we get, with the necessary simplifications:

[15i] $$(n+1)c_{n+1}=3(2n-1)c_n-(n-2)c_{n-1},\qquad n\geq 2,\quad c_1=c_2=1.$$

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
c_n	1	1	3	11	45	197	903	4279	20793	103049	518859	2646723	13648869	71039373

n	15	16	17	18	19	20
c_n	372693519	1968801519	10463578353	55909013009	300159426963	1618362158587

n	21	22	23	24	25
c_n	8759309660445	47574827600981	259215937709463	1416461675464871	7760733824437545

1.15. Relations

DEFINITION A. *An m-ary relation $\mathfrak{R}$ between $m(\geq 2)$ sets $N_1, N_2, \ldots, N_m$ is a (possibly empty) subset of the product set $N_1\times N_2\times\cdots\times N_m$. An m-tuple $(x_1, x_2, \ldots, x_m)$ is said to satisfy $\mathfrak{R}$, if and only if $(x_1, x_2, \ldots, x_m)\in\mathfrak{R}$. If $N_1=N_2=\cdots=N_m=N$, then $\mathfrak{R}$ is called an m-ary relation on N, $\mathfrak{R}\subset N^m$.*

The case that is most interesting for us, is the case of the *binary* ($m=2$) relations on N, $\mathfrak{R}\subset N^2$. In this case we denote $a\mathfrak{R}b$ [or *not* $a\mathfrak{R}b$] if $(a,b)\in\mathfrak{R}$ [or if $(a,b)\notin\mathfrak{R}$]. For N finite, a good visualization of $\mathfrak{R}$ is obtained by numbering the elements of N, $N:=\{x_1, x_2, \ldots, x_n\}$ and then make a rectangular *lattice* consisting of n vertical lines V_i, each corresponding to an $x_i \in N$, $i \in [n]$, and n horizontal lines H_j, each corresponding as well to an $x_j \in N$ (in Figure 12, $n=7$). The points of the intersections of V_i and H_j represent the points of N^2, and each point of $\mathfrak{R}$ is indicated by a little dot •. For instance, in Figure 12, $x_2 \mathfrak{R} x_3$, not $x_3 \mathfrak{R} x_2$. The points (x_i, x_i), $i \in [n]$ are the points on the *diagonal* Δ (see p. 3). The *lattice* representation thus introduced can also be applied to any relation between two sets N_1 and N_2, if we think of N_1 as the 'abscissa', and of N_2 as the 'ordinate'.

Another representation, called *matrix* representation of $\mathfrak{R} \subset N_1 \times N_2$, $|N_1| = n_1$, $|N_2| = n_2$, consists of associating with this relation an $n_1 \times n_2$ matrix of 0 and 1, defined by $a_{i,j} = 1$ if $(x_i, x_j) \in \mathfrak{R}$ and 0 otherwise, called the *incidence matrix* of $\mathfrak{R}$.

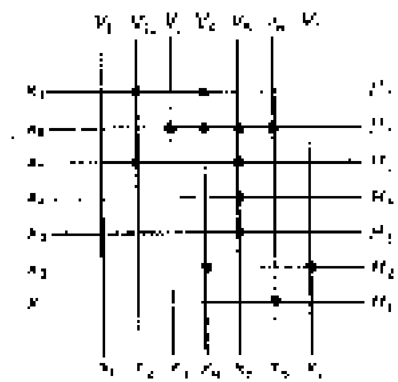


Fig. 12.

DEFINITION B. *Let* $\mathfrak{R}$ *be a binary relation on* N, $\mathfrak{R} \subset N^2$. (I) *The* reciprocal *or* inverse *relation of* $\mathfrak{R}$, *denoted* $\mathfrak{R}^{-1}$ *is defined by* $x\mathfrak{R}^{-1}y \Leftrightarrow y\mathfrak{R}x$ *(the lattice image of* $\mathfrak{R}^{-1}$ *is hence obtained from the lattice image of* $\mathfrak{R}$, *by reflection with respect to the diagonal* Δ). (II) $\mathfrak{R}$ *is called* total or complete, *if and only if for all* $(x, y) \in N^2$ $x\mathfrak{R}y$ *or* $y\mathfrak{R}x$ $(\Leftrightarrow \mathfrak{R} \cup \mathfrak{R}^{-1} = N^2)$. *A relation which is not total, is called* partial. (III) $\mathfrak{R}$ *is called* reflexive, *if and only if for all* $x \in N$, $x\mathfrak{R}x$ $(\Leftrightarrow \Delta \subset \mathfrak{R})$. $\mathfrak{R}$ *is* antireflexive *if and only if for all* $x \in N$, *not* $x\mathfrak{R}x$ $(\Leftrightarrow \Delta \cap \mathfrak{R} = \emptyset)$. (IV) $\mathfrak{R}$ *is called* symmetric *if and only if* $x\mathfrak{R}y \Rightarrow y\mathfrak{R}x$ $(\Leftrightarrow \mathfrak{R} = \mathfrak{R}^{-1})$. $\mathfrak{R}$ *is* antisymmetric *or* proper, *if and only if* $(x\mathfrak{R}y, y\mathfrak{R}x) \Rightarrow$ $\Rightarrow x = y$ $(\Leftrightarrow \mathfrak{R} \cap \mathfrak{R}^{-1} \subset \Delta)$. (V) $\mathfrak{R}$ *is called* transitive *if and only if* $(x\mathfrak{R}y, y\mathfrak{R}z) \Rightarrow x\mathfrak{R}z$. (VI) *For* $x \in N$, *the* first section, *or* vertical section *of* $\mathfrak{R}$ *along* x *is the subset* $\langle x \mid \mathfrak{R} \rangle$ *of* N *consisting of the* $y \in N$ *such that* $x\mathfrak{R}y$. *Similarly, the* second section, *or* horizontal section $\langle \mathfrak{R} \mid y \rangle$, $y \in N$ *is the set of* $x \in N$ *satisfying* $x\mathfrak{R}y$. *If* $\mathfrak{R}$ *is symmetric, then* $\langle x \mid \mathfrak{R} \rangle = \langle \mathfrak{R} \mid x \rangle$. (VII) *The* first projection *of* $\mathfrak{R}$ *on* N, *denoted by* $pr_1\mathfrak{R}$ *equals* $\{x \mid x \in N, \exists y \in N, x\mathfrak{R}y\}$. *Similarly, the* second projection *is* $pr_2\mathfrak{R} := \{y \mid y \in N, \exists x \in N, x\mathfrak{R}y\}$.

Finally, we recall the two most important binary relations.

DEFINITION C. *An* equivalence relation $\mathfrak{R}$ *on* N *is a binary relation, that is reflexive, symmetric and transitive. Then we say that* x *and* y *are* equivalent, *if and only if* $x\mathfrak{R}y$. *The section* $\langle x \mid \mathfrak{R} \rangle = \langle \mathfrak{R} \mid x \rangle$ *is called* equivalence class *of* x: *this is the set of* y *that are equivalent to* x.

The number $\varpi(n)$ of equivalence relations on N, $|N| = n$, in other words, the number of partitions of N will be extensively studied (see p. 204).

DEFINITION D. *An* order relation $\mathfrak{R}$ *on* N *is a binary relation on* N, *which is* reflexive, antisymmetric, *and* transitive. *Often* $x \leqslant y$ *is written instead of* $x\mathfrak{R}y$. *A set is said to be* ordered, *if it has been provided with an order relation; if, moreover, for all* $x, y \in N$, $x \leqslant y$ *or* $y \leqslant x$, *then the set is called* totally ordered. *The section* $\langle x \mid \mathfrak{R} \rangle = \{v \mid x \leqslant v\}$ *is called the* set of upper bounds *of* x *and the section* $\langle \mathfrak{R} \mid y \rangle = \{u \mid u \leqslant y\}$ *is called the* set of lower bounds *of* y. *For* $x, y \in N$ *the* segment $[x, y]$ *is the set of* $z \in N$ *such that* $x \leqslant z \leqslant y$. $x < y$ *means* $x \leqslant y$ *and* $x \neq y$. *A* chain with k vertices (and length $k-1$) *connecting* $x, y \in N$ *is a finite set* $z_1, z_2, \ldots, z_k$ *such that* $x = z_1 < z_2 < \cdots$ $\cdots < z_k = y$. *A* lattice *is an ordered set* N *such that for each pair* (x, y) *of elements of* N *there exist:* (1) *an element* $b \in N$, *often denoted by* $x \vee y$, *which is the smallest element of the set of upper bounds for both* x *and* y *(also called* least upper bound*), in the sense that* $x \leqslant b$, $y \leqslant b$ *and* $x \leqslant v$, $y \leqslant v \Rightarrow b \leqslant v$; (2) *an element* $a \in N$, *often denoted by* $x \wedge y$, *the largest lower bound of both* x *and* y *(also called* greatest lower bound*), in the sense that* $a \leqslant x$, $a \leqslant y$ *and* $u \leqslant x$, $u \leqslant y \Rightarrow u \leqslant a$.

The number d_n of the order relations on N, $|N| = n$, equals the number of T_0-topologies of N ([*Birkhoff, 1967, p. 117]) and the existence of a simple explicit formula seems completely impossible; even asymptotic estimates for d_n when $n \to \infty$ turns out to be a very difficult combinatorial problem ([Comtet, 1966], [Harary, 1967], [Kleitman, Rothschild, 1970], [(J.) Wright, 1972]. See also Exercise 25, p. 229).

The following is the list of known values of d_n and the numbers d^*_n of the nonisomorphic order relations (two relations are called isomorphic if one can be changed into the other by simply rearranging the numbering of the elements of N. The value d_8 due to [Erné, 1974]):

n	1	2	3	4	5	6	7	8	9
d_n	1	3	19	219	4231	130023	6129859	431723379	44511042511
d^*_n	1	2	5	16	63	318	2045		

Actually, we can introduce the numbers $D(n, k)$ of (labelled) order relations of which the longest chain has k vertices (of course, $d_n = \sum_k D(n, k)$):

$n \backslash k$	1	2	3	4	5	6	7	8
1	1							
2	1	2						
3	1	12	6					
4	1	86	108	24				
5	1	840	2310	960	120			
6	1	11642	65700	42930	9300	720		
7	1	227802	2582226	2511750	712320	90720	5040	
8	1	6284406	142259628	195426974	71243760	11481120	582642	40320

1.17. GRAPHS

Though we do not want to study graphs, we will sometimes use a little of the language of graph theory, hence this and the next section. We have to make a choice among the various current names of certain concepts, since in this field, the terminology is not yet completely standardized. Actually, this situation has some advantages, as it compels each publication on this subject to define its terms carefully. Any book on graphs can be used as a first introduction to graph theory. (For example [*Berge, 1958], [*Busacker, Saaty, 1965], [*Fiedler, 1964], [*Flament, 1965], [*Ford, Fulkerson, 1962], [*Harary, 1967a, b], [*Harary, Norman, Cartwright, 1965], [*Kaufmann, 1968a, b], [*König, 1936], [*Moon, 1968], [*Ore, 1962, 1963, 1967], [*[illegible], 1968], [*Ringel, 1959],

[*Sainte-Laguë, 1926], [*Seshu, Reed, 1961], [*Tutte, 1966], and particularly, in the viewpoint adopted here, the attractive book by [*Harary, 1969].)

Let N be a finite set. We recall that a *pair* u of N is a 2-block of N (= 2-combination, or subset of two elements, p. 7); thus $u \in \mathfrak{P}_2(N)$.

DEFINITION A. *A graph (over N) is a pair $(N, \mathscr{G})$, in which $\mathscr{G}$ is a set (possibly empty) of pairs of N, $\mathscr{G} \subset \mathfrak{P}_2(N)$. The elements of N are called the* nodes *or* vertices *of the graph, and the pairs ($\in \mathscr{G}$) are called* edges *of the graph. One often says "the graph $\mathscr{G}$" rather than "the graph $(N, \mathscr{G})$" when the set N is given once and for all.*

THEOREM A. *Giving a graph $\mathscr{G}$ on N is equivalent to giving a binary relation $\mathscr{I}$ on N, $\mathscr{I} \subset N^2$, which is* symmetric *and* antireflexive, *called* incidence relation *associated with $\mathscr{G}$.*

■ Define $\mathscr{I}$ by $x\mathscr{I}y \Leftrightarrow \{x, y\} \in \mathscr{G}$ ■

A convenient *plane* representation of a graph consists in drawing the nodes as points and the edges as straight or curved segments, and ignoring their intersections. Figure 13 represents $N = \{a, b, c, \ldots, k, l\}$ and $\mathscr{G} = \{\{a, b\}, \{b, c\}, \{c, d\}, \{c, f\}, \{d, e\}, \{d, g\}, \{e, h\}, \{e, f\}, \{f, g\}, \{f, j\}, \{h, i\}\}$.

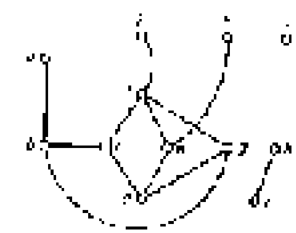


Fig. 13

DEFINITION B. *Let $\mathscr{G}(\subset \mathfrak{P}_2(N))$ be a graph over N. (1) An edge containing a node $x (\in N)$ is called* incident *with x, and $\mathscr{G}(x)$ designates the set of these edges. The number $|\mathscr{G}(x)|$ of edges incident with x, also denoted by $\delta(x)$, is called the* degree *of x. Two nodes x and y are called* adjacent *if $\{x, y\} \subset \mathscr{G}$. Similarly, two edges are called* adjacent *if they have a node in common. A node is called an* end point *or* terminal node *if its degree* equal 1, *the edge adjacent to x (which is unique) is also called* terminal. *An* isolated node *is one with degree* 0. (II) $(N', \mathfrak{G}')$ *is a* subgraph *of* $(N, \mathfrak{G})$ *if* $N' \subset N$, $\mathfrak{G}' \subset \mathfrak{G}$, $\mathfrak{G}' \subset \mathfrak{P}_2(N')$; *it is called a* complete subgraph *(or a* clique*), with support* N' *if* $\mathfrak{G}' = \mathfrak{P}_2(N')$. *An* independent set $L\ (\subset N)$ *in a graph* $\mathfrak{G}$ *is a set such that* $\mathfrak{P}_2(L) \cap \mathfrak{G} = \emptyset$; *hence is a complete subgraph of the* complementary graph, *which is the graph* $\bar{\mathfrak{G}} := \mathfrak{P}_2(N) - \mathfrak{G}$. (III) *A* path *or* chain *connecting a and b* $(\in N)$ *is a sequence of adjacent edges* $\{a, x_1\}, \{x_1, x_2\}, \ldots, \{x_{l-1}, b\}$; *this path* $\{a, x_1, x_2, \ldots, x_{l-1}, b\}$ *is said to have* length l (multiple points may occur, as in the case of the path $\{h, f, a, d, e, f, g\}$ of Figure 13). *A* cycle *or* circuit *is a closed path.* (For instance, $\{c, f, g, d, c\}$ in Figure 13.) *An* Euler circuit *is a circuit in which all edges of* $\mathfrak{G}$ *occur precisely once. A* Hamiltonian circuit *is a circuit that passes exactly once through every node.* (IV) *A graph is called* connected *if every two nodes are connected by at least one path.* (V) *A* tree *is a connected* acyclic (=*without cycles*) *graph. The* distance *between two points in a tree is the number of the edges in the (unique) path joining a with b (no repetitions of edges allowed to occur in this path).*

We indicate now a way to draw a tree $\mathfrak{A}$ of N. We choose a node $x_0\ (\in N)$. From x_0 we trace the edges connecting x_0 with the adjacent nodes (those who have distance 1 to x_0), say $x_{1,1}, x_{1,2}, \ldots$. We arrange these on a horizontal line (Figure 14). From these points, we trace the edges that connect them with the points situated at distance 2 from x_0 (hence adjacent to $x_{1,i}$ and not equal to x_0), etc. A tree in which such a special

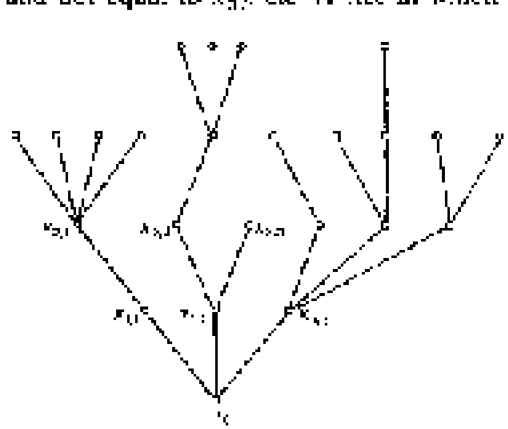


Fig. 14.

point x_0, the root, has been chosen, is also called *rooted tree*. The preceding construction proves Figure 14.

THEOREM B. *Each tree has at least two endpoints, and for* $n \geq 3$, *at least two terminal edges.*

Another characterization of trees is

THEOREM C. *Any two of the following three conditions* (1), (2) *and* (3) *imply the third, and moreover, imply that the graph* $\mathfrak{G}$ *over* N, $|N| = n$ *is a tree:* (1) $\mathfrak{G}$ *is connected;* (2) $\mathfrak{G}$ *is acyclic;* (3) $\mathfrak{G}$ *has* $(n-1)$ *edges.*

■ (1), (2) ⇒ (3). In other words, by Definition B (V), any tree with n vertices has $n-1$ edges. This is true for $n=2$. We prove the statement by complete induction, and we suppose it to be true for all trees having up to $(n-1)$ edges. In a tree $\mathfrak{A}$ with n nodes, we cut off one of the terminal nodes and its incident edge. The new graph obtained in this way is evidently a tree; hence it contains $(n-1)$ nodes, so $|\mathfrak{A}'| = n-2$ according to the induction hypothesis; hence $|\mathfrak{A}| = n-1$.

(1), (3) ⇒ (2). We reason by *reductio ad absurdum*. Suppose that there exists $(N, \mathfrak{G})$, $|N| = n$, $|\mathfrak{G}| = n-1$, which is connected, and with at least one cycle $\mathscr{C}$. We break the cycle $\mathscr{C}$ by omitting one edge. Thus we obtain a new graph $(N, \mathfrak{G}_1)$, still connected, with $|\mathfrak{G}_1| = n-2$. We repeat this operation until there are no cycles left, so we have a connected *acyclic* graph $(N, \mathfrak{G}_i)$, with $n-1-i$ edges, for some $i \geq 1$, which contradicts the statement that (1), (2) imply (3).

(2), (3) ⇒ (1). If not, there exists $(N, \mathfrak{G})$, $|N| = n$, $|\mathfrak{G}| = n-1$, with two nodes $a, b \in N$ not connected by a path of $\mathfrak{G}$. If we connect a and b by a new edge $\{a, b\}$, we obtain a new graph $(N, \mathfrak{G}_1)$, which is still acyclic, with $|\mathfrak{G}_1| = n$. Repeating this procedure, we finally obtain a *connected* acyclic graph $(N, \mathfrak{G}_l)$ with $n-1+l$ edges, for some $l \geq 1$, which again contradicts that (1), (2) imply (3). ■

Let us now prove the famous *Cayley* theorem ([Cayley, 1889]).

THEOREM D. *The number of trees over* N, $|N| = n$, *equals* n^{n-2}.

There are many proofs of this theorem. One kind, of constructive type, establishes a bijection between the set of trees over $[n]$ and the set $[n]^{n-2}$ of $(n-2)$-tuples of $[n]$, $(x_1, x_2, \dots, x_{n-2})$, $x_i \in [n]$. ([Foata, Fuchs, 1970], [Neville, 1953], [Prüfer, 1918], and, for a generalization to k-trees, [Foata, 1971]. See also p. 71.) Others follow the path of obtaining the various enumerations suggested by the problem. ([Clarke, 1958], [Dziobek, 1917], [Katz, 1955], [Mallows, Riordan, 1968], [Moon, 1963, 1967a, b], [Riordan, 1957a, 1960, 1965, 1966], [Rényi, 1959].) We give here the proof of Moon, which is of the second type.

THEOREM E. *Let* $\mathbf{T} = \mathbf{T}(N; d_1, d_2, \dots, d_n)$ *be the set of trees over* $N := \{x_1, x_2, \dots, x_n\}$ *whose node* x_i *has degree* $d_i (\geqslant 1)$, $i \in [n]$, *where* $d_1 + d_2 + \cdots + d_n = 2(n-1)$. *Then:*

$$[17a] \qquad T(n; d_1, d_2, \dots, d_n) := |\mathbf{T}(N; d_1, d_2, \dots, d_n)| = (d_1 - 1, d_2 - 1, \dots, d_n - 1)$$

(We use here the notation for the multinomial coefficients introduced in [10c], p. 27.)

It is clear that $T(n; d_1, d_2, \dots) = 0$ if $d_1 + d_2 + \cdots \neq 2(n-1)$, because every tree over N has $(n-1)$ edges (Theorem C, p. 61). We first prove three lemmas.

LEMMA A. *Let integers* $b_i \geqslant 1$, $i \in [s]$, *be given such that* $\sum_{i=1}^{s} b_i = n$. *Then:*

$$[17b] \qquad (b_1, \dots, b_s) = \sum_{k=1}^{s} (b_1, b_2, \dots, b_k - 1, \dots, b_s).$$

(So, this formula is a generalization of the binomial relation $(b, c) = (b-1, c) + (b, c-1)$, [5e], p. 10.)

■ Let be given a set M, $|M| = n$. The left hand member of [17b] enumerates the set $\mathfrak{p}$ of divisions $\mathscr{D} = (B_1, B_2, \dots, B_s)$ of M, where $|B_i| = b_i$, $i \in [s]$ (p. 27). Now we choose an $x \in M$ and we put $\mathfrak{p}_k = \{\mathscr{D} \mid \mathscr{D} \in \mathfrak{p},\ x \in B_k\}$; then [17b] follows from the fact that:

$$\mathfrak{p} = \sum_{1 \leqslant k \leqslant s} \mathfrak{p}_k, \qquad |\mathfrak{p}_k| = (b_1, b_2, \dots, b_k - 1, \dots, b_s). \quad ■$$

Then the next lemma follows immediately:

LEMMA B. *Let be given integers* $a_j \geqslant 0$, $j \in [s]$ *such that* $\sum_{j=1}^{s} a_j = n$. *Then:*

$$[17c] \qquad (a_1, a_2, \dots, a_s) = \sum_{j=1}^{s} (a_1, a_2, \dots, a_j - 1, \dots, a_s),$$

where the summation is taken over all j *such that* $a_j \geqslant 1$. (If not, then the multinomial coefficient under the summation sign equals 0 by definition. Compare with [Tainiter, 1968].)

Now we return to [17a], and we suppose that:

$$[17d] \qquad d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n.$$

This amounts to changing the numbering of the x_i.

LEMMA C. *Summing over the* i *such that* $d_i \geqslant 2$, *the following holds:*

$$[17e] \qquad T(n; d_1, d_2, \dots, d_n) = \sum_{i,\, d_i \geqslant 2} T(n-1; d_1, \dots, d_i - 1, \dots, d_{n-1}).$$

■ It follows from [17d] and from Theorem D that $d_n = 1$. Let $\mathbf{T}_i := \{\mathscr{A} \mid \mathscr{A} \in \mathbf{T},\ x_n \text{ adjacent to } x_i\}$. Hence $i \leqslant n-1$ and $d_i \geqslant 2$. Now we have the division $\mathbf{T} = \sum \mathbf{T}_i$, where we sum over all i such that $d_i \geqslant 2$. Hence [17e], if we observe that:

$$|\mathbf{T}_i| = |\mathbf{T}(N - \{x_n\}; d_1, \dots, d_i - 1, \dots, d_{n-1})|. \quad ■$$

Proof of Theorem E. We prove formula [17a] by induction. It is clearly true for $n = 3$. Suppose true for $n-1$ and smaller. Then, with [17e] and the induction hypothesis for equality (*), $d_n = 1$ for (**) and [17c] for (***):

$$\begin{aligned} T(n; d_1, d_2, \dots, d_n) &\overset{(*)}{=} \sum_{i,\, d_i \geqslant 2} (d_1 - 1, \dots, d_i - 2, \dots, d_{n-1} - 1) \\ &\overset{(**)}{=} \sum_{i,\, d_i \geqslant 2} (d_1 - 1, \dots, d_i - 2, \dots, d_n - 1) \overset{(***)}{=} [17a]. \quad ■ \end{aligned}$$

THEOREM F. *The number* $L(n, k)$ *of trees* $\mathscr{A}$ *over* N *such that a given node, say* x_1, *has degree* k, *equals:*

$$[17f] \qquad L(n, k) = \binom{n-2}{k-1} (n-1)^{n-k-1}.$$

■ We have, using [17a] for equality (*), and $c_i := d_i - 1$, $i \in [n-1]$ for (**), and [10f] (p. 23) for (***):

$$L(n,k) \overset{(*)}{=} \sum_{d_1+\cdots+d_{n-1}=2n-2-k} (d_1-1,\ldots,d_{n-1}-1,k-1)$$

$$\overset{(**)}{=} \sum_{c_1+\cdots+c_{n-1}=n-k-1} (c_1, c_2,\ldots,c_{n-1},k-1)$$

$$= \binom{n-2}{k-1} \sum_{c_1+\cdots+c_{n-1}=n-k-1} (c_1, c_2,\ldots,c_{n-1})$$

$$\overset{(***)}{=} \binom{n-2}{k-1}(n-1)^{n-k-1}. \quad ■$$

Proof of Theorem D. By Theorem E, the total number of trees over N equals:

$$\sum_{k\geq 1} L(n,k) = \sum_{1\leq k\leq n-1} \binom{n-2}{k-1}(n-1)^{n-k-1} = [1+(n-1)]^{n-2} = n^{n-2}.$$

To finish this section on graphs, we discuss the *Hasse diagram* of an order relation over N. This graph is obtained by joining a and b if and only if $a \leq b$ *and* $a \leq c \leq b \Rightarrow c = a$ or $c = b$ ($\Leftrightarrow b$ *covers* a). In this case b is placed over a. For example, Figure 15 is the Hasse diagram of the order relation $\leq$ on $N = \{a, b, c, d, e, f, g, h, i, j\}$ defined by $a \leq b$, $a \leq d$, $b \leq c$, $d \leq e$, $d \leq f$, $e \leq c$, $f \leq c$, $g \leq i$, $g \leq h$. If one wants to avoid, in this diagram, the difficulty of putting every point on different heights, then one must orient the edges; in this case one obtains a *transitive digraph*, as in Figure 16.

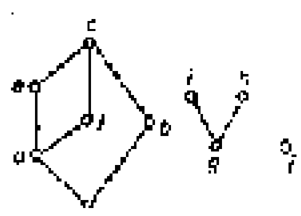


Fig. 15.

Fig. 16.

1.18. DIGRAPHS; FUNCTIONS FROM A FINITE SET INTO ITSELF

A. *Digraphs in general*

We call a 2-arrangement (x, y) of N an *ordered pair*, that is a pair in which we distinguish a first element, $(x, y) \in \mathfrak{A}_2(N)$. (see p. 6).

DEFINITION A. *A* digraph $(N, \mathscr{D})$ *or* directed graph *(over N) is a pair, such that $\mathscr{D}$ is a (possibly empty) set of ordered pairs from N, $\mathscr{D} \subset \mathfrak{A}_2(N)$. The elements of N are then called the* nodes *or* vertices *of the digraph, and the ordered pairs are called the* arcs. *One often says "digraph $\mathscr{D}$", rather than "digraph $(N, \mathscr{D})$", in case the set N is given once and for all.*

Most of the concepts introduced in the previous section have their analogue in digraphs. For instance, the *outdegree* of $x (\in N)$, denoted by $od(x)$ is the number of arcs leaving x; the *indegree*, denoted by $id(x)$ is the number of arcs entering x. An *oriented cycle* is a cycle in which the orientation of the arcs is such that of two consecutive arcs always the first one is entering their common node and the other is leaving it (or vice versa). Other definitions are adapted in the same manner.

THEOREM A. *Giving a digraph $\mathscr{D}$ over N is equivalent to giving an antireflexive binary relation $\mathfrak{I}$ on N, $\mathfrak{I} \subset N^2$, called the* incidence relation *of $\mathscr{D}$.*

■ Define $\mathfrak{I}$ by: $x\mathfrak{I}y \Leftrightarrow (x, y) \in \mathscr{D}$ ■

There is again a plane representation, analogous to the one introduced on p. 61, but with arrows added. Figure 17 shows a digraph and its associated relation. If the relation was not antireflexive, we had to introduce *loops* into the digraph. But digraphs with loops permitted and relations are the same.

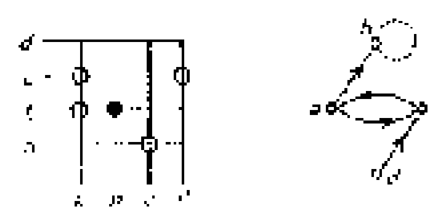


Fig. 17.

(II) *Tournaments*

DEFINITION B. *A tournament (over N) is a digraph $\mathcal{G}$ such that every pair $\{x_i, x_j\} \in \mathfrak{P}_2(N)$ is connected by precisely one arc. If the arc (x_i, x_j) belongs to $\mathcal{G}$, we say that x_i dominates x_j. The score s_i of x_i is the number of nodes x_j that are dominated by x_i. Usually, the nodes $(\in N)$ of $\mathcal{G}$ are numbered in such a way that:*

$$[18a] \quad (0 \leqslant) \; s_1 \leqslant s_2 \leqslant \cdots \leqslant s_n \; (\leqslant n-1).$$

The n-tuple $(s_1, s_2, \ldots, s_n) \in \mathbb{N}^n$ is then called the score vector of $\mathcal{G}$.

The relation $\mathcal{G}$ (the incidence relation on N) associated with $\mathcal{G}$ is hence total, antireflexive and antisymmetric. Figure 18 represents a tournament in which $s_1 = s_3 = 1$, $s_2 = s_4 = 2$.

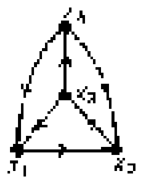


Fig. 18.

THEOREM B. *A sequence $(s_1, s_2, \ldots, s_n)$ of integers such that [18a] holds, is a score vector if and only if:*

$$[18b] \quad \sum_{i=1}^{n} s_i = \binom{n}{2}$$

$$[18c] \quad \text{For all } k \in [n], \; \sum_{i=1}^{k} s_i \geqslant \binom{k}{2}.$$

■ We only show that the condition is necessary. (For sufficiency, see the beautiful book by [*Moon, 1968] on tournaments, or the papers by [Landau, 1953] or [Ryser, 1964]. The reader is also referred to [*André, 1900] and [André, 1898–1900].) For all $x \in N$ let $\mathcal{A}(x)$ be the set of arcs issuing from x, $|\mathcal{A}(x_i)| = s_i$. [18b] follows then from considering the cardinalities in the division $\sum_{i=1}^{n} \mathcal{A}(x_i) = \mathcal{G}$. On the other hand, for all $K \subset N$, the set of $\binom{k}{2}$ arcs whose two nodes belong to K, clearly is contained in $\sum_{x \in K} \mathcal{A}(x)$; hence [18c], by considering the cardinalities of the sets involved. ■

(III) *Maps of a finite set into itself*

DEFINITION C. *A digraph over N is called* functional *if the outdegree of every node equals 0 or 1: $\forall x \in N$, $od(x) \leqslant 1$.*

There exists a bijection between the set N^N of maps φ of N into itself and the set of such digraphs $\mathcal{G}$. In fact, we may associate $\mathcal{G}$ with φ by $(x, y) \in \mathcal{G} \Leftrightarrow \varphi(x) = y$, $y \neq x$. In this case $\mathcal{G}$ is called the 'functional digraph associated with φ'. Figure 19 corresponds to a $\varphi \in [22]^{[22]}$.

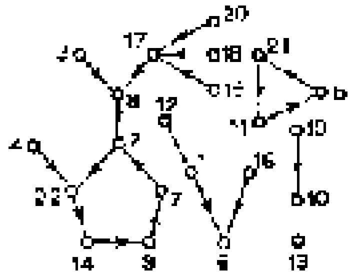


Fig. 19.

The map φ is a permutation if, moreover, for all $x \in N$, $id(x) \leqslant 1$.

THEOREM C. *The relation $\mathcal{E}$ on N defined by: $x \mathcal{E} y \Leftrightarrow \exists p \in \mathbb{N}, \exists q \in \mathbb{N}$ such that $\varphi^p(x) = \varphi^q(y)$ is an equivalence relation. The restriction of φ to each class of $\mathcal{E}$ has for associated digraph an oriented cycle, to which (possibly) some trees are attached. Such a digraph is sometimes called an 'excycle'* (Weaver).

The classes of $\mathcal{E}$ are the connected components of $\mathcal{G}$. In the case of Figure 19, there are 5 excycles. In this way each map $\varphi \in N^N$ can be decomposed into a product of disjoint excycles, this result being analogous to the decomposition of a permutation into cyclic permutations. (For other properties of N^N see, for example, [Dénes, 1959, 1968], [Harary, 1959b], [Hedrlin, 1963], [Read, 1961], [Riordan, 1962a], [Schützenberger, 1968]. For the 'probabilistic' aspect see [Katz, 1955], [[illegible], Williams, 1968].)

DEFINITION D. *A map $\varphi\in N^N$ is called acyclic if each of its connected components is a rooted tree. In other words, giving φ is equivalent to giving a rooted forest over N, i.e. a covering of N by disjoint rooted trees.*

For instance, the map φ of Figure 19 is not acyclic, but the following is: $\psi(i)=i+1$ for $i\in[21]$ and $\psi(22)=22$.

THEOREM D. *The number of acyclic maps of N into itself, that is, the number of rooted forests over N, $|N|=n$, equals $(n+1)^{n-1}$.*

■ We adjoin a point x to the set N, and we let $P:=\{x\}\cup N$, $|P|=n+1$. Each *tree* T over P becomes a *rooted forest* if we chop off the branches issuing from x. We call this rooted forest $\varphi(T)$. Its roots are just the nodes adjacent to x in T. This map establishes, evidently, a bijection between the rooted forests over N and the trees over P, hence by Theorem D (the Cayley theorem) (p. 63), $|P|^{|P|-2}=(n+1)^{n-1}$. ■

THEOREM E. *The number of acyclic maps of N into itself, with exactly k roots, equals $\binom{n-1}{k-1}n^{n-k}$.*

■ As before, by joining a $(n+1)$-th point x to each root, we get a tree with $n+1$ nodes, in which x has degree k. Then apply [17f] (p. 55). ■

(IV) *Coding functions of a finite set* ([Foata, 1971]).

After labeling, we can work with the set $[n]:=\{1,2,3,\ldots,n\}$. Let us explain how to represent any map f of $[n]$ into itself, that is to say any *function* $f\in[n]^{[n]}$, by a *word* $x=x(f)$ in the noncommutative indeterminates (or letters) $x_1, x_2, \ldots, x_n$, where each x_i is identified with the element (or *label*) $i\in[n]$.

Every *cycle* of f (p. 69) supplies letters of a word, whose *first letter*, or *label*, is its *greatest element*, the other letters following in the *opposite direction* of the arrows. For example, the cycle $(5\to 21\to 11\to 5)$ of Figure 19 gives $x_{21}x_5x_{11}$. Now, juxtaposing from left to right the preceding words by *increasing labels*, we get a word w_0 which represents the cyclic part of f. Here, $w_0=x_6x_{10}x_{14}x_{11}x_5x_{13}x_{21}x_5x_{11}x_{22}$. Considering then the *first leaf* (terminal node x of the digraph, such that $\mathrm{id}\,x=0$, $\mathrm{od}\,x=1$, with the *smallest* label), we construct a word w_1 which is the path joining this leaf to w_0, *leaf excluded, root included*, but written *from root to leaf*. Here the first leaf being 3, we have $w_1=x_2x_3$. The same operation applied to the *second leaf* (here 4) with the path joining 4 to w_0w_1 gives a word w_2 (here x_{2}). The third leaf (12) would introduce $w_3=x_8x_9$ and so on. Finally, we define $x=x(f):=w_0w_1w_2\ldots$. Here, $x=x_6x_{10}x_{14}x_{11}x_5x_{13}x_{21}x_5x_{11}x_{22}x_2x_3x_2x_8x_9\ldots$. Of course, no leaf is recorded in x, and the first repetition in x ends the cyclic part of f. So, it could be easily shown that x establishes a *bijection* between $[n]^{[n]}$ and the set $[n]^n$ of words with n letters (or n-arrangements, p. 18) on the alphabet $\{x_1, x_2, \ldots, x_n\}$.

To train the reader to code and decode, the following examples are given. (1) If f is the identity, then $x=x_1x_2\ldots x_n$. (2) $f(1)=1$, $f(2)=f(3)=\ldots=f(n)=1$; $x=x_1^n$. (3) f is circular: $f(1)=2$, $f(2)=3$, $f(3)=4,\ldots$, $f(n-1)=n$, $f(n)=1$; $x=x_nx_{n-1}\ldots x_2x_1$. (4) $f(1)=1$, $f(2)=1$, $f(3)=2$, $f(4)=3,\ldots$, $f(n)=n-1$; $x=x_1^2x_2x_3\ldots x_{n-1}$. (5) $f(1)=f(m+1)=1$, $f(2)=f(m+2)=2,\ldots$, $f(m)=f(2m)=m$; $x=(x_1x_2\ldots x_m)^2$. (6) $f(1)=f(2)=1$, $f(3)=f(m+1)=2$, $f(4)=f(m+2)=3,\ldots$, $f(m)=f(2m-2)=m-1$, $f(2m-1)=f(2m)$, $x=x_1^2x_2^2\ldots x_m^2$.

Instead of $x=x(f)$, it could be useful to introduce the *Abelian word* $t=t(f)$, that is x in which letters $x_1, x_2, \ldots$ are replaced by commutative variables $t_1, t_2, \ldots$. So, in the case of Figure 19, we get $t(f)=t_1t_2^3t_3t_4^2$ [illegible].

(V) *Enumerator of a subset of* $[n]^{[n]}$

Given $E\subset[n]^{[n]}$, it would be worthwhile to consider the *enumerator* of E, that is the (commutative) polynomial $\mathcal{T}=\mathcal{T}_E=\sum_{f\in E}t(f)$. Let us give a few examples. (1) If $E=[n]^{[n]}$, then $\mathcal{T}=(t_1+t_2+\ldots+t_n)^n$. (2) If E is the set of functions of $[n]$ for which $1, 2, 3, \ldots, k$ are *fixed points*, then $\mathcal{T}=t_1t_2\ldots t_k(t_1+t_2+\ldots+t_n)^{n-k}$. (3) If E is the set of *acyclic* functions whose fixed points (roots) are $1, 2, 3, \ldots, k$, then $\mathcal{T}=t_1t_2\ldots t_k(t_1+t_2+\ldots+t_k)(t_1+t_2+\ldots+t_n)^{n-k-1}$. Of course, $\mathcal{T}_E(1,1,\ldots,1)=|E|$.

So, the three preceding examples allow us to obtain (again) the numbers (1) n^n of functions of $[n]$, (2) n^{n-k} of functions with k given fixed points, (3) kn^{n-k-1} of trees with k given roots (especially Cayley for $k=1$). Similarly, the coefficient of $t_1^{\alpha_1}t_2^{\alpha_2}\ldots$ in $\mathscr{F}_E(t_1, t_2, \ldots)$ is the number of $f\in E$ such that $\pi(f)$ has α_1 occurrences of n_1, α_2 occurrences of n_2, etc.

For any *division* of E, $E=E_1+E_2+\cdots$, we have $\mathscr{F}_E=\mathscr{F}_{E_1}+\mathscr{F}_{E_2}+\cdots$ obviously. Finally, let us consider a division of $[n]$, $[n]=\sum_i A_i$, and a family of sets E_i of functions, $E_i\subset[n]^{[n]}$, having the following property: every $f\in E_i$ acts on A_i only, i.e. $f\in E_i\Rightarrow \forall x\notin A_i$, $f(x)=x$. Then the set $E=E_1E_2E_3\ldots$ of *all* functions which can be *factorized* $f=f_1f_2f_3\ldots$ (in the sense of the composition of functions, here commutative), where $f_1\in E_1, f_2\in E_2,\ldots$ is such that $\mathscr{F}_E=\mathscr{F}_{E_1}\mathscr{F}_{E_2}\cdots$.

SUPPLEMENT AND EXERCISES

(As far as possible we follow the order of the sections.)

1. *n points in a plane.* Let N be a set of n points or nodes in the plane such that no three among them are collinear. Moreover, we suppose that each pair among the $\binom{n}{2}$ straight lines connecting each pair of points is intersecting, and also no three among these lines have a point in common other than one of the given nodes. Show that these $\binom{n}{2}$ lines intersect each other in $\frac{1}{8}n(n-1)(n-2)(n-3)$ points different from those in N, and that they divide the plane into $\frac{1}{8}(n-1)(n^3-5n^2+18n-8)$ (connected) regions, including $n(n-1)$ unbounded regions.

***2.** *Partitions by lines, planes, hyperplanes.* (1) Let be given n lines in the plane, each two of them having a point in common but no three of them having a point in common. These lines divide the plane into $\frac{1}{2}(n^2+n+2)$ regions. [*Hint:* Show that the number a_n which is asked satisfies the relation $a_n=a_{n-1}+n$, $a_1=2$.] (2) More generally, n hyperplanes in $\mathbf{R}^k$, in general position, determine $a(n,k)$ 'regions', with $a(n,k)=\sum_{0\le i\le k}\binom{n}{i}=$ $=2^n-\sum_{0\le j\le n-k-1}2^j\binom{n-j-1}{k}$; the number of bounded regions is $\binom{n-1}{k}$.

(3) For a system $\mathscr{D}$ of n lines, satisfying the conditions of (1), let $a_{n,k}(\mathscr{D})$ be the number of regions with k sides in $\mathscr{D}$. Clearly, $\sum_{k=2}^{n} a_{n,k}(\mathscr{D})=\frac{1}{2}(n^2+n+2)$ and $\sum_{k=2}^{n} ka_{n,k}(\mathscr{D})=2n^2$. It is an open problem to find some lower and upper bounds for $a_{n,k}(\mathscr{D})$, or even better, the values taken by $a_{n,k}(\mathscr{D})$. (For more information about this problem see [*Grünbaum, 1967], pp. 390–410, and [*Grünbaum, 1972].)

3. *Circles.* n circles divide the plane into at most n^2-n+2 regions. The $\binom{n}{3}$ circles that are the circumscribed circles of all triangles whose vertices lie in a given set N of n points (in general position) in the plane, intersect each other in $\frac{1}{12}(n)_5(2n-1)$ points different from those of N.

4. *Spheres.* n spheres divide the 3-dimensional space into at most $n(n^2-3n+8)/3$ regions; n great circles divide the surface of a sphere into at most n^2-n+2 regions. More generally, n hyperspheres divide $\mathbf{R}^k$ into at most $\binom{n-1}{k}+\sum_{0\le i\le k}\binom{n}{i}$ regions.

5. *Convex polyhedra.* F, V, E stand for the number of faces, vertices and edges of a convex polyhedron. To show the famous *Euler* formula $F+V=$ $=E+2$. [*Hint:* For any open polyhedral surface the formula $F+V-E=1$ can be shown to hold by induction on the number of faces.] ([*Grünbaum, 1967] gives a thorough treatment of *polytopes* in arbitrary dimension n, with an abundance of bibliography and of open problems. See also [*Klee, 1966].)

6. *Inscribed and exscribed spheres of a tetrahedron.* Let be given a tetrahedron T, and let A_1, A_2, A_3, A_4 be the areas of its four faces. To show that the number of spheres which are tangent to all four planes that contain the faces of T (inscribed and exscribed) is equal to $8-s$, where s is the number of equalities satisfied by A_1, A_2, A_3, A_4, the equalities being taken from $A_1+A_2=A_3+A_4$, $A_1+A_3=A_2+A_4$, $A_1+A_4=A_2+A_3$ (hence $0\le s\le 3$). If possible, generalize to higher dimensions. (See [Vaughan, Gabai, 1967] and [Gerber, 1972].)

***7.** *Triangles with integer sides.* (1) The number of non-congruent triangles with integer sides and given perimeter n equals $\{\frac{1}{48}(n^2+6n+21+(-1)^{n-1}3n)\}$ ($\{x\}$ denotes here the largest integer smaller or equal to x, also called the integral part of x). (2) The number of triangles that can be constructed with n segments of lengths $1, 2, \ldots, n$ equals $\frac{1}{16}\{1+(-1)^n\}+\frac{1}{8}(n-1)+\frac{1}{4}\binom{n+2}{3}-\frac{1}{2}\binom{n-1}{3}$.

8. *Some enumeration problems related to convex polygons.* Let $A_1, A_2, \ldots, A_n$ be the n vertices of a convex polygon P in the plane. We call *diagonal* of P, any segment A_iA_j which is not a side of P. We suppose that any three diagonals have no common point, except a vertex. (1) Show that the diagonals intersect each other in $\binom{n}{4}$ interior points of the polygon, and in $\frac{1}{12}n(n-3)(n-4)(n-5)$ exterior points. (2) The sides and the diagonals divide the interior of P into $\frac{1}{24}(n-1)(n-2)(n^2-3n+12)$ convex regions (in the case of Figure 20, we have 11 such regions), and the whole plane into $\frac{1}{24}(n^4-6n^3+23n^2-26n+8)$ regions. (3) The number d_n of ways to cut up the polygon P into $(n-2)$ triangles by means of $n-3$ nonintersecting diagonals (triangulations of P) equals $(n-1)^{-1}\binom{2n-4}{n-2}$, the Catalan number a_{n-1} of p. 53; so, this number is that of well-bracketed words with $(n-1)$ letters. (The heavy lines in Figure 20 give an example of such a triangulation.) [*Hint:* Choose a fixed side, say A_nA_1; from each triangulation, remove the triangle with A_nA_1 as side; then two triangulated polygons are left; hence $d_n=d_2d_{n-1}+d_3d_{n-2}+\cdots+d_{n-1}d_2$; then check the formula, or use [10e] of p. 53.]

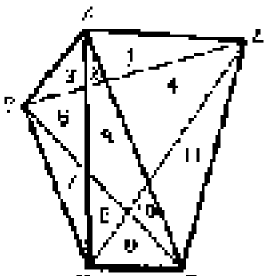


Fig. 20.

Moreover, $2(n-3)d_n=n(d_3d_{n-1}+d_4d_{n-2}+\cdots+d_{n-1}d_3)$. [*Hint:* Use the two triangulated polygons on each side of each of the $2(n-3)$ diagonal vertices A_iA_j.] ([Guy, 1967a]. Very interesting generalizations of the concept of triangulation are found in the papers by Brown, Mullin and Tutte cited in the bibliography.) Finally, there are $n2^{n-5}$ triangulations in which each triangle has at least one side which is side of P, $n\geq 4$. (4) There are $\frac{1}{d}\binom{n-3}{d-1}\binom{n}{d-1}$ ways of decomposing P into d subsets with $d-1$ diagonals that do not intersect in the interior of the polygon ([Prouhet, 1866]). (5) There are $\frac{1}{720}(n)_3(n^3+18n^2-43n+60)$ triangles in the interior of P such that every side is a side or diagonal of P. (6) Suppose n even. The number of graphs with $n/2$ edges that intersect each other outside of the polygon, equals $(n+1)^{-1}\binom{n+1}{n/2}$ (in Figure 21 the 5 graphs corresponding to $n=6$ are pictured). (See [*Yaglom, 1964] I, p. 14.) (7) The number of broken open lines without self-intersections (= the number

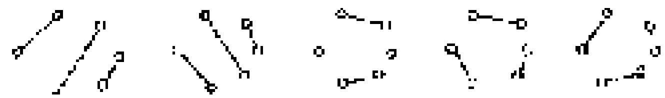


Fig. 21

of piecewise linear homeomorphic images of the segment $[0,1]$ contained in the union of P with its diagonals) whose vertices are vertices of P, equals $n2^{n-3}$. (In Figure 20, $BCAED$ is an example of such a line.) ([(Camille) Jordan, 1920].)

9. *The total number of arrangements of a set with n elements.* This number $P_n:=\sum_{k\leq n}(n)_k$ satisfies $P_n=nP_{n-1}+1$, $n\geq 1$, $P_0:=1$ and $P_n=n!\times\sum_{k\leq n}(1/k!)$. Hence P_n equals the integer closest to $e\cdot n!$. Moreover, we have as GF $\sum_{n\geq 0}P_nt^n/n!=e^t(1-t)^{-1}$.

10. *'Binomial' expansions of an integer.* Let k be an integer ≥ 1. With every integer $n\geq 1$ is associated exactly one sequence of integers b_i such that $n=\binom{b_1}{1}+\binom{b_2}{2}+\cdots+\binom{b_k}{k}$ and $0\leq b_1<b_2<\cdots<b_k$. There also exists $0\leq c_1\leq c_2\leq\cdots\leq c_k$ such that $n=\binom{c_1+1}{1}+\binom{c_2+2}{2}+\cdots+\binom{c_k+k}{k}$.

11. *Greatest common divisor of several integers.* Let $N := \{a_1, a_2, \ldots, a_n\}$ be a set of n integers $\geqslant 1$. Let P_k be the product of the $\binom{n}{k}$ LCM's of all the k-blocks of N; show that the GCD of N equals $P_1P_3P_5\ldots/P_2P_4P_6\ldots$.

12. *Partial sums of the binomial expansion.* Show that for $0 \leqslant k \leqslant n-1$:

$$\sum_{i=0}^{k}\binom{n}{i}a^{n-i}b^{i} = (n-k)\binom{n}{k}\int_{b}^{a+b} t^{k}(a+b-t)^{n-k-1}\,dt$$
$$= (n-k)\binom{n}{k}(a+b)^{n}\int_{0}^{a/b}\frac{u^{n-k-1}}{(1+u)^{n+1}}\,du$$

(See also Exercise 2, (2), p. 72).

13. *Transversals of the Pascal triangle.* Show that $\binom{n}{0}+\binom{n-1}{1}+\binom{n-2}{2}+\cdots = F_{n+1}$, the Fibonacci number (see p. 45) and $\sum_{k\leqslant n}\binom{n-k}{k}x^{k} = (B^{n+1}-A^{n+1})(B-A)^{-1}$, where $A, B = (1\pm\sqrt{1+4x})/2$. More generally, let u, v, w be integers such that $v \geqslant 0$, $w \geqslant 1$, $u \leqslant w$ and let:

$$a_n = a_n(u, v, w) := \binom{n}{v}+\binom{n+u}{v+w}+\binom{n+2u}{v+2w}+\cdots;$$

then

$$\sum_{n\geqslant 0} a_n t^{n} = \frac{t^{v}(1-t)^{w-u-v}}{(1-t)^{w}-t^{w-u}}$$

([*Riordan, 1958], p. 40. See also Exercise 26, p. 84.)

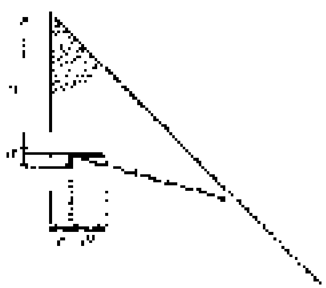


Fig. 12.

14. *The number of binomial coefficients.* For each set E of integers $\geqslant 0$,

$E \subset \mathbf{N}$, and every real number $x > 0$ we put $E(x) := |E \cap [0, x]|$. Let E be the set of values of $\binom{n}{k}$, where k and n are variables with $2 \leqslant k \leqslant n-2$ (one may even suppose that $k \leqslant n/2$). Show that $|E(x)| = \sqrt{2x} + o(\sqrt{x})$. [*Hint:* let $E_k := \left\{\binom{n}{k} \mid n \geqslant 2k\right\}$; then $E = \bigcup_{k\geqslant 2} E_k$; hence $|E_2(x)| \leqslant E(x) \leqslant |E_2(x)| + \sum_{k\geqslant 3}|E_k(x)|$. (For a generalization to multinomial coefficients, see [Erdős, Niven, 1946].)]

15. *Generalization of $\binom{p}{n} \equiv 0 \pmod{p}$ to multinomial coefficients* ([André, 1873]). Let $M := \{a_1, a_2, \ldots, a_m\}$ $(\in N^m)$ be a set of integers $\geqslant 1$, not necessarily distinct. For $n := a_1+a_2+\cdots+a_m$, Theorem A (p. 27) shows that the number $n!/(a_1!a_2!\ldots a_m!)$ is always an integer. This property can be refined as follows. We put, for each integer $d \geqslant 2$, $M(d) := \{x \mid x \in M, d \text{ divides } x\}$ and we let $\gamma(M) := \max_{d\geqslant 2}|M(d)|$. Clearly, $0 \leqslant \gamma(M) \leqslant m$, and $\gamma(M) = m$ if the a_i are not relatively prime, $\gamma(M) = 1$ if each two among a_i are relatively prime, and $\gamma(M) = 0$ if the a_i equal 1. Show then that the number $\{n-(m-\gamma(M))\}!/(a_1!a_2!\ldots a_m!)$ is always an integer (for n prime and $m=2$ we recover Theorem C, p. 14).

16. *Polynomial coefficients* ([André, 1876], [Montel, 1942]). This is the name we give to the coefficients of $f(t) = (1+t+t^2+\cdots+t^{q-1})^{x} := \sum_{k\geqslant 0}\binom{x,q}{k}t^{k}$, for q arbitrary integer $\geqslant 0$, and complex x. Evidently, $\binom{x,2}{k} = \binom{x}{k}$ and $\binom{-x,\infty}{k} = (-1)^{k}\binom{x}{k}$. (1) $\binom{x,q}{k} = \sum\binom{x}{i}\binom{-x}{j}$, where $qi+j=k$. [*Hint:* $f = (1-t^{q})^{x}(1-t)^{-x}$.] (2) If $x=n$ is an integer $\geqslant 0$, then $\binom{n,q}{k}$ is the number of k-combinations of $[n]$ having *less than* q repetitions. Generalize the most important properties of the $\binom{n}{k}$ to these combinatorial coefficients: arithmetical triangle, recurrence relations, congruences, etc., and prove the formula

$$\binom{n,q}{k} = \frac{2}{\pi}\int_{0}^{\pi/2}\left(\frac{\sin q\varphi}{\sin\varphi}\right)^{n}\cos\left(n(q-1)-2k\right)\varphi\,d\varphi.$$

Using this integral representation, find the asymptotic equivalent

$$\sup_k \binom{n,s}{k} \sim s^n \sqrt{\frac{6}{(s^2-1)\pi n}}, \qquad n\to\infty.$$

Here are the first values of *trinomial* coefficients $\binom{n,3}{k}=\binom{n-1,3}{k-2}+\binom{n-1,3}{k-1}+\binom{n-1,3}{k}$ (See also Exercise 19, p. 163):

n\k	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1												
1	1	1	1										
2	1	2	3	2	1								
3	1	3	6	7	6	3	1						
4	1	4	10	16	19	16	10	4	1				
5	1	5	15	30	45	51	45	30	15	5	1		
6	1	6	21	50	90	126	141	126	90	50	21	6	1
7	1	7	28	77	161	266	357	393	357	266	161	77	28
8	1	8	36	112	266	504	784	1016	1107	1016	784	504	266

and of *quadrinomial* coefficients $\binom{n,4}{k}=\binom{n-1,4}{k-3}+\cdots+\binom{n-1,4}{k}$:

n\k	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1												
1	1	1	1	1									
2	1	2	3	4	3	2	1						
3	1	3	6	10	12	12	10	6	3	1			
4	1	4	10	20	31	40	44	40	31	20	10	4	1
5	1	5	15	35	65	101	135	155	155	135	101	65	35
6	1	6	21	56	120	216	336	456	546	580	546	456	336
7	1	7	28	84	203	413	728	1128	1554	1918	2128	2128	1918
8	1	8	36	120	322	728	1428	2472	3823	5328	6728	7728	8092

***17.** *Arithmetic of binomial coefficients.* In the following we denote the GCD of a and b by (a,b); $c\mid d$ means "c divides d", p stands for an arbitrary prime number, and $\equiv$ means congruence modulo this p. (1) $\binom{n}{p}\equiv[n/p]$, the integral part of n/p. (2) $\binom{p+1}{k}\equiv 0$, $\binom{p-1}{k}\equiv(-1)^k$, $\binom{p-2}{k}\equiv(-1)^k(k+1)$, $\binom{p-3}{k}\equiv(-1)^k\binom{k+2}{2}$, ... (Lucas). (3) If $(k,n)=1$, then $n\mid\binom{n}{k}$ (generalization of [6g], p. 14). (4) Let $1\leqslant k\leqslant p^b$ and let a be the exponent of p in k: $p^a\mid k$, $p^{a+1}\nmid k$; then p^{b-a} divides $\binom{p^b}{k}$ and p^{b-a+1} does not divide $\binom{p^b}{k}$ ([Cartier, 1970]). (5) If $(k,n)=(k,n-1)=1$, then $n(n-1)\mid\binom{n}{k}$. If $(k+1,n+1)=(k-2,n+1)=(k-2,n+2)=1$, then $(k+1)(k+2)\mid\binom{n}{k}$ (Cesaro). For all m and n, $m!n!(m+n)!\mid(2m)!(2n)!$ (6) All $\binom{n}{k}$, n fixed, $0\leqslant k\leqslant n$, are odd if and only if $n=2^j-1$. (7) For $2\leqslant k\leqslant n-2$ the coefficient $\binom{n}{k}$ does not equal any power of a prime number ([Herting, 1968], [Stahl, 1969]). [*Hint*: The exponent of p in $n!$ equals $[n/p]+[n/p^2]+[n/p^3]+\cdots$.] (8) Let n and k be integers $\geqslant 0$, written base p as follows: $n=n_0+n_1p+n_2p^2+\cdots$ and $k_0+k_1p+k_2p^2+\cdots$. Then $\binom{n}{k}\equiv\binom{n_0}{k_0}\binom{n_1}{k_1}\binom{n_2}{k_2}\cdots$ ([Lucas, 1878]; See also [Fine, 1947], [Carlitz, 1953b, 1967], [Howard, 1971, 1973]). [*Hint*: by [6a'], p. 14, $(1+x)^p\equiv 1+x^p$, hence $(1+x)^n\equiv(1+x)^{n_0}(1+x^p)^{n_1}(1+x^{p^2})^{n_2}\cdots$.] (9) The largest exponent of p in $\binom{a+b}{a}$ equals the number of carry overs in the addition of a and b base p (Kummer). (10) If $p\geqslant 5$, $\binom{2p-1}{p-1}\equiv 1 \pmod{p^3}$ (Wolstenholme) and, more generally, $\binom{kp-1}{p-1}\equiv 1 \pmod{p^3}$ (Guérin). Many results mentioned here can be generalized to multinomial coefficients with the methods given by [[illegible], 1972]. (11) 2^n always divides $\binom{2^{n+1}}{2^n}-\binom{2^n}{2^{n-1}}$ (Fjeldstad).

18. *Maps from [k] into [n].* (1) The number of strictly increasing maps of $[k]$ into $[n]$ equals $\binom{n}{k}$. (2) The number of increasing maps (but not necessarily strictly increasing) of $[k]$ into $[n]$ equals $\left\langle{n\atop k}\right\rangle=\binom{n+k-1}{k}$. (3) The number of strictly increasing maps φ from $[k]$ into $[n]$ such that x and $\varphi(x)$ are simultaneously odd or even for all $x\in[k]$, equals $\binom{q}{k}$, where q is the largest integer $\leqslant(n+k)/2$ (the so-called *Terquem* problem; for a generalization see [Moser, Abramson, 1969], [*Netto, 1927], p. 313). (4) Compute the number of convex functions of $[k]$ into $[n]$.

19. *Sequences or 'runs'.* These are the names for intervals $S=\{i, i+1, \ldots, i+s-1\}$ contained in a given $A\subset[n]$ such that $S\subset A$ and $i-1\notin A$, $i+s\notin A$. Let $\varrho(A)$ be the number of runs of A. Then, the number of blocks $A\subset[n]$ with r runs ($|A|=a$, $\varrho(A)=r$) equals $\binom{a-1}{r-1}\binom{n-a+1}{r}$. For the circular *a*-blocks with r runs, $A\subset[\dot{n}]$, p. 24, the number is $\frac{n}{n-a}\binom{n-a}{r}\binom{a-1}{r-1}$. More generally, compute the number of divisions $A_1+A_2+\cdots+A_k=[n]$, where $|A_i|=a_i$ are fixed integers $\geqslant 1$, $i\in[k]$ and for which $\sum_{i=1}^{k}\varrho(A_i)=\gamma$.

***20.** *Generalizations of the ballot problem* (Theorem B, p. 21.) (1) Let p, q, r be integers $\geqslant 1$, with $q\geqslant rp$. Show that the number of 'minimal paths' of p. 20, joining O with the point $B(p, q)$ such that each point $M(x, y)$ satisfies $y>rx$ (instead of $y>x$ in Theorem B), equals $\frac{q-rp}{q+p}\times\binom{p+q}{q}$. (For r real >0, see [Takács, 1962].) [*Hint*: The formula evidently holds for the points $B(p, q)$ such that $p=0$ or $q=rp$; show next that if it holds for $(p-1, q)$ and $(p, q-1)$, then it holds for (p, q) as well.] (2) If in the preceding problem, the condition $y>rx$ is replaced by $y\geqslant rx$, then the number of paths becomes $\frac{q+1-rp}{q+1}\binom{p+q}{p}$. (3) More generally, let P be the probability that a path $\mathscr{C}$ of N^d joining O with the point $B(p_1, p_2, \ldots, p_d)$ is such that each of its points $M(x_1, x_2, \ldots, x_d)$ satisfies $x_1\leqslant x_2\leqslant\cdots\leqslant x_d$ (integers p_i satisfy $0\leqslant p_1\leqslant p_2\leqslant\cdots\leqslant p_d$). Then:

$$\mathrm{P}=\prod_{1\leqslant i<j\leqslant d}\left(1-\frac{p_i}{p_j+j-i}\right).$$

([*MacMahon, 1915], p. 133. See also [Narayana, 1959].)

21. *Minimal paths with diagonal steps* ([Goodman, Narayana, 1967], [Moser, Zayachkowski, 1963], [Stocks, 1967]). We generalize the concept of minimal path (p. 20) by allowing also diagonal steps. Figure 23 shows a path with 4 horizontal steps, 3 vertical steps, and 2 diagonal steps. (1) $(q-p)/(q+p-d)$ is the probability that a minimal path with d diagonal steps joining O with (p, q) satisfies $x<y$ (except in O). (2) The total number $D(p, q)$ of paths (of the preceding type) going from O to (p, q) is called *Delannoy number*. It equals $\sum_d\binom{q}{d}\binom{p+q-d}{q}$ or also $\sum_d 2^d\binom{p}{d}\binom{q}{d}$. We have $D(p, q)=D(p, q-1)+D(p-1, q-1)+D(p-1, q)$. Hence, we get the following table of the first values of $D(p, q)$:

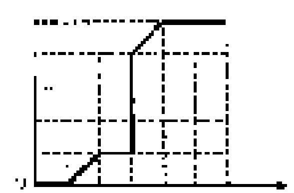


Fig. 23

q\p	0	1	2	3	4	5	6
0	1	1	1	1	1	1	1
1	1	3	5	7	9	11	13
2	1	5	13	25	41	61	85
3	1	7	25	63	129	231	377
4	1	9	41	129	321	681	1289
5	1	11	61	231	681	1683	3653
6	1	13	85	377	1289	3653	8989

The GF $\sum_{p,q\geqslant 0}D(p, q)x^py^q$ is $(1-x-y-xy)^{-1}$ and the diagonal series $\sum_{n\geqslant 0}D(n, n)t^n$ equals $(1-6t+t^2)^{-1/2}$. (3) The total number of paths joining O with (n, n), and diagonals allowed, is $P_n(3)$, where P_n is the Legendre polynomial [14], (p. 50). (4) Let q_n be the number of paths with the property of (3) and satisfying $x<y$ (except at the ends). Then $(n-2)\times q_{n+2}=3(2n+1)q_{n+1}-(n-1)q_n$, $q_1=1$, $q_2=2$. Thus show that $q_n=2s_n$ for $n\geqslant 2$, where s_n is the number of generalized bracketings (see p. 56).

†22. *Minimal paths and the diagonal; Chung-Feller theorem.* In the following 'path' will mean 'minimal path' in the sense of p. 20. (1) The number of paths joining the origin O with (n, n) equals $u_n:=\binom{2n}{n}$. Furthermore, $\sum_{n\geqslant 0}u_nt^n=(1-4t)^{-1/2}$. (2) The number of paths starting at the origin O, and of length $2n$ and such that $x\neq y$, except in O, also equals u_n. [*Hint*:

Use Theorem B, p. 21.] (3) The number of paths joining O with (n, n) and such that $x \neq y$ (except at the ends) equals $f_n := \frac{1}{(2n-1)}\binom{2n}{n} = u_n/(2n-1) = (2/n)u_{n-1}$, $n \geq 1$. Compute $\sum_{n\geq 1} f_n t^n$. (4) The number of paths starting at the origin O, of length $2n$, and with exactly r points (different from O) on the diagonal $x = y$ is equal to $2^r\binom{2n-r}{n}$. Solve an analogous problem for the paths joining the origin O with (n, n). (5) u_n and f_n are defined as in (1), (2) and (3); show that $u_n = f_1 u_{n-1} + f_2 u_{n-2} + \cdots + f_n u_0$, $n \geq 1$. (6) Let $b_{n,k}$ be the number of paths of length $2n$ with the property that $2k$ segments (of the total $2n$) lie above the diagonal $x = y$, $0 \leq k \leq n$ (in Figure 24, $n = 8$, $k = 4$). Let the abscissa of the first passage of the diagonal (different from O) be called $r (\geq 1)$ (so, in Figure 24, $r = 3$). Show that:

$$2b_{n,k} = \sum_{1 \leq r \leq k} f_r b_{n-r,k-r} + \sum_{1 \leq r \leq n-k} f_r b_{n-r,k}.$$

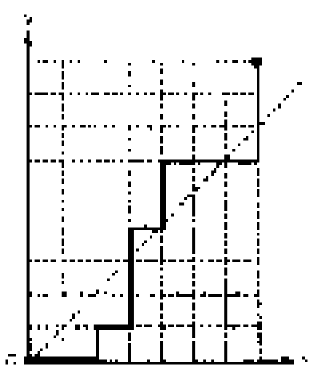


Fig. 24.

(7) Use this to show by induction (on n) that $b_{n,k} = u_k u_{n-k} = \binom{2k}{k} \times \binom{2n-2k}{n-k}$ ([Chung, Feller, 1949], [*Feller, I, 1968], p. 83). (8) Let $c_{n,k}$ be the number of paths of length $2n$ joining O with (n, n) such that $2k$ segments lie above the diagonal. Let r be as in (6) the abscissa of first passage of the diagonal. Show that $c_{n,k}$ does not depend on k, and that it equals $c_n := 1/(n+1)u_n = 1/(n+1)\binom{2n}{n}$, a *Catalan number* of p. 53

([Chung, Feller, 1949], [*Feller, I, 1968], p. 94. See also [Narayana, 1967], [Poupard, 1967].) [*Hint*: The $c_{n,k}$ satisfy the same recurrence relation as the $b_{n,k}$ in (6). Then replace the f_r by $(2/r)u_{r-1}$ (3); change the variable in the second summation: $r := n+1-r$. The value of c_n can then inductively be verified.]

23. *Multiplication table of the factorial polynomials.* We consider the polynomials $(x)_m$, $m = 0, 1, 2, \ldots$, [4f] p. 6; then the product $(x)_m (x)_n$ can be expressed as a linear combination of these polynomials, and actually equals $\sum_k \binom{m}{k}\binom{n}{k} k!(x)_{m+n-k}$, where $k \leq \min(m, n)$. [*Hint*: Use $(1+t+u+tu)^x = (1+t)^x(1+u)^x$ with [2cm] p. 41.] Same problem for the polynomials $\binom{x}{m}$, $\langle x \rangle_m$ and $\left\langle {x \atop m} \right\rangle$.

24. *Formal series and difference operator* Δ. (1) With the notations of [10e] (p. 14) show that $\sum_{n\geq 0} \Delta^k (x^n) t^n/n! = e^{xt}(e^t - 1)^k$ and that $\sum_{r\geq 0} \Delta^k (x^r) t^r/r! = e^{xt}(e^t-1)^k$. (2) If $f = \sum_{n\geq 0} f_n t^n/n!$, then, with the notations of pp. 13 and 41:

$$\sum_{n\geq 0} (\Delta^k f_n) \frac{t^n}{n!} = \sum_{h=0}^{k} (-1)^{k-h} \binom{k}{h} D^h f.$$

(3) If $f = \sum_{n\geq 0} a_n t^n$, then $\sum_{n\geq 0} (\Delta^k a_n) t^{n+k} = (1-t)^k f(t)$ and $\sum_{n\geq 0} (\Delta^n a_0) t^n = (1-t)^{-1} f(t(1-t)^{-1})$.

25. *Harmonic triangle and Leibniz numbers.* Let us define the *Leibniz numbers* by $\mathscr{L}(n, k) = (n+1)^{-1}\binom{n}{k}^{-1} = (k+1)^{-1}\binom{n+1}{k+1}^{-1} = k!\{(n+1)n(n-1)\cdots(n-k+1)\}^{-1}$ if $0 \leq k \leq n$, and $\mathscr{L}(n, k) = 0$ in the other cases. The first values are:

$n \backslash k$	0	1	2	3	4	5
0	1^{-1}					
1	2^{-1}	2^{-1}				
2	3^{-1}	6^{-1}	3^{-1}			
3	4^{-1}	12^{-1}	12^{-1}	4^{-1}		
4	5^{-1}	20^{-1}	30^{-1}	20^{-1}	5^{-1}	
5	6^{-1}	30^{-1}	60^{-1}	60^{-1}	30^{-1}	6^{-1}

Of course, $\mathscr{L}(x, k)$ could be defined for any *real* number

$x \notin \{-1, 0, 1, 2, \ldots, k-1\}$ by the same means. The "harmonic" triangle of numbers has properties very similar to those of the 'arithmetic triangle' (of binomial coefficients p. 12). (1) For $k\geq 1$, $\mathfrak{L}(n,k)+\mathfrak{L}(n,k-1)=\mathfrak{L}(n-1,k-1)$ and $\sum_{m=l}^{n}\mathfrak{L}(m,k)=\mathfrak{L}(l-1,k-1)-\mathfrak{L}(n,k-1)$. So, $\sum_{n=l}^{\infty}\mathfrak{L}(n,k)=\mathfrak{L}(l-1,k-1)$. (2) $\sum_{k=0}^{l}(-1)^k\mathfrak{L}(n,k)=\mathfrak{L}(n+1,0)+(-1)^l\mathfrak{L}(n+1,l+1)$. (3) $\Delta^k\mathfrak{L}(n,0)=(-1)^k\mathfrak{L}(n-k-1,k)$. (4) The following GF holds:

$$\sum_{0\leq k\leq n}\mathfrak{L}(n,k)\,t^{n+1}u^k=\frac{-\log((1-t)(1-ut))}{1+u(1-t)}.$$

So, $\sum_k \mathfrak{L}(n,k)\,u^k=\sum_{l=1}^{n+1}((1+u^l)/l)(u(1+u)^{-1})^{n+1-l}$ and $\sum_{n\geq k}\mathfrak{L}(n,k)\,t^{n+1}=\sum_{l=1}^{k}(-1)^{k-l}l^{-1}t^l(1-t)^{k-l}+(t-1)^k(-\log(1-t))$ (See Exercise 15, p. 294). (5) Let $l(n,k)$ be the 'inverse' of $\mathfrak{L}(n,k)$, in other words: $b_n=\sum_k\mathfrak{L}(n,k)\,a_k\Leftrightarrow a_n=\sum_k l(n,k)\,b_k$ (see p. 143). Then $l(n,k)=c(n-k)\,(k+1)!/n!$, where $(1+\frac{1}{2}t+\frac{1}{3}t^2+\cdots)^{-1}=:\sum_{n\geq 0}c(n)\,t^n$, $c(0)$, $c(1)$, $c(2),\ldots=1, -1, -1, -3, -13, -71, -461,\ldots$ (see Exercise 16, p. 294).

26. *Multisection of series.* Let f be a formal series with complex coefficients, $f=f(t)=\sum_{n\geq 0}a_nt^n$ and $\omega=\exp(2\pi i/r)$ a rth root of unity, r an integer >0. Then, for each integer a, $0\leq a<r$:

$$a_at^a+a_{a+r}t^{a+r}+a_{a+2r}t^{a+2r}+\cdots=\frac{1}{r}\sum_{0\leq j<r}\omega^{-ja}f(\omega^jt).$$

For example: $\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\cdots=\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots=2^{n-1}$; $\binom{n}{0}+\binom{n}{3}+\binom{n}{6}+\cdots=\frac{1}{3}(2^n+2\cos(n\pi/3))$, $\binom{n}{1}+\binom{n}{4}+\binom{n}{7}+\cdots=\frac{1}{3}(2^n+2\cos((n-2)\pi/3))$, $\binom{n}{2}+\binom{n}{5}+\binom{n}{8}+\cdots=\frac{1}{3}(2^n+2\cos((n+2)\pi/3))$, and, more generally,

$$\binom{n}{a}+\binom{n}{a+r}+\binom{n}{a+2r}+\cdots=\frac{1}{r}\sum_{0\leq j<r}\left(2\cos j\frac{\pi}{r}\right)^n\cos\left(j(n-2a)\frac{\pi}{r}\right).$$

(More in [*Riordan, 1968], p. 131. Cf. Exercise 13, p. 76.)

27. *p-bracketings.* Instead of computing the products of the factors pair by pair, as on p. 52, we take now p at a time, but still *adjacent*. We keep p fixed ≥ 2. Then the number $a_{n,p}$ of these p-bracketings ($a_{n,2}=a_n$ as defined on p. 52) satisfies $a_{k(p-1)+1,p}=(1/k)\binom{kp}{k-1}$, $k\geq 1$, and $a_{n,p}$ is zero if n is not of the form $k(p-1)+1$. [*Hint:* $t=y-y^p$, where $y=\sum_{n\geq 0}a_{n,p}t^n$, then use Lagrange formula, p. 148.]

28. *A multiple sum.* We sum over all systems of integers $c_1, c_2, \ldots, c_k\geq 0$ such that $c_1+c_2+\cdots+c_k=n$; show that $a_k:=\sum c_1c_2\cdots c_k=n(n^2-1^2)\cdots(n^2-(k-1)^2)/(2k-1)!$ [*Hint:* $\sum_{n\geq 0}a_nt^n=(\sum_{m\geq 0}mt^m)^k$.]

29. *Hurwitz series.* A formal series $f=\sum_{n\geq 0}f_nt^n/n!$ is called a *Hurwitz series* if all of its coefficients are integers ($\in\mathbb{Z}$). When $\mathfrak{H}$ stands for the set of all such series, show the following properties: (1) $f\in\mathfrak{H}\Rightarrow Df$ and $Pf\in\mathfrak{H}$ (D and P, the differentiation and primitivation operators are defined on p. 41). (2) $f, g\in\mathfrak{H}\Rightarrow f+g, f-g, fg\in\mathfrak{H}$. (3) $f, g\in\mathfrak{H}$, $g_0=\pm 1\Rightarrow fg^{-1}\in\mathfrak{H}$. (4) $f\in\mathfrak{H}$, $f_0=0\Rightarrow\forall m\in\mathbb{N}$, $f^m/m!\in\mathfrak{H}$. (5) $f, g\in\mathfrak{H}$, $g_0=0\Rightarrow f\circ g\in\mathfrak{H}$, where $f\circ g$ is the composition of g with f (p. 40). (6) $f\in\mathfrak{H}$, $f_0=0\Rightarrow f^{\langle\alpha\rangle}\in\mathfrak{H}$, where α is any integer ≥ 0 and $f^{\langle\alpha\rangle}$ is the αth iterate of f (p. 145), with the condition $f_1=\pm 1$ if $\alpha<0$. (7) Let us consider

$$f=f(x,y)=\sum_{k,l\geq 0}f_{k,l}\frac{x^k}{k!}\frac{y^l}{l!},$$

a *two-variable* Hurwitz series, where the Taylor coefficients $f_{k,l}$ are integers ($\in\mathbb{Z}$). If $f_{0,0}=0$ and $f_{0,1}=\pm 1$, then the *implicit* formal series $y=\varphi(x)=\sum_{n\geq 1}\varphi_nx^n/n!$ such that $f(x,\varphi(x))=0$ is also an Hurwitz series: every $\varphi_n\in\mathbb{Z}$ (see [Comtet, 1968, 1974] and p. 153).

*30. *Hadamard product.* The Hadamard product ([Hadamard, 1893]; see also [Benzaghou, 1968]) of two formal series $f:=\sum_{n\geq 0}a_nx^n$, $g:=\sum_{n\geq 0}b_nx^n$ is defined by $f\odot g:=\sum_{n\geq 0}a_nb_nx^n$. (1) The set of all formal series with complex coefficients is an *algebra* for the operations $+$ and $\odot$. (2) Now we suppose that $f(t)$ and $g(t)$ are *convergent* in a neighborhood of 0, $t\in\mathbb{C}$. Then:

$$(f\odot g)(t)=\frac{1}{2\pi i}\int f(z)\,g\left(\frac{t}{z}\right)\frac{dz}{z}=C_{z^0}f(z)\,g\left(\frac{t}{z}\right)$$

where the integration contour goes around the origin in such a way that $f(z)$ is analytic on the interior, and $g(t/z)$ is analytic on the exterior, t fixed and small. The symbol C_0 means 'coefficient of the constant term in the Laurent series'. (Compare [12g], p. 47.) (3) If f and g are expansions of rational fractions, then $f\odot g$ is too. Thus, for $f=(t^2-t+p)^{-1}$, $p\neq 0$, we have $f\odot f=(p+t)(p-t)^{-1}(p^2-t(1-2p)+t^2)^{-1}$. More generally, compute $f^{\odot n}$ in this case. (4) If f is rational, and g is algebraic, then $f\odot g$ is algebraic ([Jungen, 1931], [Schützenberger, 1962]). (5) If f and g satisfy a differential equation with polynomial coefficients, then $f\odot g$ does.

*31. *Powers of the Fibonacci numbers.* Let $\Phi_k(t):=\sum_{n\geq 0}F_n^k t^n=(\Phi(t))^{\odot k}$ with the F_n p. 45 and the preceding exercise. Then:

$$(1-2t-2t^2+t^3)\,\Phi_2(t)=1-t.$$

[Use that $F_n=(\alpha^{n+1}-\beta^{n+1})/\sqrt{5}$.] More generally determine explicitly and inductively the sequence $\Phi_k(t)$. ([Riordan, 1962b], [Carlitz, 1962c], [Horadam, 1965].)

32. *Integers generated by ch t/cos t.* We define the Salié's integers S_{2n} by:

$$\sigma(t):=\frac{\operatorname{ch} t}{\cos t}=\sum_{n\geq 0}S_{2n}\frac{t^{2n}}{(2n)!}.$$

We want to show that S_{2n} is divisible by 2^n. More precisely, there exist integers S'_{2n} such that

[4a] $\quad S_{2n}=2^nS'_{2n};$

[4b] $\quad S'_{2n}\equiv(-1)^{\binom{n}{2}}\pmod 4.$

([Carlitz, 1959, 1965c], [Gandhi, Singh, 1966]. We give the method of [Salié, 1963].) (1) The expansion $(\operatorname{ch}(ut))/\cos t=\sum S_{2n}(u)\,t^{2n}/(2n)!$ defines polynomials $S_{2n}(u)$ such that $S_{2n}=S_{2n}(1)$, satisfying $u^{2n}=\sum_k(-1)^{n-k}\binom{2n}{2k}S_{2k}(u)$. (2) Thus $(1+u^2)^n=\sum_{0\leq k\leq n}(-1)^k\binom{n}{k}\times 2^{2k}S_{2n-2k}(u)$. (3) Hence, by induction, $S_{2n}(u)=\sum_{k}2^{2n-2k}C(n,k)\times(1+u^2)^{k}$, where the $C(n,k)$ are integers. (4) Moreover, $C(n,1)=1$,

$C(n,2)=\binom{n}{2}$, $C(n,3)=\binom{n}{3}+\binom{n-1}{2}\binom{n}{4}$. (5) [4a] then follows from (3), with $u=1$. (6) Hence, by (3), $S'_{2n}=\sum_{k=1}^n 2^{k}C(n,k)$, so [4b] follows. (7) Show that $S_{2n}=\sum_k\binom{2n}{2k}|E_{2k}|$; E_{2k} is an Euler number (p. 48).

$2n$	0	2	4	6	8	10	12	14	16
S'_{2n}	1	1	3	19	217	3961	105963	3908059	190065457

33. *Generating function of min.* ([Carlitz, 1962a], where the GF of $\max(n_1,n_2,\ldots,n_k)$ is also found.) Show that:

$$\sum_{n_1,n_2,\ldots,n_k\geq 1}\min(n_1,n_2,\ldots,n_k)\,t_1^{n_1}t_2^{n_2}\ldots t_k^{n_k}=\frac{t_1t_2\ldots t_k}{(1-t_1)(1-t_2)\ldots(1-t_k)(1-t_1t_2\ldots t_k)}$$

*34. *Expansion of a rational fraction.* Let $\mathfrak{R}$ be the set of rational fractions with complex coefficients in one indeterminate t: $f\in\mathfrak{R}$ if and only if $f=P(t)/Q(t)$ where P and Q are polynomials, $Q(0)\neq 0$. Show the equivalence of the following four definitions: (1) $\mathfrak{R}$ 'is' the set of sums $W(t)=\sum_{(k,l)\in E}b_{k,l}(1-\beta_kt)^{-\alpha_{k,l}}$, where $b_{k,l}$, $\beta_k\in\mathbb{C}$, $\alpha_{k,l}$ integers ≥ 1, and E a finite subset of $\mathbb{N}^2$. (2) $\mathfrak{R}$ 'is' the set of formal series $\sum_{n\geq 0}a_nt^n$ whose coefficients satisfy a linear recurrence with constant coefficients c_j: $\sum_{j=0}^{q}c_ja_{n+j}=0$, $n\geq n_0$. (3) $\mathfrak{R}$ 'is' the set of formal series whose coefficients are of the form $a_n=\sum_{j=1}^rA_j(n)\,\beta_j^n$, $n\geq n_0$, where the A_j are polynomials, and the $\beta_j\neq 0$. (4) $\mathfrak{R}$ is the set of formal series $f=\sum_{n\geq 0}a_nt^n$ such that for each series there exist two integers d and q for which $H_{d+j}^{(q+1)}(f)=0$ for all integers $j\geq 0$, where $H_n^{(k)}(f)$ are the *Hankel* determinants of f:

$$H_n^{(k)}(f):=\begin{vmatrix}a_n & a_{n+1} & \cdots & a_{n+k-1}\\ a_{n+1} & a_{n+2} & \cdots & a_{n+k}\\ \vdots & & & \\ a_{n+k-1} & a_{n+k} & \cdots & a_{n+2k-2}\end{vmatrix}.$$

35. *Explicit values of the Chebyshev, Legendre and Gegenbauer polynomials.* Use $(1-tx)(1-2tx+t^2)^{-1}=(1-tx)(1+t^2)^{-1}(1-2tx(1+t^2)^{-1})^{-1}$

(p. 50) to show that $T_n(x)=(n/2)\sum_{0\le m\le n/2}(-1)^m(n-m-1)!\,(m!\,(n-2m)!)^{-1}(2x)^{n-2m}$ (compare Exercise 1, p. 155). Similarly, calculate the polynomials $U_n(x)$ and $C_n^\lambda(x)$ (from which $P_n(x)$ can be obtained).

Finally, establish the following expressions with determinants of order n:

$$T_n(\cos\varphi)=\cos n\varphi=\begin{vmatrix}\cos\varphi & 1 & 0 & 0 & \cdots\\ 1 & 2\cos\varphi & 1 & 0 & \cdots\\ 0 & 1 & 2\cos\varphi & 1 & \cdots\\ 0 & 0 & 1 & 2\cos\varphi & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots\end{vmatrix}$$

$$U_n(\cos\varphi)=\frac{\sin(n+1)\varphi}{\sin\varphi}=\begin{vmatrix}2\cos\varphi & 1 & 0 & 0 & \cdots\\ 1 & 2\cos\varphi & 1 & 0 & \cdots\\ 0 & 1 & 2\cos\varphi & 1 & \cdots\\ 0 & 0 & 1 & 2\cos\varphi & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots\end{vmatrix}$$

36. *Miscellaneous Taylor coefficients using Bernoulli numbers.* Use $\operatorname{th} x=(e^{2x}-1)(e^{2x}+1)^{-1}=1-2(e^{2x}-1)^{-1}+4(e^{4x}-1)^{-1}$, and [14a] (p. 48) to show that $\operatorname{th} x=\sum_{m\ge1}B_{2m}2^{2m}(2^{2m}-1)x^{2m-1}/(2m)!$. From this, obtain: $\operatorname{tg} x=x+\frac{1}{3}x^3+\frac{2}{15}x^5+\frac{17}{315}x^7+\frac{62}{2835}x^9+\frac{1382}{155925}x^{11}+\cdots=\sum_{m\ge1}B_{2m}(-1)^{m+1}2^{2m}(2^{2m}-1)x^{2m-1}/(2m)!$. (See also Exercise 11 of p. 258.) Complex variables methods can be used to show that the radius of convergence of the preceding series equals $\pi/2$.

$$\operatorname{cotg} x=x^{-1}-\tfrac{1}{3}x-\tfrac{1}{45}x^3-\tfrac{2}{945}x^5-\tfrac{1}{4725}x^7-\cdots$$
$$=x^{-1}+\sum_{m\ge1}B_{2m}(-1)^m\frac{2^{2m}x^{2m-1}}{(2m)!}.$$
$$(\sin x)^{-1}=x^{-1}+\tfrac{1}{6}x+\tfrac{7}{360}x^3+\tfrac{31}{15120}x^5+\tfrac{127}{604800}x^7+\cdots$$
$$=x^{-1}+\sum_{m\ge1}B_{2m}(-1)^{m-1}(2^{2m}-2)\frac{x^{2m-1}}{(2m)!}.$$

Use this to obtain $\log(\cos x)=\sum_{m\ge1}(-1)^mB_{2m}2^{2m-1}(2^{2m}-1)x^{2m}/m(2m)!$ and $\log((\sin x)/x)=\sum_{m\ge1}(-1)^mB_{2m}2^{2m-1}x^{2m}/m(2m)!$.

Put now $\zeta(s)=\sum_{n\ge1}n^{-s}$ with $s>1$. Use either the Fourier expansion

$$B_{2k}(x)=2(-1)^{k-1}(2k)!\sum_{n\ge1}\frac{\cos 2\pi nx}{(2\pi n)^{2k}}$$

or the expansion in rational functions

$$\operatorname{cotg} z=1/z+\textstyle\sum_{n\ge1}2z(z^2-n^2\pi^2)^{-1}$$

to show, by [14c] (p. 48), that

$$\zeta(2k)=\frac{(2\pi)^{2k}}{2(2k)!}|B_{2k}|\quad\text{or}\quad\sum_{k\ge1}\zeta(2k)t^{2k}=\tfrac{1}{2}-\tfrac{1}{2}\pi t\operatorname{cotg}\pi t.$$

Thus, $\zeta(2)=\pi^2/6$, $\zeta(4)=\pi^4/90$, $\zeta(6)=\pi^6/945$, $\zeta(8)=\pi^8/9450$.

Use this and Exercise 11, p. 258, ([Chowla, Hartung, 1972]) to obtain an *explicit formula* for the Bernoulli numbers, with only a *single* sum (p. 41) and $[x]$ the greatest integer $\le x$:

$$B_{2n}=(-1)^{n-1}\frac{1+[\varphi_n]}{2(2^{2n}-1)},\quad\text{where}\quad\varphi_n=\frac{2(2^{2n}-1)(2n)!}{2^{2n-1}\pi^{2n}}\sum_{k=1}^{5n}\frac{1}{k^{2n}}.$$

[Compare with Exercise 4, p. 220.] Finally, prove that

$$\sum_{n\ge1}\frac{1}{\binom{2n}{n}}=\frac{1}{3}+\frac{2\pi\sqrt3}{27},\quad\sum_{n\ge1}\frac{1}{n\binom{2n}{n}}=\frac{\pi\sqrt3}{9},\quad\sum_{n\ge1}\frac{1}{n^2\binom{2n}{n}}=\frac{\pi^2}{18},\quad\sum_{n\ge1}\frac{1}{n^4\binom{2n}{n}}=\frac{17\pi^4}{3240}.$$

37. *Using the Euler numbers.* We put $\beta(s)=\sum_{n\ge0}(-1)^n(2n+1)^{-s}$, with $s>0$. Then, by [14f] (p. 48), and using either the Fourier expansion

$$E_{2k}(x)=4(-1)^k(2k)!\sum_{n\ge0}\frac{\sin(2n+1)\pi x}{((2n+1)\pi)^{2k+1}}$$

or the expansion into rational functions

$$\frac{1}{\cos z}=-4\pi\sum_{n\ge0}\frac{(-1)^n(2n+1)}{4z^2-(2n+1)^2\pi^2},$$

show that

$$\beta(2k+1)=\frac{(\pi/2)^{2k+1}}{2(2k)!}|E_{2k}|.$$

Thus, $\beta(1)=\pi/4$, $\beta(3)=\pi^3/32$, $\beta(5)=5\pi^5/1536$.

38. *Sums of powers of binomial coefficients.* For any real number r, let us denote $B(n,r) := \sum_{k\geq 0}\binom{n}{k}^r$. Evidently, $B(n,0)=n+1$, $B(n,1)=2^n$, $B(n,2)=\binom{2n}{n}$ (p. 154). (1) Prove the following recurrences: $n^2B(n,3)=$ $=(7n^2-7n+2)\,B(n-1,3)+8(n-1)^2\,B(n-2,3)$ and $n^3B(n,4)=$ $2(2n-1)\,(3n^2-3n+1)\,B(n-1,4)+(4n-3)\,(4n-4)\,(4n-5)\,B(n-2,4)$ ([Franel, 1895]). (2) More generally, for every integer $r\geq 0$, the function $f_r(t) := \sum_k \binom{n}{k}^r t^k$ is algebraic and $B(n,r)$ (r fixed) satisfies a linear recurrence of which the coefficients are polynomials in n. [*Hint*: (4) Exercise 30 p. 85, and [Comtet, 1964].] (3) For any real number $\beta>0$, we have $B(\beta,2) := \sum_{k\geq 0}\binom{\beta}{k}^2 = 2^{2\beta}\pi^{-1/2}\Gamma(\beta+1/2)/\Gamma(\beta+1)$. (4) For any $r>0$, we have ([*Pólya, Szegö, 1], p. 42) the asymptotic result:

$$B(n,r) \sim \frac{2^{rn}}{\sqrt{r}}\left(\frac{2}{\pi n}\right)^{(r-1)/2}, \qquad n\to\infty.$$

39. *Transitive closure of a binary relation.* For two relations $\mathfrak{R}$ and $\mathfrak{S}$ on N, the *transitive product* $\mathfrak{R}\circ\mathfrak{S}$ is defined by $x(\mathfrak{R}\circ\mathfrak{S})\,y \Leftrightarrow \exists z\in N$, $x\mathfrak{R}z$, $z\mathfrak{S}y$. The transitive closure $\hat{\mathfrak{R}}$ of a relation $\mathfrak{R}$ is the 'smallest' transitive relation containing $\mathfrak{R}$ (= the intersection of transitive relations containing $\mathfrak{R}$). Show that $\hat{\mathfrak{R}}=\mathfrak{R}\cup\mathfrak{R}\circ\mathfrak{R}\cup\mathfrak{R}\circ\mathfrak{R}\circ\mathfrak{R}\cup\ldots$.

40. *Forests and introductions.* We consider a graph $\mathscr{G}$ over E (possibly infinite), which is a *forest*. In other words, there exist trees $(A_1,\mathscr{A}_1)$, $(A_2,\mathscr{A}_2)$ ... such that $E=A_1+A_2+\cdots$ and $\mathscr{G}=\mathscr{A}_1+\mathscr{A}_2+\cdots$.

(1) Show that E can be divided into two subsets V and W, $E=V+W$, such that $\mathfrak{P}_2(V)\subset\bar{\mathscr{G}}$ and $\mathfrak{P}_2(W)\subset\bar{\mathscr{G}}$ ($\bar{\mathscr{G}}$ means the complementary graph of $\mathscr{G}$, p. 62). [*Hint*: Choose $x_i\in A_i$, then divide A_i into V_i, W_i, where V_i is the set of $x\in A_i$ whose distance to x_i is even (p. 63); then take $V=V_1\cup V_2\cup\cdots$.]

(2) In any meeting of citizens of a city X, the number of necessary introductions is less than the number of people present at that meeting. Show that the population of X can be divided into two classes, such that in each of these two classes all people know each other.

41. *The pigeon-hole principle.* (1) If $(n+1)$ objects are distributed over n containers, then one container at least contains at least 2 objects. More generally, let $\mathscr{C}$ be a system of m subsets (not necessarily distinct) of N, $|N|=n$, $|\mathscr{C}|=m$, such that $\sum_{B\in\mathscr{C}}|B|=w$. Then a sufficient condition for h points of N to be k times covered by $\mathscr{C}$, is $w\geq(k-1)n+(h-1)\times$ $\times(m-k+1)+1$. (2) Let N be a set of $n(\geq 1)$ objects, not necessarily distinct. For one of the two following is the case: (I) $(n+1)$ objects are identical; (II) $(n+1)$ are distinct.

42. *Filter bases.* This is the name for a system $\mathscr{S}$ of N, $\mathscr{S}\subset\mathfrak{P}^*(N)$, such that for $A, B\in\mathscr{S}$ there exists a $C\in\mathscr{S}$ such that $C\subset A\cap B$. The number of filter bases of N, $|N|=n$, equals $\sum_{k\geq 0}\binom{n}{k}2^{2^k-1}$ and this is asymptotically equal to $n2^{2^n-1}$ for $n\to\infty$ ([Comtet, 1966]).

43. *Idempotents of $\mathfrak{F}(N)$ and forests of height $\leq h$.* Let $\mathfrak{F}(N)$ be the set of maps of a finite set N into itself, $\mathfrak{F}(N)=N^N$, $|N|=n$; $\mathfrak{F}(N)$ is also the *symmetric semigroup* (or *monoid*) of N. A map $f\in\mathfrak{F}(N)$ is called *idempotent* if and only if for all $x\in N$, $f(f(x))=f(x)$. (1) f is idempotent if and only if the restriction of f to its image $f(N)$ is the identity. (2) The number $i(n)$ of idempotent maps equals $\sum_{k=1}^{n}\binom{n}{k}k^{n-k}$ ([Harris, Schoenfeld, 1967], [Tainiter, 1968]).

n	1	2	3	4	5	6	7	8	9	10
$i(n)$	1	3	10	41	196	1057	6322	41393	293608	2237921

(3) Observe that $1+\sum_{n\geq 1} i(n)\,x^n/n! = \exp(xe^x)$. Use this to give an asymptotic estimate of $i(n)$. (*Hint*: Use the saddle point method ([*De Bruijn, 1961], p. 77).) (4) Let $F(n,h)$ be the number of forests (p. 70) such that the height of every rooted tree is $\leq h$ (p. 70). Show that $F(n,1)=i(n)$. Compute $F(n,h)$ ([Riordan, 1968a]).

44. *Finite geometries.* Let Δ be a projective space of dimension n over a *finite* field K (= the Galois field $GF(q)$) of $q=p^t$ elements, where p is a prime number. One often writes that Δ is a $PG(n,q)$. E is the vector space from which Δ is obtained; $\dim E=n+1$. (1) The number of nonzero vectors of E is $q^{n+1}-1$; use this to show that the number of points of S equals $(q^{n+1}-1)/(q-1)$. (2) The number of sets of $k+1$ independent points (obtained from $(k+1)$ independent vectors of E) equals $q^{\binom{k+1}{2}}(q^{n+1}-1)(q^n-1)\cdots(q^{n-k+1}-1)(q-1)^{-k-1}$. (3) Deduce that the number of projective varieties of dimension k in S equals:

$$\frac{(q^{n+1}-1)(q^n-1)\cdots(q^{n-k+1}-1)}{(q^{k+1}-1)(q^k-1)\cdots(q-1)}.$$

(For other analogous formulas, see [*Vajda, 1967a, b]. Compare also Exercise 11, p. 118.)

*45. *Bipartite trees.* Let a bipartition of a set P be given, $M+N=P$ such that $m=|M|\geqslant 1$, $n=|N|\geqslant 1$. Show that the number of *trees* over P such that each of $(m+n-1)$ edges of such a tree connects a point of M with a point of N, equals $m^{n-1}n^{m-1}$. (On this subject, see [Austin, 1960], [*Berge, 1968], p. 91, [Glicksman, 1963], [Raney, 1964], [Scoins, 1962], and especially [Knuth, 1968].)

46. *Binomial determinants.* We recall the notation $(a,b)=\binom{a+b}{a}$ (cf. p. 8). The following determinants of order r, taken from the table of binomial coefficients satisfy:

$$\begin{vmatrix} \binom{n}{k} & \binom{n}{k+1} & \cdots & \binom{n}{k+r-1} \\ \binom{n+1}{k} & \binom{n+1}{k+1} & \cdots & \binom{n+1}{k+r-1} \\ \vdots & \vdots & & \vdots \\ \binom{n+r-1}{k} & \binom{n+r-1}{k+1} & \cdots & \binom{n+r-1}{k+r-1} \end{vmatrix} = \frac{\binom{n}{k}\binom{n+1}{k}\cdots\binom{n+r-1}{k}}{\binom{k}{k}\binom{k+1}{k}\cdots\binom{k+r-1}{k}};$$

$$\begin{vmatrix} (a,b) & (a,b+1) & \cdots & (a,b+r-1) \\ (a+1,b) & (a+1,b+1) & \cdots & (a+1,b+r-1) \\ \vdots & \vdots & & \vdots \\ (a+r-1,b) & (a+r-1,b+1) & \cdots & (a+r-1,b+r-1) \end{vmatrix} = \frac{(a,b)(a+1,b)\cdots(a+r-1,b)}{(0,b)(1,b)\cdots(r-1,b)}.$$

Generalize this to determinants extracted from the table of binomial coefficients with row or column indices in arithmetic progression. (See [Zeipel, 1865] and [*Netto, 1927], p. 256.)

*47. *Equal binomial coefficients.* Determine all solutions in positive integers n, m, n', m' of $\binom{n}{m}=\binom{n'}{m'}$. *Examples:* $\binom{10}{3}=\binom{16}{2}=120$, $\binom{14}{6}=\binom{15}{5}=3003$.

PARTITIONS OF INTEGERS

The concept of partition of integers belongs to number theory as well as to combinatorial analysis. This theory was established at the end of the 18-th century by Euler. (A detailed account of the results up to ca. 1920 is found in [*Dickson, II, 1919], pp. 101–64.) Its importance was enhanced by [Hardy, Ramanujan, 1918] and [Rademacher, 1937a, b, 1938, 1940, 1943] giving rise to generalizations, which have not been exhausted yet. We will treat here only a few elementary (combinatorial and algebraical) aspects. For further reading we refer to [*Hardy, Wright, 1965], [*MacMahon, 1915–16], [Andrews, 1970, 1972b], [*Andrews, 1971], [Gupta, 1970], [Sylvester, 1884, 1886] (or *Collected Mathematical Papers*, Vol. 4, 1–83), and, for the beautiful asymptotic problems, to [*Ayoub, 1963] and [*Ostmann, 1956]. We use mostly the notations of the tables of [*Gupta, 1962], which are the most extensive ones on this matter.

2.1. DEFINITIONS OF PARTITIONS OF AN INTEGER n

DEFINITION A. *Let n be an integer ≥ 1. A partition of n is a representation of n as a sum of integers ≥ 1*, not considering the order of terms of this sum. *These terms are called* summands, *or* parts, *of the partition.*

We list all partitions of the integers 1 through 5: 1; $2=1+1$; $3=2+1=1+1+1$; $4=3+1=2+2=2+1+1=1+1+1+1$; $5=4+1=3+2=3+1+1=2+2+1=2+1+1+1=1+1+1+1+1$.

It is important to distinguish clearly between a partition of a set (p. 30) and a partition of an integer. But in the first case as well as in the second case, the order of the blocks and the order of the summands respectively does not play a role, and no block is empty, just like no summand equals zero.

Let $p(n)$ be the number of partitions of n and let $P(n, m)$ be the number of partitions of n into m summands. Thus, by the preceding list, $p(1)=1$, $p(2)=2$, $p(3)=3$, $p(4)=5$, $p(5)=7$ and $P(5,1)=P(5,4)=P(5,5)=1$,

$P(5,2)=P(5,3)=2$. Clearly, $p(n)=\sum_{m=1}^{n} P(n,m)$ and, since the order of the summands does not matter, we have:

DEFINITION B. *Each partition of n into m summands can be considered as a solution with integers $y_i \geq 1$ (= the summands of the partition) of:*

$$\text{[1a]} \qquad y_1+y_2+\cdots+y_m=n, \qquad y_1 \geq y_2 \geq \cdots \geq y_m \geq 1.$$

With such a partition, we can associate a minimal increasing path (in the sense of p. 20) starting from $W(0, 1)$, with m horizontal steps and with area contained under its graph equal to n. Figure 24' clarifies this idea for the partition $1+3+3+5$ of 12. But the interpretation related to Ferrers diagrams (p. 100) will turn out to be more rewarding.

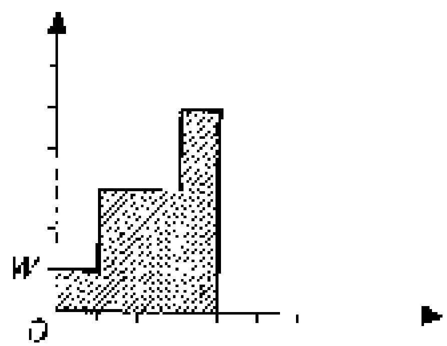


Fig. 24'.

THEOREM A. *Giving a partition of n, in other words, giving a solution of [1a], is equivalent to giving a solution with integers $x_i \geq 0$ (the number of summands equal to i) of:*

$$\text{[1b]} \qquad x_1+2x_2+\cdots+nx_n=n \quad (\text{also denoted by } x_1+2x_2+\cdots=n).$$

If the partition has m summands, we must add to [1b] the following condition:

$$\text{[1c]} \qquad x_1+x_2+\cdots+x_n=m \quad (\text{also denoted by } x_1+x_2+\cdots=m).$$

■ Evident. ■ If $(x_1, x_2, \ldots)$ are the numbers x_i in [1b], we call the corresponding partition "the partition with specification $1^{x_1}2^{x_2}\ldots$", omitting the exponents x_i which equal 1. Written in this way, the partitions of 5 become 5, 14, 23, 1^23, 12^2, 1^32, 1^5.

We write $p(n,m)$ for the number of partitions of n with *at most* m summands, or also 'distribution function' of the number of partitions of n with respect to the number of summands, $p(n,m)=\sum_{k=1}^{m}P(n,k)$, $P(n,m)=p(n,m)-p(n,m-1)$. (The analogy with a stochastic distribution function will be noted.)

THEOREM B. *If $m\geq n\geq 1$, then $p(n,m)=p(n)$, and for $n\geq m\geq 2$:*

[1d] $\quad p(n,m)=p(n,m-1)+p(n-m,m); \qquad p(n,1)=1,\ p(0,m)=1.$

■ $p(n,m)$ is the number of solutions of [1b] that satisfy $x_1+x_2+\cdots\leq m$ also. So we divide the set of solutions into two parts: first the solutions of [1b] that *also* satisfy $x_1+x_2+\cdots\leq m-1$; there are $p(n,m-1)$ of these; then the solutions of [1b] which *also* satisfy $x_1+x_2+\cdots=m$; these are just the solutions of $x_2+2x_3+\cdots=n-m$ and $x_2+x_3+\cdots\leq m$ (since $x_1\geq 0$), hence there are $p(n-m,m)$ of these. ■

The following table shows the first values of $p(n,m)$ (boldface printed: $p(n)$). [See also [*Gupta, 1962], $n\leq 400$, $m\leq 50$. For a table of $p(n,m)$ and $p(n)$ see p. 307.]

$m\backslash n$	0	1	2	3	4	5	6	7	8	9
1	1	**1**	1	1	1	1	1	1	1	1
2	1	1	**2**	2	3	3	4	4	5	5
3	1	1	2	**3**	4	5	7	8	10	12
4	1	1	2	3	**5**	6	9	11	15	18
5	1	1	2	3	5	**7**	10	13	18	23
6	1	1	2	3	5	7	**11**	14	20	26
7	1	1	2	3	5	7	11	**15**	21	28
8	1	1	2	3	5	7	11	15	**22**	29
9	1	1	2	3	5	7	11	15	22	**30**

2.2. GENERATING FUNCTIONS OF $p(n)$ AND $P(n,m)$

THEOREM A. *The generating function of the number $p(n)$ of partitions of n equals:*

[2a] $$\Phi(t):=1+\sum_{n\geq 1}p(n)t^n=\prod_{i\geq 1}(1-t^i)^{-1}=\frac{1}{(1-t)(1-t^2)(1-t^3)\cdots}$$

■ The family of formal series $a_i=(1-t^i)^{-1}=1+t^i+t^{2i}+\cdots$ is indeed multipliable, since $\omega(a_i-1)=i$ (cf. p. 40). If we let x_i stand for integers ≥ 0, we obtain:

[2b] $$\prod_{i\geq 1}(1-t^i)^{-1}=\prod_{i\geq 1}(1+t^i+t^{2i}+\cdots)=\prod_{i\geq 1}\Big(\sum_{x_i\geq 0}t^{ix_i}\Big)=\sum_{x_1,x_2,\ldots\geq 0}t^{x_1+2x_2+3x_3+\cdots},$$

and this proves that the coefficient of t^n in [2b] is just the number of solutions of [1b] p. 95, hence $p(n)$. ■

One could prove that $\Phi(t)$, written in the form [2a] as a series or as an infinite product, is convergent for $|t|<1$.

For *given* integer n, the *actual* computation of $p(n)$ by [2a] is evidently performed by just considering the *finite* product $\prod_{i=1}^{n}(1-t^i)^{-1}$.

THEOREM B. *The generating function of the number $P(n,m)$ of the partitions of n into m summands equals:*

[2c] $$\Phi(t,u):=1+\sum_{1\leq m\leq n}P(n,m)t^nu^m=\prod_{i\geq 1}(1-ut^i)^{-1}=\frac{1}{(1-ut)(1-ut^2)(1-ut^3)\cdots}$$

■ As in the preceding proof, we have:

[2d] $$\prod_{i\geq 1}(1-ut^i)^{-1}=\prod_{i\geq 1}\Big(\sum_{x_i\geq 0}u^{x_i}t^{ix_i}\Big)=\sum_{x_1,x_2,\ldots\geq 0}t^{x_1+2x_2+\cdots}u^{x_1+x_2+\cdots}.$$

Hence indeed the coefficient of t^nu^m in [2d] equals the number of solutions of [1b, c] (p. 95). ■

2.3 CONDITIONAL PARTITIONS

More generally, let $\mathrm{p}(n \mid \mathscr{P}_1, \mathscr{P}_2)$ be the number of partitions of n such that the number of summands has the property $\mathscr{P}_1$, and the value of each summand has the property $\mathscr{P}_2$; we indicate by a star $*$ the absence of a condition (notations from [*Ayoub, 1963], p. 196). Thus, $\mathrm{p}(n, m) = \mathrm{p}(n \mid \leqslant m, *)$, $P(n, m) = \mathrm{p}(n \mid m, *)$. We also denote the number of partitions of n, that satisfy $\mathscr{P}_1$ and $\mathscr{P}_2$ in the sense above, and whose summands are *all unequal*, by $\mathrm{q}(n \mid \mathscr{P}_1, \mathscr{P}_2)$. Thus, $\mathrm{q}(n \mid *, \leqslant r)$ is the number of partitions of n into unequal summands that are all $\leqslant r$.

THEOREM A *Let:*

[3a] $$\Xi(t, u) := \sum_{n, m \geqslant 0} \mathrm{p}(n \mid m, \mathscr{P})\, t^n u^m;$$

then:

[3b] $$\sum_{n \geqslant 0} \mathrm{p}(n \mid *, \mathscr{P})\, t^n = \Xi(t, 1)$$

[3c] $$\sum_{n \geqslant 0} \mathrm{p}(n \mid \text{even}, \mathscr{P})\, t^n = \tfrac{1}{2}\{\Xi(t, 1) + \Xi(t, -1)\}$$

[3d] $$\sum_{n \geqslant 0} \mathrm{p}(n \mid \text{odd}, \mathscr{P})\, t^n = \tfrac{1}{2}\{\Xi(t, 1) - \Xi(t, -1)\}$$

[3e] $$\sum_{n, m \geqslant 0} \mathrm{p}(n \mid \leqslant m, \mathscr{P})\, t^n u^m = (1-u)^{-1}\, \Xi(t, u).$$

Analogous formulas hold when everywhere in [3a, b, c, d, e] of p *is replaced by* q.

■ [3b] follows from $\mathrm{p}(n \mid *, \mathscr{P}) = \sum_{m \geqslant 0} \mathrm{p}(n \mid m, \mathscr{P})$; [3c] from $\mathrm{p}(n \mid \text{even}, \mathscr{P}) = \sum_{m \geqslant 0} \mathrm{p}(n \mid 2m, \mathscr{P})$; [3d] from $\mathrm{p}(n \mid \text{odd}, \mathscr{P}) = \sum_{m \geqslant 0} \mathrm{p}(n \mid 2m+1, \mathscr{P})$; [3e] from $\mathrm{p}(n \mid \leqslant m, \mathscr{P}) = \sum_{i=0}^{m} \mathrm{p}(n \mid i, \mathscr{P})$. ■

THEOREM B *Let $\mathfrak{A}$ be an infinite matrix consisting of 0 and 1, $\mathfrak{A} = [a_{i,j}]$, $i \geqslant 1$, $j \geqslant 0$, $a_{i,j} = 0$ or 1. Denoting by $\mathrm{p}(n \mid m, \mathfrak{A})$ the number of partitions of n into m summands such that the number of summands equal to i, equals one of the integers $j \geqslant 0$ for which $a_{i,j} = 1$. Then we have:*

[3f] $$\sum_{n, m \geqslant 0} \mathrm{p}(n \mid m, \mathfrak{A})\, t^n u^m = \prod_{i \geqslant 1} \Big(\sum_{j \geqslant 0} a_{i,j} u^j t^{ij} \Big).$$

where the (formal) variable x takes only integer values

■ The number of partitions of the indicated kind is equal to the number of solutions with integers $x_i \geqslant 0$, $i = 1, 2, \ldots$, of:

[3g] $$x_1 + 2x_2 + \cdots = n, \quad x_1 + x_2 + \cdots = m;$$
$$x_i \in \{j \mid j \geqslant 0,\ a_{i,j} = 1\} \quad (\Leftrightarrow a_{i,x_i} = 1).$$

Now, the right-hand member of [3f] can be written:

$$\prod_{i \geqslant 1} \Big(\sum_{x_i \geqslant 0} a_{i,x_i} u^{x_i} t^{i x_i} \Big) = \sum_{x_1, x_2, \ldots \geqslant 0} a_{1,x_1} a_{2,x_2} \cdots u^{x_1 + x_2 + \cdots} t^{x_1 + 2x_2 + \cdots},$$

which proves that the coefficient of $u^m t^n$ is just equal to the number of solutions of [3g]. ■

For example, if $a_{i,0} = a_{i,1} = 1$ and $a_{i,j} = 0$ if $j \geqslant 2$, then we have $\mathrm{p}(n \mid m, \mathfrak{A}) = Q(n, m) =$ the number of partitions of n into m unequal summands; hence, by [3f]:

[3h] $$\Psi(t, u) := 1 + \sum_{n, m \geqslant 1} Q(n, m)\, t^n u^m = \prod_{i \geqslant 1} (1 + u t^i).$$

Similarly, with $q(n)$—the number of partitions of n into unequal summands—$\sum_{m \geqslant 1} Q(n, m)$, we obtain with [3b, h]:

[3i] $$\Psi(t, 1) = 1 + \sum_{n \geqslant 1} q(n)\, t^n = \prod_{i \geqslant 1} (1 + t^i).$$

Here are a few values of $q(n)$:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
$q(n)$	1	1	2	2	3	4	5	6	8	10	12	15	18	22	27	32	38	46	54	64	76	89

With the same method, or otherwise, we get also:

[3j] $$1 + \sum_{n, m \geqslant 1} \mathrm{p}(n \mid m, \leqslant l)\, t^n u^m = \prod_{1 \leqslant i \leqslant l} (1 - u t^i)^{-1}.$$

[3k] $$1 + \sum_{n, m \geqslant 1} \mathrm{q}(n \mid m, \leqslant l)\, t^n u^m = \prod_{1 \leqslant i \leqslant l} (1 + u t^i).$$

2.4 FERRERS DIAGRAMS

A convenient and instructive representation of a partition of n into summands y_i such that [1a] p. 95 consists of a figure having m horizontal rows of points (the lines), the bottom one having y_1 points, the next to bottom one having y_2 points, etc., in such a way that the initial points of every line are all on one vertical line, hence the number of points on every vertical line or column decreases going from left to right. Such a figure, the *Ferrers diagram (or relation)*, clearly determines a unique partition of n. For example, Figure 25 shows the diagram of the partition $6+5+5+2+2+1$ of 21. If one considers the *columns* from left to right, the

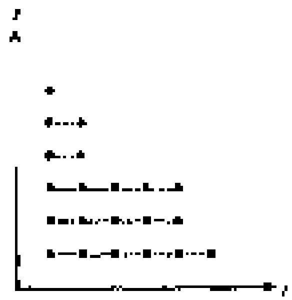


Fig. 25.

number of points in these will constitute another partition of n, with summands $z_1, z_2, \ldots$, which is called the *conjugate* partition of the partition with summands $y_1, y_2, \ldots$. In the case of the figure shown, the conjugate partition is $6+5+3+3+3+1$. Certain properties of a partition $y_1+y_2+\cdots$ have an immediate translation into terms of the conjugate partition. Thus we have:

THEOREM A. *The number of partitions of n into at most (or exactly) m summands is equal to the number of partitions of n into summands that are all $\leq m$ (or whose maximum is m); in other words, the number of partitions of $n+m$ whose maximum summand equals m.*

THEOREM B. *The number of partitions of n into unequal odd summands equals the number of 'self conjugate' partitions of n (that is, whose diagram is symmetric with respect to the line $x=y$).*

■ Theorem A is evident. For Theorem B, we associate with every

partition $(2z_1-1)+(2z_2-1)+\cdots=n$, where $z_1>z_2>\cdots$, the partition whose diagram is obtained by 'folding' the rows of the original diagram in the middle, so they form the sides of isosceles straight angled triangles, and fitting them then one by one, beginning with the largest, into each other. For instance, Figure 26 corresponds to $11+7+3+1 \to 6+5+4+4+2+1$. ■

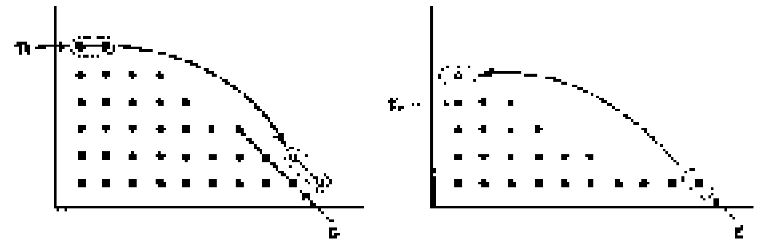
Fig. 26.

THEOREM C. *Let $q_0(n)$ (or $q_1(n)$) be the number of partitions of n into an even (or odd) number of unequal summands. Then:*

$$[4a] \qquad q_0(n)-q_1(n)=\begin{cases}(-1)^k & \text{if } n=\dfrac{3k^2\pm k}{2}\\ 0 & \text{otherwise.}\end{cases}$$

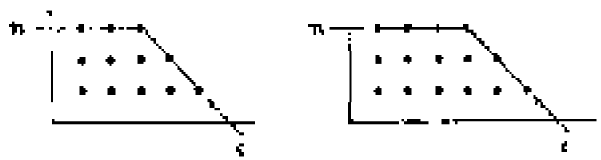
(This theorem is due to Euler; the proof given here is due to [Franklin, 1881]. See also the paper by [Andrews, 1972c] which applies the Franklin type technique to various other problems.)

■ Let $\mathfrak{q}_0=\mathfrak{q}_0(n)$ (or $\mathfrak{q}_1=\mathfrak{q}_1(n)$) be the set of Ferrers diagrams of the partitions of n into an even (or odd) number of unequal summands. For each diagram D (Figure 27a) we denote the 'northernmost' horizontal line of D by $\mathfrak{n}=\mathfrak{n}(D)$ (quite possibly $|\mathfrak{n}|=1$); we denote by $\mathfrak{e}=\mathfrak{e}(D)$ the 'easternmost' line that makes an angle of 45° with the horizontal direction ($|\mathfrak{e}|\geq 1$). Now we define $D'=\varphi(D)$ as the diagram which is obtained by sliding $\mathfrak{n}$ down to the east, if $|\mathfrak{n}|\leq|\mathfrak{e}|$ (Figure 27a), or by transporting $\mathfrak{e}$ to the north, if $|\mathfrak{n}|>|\mathfrak{e}|$ (Figure 27b). This transformation φ is defined in $\mathfrak{q}_0\cup\mathfrak{q}_1$, except if $|\mathfrak{n}\cap\mathfrak{e}|=1$, with $|\mathfrak{n}|=|\mathfrak{e}|$ (Figure 28a) or with $|\mathfrak{n}|=|\mathfrak{e}|+1$ (Figure 28b). Let $\mathfrak{a}_0$ and $\mathfrak{b}_0$ (or $\mathfrak{a}_1$ and $\mathfrak{b}_1$) be the sets of the $\mathfrak{q}_0$ (or $\mathfrak{q}_1$) corresponding to the case of Figures 28a and b.



(a) (b)

Fig. 27.



(a) (b)

Fig. 28.

Clearly α is a *bijection* of $\mathbf{q}_0 - (\mathbf{a}_0 + \mathbf{b}_0)$ onto $\mathbf{q}_1 - (\mathbf{a}_1 + \mathbf{b}_1)$. Thus:

[4b] $$q_0(n) - q_1(n) = |\mathbf{q}_0| - |\mathbf{q}_1| = |\mathbf{a}_0| + |\mathbf{b}_0| - |\mathbf{a}_1| - |\mathbf{b}_1| = a_0 + b_0 - a_1 - b_1.$$

Now, in the case of Figure 28a, n is of the form $k+(k+1)+\cdots+(2k-1) = (3k^2-k)/2$, while in the case of Figure 28b, $n=(k+1)+(k+2)+\cdots+2k=(3k^2+k)/2$, with $k:=|\varpi|=$ the number of summands of ϖ. Hence a_0, b_0, a_1, b_1 equal 0 *except* a_0 (or a_1)$=1$, if $n=(3k^2-k)/2$ and k even (or odd), and b_0 (or b_1)$=1$, if $n=(3k^2+k)/2$ and k even (or odd). This implies [4a] if we substitute these values into [4b]. ∎

The concept of a Ferrers diagram can be generalized easily to higher dimensions. We call a d-dimensional partition of n, for $d\ge 2$, any set F containing n points with integer coordinates ≥ 1 in the euclidean space $\mathbf{R}^d$ that satisfies the condition that $(a_1, a_2, \ldots, a_d)\in F$ implies that all points $(x_1, x_2, \ldots, x_d)$, where $1\le x_i\le a_i$ with $i\in[d]$, also belong to F. Let $p_d(n)$ be the number of these sets F. Clearly $p_2(n)=p(n)$. A beautiful

result of MacMahon states ([*MacMahon, II, 1916], p. 171):

[4c] $$\sum_{n\ge 0} p_3(n)\,t^n = \prod_{i\ge 1}(1-t^i)^{-i},$$

but the proof is very difficult ([Chaundy, 1931, 1932]). No other simple G.F. for $d\ge 4$ is known. ([Atkin, Bratley, Macdonald, Mackay, 1967]. See also [Gordon, Houten, 1968], [*Stanley, 1972], [Stanley, 1971a, b], [Wright, 1965a].)

2.5. SPECIAL IDENTITIES; 'FORMAL' AND 'COMBINATORIAL' PROOFS

First we prove two typical identities, which may serve as sample of many others.

THEOREM A. *The formal series introduced in* [2a, c] (p. 97) *also satisfy:*

[5a] $$\Phi(t) = 1+\sum_{n\ge 1} p(n)\,t^n \quad \left(=\prod_{i\ge 1}(1-t^i)^{-1}\right) = 1+\sum_{m\ge 1}\frac{t^m}{(1-t)(1-t^2)\cdots(1-t^m)}$$

[5b] $$\Phi(t,u) = 1+\sum_{1\le m\le n} P(n,m)\,t^n u^m \quad \left(=\prod_{i\ge 1}(1-ut^i)^{-1}\right) = 1+\sum_{m\ge 1}\frac{t^m u^m}{(1-t)\cdots(1-t^m)}$$

In the literature, often $t=q$ and $u=x$ (in honour of the elliptic functions); hence the name of 'q-identity', often given to this kind of identity. (See also Exercise 11, p. 118).

∎ *Formal proof* (also called 'algebraic' proof). We expand $\Phi(t,u)$ into a formal series in u:

[5c] $$\Phi(t,u) = \sum_{m\ge 0} C_m u^m, \qquad C_m = C_m(t).$$

The evident functional relation $\Phi(t,tu)=(1-tu)\Phi(t,u)$, which is satisfied by $\Phi(t,u)=\prod_{i\ge 1}(1-ut^i)^{-1}$, gives, when [5c] is substituted into it:

[5d] $$\sum_{n\geq 0} C_n t^n u^n = (1-tu)\sum_{n\geq 0} C_n u^n.$$

If we compare the coefficients of u^n in both members of [5d], we get $t^n C_n = C_n - tC_{n-1}$; hence:

[5e] $$C_n = \frac{t}{1-t^n}C_{n-1} = \frac{t^2}{(1-t^n)(1-t^{n-1})}C_{n-2} = \cdots = \frac{t^n}{(1-t^n)(1-t^{n-1})\cdots(1-t^2)(1-t)},$$

which, by [5c], proves [5b]. By putting $u=1$ we get [5a].

Combinatorial proof. As an example we prove [5a]. By [3j] (p. 99), the coefficient of t^k in $\{(1-t)(1-t^2)\cdots(1-t^l)\}^{-1}$ equals $p(k\mid *\leq l)$, which is the number of partitions of k into summands smaller or equal to l, here denoted by $s(k,l)$. Hence, for proving [5a], we just have to verify that the coefficients of t^n on both sides are equal; this means that we must prove that:

[5f] $$p(n) = s(n-1,1) + s(n-2,2) + \cdots.$$

By Theorem A (p. 98) $s(k,l)$ equals the number $P(k+l,l)$ of partitions of $k+l$ whose largest summand equals l. So [5f] is equivalent to $p(n) = P(n,1) + P(n,2) + \cdots$ and this last equality follows from the division of the set of partitions of n according to the value of the largest summand. ■

THEOREM B. (Sometimes called '*pentagonal theorem*' of Euler). *We have the following identity* [5g] *between formal series and the recurrence relation* [5h] *between the* $p(n)$:

[5g] $$\prod_{i\geq 1}(1-t^i) = \sum_{k\in\mathbb{Z}}(-1)^k t^{k(3k+1)/2} \overset{(*)}{=} 1 + \sum_{k\geq 1}(-1)^k\{t^{k(3k-1)/2} + t^{k(3k+1)/2}\}$$

[5h] $$p(n) = p(n-1) + p(n-2) - p(n-5) - \cdots = \sum_{k\geq 1}(-1)^{k+1}\left\{p\left(n-\frac{k(3k-1)}{2}\right) + p\left(n-\frac{k(3k+1)}{2}\right)\right\}.$$

■ *Formal proof.* Use the Jacobi identity, which is Theorem D below, and Exercise 16 (1) (p. 119).

Combinatorial proof. By using [3h] for (**), and the relations of Theorem C (p. 101), for (***), we get:

$$\prod_{i\geq 1}(1-t^i) \overset{(**)}{=} \Phi(t,-1) \overset{(***)}{=} 1 + \sum_{n\geq 1}\{e_n(n) - o_n(n)\}t^n,$$

and thus [5g (*)] follows from [4c]. For [5h], substitute [5g] into [5i] (which is equivalent to [2a], p. 97):

[5i] $$\left\{\prod_{i\geq 1}(1-t^i)\right\}\cdot\left\{1+\sum_{n\geq 1}p(n)t^n\right\} = 1,$$

and by observing that the coefficient of t^n $(n\geq 1)$ of the left hand member equals 0, we obtain the result. ■

THEOREM C. *The number of partitions of* n *into* m *unequal summands equals the number of partitions of* $n-\binom{m+1}{2}$ *into at most* m *summands (that is, into summands which are all* $\leq m$, *by Theorem A, p. 100):*

[5j] $$Q(n,m) = p\left(n-\binom{m+1}{2}, m\right) = p\left(n-\binom{m+1}{2}\,\middle|\, *\leq m\right).$$

■ *Formal proof.* This is carried out by a method analogous to the method used in the formal demonstration of Theorem A (p. 103), but this time the functional relation $\Psi(t,u) = (1+tu)\Psi(t,tu)$ is used. We get:

[5k] $$\Psi(t,u) = 1 + \sum_{1\leq m\leq n} Q(n,m)t^n u^m = \prod_{i\geq 1}(1+ut^i) = 1 + \sum_{m\geq 1}\frac{u^m t^{\binom{m+1}{2}}}{(1-t)(1-t^2)\cdots(1-t^m)}.$$

Hence, $Q(n,m)$ equals the coefficient of $t^{n-\binom{m+1}{2}}$ in $[(1-t)(1-t^2)\cdots(1-t^m)]^{-1}$, which is $p\left(n-\binom{m+1}{2}\,\middle|\, *\leq m\right)$, because of [3j] p. 99, hence equal to $p\left(n-\binom{m+1}{2}, m\right)$, by Theorem A (p. 100).

Combinatorial proof. The number of solutions of

$$[5l]\qquad y_1+y_2+\cdots+y_m=n,\quad y_1>y_2>\cdots>y_m\geq 1$$

is evidently equal to $Q(n,m)$. We put $z_1:=y_1-y_2-1,\ldots,z_{m-1}:=y_{m-1}-y_m-1$, $z_m:=y_m-1$. Hence $y_m=1+z_m$, $y_{m-1}=2+z_{m-1}+z_m,\ldots,y_1=m+z_1+z_2+\cdots+z_m$. Then equation [5l] is equivalent to:

$$[5m]\qquad z_1+2z_2+\cdots+mz_m=n-\binom{m+1}{2},\qquad z_i\geq 0,\quad i\in[m].$$

Now, the number of solutions of [5m] is clearly equal to the number of partitions of $n-\binom{m+1}{2}$ into summands not exceeding m, in other words, $p(n-\binom{m+1}{2},m)$, by Theorem A (p. 100). ■

THEOREM D. (Jacobi *identity*):

$$[5n]\qquad \prod_{i\geq 0}\{(1-t^{2i+2})(1+t^{2i+1}u)(1+t^{2i+1}u^{-1})\}=\sum_{n\in\mathbb{Z}}t^{n^2}u^n.$$

Both sides of [5n] have a generalized formal series in u, with positive and negative exponents; the theory of such series is easily developed, as on p. 43. We give here the 'formal' proof of [Andrews, 1965]. A beautiful 'combinatorial' proof is found in [Wright, 1965b]. See also [*Hermite], *Oeuvres*, Vol. II, pp. 155–56, and [Stolarsky, 1967].)

■ We replace tu by u in [5k], and u by $-u$ in [5h]. Then we get:

$$[5o]\qquad \prod_{i\geq 0}(1+t^iu)=\sum_{n\geq 0}\frac{t^{\binom{n}{2}}u^n}{(1-t)(1-t^2)\cdots(1-t^n)}$$

$$[5p]\qquad \prod_{i\geq 0}(1+t^iu)^{-1}=\sum_{n\geq 0}\frac{(-1)^nu^n}{(1-t)(1-t^2)\cdots(1-t^n)}.$$

It follows (justifications at the end) that:

$$\prod_{i\geq 0}(1+t^{2i+1}u)=$$

$$\overset{(*)}{=}\sum_{n\geq 0}\frac{t^{n^2}u^n}{(1-t^2)\cdots(1-t^{2n})}=\sum_{n\geq 0}t^{n^2}u^n\frac{\prod_{j\geq 0}(1-t^{2n+2+2j})}{\prod_{j\geq 0}(1-t^{2j+2})}=$$

$$\overset{(**)}{=}\frac{1}{\prod_{j\geq 0}(1-t^{2j+2})}\sum_{n\in\mathbb{Z}}t^{n^2}u^n\prod_{j\geq 0}(1-t^{2n+2+2j})=$$

$$\overset{(***)}{=}\frac{1}{\prod_{j\geq 0}(1-t^{2j+2})}\sum_{n\in\mathbb{Z}}\left\{t^{n^2}u^n\sum_{m\geq 0}\frac{(-1)^mt^{m^2+m+2mn}}{(1-t^2)\cdots(1-t^{2m})}\right\}=$$

$$\overset{\binom{**}{**}}{=}\frac{1}{\prod_{j\geq 0}(1-t^{2j+2})}\sum_{m\geq 0}\left\{\frac{(-1)^m(tu^{-1})^m}{(1-t^2)\cdots(1-t^{2m})}\sum_{n\in\mathbb{Z}}t^{(m+n)^2}u^{m+n}\right\}$$

$$\overset{\binom{++}{***}}{=}\frac{1}{\prod_{j\geq 0}(1-t^{2j+2})}\left\{\prod_{j\geq 0}(1+t^{2j+1}u^{-1})\right\}^{-1}\sum_{n\in\mathbb{Z}}t^{n^2}u^n.$$

(*) In [5o] replace t by t^2 and u by tu.

(**) All the terms of the summation that have negative nonzero n, are zero, because a factor 0 occurs in the product, namely when $j=-n-1$.

(***) In [5o], replace t by t^2 and u by $-t^{2n+2}$.

$\binom{**}{**}$ Interchange of summations.

$\binom{++}{***}$ In [5p], replace t by t^2 and u by tu^{-1}. ■

The natural setting for identities such as [5n] is actually the theory of elliptic functions, which is of an altogether fascinating beauty. (See, among others, [Adler, 1969], [Andrews, 1970, 1972b], and [*Bellman, 1961].) We mention here, pro memoria, the famous *Rogers-Ramanujan* identities (for a simple proof, see [Dobbie, 1962]):

$$1+\sum_{n\geq 1}\frac{t^{n^2}}{(1-t)(1-t^2)\cdots(1-t^n)}=$$

$$=\prod_{n\geq 1}\frac{1}{(1-t^{5n-1})(1-t^{5n-4})}$$

$$1+\sum_{n\geqslant 1}\frac{t^{n(n+1)}}{(1-t)(1-t^2)\cdots(1-t^n)}=$$
$$=\prod_{n\geqslant 1}\frac{1}{(1-t^{5n-2})(1-t^{5n-3})}.$$

(See also Exercises 9, 10, 11 and 12, p. [illegible].)

2.6. Partitions with forbidden summands; denumerants

Now we consider partitions of n whose summands are taken (repetitions allowed) from a sequence of *integers* $(a):=\{a_1, a_2, \ldots\}$, $1\leqslant a_1<a_2<\cdots$. As in Theorem A (p. 95), giving such a partition is equivalent to giving a solution of

[6a] $\quad a_1x_1+a_2x_2+a_3x_3+\cdots=n,\quad x_i \text{ integer} \geqslant 0.$

In other words, the matrix $\mathfrak{A}=[\alpha_{i,j}]$ (p. 96) is such that $\alpha_{i,j}=1$ for $i\in(a)$, for all $j\geqslant 0$, and $\alpha_{i,j}=0$ otherwise, except that $\alpha_{i,0}=1$. From Theorem B (p. 98) (or by direct computation) it follows immediately that:

Theorem A. *The generating function of the number $D(n;(a))=D(n;a_1,a_2,\ldots)$ of solutions of [6a], called the denumerant of n with respect to the sequence (a), equals:*

[6b] $\quad \mathfrak{D}_{(a)}(t):=1+\sum_{n\geqslant 1}D(n;(a))\,t^n=\prod_{i\geqslant 1}(1-t^{a_i})^{-1}.$

For $(a)=\{1,2,3,\ldots\}$, we find back [2a] of p. 97.

For example, in the *money changing problem*, one has as many coins of 5, 10, 20 and 50 centimes as one needs. In how many ways can one make with these a given amount of, say, 5 francs? (1 franc = 100 centimes). This is equivalent to finding the number of integer solutions of $5x_1+10x_2+20x_3+50x_4=500$, or equivalently, of $x_1+2x_2+4x_3+10x_4=100$. The solution is hence $D(100;1,2,4,10)$, which is 2691 (see p. 113).

Another example: it is immediately clear, by [5b] p. 103 and [6b] that

[6b'] $\quad D(n;1,2,3,\ldots,k)=P(n+k,k)=Q(n+k(k+1)/2,k).$

We investigate the case of a *finite* sequence $(a):=(a_1,a_2,\ldots,a_k)$,

$1\leqslant a_1<a_2<\cdots<a_k$ ($\Leftrightarrow a_i=0$, if $i>k$). The GF [6b] is then a *rational fraction*:

[6b''] $\quad \mathfrak{D}_{(a)}(t)=1+\sum_{n\geqslant 1}D(n;(a))\,t^n=\prod_{i=1}^{k}(1-t^{a_i})^{-1}.$

A first method for computing the denumerant $D(n;(a))$ is provided by a decomposition of the *rational fraction* [6b''] into *partial fractions*. For instance:

$$\mathfrak{D}_{\{1,2\}}(t)=\frac{1}{(1-t)(1-t^2)}$$
$$=\frac{1}{4}\left(\frac{1}{1+t}+\frac{1}{1-t}+\frac{2}{(1-t)^2}\right)$$
$$=\frac{1}{4}\left\{\sum_{n\geqslant 0}(-1)^n t^n+\sum_{n\geqslant 0}t^n+2\sum_{n\geqslant 0}(n+1)t^n\right\},$$

which gives as coefficient of t^n:

[6c] $\quad D(n;1,2)=\frac{1}{4}\{2n+3+(-1)^n\}.$

Similarly, for $\mathfrak{D}_{\{1,2,3\}}(t)=\{(1-t)(1-t^2)(1-t^3)\}^{-1}$ we have two decompositions. (The first one, called the *first type*, is a decomposition into ordinary partial fractions; the second one is called the *second type* or *Herschellian type*, see [Herschel, 1818].)

[6d] $$\mathfrak{D}_{\{1,2,3\}}=\frac{1}{6(1-t)^3}+\frac{1}{4(1-t)^2}+$$
$$+\frac{17}{72(1-t)}+\frac{1}{8(1+t)}+\frac{2+t}{9(1+t+t^2)}=$$
$$=\frac{1}{6(1-t)^3}+\frac{1}{4(1-t)^2}+\frac{1}{4(1-t^2)}+\frac{1}{3(1-t^3)}.$$

We denote the periodic sequence with period T (integer $\geqslant 1$), that is equal to a_i for $n\equiv i$ (mod T), $i=0,1,\ldots,T-1$, by: $(a_0,a_1,\ldots,a_{T-1})$ cir T_n (cir for *circulator*; this notation is from Herschel). If, moreover, for each divisor S of T, $1\leqslant S<T$, we have $a_R+a_{R+S}+a_{R+2S}+\cdots+a_{R+T-S}=0$ for all $R=0,1,2,\ldots,S-1$, then we rather denote the above sequence by $(a_0,a_1,\ldots,a_{T-1})$ per T_n (per stands for *prime circulator*, the notation is due to Cayley). The expansion of [6d] into a

power series gives then the following two forms for $D(n;1,2,3)$:

[6e] $$\frac{n^2}{12}+\frac{n}{2}+\frac{47}{72}+\frac{1}{8}(1,-1)\text{ per }2_n+\frac{1}{9}(2,-1,-1)\text{ per }3_n$$

[6e'] $$\frac{1}{12}(n+1)(n+5)+\frac{1}{4}(1,0)\text{ per }2_n+\frac{1}{3}(1,0,0)\text{ per }3_n.$$

For each $x\,(\in\mathbb{R})$ such that $(x-\frac{1}{2})$ is not integer, we put:

[6f] $\|x\|:=$ the integer closest to x.

By [6e] we find,

[6g] $$D(n;1,2)=\left\|\frac{2n+3}{4}\right\|.$$

A similar formula using $\|\ldots\|$ for $D(n;1,2,3)$ can be found as it follows. We transform [6e'] by grouping first the two ω's, then replacing $(n+1)(n+5)$ by $(n+3)^2-4$:

$$D(n;1,2,3)=\tfrac{1}{12}[(n+1)(n+5)+(7,0,3,4,3,0)\text{ per }6_n]$$
$$=\tfrac{1}{12}[(n+3)^2+(3,-4,-1,0,-1,-4)\text{ per }6_n].$$

Now, $\varphi(n):=\frac{1}{12}(3,-4,-1,0,-1,-4)\text{ per }6_n$, a sequence of period 6, satisfies $|\varphi(n)|\leqslant\frac{4}{12}=\frac{1}{3}<\frac{1}{2}$. Hence,

[6g'] $$D(n;1,2,3)=\|\tfrac{1}{12}(n+3)^2\|.$$

This way of writing by means of $\|\ldots\|$ is not unique. In the same way one will find for $D(n;1,2,3)$ slightly more complicated formulae $\|\frac{1}{12}(n^2+6n+7)\|$, $\|\frac{1}{12}(n+2)(n+4)\|$ and $\|\frac{1}{12}(n^2+6n+10)\|$. The first values of $D(n)=D(n;1,2,3)$ are

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$D(n)$	1	2	3	4	5	7	8	10	12	14	16	19	21	24	27	30	33	37	40	44

The following is a second method for computing the denumerant.

THEOREM B. ([Bell, 1943]). *Let A be the least common multiple of the integers $(a_1,a_2,\ldots,a_k)$, $1\leqslant a_1<a_2<\cdots<a_k$. For every integer B such that $0\leqslant B\leqslant A-1$, and every integer $m\geqslant 0$, we have:*

[6h] $$D(Am+B;a_1,a_2,\ldots,a_k)=D(Am+B;(a))=D(Am+B)=\delta(m)=c_0+c_1m+\cdots+c_{k-1}m^{k-1},$$

where the c_i $(i\in[k])$, are constants independent of m, and where the denumerant $\delta(m)$ is as defined as in Theorem A (p. 108).

■ Let α be the complex number such that $\alpha^A=1$, $\arg\alpha=2\pi/A$, then we put, with $D(n):=D(n;a_1,\ldots,a_k)$:

[6i] $$A=:a_jb_j,\quad j\in[k];\qquad P(t):=\prod_{1\leqslant j\leqslant k}(1-t^{a_j}).$$

The roots of $P(t)=0$ are hence of the form α^{b_jh}, where $0\leqslant h\leqslant a_j-1$ and $j\in[k]$. Let $\varepsilon_0\,(=1),\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_r$ be the $(r+1)$ *different* values of these roots:

[6j] $$P(t)=\left(1-\frac{t}{\varepsilon_0}\right)^k\left(1-\frac{t}{\varepsilon_1}\right)^{s_1}\cdots\left(1-\frac{t}{\varepsilon_r}\right)^{s_r},$$

where $\varepsilon_i\neq\varepsilon_j$ for $i\neq j$ and $k+s_1+s_2+\cdots+s_r=a_1+a_2+\cdots+a_k$. Necessarily, $s_1,s_2,\ldots\leqslant k$, because every root of $t^a-1=0$ is *simple*, so a multiple root of order s of $P(t)=0$ must come from s different factors $(1-t^{a_i})$, $i\in[k]$, whence $s\leqslant k$. Now we decompose the rational fraction $1/P(t)$ into partial fractions, using [6j]: there exist complex constants $C_{u,v}$ $(0\leqslant u\leqslant r,\ 1\leqslant v\leqslant k)$ such that:

[6k] $$\Phi(t)=\frac{1}{P(t)}=\sum_{\substack{0\leqslant u\leqslant r\\1\leqslant v\leqslant k}}C_{u,v}\left(1-\frac{t}{\varepsilon_u}\right)^{-v}.$$

Identifying in [6k] the coefficients of t^n, calculated by using the expansion of $(1-z)^{-v}$ in the right-hand member (see [13e'], p. 37), we obtain:

[6l] $$D(n)=\sum_{\substack{0\leqslant u\leqslant r\\1\leqslant v\leqslant k}}C_{u,v}\frac{\langle v\rangle_n}{n!}\varepsilon_u^{-n}.$$

If we put $n=Am+B$ in [6l], we get by using $\varepsilon_u^A=1$:

[6m] $$\delta(m)=\sum_{1\leqslant v\leqslant k}P_v(m)\left(\sum_{0\leqslant u\leqslant r}C_{u,v}\varepsilon_u^{-B}\right),$$

where the polynomial $P_v(m)=\langle v\rangle_n/n!=(n+1)_{v-1}/(v-1)!$ is the product of $(v-1)$ factors of the first degree in m (because $n=Am+B$), and hence of degree $(v-1)$ $(\leqslant k-1)$; [6h] follows. ■

The polynomial $\delta(n)$, of degree $(k-1)$ in n, is known by [6h], when the values $\delta(n_i)$ are known in k different points n_i, $i\in[k]$. For this, we can use either the determinant [6n], of order $(k+1)$, which eliminates the constants a_j, $(j+1)\in[k]$, from [6h]:

$$\text{[6n]}\qquad \begin{vmatrix} \delta(n) & 1 & n & n^2 & \dots & n^{k-1} \\ \delta(n_1) & 1 & n_1 & n_1^2 & \dots & n_1^{k-1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \delta(n_k) & 1 & n_k & n_k^2 & \dots & n_k^{k-1} \end{vmatrix} = 0,$$

or the Lagrange interpolation formula:

$$\text{[6o]}\qquad \delta(n) = \sum_{i=1}^{k} \delta(n_i)\,\mu_i,$$

where

$$\mu_i := \frac{(n-n_1)\cdots(n-n_{i-1})(n-n_{i+1})\cdots(n-n_k)}{(n_i-n_1)\cdots(n_i-n_{i-1})(n_i-n_{i+1})\cdots(n_i-n_k)}.$$

Particularly, for $n_i=i$, $i\in[k]$, [6o] becomes:

$$\text{[6p]}\qquad \delta(n) = \binom{n-1}{k}\sum_{1\leq i\leq k}(-1)^{k-i}\binom{k}{i}\frac{i}{n-i}\,\delta(i).$$

For example, to calculate $D(n; 1, 2, 4) := D(n)$ by means of [6p], one may use the first values of $D(n)$ (computed from $D(n; 1, 2)$, [6c]):

n	4	5	6	7	8	9	10	11	12	13	14	15
$D(n)$	4	4	6	6	9	9	12	12	16	16	20	20

This gives $D(4m)=D(4m+1)=(m+1)^2$, $D(4m+2)=D(4m+3)=(m+1)(m+2)$. It is then verified that:

$$\text{[6q]}\qquad D(n; 1, 2, 4) = [\{(n+2)(n+6)+(-1)^n n\}/16].$$

We now show, by two examples, an *efficient* practical use of Theorem B, *without* decomposition of rational fractions into partial fractions, which works particularly well if the LCM A of $(a_1, a_2, a_3, \dots)$ is not large. We abbreviate $v_k := (1-t^k)^{-1}$ and we use a point (for saving place) to denote the center of symmetry of any reciprocal polynomial. (Examples: $1+t+t^2\cdot := 1+t+t^2+t^3$, $1+2t\cdot := 1+2t+2t^2+t^3$.)

(1) We return to $D(n; 1, 2, 3)$, [5d] (p. 109). We have
$\Sigma_{1,2,3}(t) = v_1v_2v_3 = (1+t)\,v_2^2v_3 = (1+t)(1+t^2+t^4)^2(1+t^3)\,v_6^3$
$= (1+t+2t^2+3t^3+4t^4+5t^5+4\dot{t}^6+5t^7+\cdots)\sum\binom{m+2}{2}t^{6m}$.
Hence, identifying the coefficients in the first and last member, we get:
$D(6m+h; 1, 2, 3) = \alpha\binom{m+2}{2}+\beta\binom{m+1}{2}+\gamma\binom{m}{2}$, where, for $h=0, 1, 2, 3, 4, 5$, we have $\alpha=1, 1, 2, 3, 4, 5$, $\beta=4, 5, 4, 3, 2, 1$, $\gamma=1, 0, 0, 0, 0, 0$, respectively.

(2) Similarly, we compute $D(n; 1, 2, 4, 10)$, used p. 108. We have
$\Sigma_{1,2,4,10}(t) = v_1v_2v_4v_{10} = (1+t)\,v_2^2v_4v_{10} = (1+t)(1+t^2)^2v_4^3v_{10} =$
$(1+t)(1+t^2)^2(1+t^4+t^8+t^{12}+t^{16})^3(1+t^{10})(1-t^{20})^{-4} = (1+t)$
$(1+2t^2+4t^4+6t^6+9t^8+13t^{10}+18t^{12}+24t^{14}+31t^{16}+39t^{18}+$
$45t^{20}+52t^{22}+57t^{24}+63t^{26}+67t^{28}+69t^{30}+69t^{32}+\cdots+2t^{60}+t^{62})\times$
$\times\sum\binom{m+3}{3}t^{20m}$. Hence $D(20m+2h+(0\text{ or }1); 1, 2, 4, 10) = \alpha\binom{m+3}{3}+$
$+\beta\binom{m+2}{3}+\gamma\binom{m+1}{3}+\delta\binom{m}{3}$, where, for $h=0, 1, 2, \dots, 9$, we have:
$\alpha=1, 2, 4, 6, 9, 13, 18, 24, 31, 39$, $\beta=45, 52, 57, 63, 67, 69, 69, 67, 63, 57$, $\gamma=52, 45, 39, 31, 24, 18, 13, 9, 6, 4$, $\delta=2, 1, 0, 0, \dots, 0$, respectively.

Let us now give a more precise version of Theorem B (p. 110):

THEOREM C. *Supposing each pair (a_i, a_j) relatively prime, we have:*

$$\text{[6r]}\qquad D(n; a_1, a_2, \dots, a_k) := D(n) = \sum_{j=0}^{k-1} d_j n^j + V_{a_1}(n) + \cdots + V_{a_k}(n),$$

where each $V_{a_j}(n)$ is a per of period a_j, $j=1, 2, \dots, k$. So, $D(n)$ is a polynomial of degree $k-1$ in n, plus a sequence $V_A(n) := V_{a_1}(n)+\cdots+V_{a_k}(n)$, with period $A=LCM(a_1, \dots, a_k)$. Moreover, denoting $S_1 := a_1+a_2+a_3+\cdots$, $S_2 := a_1^2+a_2^2+a_3^2+\cdots$, $P := a_1a_2a_3\cdots a_k$, the following formulas hold:

$$\text{[6s]}\qquad d_{k-1} = \frac{1}{(k-1)!\,P},\qquad d_{k-2} = \frac{S_1}{2(k-2)!\,P},$$
$$d_{k-3} = \frac{3S_1^2-S_2}{24(k-3)!\,P},\qquad d_{k-4} = \frac{S_1^3-S_1S_2}{48(k-4)!\,P}.$$

■ Let us write $\mathfrak{P}_l = \mathfrak{P}_l(t)$ for any polynomial whose degree is $\leq l$. The theory of fractional decomposition implies:

$$\mathfrak{D}(t) := \sum_{n\geq 0} D(n)\,t^n = \frac{1}{(1-t^{a_1})(1-t^{a_2})\cdots}$$

$$= \frac{1}{(1-t)^k(1+t+t^2+\cdots+t^{a_1-1})(1+t+t^2+\cdots+t^{a_2-1})\cdots} =$$

$$= \frac{\mathfrak{P}_{k-1}}{(1-t)^k} + \frac{\mathfrak{P}_{a_1-2}}{1+t+t^2+\cdots+t^{a_1-1}} + \frac{\mathfrak{P}_{a_2-2}}{1+t+t^2+\cdots+t^{a_2-1}} + \cdots$$

$$= \frac{\mathfrak{P}_{k-1}}{(1-t)^k} + \frac{\mathfrak{P}_{a_1-1}}{1-t^{a_1}} + \frac{\mathfrak{P}_{a_2-1}}{1-t^{a_2}} + \cdots.$$

So, we obtain [6a], and the relations $\mathfrak{P}_{a_1-1}(1) = \mathfrak{P}_{a_2-1}(1) = \cdots = 0$ involving the numerators of the preceding line imply the per condition (concerning the sum of values which must be equal to 0, p. 109). The standard methods for determining $\mathfrak{P}_{k-1}$ give [6a]. ■ (For many other explicit formulas, see [Glaisher, 1909], [Sylvester, 1882].)

As an example, let us calculate $D(n; 3, 5, 7) := D(n)$. Here, $S_1 = 3+5+7 = 15$, $S_2 = 3^2+5^2+7^2 = 83$, $P = 3.5.7 = 105$. So, with [6c, e]:

[6f] $$D(n) = \tfrac{1}{210}n^2 + \tfrac{1}{14}n + \tfrac{74}{315} + [x_1, x_2, x_3] + [x_4, x_5, \ldots, x_8] + [x_9, \ldots, x_{15}],$$

where $[x_1, x_2, x_3]$ abbreviates (x_1, x_2, x_3) per$_3$, etc. Now, it is easy to compute $D(0), D(1), D(2), \ldots, D(11) = 1, 0, 0, 1, 0, 1, 1, 1, 1, 1, 2, 1$ by carrying out $(1-t^3)^{-1}(1-t^5)^{-1}(1-t^7)^{-1} = (1+t^3+t^6+t^9)\times(1+t^5+t^{10})(1+t^7)$ (up to degree 11) or by using the recurrence $D(n) = D(n-3)+D(n-5)+D(n-7)-D(n-8)-D(n-10)-D(n-12)+D(n-15)$. If we insert these values of $D(n)$ in [6f], we must solve the following linear system of 15 equations with unknowns $x_1, x_2, \ldots, x_{15}$ (the three last ones are the per condition, p. 109): $x_1+x_4+x_9 = 241/315$, $x_2-x_5+x_{10} = -98/315$, $x_3+x_6+x_{11} = -125/315$, $x_1+x_7+x_{12} = 160/315$, $x_2+x_8+x_{13} = -185/315$, $x_3-x_4+x_{14} = 91/315$, $x_1+x_5-x_{15} = -52/315$, $x_2+x_6+x_9 = 10/315$, $x_3+x_7+x_{10} = -35/315$, $x_1-x_8+x_{11} = -83/315$, $x_2+x_4+x_{12} = 151/315$, $x_3+x_5+x_{13} = -198/315$, $x_1+x_2+x_3 = x_4+\cdots+x_8 = x_9+\cdots+x_{15} = 0$. Solving this linear system, we find: $(x_1, x_2, \ldots, x_{15}) = (70, -35, -35;\ 126, -63, 0, 0, -63;\ 45, 0, -90, 90, -90, 0, 45)/315 = (2/9, -1/9, -1/9;\ 2/5, -1/5, 0, 0, -1/5;\ 1/7, 0, -2/7, 2/7, -2/7, 0, 1/7)$. For example, $1000 \equiv 1 \pmod 3$, $1000 \equiv 0 \pmod 5$, $1000 \equiv 6 \pmod 7$; thus, $D(1000) = 10^6/210 + 1000/14 + 74/315 + x_2(=-1/9) + x_4(=2/5) + x_{15}(=1/7) = 4834$. Here, the use of a sum of 3 Cayley's per requires only $3+5+7-3 = 12$ unknowns to find, whereas the use of one Herschel's per would require 105 unknowns, this number being the length $3.5.7$ of the oscillating term in $D(n)$.

SUPPLEMENT AND EXERCISES

1. *Recurrence relation for $P(n, m)$.* If $P(n, m)$ stands for the number of partitions of the integer n into m summands (p. 94 and table p. 307), show that $P(n, m) = P(n-1, m-1) + P(n-m, m)$, and that, for $m \geq n/2$, $P(n, m) = p(n-m)$. [*Hint*: Distinguish, in [1b, c], p. 95, the solutions with $x_1 = 0$ from those with $x_1 \geq 1$.]

2. *Recurrence relation for $Q(n, m)$.* As in the preceding exercise, prove that the number $Q(n, m)$ of partitions of the integer n into m *different* summands satisfies: $Q(n, m) = Q(n-m, m) + Q(n-m, m-1)$. Hence the first values of $Q(n, m)$ and $q(n) = \sum_m Q(n, m)$:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$q(n)$	1	1	2	2	3	4	5	6	8	10	12	15	18	22	27	32	38	46	54	64	76	89	104	122	142	165	192	222	256	296
m = 1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2			1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10	11	11	12	12	13	13	14	14
3						1	1	2	3	4	5	7	8	10	12	14	16	19	21	24	27	30	33	37	40	44	48	52	56	61
4										1	1	2	3	5	6	9	11	15	18	23	27	34	39	47	54	64	72	84	94	108
5															1	1	2	3	5	7	10	13	18	23	30	37	47	57	70	84
6																					1	1	2	3	5	7	11	14	20	26
7																												1	1	2

3. *Convexity of $p(n)$.* The number $p^*(n)$ of partitions of n into summands all > 1 equals $p(n) - p(n-1)$ and this is an increasing function of n. Deduce that the sequence $p(n)$ (= the number of partitions of n) is convex, in other words, that $\Delta^2 p(n) = p(n-2) - 2p(n-1) + p(n) \geq 0$. More generally, $\Delta^k p(n) \geq 0$ for $n \geq k \geq 1$.

4. *Some values of $P(n,m)$ and $Q(n,m)$.* For shortness, we write the sequence $(d_0, d_1, \ldots, d_{T-1})$ of T of p. 109 as $[d_0, d_1, \ldots, d_{T-1}]$. $P(n,m)$ (or $Q(n,m)$) is the number of partitions of n into m arbitrary (or unequal) summands (see p. 99). Use $P(n,m) = Q(n+\binom{m}{2}, m)$ (which can be proved combinatorially), and hence $Q(n,m) = P(n-\binom{m}{2}, m)$ to show:

$$P(n,2) = (1/4)(2n-1+[1,-1])$$

$$Q(n,2) = (1/4)(2n-3-[1,-1])$$

$$P(n,3) = (1/72)(6n^2-7-9[1,-1]+8[2,-1,-1])$$

$$Q(n,3) = (1/72)(6n^2-36n+47+9[1,-1]+ 8[2,-1,-1])$$

$$P(n,4) = (1/288)(2n^3+6n^2-9n-13+(9n+9)\times [1,-1]-32[1,-1,0]+36[1,0,-1,0])$$

$$Q(n,4) = (1/288)(2n^3-30n^2+135n-175+(9n-45)\times [1,-1]-32[1,-1,0]-36[1,0,-1,0]).$$

***5.** *Upper and lower bounds for $P(n,m)$.* Show that $P(n,m)$ and $Q(n,m)$, as defined on p. 94 and 99, satisfy:

$$Q(n,m) \leqslant \frac{1}{m!}\binom{n-1}{m-1} \leqslant P(n,m).$$

Use the fact that $Q(n,m) = P(n-\binom{m}{2}, m)$, [5i] p. 105, to prove that

$$P(n,m) \leqslant \frac{1}{m!}\binom{n+\binom{m}{2}-1}{m-1} \text{ and } P(n,m) \sim \frac{1}{m!}\binom{n-1}{m-1}$$

for $n\to\infty$ and $m = o(n^{1/3})$ ([Erdős, Lehner, 1941], [Gupta, 1945], [Rieger, 1959], [Wright, 1961].)

6. *The size of the smallest summand is given.* Let $u(n,m)$ be the number of partitions of n such that the smallest summand equals m. Then

$$\sum_{n\geqslant 0} u(n,m)\, t^n = t^m\{(1-t^m)(1-t^{m+1})\cdots\}^{-1},$$

and

$$u(n,m) = u(n-m,m) + u(n-1,m+1),$$

where

$$u(n,n) = 1, \qquad u(n,1) = p(n-1).$$

7. *Odd summands.* Let $p_I(n)$ be the number of partitions of n into summands which are all odd, then we have $\sum_{n\geqslant 0} p_I(n)\, t^n = \{(1-t)(1-t^3)\times (1-t^5)\cdots\}^{-1}$, and $p_I(n) = q(n)$ (the number of partitions into unequal summands, p. 99). Prove this by formal methods and by combinatorial methods.

8. *The summands are bounded in number and size.* Let $p(n \mid \leqslant m, \leqslant l)$ be the number of partitions of n into at most m summands all $\leqslant l$. Show that:

$$A(t,u) := \sum_{n,m\geqslant 0} p(n \mid \leqslant m, \leqslant l)\, t^n u^m = \prod_{i=0}^{l} (1-ut^i)^{-1}.$$

Use a method analogous to that on p. 98 to show that:

$$A(t,u) = 1 + \sum_{m\geqslant 1} \frac{(1-t^{l+1})(1-t^{l+2})\cdots(1-t^{l+m})}{(1-t)(1-t^2)\cdots(1-t^m)}\, u^m.$$

Deduce:

$$\sum_{n\geqslant 0} p(n \mid \leqslant m, \leqslant l)\, t^n = \frac{(1-t^{l+1})(1-t^{l+2})\cdots(1-t^{l+m})}{(1-t)(1-t^2)\cdots(1-t^m)}.$$

9. *The factorial number system.* For all $n\geqslant 1$ we have:

$$(1+t)(1+t^{2!}+t^{2\cdot 2!})\cdots(1+t^{n!}+t^{2\cdot n!}+\cdots+t^{n\cdot n!}) = 1+t+t^2+t^3+\cdots+t^{(n+1)!-1}.$$

[*Hint:* This is equivalent to $1.1!+2.2!+\cdots+n.n! = (n+1)!-1$, which can be proved either by induction or by a combinatorial interpretation.]

Use this to prove:

$$\prod_{j\geqslant 1}\ \sum_{0\leqslant k\leqslant j} t^{kj!} = (1-t)^{-1},$$

and, for every integer $n\geqslant 0$, the existence of a unique sequence of integers x_i such that

$$n = x_1.1! + x_2.2! + \cdots,$$

where $0\leqslant x_i\leqslant i$, $i = 1, 2, 3, \ldots$ (See also Exercise 4, p. 235.)

10. *With the binary number system.* (1) For all $m\geq 1$, we have:

$$(1+ut)(1+ut^2)\cdots(1+ut^{2^m})=\sum_{n=0}^{2^{m+1}-1}u^{D(n)}t^n.$$

Here, $D(n)$ stands for the number of ones in the binary (= base 2) representation of n. Consequently (generalization in [Ostrowski, 1929]):

$$\prod_{k\geq 0}(1+ut^{2^k})=\sum_{n\geq 0}u^{D(n)}t^n.$$

(2) Also prove $t(1-t)^{-1}=\sum_{k\geq 0}t^{2^k}(1-t^{2^{k+1}})^{-1}$ ([Teixeira, 1904]).

11. *q-binomial coefficients.* Let $0<q<1$. We introduce

$$((x))_k:=\frac{(1-q^x)(1-q^{x-1})\dots(1-q^{x-k+1})}{(1-q)^k}$$

$$\langle x\rangle_k:=\frac{(1-q^x)(1-q^{x+1})\dots(1-q^{x+k-1})}{(1-q)^k}$$

$$((x))=((x))_1=\langle x\rangle=\langle x\rangle_1=\frac{1-q^x}{1-q},\ ((k))!:=\langle 1\rangle_k,$$
$$((0))!:=1.$$

The q-binomial coefficients are defined by

$$\left(\binom{x}{k}\right)=\frac{((x))_k}{((k))!}=\frac{(1-q^x)(1-q^{x-1})\dots(1-q^{x-k+1})}{(1-q)(1-q^2)\dots(1-q^k)}$$

$$\left\langle\binom{x}{k}\right\rangle=\frac{\langle x\rangle_k}{((k))!}=\frac{(1-q^x)(1-q^{x+1})\dots(1-q^{x+k-1})}{(1-q)(1-q^2)\dots(1-q^k)}$$

They tend to the ordinary binomial coefficients when $q\to 1$.

(1) We have $\left(\binom{x}{k}\right)=\frac{((x))}{((k))}\left(\binom{x-1}{k-1}\right)$,

$\left(\binom{x}{k}\right)=\left(\binom{x-1}{k-1}\right)+q^k\left(\binom{x-1}{k}\right)$, $\left(\binom{-x}{k}\right)=(-1)^k q^{-kx}q^{-\binom{k}{2}}\left\langle\binom{x}{k}\right\rangle$.

(2) $\prod_{i=0}^{n-1}(1+xq^i)=\sum_{k=0}^{n}q^{\binom{k}{2}}\left(\binom{n}{k}\right)x^k$,

$\prod_{i=0}^{n-1}(1-xq^i)^{-1}=\sum_{k\geq 0}\left\langle\binom{n}{k}\right\rangle x^k$.

(Observe the analogies with the expansions of $(1+x)^n$ and $(1-x)^{-n}$...). For $n\to\infty$, we recover [5c] (p. 106) and [5b] (p. 103). (3) $b_n=\sum_{k=0}^{n}$

$\left(\binom{n}{k}\right)a_k\Leftrightarrow a_n=\sum_{k=0}^{n}(-1)^{k}q^{\binom{k}{2}}\left(\binom{n}{k}\right)b_k$. (Compare [6a, c], p. 143.) (This is a very large subject, and we only touch upon it. For a completely updated presentation, see [Goldman, Rota, 1970].)

12. *Prime numbers.* To every integer $n\geq 1$, $n=p_1^{\nu_1}p_2^{\nu_2}\dots$ its prime factor decomposition, we associate the number $\omega(n):=\nu_1+\nu_2+\cdots$, $\omega(1):=0$. Thus, $\omega(1500)=\omega(2^2 5^3 3)=2+3+1=6$. Then, for all complex numbers s and u, such that $\mathscr{R}e\, s>1$, and $|u|\leq 1$, the following equality between *functions* of s and u holds:

$$\prod_p\left(1-\frac{u}{p^s}\right)^{-1}=\sum_{n\geq 1}\frac{u^{\omega(n)}}{n^s}.$$

Here, in the infinite product, p runs through the set of all prime numbers (for $u=1$, this is the famous factorization of the *Riemann zeta function* $\zeta(s):=\sum_{n\geq 1}n^{-s}$. See also Exercise 16, p. 162).

13. *Durfee square identity for* $\sum p(n)\,t^n$. Prove the identity:

$$\frac{1}{(1-t)(1-t^2)(1-t^3)\cdots}=1+\frac{t}{(1-t)^2}+$$
$$+\frac{t^4}{(1-t)^2(1-t^2)^2}+\frac{t^9}{(1-t)^2(1-t^2)^2(1-t^3)^2}+\cdots.$$

[*Hint:* Put $\Phi(t,u):=\{(1-tu)(1-t^2u)\cdots\}^{-1}=\sum C_m(t)\,u^m$, $\prod_{i=1}^{n}(1-t^iu)^{-1}=\sum C_m(t)\,u^mE_m(t,u)$; observe that $\Phi(t,tu)=(1-tu)\,\Phi(t,u)$ and $E_n(t,u)=(1-u)^{-1}\{E_n(t,u)+t^{n+1}uE_{n-1}(t,u)\}$; obtain $C_m(t)$.]

14. *Some applications of the Jacobi identity.* If we replace x by t^k and u by $\pm t^l$ in the Jacobi identity, [5a] (p. 106), k and l integers >0, prove:

$$\prod_{i\geq 0}\{(1+t^{2ki+k-l})(1+t^{2ki+k+l})(1-t^{2ki+2k})\}=\sum_{n\in\mathbb{Z}}t^{kn^2+ln}$$

$$\prod_{i\geq 0}\{(1-t^{2ki+k-l})(1-t^{2ki+k+l})(1-t^{2ki+2k})\}=\sum_{n\in\mathbb{Z}}(-1)^n t^{kn^2+ln}.$$

(1) Use this to prove the Euler identity, [5g] (p. 104), by putting $k=\frac{3}{2}$, $l=\frac{1}{2}$.

(2) If $k=\frac{5}{2}$, $l=\frac{1}{2}$:

$$\prod_{i\geq 0}\{(1-t^{5i+1})(1-t^{5i+4})(1-t^{5i+5})\}=\sum_{n\in\mathbf{Z}}(-1)^n t^{n(5n+3)/2}.$$

(3) If $k=\frac{5}{2}$, $l=\frac{3}{2}$:

$$\prod_{i\geq 0}\{(1-t^{5i+2})(1-t^{5i+3})(1-t^{5i+5})\}=\sum_{n\in\mathbf{Z}}(-1)^n t^{n(5n+1)/2}.$$

(4) If $k=1$, $l=0$:

$$\prod_{i\geq 0}\{(1-t^{2i+1})^2(1-t^{2i+2})\}=\sum_{n\in\mathbf{Z}}(-1)^n t^{n^2}$$

$$\prod_{i\geq 0}\{(1+t^{2i+1})^2(1-t^{2i+2})\}=\sum_{n\in\mathbf{Z}}t^{n^2}.$$

15. *Use of the function* $\|x\|$, *the integer closest to* x. With the notation of [6f] (p. 110), we have, in addition to [6g, q]:

$$D(n;1,2,5)=\|(n+4)^2/20\|;$$
$$D(n;1,2,7)=\|(n+3)(n+7)/28\|;$$
$$D(n;1,3,5)=\|(n+3)(n+6)/30\|;$$
$$D(n;1,3,7)=\|(n+3)(n+8)/42\|;$$
$$D(n;1,5,7)=\|(n^2+13n+36)/70\|;$$
$$D(n;1,2,3,5)=\|(n+3)(2n^2+9)(n+9)/360\|=$$
$$=\|(n+2)(n+3)(2n+13)/360\|;$$
$$P(n,2)=Q(n+1,2)=\|(2n-1)/4\|;$$
$$P(n,3)=Q(n+3,3)=\|n^2/12\|.$$
$$P(n,4)=Q(n+6,4)=\|n^2(n+3)/144\|\ \text{for } n \text{ even},$$
$$\text{and}\ =\|(n-1)^2(n+5)/144\|\ \text{for } n \text{ odd}$$

(For plenty of other such formulas, see [Popoviciu, 1953]).

16. *Infinite power series as an infinite product.* To any sequence $(a_1, a_2, a_3, \ldots)$, let us associate $(b_1, b_2, b_3, \ldots)$ such that:

$$f(t):=1+\sum_{n\geq 1}a_n t^n=\prod_{n\geq 1}(1+b_n t^n).$$

(1) We have $a_n=\sum' b_1^{\varepsilon_1}b_2^{\varepsilon_2}b_3^{\varepsilon_3}\ldots$, where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots = 0$ or 1, and $\varepsilon_1+2\varepsilon_2+3\varepsilon_3+\cdots=n$. So, $a_1=b_1$, $a_2=b_2$, $a_3=b_3+b_1b_2$, $a_4=b_4+b_1b_3$, $a_5=b_5+b_1b_4+b_2b_3, \ldots$. Evidently, $b_1=b_2=\cdots=1$ implies $a_n=q(n)$, the number of partitions of n into unequal summands (p. 99). (2) Conversely, calculate b_n as a polynomial in $a_1, a_2, \ldots$. So, $b_1=a_1$, $b_2=a_2$, $b_3=a_3-a_2a_1$, $b_4=a_4-a_3a_1+a_2a_1^2$, $b_5=a_5-(a_4a_1+a_3a_2)+(a_3a_1^2+a_2^2a_1)-a_2a_1^3$, $b_6=a_6-(a_5a_1+a_4a_2)+(a_4a_1^2+a_3a_2a_1)-(a_3a_1^3-a_2^2a_1^2)+a_2a_1^4$, $b_7=a_7-(a_6a_1+a_5a_2+a_4a_3)+(a_5a_1^2+2a_4a_2a_1+a_3^2a_1-a_3a_2^2)-(a_4a_1^3-3a_3a_2a_1^2-a_2^3a_1)+(a_3a_1^4+2a_2^2a_1^3)-a_2a_1^5, \ldots$. If $a_1=a_2=\cdots=1$, then $b_n=0$, except $b_{2^m}=1$. (3) When $f(t)=e^{-t}$, prove the following property: $(b_n=1/n)\Leftrightarrow(n \text{ is prime})$ ([Kolberg, 1960]).

17. *Three summations of denumerants.* Verify the following summation formulas ([*Pólya, Szegö, 1, 1925], p. 3, Exercises 22, 23, 24): $\sum_{i\geq 0} D(n-i;\ 1+i, 2+i)=n+1$; $\sum_{i\geq 0} D(n-2i-1;\ 1+i, 2+i)=n+2-d(n)$, where $d(n)$ is the number of divisors of n [*Hint*: Use Exercise 16, p. 162]. $\sum_{i\geq 1} D(2i-1)n-i^2;\ i^2, (i+1)^2)=n$.

18. *Integer points.* (1) The number of points $(x_1, x_2, \ldots, x_k)\in\mathbf{Z}^k$, with integer coordinates, $x_i\in\mathbf{Z}$, such that $|x_1|+|x_2|+\cdots+|x_k|\leq p$, p integer ≥ 0, equals: $\sum_{i\geq 0}2^{k-i}\binom{k}{i}\binom{p}{k-i}$ ([*Pólya, Szegö, 1, 1925], p. 4, Exercise 39). (2) The number of solutions with integers $x_i\geq 1$, $i\in[n]$, that satisfy $(1\leq)\, x_1\leq x_2\leq\cdots\leq x_n$, $x_1\leq k+1$, $x_2\leq k+2, \ldots, x_n\leq k+n$, equals $\binom{k+2n}{n}(k+1)/(k+n+1)$. ([*Whitworth, 1901], p. 115-16, [Dartenson, 1965], [Carlitz, Roselle, Scoville, 1971].)

*19. *Rational points in a polyhedron* ([Ehrhart, 1967]). We denote the set of points in $\mathbf{R}^k$ whose coordinates are multiples of $1/n$ by $G_n^{(k)}$. The problem of the denumerants ([6a*] p. 108), which can also be written $a_1(x_1/n)+a_2(x_2/n)+\cdots+a_k(x_k/n)=1$, is hence equivalent to finding the number $I(n)$ of points of $G_n^{(k)}$ lying in the hyperplane part defined by $a_1X_1+a_2X_2+\cdots+a_kX_k=1$, $X_1, X_2, \ldots, X_k\geq 0$, whose k vertices are the points $A_1=(1/a_1, 0, 0, \ldots)$, $A_2=(0, 1/a_2, 0, \ldots)$, etc. More generally, let $\mathscr{P}$ be a polyhedral region of $\mathbf{R}^k$, whose vertices are $A_1, A_2, \ldots, A_m$ with rational coordinates, each face may or may not belong to $\mathscr{P}$. For each vertex A_j, let n_j be the LCM of the denominators of A_j. Then we denote the number of points in $\mathscr{P}\cap G_n^{(k)}$ by $I(n)$; we put $I(0):=1$. (1) There exists a polynomial $P(t)$ of degree less than $\sum a_i$ such that

$$\mathscr{F}(t) := \sum_{n \geq 0} I(n)\, t^n = \frac{P(t)}{\prod_{i=1}^{d} (1 - t^{a_i})} = \frac{P(t)}{Q(t)}.$$

[*Hint*: First treat the case of a simplex.] For example, if $\mathscr{P}$ is the open polygon in R^2 whose vertices are $A_1 = (0, 0)$, $A_2 = (1, 0)$, $A_3 = (\frac{1}{2}, \frac{1}{3})$, $A_4 = (0, 1)$, we have $a_1 = a_2 = a_4 = 1$, $a_3 = 6$. Hence $\sum_{n \geq 0} I(n)\, t^n = P(t)\,(1-t)^{-3}(1-t^6)^{-1}$, $\deg P \leq 8$. (2) The rational fraction $\mathscr{F}(t)$ can be simplified so that the exponent of the factor $(1-t)$ in the denominator is $\leq d+1$. For the preceding example we then get $\mathscr{F}(t) = P_1(t)\,(1-t)^{-2} \times (1-t^6)^{-1}$ and $P_1(t)$ can be determined by $I(0), I(1), I(2), \ldots, I(7) = -1, 0, 0, 1, 3, 6, 9, 13, 18$, respectively, which we obtain by direct inspection. Hence $P_1(t) = -1 + 2t + t^2 + t^3 + t^4 + t^5 + t^6 - 3t^7$. From this it follows that $I(n) = [n(5n-14)/12] + 1$. [*Hint*: Use the asymptotic order of $I(n)$ when $n \to \infty$.] (3) Use the preceding to prove the following values of $I(n)$ which are the numbers of integers $x, y, \ldots \in Z$ of certain relations. (1) $x + 2y + 3z - t = 3n$, [illegible] $\Rightarrow I(n) = \binom{n+2}{3} + \binom{n+3}{3}$. (2) $x + y < 3n/4$, $x - y < 3n/4$, $-x/2 < y < 2x \Rightarrow I(n) = [(9n^2 + 18$ [illegible]$)/\ldots]$ [illegible]. (3) $4x + 6y + 3z < 12n$, $x, y, z > 0 \Rightarrow I(n) = [(21n^2 + 6(-1)^n)/8] - n(17 - (-1)^n)/4 - 2$.

20. *Concerning ordinals.* Let $f(n)$ be the number of integer solutions $x_i\ (\geq 0)$ of the system $1 \leq x_1 < x_2 < \cdots < x_n$, $x_i < 2^{x_{i-1}}$, $i \in [n]$ (hence $x_1 = 1$). ([Peddicord, 1962], [Carlitz, Roselle, Scoville, 1971]; in fact, in this problem are counted the sets x of n elements such that $x \in x$ if $x \in \omega$, in the sense of the *axiomatic* set theory; cf. [*Krivine, 1969], p. 25.) (1) Let $F(n, k)$ stand for the number of solutions such that $x_n = k$, $F(n, k) = 0$ if $k < n$ or if $k \geq 2^{n}$, $f(n) = \sum_k F(n, k)$. Show that:

[α] $\qquad F(n+1) = F(n+1, 2^n)$

[β] $\qquad F(n, k) = \sum_{i < k} F(n-1, i)$.

(2) Let $\Phi(t, u) := \sum_{n,k} F(n, k)\, t^k u^n$, $\Phi_k(t) := \sum_{n \geq 0} F(n, k)\, t^n$. Then $\Phi_{2k-j}(t) = (1+t)^j\, \Phi_{2k}(t)$, $0 \leq j < 2^k$. [Use [β].] (3) Defining Ψ_k by $\Phi_{2^k}(t) = t^{k+1}\, \Psi_k(t)$, obtain from (2) a recurrence relation for the Ψ_k,

hence for $f(n)$ (use [α]), $n = 3, 4, \ldots$:

$$f(n+1) = \binom{2^n - 2}{n-1} - \sum_{j=1}^{n-2} f(j+1) \binom{2^n - 2^{j+1}}{n-j}.$$

n	1	2	3	4	5	6	7	8	9	10
$f(n)$	1	1	2	9	68	1802	15299	6423?29	105720?12	9365?62??5?8

***21.** *The number of score vectors of a tournament.* (Defined on p. 68. See [Bent, Narayana, 1964] and [*Moon, 1968], p. 66.) We want to determine the number of solutions with integers s_i of:

[α] $\qquad 1 \leq s_1 \leq s_2 \leq \cdots \leq s_n \leq n-1$

[β] $\qquad s_1 + s_2 + \cdots + s_k \geq \binom{k}{2}, \qquad k \in [n-1]$

[γ] $\qquad s_1 + s_2 + \cdots + s_n = \binom{n}{2}$.

Let $[k, l]^s$ be the number of solutions of $[\alpha, \beta, \delta]$:

[δ] $\qquad s_1 + s_2 + \cdots + s_k = l, \qquad s_k = s$.

Hence $[1, l]^s = 1$ for $s = l$ and $= 0$ if not. (1) We have $[k, l]^s = \sum_{r \leq s} [k-1, l-s]^{r}$. (2) Hence $s(n) = \sum_s [n, \binom{n}{2}]^s$. (3) Compute from this the first few values. (There is no exact formula for $s(n)$ and there is a conjecture that the ratio $s(n+1)/s(n)$ increases towards 4.)

n	1	2	3	4	5	6	7	8	9	10	11	12
$s(n)$	1	1	2	4	9	22	59	167	490	1486	4639	14805

22. *Relatively prime summands.* The number $R_k(n)$ of integer solutions $x_i \geq 1$ of $x_1 + x_2 + \cdots + x_k = n$ such that these integers are relatively prime, is such that ([Gould, 1964]. See also Exercise 16 (5), p. 161):

$$\sum_{n \geq k} R_k(n)\, \frac{t^n}{1 - t^n} = \frac{t^k}{(1-t)^k}.$$

23. *Compositions.* (1) A *composition* of the integer n into m summands, or *m-composition*, is any solution $x = (x_1, x_2, \ldots, x_m)$ of $x_1 + x_2 + \cdots + x_m = n$ with integer $x_i \geq 1$, $i \in [m]$ (the order of the summands counts!). $\mathfrak{C}_m(n)$ stands for the set of m-compositions of n. Show that $C(n, m) := |\mathfrak{C}_m(n)| =$

$-\binom{n-1}{m-1}$) has the following GF: $\sum_{n,m} L(n, m)t^n u^m = tu[1-t(1+u)]^{-1}$. (2) More generally, the number $C(n, m; A)$ of solutions of $\sum_{i=1}^{m} x_i = n$, where for all $i \in [m]$, $x_i \in A := \{a_1, a_2, a_3, \ldots\}$, $1 \leqslant a_1 < a_2 < \cdots$ is such that

$$1 + \sum_{1 \leqslant m \leqslant n} C(n, m; A)t^n u^m = \{1 - u(t^{a_1} + t^{a_2} + \cdots)\}^{-1}.$$

In how many ways can one put stamps to a total value of 30 cents on an envelope, if one has stamps of 5, 10 and 20 cents, which are glued in a single row onto the envelope (so the order of the stamps counts!). [*Answer*: 18.] More generally, for $5n$ cents (instead of 30, where $n=6$) and using notation [6f] on p. 110, the number of ways becomes $[0{,}605367\ldots(1{,}754878\ldots)^n]$. (3) Returning to (1), we endow $\mathfrak{C}_m(n)$ with an *order relation* by putting, for $x=(x_1, x_2, \ldots, x_m)$ and $x'=(x'_1, x'_2, \ldots, x'_m)$:

$$x \leqslant x' \Leftrightarrow \forall k \in [m], \quad \sum_{i=1}^{k} x_i \leqslant \sum_{i=1}^{k} x'_i.$$

Show that $\mathfrak{C}_m(n)$ becomes a *distributive lattice* in this way. (4) For each $x \in \mathfrak{C}_m(n)$ let $I_x = \{y \mid y \in \mathfrak{C}_m(n), y \leqslant x\}$, then $\sum_{x \in \mathfrak{C}_m(n)} |I_x| = (1/n)\binom{n}{m}\binom{n}{m-1}$ ([Narayana, 1955]).

24. *Decompositions with multi-indices.* For vectors $(n)=(n_1, n_2, \ldots, n_k)$ (or multi-indices, p. 36), a partition theory can be developed analogous to that given in this chapter. See for instance [*MacMahon, II, 1916], p. 54 and [Blakley, 1964a]. Let $\mathscr{S}$ be the system of k equations:

$$a_{i,1}x_1 + a_{i,2}x_2 + a_{i,3}x_3 + \cdots = n_i, \quad i \in [k],$$

where the $a_{i,j}$ are integers such that $1 \leqslant a_{i,1} < a_{i,2} < a_{i,3} < \cdots$. Show that the number $D((n); (a))$ of solutions of $\mathscr{S}$ in integers $x_i \geqslant 0$ has for GF:

$$\sum_{n_1, n_2, \ldots, n_k \geqslant 0} D((n), (a)) t_1^{n_1} t_2^{n_2} \cdots t_k^{n_k} = \prod_{j \geqslant 1} \left(1 - \prod_{i=1}^{k} t_i^{a_{i,j}}\right)^{-1}.$$

***25.** *Counting magic squares.* Let $Q(n, r)$ be the number of arrays (or matrices) of integers $a_{i,j} \geqslant 0$, $1 \leqslant i, j \leqslant n$, such that $\sum_{j=1}^{n} a_{i,j} = \sum_{j=1}^{n} a_{j,i} = r$ for all i, j. (1) $Q(1, r)=1$, $Q(2, r)=r+1$, $Q(3, r)=\binom{r+2}{2}+3\binom{r+3}{4}$, $Q(4, r)=\binom{r+3}{3}+20\binom{r+4}{5}+152\binom{r+5}{7}+352\binom{r+6}{9}$. More generally, $Q(n, r)$ is a polynomial with degree $(n-1)^2$ with respect to r. (2)

$Q(n, 1)=n!$, $Q(n, 2)=4^{-n} \sum (2n-2\alpha)!\,\alpha!\binom{n}{\alpha}^2 2^{\alpha}$, $Q(n, 3)=36^{-n} \times \sum (-1)^{\beta} \alpha!\,\beta!\,\gamma!\,(\ldots)\binom{n}{\alpha, \beta, \gamma}^2 (18)^{\beta}(12)^{\gamma}$, where $\alpha+2\beta+3\gamma=3n$, and the multinomial coefficient is denoted as in [10e'], p. 27. (3) Let $a_n = Q(n, 2)$, then $\sum_{n \geqslant 0} a_n t^n (n!)^{-2} = e^{t/2}(1-t)^{-1/2}$ and $a_n = n^2 a_{n-1} - (n-1)\binom{n}{2} a_{n-2}$. Moreover, $a_n = n!^{\ldots} d_n$, where the d_n are integers ([Anand, Dumir, Gupta, 1966], [Békéssy, 1972], [Carlitz, 1966b], [Ehrhart, 1973], [Mano, 1961], [Stanley, 1973]. Compare p. 235.)

n	0	1	2	3	4	5	6	7	8
d_n	1	1	3	[illegible]	47	257	2249	14591	[illegible]
a_n	1	1	3	21	282	[illegible]	202410	9135630	545007960

(4) Let $b_n = Q(n, 3)$ and $\varphi(x) := \sum_{k \geqslant 0} (3k)!\,(k!)^{-3} x^k$; then $\sum b_n t^n (n!)^{-2} = e^{t/2}(1-t/2)^{-1}\varphi((t/36)(1-t/2)^{-3})$. Use this to obtain for b_n a linear recurrence relation of the 6th order with coefficients that are polynomials in n.

n	0	1	2	3	4	5	6	7	8
b_n	1	1	4	55	2008	153040	20933640	4662857360	1579060245400

***26.** *Standard tableaux.* Each Ferrers diagram representing a certain partition of n can be considered in the obvious way as a 'descending wall' M, or 'profile'. Figure 29 represents the wall associated with the diagram of Figure 25 (p. 100). The 'stone' (i, j) is the one with 'abscissa' i and 'ordinate' j. We are interested in the number $\nu(M)$ of different ways in which M can be built up by piling stones one by one on top of each other, in such a way that at every stage the already constructed part is a 'descending wall'. Figure 30 gives a permissible numbering of the stones, thereby defining a so-called '*standard*' tableau, also called *Young tableau*. For a given wall M we write on each stone (i, j) the number of stones situated above and to the right of it, itself included. The table of numbers $z(i, j)$, obtained in this way, is represented in Figure 31. Hence the number of *standard tableaux* $\nu(M)$, equals $n!\{\prod_{(i,j) \in M} z(i, j)\}^{-1}$. (We refer to [Knuth, 1970, 1973a, b, 1973] for a study and a very complete bibliography of the problem, as well as for a

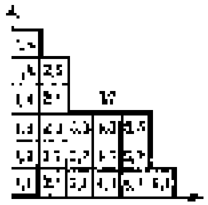


Fig. 29.

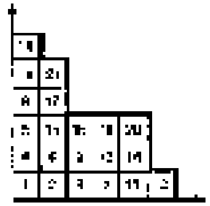


Fig. 30.

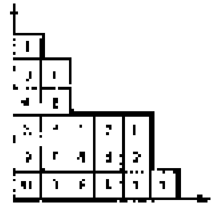


Fig. 31.

generalization to the case that part of the wall, say M', already exists, that is, it will be incorporated into M. See also [*Berge, 1968], pp. 49–59. We remark that the generalization to higher dimensions, in the sense of p. 103, is still an open problem.)

27. *Perfect partitions.* A *perfect* partition of an integer $n \geqslant 1$ is one that 'contains' precisely one partition of each integer less than n. In other words, if we consider the partition as a solution of $x_1 + 2x_2 + \cdots = n$, we call it perfect if for each integer $l \leqslant n$ there exists a *single* solution of $l_1 + 2l_2 + \cdots = l$, where $0 \leqslant l_i \leqslant x_i$, $i = 1, 2, \ldots$. So a perfect partition represents a set of weights such that each weight of l grams, $1 \leqslant l \leqslant n$, can be realized in exactly *one way*.

Show that the number of perfect partitions of n equals the number of *ordered* factorizations of $n+1$, omitting unit factors. Thus, for $n=7$, we have $8 = 4.2 = 2.4 = 2.2.2$, hence there are 4 perfect partitions: 1^7, $1^3 4$, 12^3, 124.

28. *Sums of multinomial coefficients.* Let us write $A(n)$ for the sum of the multinomial coefficients which occur in the expansion of $(x_1 + x_2 + \cdots + x_m)^n$. For example since $(x_1 + x_2 + \cdots + x_m)^3 = \sum x_i^3 + 3\sum x_i x_j^2 + 6\sum x_i x_j x_k$ (see p. 29) we have $A(3) = 1 + 3 + 6 = 10$. Prove that

$$\sum_{n \geqslant 0} A(n) \frac{t^n}{n!} = \frac{1}{\left(1 - \frac{t}{1!}\right)\left(1 - \frac{t^2}{2!}\right)\left(1 - \frac{t^3}{3!}\right)\cdots}$$

and study other properties of these numbers.

n	1	2	3	4	5	6	7	8	9	10
$A(n)$	1	3	10	47	246	1602	11481	95503	871030	8879558

IDENTITIES AND EXPANSIONS

This chapter is basically devoted to various results on formal series. The relation with counting problems is clear: for a sequence of integers with combinatorial meaning, the existence of a 'simple' formula is most frequently equivalent with the existence of a 'simple' generating function.

3.1. EXPANSION OF A PRODUCT OF SUMS; ABEL IDENTITY

The following notations slightly generalize the binomial and multinomial identities of pp. 12 and 28.

THEOREM A. *Let $\mathfrak{R}$ be a relation between two finite sets M and N* ($\mathfrak{R} \subset M \times N$, $|M| = m$, $|N| = n$), *Figure 32, and let $a(x, y)$ be a double sequence defined on $\mathfrak{R}$ and with values in a ring A (mostly $A = \mathbb{R}$ or $\mathbb{C}$). If $\langle x \mid \mathfrak{R} \rangle$ stands for the first section* (p. 59) *of $\mathfrak{R}$ by x, then we have:*

$$[1a] \qquad \prod_{x \in M} \sum_{y \in \langle x \mid \mathfrak{R} \rangle} a(x, y) = \sum_{\substack{\varphi \in N^M \\ (x, \varphi(x)) \in \mathfrak{R}}} \prod_{x \in M} a(x, \varphi(x)).$$

The summation in the second member of [1a] is taken over all maps φ of M into N, whose 'graphical representation' is a subset of $\mathfrak{R}$.

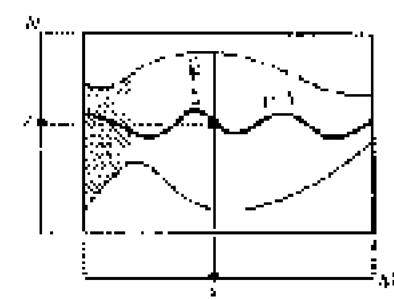


Fig. 32.

■ Let us suppose that the projection of $\mathfrak{R}$ onto M is just equal to M, because if not, then both members of [1a] equal zero. We number the elements of M and N, $M:=\{x_1, x_2, \ldots, x_m\}$, $N:=\{y_1, y_2, \ldots, y_n\}$. If $\mathfrak{N}=M\times N$, then the first member of [1a] can be written as $\prod_{i=1}^{m}\sum_{j=1}^{n} u(x_i, y_j)$. This is a product of m sums. The choice of a term in each of the m factors gives one term of the expansion, and two different choices give rise to two differently written terms. Now, any such choice is just a map φ from M into N; hence [1a]. If $\mathfrak{N}\neq M\times N$, then $u(x, y)$ can be extended to the whole of $M\times N$ by defining $u(x, y):=0$ for $(x, y)\notin\mathfrak{N}$. Then we can apply the preceding result, observing that the φ whose graph is not contained in $\mathfrak{N}$ give a contribution zero to the second member of [1a]. ■

Using [1a], the binomial and multinomial identities can easily be recovered.

We now show a deep generalization of the binomial identity.

THEOREM B. *(Abel identity)* [Abel, 1826]. *For all x, y, z we have:*

$$[1b]\qquad (x+y)^n=\sum_{k=0}^{n}\binom{n}{k}x(x-kz)^{k-1}(y+kz)^{n-k}.$$

(In a commutative ring, for instance. But [1b] also can be considered as an identity in the ring of polynomials in three indeterminates x, y, z.) For $z=0$ we recover the binomial identity [6a] (p. 12).

■ *First proof* (Lucas). We introduce the *Abel polynomials*

$$[1c]\qquad a_k(x, z):=x(x-kz)^{k-1}/k! \quad \text{for}\quad k\geq 1,\quad a_0:=1.$$

We have, successively,

$$\frac{\partial}{\partial x}a_k(x, z)=((x-kz)^{k-1}+(k-1)x(x-kz)^{k-2})/k!=$$
$$=a_{k-1}(x-z, z)$$
$$\frac{\partial^2}{\partial x^2}a_k(x, z)=\frac{\partial}{\partial x}a_{k-1}(x-z, z)=a_{k-2}(x-2z, z)$$
$$[1d]\qquad \frac{\partial^j}{\partial x^j}a_k(x, z)=a_{k-j}(x-jz, z).$$

Now, for fixed z, the $a_k(x, z)$ form a basis of the set of polynomials in x, because their degree equals $k\ (=0, 1, 2, \ldots)$. Hence, every polynomial

$P(x)$ can be uniquely expressed in the form $P(x)=\lambda_0a_0+\lambda_1a_1+\lambda_2a_2+\cdots$, where the λ_j only depend on z. Now, with [1d] for (*):

$$P^{(j)}(x)=\frac{d^j}{dx^j}P(x)=\sum_k\lambda_k\frac{\partial^j}{\partial x^j}a_k\overset{*}{=}\lambda_j+\lambda_{j+1}a_1(x-jz, z)+\cdots$$

which gives $\lambda_j=P^{(j)}(jz)$, by putting $x=jz$. So finally, for every polynomial $P(x)$ we have:

$$[1e]\qquad P(x)=\sum_{k\geq 0}a_k(x, z)P^{(k)}(kz),$$

from which [1b] follows by putting $P(x)=(x+y)^n$. ■

We still observe that if we apply [1e] to $P(x)=a_n(x+y, z)$, then we get the *convolution*

$$[1f]\qquad a_n(x+y, z)=\sum_{k=0}^{n}a_k(x, z)a_{n-k}(y, z).$$

See also [Hurwitz, 1902], [Jensen, 1902], [Kaucky, 1963], [*Riordan, 1968], p. 18–23, [Richardson, 1962], and [Salié, 1951], who gives a large bibliography.

■ *Second proof* (Françon). All the notions of p. 71 concerning the Foata coding of $[n]^{[n]}$ will be supposed known. Let $E\subset[n+2]^{[n+2]}$ be the set of functions of $[n+2]:=\{1, 2, \ldots, n, n+1, n+2\}$ such that elements $(n+1)$ and $(n+2)$ are *fixed points*. So, $\mathscr{F}_E=t_{n+1}t_{n+2}(\sum_{i=1}^{n+2}t_i)^n$. Now, consider for any set $x\subset[n]$ the set $E(x)\subset E$ of functions whose arborescence containing the element $(n+1)$ has $A_1:=x+(n+1)$ as set of nodes. Obviously, the *factorization* $E(x)=E_1E_2$ holds, where E_1 is the set of *rooted* functions *acting* on A_1 with the root $(n+1)$ only, and E_2 is the set of functions acting on $[n+2]\setminus A_1$ and having the element $(n+2)$ as a fixed point. Then

$$\mathscr{F}_{E(x)}=\mathscr{F}_{E_1}\mathscr{F}_{E_2}=t_{n+1}(t_{n+1}+\textstyle\sum_{i\in x}t_i)^{|x|-1}t_{n+2}(t_{n+2}+\sum_{i\notin x}t_i)^{n-|x|}.$$

But we have the *division* $E=\sum_{x\subset[n]}E(x)$. Therefore, $\mathscr{F}_E=\sum_{x\subset[n]}\mathscr{F}_{E(x)}$. In other words, after cancelling $t_{n+1}t_{n+2}$:

$$(t_1+t_2+\cdots+t_{n+2})^n=\sum_{x\subset[n]}t_{n+1}\Big(t_{n+1}+\sum_{i\in x}t_i\Big)^{|x|-1}\times$$
$$\times\Big(t_{n+2}+\sum_{i\notin x}t_i\Big)^{n-|x|}.$$

Now, put $t_{n+1}=x$, $t_{n+2}=y-nz$, $t_1=t_2=\cdots=t_n=-z$ to obtain [1b] after collecting the α such that $|\alpha|=k$. ∎

Of course, considering more than 2 fixed points, or other sets of functions, would give interesting other results (see Exercise 20, p. 163).

The following is an equivalent formulation of the Abel identity [1b], which generalizes [1e].

THEOREM C *For any formal series (hence for each polynomial) $f(t)$, we have:*

$$\text{[1g]} \qquad f(t)=\sum_{k\geq 0}\frac{t(t-ku)^{k-1}}{k!}f^{(k)}(ku),$$

where u is a new indeterminate, and $f^{(k)}$ the k-th derivative of f.
(For a study of the convergence of [1g], $t,u\in\mathbb{C}$, see [Halphen, 1881, 1882], [Pincherle, 1904].)

For $u=0$, we find back the ordinary (formal) Taylor formula.

∎ In fact, we have, with [1b] p. 128, $x\to t$, $y\to 0$, $z\to -u$ for $(*)$:

$$f(t):=\sum_{n\geq 0}a_n t^n \overset{(*)}{=}\sum_{n\geq 0}\Big\{a_n\sum_k\binom{n}{k}t(t-ku)^{k-1}(ku)^{n-k}\Big\}=$$

$$=\sum_{k\geq 0}\Big\{\frac{t(t-ku)^{k-1}}{k!}\sum_{n\geq k}(n)_k a_n(ku)^{n-k}\Big\}=\text{QED.} \quad ∎$$

3.2. PRODUCT OF FORMAL SERIES, LEIBNIZ FORMULA

The series used in this chapter will be always formal *Taylor series*. By definition, such a series is written as follows (for the meaning of the abbreviated notations α, k, etc., see p. 36):

$$\text{[2a]} \qquad f=f(t)=f(t_1,t_2,\ldots,t_k)=\sum_{\alpha\in\mathbb{N}^k}f_\alpha\frac{t^\alpha}{\alpha!}=$$

$$=\sum_{\alpha_1,\alpha_2,\ldots,\alpha_k\geq 0}f_{\alpha_1,\alpha_2,\ldots,\alpha_k}\frac{t_1^{\alpha_1}}{\alpha_1!}\frac{t_2^{\alpha_2}}{\alpha_2!}\cdots\frac{t_k^{\alpha_k}}{\alpha_k!}.$$

The f_α are called *Taylor coefficients of f.*

THEOREM A (*Leibniz formula*). *Let f and g be two formal series, with Taylor coefficients f_α and g_λ, $\alpha,\lambda\in\mathbb{N}^k$, and let h be the product series, $h=fg$. Then, the Taylor coefficients h_μ of h can be expressed as follows:*

$$\text{[2b]} \qquad h_\mu=h_{\mu_1,\mu_2,\ldots,\mu_k}=$$

$$=\sum\frac{\mu_1!\,\mu_2!\cdots\mu_k!}{\alpha_1!\,\lambda_1!\,\alpha_2!\,\lambda_2!\cdots\alpha_k!\,\lambda_k!}f_{\alpha_1,\alpha_2,\ldots,\alpha_k}g_{\lambda_1,\lambda_2,\ldots,\lambda_k},$$

where the summation takes place over all systems of integers $\alpha_1,\alpha_2,\ldots,\alpha_k,\lambda_1,\lambda_2,\ldots,\lambda_k$ such that $\alpha_1+\lambda_1=\mu_1$, $\alpha_2+\lambda_2=\mu_2,\ldots,\alpha_k+\lambda_k=\mu_k$. In other words:

$$\text{[2c]} \qquad h_{\mu_1,\ldots,\mu_k}=\sum_{\alpha_1,\ldots,\alpha_k}\binom{\mu_1}{\alpha_1}\cdots\binom{\mu_k}{\alpha_k}f_{\alpha_1,\ldots,\alpha_k}g_{\mu_1-\alpha_1,\ldots,\mu_k-\alpha_k},$$

or, in abbreviated notation:

$$\text{[2d]} \qquad h_\mu=\sum_{\alpha+\lambda=\mu}\frac{\mu!}{\alpha!\,\lambda!}f_\alpha g_\lambda.$$

∎ It suffices to apply definition [12g] (p. 37) of the product fg. ∎

Formula [2d] can immediately be generalized to a product h of r formal series $f_{\langle 1\rangle},f_{\langle 2\rangle},\ldots,f_{\langle r\rangle}$, $h=\prod_{i=1}^{r}f_{\langle i\rangle}$. So:

$$\text{[2e]} \qquad h_\mu=\sum\frac{\mu!}{\lambda\langle 1\rangle!\cdots\lambda\langle r\rangle!}f_{\langle 1\rangle,\lambda\langle 1\rangle}\cdots f_{\langle r\rangle,\lambda\langle r\rangle},$$

where the summation is extended over systems of multi-indices $\lambda\langle i\rangle\in\mathbb{N}^k$, $i\in[r]$ such that

$$\text{[2f]} \qquad \lambda\langle 1\rangle+\lambda\langle 2\rangle+\cdots+\lambda\langle r\rangle=\mu.$$

We observe by [2f] and Theorems B and D (p. 15), that the summation of [2e] contains $\prod_{j=1}^{k}\binom{\mu_j+r-1}{r-1}$ terms which is the number of solutions of [2f].

Actually, the exact formula [2h] allows us to calculate effectively the (partial) derivatives of a product of two functions. For each function $F(x)=F(x_1,x_2,\ldots,x_k)$ defined in a neighborhood of $a=(a_1,a_2,\ldots,a_k)\in\mathbb{R}^k$ and of class C^∞ in this point, and for any $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_k)\in\mathbb{N}^k$, we put:

$$\text{[2g]} \qquad f_\alpha:=\frac{\partial^{|\alpha|}F}{\partial x^\alpha}\Big|_{x=a}=$$

$$=\frac{\partial^{\alpha_1+\cdots+\alpha_k}}{\partial x_1^{\alpha_1}\cdots\partial x_k^{\alpha_k}}F(x_1,\ldots,x_k)\Big|_{(x_1,\ldots,x_k)=(a_1,\ldots,a_k)},$$

$$f_0=f_{0,0,\ldots,0}=F(a_1,\ldots,a_k).$$

and let:

$$[2h]\qquad f := \tau_a(F) = \sum_{\nu\in\mathbf{N}^k} f_\nu \frac{t^\nu}{\nu!}$$

be the formal Taylor series associated with the function F in a.

THEOREM B. *Let the two functions F and G be of class C^∞ in $a(\in\mathbf{R}^k)$, and let $H:=F\cdot G$. Between the three formal series [2h]: $f:=\tau_a(F)$, $g:=\tau_a(G)$, $h:=\tau_a(H)$, there exists the relation $h=fg$ in the sense of the product of formal series* ([12g], p. 37).

■ This is a well-known property of functions of class C^∞ in a point. ■ (See, for example, [*Valiron, I, 1958], p. 235.)

THEOREM C. *Let $r\,(\geq 2)$ functions $F_{(j)}=F_{(j)}(x)$, $j\in[r]$, $x\in\mathbf{R}^k$, be given, all of class C^∞ in $a\in\mathbf{R}^k$, and let $f_{(j)}:=\tau_a(F_{(j)})$, $j\in[r]$, $f_{(j)}:=\sum_{\nu\in\mathbf{N}^k} f_{(j),\nu}\, t^\nu/\nu!$, be their associated formal Taylor series* (cf. [2h]). *Then the successive derivatives h_ν of the function $H:=\prod_{j=1}^{r} F_{(j)}$ in a are given by formula* [2e] (*and particularly by* [2b, c, d] *if $k=1$*).

■ This is an immediate consequence of Theorems A and B. ■

In this way we recover for the product $H(x)=F(x)\,G(x)$ of two functions of one variable the usual Leibniz formula:

$$[2i]\qquad h_n = \sum_{l=0}^{n}\binom{n}{l} f_l\, g_{n-l}$$

where

$$f_l := \frac{d^l F(x)}{dx^l}\Big|_{x=a}$$

etc., $f_0=f(a),\ldots$ Similarly, for the product $H(x)=F_{(1)}(x)\ldots F_{(r)}(x)$ of r functions we get:

$$[2j]\qquad h_n = \sum \binom{n}{l(1),\ldots,l(r)} f_{(1),l(1)}\cdots f_{(r),l(r)}$$

where

$$l(1)+\cdots+l(r)=n,\qquad f_{(j),l} := \frac{d^l F_{(j)}(x)}{dx^l}\Big|_{x=a},\quad j\in[r].$$

Remark and example. All we said before can be summed up in the following rule: *The derivative $f_{n_1,n_2,\ldots} := \partial^{n_1+n_2+\cdots}F(x_1,x_2,\ldots)/\partial x_1^{n_1}\partial x_2^{n_2}\cdots$ of a certain function $F=F(x_1,x_2,\ldots)$ in the point $(x_1,x_2,\ldots)$ is the coefficient of $t_1^{n_1}t_2^{n_2}\cdots/n_1!\,n_2!\cdots$ in the expansion of $f=f(t_1,t_2,\ldots):=F(x_1+t_1,x_2+t_2,\ldots)$ by any known method.*

For example, if $F=(x_2+x_3)^{a_1}(x_3+x_1)^{a_2}(x_1+x_2)^{a_3}$, where a_1,a_2,a_3 are fixed real numbers, we find by abbreviating $\xi_1:=x_2+x_3$, $\xi_2:=x_3+x_1$, $\xi_3:=x_1+x_2$:

$$f=f(t_1,t_2,t_3)=(x_2+t_2+x_3+t_3)^{a_1}\times$$
$$\times(x_3+t_3+x_1+t_1)^{a_2}(x_1+t_1+x_2+t_2)^{a_3}=$$
$$=F\left(1+\frac{t_2}{\xi_1}+\frac{t_3}{\xi_1}\right)^{a_1}\left(1+\frac{t_3}{\xi_2}+\frac{t_1}{\xi_2}\right)^{a_2}\left(1+\frac{t_1}{\xi_3}+\frac{t_2}{\xi_3}\right)^{a_3},$$

that we can expand by [12m] (p. 41) (be aware of the multinomial notation, [10c'], p. 27):

$$f=F\sum_{\substack{k_1,k_1'\geq 0\\ k_2,k_2'\geq 0\\ k_3,k_3'\geq 0}}\binom{a_1}{k_1,k_1'}\binom{a_2}{k_2,k_2'}\binom{a_3}{k_3,k_3'}\times$$
$$\times\frac{t_1^{k_2'+k_3}\,t_2^{k_3'+k_1}\,t_3^{k_1'+k_2}}{\xi_1^{k_1+k_1'}\,\xi_2^{k_2+k_2'}\,\xi_3^{k_3+k_3'}}.$$

Finally, taking the coefficient of $t_1^{n_1}t_2^{n_2}t_3^{n_3}/n_1!\,n_2!\,n_3!$ we obtain:

$$f_{n_1,n_2,n_3}=\frac{\partial^{n_1+n_2+n_3}F}{\partial x_1^{n_1}\partial x_2^{n_2}\partial x_3^{n_3}}=\sum_{\substack{k_1\leq n_1\\ k_2\leq n_2\\ k_3\leq n_3}}\binom{n_1}{k_1}\binom{n_2}{k_2}\binom{n_3}{k_3}F\times$$
$$\times\frac{(a_1)_{n_2-k_2+k_3}\,(a_2)_{n_3-k_3+k_1}\,(a_3)_{n_1-k_1+k_2}}{\xi_1^{n_2-k_2+k_3}\,\xi_2^{n_3-k_3+k_1}\,\xi_3^{n_1-k_1+k_2}}.$$

3.3. BELL POLYNOMIALS

DEFINITION. *The (exponential) partial Bell polynomials are the polynomials $\mathbf{B}_{n,k}=\mathbf{B}_{n,k}(x_1,x_2,\ldots,x_{n-k+1})$ in an infinite number of variables $x_1,x_2,\ldots$, defined by the formal double series expansion:*

$$[3a]\qquad \Phi=\Phi(t,u):=\exp\left(u\sum_{m\geq 1}x_m\frac{t^m}{m!}\right)=\sum_{n,k\geq 0}\mathbf{B}_{n,k}\frac{t^n}{n!}u^k=$$
$$=1+\sum_{n\geq 1}\frac{t^n}{n!}\left\{\sum_{k=1}^{n}u^k\,\mathbf{B}_{n,k}(x_1,x_2,\ldots)\right\}$$

or, what amounts to the same, by the series expansion:

$$[3a']\qquad \frac{1}{k!}\left(\sum_{m\geq 1}x_m\frac{t^m}{m!}\right)^k=\sum_{n\geq k}\mathbf{B}_{n,k}\frac{t^n}{n!},\quad k=0,1,2,\ldots$$

The (exponential) complete Bell polynomials $\mathbf{Y}_n = \mathbf{Y}_n(x_1, x_2, \ldots, x_n)$ *are defined by:*

[3b] $$\Phi(t,1) = \exp\left(\sum_{m\ge1} x_m \frac{t^m}{m!}\right) = 1 + \sum_{n\ge1} \mathbf{Y}_n(x_1, x_2, \ldots)\frac{t^n}{n!},$$

in other words:

[3c] $$\mathbf{Y}_n = \sum_{k=1}^{n} \mathbf{B}_{n,k}; \qquad \mathbf{Y}_0 := 1.$$

([Bell, 1934], [Carlitz, 1961, 1962b, 1964, 1965a], [Frucht, 1965a, b, 1965a], [Frucht, Rota, 1965], [Kaucký, 1965].)

THEOREM A. *The partial Bell polynomials have integral coefficients, are homogeneous of degree k, and of weight n; their exact expression is:*

[3d] $$\mathbf{B}_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum \frac{n!}{c_1!\,c_2!\ldots(1!)^{c_1}(2!)^{c_2}\ldots} x_1^{c_1}x_2^{c_2}\ldots,$$

where the summation takes place over all integers $c_1, c_2, c_3, \ldots \ge 0$, such that:

[3e] $$c_1 + 2c_2 + 3c_3 + \cdots = n, \qquad c_1 + c_2 + c_3 + \cdots = k.$$

It follows that $\mathbf{B}_{n,k}$ contains $P(n,k)$ monomials, where $P(n,k)$ stands for the number of partitions of n into k summands, [1b, c] (p. 95).

■ We use the definition of the exponential series (p. 37) for relation (*), and the multinomial identity [10f] (p. 28) for (**):

[3f] $$\Phi(t,u) \overset{(*)}{=} \sum_{k\ge0}\frac{u^k}{k!}\left(\sum_{m\ge1} x_m\frac{t^m}{m!}\right)^k \overset{(**)}{=} \sum_{k\ge0}\frac{u^k}{k!}\left\{\sum_{c_1+c_2+\cdots=k}\frac{k!}{c_1!\,c_2!\ldots}\times \left(x_1\frac{t}{1!}\right)^{c_1}\left(x_2\frac{t^2}{2!}\right)^{c_2}\ldots\right\} =$$
$$= \sum_{c_1, c_2, \ldots \ge 0}\frac{u^{c_1+c_2+\cdots}\,t^{c_1+2c_2+\cdots}}{c_1!\,c_2!\ldots(1!)^{c_1}(2!)^{c_2}\ldots}x_1^{c_1}x_2^{c_2}\ldots$$

Hence [3d] follows, if we take in [3f] the coefficient of $(t^n u^k)/(n!)$. To see that the coefficients of $\mathbf{B}_{n,k}$ are integral, it suffices to observe that $(n!)/\{(1!)^{c_1}(2!)^{c_2}\ldots\}$ is the number of divisions of $[n]$ into c_1 1-parts, c_2 2-parts, etc., since $c_1 + 2c_2 + \cdots = n$ (p. 27); hence $(n!)/\{c_1!\,c_2!\ldots(1!)^{c_1}(2!)^{c_2}\ldots\}$ is the number of *unordered* divisions (or partitions of the set $[n]$), when omitting every 'empty part' corresponding to any $c_i = 0$, where the numbering of equal parts has been removed. Finally, $\mathbf{B}_{n,k}(abx_1, ab^2x_2, ab^3x_3, \ldots) = a^kb^n\mathbf{B}_{n,k}(x_1, x_2, x_3, \ldots)$ follows from [3d, e].

We have $\mathbf{B}_{0,0} = 1$, $\mathbf{B}_{1,1} = x_1$, $\mathbf{B}_{2,1} = x_2$, $\mathbf{B}_{2,2} = x_1^2$, $\mathbf{B}_{3,1} = x_3$, $\mathbf{B}_{3,2} = 3x_1x_2$, $\mathbf{B}_{3,3} = x_1^3, \ldots$, $\mathbf{B}_{n,1} = x_n$, $\mathbf{B}_{n,n} = x_1^n$. A table of the $\mathbf{B}_{n,k}$, $k \le n \le 12$, is found on p. 307. ■

THEOREM B. *The following are particular values of the* $\mathbf{B}_{n,k}$:

[3g] $\mathbf{B}_{n,k}(1, 1, 1, \ldots) = S(n,k)$ (Stirling number of the second kind, p. 50)

[3h] $\mathbf{B}_{n,k}(1!, 2!, 3!, \ldots) = \binom{n-1}{k-1}\frac{n!}{k!}$ (Lah number, p. 156)

[3i] $\mathbf{B}_{n,k}(0!, 1!, 2!, \ldots) = |s(n,k)| = \bar{s}(n,k)$ (signless Stirling number of the first kind, p. 50)

[3i'] $\mathbf{B}_{n,k}(1, 2, 3, \ldots) = \binom{n}{k}k^{n-k}$ (idempotent number, p. 92)

■ For [3g], we put $x_1 = x_2 = \cdots = 1$ in [3a]; we obtain $\Phi = \exp\{u(e^t - 1)\}$, so we get indeed the Stirling numbers of the second kind $S(n,k)$, [14q] (p. 50). For [3h], with $x_m = m!$ in [3a], we get:

[3j] $$\Phi = \exp\left(u\sum_{m\ge1} t^m\right) = \exp\{ut(1-t)^{-1}\} = \sum_{k\ge0}\frac{t^k u^k}{k!}(1-t)^{-k} = \sum_{k, l\ge0}\frac{t^{k+l}u^k}{k!\,l!}\langle k\rangle_l;$$

hence the result follows when we identify the coefficients of $u^kt^n/n!$ in the first and last member of [3j]. For [3i], $\Phi = \exp\{u\sum_{m\ge1} t^m/m\} = \exp\{-u\log(1-t)\} = (1-t)^{-u}$, which is the generating function of the absolute values of the numbers $s(n,k)$, [14p] (p. 50). Finally, [3i'] results from $\Phi = \exp(ute^t)$ here. (See Exercise 43, p. 91.) ■

The following relations can be proved easily ($n\ge 1$):

[3k] $$k\mathbf{B}_{n,k}=\sum_{l=k-1}^{n-1}\binom{n}{l}x_{n-l}\mathbf{B}_{l,k-1}$$

[3l] $$\mathbf{B}_{n,k}(x_1,x_2,\ldots)=\sum_{l=0}^{k}\binom{n}{l}x_1^l\mathbf{B}_{n-l,k-l}(0,x_2,x_3,\ldots)$$
$$=\sum_{l=0}^{k}\frac{n!}{(n-k)!\,l!}x_1^l\mathbf{B}_{n-k,k-l}\left(\frac{x_2}{2},\frac{x_3}{3},\ldots\right)$$

[3l'] $$\mathbf{B}_{n,k}\left(\frac{x_2}{2},\frac{x_3}{3},\ldots\right)=\frac{n!}{(k+n)!}\mathbf{B}_{n+k,k}(0,x_2,x_3,\ldots)$$

[3l''] $$\mathbf{B}_{n,k}\left(\frac{x_{q+1}}{\binom{q+1}{q}},\frac{x_{q+2}}{\binom{q+2}{q}},\ldots\right)=\frac{n!}{(n+qk)!}\mathbf{B}_{n+qk,k}\times(0,\ldots,0,x_{q+1},x_{q+2},\ldots)$$

[3m] $$\mathbf{B}_{n,n-a}(x_1,x_2,\ldots)=\sum_{j=a+1}^{2a}\binom{n}{j}x_1^{n-j}\mathbf{B}_{j,j-a}(0,x_2,x_3,\ldots)$$
$$=\sum_{j=a+1}^{2a}\frac{n!}{(n-j)!\,a!}x_1^{n-j}\mathbf{B}_{a,j-a}\left(\frac{x_2}{2},\frac{x_3}{3},\ldots\right)$$

[3n] $$\mathbf{B}_{n,k}(x_1+x_1',x_2+x_2',\ldots)=\sum_{\kappa\le l\le n}\binom{n}{l}\mathbf{B}_{l,\kappa}(x_1,x_2,\ldots)\mathbf{B}_{n-l,k-\kappa}(x_1',x_2',\ldots)$$

[3o'] $$\mathbf{B}_{n,k}(0,0,\ldots,0,x_j,0,\ldots)=0,\quad\text{except}\quad \mathbf{B}_{jk,k}=\frac{(jk)!}{k!\,(j!)^k}x_j^k.$$

Remark. The $\mathbf{B}_{n,k}$, as given by [3a, a'], will give a simple way of writing the Taylor coefficients (= successive derivatives) of the formal series that we now are going to study. Meanwhile, if one works with ordinary coefficients, as on pp. 36–43, it is better to use the polynomials $\hat{\mathbf{B}}_{n,k}$ (still with integral coefficients), defined by [3p, p'] instead of [3a, a'] (and tabulated on p. 309):

[3p] $$\hat{\Phi}=\hat{\Phi}(t,u):=\exp\Big(u\sum_{m\ge1}x_mt^m\Big)=\sum_{n,k\ge0}\hat{\mathbf{B}}_{n,k}(x_1,x_2,\ldots)\,t^n\frac{u^k}{k!}$$

[3p'] $$\Big(\sum_{m\ge1}x_mt^m\Big)^k=\sum_{n\ge k}\hat{\mathbf{B}}_{n,k}t^n$$

that we call *ordinary*, in contrast to the $\mathbf{B}_{n,k}$ already introduced, that we

called *exponential*. More generally, just as in the case of the c.f., [1.3e] (p. 44), let $\Omega_1,\Omega_2,\ldots$ be a *reference sequence*, $\Omega_1=1$, $\Omega_n\ne0$, given once and for all; the *Bell polynomials with respect to* Ω, $\mathbf{B}^{\Omega}_{n,k}=\mathbf{B}^{\Omega}_{n,k}(x_1,x_2,\ldots)$ are defined as follows:

[3q] $$\Omega_k\Big(\sum_{m\ge1}\Omega_mx_mt^m\Big)^k=\sum_{n\ge k}\mathbf{B}^{\Omega}_{n,k}\Omega_nt^n$$

($\Omega_n=1/n!$ in the 'exponential' case, and $\Omega_n=1$ in the 'ordinary' case). $\mathbf{B}^{\Omega}_{1,1}=x_1$; $\mathbf{B}^{\Omega}_{2,1}=x_2$, $\mathbf{B}^{\Omega}_{2,2}=x_1^2$; $\mathbf{B}^{\Omega}_{3,1}=x_3$, $\mathbf{B}^{\Omega}_{3,2}=2\Omega_2^2\Omega_3^{-1}x_1x_2$, $\mathbf{B}^{\Omega}_{3,3}=x_1^3$; ... Meanwhile, it should be perfectly clear, once and for all, that *the polynomials* $\mathbf{B}_{n,k}$ *which occur in the sequel of this book always mean the exponential Bell polynomials* ([3a] p. 134), *unless explicitly stated otherwise.*

3.4. Substitution of one formal series into another; formula of Faà di Bruno

THEOREM A. (Faà di Bruno *formula*). ([Faà di Bruno, 1855, 1857]. See also [*Bertrand, 1864] I, p. 138, [Cesàro, 1885], [Dederick, 1925], [Français, 1815], [Marchand, 1886], [Teixeira, 1880], [Wall, 1938].) *Let f and g be two formal (Taylor) series:*

[4a] $$f:=\sum_{k\ge0}f_k\frac{u^k}{k!},\quad g:=\sum_{n\ge0}g_n\frac{t^n}{n!},\quad \textit{with}\quad g_0=0,$$

and let h be the formal (Taylor) series of the composition of g by f (Theorem C, p. 40).

[4b] $$h:=\sum_{n\ge0}h_n\frac{t^n}{n!}=f\circ g=f[g].$$

Hence, the coefficients h_n *are given by the following expression:*

[4c] $$h_0=f_0,\qquad h_n=\sum_{1\le k\le n}f_k\mathbf{B}_{n,k}(g_1,g_2,\ldots,g_{n-k+1}),$$

where the $\mathbf{B}_{n,k}$ *are the exponential Bell polynomials* ([3a] p. 134).

■ By definition [4b] of h, it is clear that the h_n are linear combinations of the f_k:

[4d] $$h_n=\sum_{1\le k\le n}A_{n,k}f_k.$$

and that the $A_{n,k}$ only depend on $g_1, g_2, \dots$. Now these $A_{n,k}$ are determined by choosing for $f(u)$ the special formal series $f^*(u) := \exp(au)$, where a is a new indeterminate. Then:

$$[4e]\qquad f_k^* = \frac{d^k f^*}{du^k}\Big|_{u=0} = a^k.$$

Hence, by [3a] (p. 133), for (*), and by [4d] for (**):

$$[4f]\qquad h^* := f^*\circ g = \exp(ag) = \exp\left(a\sum_{m\geq 1} g_m\frac{t^m}{m!}\right) \overset{(*)}{=} 1+\sum_{n\geq k\geq 1} \mathbf{B}_{n,k}(g_1, g_2, \dots)\frac{t^n}{n!}a^k$$

$$[4g]\qquad = \sum_{n\geq 0} h_n^*\frac{t^n}{n!} \overset{(**)}{=} 1+\sum_{n\geq 1}\left\{\frac{t^n}{n!}\sum_{k=1}^{n} A_{n,k}f_k^*\right\} = 1+\sum_{1\leq k\leq n} A_{n,k}\frac{t^n}{n!}a^k,$$

from which it follows that $A_{n,k}=\mathbf{B}_{n,k}$ by identifying the last members of [4f] and [4g]. ■

So, we find (see p. 307): $h_1=f_1g_1$, $h_2=f_1g_2+f_2g_1^2$, $h_3=f_1g_3+3f_2g_1g_2+f_3g_1^3$, $h_4=f_1g_4+f_2(4g_1g_3+3g_2^2)+6f_3g_1^2g_2+f_4g_1^4, \dots$

By the Faà di Bruno formula we can effectively calculate the successive derivatives of a function of a function.

THEOREM B. *Let two functions $F(y)$ and $G(x)$ of a real variable be given: $G(x)$ of class C^n in $x=a$, and $F(y)$ of class C^n in $y=b=G(a)$, and let $H(x):=(F\circ G)(x)=F[G(x)]$. If we put*

$$[4h]\qquad g_m := \frac{d^mG}{dx^m}\Big|_{x=a},\qquad f_k := \frac{d^kF}{dy^k}\Big|_{y=b},\qquad h_n := \frac{d^nH}{dx^n}\Big|_{x=a},$$
$$g_0 := G(a),\qquad f_0 := F(b) = h_0 := H(a) = F[G(a)],$$

and we define the associated formal Taylor series

$$g(t) := \sum_{m\geq 1} g_m t^m/(m!),\qquad f(u) := \sum_{k\geq 0} f_k u^k/(k!),$$
$$h(t) := \sum_{n\geq 0} h_n t^n/(n!),$$

then we have formally: $h=f\circ g$. (Be careful! For g, the summation begins at $m=1$, so there is no constant term.)

■ If the Taylor expansions are convergent for a and t real, $|t|<R$, then we have: $H(a+t)=h(t)=F(b+g(t))=\sum_{k\geq 0} f_k g^k(t)/(k!)=(f\circ g)(t)$. If there is no convergence, then operate with expansions of f and g considered as asymptotic expansions. ■

THEOREM C. *Notations and hypotheses as in Theorem B for the functions F, G, H; $H=F\circ G$. Then the n-th order derivative of H in $x=a$, $n\geq 1$, equals:*

$$[4i]\qquad h_n := \frac{d^nH}{dx^n}\Big|_{x=a} = \sum_{k=1}^{n} f_k\mathbf{B}_{n,k}(g_1, g_2, \dots, g_{n-k+1}).$$

where the $\mathbf{B}_{n,k}$ are given explicitly by [3d].

■ Apply Theorems A and B. ■

Example. What is the n-th derivative of $F(x)=x^{x^a}$ ($x>0$ and a is any fixed real number $\neq 0$)? We can make the same observation as on p. 139. So, we must expand $f(t):=F(x+t)$ as a power series in t. Now, after a few manipulations:

$$f(t) = (x+t)^{(x+t)^a} =$$
$$= F(x)\cdot\exp(at\log x)\cdot\exp\left(ax\left(1+\frac{t}{x}\right)\log\left(1+\frac{t}{x}\right)\right).$$

Let us introduce the integers $b(n,k)$ such that

$$\frac{1}{k!}((1+T)\log(1+T))^k := \sum_{n\geq k} b(n,k)\frac{T^n}{n!},\qquad b(0,0):=1.$$

It is easy to verify: $b(n+1,k)=nb(n-1,k-1)-b(n,k-1)+(k-n)\times b(n,k)$, hence the following table for $b(n,k)$:

n \ k	1	2	3	4	5	6	7	8	9	10
1	1									
2	1	1								
3	−1	3	1							
4	2	−1	6	1						
5	−6	0	5	10	1					
6	24	4	−15	25	15	1				
7	−120	−28	49	−35	70	21	1			
8	720	188	196	49	0	154	28	1		
9	−5040	−1368	944	0	−231	252	294	36	1	
10	40320	11016	5340	840	1365	987	1050	510	45	1

Moreover, $b(n,k)=\sum_l \binom{l}{k} k^{l-k} s(n,l)$ with the Stirling numbers $s(n,l)$ of p. 50.

Returning to $f(t)$, we get consequently:

$$f(t)=F(x)\sum_{j\geq 0,\, k\leq m}\frac{(at\log x)^j}{j!}\, b(m,k)\,\frac{(t/a)^m}{m!}(ax)^k.$$

Finally, collecting the coefficients of $t^n/n!$ in $f(t)$ and abbreviating $\lambda:=\log x$, $\xi:=(ax)^{-1}$, we obtain the following formula for the n-th derivative:

$$f_n=\frac{d^n(x^{x^a})}{da^n}=a^n x^{x^a}\sum_{j=0}^{n}\left\{\binom{n}{j}\lambda^j\sum_{k=0}^{n-j} b(n-j,\, n-k-j)\,\xi^k\right\}.$$

For instance, $f_4=a^4x^{x^a}(1+6\xi-\xi^2+2\xi^3-4\lambda(1+3\xi-\xi^2)-6\lambda^2(1+\xi)+4\lambda^3+\lambda^4)$.

3.5 LOGARITHMIC AND POTENTIAL POLYNOMIALS

The following are three examples of applications of the Faà di Bruno formula.

THEOREM A *(successive derivatives of* $\log G$*). The* **logarithmic** *polynomials* $\mathbf{L}_n$ *defined by:*

$$\text{[5a]}\qquad \log\left(\sum_{n\geq 0} g_n\frac{t^n}{n!}\right)=\log\left(1+g_1t+g_2\frac{t^2}{2!}+\cdots\right)=\sum_{n\geq 1}\mathbf{L}_n\frac{t^n}{n!}\qquad (g_0=1),$$

which are expressions for the n-th derivative of $\log(G(x))$ *at the point* $x=a$, *equal (for the notation, cf.* [3d] *p. 134 and* [4b] *p. 138):*

$$\text{[5b]}\qquad \mathbf{L}_n=\mathbf{L}_n(g_1,g_2,\ldots,g_n)=\sum_{1\leq k\leq n}(-1)^{k-1}(k-1)!\,\mathbf{B}_{n,k}(g_1,g_2,\ldots).\qquad (\mathbf{L}_0:=0)$$

■ Use [4c, d] with $F(y):=\log y$, $f_0=1$, $f_k=(-1)^{k-1}(k-1)!$ ■

From [5a, b] the following expansion is easily deduced:

$$\text{[5c]}\qquad \log\left(g_0+g_1t+g_2\frac{t^2}{2!}+\cdots\right)=\log g_0+ \sum_{n\geq 1}\frac{t^n}{n!}\left\{\sum_{1\leq k\leq n}(-1)^{k-1}(k-1)!\,g_0^{-k}\mathbf{B}_{n,k}(g_1,g_2,\ldots)\right\},$$

where $g_0>0$. A table of logarithmic polynomials is given on p. 308. (On this subject, see also [Bouwkamp, De Bruijn, 1969].)

THEOREM B *(successive derivatives of* G^r*). The* **potential** *polynomials* $\mathbf{P}_n^{(r)}$ *defined for each complex number r by:*

$$\text{[5d]}\qquad \left(\sum_{n\geq 0} g_n\frac{t^n}{n!}\right)^r=\left(1+g_1t+g_2\frac{t^2}{2!}+\cdots\right)^r=1+\sum_{n\geq 1}\mathbf{P}_n^{(r)}\frac{t^n}{n!}\qquad (g_0=1),$$

which are expressions for the n-th derivative of $[G(x)]^r$ *at the point* $x=a$, *equal (notations as in* [3d] *p. 134, and* [4b] *p. 138):*

$$\text{[5e]}\qquad \mathbf{P}_n^{(r)}=\mathbf{P}_n^{(r)}(g_1,g_2,\ldots,g_n)=\sum_{1\leq k\leq n}(r)_k\,\mathbf{B}_{n,k}(g_1,g_2,\ldots).\qquad (\mathbf{P}_0^{(r)}:=1)$$

■ Use [4c, d] with $F(y):=y^r$, $f_0=1$, $f_k=(r)_k$. ■

From [5d, e] we obtain easily the expansion

$$\text{[5f]}\qquad \left(g_0+g_1t+g_2\frac{t^2}{2!}+\cdots\right)^r=g_0^r+\sum_{n\geq 1}\frac{t^n}{n!}\left\{\sum_{1\leq k\leq n}(r)_k\,g_0^{r-k}\mathbf{B}_{n,k}(g_1,g_2,\ldots)\right\},$$

where $g_0>0$ for r an arbitrary real or complex number, $g_0\neq 0$ for r an arbitrary integer, and g_0 arbitrary for r an integer ≥ 0. When $g_0=0$ in [5f], and r is an integer ≥ 0, then we find back [3a'] (p. 133), and when r is integer ≤ 0, we get the following Laurent series, whose expansion is given by [5d] ($g_1\neq 0$):

$$\text{[5g]}\qquad \left(g_1t+g_2\frac{t^2}{2!}+\cdots\right)^r=(g_1t)^r\left(1+\frac{g_2}{2g_1}\frac{t}{1!}+\frac{g_3}{3g_1}\frac{t^2}{2!}+\cdots\right)^r$$

Finally, by [3l'], one may show that for all integers l and $q\geq 0$, we have

$$\left(g_q\frac{t^q}{q!}+g_{q+1}\frac{t^{q+1}}{(q+1)!}+\cdots\right)^{-l}=$$
$$=\frac{(q!)^l}{(g_q)^l}\frac{1}{t^{ql}}\sum_{n\geq 0}t^n\sum_{j\leq n}\frac{(-1)^j(q!)^j\langle l\rangle_j}{(n+qj)!\,(g_q)^j}B_{n+qj,\,j}(\underbrace{0,0,\ldots,0}_{q\text{ zeros}},g_{q+1},g_{q+2},\ldots).$$

Theorem C. *For any complex number r, we have:*

$$\text{[5h]}\qquad P_n^{(-r)}=r\binom{n+r}{n}\sum_{1\leq j\leq n}(-1)^j\frac{1}{r+j}\binom{n}{j}P_n^{(j)}.$$

In other words, for $G(x)=C^\infty$ in the point a, $g_0=G(a)=1$:

$$\text{[5i]}\qquad \frac{d^n}{dx^n}G^{-r}(x)\bigg|_{x=a}=r\binom{n+r}{n}\sum_{1\leq j\leq n}(-1)^j\frac{1}{r+j}\binom{n}{j}\frac{d^n}{dx^n}G^j(x)\bigg|_{x=a}.$$

■ Let $g=1+\sum_{n\geq 1}g_nt^n/(n!)$; then we get

$$\text{[5j]}\qquad g^{-r}=1+\sum_{n\geq 1}P_n^{(-r)}\frac{t^n}{n!}=\{1+(g-1)\}^{-r}=$$
$$=\sum_{k\geq 0}\binom{-r}{k}(g-1)^k.$$

Now t^k divides $(g-1)^k=(g_1t+g_2t^2/2!+\cdots)^k$; hence, by virtue of [5j], $P_n^{(-r)}$ equals the coefficient of $t^n/(n!)$ in:

$$\sum_{k=0}^{n}\binom{-r}{k}(g-1)^k=\sum_{0\leq j\leq k\leq n}\binom{-r}{k}\binom{k}{j}(-1)^{k-j}g^j.$$

Hence

$$P_n^{(-r)}=\sum_{0\leq j\leq k\leq n}\binom{-r}{k}\binom{k}{j}(-1)^{k-j}P_n^{(j)}=$$
$$=\sum_{0\leq j\leq n}(-1)^jP_n^{(j)}\cdot\gamma,$$

where, using [7e] (p. 17), for (*):

$$\gamma=\sum_{j\leq k\leq n}\binom{r+k-1}{k}\binom{k}{j}=\binom{r+j-1}{j}\sum_{j\leq k\leq n}\binom{r+k-1}{k-j}\overset{(*)}{=}$$
$$\overset{(*)}{=}\binom{r+j-1}{j}\binom{r+n}{n-j}=\frac{r}{r+j}\binom{n+r}{n}\binom{n}{j}.\ ■$$

3.6 INVERSION FORMULAS AND MATRIX CALCULUS

We just treat two examples and for the rest we refer to [*Riordan, 1968], pp. 43–127, for a very extensive study of the subject.

(I) *Binomial coefficients*

Let two sequences be given, consisting for instance of real numbers (more generally, in a commutative ring with identity) such that:

$$\text{[6a]}\qquad f_n=\sum_{0\leq k\leq n}\binom{n}{k}g_k,\qquad n\geq 0.$$

We want to express g_n as a function of the f_k.

The simplest method consists of observing that [6a] means that:

$$\text{[6b]}\qquad \mathbf{F}=\mathbf{PG},$$

where $\mathbf{F}$, $\mathbf{G}$ are matrices consisting of a single (infinite) column, and $\mathbf{P}$ the (infinite triangular) Pascal matrix:

$$\text{[6c]}\qquad \mathbf{F}:=\begin{bmatrix}f_0\\f_1\\f_2\\\vdots\end{bmatrix},\qquad \mathbf{G}:=\begin{bmatrix}g_0\\g_1\\g_2\\\vdots\end{bmatrix},\qquad \mathbf{P}:=\begin{bmatrix}1&&&\\1&1&&\\1&2&1&\\\vdots&\vdots&\vdots&\ddots\end{bmatrix}.$$

We take for $\mathbf{F}$ and $\mathbf{G}$ special matrices such that $f_n=y^n$, $g_n=x^n$, in this case we get, by [6a], $y=1+x$. Hence $x^n=(y-1)^n=\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}y^k$; consequently:

$$\text{[6d]}\qquad \mathbf{P}^{-1}=\left[(-1)^{n-k}\binom{n}{k}\right]_{n,k\geq 0}=\begin{bmatrix}1&&&&\\-1&1&&&\\1&-2&1&&\\-1&3&-3&1&\\\vdots&\vdots&\vdots&\vdots&\ddots\end{bmatrix}$$

So, $\mathbf{P}^{-1}$ is the same as $\mathbf{P}$, except that signs $-$ appear in a checkerboard pattern. (Because $\mathbf{P}$ is triangular, [6d] also holds, if the matrices are cut off at the nth line, and thus turned into finite matrices.) Finally, if we take into account that $\mathbf{G}=\mathbf{P}^{-1}\mathbf{F}$:

$$[6e]\qquad g_n=\sum_{0\le k\le n}(-1)^{n-k}\binom{n}{k}f_k.$$

(II) *Stirling numbers*

We now show that the matrix $\mathbf{s}:=[s(n,k)]_{n,k\ge 0}$ consisting of the Stirling numbers of the first kind, is the inverse of the matrix $\mathbf{S}:=[S(n,k)]_{n,k\ge 0}$ of the Stirling numbers of the second kind; this means, like in the preceding case of the binomial coefficients:

$$[6f]\qquad f_n=\sum_k S(n,k)\,g_k \Leftrightarrow g_n=\sum_k s(n,k)\,f_k.$$

Now, using [14s] (p. 51) for (*), and using the notation:

$$f:=\sum_{n\ge 0} f_n t^n/n!,\qquad g:=\sum_{n\ge 0} g_n t^n/n!,$$

we get:

$$[6g]\qquad f=f(t)=\sum_{n\ge 0}\frac{t^n}{n!}\Big(\sum_k S(n,k)\,g_k\Big)=\sum_{k\ge 0} g_k\Big(\sum_{n\ge k} S(n,k)\frac{t^n}{n!}\Big)\overset{(*)}{=}\sum_{k\ge 0} g_k\frac{(e^t-1)^k}{k!}=g(e^t-1).$$

Putting $u:=e^t-1$, let $t=\log(1+u)$. Then [6g] gives, with [14r] (p. 51) for (**):

$$[6h]\qquad g=g(u)=f(\log(1+u))=\sum_{k\ge 0} f_k\frac{\log^k(1+u)}{k!}\overset{(**)}{=}\sum_{k\ge 0} f_k\Big(\sum_{n\ge k} s(n,k)\frac{u^n}{n!}\Big)=\sum_{n\ge 0}\frac{u^n}{n!}\Big(\sum_k s(n,k)\,f_k\Big),$$

which proves [6f], if we identify the coefficients of $u^n/n!$ of the first and the last member of [6h].

3.7. FRACTIONARY ITERATES OF FORMAL SERIES

The Faà di Bruno formula, [4c] (p. 137), with $f=g$, gives the coefficients or derivatives of $f\circ f$, and more generally, it also gives the coefficients of

the iterate of order m of the formal series f (when $f_0=0$, m integer ≥ 1), denoted by $f^{\langle m\rangle}$, and defined as follows:

$$[7a]\qquad f^{\langle 1\rangle}=f,\quad f^{\langle 2\rangle}=f\circ f,\ \ldots,\ f^{\langle m\rangle}=f\circ f^{\langle m-1\rangle}.$$

We now want to define the iterate (analytical or fractionary) of order α of f, also denoted by $f^{\langle\alpha\rangle}$, for any α from the field of the coefficients of f; in the case we consider, this will be the field of the complex numbers (this constitutes no serious loss of generality). In this section every formal series f is supposed to be of the form:

$$[7b]\qquad f=\sum_{r\ge 1}\Omega_r f_r t^r,$$

where $\Omega_1,\Omega_2,\ldots$ is a reference sequence, given once and for all, $\Omega_1=1$, $\Omega_n\ne 0$ (p. 44); in this way we treat at the same time the case of 'ordinary' coefficients of f ($\Leftrightarrow\Omega_n=1$), and the case of 'Taylor coefficients' ($\Leftrightarrow\Omega_n=1/n!$).

With every series f we associate the infinite lower *iteration* matrix (with respect to Ω):

$$[7c]\qquad \mathbf{B}=\mathbf{B}(f):=\begin{pmatrix}\mathrm{B}_{1,1} & 0 & 0 & \ldots\\ \mathrm{B}_{2,1} & \mathrm{B}_{2,2} & 0 & \ldots\\ \mathrm{B}_{3,1} & \mathrm{B}_{3,2} & \mathrm{B}_{3,3} & \ldots\\ \vdots & \vdots & \vdots & \end{pmatrix},$$

where $\mathrm{B}_{n,k}=\mathrm{B}^{\Omega}_{n,k}(f_1,f_2,\ldots)$ is the Bell polynomial with respect to Ω ([3p*] p. 137), defined as follows:

$$[7d]\qquad \Omega_k f^k=\sum_{n\ge k}\mathrm{B}_{n,k}\Omega_n t^n.$$

Thus, the matrix of the binomial coefficients is the iteration matrix for $f=t(1-t)^{-1}$, $\Omega_n=1$, and the matrix of the Stirling numbers of the second kind $S(n,k)$ is the iteration matrix for $f=e^t-1$, $\Omega_n=1/n!$.

THEOREM A. *For three sequences f, g, h (written as in [7b]) $h=f\circ g$ is equivalent to the matrix equality:*

$$[7e]\qquad \mathbf{B}(h)=\mathbf{B}(g).\mathbf{B}(f).$$

(Jabotinsky, 1947, 1949, 1963.) If we transpose the matrices, we get: $h=f\circ g\Leftrightarrow {}^t\mathbf{B}(h)={}^t\mathbf{B}(f).{}^t\mathbf{B}(g)$, which looks better. However, the classical combinatorial matrices, as the binomial and the Stirling matrices, are most frequently denoted as lower triangular matrices, hence our choice.)

■ For each integer $k\geq 1$, we have, with [7d] for (*):

[7f] $$\sum_{n\geq k}\mathbf{B}_{n,k}(u_1,u_2,\ldots)\Omega_n t^n \stackrel{(*)}{=} \Omega_k u^k = \Omega_k(f(g))^k \stackrel{(*)}{=} \sum_{l\geq k}\mathbf{B}_{l,k}(f_1,f_2,\ldots)\Omega_l g^l \stackrel{(*)}{=} \sum_{n\geq l\geq k}\mathbf{B}_{n,l}(g_1,g_2,\ldots)\mathbf{B}_{l,k}(f_1,f_2,\ldots)\Omega_n t^n,$$

from which [7e] follows if we collect the coefficient of $\Omega_n t^n$ at both 'ends' of [7f]. ■

If we consider in [7e] the first column of $\mathbf{B}(h)$ only, we obtain again the formula of Faà di Bruno ([4i] p. 139), if we take $\Omega_n = 1/n!$. More generally, if we have α series $f_{(1)}, f_{(2)}, \ldots, f_{(\alpha)}$, then [7e] gives the matrix equality $\mathbf{B}(f_{(\alpha)}\circ\cdots\circ f_{(2)}\circ f_{(1)}) = \mathbf{B}(f_{(1)})\mathbf{B}(f_{(2)})\ldots\mathbf{B}(f_{(\alpha)})$. In other words, if we consider again the first column only, we obtain a generalized Faà di Bruno formula for the n-th derivative of the composite of α functions (again, we must take $\Omega_n = 1/n!$). Similarly, $\mathbf{B}(f^{\langle\alpha\rangle}) = (\mathbf{B}(f))^\alpha$ for all integers $\alpha\geq 1$, which leads to an explicit formula for integral order iterates ([Tambs, 1927]).

Now we suppose that the coefficient of t in f equals 1, $f_1 = 1$; shortwise, we say that f is *unitary*. Furthermore, we assign values to $\mathbf{B}^\alpha = (\mathbf{B}(f))^\alpha$, α complex, in the following way: denoting the unit matrix by $\mathbf{I}$, and putting $\mathscr{B} := \mathbf{B} - \mathbf{I}$ (which is $\mathbf{B}$ with all 1's on the diagonal erased), we define:

[7g] $$\mathbf{B}^\alpha = (\mathbf{I}+\mathscr{B})^\alpha = \sum_{j\geq 0}\binom{\alpha}{j}\mathscr{B}^j.$$

In other words, between the coefficients of $\mathbf{B}^\alpha$, denoted by $\mathbf{B}^{(\alpha)}_{n,k}$ (n is the row number and k is the column number), and the coefficients of $\mathscr{B}^j$, denoted by $[\mathscr{B}^j]_{n,k}$, the following relation holds:

[7h] $$\mathbf{B}^{(\alpha)}_{n,k} = \sum_{0\leq j\leq n-k}\binom{\alpha}{j}[\mathscr{B}^j]_{n,k},$$

by which the matrix $\mathbf{B}^\alpha$ can actually be computed. For all α, α', the reader will verify the matrix equalities

[7i] $$\mathbf{B}^\alpha\mathbf{B}^{\alpha'} = \mathbf{B}^{\alpha+\alpha'} = \mathbf{B}^{\alpha'}\mathbf{B}^\alpha, \qquad (\mathbf{B}^\alpha)^{\alpha'} = \mathbf{B}^{\alpha\alpha'} = (\mathbf{B}^{\alpha'})^\alpha.$$

DEFINITION. *For each complex number α, the α-th order fractionary iterate $f^{\langle\alpha\rangle}$ of the unitary series f is the unitary series, whose iteration matrix is $\mathbf{B}^\alpha$. In other words, $f^{\langle\alpha\rangle} = \sum_{n\geq 1} f_n^{\langle\alpha\rangle}\Omega_n t^n$, where the coefficients $f_n^{\langle\alpha\rangle}$ have the following expression, using $\mathbf{b}_{n,j} := [\mathscr{B}^j]_{n,1}$, $n\geq 2$:*

[7j] $$f_n^{\langle\alpha\rangle} = \mathbf{B}^{(\alpha)}_{n,1} = \sum_{1\leq j\leq n-1}\binom{\alpha}{j}\mathbf{b}_{n,j}, \quad n\geq 2, \quad f_1^{\langle\alpha\rangle} = 1.$$

Series $f^{\langle\alpha\rangle}$, thus defined, does not depend on the reference sequence Ω_n.

Evidently, $f^{\langle 0\rangle}$ is the 'identity' series, $f^{\langle 0\rangle}(t) = t$. In the case of 'Taylor coefficients', $\Omega_n = 1/n!$, we obtain, by computing the powers $\mathscr{B}^j$, the following first values for the *iteration polynomials* $\mathbf{b}_{n,j}$:

$\mathbf{b}_{2,1} = f_2$; $\mathbf{b}_{3,1} = f_3$, $\mathbf{b}_{3,2} = 3f_2^2$; $\mathbf{b}_{4,1} = f_4$, $\mathbf{b}_{4,2} = 10f_2f_3 + 3f_2^3$, $\mathbf{b}_{4,3} = 18f_2^3$; $\mathbf{b}_{5,1} = f_5$, $\mathbf{b}_{5,2} = 15f_2f_4 + 10f_3^2 + 25f_2^2f_3$, $\mathbf{b}_{5,3} = 150f_2^2f_3 + 75f_2^4$, $\mathbf{b}_{5,4} = 180f_2^4$; $\mathbf{b}_{6,1} = f_6$, $\mathbf{b}_{6,2} = 21f_2f_5 + 35f_3f_4 + 60f_2^2f_4 + 70f_2f_3^2 + 15f_2^3f_3$, $\mathbf{b}_{6,3} = 270f_2^2f_4 + 350f_2f_3^2 + 1065f_2^3f_3 + 180f_2^5$, $\mathbf{b}_{6,4} = 2310f_2^3f_3 + 1935f_2^5$, $\mathbf{b}_{6,5} = 2700f_2^5$; $\mathbf{b}_{7,1} = f_7$, $\mathbf{b}_{7,2} = 28f_2f_6 + 56f_3f_5 + 35f_4^2 + 126f_2^2f_5 + 350f_2f_3f_4 + 70f_3^3 + 105f_2^2f_3^2 + 105f_2^3f_4$, $\mathbf{b}_{7,3} = 501f_2^2f_5 + 1610f_2f_3f_4 + 340f_3^3 + 1255f_2^3f_4 + 5135f_2^2f_3^2 + 4935f_2^4f_3 + 315f_2^6$, $\mathbf{b}_{7,4} = 6300f_2^3f_4 + 11900f_2^2f_3^2 + 42420f_2^4f_3 + 13545f_2^6$, $\mathbf{b}_{7,5} = 54810f_2^4f_3 + 59535f_2^6$, $\mathbf{b}_{7,6} = 56700f_2^6$.

From these values we obtain immediately, by [7j], the expressions for the first derivatives $f_n^{\langle\alpha\rangle}$ of the iterate $f^{\langle\alpha\rangle}$. For example, the fractionary iterate of $f(t) = e^t - 1 = \sum_{n\geq 1} t^n/n!$ is $f^{\langle\alpha\rangle}(t) = t + \sum_{n\geq 2} f_n^{\langle\alpha\rangle} t^n/n!$, where $f_n^{\langle\alpha\rangle} = \sum_{j=1}^{n-1}\binom{\alpha}{j}\mathbf{b}_{n,j}$ for $n\geq 2$; the first few values of $\mathbf{b}_{n,j}$ are:

$n \backslash k$	1	2	3	4	5	6
2	1					
3	1	5				
4	1	13	19			
5	1	50	205	183		
6	1	201	1865	4245	2700	
7	1	875	15674	74165	114545	55705
8	1	4138	155477	1209830	3394790	3919860	1587600

Evidently, the alternating row sums $\sum_{k=1}^{n-1}(-1)^k b_{n,k}$ equal $(-1)^{n-1} \times (n-1)!$, since $f_{\circ}^{(-1)}(t) = \log(1+t)$.

Theorem B. *For all complex numbers α, α', the fractionary iterates of the unitary series f satisfy:*

[7k] $$f^{\langle\alpha\rangle} \circ f^{\langle\alpha'\rangle} = f^{\langle\alpha+\alpha'\rangle} = f^{\langle\alpha'\rangle} \circ f^{\langle\alpha\rangle},$$
$$(f^{\langle\alpha\rangle})^{\langle\alpha'\rangle} = f^{\langle\alpha\alpha'\rangle} = (f^{\langle\alpha'\rangle})^{\langle\alpha\rangle}.$$

■ This follows immediately from [7i]. ■

3.8. INVERSION FORMULA OF LAGRANGE

For every formal series $f = \sum_{n \geq 0} a_n t^n$, we denote the derivative by f' or Df, or df/dt; let furthermore:

[8a] $\mathbb{C}_n f = a_n =$ the coefficient of t^n in f.

Supposing $a_0 = 0$, $a_1 \neq 0$, we are going to compute the coefficients $a_n^{\langle-1\rangle}$ of the *reciprocal* series, which is:

$$f^{\langle-1\rangle} = \sum_{n \geq 1} a_n^{\langle-1\rangle} t^n,$$

such that $f \circ f^{\langle-1\rangle} = f^{\langle-1\rangle} \circ f = t$ (inversion problem for formal series).

Theorem A. *(Inversion formula of Lagrange). With the condition [8a], we have, for all integers k, $1 \leq k \leq n$:*

[8b] $$\mathbb{C}_n (f^{\langle-1\rangle})^k = \frac{k}{n} \mathbb{C}_{n-k} \left(\frac{f(t)}{t}\right)^{-n}.$$

([Lagrange, 1770]. See also [Lagrange, Legendre (Bürmann), 1798]. The formal demonstration given here is due to [Henrici, 1964]. There is an immense literature on this problem, and we mention only [Blakley, 1964a, b, c], [Brun, 1955], [Good, 1960, 1965], [*Gröbner, 1960] p. 50–65, [Perron, 1954], [Raney, 1960, 1961], [Sack, 1965a, b, 1966], [Stieltjes, 1885], [Tyurin, 1962].) In (8b), $(f/t)^{-n}$ means evidently $a_1^{-n}(1 + (a_2/a_1)t + (a_3/a_1)t^2 + \cdots)^{-n}$.

■ According to Theorem A (p. 145), all we need to prove is that the product of the matrix whose n-th row-k-th column coefficient is the right-hand member of [8b], by the matrix whose n-th row-k-th column coefficient is $\mathbb{C}_n f^k$ (this is the matrix $B(f)$, with respect to $\Omega_n = 1$, [7c], p. 145) equals the identity matrix I. Now, the coefficient on the n-th row and k-th column, say $\pi_{n,k}$, of this product matrix, is by definition equal to:

$$\pi_{n,k} = \sum_{k \leq l \leq n} \left\{\frac{l}{n} \mathbb{C}_{n-l} \left(\frac{f(t)}{t}\right)^{-n} \cdot \mathbb{C}_l f^k\right\}.$$

So we only have to prove that $\pi_{n,k} = 1$ for $n = k$ and $= 0$ for $n \neq k$. For this we observe that $l\mathbb{C}_l f^k = \mathbb{C}_l(tD(f^k)) = k\mathbb{C}_l(tf^{k-1}f')$. Hence, with [17g] (p. 37) for (∗):

$$\pi_{n,k} = \frac{k}{n} \sum_l \{\mathbb{C}_{n-l}(f/t)^{-n} \cdot \mathbb{C}_l(tf^{k-1}f')\} =$$
$$\overset{(*)}{=} \frac{k}{n} \mathbb{C}_n\{(f/t)^{-n} tf^{k-1}f'\} = \frac{k}{n} \mathbb{C}_0(t^{n+1}f^{-n+k-1}f'),$$

which implies immediately that $\pi_{n,n} = 1$, for $n = 1, 2, \ldots$ For $n \neq k$, on the other hand, we have:

$$\pi_{n,k} = \frac{k}{n} \mathbb{C}_{-1}\left\{tD\left(\frac{f^{-n+k}}{-n+k}\right)\right\},$$

where the series following the differentiation sign D is now a *Laurent series* (p. 43). In the derivative of such a series terms t^{-1} cannot occur, so indeed $\pi_{n,k} = 0$. ■

Here are other forms of the Lagrange formula [8b].

Theorem B. *With notations as above, and $g := f^{\langle-1\rangle}(t)$ we have for any* *formal series* Φ:

$$[8c] \qquad \Phi(u)=\Phi(0)+\sum_{n\geq 1}\frac{t^n}{n}\,\mathbf{C}_{n-1}\,\Phi'(t)\left(\frac{f(t)}{t}\right)^{-n}$$

or, if one likes that more:

$$[8c'] \qquad n\,\mathbf{C}_n\,\Phi(f^{\langle -1\rangle}(t))=\mathbf{C}_{n-1}\,\Phi'(t)\left(\frac{f(t)}{t}\right)^{-n}.$$

■ Let $\Phi(t)=\sum_{k\geq 1}\varphi_k t^k$; it suffices to show [8c] for t^k; but this is just [8b]. ■

THEOREM C. *Let* $y=y_0+xF(y)$ *determine* y *as a series in* x, *with constant term* y_0. *Then:*

$$[8c''] \qquad \Xi(y)=\Xi(y_0)+\sum_{n\geq 1}\frac{x^n}{n!}\frac{d^{n-1}}{dy_0^{n-1}}\left(\Xi'(y_0)F^n(y_0)\right).$$

■ Writing $y=y_0+u$, we get $x=u(F(y_0+u))^{-1}:=f(u)$. Then apply [8c], with $t=x$, $\Phi(u)=\Xi(y_0+u)$. ■

THEOREM D. ([Hermite, 1891]). *With notations as above, and* $u=f^{\langle -1\rangle}(t)$, *we have for all formal series* Ψ:

$$[8d] \qquad \frac{t\Psi(u)}{uf'(u)}=\sum_{n\geq 0}t^n\,\mathbf{C}_n\,\Psi(t)\left(\frac{f(t)}{t}\right)^{-n}.$$

In other words,

$$[8d'] \qquad \mathbf{C}_n\frac{t\Psi(u)}{uf'(u)}=\mathbf{C}_n\,\Psi(t)\left(\frac{f(t)}{t}\right)^{-n}.$$

■ If we take the derivative of [8c] with respect to t, then, using $t=f(u)$, $du/dt=1/f'(u)$, we get

$$[8e] \qquad \Phi'(u)\frac{du}{dt}=\frac{\Phi'(u)}{f'(u)}=\sum_{n\geq 0}t^n\,\mathbf{C}_n\,\Phi'(t)\left(\frac{f}{t}\right)^{-n-1}.$$

So we only need to substitute $\Psi(u)=u\Phi'(u)/f(u)$ into [8e]. ■

THEOREM E. *The Taylor coefficients of the formal series* $f^{\langle -1\rangle}=\sum_{n\geq 1}f_n^{\langle -1\rangle}t^n/n!$, *which is the reciprocal of* $f=\sum_{n\geq 1}f_n t^n/n!$ *can be expressed as function of the Taylor coefficients* f_n *of* f *in the following manner:*

$$[8f] \qquad f_n^{\langle -1\rangle}=\sum_{k=1}^{n-1}(-n)_k f_1^{-n-k}\,\mathbf{B}_{n-1,k}\left(\frac{f_2}{2},\frac{f_3}{3},\dots\right)$$

$$[8g] \qquad =\sum_{k=1}^{n-1}(-1)^k f_1^{-n-k}\,\mathbf{B}_{n+k-1,k}(0,f_2,f_3,\dots)$$

with $f_1^{\langle -1\rangle}=1/f_1$, *and with* $\mathbf{B}_{n,k}$ *the exponential Bell polynomials.* ([3a], p. 134. For this problem see also [Bödewadt, 1942], [Kamber, 1946], [Ostrowski, 1957] and [*1966], p. 255, [*Riordan, 1968], pp. 145 and 177.)

■ [8f] is an immediate consequence of [8b], with $k=1$, where the right-hand member is expressed by means of [5f] (p. 141); then [8g] follows from [3l'] (p. 136). ■

The first values of $f_n^{\langle -1\rangle}$ are: $f_1^{\langle -1\rangle}=f_1^{-1}$; $f_2^{\langle -1\rangle}=-f_1^{-3}f_2$; $f_3^{\langle -1\rangle}=-f_1^{-4}f_3+3f_1^{-5}f_2^2$; $f_4^{\langle -1\rangle}=-f_1^{-5}f_4+10f_1^{-6}f_2f_3-15f_1^{-7}f_2^3$; $f_5^{\langle -1\rangle}=-f_1^{-6}f_5+f_1^{-7}(15f_4f_2+10f_3^2)-105f_1^{-8}f_3f_2^2+105f_1^{-9}f_2^4$; $f_6^{\langle -1\rangle}=-f_1^{-7}f_6+f_1^{-8}(21f_5f_2+35f_4f_3)-f_1^{-9}(210f_4f_2^2+280f_3^2f_2)+1260f_1^{-10}f_3f_2^3-945f_1^{-11}f_2^5$; $f_7^{\langle -1\rangle}=-f_1^{-8}f_7+f_1^{-9}(28f_6f_2+56f_5f_3+35f_4^2)-f_1^{-10}(378f_5f_2^2+1260f_4f_3f_2+280f_3^3)+f_1^{-11}(3150f_4f_2^3+6300f_3^2f_2^2)-17325f_1^{-12}f_3f_2^4+10395f_1^{-13}f_2^6$; $f_8^{\langle -1\rangle}=-f_1^{-9}f_8+f_1^{-10}(36f_7f_2+84f_6f_3+126f_5f_4)-f_1^{-11}(630f_6f_2^2+2520f_5f_3f_2+1575f_4^2f_2+2100f_4f_3^2)+f_1^{-12}(6930f_5f_2^3+34650f_4f_3f_2^2+15400f_3^3f_2)-f_1^{-13}(51975f_4f_2^4+138600f_3^2f_2^3)+270270f_1^{-14}f_3f_2^5-135135f_1^{-15}f_2^7$.

To check this table, observe that the coefficient of $(-1)^k f_1^{-n-k}$, when $f_2=f_3=\dots=1$, is exactly $S_2(k+n-1,k)$ of p. 222.

THEOREM F. *Let* a *be an integer* ≥ 1. *For* $f(t)=t(1-\sum_{n\geq 1}x_n t^{an}/n!)$, *we have* $f^{\langle -1\rangle}(t)=t(1+\sum_{n\geq 1}y_n t^{an}/n!)$, *where*

$$[8h] \qquad y_n=\sum_{k=1}^{n}(an+k)_{k-1}\,\mathbf{B}_{n,k}(x_1,x_2,\dots).$$

■ Apply [8b] (p. 148). ■

Formula [8h] could save time and place. For example, if we want to invert $f(t)=(1/2)(\sinh t\cos t+\cosh t\sin t)=t(1-\sum_{n\geq 1}(-4)^n t^{4n}/(4n+1)!)$, up to t^{12}, we need the $B_{n,k}$ up to $n=12$ by [8f], and only up to $n=7$ by [8c]. So, $f^{\langle -1\rangle}(t)=t-t^5/30+t^9/22680-t^{13}/97297200+\cdots$ (Zyczkowski, 1965).)

Theorem G. *We have the following formula, using only coefficients of powers of $f(t)$ with positive integral exponents ($f(t)=a_1t+a_2t^2+\cdots$, $a_1\neq 0$):*

$$\text{[8i]}\qquad C_n(f^{\langle -1\rangle}(t))^k=k\binom{2n-k}{n}\sum_{j=1}^{n-k}\frac{(-1)^j}{n+j}\binom{n-k}{j}\times a_1^{-n-j}C_{n-k+j}(f(t))^j.$$

■ Use [5h] (p. 148) and [5h] (p. 142). ■

Remark. The correspondence between a formal series and its iteration matrix was already used when we inverted the Stirling matrix S (p. 144): we took the inverse function of $f(t):=e^t-1$, whose iteration matrix was S (with respect to $\Omega_n=1/n!$).

Applications

(I) The most classical example is undoubtedly that of computing the coefficients of the inverse function $f^{\langle -1\rangle}(t)$ for the case $f(t)=te^{-t}$. By [8b] (p. 148), $k=1$, we get:

$$C_n f^{\langle -1\rangle}=\frac{1}{n}C_{n-1}\left(\frac{te^{-t}}{t}\right)^{-n}=\frac{1}{n}C_{n-1}e^{nt}=\frac{1}{n}\cdot\frac{n^{n-1}}{(n-1)!}.$$

Hence $f^{\langle -1\rangle}(t)=\sum_{n\geq 1}n^{n-1}t^n/n!$. (See also Exercise 13, p. 163.)

(II) For given fixed complex z, what is the 'value' of the series $F(t):=\sum_{n\geq 0}\binom{nz}{n}t^n$? Since

$$F(t)=\sum_{n\geq 0}t^nC_n(1+t)^{nz},$$

we can apply [8d] with $f(t):=t(1+t)^{-z}$ and $\Psi(t)=1$. After simplifications, we obtain $F(t)=(1+u)\{1-(z-1)u\}^{-1}$, where $u:=f^{\langle -1\rangle}(t)$ is the reciprocal of $f(t)$. (For $z=2$ we find back (1) of Exercise 22, p. 81.)

(III) *Calculate the n-th derivative of an implicit function.* We consider a Taylor formal expansion in two variables: $f(x,y)=\sum_{m,n\geq 0}f_{m,n}x^my^n/(m!n!)$, where $f_{0,0}=0$, $f_{0,1}\neq 0$. Therefore, $f(x,y)=\sum_{n\geq 0}\varphi_n(x)y^n/n!$, with $\varphi_n(x)=\sum_{m\geq 0}f_{m,n}x^m/m!$. We want to find a formal series $y=\sum_{n\geq 1}y_nx^n/n!$ such that $f(x,y)=0$ ("problem of implicit functions"). For that, we solve $\sum_{n\geq 1}\varphi_ny^n/n!=-\varphi_0$ by the Lagrange formula, where the variable is $-\varphi_0$, the unknown function is y, all the $\varphi_1,\varphi_2,\varphi_3,\ldots$ being temporarily considered as constants, and collect afterwards the terms in $x^n/n!$ in the expression of y just found, where $\varphi_0=\varphi_0(x)$, $\varphi_1=\varphi_1(x)$, etc. Putting $a:=f_{1,0}$, $b:=(f_{0,1})^{-1}$, we find ([Comtet, 1968], [David, 1887], [Goursat, 1904], [Sack, 1966], [Teixeira, 1904], [Wronski, 1894] and p. 175): $y_1=-ab$ (this is the well-known formula $y'=-f'_x/f'_y$), $y_2=-b(f_{2,0}-2abf_{1,1}+a^2b^2f_{0,2})$, $y_3=-b\{f_{3,0}-3bf_{2,0}f_{1,1}-3abf_{2,1}+ab^2(6f_{1,1}^2+3f_{2,0}f_{0,2})+3a^2b^2f_{1,2}-9a^2b^3f_{1,1}f_{0,2}-a^3b^3f_{0,3}+3a^3b^4f_{0,2}^2\}$, $y_4=-b\{f_{4,0}+b(-4f_{3,0}f_{1,1}-6f_{2,0}f_{2,1})+b^2(12f_{2,0}f_{1,1}^2+3f_{2,0}^2f_{0,2})-4abf_{3,1}+ab^2\times(12f_{2,0}f_{1,2}+24f_{1,1}f_{2,1}+4f_{3,0}f_{0,2})-ab^3(24f_{1,1}^3+36f_{2,0}f_{1,1}f_{0,2})+6a^2b^2f_{2,2}-a^2b^3(36f_{1,1}f_{1,2}+18f_{2,1}f_{0,2})+a^2b^4(72f_{1,1}^2\times f_{0,2}+18f_{2,0}f_{0,2}^2)-4a^3b^3f_{1,3}+a^3b^4(24f_{0,2}f_{1,2}+12f_{1,1}f_{0,3})-60a^3b^5f_{1,1}f_{0,2}^2+a^4b^4f_{0,4}-10a^4b^5f_{0,3}f_{0,2}+15a^4b^6f_{0,2}^3\}$. ■

(IV) Solve the equation $y=x+x^py^q$, where p and q are integers ≥ 0. We have $x=y(1-x^py^{q-1})=f(y)$. So, with [8b] p. 148, $y=\sum_{n\geq 1}b_nx^n$, where $b_n=b_n(x)=(1/n)C_{n-1}(1-x^py^{q-1})^{-n}$. Therefore,

$$y=x\sum_{k\geq 0}\frac{1}{kq+1}\binom{kq+k}{k}x^{k(p+q-1)},\qquad |x|<\ldots$$

(V) Let us give another proof of *Abel formula* ([11b] p. 128). For that, take $f(t):=te^{-t}$, $\Phi(t)=e^{xt}$ in [8c]. Then $\Phi(u)=e^{xu}=1+\sum_{k\geq 1}(t^k/k!)\times\times C_{k-1}(xe^{xt})(e^{kt})=\ldots\sum_{k\geq 0}t^kx(x+kz)^{k-1}/k!$. Now, multiply the preceding by e^{yu}, replace t by $t=f(u)=ue^{-zu}$, and take coefficient of $u^n/n!$.

3.9. Finite Summation Formulas

Now we want, in the simplest cases, to express a sum $A:=\sum_{k=1}^n a(k)$ by means of an *explicit* (or *closed*) formula, called a *summation formula*, that is an expression in which the summation sign $\sum$ does not occur anymore (neither 'hidden' dots!).

Example 1. Show that $A:=\sum_{k=0}^n\binom{n}{k}=2^n$. In fact, $A=(1+1)^n$, because of the binomial formula.

Example 2. Compute $A_n(x):=\sum_k k\binom{n}{k}x^k$. We have $\sum_k\binom{n}{k}x^k=(1+x)^n$. Taking the derivative, we get $\sum_k k\binom{n}{k}x^{k-1}=n(1+x)^{n-1}$.

Hence $A_n(x)=nx(1+x)^{n-1}$. Particularly, $A_n(1)=\sum k\binom{n}{k}=n2^{n-1}$ and $A_n(-1)=\sum(-1)^k k\binom{n}{k}=0$, except $A_1(-1)=-1$.

Example 3. *Compute* $A:=\sum_{k=0}^{n}\binom{n}{k}^2$. Observe that $A=\sum_{k=0}^{n}\binom{n}{k}\times\binom{n}{n-k}$, which means that A equals the coefficient of t^n in the product of $(1+t)^n$ with itself:

$$A=C_{t^n}(1+t)^n(1+t)^n=C_{t^n}(1+t)^{2n}=\binom{2n}{n}.$$

(See Exercise 28, p. 90.) More generally, we have the *convolution identity of Vandermonde*:

[9a] $$\sum_h\binom{m}{h}\binom{n-m}{k-h}=\binom{n}{k},\quad 0\leq m\leq n,$$

which follows from p. 28 or [13c] on p. 44, or also, as before, from:

$$\binom{n}{k}=C_{t^k}(1+t)^n=C_{t^k}(1+t)^m(1+t)^{n-m}.$$

In other cases, $A=A(n)=\sum_{k=1}^{n}a(k)$ and a summation formula expresses now that $A=\sum_{l=1}^{m}b(l)$, where $b(l)$ is another sequence. If $m<n$, we save making additions in this way. More generally, a *summation formula* is an equality between two expressions, one of which contains one or more summations. A summation formula is interesting if it establishes a connection between expressions which are built up from known or tabulated expressions.

Example 4. *Use the Bernoulli polynomials* ([14a], p. 48), *to compute for each integer* $r\geq 0$:

[9b] $$Z=Z(n,r):=\sum_{1\leq k\leq n}k^r=1^r+2^r+\cdots+n^r.$$

For this we consider the formal series:

$$f_n(t):=\sum_{r\geq 0}\{Z(n,r)\,t^{r+1}/r!\}.$$

We get, by [14e] (p. 48), for (*):

[9c] $$f_n(t)=t\sum_{\substack{r\geq 0\\1\leq k\leq n}}k^r\frac{t^r}{r!}=t\sum_{1\leq k\leq n}\left\{\sum_{r\geq 0}\frac{(kt)^r}{r!}\right\}=t\sum_{1\leq k\leq n}e^{kt}=$$

$$=t\frac{e^{(n+1)t}-e^t}{e^t-1}=t\frac{e^{(n+1)t}}{e^t-1}-\frac{t}{e^t-1}-t\overset{(*)}{=}$$

$$\overset{(*)}{=}\sum_{r\geq 0}\frac{t^r}{r!}B_r(n+1)-\sum_{r\geq 0}\frac{t^r}{r!}B_r-t.$$

Hence, by identification of the coefficients of $t^{r+1}/r!$ in the first and last member of [9c], we get, by [14g] (p. 48), for (**), $r\geq 1$ ($Z(n,0)=n$):

[9d] $$Z(n,r)=\frac{1}{r+1}\{B_{r+1}(n+1)-B_{r+1}\}\overset{(**)}{=}\frac{1}{r+1}\sum_{0\leq k\leq r}B_k\binom{r+1}{k}(n+1)^{r+1-k}.$$

Thus we find, by the table on p. 49 (a table of the $Z(n,r)$, $r\leq 10$, $n\leq 100$ is found in [*Abramowitz, Stegun, 1964], pp. 813–817; see also [Carlitz, Riordan, 1963], Exercise 4, p. 220 and Exercise 31, p. 165):

$Z(n,1)=n(n+1)/2,$

$Z(n,2)=n(n+1)(2n+1)/6,$

$Z(n,3)=n^2(n+1)^2/4,$

$Z(n,4)=n(n+1)(2n+1)(3n^2+3n-1)/30,$

$Z(n,5)=n^2(n+1)^2(2n^2+2n-1)/12,$

$Z(n,6)=n(n+1)(2n+1)(3n^4+6n^3-3n+1)/42,$

$Z(n,7)=n^2(n+1)^2(3n^4+6n^3-n^2-4n+2)/24,$

$Z(n,8)=n(n+1)(2n+1)(5n^6+15n^5+5n^4-15n^3-n^2+9n-3)/90.$

As additional properties of $Z(n,r)$, we have:

(1) $Z(n,r)=r\int_0^n Z(x,r-1)\,dx+B_r n$.

(2) $Z(n,2)$ divides $Z(n,2k)$ and $Z(n,3)$ divides $Z(n,2k+1)$, $k\geq 1$.

SUPPLEMENT AND EXERCISES

1. *Two relatives of the binomial identity.* Show that:

$$(x+y)^{2n}=\sum_{0\leq k\leq n}\binom{2n-k-1}{n-1}(x^k+y^k)(x+y)^k(xy)^{n-k}$$

$$x^n + y^n = \sum_{0\le 2k\le n} (-1)^k \frac{n}{n-k}\binom{n-k}{k}(xy)^k (x+y)^{n-2k}.$$

[*Hint*: Induction. See also Exercise 35, p. 87 and p. 158.]

2. *Lah numbers* ([*Riordan, 1958], p. 44). These are the numbers $L_{n,k} := (-1)^n \binom{n-1}{k-1} n!/k!$ which appeared in [3h] (p. 135), $\exp\{ux(1-t)^{-1}\} = 1 + \sum_{1\le k\le n} L_{n,k}(-t)^n u^k/n!$. (1) $L_{n+1,k} = -(n+k)L_{n,k} - L_{n,k-1}$.

n\k	1	2	3	4	5	6	7	8	9	10
1	−1									
2	2	1								
3	−6	−6	−1							
4	24	36	12	1						
5	−120	−240	−120	−20	−1					
6	720	1800	1200	300	30	1				
7	−5040	−15120	−12600	−4200	−630	−42	−1			
8	40320	141120	141120	58800	11760	1176	56	1		
9	−362880	−1451520	−1693440	−846720	−211680	−28224	−2016	−72	−1	
10	3628800	16329600	21772800	12700800	3810240	635040	60480	3240	90	1

$-L_{n,k-1}$. (2) $(-x)_n = (-1)^n\langle x\rangle_n = \sum_{k=1}^n (x)_k L_{n,k}$. (3) $a_n = \sum L_{n,k} b_k$ is equivalent to $b_n = \sum L_{n,k} a_k$. (4) $L_{n,k} = \sum (-1)^l s(n,l) S(l,k)$, where $s(n,l)$ and $S(l,k)$ are the Stirling numbers of the first and second kind.

3. *Bell potential and logarithmic polynomials.* (1) Show that $k!\mathbf{B}_{n,k} = \sum_{r\le k}\binom{k}{r}(-1)^{k-r}\mathbf{P}_n^{(r)}$. Which property of derivatives does this formula give when combined with the Faà di Bruno formula of p. 137? (2) Use $\log(1+g) = \sum_{r\ge 1}(-1)^{r-1}r^{-1}g^r$, where $g = \sum_{n\ge 1} g_n t^n/n!$, to show that $\mathbf{L}_n = \sum_{1\le r\le n}(-1)^{r-1}r^{-1}\mathbf{P}_n^{(r)}$. Translate this formula in terms of derivatives. Similarly, with $s(l,k)$, the Stirling number of the first kind:

$$C_n \frac{\log^k(1+g)}{k!} = \sum_{l=k}^{n} \frac{s(l,k)}{l!} C_n g^l.$$

4. $\mathbf{P}_n^{(r)}$ *as a function of a single Bell polynomial when r is integer.* If r is a positive integer, show that:

$$\mathbf{P}_n^{(r)} = \binom{n+r}{r}^{-1} \mathbf{B}_{n+r,r}(1, 2g_1, 3g_2, \ldots).$$

[*Hint*: We get $(1 + g_1 t + g_2 t^2/2! + \cdots)^r = t^{-r}(t + 2g_1 t^2/2! + 3g_2 t^3/3! + \cdots)^r$, by [3g], p. 141.]

5. *Determinantal expressions.* (1) Let $f := \sum_{n\ge 0} a_n t^n$, $a_0 \ne 0$, and $g := \sum_{n\ge 0} b_n t^n := f^{-1}$. Then $b_n = (-1)^n a_0^{-n-1}\det[\alpha_{i,j}]$, where $\alpha_{i,j} := a_{i-j+1}$, $1\le i, j\le n$, $a_k := 0$ for $k<0$. (This gives a determinantal expression for $\mathbf{P}_n^{(-1)}$.) (2) The Faà di Bruno formula ([4b] p. 139) can be restated operationally in the following form ([Ivanoff, 1958]), using the Pascal triangle of dimension n, with an upper diagonal of -1:

$$f_n = \begin{vmatrix} g_1 D & -1 & 0 & 0 & \cdots \\ g_2 D & g_1 D & -1 & 0 & \cdots \\ g_3 D & 2g_2 D & g_1 D & -1 & \cdots \\ g_4 D & 3g_3 D & 3g_2 D & g_1 D & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{vmatrix} f,$$

where $D^k f = f_k$. For example,

$$f_2 = \begin{vmatrix} g_1 D & -1 \\ g_2 D & g_1 D \end{vmatrix} f = \left(g_1^2 D^2 + g_2 D\right) f = g_1^2 f_2 + g_2 f_1.$$

6. *Successive derivatives of $F(\log x)$ and $F(e^x)$.* Expressed as a function of the Stirling numbers of the first kind $s(n,k)$ and of the second kind $S(n,k)$ we have:

$$\frac{d^n}{dx^n} F(\log x) = x^{-n}\sum_{k=1}^{n} s(n,k) F^{(k)}(\log x)$$

$$\frac{d^n}{dx^n} F(e^x) = \sum_{k=1}^{n} S(n,k)\, e^{kx} F^{(k)}(e^x)$$

Moreover, for $y = x_1 x_2 \ldots x_n$, we have

$$\frac{\partial^n F(y)}{\partial x_1 \partial x_2 \ldots \partial x_n} = \sum_{k=1}^{n} S(n,k)\, y^{k-1} F^{(k)}(y).$$

7. *Successive derivatives of $F(x^a)$.* Let a be a real constant and $F(x)$ a function of class C^∞ in the point $x = x\,(>0)$. Using the notations of [4h]

(p. 138), and the Faà di Bruno formula [1] of p. 139, show that the n-th derivative of $H(x) := F(x^r)$ in the point $x = a$ equals $h_n = \sum_{k=1}^{n} f_k a^{rk-n} Z_{n,k}(r)$, where the $Z_{n,k}(r)$ are generated by $((1+T)^r - 1)^k/k! = \sum_{n\geq k} Z_{n,k}(r) T^n/n!$ (See Exercise 21, p. 163.)

Deduce the well-known formulas:

$$Z_{n,k}(-1) = (-1)^n \frac{n!}{k!}\binom{n-1}{k-1}$$

$$Z_{n,k}(\tfrac{1}{2}) = (-1)^{n-k}\frac{(n-1)!}{(k-1)!}\binom{2n-k-1}{n-1}\frac{1}{2^{2n-k}}$$

$$Z_{n,k}(2) = \frac{n!}{k!}\binom{k}{n-k}2^{2k-n}.$$

8. *Expansions of the coordinates with respect to the Frenet-Serret trihedron in terms of arc length.* Let $\rho = \rho(s)$ be the curvature of a plane curve $M = M(s)$ as a function of the length s of the arc with origin $M(0)$ (intrinsic equation). We introduce the Frenet-Serret trihedron $(M(0), \vec{t}, \vec{n})$, where $\vec{t} = dM/ds\,|_{s=0}$, $\rho\vec{n} = d\vec{t}/ds\,|_{s=0}$, $\rho > 0$, and $\overrightarrow{M(0)M(s)} = x\vec{t} + y\vec{n}$, $x = \sum_{n\geq 1} x_n s^n/n!$, $y = \sum_{n\geq 1} y_n s^n/n!$. Putting $\rho_k = d^k\rho/ds^k\,|_{s=0}$, $\rho_0 = \rho(0)$, $\mathbf{B}_{n,k} = B_{n,k}(\rho_0, \rho_1, \rho_2, \ldots)$, we have:

$$x_{n+1} = \sum_p (\mathbf{B}_{n,4p} - \mathbf{B}_{n,4p+2}),$$

$$y_{n+1} = \sum_q (\mathbf{B}_{n,4q+1} - \mathbf{B}_{n,4q+3}).$$

For example, $x_1 = 1, x_2 = 0, x_3 = -\rho_0^2, x_4 = -3\rho_0\rho_1, x_5 = -4\rho_0\rho_2 - 3\rho_1^2, \ldots$, $y_1 = 0, y_2 = \rho_0, y_3 = \rho_1, y_4 = \rho_2 - \rho_0^3, y_5 = \rho_3 - 6\rho_0^2\rho_1, \ldots$

* Find similar formulas for a space curve with respect to the curvature $\rho = \rho(s)$ and the torsion $\tau = \tau(s)$.

9. *Symmetric functions.* A symmetric function, abbreviated SF, is a polynomial $P(x_1, x_2, \ldots, x_n)$ in the n variables $x_1, x_2, \ldots, x_n$ with coefficients in a field K (often $= \mathbb{R}$ or $\mathbb{C}$), and which is invariant under any permutation of the variables: for any $\sigma \in \mathfrak{S}(n)$, $P(x_1, x_2, \ldots, x_n) = P(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. A *monomial symmetric function* (abbreviated MSF) is a symmetric function of the form:

$$f = \sum x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\nu}^{\alpha_\nu}, \quad \text{also denoted by} \quad \sum\nolimits^{(n)} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_\nu^{\alpha_\nu},$$

where the α_i are given integers such that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_\nu \geq 1$, and where the above summation takes place over all arrangements $(i_1, i_2, \ldots, i_\nu)$ of $[n]$ such that the corresponding monomials (in the summation) are *all distinct*. Thus $\sum^{(3)} x_1^2 x_2 x_3 = x_1^2x_2x_3 + x_2^2x_1x_3 + x_3^2x_1x_2$. The MSF a_r and s_r, $a_r = \sum^{(n)} x_1 x_2 \cdots x_r$, $s_r = \sum^{(n)} x_1^r$, are called '*elementary* SF' and the '*sum of r-th powers* SF', respectively. (1) Every SF is a linear combination of MSF (detailed tables in [*David, Kendall, Barton, 1966]). Particularly $(x_1 + x_2 + \cdots + x_n)^m$ is a linear combination of MSF. In this summation occur $p(m)$ such MSF, which is the number of partitions of m (pp. 94 and 126). (2) The a_r have for GF: $F(t) := \sum_{r\geq 0} a_r t^r = \prod_{i=1}^{n} (1 + x_i t)$. (3) $s_r = (-1)^{r-1}/(r-1)!\, \mathbf{L}_r(a_1, 2!\,a_2, 3!\,a_3, \ldots)$. [*Hint:* Take $\log F(t) = \sum_{i=1}^{n} \log(1 + x_i t) = \sum_{r\geq 1} (-1)^{r-1}(t^r/r)s_r$.] (4) $a_r = \mathbf{Y}_r(s_1, -1!\,s_2, 2!\,s_3, -3!\,s_4, \ldots)/r!$.

10. *Bell polynomials and partitions.* From identity [5b] (p. 108) follows after replacing x by u:

$$((1-u)(1-tu)(1-t^2u)\cdots)^{-1} = 1 + \sum_{k\geq 1} u^k((1-t)(1-t^2)\cdots(1-t^k))^{-1}.$$

If we put $x_k = (1-t^k)^{-1}$, and use $1 + \sum_{k\geq 1} u^k x_1 x_2 \cdots x_k = \exp\{-\sum_{m\geq 0} \log(1 - t^m u)\}$, show that $k!\,x_1 x_2 \cdots x_k = \mathbf{Y}_k(x_1, 1!\,x_2, 2!\,x_3, \ldots)$. For example: $2x_1x_2 = x_2 + x_1^2$, $4x_1^2x_2 = x_2 + x_1^2 + 2x_1^4$, $8x_1x_2^2 = 4x_2^2 + x_2 + x_1^2 + 2x_1^4$, $12x_1x_2x_3 = 4x_3 + 3x_2 + 3x_1^2 + 2x_1^4$. Obtain from this the (Herschel) expansions of $\{(1-t)(1-t^2)\}^{-1}$, $\{(1-t)^2(1-t^2)\}^{-1}$, $\{(1-t)(1-t^2)^2\}^{-1}$, $\{(1-t)(1-t^2)(1-t^3)\}^{-1}$. To which generalization of the notion of denumerant do the second and third example correspond?

Finally, give formulas and recurrences for the D'Arcais numbers $A(n,k)$ defined by $((1-t)(1-t^2)(1-t^3)\cdots)^{-u} = \sum_{n,k} A(n,k) u^k t^n/n!$ ([D'Arcais, 1913]), of which the first values are:

n\k	1	2	3	4	5	6	7	8
1	1							
2	3	1						
3	8	9	1					
4	42	59	18	1				
5	144	450	215	30	1			
6	1440	3394	2475	565	45	1		
7	5760	30912	28294	9345	1225	63	1	
8	75620	293292	340116	147889	27720	2338	84	1

11. *Characteristic numbers for a random variable.* Let be given a probability space $(\Omega, \mathscr{A}, \mathbf{P})$ and a real random variable $X:\Omega\to\mathbf{R}$ (abbreviated RV) with distribution function $F(x):=\mathbf{P}(X<x)$. Let μ_n (or μ'_n) be the *central* (or *noncentral*) *moments* of X: $\mu'_n=\mathbf{E}(X^n)=\int_{-\infty}^{\infty}x^n\,dF(x)$, $\mu_n=\mathbf{E}(X-\mu)^n$, where $\mu=\mu'_1=\mathbf{E}(X)$ is the *expectation* of X (then $\mu_1=0$). We define furthermore for X the *variance* $\mu_2=\mathbf{E}(X-\mu)^2$ (also denoted by $\operatorname{var}X$) and the *standard deviation* $\mathbf{D}(X):=\sqrt{\operatorname{var}X}$; the *GF of the moments*:

$$\Psi(t):=1+\sum_{n\geqslant 1}\mu'_n t^n/n!=\mathbf{E}(e^{tX});$$

the *generating function of the central moments*:

$$\Psi^*(t):=1+\sum_{n\geqslant 2}\mu_n t^n/n!=\mathbf{E}(e^{t(X-\mu)})=e^{-\mu t}\Psi(t);$$

and the *GF of the cumulants* $\varkappa_n$:

$$\chi(t):=\log\Psi(t)=\sum_{n\geqslant 1}\varkappa_n t^n/n!.$$

If the RV is *discrete* ($\subset X(\Omega)\subset\mathbf{N}$), $p_k=\mathbf{P}(X=k)$, then we have the *GF of the probabilities*: $\varphi(u):=\sum_{k\geqslant 0}p_k u^k$, hence $\varphi(e^t)=\Psi(t)$, $\log\varphi(e^t)=\chi(t)$. (1) $\mu_n=\sum\binom{n}{k}(-1)^k\mu^k\mu'_{n-k}$, $\mu'_n=\sum\binom{n}{k}\mu^k\mu_{n-k}$, where $0\leqslant k\leqslant n$, $\mu'_0=\mu_0:=1$. (2) $\mu'_n=\mathbf{Y}_n(\varkappa_1,\varkappa_2,\ldots)$, $\mu_n=\mathbf{Y}_n(0,\varkappa_2,\varkappa_3,\ldots)$, $\varkappa_n=\mathbf{L}_n(\mu'_1,\mu'_2,\ldots)=\mathbf{L}_n(0,\mu_2,\mu_3,\ldots)$. (3) Let $X_1,X_2,X_3,\ldots$ be *independent* Bernoulli RV's with the same distribution law, $\mathbf{P}(X_i=0):=q$, $\mathbf{P}(X_i=1):=p$, $p,q>0$, $p+q=1$. Then $\mathbf{E}(X_1+X_2+\cdots+X_n)^r=\sum_k(n)_k p^k S(r,k)$. (4) Let X be a *Poisson* RV, $p_k:=\mathbf{P}(X=k):=e^{-\lambda}\lambda^k/k!$ ($\lambda>0$ is called the *parameter* of X). Then $\mu'_n=\sum_k S(n,k)\lambda^k$; $\varkappa_1=\mu_2=\mu_3=\lambda$, $\mu_4=\lambda+3\lambda^2$, $\mu_5=\lambda+10\lambda^2$, $\mu_6=\lambda+25\lambda^2+15\lambda^3,\ldots$.

12. *Factorial moments of a RV.* With the notations of Exercise 11, we define for each *discrete* RV, $p_k:=\mathbf{P}(X=k)$, the factorial moments $\mu_{(m)}=\mu_{(m)}(X):=\sum_k p_k(k)_m$, $(k)_m=k(k-1)\ldots(k-m+1)$, $\mu_{(0)}=1$, $m=1,2,3,\ldots$. Show that $\mu_{(m)}=\sum_k s(m,k)\mu'_k$, $\mu'_m=\sum S(m,k)\mu_{(k)}$, and that $\varphi(1+t)=\sum_{m\geqslant 0}\mu_{(m)}t^m/m!$.

13. *Random formal series.* Let $X_1,X_2,\ldots$ be Bernoulli random variables with the same distribution function, $\mathbf{P}(X_i=1)=p$, $\mathbf{P}(X_i=0)=1-p$, $0<p<1$. Let $V_1,V_2,\ldots,W_1,W_2,\ldots$ be the RV defined by $\exp(X_1t+X_2t^2+\cdots)=1+V_1t+V_2t^2+\cdots$ and $(1-X_1t-X_2t^2-\cdots)^{-1}=1+W_1t+W_2t^2+\cdots$, where $p>0$ is given. Show that the *expectations* $\mathbf{E}(V_n)$ and $\mathbf{E}(W_n)$ tend to infinity with n.

***14.** *Distribution of a sum of uniformly distributed RV.* Let $X_1,X_2,\ldots,X_n$ be independent symmetrical RV with uniform distribution function. In other words, there exist $a_i>0$, $i=1,2,\ldots,n$ such that $|X_i|\leqslant a_i$, and, for $x\in[-a_i,a_i]$, $\mathbf{P}(X_i<x)=(a_i+x)/(2a_i)$. Determine the distribution function of $S:=X_1+X_2+\cdots+X_n$. (In other words $\mathbf{P}(S<x)$.) [Ostrowski, 1952].

15. *A formula of Halphen* ([Halphen, 1879]). Use [4b] (p. 146) or some other way to show that:

$$\frac{d^n}{dx^n}\left\{x^{n-1}F\left(\frac{1}{x}\right)\right\}=\frac{(-1)^n}{x^{n+1}}F^{(n)}\left(\frac{1}{x}\right),$$

where $F^{(n)}(1/x)$ stands for the n-th derivative of F taken in the point $1/x$. Thus $(d^n/dx^n)(x^{n-1}\log x)=(n-1)!/x$, $(d^n/dx^n)(x^n\log x)=n!\,(\log x+1+\frac{1}{2}+\cdots+1/n)$, $(d^n/dx^n)(x^{n-1}e^{1/x})=(-1)^n e^{1/x}x^{-n-1}$. More generally:

$$\left\{F\left(\frac{1}{x}\right)G(x)\right\}^{(n)}=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{x^k}F^{(k)}\left(\frac{1}{x}\right)\cdot\left\{\frac{G(x)}{x^k}\right\}^{(n-k)}.$$

***16.** *Lambert series and the Möbius function.* Let $f(t):=\sum_{n\geqslant 1}a_nt^n$, and $g(t):=\sum_{n\geqslant 1}a_nt^n(1-t^n)^{-1}$, which is called the *Lambert* GF of the sequence a_n. (1) We have $g(t)=\sum_{m\geqslant 1}f(t^m)$. (2) Defining the *Möbius function* (= sequence) $\mu(n)$ by $t=\sum_{n\geqslant 1}\mu(n)t^n(1-t^n)^{-1}$, show that $b_n=\sum_{d\mid n}a_d$ and that $a_n=\sum_{d\mid n}\mu(d)b_{n/d}$ (the notation $d\mid n$ means d divides n). (3) $\mu(1)=1$; furthermore, for $n=p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_k^{\alpha_k}$, where the p_i are distinct prime factors of n, we have $\mu(n)=(-1)^k$ if *all* α_i equal 1 (such numbers n are called squarefree), and $\mu(n)=0$ in the other cases. It follows that $\mu(n)$ is multiplicative, in the sense that when a and b are relatively prime, then $\mu(ab)=\mu(a)\mu(b)$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\mu(n)$	1	−1	−1	0	−1	1	−1	0	0	1	−1	0	−1	1	1	0	−1	0	−1	0

Show that $t+t^2-t^4-t^6-\cdots=\sum_{m\geq 0}\mu(2m+1)\,t^{2m+1}(1-t^{2m+1})^{-1}$. (4) Let $d(n)$ be the number of divisors of n, in other words the number of solutions with integers x and $y\geq 1$ of the equation $xy=n$. Then $\sum_{n\geq 1}d(n)\,t^n=\sum_{n\geq 1}t^n(1-t^n)^{-1}=\sum_{n\geq 1}t^{n^2}(1+t^n)(1-t^n)^{-1}$. (5) If $\varphi(n)$ is the indicator function of Euler, [6c] p. 193, then we have $t(1-t)^{-2}=\sum_{n\geq 1}\varphi(n)\,t^n(1-t^n)^{-1}$. Moreover, $\sum_{n\geq 1}\varphi(n)\,t^n(1+t^n)^{-1}=t(1+t^2)(1-t^2)^{-2}-\sum_{m\geq 0}\varphi(2m+1)\,t^{4m+2}(1-t^{4m+2})^{-1}$. (6) Also prove:

$$\sum_{n\geq 1}(-1)^{n-1}t^n(1-t^n)^{-1}=\sum_{n\geq 1}t^n(1+t^n)^{-1}$$
$$\sum_{n\geq 1}nt^n(1-t^n)^{-1}=\sum_{n\geq 1}t^n(1-t^n)^{-2}$$
$$\sum_{n\geq 1}(-1)^{n-1}nt^n(1-t^n)^{-1}=\sum_{n\geq 1}t^n(1+t^n)^{-2}$$
$$\sum_{n\geq 1}(1/n)\,t^n(1-t^n)^{-1}=\sum_{n\geq 1}\log\{(1-t^n)^{-1}\}.$$

(A generalization of Lambert series is found in [Touchard, 1960].) (7) Let $r(n)$ be the number of solutions of $n=x^2+y^2$ with integers $x,y\gtrless 0$ (representation of n as sum of two squares). Thus, $r(0)=1$, $r(1)=4$, because $1=(\pm 1)^2+0^2=0^2+(\pm 1)^2$, $r(5)=8$, because $5=(\pm 2)^2+(\pm 1)^2=(\pm 1)^2+(\pm 2)^2$. Then:

$$\sum_{n\geq 1}r(n)\,t^n=4\sum_{n\geq 1}(-1)^{n-1}t^{2n-1}(1-t^{2n-1})^{-1}.$$

(8) With the notations of (3) and $\omega_k:=\alpha_1+\alpha_2+\cdots+\alpha_k$,

$$\sum_{n\geq 1}(-1)^{\omega_n}\frac{t^n}{1-t^n}=\sum_{n\geq 1}t^{n^2}.$$

(See also Exercise 12, p. 119). (9) Finally prove

$$\left(\sum_{n\geq 0}t^{\binom{n+1}{2}}\right)^4=\sum_{n\geq 0}\frac{(2n+1)\,t^n}{1-t^{2n+1}}.$$

17. *Ordinary Bell polynomials with rational variables.* Let all α_m be rational, $\alpha_m\in\mathbb{Q}$, and let the numbers c_n be defined formally by $g(x):=\exp(\sum_{m\geq 1}\alpha_m x^m)=\sum_{n\geq 0}c_n x^n$. A necessary and sufficient condition that all numbers c_n are rational *integers*, $c_n\in\mathbb{Z}$, is that *for all* $k\geq 1$, we have $\sum_{r\mid k}r\alpha_r\mu(k/r)\equiv 0\ (\mathrm{mod}\ k)$. (See [Carlitz, 1958b, 1968h], [Dieudonné, 1957].) [*Hint.* The c_n are integers if and only if the b_m, defined inductively by $g(x):=\prod_{m\geq 1}(1-x^m)^{-b_m}$, are all integers. Consider then $\log g(x)$, and expand: $k\alpha_k=\sum_{d\mid k}db_d$. Then apply the Möbius inversion formula (2) of Exercise 16.]

18. *With the Lagrange formula.* (1) Deduce from $x=y\exp(-y)$ that $\exp(\alpha y)=1+\sum_{n\geq 1}\alpha(\alpha+n)^{n-1}x^n/n!$ and $(1-y)^{-1}\exp(\alpha y)=\sum_{n\geq 0}(\alpha+n)^n x^n/n!$. (2) Supposing $f(t)=t+a_2t^2+a_3t^3+\cdots$ $(a_1=1)$, prove that, for every complex number α, with $k\leq n$:

$$\mathrm{C}_{t^{n-\alpha}}(f^{\langle -1\rangle}(t))^{k-\alpha}=\frac{k-\alpha}{n-\alpha}\,\mathrm{C}_{t^{n-k}}\left(\frac{f(t)}{t}\right)^{-n+\alpha}.$$

19. *Middle trinomial coefficients.* These are $a_n=\mathrm{C}_{t^n}(1+t+t^2)^n$ (p. 77):

n	0	1	2	3	4	5	6	7	8	9	10	11
a_n	1	1	3	7	19	51	141	393	1107	3139	8953	25653

(1) The integer a_n is the number of distributions of indistinguishable balls into n different boxes, each box containing at most 2 balls. (2) $(n+1)a_{n+1}=(2n+1)a_n+3na_{n-1}$. (3) $\sum_{n\geq 0}a_nt^n=(1-2t-3t^2)^{-1/2}$. (4) Using the notation [6f] (p. 110) $\sum_{i=0}^{n}a_ia_{n-i}=|3^{n+1}/4|$. (5) For $n\to\infty$, we have the asymptotic equivalent $a_n\sim 3^n\sqrt{3/(4\pi n)}$. (6) For each prime number p, then $a_p\equiv 1\ (\mathrm{mod}\ p)$ holds.

20. *Hurwitz identity* ([Hurwitz, 1902]). Considering the set E of *acyclic* functions of $[n+2]$ whose set of roots is $\{n+1, n+2\}$, prove, by an argument similar to that of p. 125:

$$\begin{aligned}(x+y)(x+y+z_1+z_2+\cdots+z_n)^{n-1}=\\=\sum x(x+\varepsilon_1z_1+\cdots+\varepsilon_nz_n)^{\varepsilon_1+\cdots+\varepsilon_n-1}\,y(y+\bar\varepsilon_1z_1+\cdots\\+\cdots+\bar\varepsilon_nz_n)^{\bar\varepsilon_1+\cdots+\bar\varepsilon_n-1},\end{aligned}$$

where the summation is over all 2^n choices of $\varepsilon_1,\ldots,\varepsilon_n$ independently taking the values 0 and 1, and $\bar\varepsilon_i=1-\varepsilon_i$. Generalize for more than 2 roots.

21. *Expansions related to* $1-(1-ax)^k$. (1) When k and l are given integers ≥ 1, express the Taylor coefficients of $f:=((1+x)^{1/k}-1)^l$ in the point $x=0$ by an exact formula of rank $(l-2)$, (as defined on p. 216. Such a formula is apparently only useful if $k>l$.) [*Hint*: Putting $y:=(1+x)^{1/k}-1$, we have $x=(y+1)^k-1$ and $f=y^l$; hence [8d] (p. 150) can be applied.] (2) For any real number a,

$$\left(\frac{1+\sqrt{1-4t}}{2}\right)^{-a}=\left(\frac{1-\sqrt{1-4t}}{2t}\right)^{a}=1+a\sum_{n\geq 1}\binom{a+2n-1}{n-1}\frac{t^n}{n}.$$

(3) Using Hermite's formula ([8d] p. 150), prove that for any a:

$$C_n\left(\frac{1-(1-t)^a}{at}\right)^{-n-1}=\binom{1-1/a}{n}a^n.$$

22. *Three special triangular matrices.* (Obviously, the three following computations of infinite lower triangular matrices give the same result if the matrices are truncated at the n-th row and column, so that they become square $n\times n$ matrices.) We let $\mu(n,k)$ denote the coefficient on the n-th row and the k-th column of the matrix $\mathbf{M}$, and we let $\mu^{(r)}(n,k)$ denote the corresponding coefficient in the matrix $\mathbf{M}^r$ (in the sense of [7g] p. 145). (1) Let $\mu(n,k):=\binom{n+s}{n-k}$ for $0\leq k\leq n$ and $:=0$ otherwise. (This is the coefficient of $(-1)^k x^k/k!$ in the Laguerre polynomial $L_n^{(s)}(x)$ of p. 50.) Then $\mu^{(-1)}(n,k)=(-1)^{n-k}\binom{s+1}{n-k}$. [*Hint*: Straightforward verification, or the method of [7f], p. 144.] (2) Let $\mu(n,k):=\binom{n}{k}k^{n-k}$ for $1\leq k\leq n$ and $:=0$ otherwise. Then $\mu^{(-1)}(n,k)=(-1)^{n-k}\binom{n-1}{k-1}n^{n-k}$. [*Hint*: [8b], p. 148. See also Exercise 43, p. 91.] (3) Let $f(t)=\sum_{n\geq 0}a_nt^n$. We put $\mu(n,k):=a_{n-k}$ for $0\leq k\leq n$ and $:=0$ otherwise. Then $\mu^{(r)}(n,k)=b_{n-k}$ for $0\leq k\leq n$ and $:=0$ otherwise, where the b_n are defined by $f^r(t)=\sum_{n\geq 0}b_nt^n$.

23. *'Inversion' of some polynomials.* $B_n(x)$, $P_n(x)$ and $H_n(x)$ denote the Bernoulli ([14a] p. 48), the Legendre ([14i] p. 50), and the Hermite ([14c] p. 50) polynomials, respectively. Show that:

$$x^n=\sum_k\binom{n}{k}(n-k+1)^{-1}B_k(x)$$

$$x^n=n!\,2^{-n}\sum_{0\leq k\leq n/2}(2n-4k+1)\{k!(\tfrac{3}{2})_{n-k}\}^{-1}P_{n-2k}(x)$$

$$x^n=n!\,2^{-n}\sum_{0\leq k\leq n/2}\{k!(n-2k)!\}^{-1}H_{n-2k}(x).$$

It is somewhat more difficult to invert the Gegenbauer and Laguerre polynomials of p. 50. [*Hint*: Lagrange formula.]

24. *Coverings of a finite set.* A *covering* $\mathscr{B}$ of N, $|N|=n$, is an unordered system of blocks of N, $\mathscr{B}\in\mathfrak{P}(\mathfrak{P}^*(N))$, whose union equals N: $\bigcup_{B\in\mathscr{B}}B=N$. The number γ_n of coverings of N equals $\sum_k(-1)^k\times\binom{n}{k}2^{2^{n-k}-1}$; $\gamma_1=1$, $\gamma_2=5$, $\gamma_3=109$, $\gamma_4=32297$, $\gamma_5=2147321017$. [*Hint*: $|\mathfrak{P}(\mathfrak{P}^*(N))|=2^{2^n-1}=\sum_k\binom{n}{k}\gamma_k$, and [5a, c], p. 143.] Also compute the number $\gamma_{n,m}$ of coverings with m blocks, $|\mathscr{B}|=m$, and the number $\gamma_n^{(h)}$ of coverings with h-blocks ($B\in\mathscr{B}\Rightarrow|B|=h$). ([Comtet, 1966]. See also Exercise 40, p. 303.)

25. *Regular chains* ([Schröder, 1870]). Let q be an integer ≥ 2, and N a finite set, $|N|=n$. We 'chain' now q elements of N together in a q-block $A_1(\subset N)$. Let N_1 be the set, whose $(n-q+1)$ elements are the $(n-q)$ elements of $N\setminus A_1$ and the block A_1. Then we chain again q elements of N_1 together into a block A_2, from which we obtain a new set N_2, etc. We want now to compute the total number of such chains, called regular chains, *not taking the order of the chaining into account*. Show first that:

$$\varphi_n=\frac{1}{q!}\sum_{\substack{k_1+k_2+\cdots+k_q=n\\k_1,k_2,\ldots,k_q\geq 1}}\frac{n!}{k_1!\,k_2!\cdots k_q!}\varphi_{k_1}\varphi_{k_2}\cdots\varphi_{k_q},$$

where $\varphi_0=0$, $\varphi_1=1$, $\varphi_2=\varphi_3=\cdots=\varphi_{q-1}=0$, $\varphi_q=1$. [*Hint*: Consider the q-blocks in existence just before the last chaining operation, in the case they are of size $k_1, k_2, \ldots, k_q$.] Obtain from this $\Phi(t)$ ...

$:=\sum_{n\geq 1} c_n(t^n/n!)=t+C(t)/a)$, and also obtain the value of c_n by applying the inversion formula of Lagrange.

26. *The number of connected graphs* ([Riddell, Uhlenbeck, 1953], [Gilbert, 1956b]). A *connected* graph over N, $|N|=n$, is a graph such that any two of its points are connected by at least one path (Definition II, p. 62). Let $g(n,k)$ be the total number of graphs with n nodes and k edges, and $\gamma(n,k)$ the number of those among them that are connected. Clearly, $g(n,k)=\binom{\binom{n}{2}}{k}$. The *connected component* $C(p)$ of a vertex $p\in N$ is the set of all $x\in N$ 'connected' to p by at least one path. Now we choose $x\in N$, and let $M:=N\setminus\{x\}$. Giving a graph on N is equivalent to giving the trace V of $C(x)$ on M ($C(x)=\{x\}+V$), and to giving, moreover, a graph on $M\setminus V$; show that:

$$g(n,k)=\sum_{i,m\geq 0}\binom{n-1}{i}\gamma(i+1,m)\,g(n-1-i,k-m).$$

Deduce from this:

$$\sum_{n,k\geq 0}\gamma(n,k)\frac{t^n}{n!}u^k=\log\Big\{1+\sum_{n\geq 1}(1+u)^{\binom{n}{2}}\frac{t^n}{n!}\Big\}.$$

More generally, let $g_{\mathscr{P}}(n,k)$ be the number of graphs with n vertices and k edges such that each connected component has the property $\mathscr{P}$, and let $\gamma_{\mathscr{P}}(n,k)$ be the number of those among them that, moreover, are connected. Then:

$$\sum_{n,k\geq 0}\gamma_{\mathscr{P}}(n,k)\frac{t^n}{n!}u^k=\log\Big\{1+\sum_{l,m}g_{\mathscr{P}}(m,l)u^l\frac{t^m}{m!}\Big\}.$$

27. *Generating functions and computation of integrals* ([Comtet, 1967]). (1) Let $J_m:=\int_0^{\pi/2}(A^2\cos^2\varphi+B^2\sin^2\varphi)^{-m}\,d\varphi$. Then $\sum_{m\geq 1}J_m t^{m-1}=\int_0^{\pi/2}(A^2\cos^2\varphi+B^2\sin^2\varphi-t)^{-1}\,d\varphi=(\pi/2)[(A^2-t)(B^2-t)]^{-1/2}$. By expanding this last function into a power series, deduce that $J_{m+1}=\pi(2^{2m+1}AB\,m!)^{-1}\sum_{k=0}^{m}a_{m,k}A^{-2k}B^{-2m+2k}$, where the coefficients $a_{m,k}=\sum_{j=0}\binom{j}{k}\binom{2m-2k}{j}\binom{2m-2k-j}{m-k}$ satisfy the recurrence relation $a_{m+1,k}=(2m+3)(a_{m,k-1}+a_{m,k})-4(m+1)^2a_{m,k-1}$. The first few values of the $a_{m,k}$ are:

$k \backslash m$	0	1	2	3	4	5	6	7	8	9
0	1	1	3	15	105	945	10395	135135	2027025	34459425
1		1	2	9	60	525	5670	72765	1081080	18243225
2			3	9	54	450	4725	59535	873180	[illegible]
3				15	60	450	4500	55125	793800	[illegible]
4					105	525	4725	55125	771750	[illegible]

(2) Compute

$$\int_0^{\infty}\{(x^2+a^2)(x^2+b^2)\}^{-m}\,dx \quad\text{and}\quad \int_{-\infty}^{\infty}\Big\{\prod_{i=1}(x^2+a_i^2)\Big\}^{-m}dx$$

(3) Compute $A_n:=\int_0^{\pi/2}(\log\sin^\alpha\varphi\cos^\beta\varphi)^n\,d\varphi$, where α and β are ≥ 0 ([Chaudhuri, 1967]). [*Hint:*

$$\sum_{n\geq 0}A_n\frac{t^n}{n!}=\int_0^{\pi/2}\sin^{\alpha t}\varphi\,.\cos^{\beta t}\varphi\,.\,d\varphi=$$

$$=\frac{1}{2}\frac{\Gamma((1+\alpha t)/2)\,\Gamma((1+\beta t)/2)}{\Gamma(1+(\alpha+\beta)t/2)}.\Big]$$

(4) Compute $I(p,q):=\int_0^1(\log x)^p(1-x^2)^{-q}\,dx$, where p and q are positive integers. [*Hint:* $\sum_{p,q\geq 0}I(p,q)t^p(u^q/p!)=\int_0^\infty x^t(1+x^2-u)^{-1}\,dx$, to be associated with the well-known result $\int_0^\infty x^{s-1}(x+1)^{-1}\,dx=\pi(\sin\pi s)^{-1}$.]

28. *A multiple series.* Let S be the convergent series of order k defined by $\sum[a_1a_2\cdots a_k(a_1+a_2+\cdots+a_k)]^{-1}$, where the summation is taken over all systems of integers $a_1, a_2,\ldots, a_k$ which are all ≥ 1 and relatively prime. Then $S=k!$ (*AMM* 73 (1966) 1025).

29. *Expansion of* $(\arcsin t)^k$. Use the Cauchy formulas

$$\sin uz=u\sum_{n\geq 0}(-1)^n(u^2-1^2)(u^2-3^2)\cdots$$
$$\cdots(u^2-(2n-1)^2)\frac{\sin^{2n+1}z}{(2n+1)!}$$
$$\cos uz=\sum_{n\geq 0}(-1)^n u^2(u^2-2^2)(u^2-4^2)\cdots$$
$$\cdots(u^2-(2n-2)^2)\frac{\sin^{2n}z}{(2n)!}.$$

where $x = \arcsin u$ has to be substituted ([Teixeira, 1896]). Use the same formula to prove:

$$\frac{\sin ux}{\cos x} = u \sum_{n \geq 0} (-1)^n (u^2 - 2^2) \ldots (u^2 - (2n)^2) \frac{\sin^{2n+1} x}{(2n+1)!}$$

$$\frac{\cos ux}{\cos x} = \sum_{n \geq 0} (-1)^n (u^2 - 1^2)(u^2 - 3^2) \cdots$$
$$\cdots (u^2 - (2n-1)^2) \frac{\sin^{2n} x}{(2n)!}.$$

30. *Some summation formulas and interesting combinatorial identities.*

$$\sum_{k=0}^{n} \frac{k}{(k+1)!} = 1 - \frac{1}{(n+1)!};$$

$$\sum_{0 \leq k \leq n/2} (-1)^k \binom{n-k}{k} 2^{n-2k} = n + 1;$$

$$\sum_{k=0}^{n} k \binom{n}{k}^2 = (2n-1) \binom{2n-2}{n-1} \quad \text{(see Exercise 12, p. 225);}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 \binom{2k}{k}^{-1} = \binom{2n}{n}^{-1};$$

$$\sum_{k=1}^{n} (-1)^{k+1} k^{-1} \binom{n}{k} = \sum_{i=1}^{n} i^{-1};$$

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} \binom{n+i}{k} = (-1)^k;$$

$$\sum_{1 \leq m \leq M,\ 1 \leq n \leq N} \min(m, n) = \tfrac{1}{6} N(N+1)(3M - N + 1);$$

$$\sum_{1 \leq m \leq M,\ 1 \leq n \leq N} \max(m, n) = \tfrac{1}{6} N(N^2 - 1) + \tfrac{1}{2} MN(M+1);$$

$$\sum_{k=1}^{n} k \cdot k! = (n+1)! - 1, \text{ and its generalization (of Gould):}$$

$$\sum_{k=0}^{n} \binom{x}{k}^r \left(\frac{k!}{x^{k+1}}\right)^r [(x-k)^r - x^r] =$$
$$= \binom{x}{n+1}^r \left(\frac{(n+1)!}{x^{n+1}}\right)^r - 1;$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 x^k = \sum_{i=0}^{n} \binom{n}{i} \binom{2n-i}{n} (x-1)^i;$$

$$\sum_{i=1}^{k} \binom{x}{i} \binom{-x}{n-i} = \frac{k}{n} \binom{x-1}{k} \binom{-x}{n-k},$$

$$\sum_{i=0}^{k} \binom{x}{i} \binom{1-x}{n-i} = \frac{(n-1)(1-x)-k}{n(n-1)} \binom{x-1}{k} \binom{-x}{n-k-1}$$

([Anderson, 1953]). Finally, all the $\xi_1, \xi_2, \xi_3, \ldots$ being $\neq 0$, let us write

$$\binom{x}{j}_\xi := \frac{x(x-\xi_1)(x-\xi_2)\cdots(x-\xi_{j-1})}{\xi_1 \xi_2 \cdots \xi_j}, \qquad \binom{x}{0}_\xi := 1.$$

Then, we have (see [5b], p. 13):

$$\sum_{j=0}^{k} (-1)^j \binom{x}{j}_\xi = (-1)^k \frac{\xi_{k+1}}{x} \binom{x}{k+1}_\xi.$$

The reader will find in [*Gould, 1972] plenty of very fine results and sources concerning binomial identities.

31. *Sum of the r-th powers of the terms of an arithmetic progression.* Let $S_r := \sum_{k=1}^{n} (a + (k-1)b)^r$. By a method analogous to that used on p. 154, find the value of S_r as a function of the Bernoulli numbers. One can also establish the recurrence relation $(a+nb)^{r+1} - a^{r+1} = \sum_{l=1}^{r+1} \binom{r+1}{l} b^l S_{r+1-l}$, where $S_0 := n$. [*Hint*: Consider $\sum_{k=1}^{n} (a+kb)^{r+1}$ and expand then $(a+kb)^{r+1} = \{b + (a+(k-1)b)\}^{r+1}$ using the binomial identity.] As examples, for $t_r := 1^r + 3^r + 5^r + \cdots + (2n-1)^r$, we find $t_1 = n^2$, $t_2 = \binom{2n+1}{3}$, $t_3 = n^2(2n^2 - 1)$.

32. *Four trigonometric summation formulas* ([Hofmann, 1959]). For r integer ≥ 1, we have:

$$\sum_{k=1}^{n} \sin^{2r} kx = 2^{-2r-1} \left\{ (2n+1) \binom{2r}{r} + \right.$$
$$\left. + 2 \sum_{k=1}^{r} (-1)^k \binom{2r}{r-k} \frac{\sin[k(2n+1)x]}{\sin kx} \right\};$$

$$\sum_{k=1}^{n} \sin^{2r-1} kx = 2^{2-2r} \sum_{k=1}^{r} (-1)^k \left\{ \binom{2r-1}{r-k} \times \right.$$
$$\left. \times \frac{\sin(2k-1)(n+1)x/2 \cdot \sin(2k-1)nx/2}{\sin(2k-1)x/2} \right\};$$

$$\sum_{k=1}^{n} \cos^{2r} kx = -\tfrac{1}{2} + 2^{-2r-1}\left\{(2n+1)\binom{2r}{r} + 2\sum_{k=1}^{r}\binom{2r}{r-k}\frac{\sin[k(2n+1)x]}{\sin kx}\right\};$$

$$\sum_{k=1}^{n} \cos^{2r+1} kx = -\tfrac{1}{2} + 2^{-2r-1}\sum_{k=0}^{r}\left\{\binom{2r+1}{r-k}\times \frac{\sin(2k+1)(2n+1)x/2}{\sin(2k+1)x/2}\right\}.$$

33. *On the roots of* $ax=\operatorname{tg} x$. For computing the root x which lies between $n\pi$ and $(n+1)\pi$, insert $x=n\pi+\pi/2-u$, $|u|<\pi/2$, in $ax=\operatorname{tg} x$. Then, $t:=(a\pi(n+\frac{1}{2}))^{-1}=(\operatorname{tg} u)(1+au\operatorname{tg} u)^{-1}:=f(u)$, which can be (formally) inverted by the Lagrange formula: $u=f^{\langle -1\rangle}(t)$. Returning to x, the following purely asymptotic expansion holds:

$$x \approx (n+\tfrac{1}{2})\pi - \sum_{m\geq 0}\frac{t^{2m+1}}{(2m+1)!}\left\{\sum_{k=0}^{m}(-1)^{m-k}C(m,k)\,a^{k}\right\},$$

where the $C(m,k)$, closely related to arctangent numbers (p. 260), satisfy:

$$C(m,k)=\frac{(2m-1)\,2m\,(2m+1)}{(2m-k)(2m-k+1)}\{C(m-1,k-1)+C(m-1,k)\}.$$

Here is a table of the $C(m,k)$:

m\k	0	1	2	3	4	5	6	7
0	1							
1	1	3						
2	3	20	30					
3	15	161	525	525				
4	105	1584	8232	17640	13230			
5	945	18579	134970	457380	727650	436590		
6	10395	253812	2395364	11294140	28243215	35675640	17837820	
7	135135	3963105	46360580	283245262	981245265	1938871935	2029052025	869593725

Of course, when $a=1$, $x=\operatorname{tg} x$, the alternating horizontal sums extend Euler's result: $x=(n+\frac{1}{2})\pi-\sum_{m\geq 0}c_m t^{2m+1}/(2m+1)!$, where $t=(\pi(n+\frac{1}{2}))^{-1}$ and

m	0	1	2	3	4	5	6	7
c_m	1	2	13	146	2343	48984	1264248	38947730

(c_m, $m\leq 4$, due to [*Euler, 1748 II, p. 322].)

34. *About the (unruly) formal series* $\varphi(t)=\sum_{n\geq 1} n!\,t^n$. Let us define the integers $A(n,k)$ by $(\varphi(t))^k=\sum_{n\geq k}A(n,k)\,t^n$. (1) These numbers satisfy the following recurrence: $A(n,k)=A(n-1,k-1)+((n+k-1)/k)\times A(n-1,k)$. [*Hint*: Use $t^2\varphi'=(1-t)\varphi-t$.] (2) Also find a triangular recurrence for the $a(n,k)$ defined by $(\varphi^{\langle -1\rangle}(t))^k=\sum_{n\geq k}a(n,k)\,t^n$, and verify the following tables (of course, $a=A^{-1}$):

$A(n,k)$

n\k	1	2	3	4	5	6	7	8
1	1							
2	2	1						
3	6	4	1					
4	24	16	6	1				
5	120	72	30	8	1			
6	720	372	152	48	10	1		
7	5040	2208	828	272	70	12	1	
8	40320	14976	4968	1576	440	96	14	1

$a(n,k)$

n\k	1	2	3	4	5	6	7	8
1	1							
2	−2	1						
3	2	−4	1					
4	−4	8	−6	1				
5	−4	−16	18	−8	1			
6	−48	12	−44	32	−10	1		
7	−336	−96	72	−96	50	−12	1	
8	−2928	−480	−216	216	−180	72	−14	1

(3) Prove that $\Delta^j A(k,k+j)=\Delta^j a(k,k+j)=3^j$ (see also Exercises **14** p. 261, 15 and 16 p. 294).

35. *Fermat matrices*. Let $\mathbf{F}_n$ be the n-th section of the Fermat matrix $\mathbf{F}$ composed of the binomial coefficients $(a,b)=\binom{a+b}{a}$, in the symmetric notation of p. 8, $0\le a, b\le n$. So

$$\mathbf{F}_0=(1),\quad \mathbf{F}_1=\begin{pmatrix}1&1\\1&2\end{pmatrix},\quad \mathbf{F}_2=\begin{pmatrix}1&1&1\\1&2&3\\1&3&6\end{pmatrix},$$
$$\mathbf{F}_3=\begin{pmatrix}1&1&1&1\\1&2&3&4\\1&3&6&10\\1&4&10&20\end{pmatrix},\ldots.$$

Prove that $\mathbf{F}_n=\mathbf{P}\,{}^t\mathbf{P}$, where $\mathbf{P}$ is the Pascal matrix (p. 143) and ${}^t\mathbf{P}$ its transpose. (2) So, $\det(\mathbf{F}_n)=1$ (cf. Exercise 46, p. 92) and all coefficients of $\mathbf{F}_n^{-1}$ are integers: $f_n(i,j)=(-1)^{i+j}\sum_{l\le n}\binom{l}{i}\binom{l}{j}$. (3) The unsigned coefficients $C_n(i,j):=|f_n(i,j)|$ satisfy: $C_n(i,j)=C_{n-1}(i-1,j-1)+C_{n-1}(i-1,j)+C_{n-1}(i,j-1)+C_{n-1}(i,j)$, with $C_n(i,j):=0$ if $i<0$ or $j<0$, except $C_n(-1,-1):=1$.

$$\mathbf{C}_0=(1),\quad \mathbf{C}_1=\begin{pmatrix}2&1\\1&1\end{pmatrix},\quad \mathbf{C}_2=\begin{pmatrix}3&3&1\\3&5&2\\1&2&1\end{pmatrix},$$
$$\mathbf{C}_3=\begin{pmatrix}4&6&4&1\\6&14&11&3\\4&11&10&3\\1&3&3&1\end{pmatrix},\quad \mathbf{C}_4=\begin{pmatrix}5&10&10&5&1\\10&30&35&19&4\\10&35&46&27&6\\5&19&27&17&4\\1&4&6&4&1\end{pmatrix},\ldots$$

(4) $C_n(k,0)=C_n(0,k)=\binom{n+1}{k+1}$, $C_n(k,1)=\binom{n+1}{k+1}((k+1)(n-k)-1)/(k+2),\ldots$, and $\sum_{i,j}C_n(i,j)=(4^{n+1}-1)/3$.

***36.** *Simple and double summations.* Prove the equality ([Carlitz, 1964a]):

$$\sum_{i+j+k=n}\binom{i+j}{i}\binom{j+k}{j}\binom{k+i}{k}=\sum_{0\le l\le n}\binom{2l}{l}.$$

37. *Two multiple summations.* (1) The summation $\sum(x_1x_2\ldots x_l)^{-1}$, taken over all systems of integers $x_i\ge 1$, $i\in[l]$, such that $x_1+x_2+\cdots+x_l=n$, equals $(l!/n!)\,s(n,l)$, where $s(n,l)$ is the Stirling number of the first kind, [3d] (p. 213). (2) The summation $\sum(x_1^n+x_2^n+\cdots+x_l^n)=\sigma_{l,n}(p)$,

taken over all systems of integers $x_i\ge 0$, such that $x_1+x_2+\cdots+x_l=p$, equals $l\sum_{k=1}^{n}k!\,S(n,k)\binom{l+p-1}{p-k}$, where $S(n,k)$ is the Stirling number of the second kind, [14a] (p. 31). [*Hint.* Consider $\sum_{p\ge 0}\sigma_{l,n}(p)t^p$.]

38. *The formula of Li Jen-Shu* (see, for instance, [Kaucky, 1964]).

$$\sum_{0\le j\le k}\binom{k}{j}^2\binom{n+2k-j}{2k}=\binom{n+k}{k}^2.$$

39. *A formula of Riordan* ([Riordan, 1962a], [Gould, 1963a]).

$$\sum_{0\le k\le n}\binom{n-1}{k}n^{n-k-1}(k+1)!=n^n.$$

40. *A formula of Gould.* If we put $A_k(a,b):=a(a+bk)^{-1}\binom{a+bk}{k}$, then we have

$$\sum_{0\le k\le n}A_k(a,b)\,A_{n-k}(a',b)=A_n(a+a',b).$$

([Gould, Kaucky, 1965], and for a 'combinatorial' proof, [Blackwell, Dubins, 1966]. We already met similar numbers in [9b], p. 24.)

***41.** *The 'Master Theorem' of MacMahon.* The $a_{i,j}$, $i,j\in[n]$ being constants (complex, for instance), let us consider the n linear forms:

$$X_i:=\sum_{j=1}^{n}a_{i,j}x_j,\quad i\in[n].$$

The 'Master Theorem' asserts that the coefficient of the monomial $x_1^{m_1}x_2^{m_2}\ldots x_n^{m_n}$ (where $m_1,m_2,\ldots,m_n$ are integers ≥ 0) in the polynomial $X_1^{m_1}X_2^{m_2}\ldots X_n^{m_n}$ is equal to the coefficient of the same monomial in D^{-1}, where D is the determinant:

$$D:=\begin{vmatrix}1-a_{11}x_1 & -a_{12}x_1 & \cdots & -a_{1n}x_1\\ -a_{21}x_2 & 1-a_{22}x_2 & \cdots & -a_{2n}x_2\\ \vdots & & & \\ -a_{n1}x_n & -a_{n2}x_n & \cdots & 1-a_{nn}x_n\end{vmatrix}.$$

In other words, if the identity matrix is denoted I, if A is $[a_{i,j}]_{i,j\in[n]}$, if the column matrix of the x_i, $i\in[n]$, is X, if the *diagonal* matrix of the x_i is X, then we have (with the notation [3a], p. 148):

$$C_{x_1^{m_1}\dots x_n^{m_n}}\prod_{i=1}^{n}(AX)_i^{m_i}=C_{x_1^{m_1}\dots x_n^{m_n}}(\det(I-XA))^{-1}.$$

([*MacMahon, I, 1915], p. 98. See [Foata, 1964, 1965], [*Cartier, Foata, 1969], pp. 54–60, for a noncommutative generalization, [Good, 1962], from whom we borrow the proof, and [Wilf, 1968b].) [*Hint*: Put $Y_i=1+X_i$, then the required coefficient is equal to the coefficient of $x_1^{m_1}\dots x_n^{m_n}$ in $Y_1^{m_1}\dots Y_n^{m_n}$, hence, by the Cauchy theorem:

$$(2\pi i)^{-n}\int\int\dots\int\frac{Y_1^{m_1}\dots Y_n^{m_n}}{x_1^{m_1+1}\dots x_n^{m_n+1}}\,dx_1\dots dx_n,$$

where the integration contours are circles around the origin. Then perform the change of variables $w_i:=x_iY_i$, $i\in[n]$, whose Jacobian causes D to appear.]

42. *Dixon formula.* This famous identity can be stated as follows:

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^3=(-1)^n\frac{(3n)!}{(n!)^3}.$$

This is a special case ($a=b=c=n$) of:

$$S:=\sum_s(-1)^s\binom{b+c}{b+s}\binom{c+a}{c+s}\binom{a+b}{a+s}=\frac{(a+b+c)!}{a!\,b!\,c!}.$$

[*Hint*: Observe that $S=(-1)^{a+b+c}C_{x^{2a}y^{2b}z^{2c}}(y-z)^{b+c}(z-x)^{c+a}\times(x-y)^{a+b}$, and apply then the 'Master Theorem' of Exercise 41.] ([Dixon, 1891]. See also [*De Bruijn, 1961], p. 72, [*Cartier, Foata, 1969], [Good, 1962], [Gould, 1959], [Kolberg 1957], [Nanjundiah, 1958], [Toscano, 1965].)

43. *A beautiful identity concerning the exponential.* Show that:

$$\exp\left\{\sum_{n\geq 1}n^{n-1}\frac{t^n}{n!}\right\}=1+\sum_{n\geq 1}(n+1)^{n-1}\frac{t^n}{n!}.$$

44. *The number of terms in the derivatives of implicit functions* ([Comtet,

1974]) The number $a(n)$ of *different* monomials $Af_{i_1j_1}^{k_1}f_{i_2j_2}^{k_2}\dots$ in the expression of $y_n=\varphi^{(n)}(x)$, where $F(x,y)=0$ (see p. 153) is such that

$$a(n)=C_{t^nu^{2n-1}}\prod_{(i,j)\in E}\frac{1}{1-t^iu^j},$$

A3262

with $E:=\mathbb{N}^2\setminus\{(0,0),(0,1)\}$. The first values of $a(n)$ are:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$a(n)$	1	2	9	24	76	145	383	722	1565	3347	6584	12097	25575	48117	91129

45. *Some expansions related to the derivatives of the gamma function.* In the sequel, we write $\gamma=0.577\dots$ for the Euler constant, $\zeta(s)=\sum_{n\geq 1}n^{-s}$ (see Exercise 36, p. 89), $\zeta(s,a)=\sum_{n\geq 0}(a+n)^{-s}$, $s_k=(-1)^k(k-1)!\,\zeta(k)$, and Y_n for the Bell polynomials, [3c] p. 134. (1) We have:

$$\Gamma(1+t)=\exp\{-\gamma t+\zeta(2)t^2/2-\zeta(3)t^3/3+\cdots\}.$$

Consequently,

$$\Gamma^{(n)}(1)=Y_n(-\gamma,s_2,s_3,\dots)=\int_0^\infty e^{-x}\log^n x\,dx.$$

(2) Hence,

$$\frac{1}{\Gamma(t)}=\sum_{n\geq 0}\frac{t^{n+1}}{n!}Y_n(\gamma,-s_2,-s_3,\dots).$$

(3) Find similar expansions for $\Gamma(a+t)$ using $\zeta(s,a)$.

SIEVE FORMULAS

This chapter solves the following problem: let be given a system $(A_1, A_2, \dots, A_p)$ of p subsets of a set N, whose mutual relations are somehow known, compute the cardinal of each subset of N that can be formed by taking intersections and unions of the given subsets or their complements.

In the sequel, we will denote the intersection of A and B by AB as well as by $A \cap B$, similarly the complement of A by $\bar{A}$ or $\complement A$. Each subset of $[p] := \{1, 2, \dots, p\}$ will be denoted by a lower case Greek letter.

4.1. NUMBER OF ELEMENTS OF A UNION OR INTERSECTION

We want to generalize the following formula:

[1a] $$|A \cup B| = |A| + |B| - |AB|, \qquad AB := A \cap B,$$

where A, B are subsets of N, and that follows (notations [10a], p. 35, and [10d], p. 38) from:

$$A \cup B = A + (B - AB) \Rightarrow |A \cup B| = |A| + |B - AB| = |A| + |B| - |AB|.$$

The interpretation of [1a] in Figure 33 is intuitively clear.

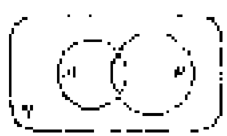


FIG. 33.

THEOREM A. (*Sieve formula, or inclusion-exclusion principle*). *Let $\mathscr{A}$ be a p-system of N, in other words a sequence of p subsets $A_1, A_2, \dots, A_p$ of*

N, among which some may be empty or coinciding with each other. Then:

[1b] $$|A_1 \cup A_2 \cup \dots \cup A_p| = \sum_{1 \le i \le p} |A_i| - \sum_{1 \le i_1 < i_2 \le p} |A_{i_1}A_{i_2}| + \sum_{1 \le i_1 < i_2 < i_3 \le p} |A_{i_1}A_{i_2}A_{i_3}| - \dots + (-1)^{p-1} |A_1 A_2 \dots A_p|.$$

(Formula [1b] is also known as formula of [Da Silva, 1854], [Sylvester, 1883]; it holds whether N is finite or not.)

First, we indicate two other ways, [1d, f], to write [1b]:

(1) Using Exercise 9 (p. 158) for (*) and introducing

[1c] $$S_k := \sum_{1 \le i_1 < i_2 < \dots < i_k \le p} |A_{i_1}A_{i_2} \dots A_{i_k}| \overset{(*)}{=} \sum^{(k)} |A_1 A_2 \dots A_k|,$$

formula [1b] becomes:

[1d] $$|A_1 \cup A_2 \cup \dots \cup A_p| = \sum_{1 \le k \le p} (-1)^{k-1} S_k = S_1 - S_2 + S_3 - \dots + (-1)^{p-1} S_p.$$

(2) Let α be a subset of $[p] := \{1, 2, \dots, p\}$, $\alpha \neq \emptyset$. We introduce the following notations:

[1e] $$A_\alpha := \bigcap_{i \in \alpha} A_i, \qquad A_\emptyset := \bigcap_{i \in \emptyset} A_i = N, \qquad \bigcup_{i \in \emptyset} A_i = \emptyset.$$

Formula [1b] becomes (with $\mathfrak{P}'[p] := \mathfrak{P}'(p) =$ the set of blocks = the set of nonempty subsets of $[p]$):

[1f] $$|A_1 \cup A_2 \cup \dots \cup A_p| = \sum_{\alpha \in \mathfrak{P}'[p]} (-1)^{|\alpha|-1} |A_\alpha|.$$

■ We argue by induction on p. Because of [1a] for equality (*), we get:

[1g] $$\Big|\bigcup_{1 \le i \le p+1} A_i\Big| = \Big|A_{p+1} \cup \Big(\bigcup_{1 \le i \le p} A_i\Big)\Big| \overset{(*)}{=} |A_{p+1}| + \Big|\bigcup_{1 \le i \le p} A_i\Big| - \Big|\bigcup_{1 \le i \le p} (A_{p+1}A_i)\Big|,$$

where, if [1f] is supposed to hold, we have (using the notation $\mathfrak{P}_{\ge 2}[p] := \{\sigma \mid \sigma \subset [p], |\sigma| \ge 2\}$):

[1h] $$\Big|\bigcup_{1 \le i \le p} A_i\Big| = \sum_{1 \le i \le p} |A_i| + \sum_{\sigma \in \mathfrak{P}_{\ge 2}[p]} (-1)^{|\sigma|-1} |A_\sigma|$$

[1i] $$\Big|\bigcup_{1\leqslant i\leqslant n}(A_{n+1}A_i)\Big|=\sum_{\{n+1\}\subset K\subset [n+1]}(-1)^{|K|-2}|A_K|.$$

Substituting [1h, i] into [1g] gives then:

$$\Big|\bigcup_{i=1}^{n+1}A_i\Big|=\sum_{i=1}^{n+1}|A_i|+\sum_{K\subset[n+1]}(-1)^{|K|-1}|A_K|-$$

$$-\sum_{K\subset[n+1]}(-1)^{|K|-1}|A_K|.\ \blacksquare$$

THEOREM B. *Conditions and notations as in Theorem A; for* $S_0:=|N|$, *N being finite, we have:*

[1j] $$|\bar A_1\bar A_2\ldots\bar A_p|=\sum_{K\subset[p]}(-1)^{|K|}|A_K|=$$

$$=\sum_{0\leqslant k\leqslant p}(-1)^k S_k=S_0-S_1+\cdots+(-1)^p S_p.$$

■ Follows from $|\bar A_1\bar A_2\ldots|=|\complement(A_1\cup A_2\cup\cdots)|=|N|-|A_1\cup A_2\cup\cdots|$, and from [1d, f]. ■

■ *Two examples*

(I) *The 'sieve of Eratosthenes'.* Let $p_1(=2)$, $p_2(=3)$, $p_3(=5)$, ... be the increasing sequence of prime numbers, and let $\pi(x)$ stand for the number of prime numbers that are $\leqslant x$, for x real >0. Let A_i be the set of the multiples of p_i that belong to $N:=\{2, 3, \ldots, n\}$. If $q\in\bar A_1\bar A_2\ldots\bar A_k$, where $k:=\pi(\sqrt{n})$, then this means that each prime factor of q is larger than p_k; hence q is a prime number such that $\sqrt{n}<q\leqslant n$. Thus $|\bar A_1\bar A_2\ldots\bar A_k|=\pi(n)-\pi(\sqrt{n})$. On the other hand, for $1\leqslant i_1<i_2<\cdots<i_h\leqslant k$, the fact that r belongs to $A_{i_1}A_{i_2}\ldots A_{i_h}$ means that $r(\leqslant n)$ is a multiple of $p_{i_1}p_{i_2}\ldots p_{i_h}$, hence $|A_{i_1}A_{i_2}\ldots A_{i_h}|=E(n/(p_{i_1}p_{i_2}\ldots p_{i_h}))$, where $E(x)$ means the largest integer $\leqslant x$, called the *integral part* of x, and also denoted by $[x]$. So we obtain as result, by [1j] (and with $|N|=n-1$):

[1k] $$\pi(n)-\pi(\sqrt{n})=(n-1)-\sum_{1\leqslant i\leqslant k}E\binom{n}{p_i}+$$

$$+\sum_{1\leqslant i_1<i_2\leqslant k}E\left(\frac{n}{p_{i_1}p_{i_2}}\right)-\cdots+(-1)^k E\left(\frac{n}{p_1p_2\ldots p_k}\right).$$

This formula allows us to compute theoretically $\pi(n)$ if we know all prime numbers $\leqslant\sqrt{n}$.

(II) *Chromatic polynomials.* Let $\mathscr{G}\subset\mathfrak{P}_2[n]$ be a graph on the set (of *nodes*) $[n]=\{1, 2, \ldots, n\}$, and let λ be an integer $\geqslant 0$. The *chromatic polynomial of* $\mathscr{G}$ is the number $P_{\mathscr{G}}(\lambda)$ of ways to colour the nodes in λ (or fewer) colours such that two adjacent nodes have different colours. Indeed, any colouring is a map of $[n]$ into $[\lambda]$, say $f\in[\lambda]^{[n]}$, such that $\{i, j\}\in\mathscr{G}\Rightarrow f(i)\neq f(j)$. For instance, if $\mathscr{G}=\{\{1, 2\}, \{2, 3\}, \{3, 4\}, \ldots, \{n-1, n\}\}$, we find $P_{\mathscr{G}}(\lambda)=\lambda(\lambda-1)^{n-1}$ by successively choosing the colours of the nodes $\{1\}, \{2\}, \{3\}, \ldots$. In the same manner, if $\mathscr{G}=\mathfrak{P}_2[n]$, we find $P_{\mathscr{G}}(\lambda)=(\lambda)_n=\lambda(\lambda-1)\ldots(\lambda-n+1)$. Evidently, $P_{\mathscr{G}}(0)=P_{\mathscr{G}}(1)=0$. Let us prove that $P_{\mathscr{G}}(\lambda)$ is always a *polynomial* in λ. For each edge $E_i\in\mathscr{G}$, $1\leqslant i\leqslant q:=|\mathscr{G}|\leqslant\binom{n}{2}$, let $A_i\subset[\lambda]^{[n]}$ be the set of colourings which give the same colour to the two nodes of E_i. Then, by [1j], $P_{\mathscr{G}}(\lambda)=|\bar A_1\bar A_2\ldots\bar A_q|=$ $=\lambda^n-(|A_1|+|A_2|+\cdots)+(|A_1A_2|+|A_1A_3|+\cdots)-\cdots$. Now, $|A_1|=|A_2|=\cdots=\lambda^{n-1}$, $|A_1A_2|=|A_2A_3|=\cdots=\lambda^{n-2}$, and any other $|A_{i_1}A_{i_2}\ldots A_{i_h}|$, $h\geqslant 3$, is a polynomial in λ with degree $\leqslant n-2$, as can be seen easily. Consequently, $P_{\mathscr{G}}(\lambda)=\lambda^n-q\lambda^{n-1}+a_2\lambda^{n-2}-a_3\lambda^{n-3}+-\cdots+(-1)^{n-1}a_{n-1}\lambda$, where the a_i are integers, which all can be proven to be $\geqslant 0$.

The following pretty results are worthwhile: (I) if the graph $\mathscr{G}$ has connected components $\mathscr{G}_1, \mathscr{G}_2, \ldots$, then $P_{\mathscr{G}}=P_{\mathscr{G}_1}P_{\mathscr{G}_2}\ldots$. (II) $\mathscr{G}$ is a tree if and only if $P_{\mathscr{G}}(\lambda)=\lambda(\lambda-1)^{n-1}$. (III) If $\mathscr{G}$ is a polygon (i.e. circuit), then $P_{\mathscr{G}}(\lambda)=(\lambda-1)^n-(-1)^n(\lambda-1)$. (IV) If $\mathscr{G}$ is the complete bipartite graph with parts M and N (i.e. $\{x, y\}\in\mathscr{G}\Leftrightarrow x\in M, y\in N$), then $P_{\mathscr{G}}(\lambda)=$ $=\sum_k S(m, k)S(n, l)(\lambda)_{k+l}$ (see p. 204). (V) If $\mathscr{G}$ is connected, then $P_{\mathscr{G}}(\lambda)\leqslant\lambda(\lambda-1)^{n-1}$ for every integer $\lambda\geqslant 0$. (VI) The smallest number r such that λ^r has a nonzero coefficient in $P_{\mathscr{G}}(\lambda)$ is the number of components of $\mathscr{G}$. (See, for instance, the introductory survey of [Read, 1968].) Finally, let us mention as still unsolved problems: (I) the characterization of chromatic polynomials; (II) the unimodality (p. 269) of the coefficients $1, q, a_2, a_3, a_4, \ldots$; (III) the condition for two graphs to have the same chromatic polynomial.

DEFINITION. *A system* $\{A_1, A_2, \ldots, A_p\}$ *of subsets of* N *is called* interchangeable *if and only if the cardinality of any intersection of* k

arbitrary subsets among them depends only on k, for all $k\subset[p]$.

THEOREM C. *Let be given an interchangeable system of subsets of N, say* $(A_1, A_2, \ldots, A_p)$; *then we have:*

$$[1l]\quad |A_1\cup A_2\cup\cdots\cup A_p| = p|A_1| - \binom{p}{2}|A_1A_2| + \binom{p}{3}|A_1A_2A_3| - \cdots = \sum_{1\le k\le p}(-1)^{k-1}\binom{p}{k}|A_{[k]}|$$

$$[1m]\quad |A_1'A_2'\cdots A_p'| = |N| - \binom{p}{1}|A_1| + \binom{p}{2}|A_1A_2| - \cdots = \sum_{0\le k\le p}(-1)^k\binom{p}{k}|A_{[k]}|.$$

This is an immediate consequence of the definition of interchangeable systems and of [1b, j].

4.2. THE 'PROBLÈME DES RENCONTRES'

DEFINITION. *A permutation (Definition B, p. 7) σ of N, |N| = n, is called a* **derangement**, *if it does not have a fixed point, or rencontre, or coincidence, in the sense that for all* $x\in N$, $\sigma(x)\neq x$.

For example, the permutation $\sigma_1 := \begin{pmatrix} abcde \\ cedab\end{pmatrix}$ does not have a coincidence, while $\sigma_2 := \begin{pmatrix} abcde \\ abecd\end{pmatrix}$ has 2. The famous 'problème des rencontres' (*Montmort, 1708]) consists of computing the number $d(n)$ of derangements of N, $n=|N|$.

THEOREM A. *The number* $d(n)$ *of derangements of N, n = |N|, equals:*

$$[2a]\quad d(n) = \sum_{0\le k\le n}(-1)^k\frac{n!}{k!} = n!\left(1-\frac{1}{1!}+\frac{1}{2!}-\cdots+\frac{(-1)^n}{n!}\right)$$

or also, for $n\geq 1$, *the integer closest to* $n!e^{-1}$:

$$[2a']\quad d(n) = [n!\,e^{-1}]$$

(Because of [2a], Chrystal has suggested the name of subfactorial for $d(n)$, and the notation $n¡$.)

■ If we identify N with $[n] := \{1, 2, \ldots, n\}$, we denote the set of permutations of $[n]$ by $\mathfrak{S}[n]$, and the subset of $\mathfrak{S}[n]$ consisting of permutations σ such that $\sigma(i)=i$, $i\in[n]$, by $\mathfrak{S}_i=\mathfrak{S}_i[n]$, and the set of derangements of $[n]$ by $\mathfrak{D}[n]$. Clearly $\mathfrak{S}[n]=\mathfrak{D}[n]+\bigcup_{i=1}^n\mathfrak{S}_i$. Hence, by Theorem B (p. 7), for (*):

$$[2b]\quad n! \overset{(*)}{=} |\mathfrak{S}[n]| = d(n) + \Big|\bigcup_{1\le i\le n}\mathfrak{S}_i\Big|$$

Now the $\mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_n$ are interchangeable (Definition p. 179), since giving a $\sigma\in\mathfrak{S}_{i_1}\mathfrak{S}_{i_2}\cdots\mathfrak{S}_{i_k}$ is equivalent to giving one of the permutations of $[n]-\{i_1, i_2, \ldots, i_k\}$, whose total number is $(n-k)!$ $(i_1<i_2<\cdots<i_k)$. Thus, [2a] follows from [1l] applied to $|\bigcup_{i=1}^n\mathfrak{S}_i|$ in [2b]. Finally, for [2a'], use in (*) the well-known inequality that relates the rest of an alternating series to the first neglected term:

$$|n!\,e^{-1}-d(n)| = n!\left|\sum_{k=n+1}^{\infty}\frac{(-1)^k}{k!}\right| \le n!\,\frac{1}{(n+1)!} = \frac{1}{n+1} < \frac{1}{2}.\quad ■$$

In particular, [2a] shows that $\lim_{n\to\infty}\{d(n)/n!\}=1/e$. The way the number e intrudes here into a combinatorial problem has strongly appealed to the imagination of the geometers of the 18th century. In more colourful terms, if the guests to a party leave their hats on hooks in the cloakroom, and grab at good luck a hat when leaving, then the probability that nobody gets back his own hat is (approximately) $1/e$.

Another method of computing $d(n)$ consists of observing that the set $\mathfrak{S}_K[n]$ of permutations of $[n]$ for which $K(\subset[n])$ is the set of fixed points has for cardinality $d(n-|K|)$. So:

$$\mathfrak{S}[n] = \sum_{K\subset[n]}\mathfrak{S}_K[n] = \sum_{k=0}^{n}\Big(\sum_{|K|=k}\mathfrak{S}_K[n]\Big).$$

Hence $n! = \sum_{0\le k\le n}\binom{n}{k} d(n-k) = \sum_{0\le h\le n}\binom{n}{h} d(h)$, from which [2a] follows by the inversion formula [6a, e] (p. 143).

THEOREM B. *The number $d(n)$ of derangements of $[n]$ has for generating function:*

[2c] $$\mathscr{D}(t) := \sum_{n\ge 0} d(n)\frac{t^n}{n!} = e^{-t}(1-t)^{-1}.$$

■ In fact, using [2a] for (*) and $h = n-k$ for (**):

$$\sum_{n\ge 0} d(n)\frac{t^n}{n!} \overset{(*)}{=} \sum_{n\ge 0} t^n\Big(\sum_{0\le k\le n}\frac{(-1)^k}{k!}\Big)$$

$$\overset{(**)}{=} \sum_{h,k\ge 0}(-1)^k\frac{t^{h+k}}{k!} = \Big(\sum_{h\ge 0} t^h\Big)\Big(\sum_{k\ge 0}\frac{(-t)^k}{k!}\Big). \quad ■$$

THEOREM C. *The number $d(n)$ of derangements of $[n]$ satisfies the following recurrence relations:*

[2d] $$d(n+1) = (n+1)\,d(n) + (-1)^{n+1};$$

[2d'] $$d(n+1) = n\{d(n) + d(n-1)\}.$$

■ Taking the derivative of $e^{-t} = (1-t)\mathscr{D}$, we get $-e^{-t} \overset{(*)}{=} -\mathscr{D} + (1-t)\mathscr{D}' \overset{(**)}{=} -(1-t)\mathscr{D}$, and then we equate coefficients in (*) to obtain [2d], and in (**) to obtain [2d'] (combinatorial proofs are also easy to find!).

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$d(n)$	1	0	1	2	9	44	265	1854	14833	133496	1334961	14684570	176214841

We discuss now a natural generalization of the 'problème des rencontres'. A $(k\times n)$ *latin rectangle* will be any rectangular matrix with k rows and n columns consisting of integers $\in [n]$, and such that all integers occurring in any one given row or column are all different $(k\le n)$. We suppose that the first row is $\{1, 2, 3, \ldots, n\}$ in this order (and we say that the rectangle is *reduced* then). We give an example of a (3×5) latin rectangle:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 5 & 2 \\ 5 & 3 & 1 & 2 & 4 \end{pmatrix}.$$

The number K_n of (reduced latin) $(3\times n)$-rectangles satisfies several recurrence relations (see, for instance, [Jacob, 1930], [Kerawala, 1941], [*Riordan, 1958], p. 204) and today there are asymptotic expansions known for it ([*Riordan, 1958], p. 209). The first values are (taken from tables of Kerawala, $n\le 15$):

n	3	4	5	6	7	8
K_n	2	24	552	21280	1073760	70299264

No known recurrence relations exist for the number $L(n, k)$ of $(k\times n)$ rectangles, $k\ge 4$, but a nice asymptotic formula is known ([Erdős, Kaplansky, 1946], [Kerawala, 1947a], [Yamamoto, 1951]): $L(n, k) \sim (n!)^k \exp\left(-\binom{k}{2}\right)$, for $k < n^{1/3-\epsilon}$ and $\epsilon > 0$ arbitrary. As far as the number of latin *squares* ($n\times n$-rectangles) is concerned, only the first 8 values are known precisely: if l_n stands for the number of *normalized* latin squares (first row and column consist of $\{1, 2, \ldots, n\}$, in this order) then we have:

n	2	3	4	5	6	7	8
l_n	1	1	4	56	9408	16942080	535281401856

l_7 being due to [Norton, 1939], [Sade, 1948b, 1951] and l_8 to [Wells, 1967], ([J. W. Brown, 1968]). Estimates for l_n when $n\to\infty$ seems to be an extremely difficult combinatorial problem.

4.3. THE 'PROBLÈME DES MÉNAGES'

This is the following problem: *What is the number of possible ways one can arrange n married couples (= n ménages) around a table such that men and women alternate, but no woman sits next to her husband.* (Posed, solved and popularized by [*Lucas, 1891]. See also [Cayley, 1878a, b]; [Moser, 1967] gives an interesting generalization.)

We suppose the wives already placed around the table ($2\cdot n!$ pos-

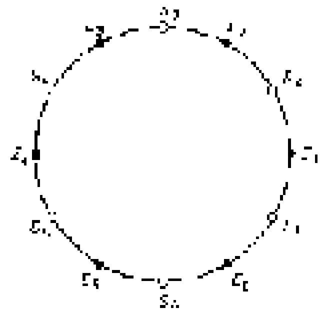


Fig. 34

sibilities). We number them 1, 2, ..., n in the ordinary (counterclockwise) direction, starting from one of them: $H_1, H_2, \ldots, H_n$ (Figure 34, $n=6$). We assign to every husband the number of his wife: $M_1, M_2, \ldots, M_n$, and to every empty seat the number of the wife to the right: $S_1, S_2, \ldots, S_n$. The problem consists of counting the number of possible admissible assignments of seats to husbands. Such an assignment is tantamount to giving a permutation σ of $[n]=\{1, 2, \ldots, n\}$, where $\sigma(i)$ stands for the seat number assigned to husband M_i, $i\in[n]$. This number should satisfy:

$$[3a] \qquad \sigma(i)\neq i, \quad \sigma(i)\neq i+1 \quad \text{for} \quad i\in[n-1]; \qquad \sigma(n)\neq n, \quad \sigma(n)\neq 1.$$

Let $\mu(n)$ be the number of permutations such that [3a] holds, this is usually called the '*reduced number of ménages*'. The *total* number $\mu^*(n)$ of placements of *ménages* is hence equal to $2n!\,\mu(n)$, if we take into account the $2n!$ possibilities of arranging the wives. We concentrate now on computing $\mu(n)$. The main idea consists of connecting this problem with the theorem on p. 24. To carry this out, we put:

$$[3b] \qquad A_{2i-1} := \{\sigma \mid \sigma(i)=i\}, \qquad i\in[n],$$
$$A_{2i} := \{\sigma \mid \sigma(i)=i+1\}, \quad i\in[n-1];$$
$$A_{2n} := \{\sigma \mid \sigma(n)=1\}.$$

Clearly, by [1a, b] for (*), and [1j] (p. 178), for (**):

$$[3c] \qquad \mu(n) \overset{(*)}{=} \Big|\bigcap_{1\leqslant i\leqslant 2n} \bar{A}_i\Big| \overset{(**)}{=} \sum_{\beta\subset[2n]} (-1)^{|\beta|}\,|A_\beta|.$$

Now, $|A_\beta| := |\bigcap_{i\in\beta} A_i|$ is evidently equal to 0 if β contains two consecutive elements of the 'circle' (1, 2, 3, ..., 2n, 1). In the opposite case, $|A_\beta|$ equals $(n-|\beta|)!$ and, according to the Theorem on p. 24, such β happen $g_c(2n, k)$ times; hence:

$$\mu(n)=\sum_{k=0}^{n} (-1)^k\,(n-k)!\,g_c(2n, k).$$

Finally, we obtain:

THEOREM. *The number $\mu(n)$ of reduced solutions to the 'ménages' problem, defined above, equals:*

$$[3d] \qquad \mu(n)=\sum_{0\leqslant k\leqslant n} (-1)^k \frac{2n}{2n-k}\binom{2n-k}{k}(n-k)!.$$

This beautiful formula (due to [Touchard, 1953]) is perhaps not the best for the actual computation of the $\mu(n)$; several recurrence relations for more efficient computations are known. (See [*Riordan, 1958], pp. 195-201, [Carlitz, 1952a, 1954a], [Gilbert, 1956a], [Kaplansky, Riordan, 1946], [Kerawala, 1947b], [Riordan, 1952a], [Schöbe, 1943, 1961], [Touchard, 1934].) The first values of $\mu(n)$ (taken from the tables of [Moser, Wyman, 1955a], $n\leqslant 65$), are:

n	2	3	4	5	6	7	8	9
$\mu(n)$	0	1	2	13	80	579	4738	43387

n	10	11	12	13	14
$\mu(n)$	439792	4890741	59216642	775596313	10927434464

4.4. BOOLEAN ALGEBRA GENERATED BY A SYSTEM OF SUBSETS

Let $\mathscr{A} := (A_1, A_2, \ldots, A_p)$ be a system of subsets of a set N, $A_i\subset N$, $i\in[p]$, among which there may be identical or empty subsets.

DEFINITION A. *The Boolean algebra (of subsets) generated by $\mathscr{A}$, denoted by $\mathfrak{B}(\mathscr{A})$, is the set of subsets of N that can be obtained by means of a finite number of the set operations: union, intersection and complementation. Each of the elements of $\mathfrak{B}(\mathscr{A})$ will be called* Boolean function generated *by $\mathscr{A}$.*

It can be immediately verified that, for the operations $\cap$, $\cup$ and

$\mathscr{B} = (\complement\mathscr{B}, \mathfrak{b}(\mathscr{A}))$ is actually a Boolean algebra in the sense of p. 2. The following are two examples of Boolean functions generated by (A_1, A_2, A_3) (we recall that the notation ST means $S \cap T$):

[4a] $f_1 = (A_1A_2) \cup \bar{A}_3, \quad f_2 = A_1 \cup (\overline{A_2 \cup A_3})A_3.$

As for polynomials, it is sometimes very interesting to interpret any Boolean function $f \in \mathfrak{b}(\mathscr{A})$ as a *purely formal expression* of the 'variables' $(A_1, A_2, \ldots, A_n)$ and to introduce an equivalence relation on the set of these expressions by putting $f \sim g$ when g can be obtained from f by the rules of computation in any Boolean algebra (see p. 2). For example, $f_1 = \complement(A_1 \cup \complement A_2) \sim g = \complement(\bar A_1 \cap A_2)$ is true, but $f_1 = A_1A_2 \sim g = A_1 \cup A_2$ is not true.

DEFINITION B. *The* complete products *of $\mathscr{A}$ are the 2^n Boolean functions of the form* (*see notation* [1e], p. 171):

[4b] $A_\pi \bar{A}_{\bar\pi} = (\bigcap_{i \in \pi} A_i) \cap (\bigcap_{j \in \bar\pi} \bar{A}_j), \quad$ *where* $\pi \subset [n]$.

The set of complete products is called $\mathfrak{p}(\mathscr{A})$.

For instance, the 8 complete products of $\mathscr{A} = (A_1, A_2, A_3)$ are:

[4c] $A_1A_2A_3, \quad A_1A_2\bar{A}_3, \quad A_1\bar{A}_2A_3, \quad \bar{A}_1A_2A_3,$
$\quad A_1\bar{A}_2\bar{A}_3, \quad \bar{A}_1A_2\bar{A}_3, \quad \bar{A}_1\bar{A}_2A_3, \quad \bar{A}_1\bar{A}_2\bar{A}_3.$

DEFINITION C. *The* conjunctions *of $\mathscr{A}$ are the 2^n Boolean functions of the form:*

[4d] $A_\lambda := \bigcap_{i \in \lambda} A_i, \quad$ where $\lambda \subset [n]$.

The set of conjunctions can be denoted by $\mathfrak{c}(\mathscr{A})$.

For instance, the 8 conjunctions of $\mathscr{A} = (A_1, A_2, A_3)$ are $\Omega, A_1, A_2, A_3, A_1A_2, A_1A_3, A_2A_3, A_1A_2A_3$.

THEOREM A. *Each Boolean function has a unique representation as a union of complete products (up to order). Hence (with the notation $\sum$ of* [10a] *of p. 25 for the disjoint union):*

[4e] $\forall f \in \mathfrak{b}(\mathscr{A}), \quad \exists!\mathscr{M} \subset \mathfrak{p}(\mathscr{A})$ such that $f = \sum_{M \in \mathscr{M}} M$.

We say in this case that f is put in the canonical disjunctive *form.*

From this theorem it follows that there are 2^{2^n} different (not equivalent) Boolean functions in $\mathfrak{b}(\mathscr{A})$. We give a sketch of proof of the theorem.

■ (1) The proposition is evidently true for all $A_i \in \mathscr{A}$, because $A_i = \sum_{i \in \pi \subset [n]} A_\pi \bar{A}_{\bar\pi}$.

(2) If $f, g \in \mathfrak{b}(\mathscr{A})$ are brought into the canonical disjunctive form, then $f \cup g$ can be brought into canonical form too, because for $f = \bigcup_{B \in \mathscr{M}} B$, $g = \bigcup_{C \in \mathscr{N}} C$, where $\mathscr{M}, \mathscr{N} \subset \mathfrak{p}(\mathscr{A})$, we have $f \cup g = \bigcup_{D \in \mathscr{M} \cup \mathscr{N}} D$.

(3) Similarly, for

$$f \cap g = (\bigcup_{B \in \mathscr{M}} B) \cap (\bigcup_{C \in \mathscr{N}} C) \overset{(*)}{=} \bigcup_{B \in \mathscr{M}, C \in \mathscr{N}} (BC) = \bigcup_{D \in \mathscr{M} \cap \mathscr{N}} D,$$

by means of [1g] (p. 3), for $(*)$.

(4) Finally, for the passage to the complement, we have:

$$\bar f = \complement(\bigcup_{B \in \mathscr{M}} B) \overset{(**)}{=} \bigcap_{B \in \mathscr{M}} \bar B = \bigcup_{C \in \mathfrak{p}(\mathscr{A}) - \mathscr{M}} C,$$

with [1i] (p. 3), for $(**)$.

(1), (2), (3), (4) make it hence possible to reduce any $f \in \mathfrak{b}(\mathscr{A})$ step by step. ■

By way of example, we show the reduction of the functions [4a]:

$$f_1 = A_1A_2 \cup \bar{A}_3 = (A_1A_2A_3 \cup A_1A_2\bar{A}_3) \cup$$
$$\cup (A_1A_2\bar{A}_3 \cup A_1\bar{A}_2\bar{A}_3 \cup \bar{A}_1A_2\bar{A}_3 \cup \bar{A}_1\bar{A}_2\bar{A}_3)$$
$$= A_1A_2A_3 \cup A_1A_2\bar{A}_3 \cup A_1\bar{A}_2\bar{A}_3 \cup \bar{A}_1A_2\bar{A}_3 \cup \bar{A}_1\bar{A}_2\bar{A}_3,$$
$$= A_1A_2A_3 + \underline{A_1A_2\bar{A}_3} + \bar{A}_1\bar{A}_2\bar{A}_3$$
$$f_2 = A_1 \cup (\overline{A_2 \cup A_3})A_3 = A_1 \cup \{(\bar{A}_2 \cup A_3) \cup \bar{A}_3\}$$
$$= A_1 \cup A_2 \cup \bar{A}_3 = \complement(\bar{A}_1\bar{A}_2A_3) = A_1A_2A_3 + \bar{A}_1A_2A_3 +$$
$$+ A_1\bar{A}_2A_3 + A_1A_2\bar{A}_3 + \bar{A}_1A_2\bar{A}_3 + A_1\bar{A}_2\bar{A}_3 + \bar{A}_1\bar{A}_2\bar{A}_3.$$

We have already met, on pp. 25 and 28, in the set $\mathfrak{P}(N)$ of subsets of N, the operations $+$ and $-$, whose definition we recall now. For $A, B, C, D \subset N$, we put:

[4f] $C = A + B \Leftrightarrow C = A \cup B, \quad A \cap B = \emptyset$

[4g] $D = A - B \Leftrightarrow A = B + D \Leftrightarrow D = A \setminus B, B \subset A.$

It follows then for the cardinalities:

[4h] $|A+B| = |A| + |B|, \qquad |A-B| = |A| - |B|$

and for the rules of computation:

[4i]
(I) $(A+B)+C = A+(B+C)$,
(II) $A+B = B+A$,
(III) $A+\emptyset = \emptyset + A = A$,
(IV) $A+\bar{A} = N$,
(V) $A(B+C) = AB+AC$,
(VI) $A-\emptyset = A$,
(VII) $A(B-C) = AB-AC$,
(VIII) $A-(B-C) = (A-B)+C$ (provided the two pairs of brackets make sense according to [4g]).

THEOREM B. *The cardinal number $|f|$ of every Boolean function $f \in b(\mathscr{A})$ can be expressed as a linear combination with integer coefficients ($\in Z$) of the cardinals of the conjunctions of $\mathscr{A}$:*

[4j] $\forall f \in b(\mathscr{A}), \quad \exists \{l_1, l_2, \ldots, l_s\} \subset Z,$
$\exists \{C_1, C_2, \ldots, C_s\} \subset c(\mathscr{A}), \qquad |f| = \sum_{1 \leq i \leq s} l_i |C_i|.$

■ According to [4e], it suffices to prove [4j] for each complete product M, because $|f| = \sum_{M \subset f} |M|$; this fact is proved in the following theorem. ■

THEOREM C. *Let $B \in b(\mathscr{A})$ be a subset of N which is the intersection of some A_i and $\bar{A}_j$:*

[4k] $B = (\bigcap_{i \in \lambda} A_i) (\bigcap_{j \in \mu} \bar{A}_j)$, where $\lambda + \mu = [r]$.

Then, the cardinal $|B|$ can be computed by performing successively the following operations:

(1) *Replace in* [4k] *the $\bar{A}_j$ by $1 - A_j$.*

(2) *Expand the new form, thus obtained, of* [4k] *into a polynomial in the variables A_i, A_j, $i \in \lambda$, $j \in \mu$, the $\cap$ being considered as* product operation.

(3) *Replace every monomial by its cardinal number and replace the monomial 1 (if it occurs) by n ($= |N|$).*

We illustrate this rule by computing the cardinal of $P\bar{Q}\bar{R}$:

$$P\bar{Q}\bar{R} \overset{(1)}{\to} P(1-Q)(1-R) \overset{(2)}{\to} P - PQ - PR + PQR \to$$
$$\overset{(3)}{\to} |P| - |PQ| - |PR| + |PQR| = |P\bar{Q}\bar{R}|.$$

■ We use $|V\bar{W}| = |V| - |VW|$, this formula is evident. Then we put $V := \bigcap_{i \in \lambda} A_i$ and $\bar{W} := \bigcap_{j \in \mu} \bar{A}_j$. Then $|B| = |V\bar{W}| = |V| - |V(\bigcup_{j \in \mu} A_j)| = |V| - |\bigcup_{j \in \mu} VA_j|$. In other words, by [1b]:

$$|B| = |V| - \sum_{j \in \mu} |VA_j| + \sum_{j_1 < j_2,\ j_1, j_2 \in \mu} |VA_{j_1}A_{j_2}| - \ldots \text{ etc.} \quad ■$$

So, in example [4a], $f_1 = (A_1A_2\bar{A}_3 \cup A_1A_2A_3) \cup \bar{A}_3 = A_1A_2A_3 + \bar{A}_3$, hence $|f_1| = n - |A_3| + |A_1A_2A_3|$. Similarly, $f_2 = \bar{A}_1 \cup (\overline{A_1 \cup A_2}) A_3 = \bar{A}_1 \cup (\bar{A}_1\bar{A}_2)A_3 = \bar{A}_1 + \bar{A}_1\bar{A}_2A_3$; hence, with the example $P\bar{Q}\bar{R}$ above, $|f_2| = |\bar{A}_1| + |A_3| - |A_1A_3| - |A_2A_3| + |A_1A_2A_3|$, or $|f_2| = n - |A_1| + |A_3| - |A_1A_3| - |A_2A_3| + |A_1A_2A_3|$. (On this section see also [*Loève, 1963], p. 44.)

4.5. THE METHOD OF RÉNYI FOR LINEAR INEQUALITIES

DEFINITION A. *Let f be a (set) function mapping a certain Boolean algebra of subsets of N, say $\mathfrak{B}$, into a set of real numbers ≥ 0: $f \in [0, \infty)^{\mathfrak{B}}$. We say that f is a measure on $(N, \mathfrak{B})$, and we denote $f \in \mathfrak{M} = \mathfrak{M}(N, \mathfrak{B})$ if and only if f is additive, in the sense that for each pair (B_1, B_2) of disjoint subsets of N ($\Leftrightarrow B_1 + B_2 \subset N$), belonging to $\mathfrak{B}$, we have:*

[5a] $f(B_1 + B_2) = f(B_1) + f(B_2)$.

The triple $(N, \mathfrak{B}, f)$ is then called a measure space.

(So $\mathfrak{B}$ is a system of subsets of N, containing $\emptyset$ and N, and closed under the operations of complementation, finite union and finite intersection, [1d], p. 2.)

Hence, for each measure f, we have $f(\emptyset) = 0$, and for all pairwise disjoint $B_1, B_2, \ldots, B_r \in \mathfrak{B}$:

[5b] $f(\sum_{i=1}^{r} B_i) = \sum_{i=1}^{r} f(B_i)$.

DEFINITION B. *The measure space* $(N, \mathfrak{B}, f)$ *is said to be a probability space, if* $f(N)=1$. *In this case* f *is called a probability measure, or probability, and will often be denoted by* P; *each set* $B\in\mathfrak{B}$ *is called an event,* N *is the certain event, mostly denoted* Ω. *Each point* $\omega\in\Omega$ *is called a sample.*

DEFINITION C. *An atom of the Boolean algebra* $\mathfrak{b}(\mathscr{A})$ *generated by* $\mathscr{A}=(A_1, A_2,\ldots, A_r)$ *(Definition A, p. 185) is a nonempty complete product (Definition B, p. 185). We denote the set of atoms of* $\mathfrak{b}(\mathscr{A})$ *by* $\mathfrak{a}(\mathscr{A})$.

THEOREM A. *A probability measure* f *on* $\mathfrak{b}(\mathscr{A})$ *is completely determined by the values* $(\geqslant 0)$ *of* f *on each atom* $C\in\mathfrak{a}(\mathscr{A})$ *(the set of values of* f *on the atoms is only subject to the restriction* $\sum_{C\in\mathfrak{a}(\mathscr{A})} f(C)=1$*).*

■ This follows from the fact that $\mathfrak{a}(\mathscr{A})$ is a partition of N, and that every subset $B\in\mathfrak{b}(\mathscr{A})$ is a union of disjoint atoms (Theorem A, p. 185). ■

THEOREM B. *Let* $\mathscr{A}=(A_1, A_2,\ldots, A_r)$ *be a system of subsets of* N, $A_i\subset N$, $i\in[r]$, *and let* $\mathfrak{M}$ *be the set of measures on the Boolean algebra* $\mathfrak{b}(\mathscr{A})$ *generated by* $\mathscr{A}$. *Let* $\mathfrak{M}^*$ *be the subset of* $\mathfrak{M}$ *consisting of the measures* g *which are zero on all atoms* C *of* $\mathfrak{a}(\mathscr{A})$ *except one,* C_0, *called supporting atom, for which* $g(C_0)=1$. *(Here,* C_0 *runs through* $\mathfrak{a}(\mathscr{A})$.*) Then for every sequence of* l *real numbers say* $(b_1, b_2,\ldots, b_l)$, *and every sequence of* l *subsets taken from* $\mathfrak{b}(\mathscr{A})$, *say* $(B_1, B_2,\ldots, B_l)$, *the following conditions* [5c] *and* [5d] *are equivalent:*

[5c] For all $f\in\mathfrak{M}$, $\quad \sum_{1\leqslant k\leqslant l} b_k f(B_k)\geqslant 0$

[5d] For all $g\in\mathfrak{M}^*$, $\quad \sum_{1\leqslant k\leqslant l} b_k g(B_k)\geqslant 0$.

([Rényi, 1958] and [*, 1966], pp. 30–33. See also [Galambos, 1966]. For a generalization to certain quadratic and cubic, etc., inequalities, see [Galambos, Rényi, 1968].)

■ The fact that [5c] implies [5d] follows from the fact that $\mathfrak{M}^*\subset\mathfrak{M}$. Conversely, let $g\in\mathfrak{M}^*$, so there exists a $C_0\in\mathfrak{a}(\mathscr{A})$ such that:

[5e] $\quad g(C_0)=1, \quad\text{and}\quad g(C)=0 \quad\text{if}\quad C\in\mathfrak{a}(\mathscr{A}),\quad C\neq C_0.$

Now, according to [5d], with $\mathfrak{a}:=\mathfrak{a}(\mathscr{A})$ for short, [5a] for (*), a permutation of the summation order for (**) and [5e] for (***):

$$0\leqslant \sum_{1\leqslant k\leqslant l} b_k g(B_k) = \sum_{1\leqslant k\leqslant l} b_k g\Big(\sum_{\substack{C\subset B_k\\ C\in\mathfrak{a}}} C\Big)$$
$$\overset{(*)}{=} \sum_{1\leqslant k\leqslant l} b_k\Big\{\sum_{C\subset B_k} g(C)\Big\}$$
$$\overset{(**)}{=} \sum_{C\in\mathfrak{a}} g(C)\Big\{\sum_{\substack{C\subset B_k\\ 1\leqslant k\leqslant l}} b_k\Big\} \overset{(***)}{=} \sum_{C_0\subset B_k} b_k .$$

Because the measure $g\in\mathfrak{M}^*$ is arbitrary, it follows that for each atom $C(=C_0$ from above), we have:

[5f] $\quad \sum_{C\subset B_k} b_k\geqslant 0$

Let us now consider [5c]. We can compute by the same way, now using [5f] for (+):

$$\sum_{1\leqslant k\leqslant l} b_k f(B_k) = \sum_{1\leqslant k\leqslant l} b_k\Big(\sum_{\substack{C\subset B_k\\ C\in\mathfrak{a}}} f(C)\Big)$$
$$= \sum_{C\in\mathfrak{a}} f(C)\Big\{\sum_{C\subset B_k} b_k\Big\}\overset{(+)}{\geqslant} 0. \quad ■$$

THEOREM C. *Notations as in Theorem B. The conditions* [5c] *and* [5d] *remain equivalent if all '*$\geqslant 0$*' signs are simultaneously replaced by '*$\leqslant 0$*' or by '*$=0$*'.*

■ In the first case, replace the sequence $(b_1, b_2,\ldots, b_l)$ of Theorem B by $(-b_1, -b_2,\ldots, -b_l)$. In the second case, observe that $x=0 \Leftrightarrow x\geqslant 0$ and $x\leqslant 0$. ■

Examples of applications of Rényi's method follow now.

4.5. POINCARÉ FORMULA

The method of the preceding section will enable us to show very quickly various equalities and inequalities concerning measures f associated with a finite system $(A_1, A_2,\ldots, A_r)$ of subsets of N.

With every measure f on $(N, \mathfrak{b}(\mathscr{A}))$ [Definition A, pp. 185 and

1.9) and every integer $k\in[p]$ we associate, as in [1c, p. 177] (using the notation [1c], p. 177, for (*)):

$$\text{[6a]}\qquad S_k=S_k(\mathscr{A})=S_k(f,\mathscr{A})=\sum^{(*)} f(A_{i_1}A_{i_2}\dots A_{i_k}):=$$
$$:=\sum_{1\leqslant i_1<i_2<\dots<i_k\leqslant p} f(A_{i_1}A_{i_2}\dots A_{i_k})\overset{(*)}{=}\sum_{K\in\mathfrak{P}_k[p]} f(A_K)$$
$$S_0:=f(N).$$

THEOREM. *For every measure $f\in\mathfrak{M}(N,\mathfrak{b}(\mathscr{A}))$, where $\mathscr{A}=(A_1,A_2,\dots,A_p)$, $A_i\subseteq N$, $i\in[p]$, the S_k being defined by [6a], we have:*

$$\text{[6b]}\qquad f(A_1\cup A_2\cup\dots\cup A_p)\overset{(*)}{=}\sum_{K\in\mathfrak{P}[p]}(-1)^{|K|-1}f(A_K)=$$
$$\overset{(*)}{=}\sum_{1\leqslant k\leqslant p}(-1)^{k-1}S_k$$

$$\text{[6c]}\qquad f(\bar{A}_1\bar{A}_2\dots\bar{A}_p)=\sum_{K\in\mathfrak{P}[p]}(-1)^{|K|}f(A_K)=\sum_{0\leqslant k\leqslant p}(-1)^kS_k.$$

In the case that f is a probability, [6b] is often called the "*Poincaré formula*". If f stands for the *cardinal*, $f(B)=|B|$, we obtain [1b, c, d], p. 177.

■ [6c] follows from the application of [6b] to $f(\bar{A}_1\bar{A}_2\dots)=$ $=f(\complement(A_1\cup A_2\cup\cdots))=f(N)-f(A_1\cup A_2\cup\cdots)$. Equality [6b(**)] follows from collecting in the above summation all terms with K, $|K|=k$. Proving [6b(*)] is equivalent to proving it for all $g\in\mathfrak{M}^*=\mathfrak{M}^*(N,\mathfrak{b}(\mathscr{A}))$, according to Theorem B (p. 190). When C_g denotes the support (atom) of g, we let $L(\subseteq[p])$ be the set of indices i such that $C_g\subseteq A_i$ and $l:=|L|$. If $L=\emptyset$, all terms of [6b] are zero. If $l\geqslant 1$, the first member $g(A_1\cup A_2\cup\cdots)$ of [6b] equals 1. On the other hand:

$$\text{[6d]}\qquad g(A_K)=g\Big(\bigcap_{i\in K}A_i\Big)=\begin{cases}1 & \text{if } K\subseteq L\\ 0 & \text{otherwise.}\end{cases}$$

The second member of [6b] is hence equal to 1, too, since with [6d] for (*):

$$\sum_{K\in\mathfrak{P}[p]}(-1)^{|K|-1}g(A_K)\overset{(*)}{=}\sum_{K\in\mathfrak{P}(L)}(-1)^{|K|-1}$$
$$=\sum_{1\leqslant k\leqslant l}(-1)^{k-1}\binom{l}{k}=1-(1-1)^l=1.\ ■$$

Example: Euler function. For any integer $n\geqslant 1$, let $\Phi=\Phi(n)$ be the set of positive integers x which do not exceed n, and relatively prime with respect to n, $1\leqslant x\leqslant n$, GCD $(x,n)=1$. The number $\varphi(n)=|\Phi|$ is called the *Euler function* of n and we are going to compute it now. Let the decomposition of n into prime factors be $n=p_1^{\alpha_1}p_2^{\alpha_2}\dots p_r^{\alpha_r}$ and let M_i be the set of multiples of p_i which are smaller than or equal to n. Clearly, $\Phi=\bar{M}_1\bar{M}_2\dots\bar{M}_r$. Hence, for each measure f on $[n]$, we get by [6c]: $f(\Phi)=f([n])-\sum^{(*)}f(M_i)+\sum^{(*)}f(M_iM_j)-\cdots$. First we take for f the cardinal number function. Then $f([n])=n$, $f(M_i)=n/p_i$, $f(M_iM_j)=$ $=n/p_ip_j$, ..., from which we obtain, after an evident factorization:

$$\text{[6e]}\qquad \varphi(n)=n\Big(1-\frac{1}{p_1}\Big)\Big(1-\frac{1}{p_2}\Big)\cdots\Big(1-\frac{1}{p_r}\Big).$$

If we had defined f by $f(X)=\sum_{x\in X}x$, where $X\subseteq[n]$, then we would have found $f(M_i)=p_i+2p_i+\cdots+(n/p_i)p_i=n^2/2p_i+n/2$, $f(M_iM_j)=p_ip_j+$ $+2p_ip_j+\cdots+(n/p_ip_j)p_ip_j=n^2/2p_ip_j+n/2$, ...; hence, after simplifications: $f(\Phi)=\sum_{x\in\Phi}x=(n/2)\,\varphi(n)$. Here is a table for $\varphi(n)$:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\varphi(n)$	1	1	2	2	4	2	6	4	6	4	10	4	12	6	8

n	16	17	18	19	20	21	22	23	24	25	26	27	28
$\varphi(n)$	8	16	6	18	8	12	10	22	8	20	12	18	12

4.7. BONFERRONI INEQUALITIES

DEFINITION. *Let R be an alternating sum of $a_k\geqslant 0$, $k\in[r]$:*

$$\text{[7a]}\qquad R=\sum_{1\leqslant k\leqslant r}(-1)^{k-1}a_k=a_1-a_2+\cdots+(-1)^{r-1}a_r.$$

We say that [7a] satisfies the alternating inequalities, if and only if $(-1)^k\{R+\sum_{i=1}^{k}(-1)^i a_i\}\geqslant 0$ for all $k\in[r]$. In other words:

$$\text{[7b]}\qquad R\leqslant a_1,\qquad R\geqslant a_1-a_2,\qquad R\leqslant a_1-a_2+a_3,\ \dots$$

THEOREM ([Bonferroni, 1936]). *Let the S_k be defined by [6a] (p. 192); then for all measures $f\in\mathfrak{M}(N,\mathfrak{b}(\mathscr{A}))$, the sum $\sum_{k=1}^{p}(-1)^{k-1}S_k$, introduced in [6b] (p. 192), satisfies the alternating inequalities. Hence, for each*

$k\in[p]$, we have:

$$[7c]\qquad (-1)^k\{f(A_1\cup A_2\cup\cdots\cup A_p)+\sum_{1\le j\le k}(-1)^j S_j\}\geq 0.$$

Quite similarly, with [6e], we have:

$$[7c']\qquad (-1)^{k-1}\{f(\bar A_1\bar A_2\ldots\bar A_p)-\sum_{0\le j\le k}(-1)^j S_j\}\geq 0.$$

Particularly, for $f(W)=|W|=$ the cardinal of W, we obtain (cf. [1c], p. 177):

$$[7d]\qquad |A_1\cup\cdots\cup A_p|\leq\sum_{1\le i\le p}|A_i|\quad\text{(Boole inequality)}$$

$$|A_1\cup\cdots\cup A_p|\geq\sum_{1\le i\le p}|A_i|-\sum_{1\le i<j\le p}|A_iA_j|,\quad\text{etc.,}$$

and the analogous inequalities in the case that $f=P$ is a probability.

■ According to Theorem B (p. 190) it suffices to prove [7c] for an arbitrary measure $g\in\mathfrak{M}^*$. Let λ have the sense given in the proof of the Theorem, on p. 192, then the first member of [7c] is evidently equal to 0 if $\lambda=\emptyset$. Otherwise, we get, with $l:=|\lambda|\geq 1$, and [6d] (p. 192, where x is replaced by g) for (*):

$$g(A_1\cup A_2\cup\cdots\cup A_p)+\sum_{1\le j\le k}(-1)^j S_j(g)$$

$$=g(A_1\cup\cdots\cup A_p)+\sum_{K\in\mathfrak{P}(p),\,1\le|K|\le k}(-1)^{|K|}g(A_K)$$

$$\overset{(*)}{=}1+\sum_{K\in\mathfrak{P}(\lambda),\,1\le|K|\le k}(-1)^{|K|}g(A_K)$$

$$=1-\binom{l}{1}+\binom{l}{2}-\cdots+(-1)^k\binom{l}{k}:=W_k.$$

Now, by applying the Taylor formula of order k in $x=0$ to the function $(1-x)^l$, $k\leq l-1$, we get for all $x\in\mathbb{R}$, $0<\theta(x)<1$:

$$[7e]\qquad (1-x)^l=1-\binom{l}{1}x+\cdots+(-1)^k\binom{l}{k}x^k+$$

$$+(-1)^{k+1}\binom{l}{k+1}x^{k+1}(1-x\theta(x))^{l-k-1}.$$

If we put $x=1$ in [7e], we find $(-1)^k W_k=\binom{l}{k+1}(1-\theta(1))^{l-k-1}\geq 0$, in other words, [7c] for all $g\in\mathfrak{M}^*$. ■

4.8. FORMULAS OF CH. JORDAN

THEOREM A ([Charles Jordan, 1926, 1927, 1934, 1939]). *Let $N_r(\mathscr{A})$ stand for the set of points of N that are covered by exactly r subsets of the system $\mathscr{A}=\{A_1, A_2, \ldots, A_p\}$, then we have for every measure $f\in\mathfrak{M}(N, \mathfrak{B}(\mathscr{A}))$:*

$$[8a]\qquad f(N_r(\mathscr{A}))=\sum_{K\in\mathfrak{P}(p),\,|K|\ge r}(-1)^{|K|-r}\binom{|K|}{r}f(A_K)$$

$$=\sum_{r\le k\le p}(-1)^{k-r}\binom{k}{r}S_k,$$

where the S_k are defined by [6a]. Moreover, [8a] satisfies the alternating inequalities.

For $r=0$ we have a formula analogous to [6e] (p. 192).

■ We use Theorem B (p. 190) once more. For all $g\in\mathfrak{M}^*$, with supporting atom C_λ contained in the A_i such that $i\in\lambda\ (\subset[p])$, $l:=|\lambda|$, we have evidently:

$$[8b]\qquad g(N_r(\mathscr{A}))=0\quad\text{if}\quad r\neq l,\quad\text{and}\quad =1\quad\text{if}\quad r=l.$$

Now the second member of [8a], with f replaced by g, and [6d] (p. 192) for (*), can be written:

$$\sum_{K\in\mathfrak{P}(p)}(-1)^{|K|-r}\binom{|K|}{r}g(A_K)=$$

$$\overset{(*)}{=}\sum_{K\in\mathfrak{P}(\lambda)}(-1)^{|K|-r}\binom{|K|}{r}g(A_K)$$

$$=\sum_{r\le k\le l}(-1)^{k-r}\binom{k}{r}\binom{l}{k}$$

$$=\binom{l}{r}\sum_k(-1)^{k-r}\binom{l-r}{k-r}=\binom{l}{r}0^{l-r}.$$

which is indeed equal to [8b]. The alternating inequalities for [8a] follow from the fact that they hold for $\sum_k (-1)^{k-r}\binom{l-r}{k-r}$, according to [7c] (p. 194). ■ (The interested reader is referred to [*Fréchet, 1940, 1943], as well as to [Takács, 1967], which has a very extensive bibliography.)

We can prove by a similar method:

Theorem B. *Let $N_{\geq r}(\mathcal{A})$ stand for the set of points of N that are covered by at least r subsets of $\mathcal{A}$, then we have:*

$$[8c]\qquad f(N_{\geq r}(\mathcal{A})) = \sum_{K\in\mathfrak{P}_{\geq r}(M)} (-1)^{|K|-r}\binom{|K|-1}{r-1} f(A_K) = \sum_{r\leq k\leq n} (-1)^{k-r}\binom{k-1}{r-1} S_k,$$

with the alternating inequalities.

4.9. PERMANENTS

Definition. *Let $B := [b_{i,j}]_{i\in[m],\, j\in[n]}$ be a rectangular matrix with m rows and n columns, $m\leq n$, with coefficients b_{ij} in a commutative ring Ω. The permanent of B, denoted by per B, equals, by definition:*

$$[9a]\qquad \operatorname{per} B = \sum_{\varphi\in\mathfrak{A}_m[n]} b_{1,\varphi(1)}\, b_{2,\varphi(2)} \cdots b_{m,\varphi(m)},$$

where the summation is taken over all m-arrangements of $[n]$ (p. 6). (For the main properties and an extensive bibliography see [Marcus, Minc, 1965].)

For example, $\operatorname{per}\begin{pmatrix} 2 & 3 & 1\\ 5 & 0 & 4\end{pmatrix} = 2.0 + 5.3 + 2.4 + 5.1 + 3.4 + 0.1 = 40.$

Hence there are $(n)_m$ terms in the summation [9a]. If $m=n$, the terms of $\operatorname{per}(B)$ are, up to sign, those of $\det(B)$, and for the permanent there are properties similar to those of the determinants; however, $\operatorname{per}(AB) \neq \operatorname{per}(A).\operatorname{per}(B)$, in general.

For each matrix $A := [a_{i,j}]_{i\in[r],\, j\in[s]}$, $a_{i,j}\in\Omega$, let $w(A)$ be the product

of the r sums of elements of each row of A:

$$[9b]\qquad w(A) := \prod_{i=1}^{r} \sum_{j=1}^{s} a_{i,j},$$

and for every subset $\lambda\subset[s]$ let $A(\lambda)$ be the matrix obtained by keeping in A precisely those *columns* whose index *belongs* to λ. For example, if $A = \begin{pmatrix} 1 & 3 & 2 & 3\\ -2 & 4 & 1 & 0\end{pmatrix}$, then $w(A) = 9\times 3 = 27$ and $A(\{1,3\}) = \begin{pmatrix} 1 & 2\\ -2 & 1\end{pmatrix}$.

Theorem (*Ryser formula*, [*Ryser, p. 26]). *With the above notations, and $w(B(\emptyset)) = 0$, $\operatorname{per}(B)$ is also equal to:*

$$[9c]\qquad \sum_{\lambda\subset[n]} (-1)^{m-|\lambda|}\binom{n-|\lambda|}{n-m} w(B(\lambda)),$$

that is to say

$$[9d]\qquad \sum_{\lambda\in\mathfrak{P}_m[n]} w(B(\lambda)) - \binom{n-m+1}{n-m}\sum_{\lambda\in\mathfrak{P}_{m-1}[n]} w(B(\lambda)) + \cdots + (-1)^{m-1}\binom{n-1}{n-m}\sum_{\lambda\in\mathfrak{P}_1[n]} w(B(\lambda)).$$

Particularly, for a square matrix, $m=n$:

$$[9e]\qquad \operatorname{per} B = \sum_{\lambda\subset[n]} (-1)^{n-|\lambda|} w(B(\lambda)) = \sum_{k\in[n]} (-1)^{n-k} \sum_{\lambda\in\mathfrak{P}_k[n]} w(B(\lambda)).$$

■ We use [5a], p. 193. The role of N is played here by the set of maps of $[m]$ into $[n]$, so $N = [n]^{[m]}$ (caution! $|N| = n^m$), with its system $\mathcal{A} = \{A_1, A_2, \ldots\}$:

$$[9f]\qquad A_i := \{\varphi \mid \varphi\in[n]^{[m]};\ \exists j\in[m],\ \varphi(j) = i\},\quad i\in[n].$$

Now we suppose first that all $b_{i,j}$ are real nonnegative. We define the measure f for each subset $X\subset[n]^{[m]}$ by:

$$[9g]\qquad f(X) := \sum_{\varphi\in X} f(\varphi),\quad \text{where}\quad f(\varphi) := \prod_{i=1}^{m} b_{i,\varphi(i)}.$$

Now φ is injective ($\in\mathfrak{A}_m[n]$) if and only if the image of $[m]$ under φ has cardinality m, in other words, $\varphi\in N_m(\mathcal{A})$ in the notation of Theorem A.

(p. 195). Hence, by [9g]:

[9h] $\operatorname{per} B = f(K_m(\mathscr{A}))$.

To this expression we will apply now [6a] (p. 195). Let $\mathscr{A} := \{l_1, l_2, \ldots, l_n\} = [n]$. Then we have:

$$f(A_x) := \sum_{\varphi \in \mathfrak{F}_m(A_x)} b_{1,\varphi(1)} b_{2,\varphi(2)} \cdots = \sum_{\varphi \in \mathfrak{F}} b_{1,\varphi(1)} b_{2,\varphi(2)} \cdots,$$

where $\mathfrak{F}$ stands for the set of maps of $[m]$ into $[n]-x$. Hence, by Theorem A (p. 127), and the notation of [9b]:

[9i] $f(A_x) = \pi(R([n]-x))$.

Then [9c] follows by putting $X := [n]-x$ in [9i] and [6a] (p. 195). Since [9c] is true for all $b_{i,j} \geqslant 0$, it is also true in a commutative ring, since the term-by-term expansion [9a] is the same in both cases. ■ (For other expressions of per B, see [*Cartier, Foata, 1969], p. 76, [Crapo, 1968], [Wilf, 1968a, b]).

If per B can be directly computed, then [9c] gives, together with [9c, d, e], a 'remarkable' identity. For example, when B is the square matrix of order n consisting entirely of 1, $b_{i,j}=1$, then evidently per $B = n!$; hence by [9e]: $n! = \sum_{j=1}^{n} (-1)^{n-j} \binom{n}{j} j^n$. Thus we find back the evident property $S(n,n)=1$ for the Stirling numbers ([1b] p. 204). If we take next $b_{i,j}=2^{j-1}$, we find $2^{\binom{n}{2}} n! = \sum (-1)^{n-D(j)} j^n$, where $1 \leqslant j \leqslant 2^n-1$, and where $D(j)$ stands for the number of digits 1 in the binary form (= base 2) of j. Finally, if all b_{ij} equal 0, except $b_{1,1}=b_{2,2}=\cdots=b_{n,n}=x$ and $b_{1,2}=b_{2,3}=\cdots=b_{n-1,n}=b_{n,1}=y$, we find, using [9b] (p. 24):

$$x^n + y^n = \sum_{k \leqslant n/2} (-1)^k \frac{n}{n-k} \binom{n-k}{k} (xy)^k (x+y)^{n-2k},$$

to be compared with Exercise 1, p. 155.

SUPPLEMENT AND EXERCISES

1. *Variegated words.* Using 2 letters a_1, 2 letters $a_2, \ldots,$ 2 letters a_n, how many words of length $2n$ can be formed in which no two identical letters are adjacent? (For instance, for $n=3$, the word $a_1a_2a_3a_1a_2a_3$.) [*Hint.* When A_i stands for the set of words in which the two letters a_i are adjacent, then the required number is equal to $N(\bar{A}_1\bar{A}_2\ldots\bar{A}_n)$.] Now generalize. (Cf. Exercise 1, p. 219, and Exercise 21 (3), p. 265.)

2. *Sums of the type of the Euler function.* If in the following the summation is taken over all integers $x \leqslant n$ which are prime relatively to n, $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$, then show that $\sum x^2 = (n^2/3)\varphi(n) + (-1)^r (1/6) p_1 \ldots p_r \varphi(n)$. Generalize to $\sum x^k$.

3. *Jordan function.* This is the following double sequence:

$$J_k(n) := n^k \prod_{p|n} (1 - p^{-k}),$$

p is a prime number, and where $p \mid n$ means 'p divides n'. It is a generalization of the Euler function ([6c] p. 193): $J_1(n) = \varphi(n)$. For any integer $k \geqslant 1$, show that $J_k(n)$ is equal to the number of $(k+1)$-tuples $(x_1, x_2, \ldots, x_k, n)$ of integers $x_i \in [n]$, $i \in [k]$, whose GCD equals 1. Show that $\sum_{d|n} J_k(d) = n^k$ and deduce from this the Lambert GF (Exercise 16, p. 161) $\sum_{n \geqslant 1} J_k(n)\, t^n (1-t^n)^{-1} = A_k(t)(1-t)^{-k-1}$, where the $A_k(t)$ are the Eulerian polynomials of p. 244.

4. *Other properties of the number $d(n)$ of derangements.* (1) We have $d(n) = \Delta^n 0!$, Δ being the difference operator (p. 13). (2) $f := \sum d(n)\, t^n$ satisfies the differential equation $(t^2+t^3)f' + (t^2-1)f + 1 = 0$. Use this to prove $f = -t^{-1}\exp(-t^{-1}) \int \exp(t^{-1})(t+t^2)^{-1}\, dt$, formally. (3) The number $d_k(n)$ of permutations of $[n]$ with k fixed points (13f, p. 231) has as $\sum_{k,n \geqslant 0} d_k(n)\, u^k t^n/n! = (1-t)^{-1}\exp(-t(1-u))$.

***5.** *Other properties of the reduced ménages numbers $\mu(n)$.* (1) The following recurrence relation holds: $(n-2)\mu(n) = n(n-2)\mu(n-1) + n\mu(n-2) + 4(-1)^{n+1}$ ([*Lucas, 1891], p. 495). (2) When n tends to infinity, $\mu(n) \sim n!\, e^{-2}$. (3) $n! = \sum_{k=0}^{n} \binom{2n-k}{k} \mu(n-k)$, $\mu(0) := 1$, $\mu(1) := -1$ (Riordan). (4) $\mu(n) = [n\, e^{-2} \sum (-1)^k (n-k-1)!/k!]$, where $0 \leqslant k \leqslant (n-1)/2$, with the notation [6f] (p. 110) (Schöbe). (5) $\sum_{n \geqslant 0} \mu(n)\, t^n =$

$=(t^2-1)t^{-4}\exp(-t-t^{-1})\int t^2(t+1)^{-2}\exp(t+t^{-1})\,dt$, formally ([Cayley, 1874b]).

6. *Random integers.* Repetitions being allowed, n integers ≥ 1 are independently drawn at random, say $\omega_1, \omega_2, \dots, \omega_n$. What is the probability that the product $\pi_n := \omega_1\omega_2\dots\omega_n$ has last digit (the number of units, hence) equal to 5? More generally, compute the probability that a given integer $k \geq 1$ divides π_n.

7. *Knock-out tournaments.* A set of 2^s players of equal strength is at random arranged into 2^{s-1} disjoint pairs. They play one round, and 2^{s-1} are eliminated. The same operation is repeated with the remaining 2^{s-1} players, until a champion remains after the s-th round. Show that the probability that a player takes part in exactly k rounds equals 2^{-k} for $1 \leq k \leq s-1$ and 2^{-k+1} if $k=s$. ([Narayana, 1968], and [Narayana, Zidek, 1968] for other results and generalizations. See also [*André, 1900].)

8. *A determinant.* Let A be a square matrix of order n, $A := [a_{i,j}]$, $i,j \in [n]$, where the $a_{i,j}$ belong to a commutative ring Ω. For each subset $x \subset [n]$, let $D(x)$ be the determinant of the matrix that is obtained by deleting from A all rows and columns whose index does not belong to x, $D(\emptyset)=1$. Then, for $x_1, x_2, \dots \in \Omega$:

$$\begin{vmatrix} a_{1,1}+x_1 & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2}+x_2 & \dots & a_{2,n} \\ \vdots & & & \\ a_{n,1} & a_{n,2} & \dots & a_{n,n}+x_n \end{vmatrix} = \sum_{x \subset [n]} \{D(x)\prod_{i \notin x} x_i\}.$$

9. *Inversion of the Jordan formula.* In [8a] (p. 195) we put $\mathrm{T}_r := f(N(\mathscr{A}))$, $\mathrm{T}_r = \sum_k (-1)^{k+r}\binom{k}{r}\mathrm{S}_k$. Now show that $\mathrm{S}_r = \sum_k \binom{k}{r}\mathrm{T}_k$.

10. *Inequalities satisfied by the* S_k. Show that the S_k, as defined by [6a] (p. 192), satisfy the *Fréchet* inequalities ([*Fréchet, 1940]):

$$\mathrm{S}_k\Big/\binom{p}{k} \leq \mathrm{S}_{k-1}\Big/\binom{p}{k-1}, \quad k \in [p].$$

and the *Gumbel* inequalities, $k \in [p-1]$:

$$\left\{\binom{p}{k-1}-\mathrm{S}_{k-1}\right\}\Big/\binom{p-1}{k} \leq \left\{\binom{p}{k}-\mathrm{S}_k\right\}\Big/\binom{p-1}{k-1}.$$

11. *The number of systems of distinct representatives.* Let $\mathscr{B} := (B_1, B_2, \dots, B_m)$ be a system of not necessarily distinct blocks of $[n]$, $B_i \subset [n] := \{1, 2, \dots, n\}$, $1 \leq m \leq n$, and let $\mathbf{B}=[b_{i,j}]$ be the incidence matrix of $\mathscr{B}$ defined by $b_{i,j}=1$ if $j \in B_i$ and $=0$ otherwise, $i \in [m]$, $j \in [n]$. Show that the number of systems of distinct representatives (Exercise 32, p. 300) of $\mathscr{B}$ equals per$(\mathbf{B})$.

12. *Permanent of a stochastic matrix.* Let $A := [a_{i,j}]$ be a $n \times n$ square *doubly stochastic* matrix. This means:

$$a_{i,j} \geq 0, \quad \sum_{j=1}^{n} a_{i,j} = 1, \quad \sum_{i=1}^{n} a_{i,j} = 1, \quad i,j \in [n].$$

Let n boxes contain each a ball. At a certain moment, each ball jumps out of its box, and falls back into a random box (perhaps the same) such that the ball from box i goes to box j with a probability of $a_{i,j}$, $i,j \in [n]$. Then per A represents the probability that after the transfer there still is one ball in each box.

13. *The number of permutations with forbidden positions.* Let I stand for the $n \times n$ unit matrix, and let J be the $n \times n$ matrix, all whose entries equal 1. Then show that per$(J-I) = d(n)$, the number of derangements of $[n]$ (p. 180). Use this to obtain (by [8e] p. 197):

$$d(n) = \sum_{r=0}^{n-1}(-1)^r\binom{n}{r}(n-r)^r(n-r-1)^{n-r}.$$

More generally, let $\mathfrak{B}$ be a relation in $[n]$, $\mathfrak{B} \subset [n] \times [n]$, and let $\mathfrak{S}_{\mathfrak{B}}(n)$ be the set of permutations σ of $[n]$ such that $(i, \sigma(i)) \in \mathfrak{B}$. Let also $B = [b_{i,j}]$ be the $n \times n$ square matrix such that $b_{i,j} = 1$ for $(i,j) \in \mathfrak{B}$, and $=0$ otherwise. Then $|\mathfrak{S}_{\mathfrak{B}}(n)| = \mathrm{per}(B)$. (There is in [*Riordan, 1958], pp. 163–237, a very complete treatise on this subject. See also [Foata, Schützenberger, 1970].)

14. *Vector spaces.* Let $A_1, A_2, \dots$ be finite-dimensional vector subspaces

with dimensions $\delta(A_1), \delta(A_2), \ldots$. We denote A_1A_2 for $A_1 \cap A_2$. Then (1) $\delta(A_1+A_2)=\delta(A_1)+\delta(A_2)-\delta(A_1A_2)$, where A_1+A_2 stands for the subspace spanned by $A_1 \cup A_2$. (2) $\delta(A_1+A_2+A_3)\leqslant\delta(A_1)+\delta(A_2)+\delta(A_3)-\delta(A_2A_3)-\delta(A_3A_1)-\delta(A_1A_2)+\delta(A_1A_2A_3)$. (3) This inequality cannot be generalized to more than three subspaces; but we always have: $\delta(A_1A_2\ldots A_n)\leqslant\delta(\sum_{i=1}^n A_i)\leqslant\sum_{i=1}^n\delta(A_i)$.

***15.** *Möbius function.* Let P be a partially ordered set, in other words, there is an order relation $\leqslant$ given on P (Definition D, p. 59). Moreover, P is supposed to be *locally finite*, in the sense that each *segment* $[x,y]:=\{u \mid x\leqslant u\leqslant y\}$ is finite. A stands for the set of functions $f(x,y)$, $x, y\in P$, real-valued, which are zero if $x\not\leqslant y$ ($\Leftrightarrow$ not $x\leqslant y$). (1) We define the (convolution) product h of f by g, denoted by $h=f*g$, by:

$$h(x,y):=\sum_{x\leqslant u\leqslant y} f(x,u)\,g(u,y).$$

Show that with this multiplication, A becomes a group, with unit element δ defined by $\delta(x,y):=1$ for $x=y$, and $:=0$ otherwise. (2) The *zeta function* ζ of P is such that $\zeta(x,y):=1$ if $x\leqslant y$ and $:=0$ otherwise. The inverse μ of ζ, which satisfies $\mu*\zeta=\zeta*\mu=\delta$, is called the *Möbius function* of P. If we suppose that P has a universal lower bound denoted by 0, verify the following 'Möbius inversion formula' for $f, g\in A$:

$$[*]\qquad g(x)=\sum_{y\leqslant x} f(y)\Leftrightarrow f(x)=\sum_{y\leqslant x} g(y)\,\mu(y,x).$$

(3) Let $P:=\{1,2,3,\ldots\}$ be ordered by divisibility: $x\leqslant y\Leftrightarrow x\mid y\Leftrightarrow x$ divides y. Show that $\mu(x,x)=1$, $\mu(x,y)=(-1)^k$ if x divides y and the quotient equals $p_1p_2\ldots p_k$, where the prime numbers p_i are all *different*; $\mu(x,y)=0$ in the other case. Hence $\mu(x,y)=\bar{\mu}(y\mid x)$, where $\bar{\mu}(n)$ is the ordinary arithmetical Möbius function (Exercise 16, p. 161). What does the inversion formula (*) give us in this case? (4) We order the set $P:=\mathfrak{P}(N)$ of subsets of a finite set N by inclusion. Then $\mu(x,y)=(-1)^{|y|-|x|}$ if $x\leqslant y\Leftrightarrow(x\subset y)$. What does (*) give in this case? (5) Let P now stand for the set of *partitions* of a finite set N ordered as in Exercise 3 (p. 220). Then, for $x\leqslant y$ with $y=\{B_1, B_2,\ldots, B_k\}$, $B_1+B_2+\cdots+B_k=N$, we have $\mu(x,y)=(-1)^{|x|+|y|}(n_1-1)!(n_2-1)!\ldots(n_k-1)!$, where n_i is the number of blocks of x contained in B_i, $i\in[k]$. (This formula is due to [Schützenberger, 1954]. For a recent study of all

these questions see [Rota, 1964b] and [*Cartier, Foata, 1969], pp. 19–23. See also [Weisner, 1935], [Mullin, Rota, 1970], [Crapo, 1966, 1968], [Smith, 1967, 1969].)

***16.** *Jordan and Bonferroni formulas in more variables.* Let $A_1, A_2,\ldots, A_r$ and $B_1, B_2,\ldots, B_s$ be subsets of N, and let $N_{p,q}$ be the set of points of N belonging to p sets A_i and to q sets B_j. For each measure f on N, we put $S_{k,l}:=\sum f(A_KB_L)$ where $K\in\mathfrak{P}_k[r]$ and $L\in\mathfrak{P}_l[s]$, with notation [1e] of p. 177.

$$(1)\qquad f(N_{p,q})=\sum_{k=p}^{r}\sum_{l=q}^{s}(-1)^{k+l+p+q}\binom{k}{p}\binom{l}{q}S_{k,l}.$$

(2) With a notation analogous to that of Theorem B (p. 196):

$$f(N_{\geqslant p,\geqslant q})=\sum_{k=p}^{r}\sum_{l=q}^{s}(-1)^{k+l+p+q}\binom{k-1}{p-1}\binom{l-1}{q-1}S_{k,l}.$$

(3) With respect to the first summations in (1) and (2) the alternating inequalities hold ([Meyer, 1969]).

(4) Generalize to more than two systems of subsets of N.

***17.** *A beautiful determinant.* Let (i,j) be the GCD of the integers i and j, and let $\varphi(n)$ be the Euler function (p. 193). Show that:

$$\begin{vmatrix}(1,1) & (1,2) & \ldots & (1,n)\\ (2,1) & (2,2) & \ldots & (2,n)\\ \vdots & \vdots & & \vdots\\ (n,1) & (n,2) & \ldots & (n,n)\end{vmatrix}=\varphi(1)\,\varphi(2)\ldots\varphi(n)$$

([Smith, 1875], [Catalan, 1878]).

More generally, if we replace in the preceding every (i,j) by $(i,j)^r$, then the determinant equals $\prod_{k=1}^{n} J_r(k)$, where $J_r(k)$ is the Jordan function of Exercise 3 (p. 199).

STIRLING NUMBERS

Let us give a survey of the three most frequently occurring notations: numbers of the first kind $= s(n,k)$ (Riordan, and also this book, ...) $= S_n^k$ (Jordan, Mitrinović, ...) $= (-1)^{n-k} S_1(n-1, n-k)$ (Gould, Hansen, ...); numbers of the second kind $= S(n,k) = \mathfrak{S}_n^k = S_2(k, n-k)$.

5.1. STIRLING NUMBERS OF THE SECOND KIND $S(n,k)$ AND PARTITIONS OF SETS

DEFINITION A. *The number $S(n,k)$ of k-partitions (partitions in k blocks, Definition C, p. 30) is called Stirling number of the second kind. Hence $S(n,k)>0$ for $1 \leqslant k \leqslant n$ and*

[1a] $\quad S(n,k)=0 \quad \text{if} \quad 1 \leqslant n < k.$

We put $S(0,0)=1$ and $S(0,k)=0$ for $k \geqslant 1$.

In other words, $S(n,k)$ is the number of *equivalence relations with k classes on N*. It is also the number of distributions of n distinct balls into k indistinguishable boxes (the order of the boxes does not count) *such that no box is empty.*

On p. 206 we will prove that the $S(n,k)$ are indeed the numbers previously introduced on p. 50.

THEOREM A. *The Stirling number of the second kind $S(n,k)$ equals:*

$$[1b] \qquad S(n,k) \overset{(*)}{=} \frac{1}{k!} \sum_{0 \leqslant j \leqslant k} (-1)^j \binom{k}{j} (k-j)^n =$$

$$= \frac{1}{k!} \sum_{1 \leqslant i \leqslant k} (-1)^{k-i} \binom{k}{i} i^n \overset{(**)}{=} \frac{1}{k!} \Delta^k 0^n \quad (1 \leqslant k \leqslant n),$$

[1c] *and this formula is still true for $k > n$ ($\Rightarrow S(n,k)=0$, [1a]).*

■ For the proof of [1b, (*)] we apply the *sieve* method of p. 177. Let E be the set of maps of N into $[k] := \{1, 2, \ldots, k\}$ and let F be the subset of E consisting of the *surjective* maps:

$$[1d] \qquad |E| \overset{(1)}{=} k^n, \qquad |F| \overset{(2)}{=} k!\, S(n,k).$$

(1) follows from [3a] (p. 4) and (2) from the fact that any $f \in F$ corresponds to precisely one *partition* of N, namely the partition consisting of the k *pre-images* $f^{-1}(i)$, $i \in [k]$ (p. 30), together with a numbering of this partition. Let now B_i be the set of $f \in E$ that do not have i in their image: $\forall x \in N$, $f(x) \neq i$. Evidently $F = \bar{B}_1 \bar{B}_2 \ldots \bar{B}_k$ and for the *interchangeable* system of the B_i (p. 179), we have $|B_{i_1} B_{i_2} \cdots B_{i_j}| = |B_1 B_2 \cdots B_j| = (k-j)^n$. Hence, by [1m] (p. 180), for (2):

$$[1e] \qquad k!\, S(n,k) = |F| = |\bar{B}_1 \bar{B}_2 \ldots| =$$

$$\overset{(1)}{=} |E| - \binom{k}{1} |B_1| + \binom{k}{2} |B_1 B_2| - \cdots \quad \text{CQFD.}$$

As far as [1b(**)] is concerned, this is formula [6d] (p. 14). Finally, if $n<k$, then $|F|$ is clearly equal to 0 and the sieve formula can still be applied, hence [1c]. ■

Thus we find $S(n,1)=1$, $S(n,2)=2^{n-1}-1$, $S(n,3)=(3^{n-1}+1)/2-2^{n-1}$, ... Another way to prove [1b] would be to observe that any map f ($\in E$) is *surjective* from N onto $L = f(N)$. So, putting $u_l := l!\, S(n,l)$,

$$u_k + \ldots = |E| = k^n = \sum_{L \subset [k]} u_{|L|} = \sum_{0 \leqslant l \leqslant k} \binom{k}{l} u_l,$$

[1e'] which gives u_k (consequently $S(n,k)$) by the inversion formula [6c] p. 144.

DEFINITION B. *A partition $\mathscr{P}$ of a set N is said to be of type $[c] = [c_1, c_2, \ldots, c_n]$, where the integers $c_i \geqslant 0$ satisfy $c_1 + 2c_2 + \cdots + nc_n = n$ $(=|N|)$, if and only if $\mathscr{P}$ has c_i i-blocks, $i \in [n]$. (So we have $c_1 + c_2 + \cdots + c_n = |\mathscr{P}|$.)*

THEOREM B. *The number of partitions of type $[c]$ is equal to $n!/c_1!\, c_2! \cdots (1!)^{c_1} (2!)^{c_2} \cdots$.*

■ Giving such a partition is equivalent to first giving a division of N into c_1 1-blocks, c_2 2-blocks, ...; of these there are $z = n!/((1!)^{c_1}(2!)^{c_2} \ldots)$, [10c] (p. 27), and to consequently erasing the numbering of blocks with equal size; so we must divide the number z by $c_1!\, c_2! \ldots$ ■

5.2. GENERATING FUNCTIONS FOR $S(n, k)$

The following theorem shows that the Stirling numbers defined in [1b] are indeed the numbers which were introduced for the first time in [14s] (p. 51).

THEOREM A. *The Stirling numbers of the second kind $S(n, k)$ have as "vertical" GF:*

[2a] $$\Phi_k(t) := \sum_{n\geq k} S(n,k)\frac{t^n}{n!} = \frac{1}{k!}(e^t-1)^k, \quad k\geq 0$$

where $n\geq k$ can be replaced by $n\geq 0$, and for "double" GF:

[2b] $$\Phi(t,u) := \sum_{n,k\geq 0} S(n,k)\frac{t^n}{n!}u^k = 1+\sum_{n\geq 1}\frac{t^n}{n!}\Big(\sum_{1\leq k\leq n} S(n,k)u^k\Big) = \exp\{u(e^t-1)\}.$$

■ Using [1c] (p. 204) for (*), and [1b] for (**):

$$\Phi_k(t) \overset{(*)}{=} \sum_{n\geq 0} S(n,k)\frac{t^n}{n!} \overset{(**)}{=} \frac{1}{k!}\sum_{\substack{n\geq 0\\ 0\leq j\leq k}} (-1)^j\binom{k}{j}(k-j)^n\frac{t^n}{n!} = \frac{1}{k!}\sum_{0\leq j\leq k}\Big\{(-1)^j\binom{k}{j}\sum_{n\geq 0}\frac{(k-j)^n t^n}{n!}\Big\} = \frac{1}{k!}\sum_{0\leq j\leq k}\binom{k}{j}(-1)^j(e^t)^{k-j} = \frac{1}{k!}(e^t-1)^k.$$

Similarly, with [2a] for (***):

$$\Phi(t,u) = \sum_{k\geq 0}\Big\{u^k\sum_{n\geq k} S(n,k)\frac{t^n}{n!}\Big\} \overset{(***)}{=} \sum_{k\geq 0}\frac{1}{k!}u^k(e^t-1)^k = \exp\{u(e^t-1)\}.$$ ■

THEOREM B. *The $S(n, k)$ have for "horizontal" GF (which is often taken as definition of the $S(n, k)$):*

[2c] $$x^n = \sum_{0\leq k\leq n} S(n,k)(x)_k,$$

where $(x)_k := x(x-1)\cdots(x-k+1)$, $(x)_0 := 1$.

■ Identify the coefficients of $t^n/n!$ in the first and last member of:

$$\sum_{n\geq 0} x^n\frac{t^n}{n!} = e^{xt} = \{1+(e^t-1)\}^x = \overset{(*)}{=} \sum_{k\geq 0}(x)_k\frac{(e^t-1)^k}{k!} \overset{(**)}{=} \sum_{0\leq k\leq n}(x)_k S(n,k)\frac{t^n}{n!},$$

where (*) follows from [12c] (p. 37), and (**) from [2a]. ■

THEOREM C. *The $S(n, k)$ have the following rational GF:*

[2d] $$\psi_k := \sum_{n\geq k} S(n,k)u^n = \frac{u^k}{(1-u)(1-2u)\cdots(1-ku)}, \quad k\geq 1.$$

(According to [1a], $n\geq k$ can be replaced by $n\geq 0$.)

■ If we decompose the rational fraction ψ_k into partial fractions, we obtain equality (*), and for (**) we use [1b]. Then we get:

$$\psi_k = \frac{u^k}{(1-u)\cdots(1-ku)} \overset{(*)}{=} \sum_{0\leq j\leq k}\frac{(-1)^j}{k!}\binom{k}{j}\frac{1}{1-(k-j)u} = \sum_{0\leq j\leq k}\Big\{\frac{(-1)^j}{k!}\binom{k}{j}\sum_{n\geq 0}(k-j)^n u^n\Big\} = \sum_{n\geq 0}\Big\{u^n\frac{1}{k!}\sum_{0\leq j\leq k}(-1)^j\binom{k}{j}(k-j)^n\Big\} \overset{(**)}{=} \sum_{n\geq k} S(n,k)u^n.$$ ■

THEOREM D. *The following explicit formula holds:*

[2e] $$S(n,k) = \sum_{c_1+c_2+\cdots+c_k=n-k} 1^{c_1}2^{c_2}\cdots k^{c_k}.$$

In other words, the Stirling number of the second kind $S(n,k)$ is the sum of all products of $n-k$ not necessarily distinct integers from $[k] = \{1,2,\dots,k\}$ (there are $\binom{n-1}{k-1}$ such products).
For instance, $S(5,2) = 1^3+1^2.2+1.2^2+2^3=15$. Thus the numbers $S(n,k)$ are the *symmetric monomial functions* of degree $(n-k)$ of the first k integers (Exercise 9, p. 158). This is the same thing as expanding $(1+2+\dots+k)^{n-k}$ by the multinomial theorem and afterwards suppressing every multinomial coefficient. (This procedure applied to $(a_1+a_2+\dots+a_k)^n$ gives the so-called *Wronski alephs*.)

■ After expanding φ_k [2d], identify the coefficients of u^{n-k} of the first and last member of:

$$\varphi_k/u^k = \prod_{1\le j\le k}(1-ju)^{-1} = \prod_{1\le j\le k}\sum_{r_j\ge 0} j^{r_j}u^{r_j} = \sum_{r_1,r_2,\dots,r_k\ge 0}(1^{r_1}2^{r_2}\dots k^{r_k})u^{r_1+r_2+\dots+r_k}.$$ ■

5.3. RECURRENCE RELATIONS BETWEEN THE $S(n,k)$

THEOREM A. *The Stirling numbers of the second kind $S(n,k)$ satisfy the 'triangular' recurrence relation:*

[3a] $S(n,k) = S(n-1,k-1)+kS(n-1,k), \quad n,k\ge 1;$
$S(n,0)=S(0,k)=0, \quad \text{except} \quad S(0,0)=1.$

This is a quick tool for computing the first values of $S(n,k)$ (see table on p. 310).

■ We give two proofs of [3a].
(1) *Analytical.* Equate the coefficients of $(x)_k$ in the first and last members of [3b]:

[3b] $$\sum_k S(n,k)(x)_k = x^n = x.x^{n-1} = x\sum_k S(n-1,k)(x)_k = \sum_k S(n-1,k)\{(x)_{k+1}+k(x)_k\},$$

since the $(x)_k$ form an independent system of vectors in the linear space of polynomial functions.

(2) *Combinatorial.* We return to Definition A (p. 204) of the $S(n,k)$. Let $x\in N$ be a fixed point, and let $M := N-\{x\}$, $|N| = n\ge 2$. We partition the set $s = s(N,k)$ of the k-partitions of N into s' and s'', s' is the set of partitions in which the block $\{x\}$ occurs, and $s'' = s-s'$. For all $\mathscr{P}\in s$, let $r(\mathscr{P}) := \{B\cap M \mid B\in\mathscr{P}, B\cap M\neq\emptyset\}$ be the *trace* of $\mathscr{P}$ on M. If $\mathscr{P}\in s'$, $r(\mathscr{P})\in s(M,k-1)$, and we see clearly that r is bijective; hence $|s'| = |s(M,k-1)| = S(n-1,k-1)$. If $\mathscr{P}\in s''$, $r(\mathscr{P})\in s(M,k)$, and for each partition $\mathscr{Q}\in s(M,k)$, $|r^{-1}(\mathscr{Q})|$ equals the number of possible choices of joining x to one of the blocks of $\mathscr{Q}$, which is k; hence $|s''| = k|s(M,k)| = kS(n-1,k)$. Finally [3a] follows from $|s| = |s'|+|s''|$. ■

THEOREM B. *The $S(n,k)$ satisfy the 'vertical' recurrence relations:*

[3c] $$S(n,k) = \sum_{k-1\le l\le n-1}\binom{n-1}{l}S(l,k-1),$$

[3d] $$S(n,k) = \sum_{k\le l\le n} S(l-1,k-1)k^{n-l}.$$

■ For [3c], we differentiate [2a] (p. 206) with respect to t, and we identify the coefficients of $t^{n-1}/(n-1)!$ in the first and last member of:

$$\sum_{n\ge k} S(n,k)\frac{t^{n-1}}{(n-1)!} = \frac{d\Phi_k}{dt} = e^t\Phi_{k-1} = \sum_{l,m\ge 0} S(l,k-1)\frac{t^{l+m}}{l!\,m!}.$$

For [3d], use [2d] (p. 207):

$$\sum_{n\ge k} S(n,k)u^n = \varphi_k = u(1-ku)^{-1}\varphi_{k-1} = \sum_{l,m\ge 0} S(l-1,k-1)k^m u^{l+m}.$$ ■

THEOREM C. *The $S(n,k)$ satisfy the 'horizontal' recurrence relations:*

[3e] $$S(n,k) = \sum_{0\le j\le n-k}(-1)^j\langle k+1\rangle_j S(n+1,k+j+1)$$

where $\langle x\rangle_j := x(x+1)\dots(x+j-1), \quad \langle x\rangle_0 := 1.$

[3f] $$k!\,S(n,k) = k^n - \sum_{j=1}^{k-1}(k)_j\,S(n,j).$$

■ It suffices, by [3a], to replace $S(n+1,k+j+1)$ of [3e] by $S(n,k+j)+$

$+(k-i+1)S(n,k+i-1)$, and then to expand: after simplification only $S(n,k)$ is left. For [3f], this is formula [1e], p. 205.

5.4. THE NUMBERS $\varpi(n)$ OF PARTITIONS OR EQUIVALENCE RELATIONS OF A SET WITH n ELEMENTS

The number $\varpi(n)$ of *all* partitions of a set N, often called *exponential number* or *Bell number* ([Becker, Riordan, 1948], [Touchard, 1956]) apparently equals, by Definition A (p. 204):

$$[4a]\qquad \varpi(n)=\sum_{1\leqslant k\leqslant n} S(n,k),\qquad n\geqslant 1.$$

So it is also equal to the number of *equivalence relations* on N.

THEOREM A. *The numbers $\varpi(n)$ have the following GF:*

$$[4b]\qquad \sum_{n\geqslant 0}\varpi(n)\frac{t^n}{n!}=\exp(e^t-1),\qquad \varpi(0):=1.$$

They satisfy the recurrence relations ([Aitken, 1933]):

$$[4c]\qquad \varpi(n+1)=\sum_{0\leqslant k\leqslant n}\binom{n}{k}\varpi(k),\qquad n\geqslant 0,$$

and they can be given in the form of a convergent series ([Dobinski, 1877]):

$$[4d]\qquad \varpi(n)=\frac{1}{e}\sum_{k\geqslant 0}\frac{k^n}{k!}\stackrel{(*)}{=}\left\lceil\frac{1}{e}\sum_{k\leqslant 2n}\frac{k^n}{k!}\right\rceil\qquad (n\geqslant 1,\ [3f]\ \text{p. }110)$$

■ Taking into account [4a], the first member of [4b] equals $\Phi(t,1)$; then, by [2b] (p. 206) the result follows.

For [4c], as for [3a], there are two ways again. *Analytically*, identify the coefficients of $t^n/n!$ in $d\Phi(t,1)/dt=e^t\Phi(t,1)$. *Combinatorially*, let $s(P)$ be the set of all partitions of P, $|P|=n+1$ and let $x\in P$ be a fixed point, $N:=P-\{x\}$, $|N|=n$. For $K\subset N$, let $s_K(P)$ be the set of partitions of P such that the block containing x is $\{x\}\cup K$. Then we have evidently a bijection between $s(N-K)$ and $s_K(P)$. Hence, by virtue of the division $s(P)=\sum_{K\subset N}s_K(P)$ and by passage to the cardinals, we have:

$$\varpi(n+1)=|s(P)|=\sum_{K\subset N}|s_K(P)|=\sum_{K\subset N}|s(N-K)|=$$
$$=\sum_{0\leqslant k\leqslant n}\ \sum_{|K|=k}|s(N-K)|=\sum_{0\leqslant k\leqslant n}\binom{n}{k}\varpi(n-k).$$

Finally, for [4d], we identify the coefficients of $t^n/n!$ in the first and last member of [4e] in which the series are power series converging for each complex number t:

$$[4e]\qquad \sum_{n\geqslant 0}\varpi(n)\frac{t^n}{n!}=\frac{1}{e}\exp(e^t)=\frac{1}{e}\sum_{k\geqslant 0}\frac{e^{kt}}{k!}=\frac{1}{e}\sum_{k\geqslant 0}\left(\frac{1}{k!}\sum_{n\geqslant 0}\frac{k^nt^n}{n!}\right).$$

We are leaving [4d] (*) to the reader as a gift. ■

See [Rota, 1964a] and its bibliography. (For the asymptotic study of $\varpi(n)$ see [Moser, Wyman, 1955b], [Binet, Szekeres, 1957], and [*De Bruijn, 1961], pp. 102-8. See also Exercise 23, p. 296.) A table of $\varpi(n)$ is found on p. 310.

We show now a method of computation of the $\varpi(n)$ without using the $S(n,k)$.

THEOREM B. ([Aitken, 1933]). *In the sense of p. 14 we have:*

$$\varpi(n)=\Delta^n\varpi(1).$$

■ In fact, by [6e] (p. 13) (here, $x=1$) for (*), and by [4c] (p. 210) for (**) we have:

$$\Delta^n\varpi(1)\stackrel{(*)}{=}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\varpi(k+1)=$$
$$\stackrel{(**)}{=}\sum_{k,l}(-1)^{n-k}\binom{n}{k}\binom{k}{l}\varpi(l)=\sum_{l=0}^{n}\varpi(l)A(n,l),$$

where

$$A(n,l)=\sum_k(-1)^{n-k}\binom{n}{n-k}\binom{k}{l}=$$
$$=C_{t^n}(1-t)^n t^l(1-t)^{-l-1}=C_{t^{n-l}}(1-t)^{n-l-1}=$$
$$=0,\quad\text{except}\quad A(n,n)=1,\quad\text{QED.}\ ■$$

More generally, the same method enables us to prove that the polynomials $S_n(x):=\sum_{k=1}^{n}S(n,k)x^k$ satisfy $xS_n(x)=\Delta^n S_1(x)$ $(\varpi(n)=S_n(1))$. In practice, the computation of the $\varpi(n)$ by way of this property proceeds as in the table shown. One goes from left to right, upward under an angle of 45°, starting from the table obtained for $\varpi(n-1)$. Then, after having arrived at the value of $\varpi(n)$, it is brought down to the bottom of the first column, and one starts again. In the table is shown the computation of $\varpi(6)$, starting from the table obtained by computing $\varpi(5)$: $52+15=67$, $67+20=87$, etc....

n	1	2	3	4	5	6	7
$\varpi(n)$	1	2	5	15	52	203	877
$\Delta\varpi(n)$	1	3	10	37	151	674	
$\Delta^2\varpi(n)$	2	7	27	114	523		
$\Delta^3\varpi(n)$	5	20	87	409			
$\Delta^4\varpi(n)$	15	67	322				
$\Delta^5\varpi(n)$	52	255					
$\Delta^6\varpi(n)$	203						

5.5. STIRLING NUMBERS OF THE FIRST KIND $s(n,k)$ AND THEIR GENERATING FUNCTIONS

We have already met two definitions of the *Stirling numbers of the first kind* $s(n,k)$:

(1) The $s(n,k)$ have for 'double' GF ([14p], p. 50):

[5a] $$\Phi(t,u)=\sum_{n,k\geq 0} s(n,k)\frac{t^n}{n!}u^k = 1+\sum_{n\geq 1}\frac{t^n}{n!}\Big\{\sum_{1\leq k\leq n} s(n,k)\,u^k\Big\}=(1+t)^u$$

or for 'vertical' GF ([14r], p. 51):

[5b] $$\Phi_k(t)=\sum_{n\geq k} s(n,k)\frac{t^n}{n!}=\frac{1}{k!}\log^k(1+t);$$

hence $s(n,k)=0$ if not $1\leq k\leq n$ *except* $s(0,0)=1$.

(2) The infinite (lower) triangular matrix of the $s(n,k)$ is the inverse

of the matrix of the $S(n,k)$, [6c] (p. 144):

[5c] $$\|s(n,k)\| = \|S(n,k)\|^{-1}.$$

The $s(n,k)$ are not all positive, their sign is given by: $|s(n,k)| = (-1)^{n-k}s(n,k)$, which follows from [5a], if one replaces t, u by $-t, -u$. On p. 233 we will give the combinatorial interpretation of $|s(n,k)|$, the *unsigned* or *absolute* Stirling number of the first kind, which may be denoted by $\bar{s}(n,k)$:

[5d] $$\bar{s}(n,k) := |s(n,k)| = (-1)^{n+k}s(n,k).$$

THEOREM A. *The $s(n,k)$ have for 'horizontal' GF (this is often taken as definition of the $s(n,k)$):*

[5e] $$(x)_n = \sum_{0\leq k\leq n} s(n,k)\,x^k,$$

[5f] $$\langle x\rangle_n = \sum_{0\leq k\leq n} \bar{s}(n,k)\,x^k,$$

where $(x)_n = x(x-1)\cdots(x-n+1)$, $\langle x\rangle_n = x(x+1)\cdots(x+n-1)$, $(x)_0 = \langle x\rangle_0 = 1$.

■ It suffices, by [12b, c] (p. 50) to identify the coefficients of $t^n/n!$ in:

$$\sum_{n,k\geq 0} s(n,k)\frac{t^n}{n!}x^k = (1+t)^x = \sum_{n\geq 0}(x)_n\frac{t^n}{n!},$$

$$\sum_{n,k\geq 0} \bar{s}(n,k)\frac{t^n}{n!}x^k = (1-t)^{-x} = \sum_{n\geq 0}\langle x\rangle_n\frac{t^n}{n!}.$$ ■

THEOREM B. *The $s(n,k)$ have for 'horizontal' GF:*

[5g] $$\Psi_n(u) := \sum_{1\leq k\leq n} s(n,k)\,u^{n-k} = (1-u)(1-2u)\cdots(1-(n-1)u),$$

[5h] $$\Psi_n(-u) = \sum_{1\leq k\leq n} \bar{s}(n,k)\,u^{n-k} = (1+u)(1+2u)\cdots(1+(n-1)u).$$

■ Replace x by u^{-1} in [5e, f], and simplify. ■

THEOREM C. *The $s(n+1,k+1)$, for n fixed and variable k, are the elements* *tary symmetric functions of the first n integers. In other words, for* $l=1, 2, \ldots, n$:

[5i] $$s(n+1, n+1-l) = \sum_{1\leq i_1<i_2<\cdots<i_l\leq n} i_1 i_2 \cdots i_l.$$

Differently formulated, the unsigned Stirling number of the first kind $\bar{s}(n, k)$ *appears here as the sum of all products of* $n-k$ *different integers taken from* $[n-1]=\{1, 2, \ldots, n-1\}$. (There are $\binom{n-1}{k-1}$ such products.)

For instance, $\bar{s}(6, 2)=s(6, 2)=1.2.3.4+1.2.3.5+1.2.4.5+1.3.4.5+2.3.4.5=274$.

■ This is clear from [5h], or if one prefers:

[5j] $$(x+1)(x+2)\cdots(x+n) = \sum_{0\leq k\leq n} \bar{s}(n+1, k+1)\, x^k$$

[5k] $$(1+u)(1+2u)\cdots(1+nu) = \sum_{0\leq l\leq n} \bar{s}(n+1, n+1-l)\, u^l. \blacksquare$$

(For generalizations, see [Toscano, 1939], [Strom, 1946].)

5.6. RECURRENCE RELATIONS BETWEEN THE $s(n, k)$

THEOREM A. *The Stirling numbers of the first kind* $s(n, k)$ *satisfy the "triangular" recurrence relations:*

[6a] $$s(n, k) = s(n-1, k-1) - (n-1)\, s(n-1, k), \quad n, k \geq 1;$$
$$s(n, 0) = s(0, k) = 0, \quad \text{except } s(0, 0) = 1.$$

For the unsigned numbers, this can be written

[6a'] $$\bar{s}(n, k) = \bar{s}(n-1, k-1) + (n-1)\, \bar{s}(n-1, k).$$

This is a means for a quick computation of the first values of the $s(n, k)$ (see table on p. 310 and Exercise 16, p. 226); particularly:

[6b] $$s(n, 1) = (-1)^{n-1}(n-1)!,$$
$$s(n, n-1) = -\binom{n}{2}, \quad s(n, n) = 1.$$

■ Equate the coefficients of x^k in the first and last member of [6c]:

[6c] $$\sum_k s(n, k)\, x^k = (x)_n = (x-(n-1))\,(x)_{n-1} = \{x-(n-1)\}\sum_k s(n-1, k)\, x^k. \blacksquare$$

THEOREM B. *The* $s(n, k)$ *satisfy the "vertical" recurrence relations:*

[6d] $$k\, s(n, k) = \sum_{k-1\leq l\leq n-1} (-1)^{n-l+1} \binom{n}{l} s(l, k-1),$$

[6e] $$s(n+1, k+1) = \sum_{k\leq l\leq n} (-1)^{n-l} (l+1)(l+2)\cdots(n)\, s(l, k).$$

■ For [6d], equate the coefficients of $u^{k-1}t^n/n!$ in $\partial\Psi/\partial u = \Psi \log(1+t)$, [5a] (p. 212). For [6e], use in an analogous way $\partial\Psi/\partial t = u(1+t)^{-1}\Psi$. ■

THEOREM C. *The* $s(n, k)$ *satisfy the "horizontal" recurrence relations* ([Lagrange, 1771]):

[6f] $$(n-k)\, s(n, k) = \sum_{k+1\leq l\leq n} (-1)^{l-k} \binom{l}{k-1} s(n, l)$$

[6g] $$s(n, k) = \sum_{k\leq l\leq n} s(n+1, l+1)\, n^{l-k}.$$

■ For [6f], equate the coefficients of x^k in the expressions to the right of (*) and (**):

$$x(x-1)_n = x\sum_l s(n, l)(x-1)^l$$
$$\overset{(*)}{=} \sum_{k, l} (-1)^{l-k} s(n, l) \binom{l}{k} x^{k+1} = (x-n)(x)_n$$
$$\overset{(**)}{=} \sum_l s(n, l)\, x^{l+1} - n\sum_l s(n, l)\, x^l.$$

For [6g], equate the coefficients of u^{n-k} in $\Psi_{n-1} = \{1-(n-1)u\}^{-1}\Psi_n$, [5g] (p. 213). ■

Figure [illegible] shows the diagrams of the recurrence relations established in the preceding section (See p. 212). Analogous diagrams hold evidently as well for the recurrence relations of (p. 208 and 209).

Fig. 55.

5.7. THE VALUES OF $s(n, k)$

According to [1b] (p. 204) the Stirling number of the second kind $S(n, k)$ can be expressed as a single summation of *elementary* terms, that is, which are themselves *products and quotients of factorials and powers*. There does not exist an analogous formula for the numbers of the first kind, the 'shortest formula' [7a, a'] below being a *double* summation of *elementary* terms. Shortwise, we will say that $S(n, k)$ is of *rank one* and that $s(n, k)$ is of *rank two*.

THEOREM A. ([Schlömilch 1852]). *The 'exact' value of $s(n, k)$ is:*

$$[7a] \qquad s(n,k) = \sum_{0\le h\le n-k} (-1)^h \binom{n-1+h}{n-k+h}\binom{2n-k}{n-k-h} S(n-k+h, h)$$

$$[7a'] \qquad = \sum_{0\le j\le h\le n-k} (-1)^{j+h} \binom{h}{j}\binom{n-1+h}{n-k+h}\binom{2n-k}{n-k-h}\frac{(h-j)^{n-k+h}}{h!}.$$

■ We use the Lagrange formula (p. 148). Let $f(t) := e^t - 1$ and its inverse function $f^{\langle -1\rangle}(t) = \log(1+t)$. We get, by [3b] (p. 212), for (I), [8b] (p. 149), for (II), [5b] (p. 148) for (III) and [2a] (p. 206), for (IV):

$$\frac{k!}{n!} s(n,k) \overset{(I)}{=} C_n \log^k(1+t) \overset{(II)}{=} \frac{k}{n} C_{n-k}\left(\frac{e^t-1}{t}\right)^{-n}$$

$$\overset{(III)}{=} \frac{k}{n} \binom{2n-k}{n}\sum_{h=0}^{n-k}(-1)^h\frac{1}{n+h}\binom{n-k}{h} C_{n-k}\left(\frac{e^t-1}{t}\right)^h$$

$$\overset{(IV)}{=} k\binom{2n-k}{n}\sum_{h=0}^{n-k}(-1)^h\frac{1}{n+h}\binom{n-k}{h}\frac{h!\,S(n-k+h,h)}{(n-k+h)!};$$

hence [7a] follows after simplifications. If we substitute the exact value [1b] (p. 204) into [7a], we obtain [7a'].

For small values of k, [7a, a'] is perhaps less convenient than expression [7b] below.

THEOREM B. *We have:*

$$[7b] \qquad s(n+1, k+1) = \frac{n!}{k!} Y_k(\zeta_n(1), -1!\,\zeta_n(2), 2!\,\zeta_n(3), \ldots),$$

where Y_k stands for the Bell polynomial (complete exponential, [3b, c] (p. 134), tabulated on p. 307) *and* $\zeta_n(s) := \sum_{1\le j\le n} j^{-s}$.

■ In fact, by [5j] (p. 214) and (*):

$$\sum_k s(n+1,k+1)x^k \overset{(*)}{=} n!(1+x)\left(1+\frac{x}{2}\right)\cdots\left(1+\frac{x}{n}\right)$$

$$= n! \exp\Big\{\sum_{j=1}^n \log(1+xj^{-1})\Big\}$$

$$= n!\exp\Big\{\sum_{j=1}^n\sum_{r\ge1}(-1)^{r-1} x^r r^{-1} j^{-r}\Big\}$$

$$= n!\exp\Big\{\sum_{r\ge1}(-1)^{r-1}x^r r^{-1}\zeta_n(r)\Big\},$$

and then we apply definition [3c] (p. 134) of the Y_k. ■

(There is an analogous formula for each elementary symmetric function, Exercise 9, (5) p. 158. See also Exercises 16, p. 226, and 9, p. 294.)

Thus:

$$s(n+1,2) = n!\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) = n!\,H_n,$$

where H_n denotes the harmonic number.

$$s(n+1,3) = \frac{n!}{2}\left\{H_n^2 - \left(1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}\right)\right\},$$

$$s(n+1,4) = \frac{n!}{6}\left\{H_n^3 - 3H_n\left(1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}\right) + 2\left(1+\frac{1}{2^3}+\cdots+\frac{1}{n^3}\right)\right\}.$$

5.8. CONGRUENCE PROBLEMS

It is interesting to know in advance or to discover some congruences in any table of a sequence of combinatorial integers. This is a rapid way of checking computations, and an attractive connection between Combinatorial Analysis and Number Theory. We show two typical examples in this matter.

Let two *polynomials* be given:

$$f(x) = \sum_k a_k x^k, \qquad g(x) = \sum_k b_k x^k$$

with *integer* coefficients, $a_k, b_k \in \mathbf{Z}$. We often write, when $a_k \equiv b_k \pmod m$ for all k:

[8a] $\quad f(x) \equiv g(x) \pmod m$

and we say '*f congruent g modulo m*'.

THEOREM A (*Lagrange*). *For each prime p, we have in the sense of* [8a]:

[8b] $\quad (x)_p := x(x-1)\cdots(x-p+1) \equiv x^p - x \pmod p.$

In other words, the Stirling numbers of the first kind satisfy:

[8c] $\quad s(p, k) \equiv 0 \pmod p,$

except $s(p, p) = 1$, *and* (Wilson *theorem*)

[8d] $\quad s(p, 1) = (p-1)! \equiv -1 \pmod p.$

■ For p fixed, we argue by induction, on k *decreasing* from $p-1$. By [6b] (p. 214), for (*), and Theorem C (p. 15) for (**), [8c] is true when $k = p-1$: $s(p, p-1) \overset{*}{=} -\binom{p}{2} \overset{**}{\equiv} 0 \pmod p$. Now, by [6f] (p. 215):

[8e] $\quad (p-k)\, s(p, k) = \sum_{k+1 \leqslant l \leqslant p} (-1)^{l-k} \binom{l}{k-1} s(p, l).$

Assume that [8c] is true, thus $s(p, l) \equiv 0 \pmod p$ for $(3 \leqslant)\, k+1 \leqslant l \leqslant p-1$. Then, [8e] implies, by Theorem C (p. 14) for (*):

$$-k s(p, k) - (-1)^{p-k} \binom{p}{k-1} s(p, p) \overset{*}{\equiv} 0 \pmod p,$$

from which [8c] follows, since $2 \leqslant k \leqslant p-2$.

For [8d], [6f] (p. 215) gives in the case that $k = 1$, by [8c]: $(p-1) \times s(p, 1) \equiv 1 \pmod p$; hence, since $s(p, 1) = (-1)^{p-1}(p-1)!$, [6c]:

$$-1 \equiv (p-1)(p-1)! = p! - (p-1)! \equiv -(p-1)!$$
$$\pmod p \quad ■$$

(For generalizations see [Bell, 1937], [Touchard, 1956], [Carlitz, 1966a, b].)

COROLLARY (*Fermat theorem*). *For all integers* $a \geqslant 0$, *and each prime number* p,

[8f] $\quad a^p \equiv a \pmod p.$

Put $x = a$ in [8b], then $(a)_p \equiv 0 \pmod p$ because, among p consecutive integers, at least one is a multiple of p.

THEOREM B. *For each prime number p, the Stirling numbers of the second kind satisfy:*

[8g] $\quad S(p, k) \equiv 0 \pmod p, \quad$ except $\quad S(p, 1) = S(p, p) = 1.$

■ In fact, by [1d] (p. 204), for (*), [8f] for (**), and Example 2 (p. 153), for (***), we have for $k \geqslant 2$:

$$k!\, S(p, k) \overset{(*)}{=} \sum_l (-1)^{k-l} \binom{k}{l} l^p \overset{(**)}{\equiv} \sum_l (-1)^{k-l} \binom{k}{l} l \overset{(***)}{=} 0.$$

Thus p divides $k!\,S(p, k)$, hence $S(p, k)$ when $k \leqslant p-1$, because then p is relatively prime with respect to $k!$. ■

Note. One can prove this also by induction, using [3f], p. 209, as in the proof of Theorem A.

SUPPLEMENT AND EXERCISES

1. *Banners and chromatic polynomials*. (1) Show that the number $d(n, k)$ of banners with n vertical bands and k colours, two adjacent bands of different colour, equals $k!\,S(n-1, k-1)$. (2) Moreover, for every *tree* A over N, $|N| = n$, $d(n, k)$ is also the number of colourings of the n nodes with k colours such that two adjacent nodes have a different colour. (Compare with Exercise 1, p. 158.) (3) More generally, considering a

graph $\mathscr{G}$ with n nodes and introducing the number $a(\mathscr{G}, k)$ of colourings of these nodes with k colours, having the preceding property, show that the chromatic polynomial (cf p. 179) satisfies: $P_{\mathscr{G}}(\lambda)=\sum_{k\geq 1} a(\mathscr{G}, k)\times\binom{\lambda}{k}$. *(4) What is the number of checkerboards of dimensions $(m\times n)$ with k colours? [Two squares with a side in common must be coloured differently.]

2. *Lie derivative and operational calculus.* Let $\lambda(t)$ be a formal series. We define the operator λD (Lie derivative) by $(\lambda D)f := \lambda Df=\lambda f'$, where D is the usual derivation (p. 41). Similarly, $(D\lambda)f := D(\lambda f)=\lambda' f+\lambda f'$. (1) $(tD)^n=\sum_{1\leq l\leq n} S(n,l)\,t^lD^l$ and $(Dt)^n=\sum_{l\geq 0} S(n+1,l+1)\,t^lD^l$. (2) $(e^{at}D)^n=e^{nat}\sum_{l\geq 1} s(n,l)\,a^{n-l}D^l$. (3) $(t^{a+1}D)^n=t^{an}\sum_{l=1}^{n} P_{n,l}(a)\,t^lD^l$, where $\sum_{n\geq l} P_{n,l}\,t^n/n! = (1/l!)\{(1-at)^{-1/a}-1\}^l$. (4) Find an explicit formula for $(\lambda D)^n$ and $(D\lambda)^n$ ([Comtet, 1973]). (5) The following result of *Pourchet* shows that this problem is closely connected with the Faà di Bruno formula:

$$\left(\lambda(x)\frac{d}{dx}\right)^n f(x)=\frac{d^n}{dw^n}f(x(w)),\quad \text{where}\quad \frac{dx}{dw}=\lambda(x),\quad x=x(w).$$

Apply this method to prove: $(x\log x\, D)^n=\sum_{0\leq k\leq l\leq n} s(n,l)\,S(l,k)\,(\log x)^l x^k D^k$.

3. *The lattice of the partitions of a set.* Let be given two partitions $\mathscr{P}, \mathscr{F}$ of a set N. Then we say that $\mathscr{P}$ is *finer* than $\mathscr{F}$ or that $\mathscr{P}$ is a *subpartition* of $\mathscr{F}$, denoted by $\mathscr{P}\leq\mathscr{F}$, if and only if each block of $\mathscr{P}$ is contained in a block of $\mathscr{F}$. Show that this order relation on the set of partitions of N makes it into a *lattice* (Definition D, p. 59).

4. *Bernoulli and Stirling numbers and sums of powers.* We write the GF of the Bernoulli numbers B_n, [14a] (p. 48), in the form:

$$[*]\qquad \sum_{n\geq 0} B_n\frac{t^n}{n!}=\frac{t}{e^t-1}=\frac{\log(1+(e^t-1))}{e^t-1}.$$

Show that $B_n=\sum_{k\geq 0}(-1)^k k!\,S(n,k)/(k+1)$. Use this to obtain the value of B_n, expressed as a double sum. Show also, by substituting $u:=e^t-1$ into [*], that $\sum_k s(n,k)\,B_k=(-1)^n n!/(n+1)$. Verify the formula $B_n=\sum_j(-1)^j\binom{n+1}{j+1}Z(j,n)/(j+1)$, where $Z(j,n)=1^n+2^n+\cdots+j^n$ ([Bergmann, 1967] and p. 155. See also [Gould, 1972] which gives other explicit formulas for the Bernoulli numbers). Show that $Z(n,r)=\sum_{j=1}^{r+1}(j-1)!\,S(r+1,j)\binom{n}{j}$.

5. *A transformation of formal series.* For each integer $k\geq 1$, let T_k be the transformation of formal series defined by: $f=\sum_{n\geq 0} a_nt^n \mapsto T_kf=\sum_{n\geq 0} n^ka_nt^n$. (1) Show that $T_kf=\sum_{l=1}^{k} S(k,l)\,t^lD^lf$ (D is the differentiation operator, p. 41). (2) Deduce from this the value of $\sum_{n\geq 0} n^kt^n$ in the form of a rational fraction, and also that of $\sum_{n=0}^{N} n^kt^n$. (3) Furthermore, with the Eulerian numbers $A(k,h)$ (pp. 51 and 242) we have $\Phi_k(t):=(1-t)^{k+1}\sum_{n\geq 0} n^kt^n=\sum_{h=1}^{k} A(k,h)\,t^h$. [*Hint:* Apply [14v] (p. 51) to $\sum_{k\geq 0}\Phi_k(t)\,u^k/k!$.] (4) Express $\sum_{n\geq 0} n^kt^n/n!$ in the form of a product of e^t with a polynomial. (5) Solve analogous problems for $\sum_{n\geq 0} n^k\langle\alpha\rangle_n t^n/n!$ (and $\sum_{n=0}^{N}$), where α is a complex number. (6) Study the transformation $T_{k,c}$, with c a given integer ≥ 0, such that $T_{k,c}f:=\sum_{n\geq 0}(n+c)^ka_nt^n$.

6. *The Taylor-Newton formula.* For each polynomial $P(x)$ we have (Δ is the *difference* operator defined on p. 13):

$$P(x)=\sum_{k\geq 0}\frac{(x-a)_k}{k!}\Delta^kP(a)=(1+\Delta)^xP(0).$$

More generally, let be given a sequence $\alpha_0, \alpha_1, \alpha_2, \ldots$ of *different* complex numbers, f a formal series (with complex coefficients) and t, x two indeterminates. We put $(x)_n=(x-\alpha_0)(x-\alpha_1)\cdots(x-\alpha_{n-1})$ and $(\alpha_k)'_l=\prod_{j=0,\,j\neq k}^{l}(\alpha_k-\alpha_j)$ for $k\leq l$. Prove the Newton *interpolation formula*:

$$f(tx)=\sum_{n\geq 0}(x)_n\sum_{j=0}^{n}\frac{f(t\alpha_j)}{(\alpha_j)'_n}.$$

Use this to recover the formulas of Exercise 29 (p. 167).

7. *Associated Stirling numbers of the second kind.* For r integer ≥ 1, let $S_r(n,k)$ be the number of partitions of the set N, $|N|=n$, into k blocks, all of cardinality $\geq r$. We call this number the r-associated Stirling number

of the second kind. In particular, $S_1(n,k)=S(n,k)$. Then we have the GF:

$$\sum_{n,k\geq 0} S_r(n,k)\,u^k\frac{t^n}{n!}=\exp\left\{u\left(\frac{t^r}{r!}+\frac{t^{r+1}}{(r+1)!}+\cdots\right)\right\},$$

and the 'triangular' recurrence relations:

$$S_r(n+1,k)=kS_r(n,k)+\binom{n}{r-1}S_r(n-r+1,k-1).$$

Moreover, $S_2(n,k)\equiv 0 \pmod{1\cdot 3\cdot 5\dots(2k-1)}$ and, for $l\geq 1$, $(-1)^l l! =\sum_{m=1}^{l}(-1)^m S_2(l+m,m)$. The first values of $S_2(n,k)$ are:

$k\backslash n$	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1	1
2			3	10	25	56	119	246	501	1012	2035
3					15	105	490	1918	6825	22935	74316
4							105	1260	9450	56980	302995
5									945	17325	190575
6											10395

$n\backslash k$	13	14	15	16	17	18
1	1	1	1	1	1	1
2	4082	8177	16368	32751	65518	131053
3	235092	731731	2252341	6879678	20900922	63259533
4	1487200	6914908	30950920	134779645	575156036	2417578670
5	1636635	12122110	81431350	510880370	3046610676	17539336815
6	270270	4099095	47507460	466876410	4104160060	33309926040
7		135135	4729725	94594500	1422280860	17892864990
8				2027025	91891800	2343240900
9						34459425

Let $P_r(t)=\sum_{k=0}^{r-1} t^k/k!$. Use the $S_r(n,k)$ to expand $(P_r(t))^u$, $P_r(t)$, $P_r(u)$ and $\log(P_r(t))$.

8. *Distributions of balls in boxes.* The number of distributions of n balls into k boxes equals: (1) k^n if all balls and all boxes are different; $k!\,S(n,k)$ if no box is allowed to be empty. (2) $\binom{n+k-1}{n}$ if the balls are indistinguishable, and all the boxes different; $\binom{n-1}{k-1}$ if, moreover, no box is allowed to be empty (Theorem C, p. 15). (3) Suppose the boxes are all different, and the balls of equal size, but painted in different colours. Balls of the same colour are supposed to be not distinguishable. In this way we define a partition of the set N of balls. If there are in this partition c_i i-blocks, $i=1,2,3,\dots$, then the number of distributions is equal to $\binom{k}{1}^{c_1}\binom{k+1}{2}^{c_2}\binom{k+2}{3}^{c_3}\cdots$, $c_1+2c_2+\cdots=n$ [use (2)]. (4) What do we get for all the preceding answers when the boxes and balls are put in rows? (For all these problems, see especially [*MacMahon, 1915–16]. Good information is also found in [*Riordan, 1958], pp. 90–106.)

9. *Return to the Bell polynomials. Applications to rational functions.* The exponential partial Bell polynomials $\mathbf{B}_{n,k}$ are a generalization of the Stirling numbers, because $\mathbf{B}_{n,k}(1,1,1,\dots)=S(n,k)$, [3g] (p. 135). (1) Let $a_1,a_2,\dots$ be integers ≥ 0. Show that $\mathbf{B}_{n,k}(a_1,a_2,\dots)$ equals the number of partitions of N, $|N|=n$, into k blocks, the i-blocks being painted with colours taken from a stock A_i given in advance, and with a_i colours in the stock A_i, $i=1,2,3,\dots$. (It is not compulsory to use all colours of each stock.) (2) We denote the value of the n-th derivative in the point $x=a$ of $F(x)$ [or $G(x)$] by f_n [or g_n], f_0, $g_0=F(a)$, $G(a)$. Suppose that $x=a$ is a multiple root of order k of $G(x)=0$, and that $F(x)/G(x)$ has the singular part $\sum_{h=1}^{k}\gamma_h(x-a)^{-h}$. Show that the coefficients γ_h equal

$$\sum_{0\leq\nu\leq\mu\leq k-h}\frac{(-1)^\nu\,\nu!\,k!\,f_{k-h-\mu}}{\mu!\,(k-h-\mu)!\,g_k^{\nu+1}}\,\mathbf{B}_{\mu,\nu}\left(\frac{g_{k+1}}{\binom{k+1}{k}},\frac{g_{k+2}}{\binom{k+2}{k}},\dots\right).$$

(For $k=1$, we recover $\gamma_1=f_0/g_1=F(a)/G'(a)$, that is the residue of F/G when $x=a$ is a simple pole.) (3) Now take F and G to be *polynomials*, $G=\prod_{j=1}^{s}(x-a_j)^{k_j}$, with all different a_j. Express the $\gamma_{h,j}$ by an 'exact' *formula of rank* $\leq n-2$.

10. *The Schröder problem.* ([Schröder, 1870], see also [Carlitz, Riordan, 1955], [Comtet, 1970], [Knödel, 1951]). Let N be a finite set, $|N|=n$, and let us use the name 'Schröder system' for any system (of blocks of N)

$\mathscr{S} \in \mathfrak{P}'(N)$ such that: (α) Every 1-block of N belongs to $\mathscr{S}$: $\mathfrak{P}_1(N) \subset \mathscr{S}$. (β) N does not belong to it: $N \notin \mathscr{S}$. (γ) $B, B' \in \mathscr{S} \Rightarrow B \subset B'$ or $B' \subset B$ or $B \cap B' = \emptyset$. We denote the family of all Schröder systems of N by $\mathbf{s}(N)$, and the problem is now to compute its cardinal $s_n := |\mathbf{s}(N)|$. (1) Let the number k_i of maximal i-blocks be fixed, $i \in [n-1]$. (Maximal block is a block contained in no other.) Then we have:

[a] $$k_1 + 2k_2 + \cdots + (n-1)k_{n-1} = n,$$

and the number of the corresponding $\mathscr{S} \in \mathbf{s}(N)$ equals: $n!\, s_1^{k_1} s_2^{k_2} \cdots (1!)^{-k_1}(2!)^{-k_2} \cdots (k_1!)^{-1}(k_2!)^{-1} \cdots$. (2) Observe that the condition [a] is equivalent to the two conditions $k_1 + 2k_2 + \cdots + nk_n = n$, $k_1 + k_2 + \cdots + k_n \geq 2$. Show that the GF $y := \sum_{n \geq 1} s_n t^n/n!$ satisfies:

[aa] $$e^y - 2y - 1 + t = 0.$$

(3) We have $s_n = \sum_{k=0}^{n-2} S_2(n+k, k+1)$ (= diagonal sums of the associated Stirling numbers of Exercise 7 (p. 221)). So, $s_4 = 1 + 10 + 15 = 26$. Hence the table of values:

n	1	2	3	4	5	6	7	8	9	10
s_n	1	1	4	26	236	2752	39208	660032	12818912	282137824

(4) $s_p \equiv 1 \pmod p$, for p prime. (5) $s_n = \sum_{k \geq 1} 2^{-n-k} S_2(n+k-1, k)$ (style Dobinski). (6) Explicitly,

$$s_n = \sum_{1 \leq j \leq k \leq n-1} \frac{(-1)^{j-k+1}}{k!} \binom{k}{j} \binom{n+k-1}{n+j-1} (k-j)^{n+k-1}.$$

(7) Asymptotically,

$$s_n \approx \frac{1}{2}\sqrt{\frac{A}{\pi n}}\, \frac{(n-1)!}{A^n} \left\{1 - \sum_{k \geq 1} \frac{d_k}{n^k}\right\}, \quad n \to \infty,$$

where $A = 2\log 2 - 1 = 0.386294\ldots$ and d_k are polynomials in A: $d_1 = (9 - A)/24$, $d_2 = (225 - 90A + A^2)/1152, \ldots$

11. *Congruences of the (Bell) number of partition $\varpi(n)$.* Let p be a prime number. Modulo p, we have $\varpi(p) \equiv 2$, $\varpi(p+1) \equiv 3$ and, more generally, $\varpi(p^r + h) \equiv r\varpi(h) + \varpi(h+1)$. Modulo p^2, we have $\varpi(2p) - 2\varpi(p+1) - 2p\varpi(p) + p + 3 \equiv 0$ ([Touchard, 1956]).

12. *Generalization of* $\sum \binom{n}{k}^2 = \binom{2n}{n}$. Let $P_{n,r}(x) = \sum_{k=0}^{n} k^r \binom{n}{k}^2 x^{n-k}$, where r is integer ≥ 0. Use Exercise 6 (1) (p. 221) to show that

$$P_{n,r}(x) = \sum_q S(r,q)(n)_q [t^{n-q}] (1+xt)^n (1+t)^{n-q}.$$

Thus,

$$A(n,r) := P_{n,r}(1) = \sum_{k=0}^{n} k^r \binom{n}{k}^2 = \sum_q S(r,q)(n)_q \binom{2n-q}{n}.$$

Particularly, $A(n,0) = \binom{2n}{n}$, $A(n,1) = (2n-1)\binom{2n-2}{n-1}$, $A(n,2) = n^2\binom{2n-2}{n-1}$, $A(n,3) = n\binom{n+1}{2}\binom{2n-2}{n-1}$. Similarly,

$$\sum_{k=0}^{n} (-1)^k k^r \binom{n}{k}^2 = \sum_{q+2j=n} (-1)^j S(r,q)(n)_q \binom{n-q}{j}\binom{q}{j}.$$

13. *A 'universal' generating function.* The following solves, for partitions of a set, a problem analogous to the problem for partitions of integers, which is solved by Theorem B (p. 98). Let $\mathfrak{A}$ be an infinite matrix consisting of 0 and 1, $\mathfrak{A} = [a_{i,j}]$, $i \geq 1$, $j \geq 0$, $a_{i,j} = 0$ or 1. Let $s(n, k, \mathfrak{A})$ be the number of partitions of a set N into k blocks such that the number of blocks of size (= cardinal number) i equals to one of the integers $j \geq 0$ for which $a_{i,j} = 1$. Then we have the 'universal' GF:

$$\sum_{n,k \geq 0} s(n,k \mid \mathfrak{A})\, u^k \frac{t^n}{n!} = \prod_{i \geq 1} \left\{ \sum_{j \geq 0} \frac{a_{i,j}}{j!} \left(u \frac{t^i}{i!}\right)^j \right\}.$$

In particular we obtain the following table of GF, where ∗ means no condition ($a_{i,j} = 1$ provides the 'total' GF)

Number of blocks	Size of each block	GF by 'number of blocks'	Total GF
*	*	$\exp(u(e^t-1))$	$\exp(e^t-1)$
*	odd	$\exp(u\,\mathrm{sh}\,t)$	$\exp(\mathrm{sh}\,t)$
*	even	$\exp(u(\mathrm{ch}\,t-1))$	$\exp(\mathrm{ch}\,t-1)$
odd	*	$\mathrm{sh}(u(e^t-1))$	$\mathrm{sh}(e^t-1)$
even	*	$\mathrm{ch}(u(e^t-1))$	$\mathrm{ch}(e^t-1)$
odd	odd	$\mathrm{sh}(u\,\mathrm{sh}\,t)$	$\mathrm{sh}(\mathrm{sh}\,t)$
odd	even	$\mathrm{sh}(u(\mathrm{ch}\,t-1))$	$\mathrm{sh}(\mathrm{ch}\,t-1)$
even	odd	$\mathrm{ch}(u\,\mathrm{sh}\,t)$	$\mathrm{ch}(\mathrm{sh}\,t)$
even	even	$\mathrm{ch}(u(\mathrm{ch}\,t-1))$	$\mathrm{ch}(\mathrm{ch}\,t-1)$

14. *'Stackings' of x.* Let $f_l(x)$ be the sequence of functions defined by $f_1=x, f_2=x^x, \dots, f_l=x^{f_{l-1}}, l\geq 2$. Determine and study the coefficients of the expansion $f_l(x)=\sum_{p,q\geq 0} a_l(p,q)\,x^p \log^q x$. (See also p. 139.)

***15.** *The number of 'connected' relations.* Let p and q be two integers ≥ 1. A relation $\mathscr{A}\subset[p]\times[q]$ is called 'connected' if $\mathrm{pr}_1\mathscr{A}=[p]$, $\mathrm{pr}_2\mathscr{A}=[q]$ (p. 59), and if any two points of $\mathscr{A}$ can be connected by a polygonal path with unit sides in horizontal or vertical direction, all whose vertices are in $\mathscr{A}$. We say also that $\mathscr{A}$ is $(p\times q)$-*animal*. Thus, in Figure 36, (I) is an animal, but (II) is not. Compute or estimate the

(I) (II)

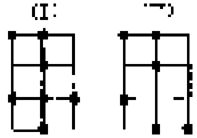


Fig. 36

number $A(n;p,q)$ of the $\mathscr{A}$ such that $|\mathscr{A}|=n$, also called 'animals' (This term is taken from [*Golomb, 1966]. For an approach to this problem, see [Kreweras, 1969] and [Read, 1962a]). Analogous question for dimension $d\geq 3$, $\mathscr{A}\subset[p_1]\times[p_2]\times\cdots\times[p_d]$.

16. *Values of $S(n,n-\alpha)$ and $s(n,n-\alpha)$.* (1) We have

$$S(n,n-\alpha)=\sum_{j=\alpha+1}^{2\alpha}\binom{n}{j}S_2(j,j-\alpha),$$

S_2 as defined in Exercise 7 (p. 221). Thus, $S(n,n)=1$,

$S(n,n-1)=\binom{n}{2}$, $S(n,n-2)=\binom{n}{3}+3\binom{n}{4}=\frac{1}{4}\binom{n}{3}(3n-5)$, $S(n,n-3)=\binom{n}{4}+10\binom{n}{5}+15\binom{n}{6}=\frac{1}{2}\binom{n}{4}(n^2-5n+6)$. (2) Similarly, we have $s(n,n-\alpha)=\sum_{j=\alpha+1}^{2\alpha}\binom{n}{j}s_2(j,j-\alpha)$, where the s_2 are defined by $\sum_{n,k}s_2(n,k)u^k t^n/n!=e^{-tu}(1+t)^u$ (Exercise 7, p. 256; Exercise 20, p. 295). Thus, $s(n,n)=1$, $s(n,n-1)=-\binom{n}{2}$, $s(n,n-2)=2\binom{n}{3}+3\binom{n}{4}=\frac{1}{4}\times\binom{n}{3}(3n-1)$, $s(n,n-3)=-6\binom{n}{4}-20\binom{n}{5}-15\binom{n}{6}=-\frac{1}{2}\binom{n}{4}\times n(n-1)$, $s(n,n-4)=\frac{1}{48}\binom{n}{5}(15n^3-30n^2+5n+2)$. (Other 'exact' formulas in [Mitrinović, 1960, 1961, 1963]. See also Exercise 9, p. 293.)

17. *Stirling numbers and Vandermonde determinants.* The value of the unsigned number of the first kind $s(n+1,k)$ is the quotient of the n-th order determinant obtained by omitting the k-th column of the matrix

$$\begin{bmatrix}1 & 1 & 1 & \dots & 1\\ 1 & 2 & 2^2 & \dots & 2^n\\ 1 & 3 & 3^2 & \dots & 3^n\\ \dots & \dots & \dots & \dots & \dots\\ 1 & n & n^2 & \dots & n^n\end{bmatrix},$$

by $1!2!\dots(n-1)!$. The number of the second kind $S(n,k)$ can be expressed using a determinant of order k:

$$k!\,S(n,k)=\begin{vmatrix}1 & 1 & 1 & \dots & 1 & 1 & 1\\ 1 & 2 & 2^2 & \dots & 2^{k-3} & 2^{k-2} & 2^n\\ 1 & 3 & 3^2 & \dots & 3^{k-3} & 3^{k-2} & 3^n\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 1 & k & k^2 & & k^{k-3} & k^{k-2} & k^n\end{vmatrix}.$$

18. *Generalized Bernoulli numbers.* These are the numbers $B_n^{(r)}$ defined for every complex number r:

$$\left(\frac{t}{e^t-1}\right)^r=\sum_{n\geq 0}B_n^{(r)}\frac{t^n}{n!}.$$

Evidently $B_n^{(1)} = B_n$ [(4a)] (p. 48). Show (with [5h] p. 142) that $B_n^{(r)} = \sum_{j\geq 0} (-1)^j \binom{n+1}{j+1} B_n^{(r+j)}$. Moreover, for all pairs of integers (n, p) such that $0 \leq n \leq p-1$ we have $B_n^{(p)} = \binom{p-1}{n}^{-1} s(p, p-n)$. Besides, $B_n^{(r)} = \mathbf{P}_n^{(r)}(\frac{1}{2}, \frac{1}{3}, \ldots)$, by [5c] (p. 141); $B_0^{(r)} = 1$, $B_1^{(r)} = -r/2$, $B_2^{(r)} = \frac{1}{12} r(3r-1)$, $B_3^{(r)} = -\frac{1}{8} r^2 (r-1), \ldots$. Finally, determine an 'exact' formula of minimal rank for $B_n^{(r)}$ (p. 216).

19. *Diagonal differences.* Show that $\Delta^{2j} S(k, k-j) = \Delta^{2j} s(k, k+j) = 1.3.5 \ldots (2j-1)$.

20. *The number of 'Fubini formulas'.* Let a_m be the number of possible ways to write the Fubini formula ([13] p. 34) for a summation of integration of order m. Evidently, $a_1 = 1$, $a_2 = 3$, $a_3 = 13$, because $\sum_{x_1, x_2, x_3} = \sum_{x_1}(\sum_{x_2, x_3}) = \sum_{x_2}(\sum_{x_1, x_3}) = \sum_{x_3}(\sum_{x_1, x_2}) = \sum_{x_1, x_2}(\sum_{x_3}) = \sum_{x_1, x_3}(\sum_{x_2}) = \sum_{x_2, x_3}(\sum_{x_1}) = \sum_{x_1}(\sum_{x_2}(\sum_{x_3})) = \sum_{x_1}(\sum_{x_3}(\sum_{x_2})) = \sum_{x_2}(\sum_{x_1}(\sum_{x_3})) = \sum_{x_2}(\sum_{x_3}(\sum_{x_1})) = \sum_{x_3}(\sum_{x_1}(\sum_{x_2})) = \sum_{x_3}(\sum_{x_2}(\sum_{x_1}))$. Show that $a_m = \sum_{k=1}^{m} k! S(m, k)$ and that $\sum_{m\geq 0} a_m t^m / m! = (2 - e^t)^{-1}$.

m	1	2	3	4	5	6	7	8	9	10
a_m	1	3	13	75	541	4683	47293	545835	7087261	102247563

Moreover, $a_n = \sum_k A(n, k) 2^{k-1}$, as a function of the Eulerian numbers of p. 51 or 242, and $a_n = \| n! (\ln 2)^{-n-1} 2^{-1} \|$ (notation [6f], p. 110).

21. *A beautiful determinant.* Let s be the unsigned Stirling numbers of the first kind (p. 213). Then,

$$\begin{vmatrix} s(n+1, 1) & s(n+1, 2) \ldots s(n+1, k) \\ s(n+2, 1) & s(n+2, 2) \ldots s(n+2, k) \\ \ldots\ldots & \ldots\ldots \\ s(n+k, 1) & s(n+k, 2) \ldots s(n+k, k) \end{vmatrix} = (n!)^k.$$

***22.** *Inversion of $y^\alpha e^y$ and $y \log^\beta y$ in a neighbourhood of infinity* ([Comtet, 1970]). The equations $y^\alpha e^y = x$ and $y \log^\beta y = x$, where α and β are constants $\gtrless 0$, have solutions $y = \Phi_\alpha(x)$ and $y = \Psi_\beta(x)$ that tend to infinity for x tending to infinity. Then, with $L_1 := \log x$ and $L_2 := \log \log x$, we have:

$$\Phi_\alpha(x) = L_1 - \alpha L_2 + \sum_{h \geq 1} \left\{ \frac{(-\alpha)^{h-1}}{L_1^h} \sum_{m=1}^{h} s(h, h-m+1) \frac{L_2^m}{m!} \right\},$$

$$\Psi_\beta(x) = \frac{x}{L_1^\beta} \left\{ 1 + \sum_{h \geq 1} \frac{(-\beta)^h}{L_1^h} \sum_{m=1}^{h} \frac{(-L_2)^m}{m!} Q_{h,m}(\beta) \right\},$$

the polynomial $Q_{h,m}(\beta)$ being $\sum_{k=0}^{m-1} \binom{h-m+k-1}{h-m} s(h, h-m+k) \beta^k$.

***23.** *Congruences of the Stirling numbers.* Let p be prime. We denote 'a divides b' by $a \mid b$. (1) $p^2 \mid s(p, 2k)$ for $2 \leq 2k \leq p-3$ and $p \geq 5$ (Nielsen). Particularly, the numerator of the harmonic number $H_{p-1} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + (1/(p-1))$ is divisible by p^2. (2) $p \mid S(p+1, k)$ for $3 \leq k \leq p$ and $p \mid (S(p+1, 2) - 1)$.

24. *An asymptotic expansion for the sum of factorials.* If $n \to \infty$, we have:

$$\frac{1}{n!} \sum_{k=0}^{n} k! \approx 1 + \sum_{k \geq 0} \frac{\varpi(k)}{n^{k+1}} = 1 + \frac{1}{n} + \frac{1}{n^2} + \frac{2}{n^3} + \frac{5}{n^4} + \cdots$$

25. *The number of topologies on a set of n elements.* This number t_n equals $\sum_k S(n, k) d_k$, the d_k being the number of order relations defined on p. 60 ([Comtet, 1966]).

n	1	2	3	4	5	6	7	8	9
t_n	1	4	29	355	6942	209527	9535141	642779354	63260289423

PERMUTATIONS

6.1. THE SYMMETRIC GROUP

We recall that a permutation σ of a finite set N, $|N|=n$, is a bijection of N onto itself.

Actually, as N is finite, we could as well have said 'surjection' or 'injection' instead of 'bijection'.

A permutation σ can be represented by writing the elements of the set N on a top row, and then underneath each element its image under the mapping σ. Thus $\begin{pmatrix} abcdefg \\ caedbgf \end{pmatrix}$ represents a $\sigma \in \mathfrak{S}(N)$, where $N := \{a, b, c, d, e, f, g\}$, $\sigma(a)=c$, $\sigma(b)=a$, $\sigma(c)=e$, $\sigma(d)=d$, $\sigma(e)=b$, $\sigma(f)=g$, $\sigma(g)=f$.

Another way of representing σ consists of associating with it a digraph $\mathcal{G}$ (p. 67), where it is understood that an arc $\overrightarrow{xy}$ is drawn if and only if $y=\sigma(x)$, $y \neq x$. Figure 37 corresponds in this way with the above permutation.

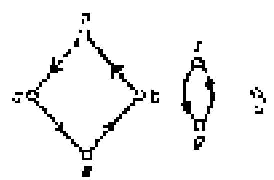


Fig. 37.

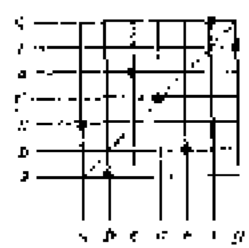


Fig. 38.

One can also represent σ by a relational lattice, as on p. 58. Then Figure 38 corresponds with the permutation of Figure 37. Clearly a binary relation on N is associated with a permutation in this way if and only if all its horizontal and vertical sections have one element.

Finally, σ can be represented by a square matrix, say $B=[b_{i,j}]$, defined

by $b_{i,j}=1$ if $j=\sigma(i)$, and $b_{i,j}=0$ otherwise. Such a matrix is called a *permutation matrix*.

We denote the group of permutations of N (with composition of maps as operation) by $\mathfrak{S}(N)$. This group is also called the symmetric group of N. The unit element of this group is the identity permutation, denoted by e: $\forall x \in N$, $e(x)=x$. Evidently, $|\mathfrak{S}(N)|=n!$ (p. 7).

We recall some notions about permutations $\sigma \in \mathfrak{S}(N)$.

The *orbit* of $x (\in N)$ for a permutation σ is the subset of N consisting of the points $x, \sigma(x), \sigma^2(x), \ldots, \sigma^{k-1}(x)$, where k, the *length* of the orbit, is the smallest integer ≥ 1 such that $\sigma^k(x)=x$. If $k=1$, $\sigma(x)=x$, then x is a *fixed point* of σ (See p. 180).

Let $x_1, x_2, \ldots, x_k$ be k different points of N, $1 \leq k \leq n$. The *cycle* $\gamma = (x_1, x_2, \ldots, x_k)$ is the following permutation: $\gamma(x_1)=x_2$, $\gamma(x_2)=x_3, \ldots, \gamma(x_{k-1})=x_k$, $\gamma(x_k)=x_1$ and $\gamma(x)=x$ if $x \neq x_i$. We say that γ has *length* k (also denoted by $|\gamma|$) and has the set $\{x_1, x_2, \ldots, x_k\}$ for *domain* (or orbit). Evidently, there are $(n)_k/k$ cycles of length k because each cycle $(x_1, x_2, \ldots, x_k)$ is given by any one of the following k-arrangements: $(x_1, x_2, \ldots, x_k)$, $(x_2, x_3, \ldots, x_k, x_1)$, $\ldots$, $(x_k, x_1, \ldots, x_{k-1})$, and only by those.

A *circular permutation* is a cycle of length n $(=|N|)$. So there are $(n)_n/n=(n-1)!$ such permutations. A *transposition* τ is a cycle of length 2; in other words, there exist two points a and b, $a \neq b$, such that $\tau(a)=b$, $\tau(b)=a$. There are exactly $\binom{n}{2}$ transpositions of N.

We recall that each permutation can be written as a *product* of cycles, with disjoint domains, this decomposition being unique up to order. For example, the permutation of p. 230 can be written as $(a, c, e, b)(f, g)(d) = (a, c, e, b)(f, g)$ (the cycles of length 1 are often omitted). Similarly, $e=(x_1)(x_2)\ldots(x_n)$. Commonly, the cycles in the sense of graphs (p. 62) and cycles in the sense of permutations will be identified, as in Figure 37. Each cycle is product of transpositions; in fact, $(x_1)=(x_1, x_2)(x_2, x_1)$ and $(x_1, x_2, \ldots, x_k)=(x_1, x_k)(x_1, x_{k-1})\ldots(x_1, x_2)$ for $k \geq 2$. Hence, this holds for each permutation, because they are products of cycles.

It follows that the set $\mathfrak{T}=\mathfrak{T}(N)$ of transpositions of N, $|\mathfrak{T}|=\binom{n}{2}$, generates the group $\mathfrak{S}(N)$. In fact, $\mathfrak{S}(N)$ can be generated by a much smaller set of transpositions. To make this more precise, let us associate with every set of transpositions $\mathfrak{U} \subset \mathfrak{T}$ the *graph* $\mathcal{G}(\mathfrak{U})$ defined as follows:

$\{x, y\}$ is an edge of $g(\mathfrak{U})$ if and only if the transposition $(x, y) \in \mathfrak{U}$.

THEOREM. *A set $\mathfrak{U}\,(\subset \mathfrak{T})$ of $(n-1)$ transpositions of N generates $\mathfrak{S}(N)$ if and only if $g(\mathfrak{U})$ is a tree (Definition B, p. 62).*

■ If $g(\mathfrak{U})$ is a tree over N, then for all $a, b \in N$, $a \neq b$ there exists a unique path $x_1(=a), x_2, \ldots, x_k(=b)$ such that $\{x_i, x_{i+1}\}$ is an edge of $g(\mathfrak{U})$, hence the transposition $(x_i, x_{i+1}) \in \mathfrak{U}$, $i \in [k-1]$. Now it is easily verified that the transposition (a, b) can be factored as follows in the group $\mathfrak{S}(N)$:

$$(a, b) = (x_1, x_k) = (x_{k-1}, x_k)(x_{k-2}, x_{k-1}) \cdots (x_2, x_3) \times \\ \times (x_1, x_2) \cdots (x_{k-2}, x_{k-1})(x_{k-1}, x_k).$$

Thus, as each $(a, b) \in \mathfrak{T}$ is generated by $\mathfrak{U}$, $\mathfrak{S}(N)$ is too (cf. p. 231).

Now we suppose conversely that $\mathfrak{U}$ generates $\mathfrak{S}(N)$, but that $g(\mathfrak{U})$ is not a tree. Because $g(\mathfrak{U})$ has $(n-1)$ edges, there exist a and b not connected by a path (Theorem C, p. 62); this implies that the transposition (a, b) is not equal to any product of transpositions belonging to $\mathfrak{U}$, etc. ■ (For other properties related to representing a set of permutations by a graph, see [Dénes, 1959], [Eden, Schützenberger, 1962], [Eden, 1967], and [*Berge, 1968], pp. 117–23.)

For two decompositions into a product of transpositions of a given permutation, $\sigma = \varphi_1 \varphi_2 \ldots \varphi_r = \psi_1 \psi_2 \ldots \psi_s$, the numbers r and s have the *same* parity. This can be quickly seen by observing first that the product $\tau\sigma$ of the transposition $\tau = (a, b)$ and a permutation σ with k cycles is a permutation with $k+1$ cycles if a and b are in the same orbit, and with $k-1$ cycles if a and b are in different orbits of σ. Hence it follows that $\varphi_1 \varphi_2 \ldots \varphi_r$ and $\psi_1 \psi_2 \ldots \psi_s$ have a number of cycles equal to $1 \pm 1 \pm 1 \pm \pm \cdots \pm 1$, $(r-1)$ times ± 1, and $1 \pm 1 \pm 1 \pm \cdots \pm 1$, $(s-1)$ times ± 1, respectively. The equality of these two numbers implies the above mentioned property. (This is the proof by [Cauchy, 1815]. See also [*Serret, 1866], II, p. 248.)

A permutation is called *even* (respectively *odd*) if it can be decomposed into an *even* (respectively *odd*) numbers of transpositions. Suppose $\sigma = \gamma_1 \gamma_2 \ldots \gamma_k$, a product of k cycles. The parity of σ is equivalent to the parity of the integer $n - k\,(= \sum (|\gamma_i| - 1))$ because of the decomposition of each cycle of length l into $l-1$ transpositions (see above). Thus a permutation is even (respectively odd) if it has an even (respectively odd) number of cycles of even length.

The *sign* $\chi(\sigma)$ of a permutation σ is defined by $\chi(\sigma) = +1$ (-1 respectively) if σ is even (respectively odd). From the decomposition into transpositions it follows immediately that for each two permutations σ and σ':

$$\chi(\sigma\sigma') = \chi(\sigma)\,\chi(\sigma').$$

The *alternating subgroup* of $\mathfrak{S}(N)$ consists of the *even* permutations of N.

The *order* of a permutation σ is the smallest integer $k \geq 1$ such that $\sigma^k = \iota$. This is clearly the LCM of the system of integers consisting of lengths of the cycles occurring in the decomposition of σ.

6.2. COUNTING PROBLEMS RELATED TO DECOMPOSITION INTO CYCLES; STIRLING NUMBERS OF THE FIRST KIND

DEFINITION. *Let $c_1, c_2, \ldots, c_n$ be integers ≥ 0 such that:*

[2a] $\qquad c_1 + 2c_2 + \cdots + nc_n = n.$

A permutation $\sigma \in \mathfrak{S}(N)$, $|N| = n$ is said to be of type $[c] = [c_1, c_2, \ldots, c_n]$ if its decomposition into disjoint cycles contains exactly c_i cycles of length i, $i = 1, 2, 3, \ldots, n$. In other words, the partition of N given by the orbits of σ is of type $[c_1, c_2, \ldots]$ (Definition B, p. 205).

THEOREM A. *A permutation $\sigma \in \mathfrak{S}(N)$ of type $[c]$ is even (or odd), if and only if $c_2 + c_4 + c_6 + \cdots$ is even (or odd).*

■ We have already seen this on p. 231. ■

THEOREM B. *The number of permutations of type $[c] = [c_1, c_2, \ldots]$ equals:*

[2b] $\qquad p(n; c_1, c_2, \ldots) = \dfrac{n!}{c_1!\, c_2! \ldots c_n!\, 1^{c_1} 2^{c_2} \ldots n^{c_n}} \qquad (0! = 1^0 = 1).$

■ Giving such a permutation of type $[c]$ is equivalent to giving first a *division* of N into the c_i orbits of length i of the permutation, with $i = 1, 2, 3, \ldots$; then to *erasing for all i* the order in the set of c_i orbits of length i, and finally to equipping each orbit with a cyclic permutation of its own.

Thus:

$$p(n;c_1,c_2,\ldots)=\frac{n!}{(1!)^{c_1}(2!)^{c_2}\cdots c_1!\,c_2!\cdots}\times\{(2-1)!\}^{c_2}\{(3-1)!\}^{c_3}\cdots$$

which gives [2b] after cancellations. ■

THEOREM C. *Let $p(n,k;c_1,c_2,\ldots)$ be the number of permutations of N, $|N|=n$, of type $[c_1,c_2,\ldots]$, whose total number of orbits (= number of cycles in the decomposition) equals k, $c_1+c_2+\cdots=k$. Then we have the following GF in an infinite number of variables $t,u,x_1,x_2,\ldots$:*

[2c]
$$\Phi=\Phi(t,u;x_1,x_2,\ldots):=\sum_{n,k,c_1,c_2,\ldots\geq 0}p(n,k;c_1,c_2,\ldots)u^k\frac{t^n}{n!}x_1^{c_1}x_2^{c_2}\cdots=\exp\left\{u\left(x_1t+x_2\frac{t^2}{2}+x_3\frac{t^3}{3}+\cdots\right)\right\}.$$

■ In fact, $p(n,k;c_1,c_2,\ldots)=p(n;c_1,c_2,\ldots)$ if $c_1+c_2+\cdots=k$ and $c_1+2c_2+\cdots=n$; if not, $p(n,k;c_1,c_2,\ldots)=0$. Hence, by [2b]:

$$\Phi=\sum_{n,c_1,c_2,\ldots\geq 0}\frac{n!}{c_1!\,c_2!\cdots 1^{c_1}2^{c_2}\cdots}u^{c_1+c_2+\cdots}\frac{t^{c_1+2c_2+\cdots}}{n!}x_1^{c_1}x_2^{c_2}\cdots$$
$$=\sum_{c_1,c_2,\ldots\geq 0}\frac{1}{c_1!}(tux_1)^{c_1}\frac{1}{c_2!}\left(\frac{t^2}{2}ux_2\right)^{c_2}\cdots$$
$$=\prod_{i\geq 1}\sum_{c_i\geq 0}\frac{((t^i/i)ux_i)^{c_i}}{c_i!}=\prod_{i\geq 1}\exp\left(\frac{t^i}{i}ux_i\right)=\text{QED.}\ ■$$

THEOREM D. *The number of permutations of N with k orbits (whose decomposition has k cycles) equals the unsigned Stirling number of the first kind $\bar{s}(n,k)$.*

■ The required number, say $a(n,k)$, equals the sum of the $p(n,k;c_1,c_2,\ldots)$, taken over all systems of integers $c_1,c_2,\ldots$ such that $c_1+c_2+\cdots=k$ and $c_1+2c_2+\cdots=n$. Hence, by [2c]:

$$\sum_{n,k\geq 0}a(n,k)\frac{t^n}{n!}u^k=\Phi(t,u;1,1,1,\ldots)$$

$$=\exp\left\{u\left(t+\frac{t^2}{2}+\frac{t^3}{3}+\cdots\right)\right\}$$
$$=\exp\{-u\log(1-t)\}=(1-t)^{-u}.$$

Hence $a(n,k)=\bar{s}(n,k)$ by [5a, d] (p. 212). ■

6.3. MULTIPERMUTATIONS

We show now an immediate generalization of the concept of permutation, suggested by the matrix notation of p. 230. For each integer $k\geq 0$, a relation $\mathfrak{R}$ will be called a k-permutation (of $[n]$) when all vertical sections *and* all horizontal sections all have k elements. Let $P(n,k)$ be the number of these relations. Evidently, $P(n,k)=0$ if $k>n$, and otherwise $P(n,k)=P(n,n-k)$. We have $P(n,0)=P(n,n)=1$ and we recover the ordinary permutations for $k=1$: $P(n,1)=P(n,n-1)=n!$

THEOREM A. *Let $k_1,k_2,\ldots,k_n$ and $l_1,l_2,\ldots,l_n$ be $2n$ integers, all ≥ 0. The number of relations $\mathfrak{R}$ such that the i-th vertical section has k_i elements, and the j-th horizontal section has l_j elements, is given by the following coefficient:*

[3a]
$$P_{k_1,\ldots,k_n;l_1,\ldots,l_n}=C_{u_1^{k_1}\cdots u_n^{k_n}v_1^{l_1}\cdots v_n^{l_n}}\prod_{\substack{i\in[n]\\ j\in[n]}}(1+u_iv_j).$$

■ It suffices to expand the product in [3a] and to observe that the coefficient under consideration is the number of solutions with $x_{i,j}=0$ or 1 of the system of $2n$ equations:

$$\sum_{j=1}^{n}x_{i,j}=k_i,\quad i\in[n],\qquad \sum_{i=1}^{n}x_{i,j}=l_j,\quad j\in[n],$$

in other words, the number of relations we want to find. ■

We now investigate the number $P(n,2)$ of *bipermutations*, short notation P_n.

THEOREM B. *We have:*

[3b]
$$P_n=\frac{1}{4^n}\sum_{\alpha=0}^{n}(-1)^{\alpha}(2n-2\alpha)!\,\alpha!\binom{n}{\alpha}^2 2^{\alpha},$$

[3d] $$f(t) := \sum_{n\geq 0} P_n \frac{t^n}{n!^2} = \frac{e^{-t/2}}{\sqrt{1-t}}$$

[3e] $$P_n = \binom{n}{2}(2P_{n-1} + (n-1)P_{n-2})$$

■ By [3c], $P_n = C_{x_1^2 x_2^2 \cdots x_n^2} \prod (1+x_ix_j) = C_{x_1^2\cdots x_n^2}(\sum_{i<j} x_ix_j)^n = (\frac{1}{2})^n \times C_{x_1^2\cdots x_n^2}\{(x_1+\cdots+x_n)^2-(x_1^2+\cdots+x_n^2)\}^n = (\frac{1}{2})^n C_{x_1^2\cdots x_n^2}\sum_{\alpha=0}^{n}(-1)^\alpha \times \binom{n}{\alpha}(x_1^2+\cdots+x_n^2)^\alpha(x_1+\cdots+x_n)^{2(n-\alpha)} = (\frac{1}{2})^n\sum_{\alpha=0}^{n}(-1)^\alpha\binom{n}{\alpha}(n)_\alpha \times (2n-2\alpha)!/2^{n-\alpha}$. Q.E.D. The GF [3d] follows then from the explicit formula [3b]. As for the recurrence relation [3e], this follows from the differential equation $2(1-t)f' = tf$. ■

By Theorem A, one can deduce for $P(n,k)$ more and more complicated formulas. For instance,

$$P(n,3) = \frac{n!^2}{6^n}\sum_{\alpha_1+\alpha_2+\alpha_3=n}(-1)^{\alpha_2}(2\alpha_2+3\alpha_3)! \times$$
$$\times \alpha_1! [illegible] (\alpha_1!\alpha_2!\alpha_3!)^{-2} 18^{\alpha_2} 12^{\alpha_3},$$

from which one may deduce a linear recurrence relation for $P(n,3)$ with coefficients that are polynomials in n. There is little known about $P(n,k)$ except the asymptotic result $P(n,k)\sim(kn)!(k!)^{-2n}e^{-(k-1)^2/2}$ for fixed k and $n\to\infty$ ([Everett, Stein, 1971]). The first values of $P(n,k)$ are:

$n \backslash k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	2	1					
3	1	6	6	1				
4	1	24	90	24	1			
5	1	120	2040	2040	120	1		
6	1	720	67950	297200	67950	720	1	
7	1	5040	3110940	68938800	68938800	3110940	5040	1

6.4. INVERSIONS OF A PERMUTATION OF $[n]$

In Sections 6.4 and 6.5 we study the permutations of a *totally ordered* set N, which will be identified with $[n] := \{1, 2, \ldots, n\}$. We make the following abbreviations:

[4a] $$\mathfrak{S}[n] := \mathfrak{S}([n]), \qquad \mathfrak{P}_k[n] := \mathfrak{P}_k([n]).$$

It is often convenient to represent a permutation $\sigma\in\mathfrak{S}[n]$ by a *polygon* whose sides are segments A_iA_{i+1}, $i\in[n-1]$, such that A_i has i for abscissa and $\sigma(i)$ for ordinate. The heavy line in Figure 39 represents the polygon of $\sigma\in\mathfrak{S}[7]$, defined by the cycle $(1, 3, 5, 2)$, in the sense of p. 231; hence, the points 4, 6, 7 are fixed points.

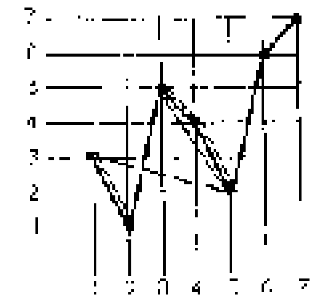


Fig. 39.

DEFINITION. *An inversion of a permutation $\sigma\in\mathfrak{S}[n]$ is a pair (i, j) such that $1\leq i<j\leq n$ and $\sigma(i)>\sigma(j)$. In this case we say that σ has an inversion at (i, j).*

Hence, in the associated polygon, an inversion is a segment A_iA_j, $1\leq i<j\leq n$, with negative slope. The permutation which is represented in Figure 39 induces 5 inversions, whose corresponding segments are indicated by thin lines.

Let I_σ be the number of inversions of $\sigma\in\mathfrak{S}[n]$. Clearly, $0\leq I_\sigma\leq\binom{n}{2}$, with $I_\sigma = 0 \Leftrightarrow \forall i\in[n], \sigma(i)=i$ and $I_\sigma=\binom{n}{2} \Leftrightarrow \forall i\in[n], \sigma(i)=n-i+1$.

THEOREM A. *The sign $\varepsilon(\sigma)$ (see p. 233) of a permutation $\sigma\in\mathfrak{S}[n]$ equals $(-1)^{I_\sigma}$.*

■ We abbreviate $e(\sigma) := (-1)^{I_\sigma}$ and $[n]_2 := \mathfrak{P}_2[n]$. Then:

$$e(\sigma) = \prod_{\{i,j\}\in[n]_2}\frac{\sigma(i)-\sigma(j)}{i-j}.$$

Hence, for σ and $\tau\in\mathfrak{S}[n]$, we obtain by change of variable $i'=\tau(i)$

$j' = \beta(j)$ in (*):

[4b] $$\varrho(\alpha\beta) = \prod_{\{i,j\}\in[n]_2} \frac{(\alpha\beta)(i)-(\alpha\beta)(j)}{i-j}$$
$$= \prod_{\{i,j\}\in[n]_2} \frac{\alpha(\beta(i))-\alpha(\beta(j))}{\beta(i)-\beta(j)}\cdot\frac{\beta(i)-\beta(j)}{i-j}$$
$$\overset{(*)}{=} \prod_{\{i',j'\}\in[n]_2}\frac{\alpha(i')-\alpha(j')}{i'-j'}\cdot\prod_{\{i,j\}\in[n]_2}\frac{\beta(i)-\beta(j)}{i-j}$$
$$= \varrho(\alpha)\cdot\varrho(\beta).$$

Moreover, the number of inversions I_τ of a transposition $\tau=(a,b)$, which interchanges a and b, $1\le a<b\le n$, can be read off from the polygon of τ, and it equals $2(b-a)-1$, hence $\varrho(\tau)=-1$. Thus, if we write an arbitrary $\sigma\in\mathfrak{S}[n]$ as a product of transpositions, it follows, with [4b] for (*) and p. 233 for (**) that:

$$(-1)^{I_\sigma} = \varrho(\sigma) = \varrho(\tau_1\tau_2\cdots\tau_s)$$
$$\overset{(*)}{=} \varrho(\tau_1)\varrho(\tau_2)\cdots\varrho(\tau_s) = (-1)^s \overset{(**)}{=} \chi(\sigma). \quad ■$$

Theorem B. *The number $b(n,k)$ of permutations of $[n]$ with k inversions satisfies the recurrence relations* ([Bourget, 1871]):

[4c] $$b(n,k) = \sum_{0,\, k-n+1\le j\le k} b(n-1,j) \quad \text{if } n\ge 1;$$
$$b(n,0)=1; \qquad b(0,k)=0 \quad \text{if } k\ge 1.$$

■ Let $\mathfrak{b}(n,k)$ be the set of permutations of $[n]$ that induce k inversions, $b(n,k)=|\mathfrak{b}(n,k)|$, and let $\mathfrak{b}_i(n,k)$ be the set of the $\sigma\in\mathfrak{b}(n,k)$ such that $\sigma(1)=i$, $i\in[n]$. Then we have the division:

[4d] $$\mathfrak{b}(n,k) = \sum_{1\le i\le n}\mathfrak{b}_i(n,k).$$

Let f be the map of $\mathfrak{b}_i(n,k)$ into $\mathfrak{b}(n-1,k-i+1)$ defined by:

[4e] $$f(\sigma)(j) = \begin{cases}\sigma(j+1), & \text{if } \sigma(j+1)<i\\ \sigma(j+1)-1, & \text{if } \sigma(j+1)>i\end{cases} \quad j\in[n-1].$$

It is clear that f is a bijection. Hence, if we use the *convention*:

[4f] $$b(u,v)=0, \quad \text{if } v<0 \quad \text{or if } v>\binom{u}{2},$$

we get, by passing to the cardinalities in [4d]:

[4g] $$b(n,k) = \sum_{1\le i\le n}|\mathfrak{b}_i(n,k)| = \sum_{1\le i\le n} b(n-1,k-i+1),$$

In other words, we just obtain [4c], if we do not use the convention [4f] and if we change the summation variable to $j=k-i+1$. ■

Theorem C ([Muir, 1898]). *The numbers $b(n,k)$ have as GF:*

[4h] $$\Phi_n(u) := \sum_{0\le k\le\binom{n}{2}} b(n,k)u^k = \prod_{1\le i\le n}\frac{1-u^i}{1-u} =$$
$$=(1+u)(1+u+u^2)\cdots(1+u+u^2+\cdots+u^{n-1}).$$

■ Using [4c] for (*) and putting $i=k-j+1$ for (**), we get:

$$\Phi_n(u) \overset{(*)}{=} \sum_{0\le k\le\binom{n}{2}}\Big(u^k\sum_{j\le k} b(n-1,j)\Big)$$
$$\overset{(**)}{=} \sum_{\substack{1\le i\le n\\ 0\le j\le\binom{n-1}{2}}} u^{i+j-1} b(n-1,j)$$
$$=\Big(\sum_{1\le i\le n}u^{i-1}\Big)\Big(\sum_{0\le j\le\binom{n-1}{2}} b(n-1,j)u^j\Big)$$
$$=(1+u+\cdots+u^{n-1})\,\Phi_{n-1}(u),$$

from which [4h] easily follows. ■

Theorem D. *The numbers $b(n,k)$ satisfy the following relations:*

(I) $$b(n,k) = b(n,k-1)+b(n-1,k), \quad \text{if } k<n.$$

(II) $$\sum_{k=0}^{\binom{n}{2}} b(n,k) = n!$$

(III) $$\sum_{k=0}^{\binom{n}{2}} (-1)^k b(n,k) = 0.$$

(IV) $b(n,k) = b\left(n, \binom{n}{2} - k\right)$

(V) $\sum_{k=0}^{\binom{n}{2}} k\, b(n,k) = \frac{1}{2}\binom{n}{2} n! = \sum_{\sigma} I_\sigma$ ([*Henry, 1881]).

■ (I) From [4f] follows $(1-u)\Phi_n = (1-u^n)\Phi_{n-1}$, where the coefficients of u^k must be identified. (II) Put $u=1$ in [4f]. (III) Put $u=-1$ in [4f]. (IV) Observe that the polynomial $\Phi_n(u)$ is reciprocal. (V) Put $u=1$ in $d\Phi_n/du$. ■

N.B. Find also combinatorial proofs of Theorem D.

Table of $b(n,k) = b\left(n, \binom{n}{2} - k\right)$

$n \backslash k$	0	1	2	3	4	5	6	7	8	9
1	1									
2	1	1								
3	1	2	2	1						
4	1	3	5	6	5	3	1			
5	1	4	9	15	20	22	20	15	9	4
6	1	5	14	29	49	71	90	101	101	90
7	1	6	20	49	98	169	259	359	455	531
8	1	7	27	76	174	343	602	961	1415	1940
9	1	8	35	111	285	628	1230	2191	3606	5545
10	1	9	44	155	440	1068	2298	4489	8095	13640

$n \backslash k$	10	11	12	13	14	15	16
5	1						
6	71	49	29	14	5	1	
7	573	573	531	455	359	259	169
8	2493	3017	3450	3736	3836	3736	3450
9	8031	11021	14395	17957	21450	24584	27073
10	21670	32683	47043	64889	86054	110010	135853

([*David, Kendall, Barton, 1966], p. 241, for $n \leqslant 16$.)

6.5. Permutations by number of rises; Eulerian numbers

Definition. *A permutation* $\sigma \in \mathfrak{S}[n]$ *induces a* rise [*or a* fall] *in* $i \in [n-1]$ *if* $\sigma(i) < \sigma(i+1)$ [*or* $\sigma(i) > \sigma(i+1)$].

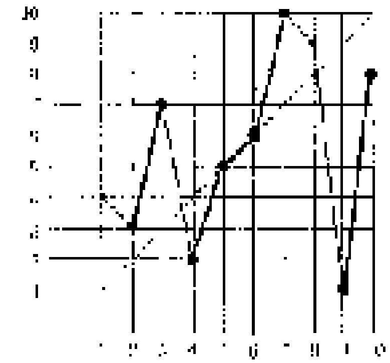


Fig. 40.

Thus, in Figure 40 the 5 rises [4 falls] of a permutation of [10] are indicated by a heavy [thin] line.

Let A_σ be the number of rises of σ, in other words, the number of sides with positive slope of the associated polygon. Clearly, $0 \leqslant A_\sigma \leqslant n-1$, and $A_\sigma = 0 \Leftrightarrow \forall i \in [n], \sigma(i) = n-i+1$, and $A_\sigma = n-1 \Leftrightarrow \forall i \in [n], \sigma(i) = i$. Moreover, the number of *falls* of σ is evidently equal to:

[5a] $n - 1 - A_\sigma$.

Theorem A. *The number* $a(n,k)$ *of permutations of* $[n]$ *with* k *rises satisfies the following recurrence relation:*

[5b] $a(n,k) = (n-k)\, a(n-1,k-1) + (k+1)\, a(n-1,k)$

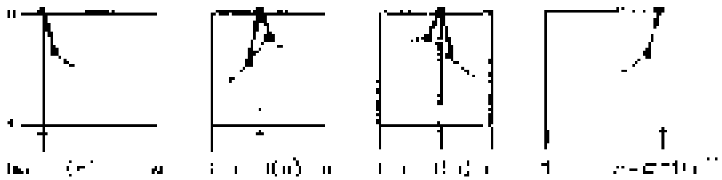
for $n, k \geqslant 1$, *with* $a(n,0) = 1$ *for* $n \geqslant 0$, *and* $a(0,k) = 0$ *for* $k \geqslant 1$.

■ Let $\mathfrak{A}(n,k)$ be the set of permutations of $[n]$ that induce k rises. The number $a(n,k) = |\mathfrak{A}(n,k)|$ is also the number of permutations of $[n]$ that induce k falls, which can be seen by associating with $\sigma \in \mathfrak{S}[n]$ the permutation $i \mapsto \sigma(n-i+1)$. Hence

[5c] $a(n,k) = a(n, n-k-1)$.

Now we define the map φ of $\mathfrak{A}(n,k)$ into $\mathfrak{S}[n-1]$ by:

[5d] $\sigma' = g(\sigma) \Leftrightarrow \sigma'(i) = \begin{cases}\sigma(i) & \text{if } i < \sigma^{-1}(n) \\ \sigma(i+1) & \text{if } i \geq \sigma^{-1}(n)\end{cases}$



(a) (b) (c) (d)

FIG. 41.

It is clear that $\sigma' \in \mathfrak{a}(n-1,k)$ in the case of Figures 41a, b, and that $\sigma' \in \mathfrak{a}(n-1,k-1)$ in the case of Figures 41c, d.

Conversely, if $\sigma' \in \mathfrak{a}(n-1,k)$, some reflection shows that $|g^{-1}(\sigma')| =$ the number of rises of σ' (see Figure 41b) + 1 (see Figure 41a) $= k+1$; if $\sigma' \in \mathfrak{a}(n-1,k-1)$ we have, similarly, with [5a] for (a): $|g^{-1}(\sigma')| =$ the number of falls of σ' (see Figure 41c) + 1 (see Figure 41d) $\overset{(a)}{=} \{(n-1)-1-(k-1)\}+1 = n-k$. Hence,

$$|\mathfrak{a}(n,k)| = \sum_{\sigma' \in \mathfrak{a}(n-1,k)} |g^{-1}(\sigma')| + \sum_{\sigma' \in \mathfrak{a}(n-1,k-1)} |g^{-1}(\sigma')| = (k+1)\,|\mathfrak{a}(n-1,k)| + (n-k)\,|\mathfrak{a}(n-1,k-1)|. \blacksquare$$

THEOREM D. *Let $A(n,k)$ denote the Eulerian numbers (introduced in [14f], p. 51); then we have:*

[5e] $a(n,k-1) = A(n,k) \overset{(2)}{=} A(n,n-k+1)$.

■ In fact, if we put $\bar{A}(n,k) := a(n,k-1)$, then the recurrence relation [5h] becomes exactly [14u] (p. 51), where $A(n,k)$ is replaced by $\bar{A}(n,k)$, *including the initial conditions*. Hence $\bar{A}(n,k) = A(n,k)$. Equality [5e] (2) follows then from [5c]. ■

Evidently, $\sum_k A(n,k) = n!$ and, by [5h]:

[5e'] $A(n,k) = (n-k+1)\,A(n-1,k-1) + kA(n-1,k)$.

Table of Eulerian numbers $A(n,k)$

$n \backslash k$	1	2	3	4	5	6	7	8	9
2	1	1							
3	1	4	1						
4	1	11	11	1					
5	1	26	66	26	1				
6	1	57	302	302	57	1			
7	1	120	1191	2416	1191	120	1		
8	1	247	4293	15619	15619	4293	247	1	
9	1	502	14608	88234	156190	88234	14608	502	1
10	1	1013	47840	455192	1310354	1310354	455192	47840	1013
11	1	2036	152637	2203488	9738114	15724248	9738114	2203488	152637
12	1	4083	478271	10187685	66318474	162512286	162512286	66318474	10187685

([*David, Kendall, Barton, 1966], p. 260, and [6].)

THEOREM E. *The Eulerian numbers $A(n,k)$ have the value:*

[5f] $$A(n,k) = \sum_{0 \le j \le k} (-1)^j \binom{n+1}{j} (k-j)^n.$$

■ Use the GF [14v] of p. 51, and equate the coefficients in the first and last member of [5g] of $u^k t^n / n!$:

[5g] $$1 + \sum_{1 \le k \le n} A(n,k) \frac{t^n}{n!} u^k = \frac{1-u}{1-u e^{t(1-u)}} = (1-u) \sum_{l \ge 0} u^l e^{lt(1-u)} = \sum_{l,n \ge 0} \frac{t^n}{n!} l^n u^l (1-u)^{n+1} = \sum_{l,n,h \ge 0} (-1)^h \frac{t^n}{n!} \binom{n+1}{h} l^n u^{l+h}. \blacksquare$$

If $k > n$, then $A(n,k) = 0$, and [5f] implies an interesting identity in that case.

THEOREM F. *The Eulerian numbers $A(n,k)$ satisfy:*

[5h] $$x^n = \sum_{1 \le k \le n} A(n,k) \binom{x+k-1}{n}.$$

([Worpitzky, 1883].) For other properties and generalizations see [Abramson, Moser, 1967], [André, 1906], [Carlitz, 1952b, 1959, 1960a, 1963a], [Carlitz, Riordan, 1953], [Carlitz, Roselle, Scoville, 1966], [Cesàro, 1886], [Dillon, Roselle, 1968], [Foata, 1965], [Frobenius, 1910], [Poussin, 1968], [Roselle, 1968], [Schrutka, 1941], [Shanks, 1951], [Toscano, 1950], [Toscano, 1965]. [*Foata, Schützenberger, 1970] contains a very exhaustive and completely new treatment of this subject.]

■ As identity [5h] is polynomial in x, of degree n, it suffices to verify it for $x=0, 1, 2, \ldots, n$, which comes down to 'inverting' [5f] in the sense of p. 143. By [5f], for (*), we get (cf. Exercise 5 (3), p. 221):

$$\sum_k A(n,k)\,t^k \overset{(*)}{=} \sum_{0\leq j\leq k} (-1)^{k-j}\binom{n+1}{k-j} j^n t^k$$
$$= \sum_{j\geq 0}\left\{ j^n t^j \sum_{k\geq j}\binom{n+1}{k-j}(-t)^{k-j}\right\} =$$
$$= (1-t)^{n+1}\sum_{j\geq 0} j^n t^j.$$

Hence $\sum_{j\geq 0} j^n t^j = (1-t)^{-n-1}\sum_{k=1}^{n} A(n,k)\,t^k$. In other words we have for the coefficient of t^i: $i^n = \sum_k A(n,k)\binom{n+i-k}{n}$. Hence [5h] with $x=i$ and [5c] (*). ■

We now introduce the Eulerian polynomials $A_n(u) = \sum_k A(n,k)\,u^k$; $A_0(u)=1$, $A_1(u)=u$, $A_2(u)=u+u^2$, $A_3(u)=u+4u^2+u^3, \ldots$. Taking [14v] p. 51 into account for [5i], and [14i] p. 51 for [5j], we have the following GF:

[5i] $$\sum_{n\geq 0} A_n(u)\frac{t^n}{n!} = \frac{1-u}{1-ue^{t(1-u)}}$$

[5j] $$1+\sum_{n\geq 1}\frac{A_n(u)}{u}\cdot\frac{t^n}{n!} = \frac{1-u}{e^{t(u-1)}-u},$$

[5k] $$\sum_{n\geq 0}\frac{A_n(u)}{u(u-1)^n}\cdot\frac{t^n}{n!} = \frac{u-1}{e^t-u},$$

the last one, [5k], follows from [5i], where t is replaced by $t/(u-1)$.

THEOREM E (Frobenius). *The Eulerian polynomials are equal to:*

[5l] $$A_n(u) = u\sum_{k=1}^{n} k!\,S(n,k)\,(u-1)^{n-k}$$

[5m] $$= \sum_{k=0}^{n} k!\,S(n+1,k+1)\,(u-1)^{n-k}.$$

■ By [5k] for (*): $\sum_{n\geq 0} A_n(u)\,t^n/(n!\,u(u-1)^n) \overset{(*)}{=} (1-(e^t-1)/(u-1))^{-1} = \sum_{k\geq 0}(e^t-1)^k(u-1)^{-k}$. Hence $A_n(u) = u\sum_{k\geq 0}(u-1)^{n-k}\,\mathrm{C}_{t^n/n!}(e^t-1)^k$, in other words, [5l]. Then [5m] follows, if we replace $u(u-1)^{n-k}$ in [5l] by $(u-1)^{n+1-k}+(u-1)^{n-k}$, and if we use [3a] of p. 208. ■

The historical origin of the Eulerian polynomials is the following summation formula:

THEOREM F. *For each integer $n\geq 0$, the power series with coefficients 'n-th powers' equals:*

[5n] $$\sum_{l\geq 0} l^n u^l = \frac{A_n(u)}{(1-u)^{n+1}}.$$

■ See the proof of Theorem D above. (Cf. Exercise 5, p. 221.) ■

Examples. For $n=0, 1, 2, 3$ we get respectively:

$$1+u+u^2+u^3+\cdots = \frac{1}{1-u}$$
$$u+2u^2+3u^3+4u^4+\cdots = \frac{u}{(1-u)^2}$$
$$u+2^2u^2+3^2u^3+4^2u^4+\cdots = \frac{u+u^2}{(1-u)^3}$$
$$u+2^3u^2+3^3u^3+4^3u^4+\cdots = \frac{u+4u^2+u^3}{(1-u)^4}.$$

The above-mentioned GF of the Eulerian numbers, namely

[5o] $$\mathfrak{A}(t,u) = 1+\sum_{1\leq k\leq n} A(n,k)\frac{t^n}{n!}u^{k-1} = \frac{1-u}{e^{t(u-1)}-u}$$

[5p] $$\mathfrak{A}_1(t,u) = \sum_{0\leq k\leq n} A(n,k)\frac{t^n}{n!}u^k = \frac{1-u}{1-ue^{t(1-u)}}$$

have the disadvantage of being asymmetric. Everything becomes easier if we introduce the *symmetric Eulerian numbers* $\tilde{A}(l,m)$ defined by:

[5q] $$\tilde{A}(l,m) = A(l+m+1, m+1).$$

The table of these is obtained from the table on p. 243 by sliding all columns upward:

$l \setminus m$	0	1	2	3
0	1	1	1	1
1	1	4	11	26
2	1	11	66	302
3	1	26	302	2416

THEOREM G ([Carlitz, 1960]). *We have the following GF:*

$$[5r] \qquad \sum_{l,m\geq 0} A(l,m)\frac{x^l y^m}{(l+m+1)!} = \frac{e^x - e^y}{xe^y - ye^x}.$$

■ In fact, by [5p] for (*), the left-hand member of [5r] equals: $\sum_{l,m\geq 0} A(l+m+1, m+1)\, x^l y^m/(l+m+1)! = (1/y) \sum_{n\geq 0,\, 0\leq k\leq n} A(n, k+1) \times x (y/x)^{k+1} x^n/n! \overset{(*)}{=} (1/y)(-1+\mathfrak{A}_1(y, y/x))$, providing the second member of [5r] after simplifications. ■

The following is a generalization of the problem of the rises, often called the 'problem of Simon Newcomb'. Instead of permuting the set $[n]$, one permutes a set P, $|P|=p$, consisting of c_1 numbers 1, c_2 numbers 2, ..., c_n numbers n, $c_1+c_2+\cdots+c_n=p$, and we want to find the number of permutations with $k-1$ rises. ([Kreweras, 1965, 1966b, 1967], [*Riordan, 1958], p. 216; cf. Exercise 21, p. 255.) In more concrete terms, one draws from a set of 52 playing cards all cards, one by one, stacking them on piles in such a way that one starts a new pile each time a card appears that is 'higher' than its predecessor. In how many ways can one obtain $k-1$ piles? (here $c_1=c_2=\cdots=c_{13}=4$).

6.6. GROUPS OF PERMUTATIONS; CYCLE INDICATOR POLYNOMIAL; BURNSIDE THEOREM

DEFINITION A. *A group $\mathfrak{G}$ of permutations of a finite set N is a subgroup of the group $\mathfrak{S}(N)$ of all permutations of N. We denote $\mathfrak{G} \leq \mathfrak{S}(N)$. $|\mathfrak{G}|$ is called the order of $\mathfrak{G}$, and $|N|$ its degree.*

Thus, the alternating group is a permutation group of N, of order $n!/2$

For each permutation $\sigma \in \mathfrak{S}(N)$, $|N|=n$, we denote:

$$[6a] \qquad c_i(\sigma) := \text{the number of orbits of length } i \text{ of } \sigma, \; i \in [n],$$

and, for each group of permutations $\mathfrak{G} \leq \mathfrak{S}(N)$ and each sequence $(c_1, c_2, \dots, c_n)$ of integers ≥ 0 such that $c_1+2c_2+\cdots=n$ we denote, with the definition on p. 233:

$$[6b] \qquad \mathfrak{G}(c_1, c_2, \dots, c_n) := \{\sigma \mid \sigma \in \mathfrak{G},\ \sigma \text{ is of type } [[c_1, c_2, \dots]]\}.$$

DEFINITION B. *The cycle indicator polynomial $Z(x)$ of a group of permutations $\mathfrak{G}$ of N, $\mathfrak{G} \leq \mathfrak{S}(N)$, also denoted by $Z(\mathfrak{G}, x)$ or by $Z(x_1, x_2, \dots, x_n)$ is by definition* (cf. [6a, b]):

$$[6c] \qquad Z(x) := \frac{1}{|\mathfrak{G}|} \sum_{\sigma \in \mathfrak{G}} x_1^{c_1(\sigma)} x_2^{c_2(\sigma)} \dots x_n^{c_n(\sigma)}$$

$$[6d] \qquad = \frac{1}{|\mathfrak{G}|} \sum |\mathfrak{G}(c_1, c_2, \dots, c_n)|\, x_1^{c_1} x_2^{c_2} \dots x_n^{c_n},$$

where the last summation takes place over all integers $c_i \geq 0$ such that $c_1+2c_2+\cdots=n=|N|$.

The fact that the expressions [6c] and [6d] are equal follows from [6b]. The polynomial $Z(x)$ has at most $p(n)$ terms ([1e], p. 95) and the weight is n: $Z(\lambda x_1, \lambda^2 x_2, \dots) = \lambda^n Z(x_1, x_2, \dots)$. The following are a few examples.

(1) If $\mathfrak{G}$ consists of the identity permutation e only, then $Z(x) = x_1^n$.

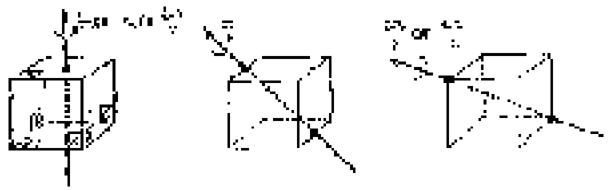
(2) If $\mathfrak{G} = \mathfrak{S}(N)$ (the symmetric group of N), we get, by [2b] (p. 233), applied to the form [6d] of $Z(x)$, and also, by [3b, c] (p. 134) for (*):

$$[6e] \qquad Z(x) = \sum_{c_1+2c_2+\cdots=n} \frac{1}{c_1!\,c_2!\dots}\left(\frac{x_1}{1}\right)^{c_1}\left(\frac{x_2}{2}\right)^{c_2}\dots$$

$$\overset{(*)}{=} \frac{1}{n!} Y_n(x_1, 1!\,x_2, 2!\,x_3, \dots).$$

(3) Let N be the set of the 6 faces of a cube, $N := \{A, B, C, D, E, F\}$ (Figure 42), and let $\mathfrak{G}$ be the group of permutations of N induced by the rotations of the cube. For instance, a rotation of $+\pi/2$ (around the axis, in Figure 42) gives the permutation $\sigma = \begin{pmatrix} A & B & C & D & E & F \\ B & C & D & A & E & F \end{pmatrix}$ for which we have,

by [6a]: $c_1(\sigma)=2$, $c_2(\sigma)=c_3(\sigma)=0$, $c_4(\sigma)=1$, $c_5(\sigma)=c_6(\sigma)=0$, hence the monomial $\frac{1}{24}x_1^2x_4$ in $Z(x)$. There are 6 kinds of rotations, which can be described by Figures 42a, b, c, namely, a rotation of $\pi/2$ or π or $3\pi/2$ around a line joining the centers of opposite faces (Figure 42a),



(a) (b) (c)

Fig. 42.

a rotation of π around a line joining the centers of opposite edges (Figure 42b) and rotations of $2\pi/3$ or $4\pi/3$ around a line joining opposite vertices (Figure 42c). Making up the list of permutations of each kind, we finally find, by [6e]:

$$[6f] \qquad Z(x)=\frac{1}{24}(x_1^6+3x_1^2x_2^2+6x_1^2x_4+6x_2^3+8x_3^2).$$

DEFINITION C. *The stabilizer of x ($\in N$) with respect to $\mathfrak{G}$ ($\leq\mathfrak{S}(N)$), denoted by $\mathfrak{G}(x)$, is the set of permutations $\sigma\in\mathfrak{G}$ for which $\sigma(x)=x$.*

It is clear that $\mathfrak{G}(x)$ is a subgroup of $\mathfrak{G}$.

DEFINITION D. *For $\mathfrak{G}\leq\mathfrak{S}(N)$, the orbit of x ($\in N$) under $\mathfrak{G}$, denoted by $x^{\mathfrak{G}}$, is the set of all $y\in N$ for which there exists $\sigma\in\mathfrak{G}$ such that $y=\sigma(x)$.*

In particular, the orbit of x under the subgroup $\langle\sigma\rangle$ generated by σ, $\langle\sigma\rangle=\{e,\sigma,\sigma^2,\ldots\}$ is just $x^{\langle\sigma\rangle}=\{x,\sigma(x),\sigma^2(x),\ldots\}$ (see p. 231). For $x\neq x'$ either $x^{\mathfrak{G}}=x'^{\mathfrak{G}}$ or $x^{\mathfrak{G}}\cap x'^{\mathfrak{G}}=\emptyset$. The set Ω of all (different) orbits is hence a *partition* of N, $N=\sum_{\omega\in\Omega}\omega$.

THEOREM A *(on the stabilizer). For every $x\in N$ and every group $\mathfrak{G}\leq\mathfrak{S}$, the order of $\mathfrak{G}$ equals the product of the order of the stabilizer $\mathfrak{G}(x)$ by*

the size of the orbit $x^{\mathfrak{G}}$:

$$[6g] \qquad |\mathfrak{G}(x)|\cdot|x^{\mathfrak{G}}|=|\mathfrak{G}|.$$

In other words, denoting by Ω the set of orbits, $\sum_{\omega\in\Omega}\omega=N$:

$$[6h] \qquad x\in\omega\in\Omega\Rightarrow|\mathfrak{G}(x)|\cdot|\omega|=|\mathfrak{G}|.$$

■ It is clear that for each permutation $\alpha\in\mathfrak{G}$:

$$[6i] \qquad |\alpha\mathfrak{G}(x)|=|\mathfrak{G}(x)|,$$

where $\alpha\mathfrak{G}(x):=\{\alpha\beta\mid\beta\in\mathfrak{G}(x)\}$ is a *left coset* of the subgroup $\mathfrak{G}(x)$ of $\mathfrak{G}$.

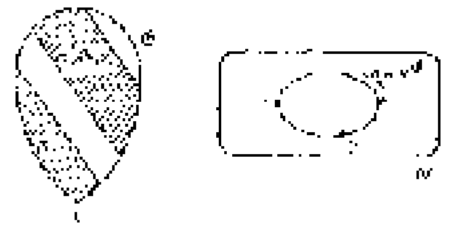


Fig. 43

For each y of the orbit of x, $y\in x^{\mathfrak{G}}:=\omega$ (Figure 43) we choose one single permutation $\alpha=\alpha_y\in\mathfrak{G}$ such that $y=\alpha(x)$, and we consider the map $f: y\mapsto\alpha_y\mathfrak{G}(x)$. It is easily verified that f is a *bijection* of ω onto the set of left cosets of $\mathfrak{G}(x)$. All these cosets have the same number of elements, [6i], and since they constitute together a partition of $\mathfrak{G}$, we get: $|\mathfrak{G}|=$ the number of elements in every class $\times$ the number of classes $=|\mathfrak{G}(x)|\cdot|\omega|$. ■

THEOREM B (Burnside-Frobenius). *Let Ω stand for the set of orbits of $\mathfrak{G}$. Then we have:*

$$[6j] \qquad |\Omega|=\frac{1}{|\mathfrak{G}|}\sum_{\sigma\in\mathfrak{G}}|N_0(\sigma)|,$$

where $N_0(\sigma)$ is the set of fixed points of σ.

■ Let E be the set of pairs (x,σ), $\sigma\in\mathfrak{G}$ such that $\sigma(x)=x$. Clearly, we have the following divisions:

$$[6k] \qquad E=\sum_{\sigma\in\mathfrak{G}}\{(x,\sigma)\mid\sigma(x)=x\}=\sum_{x\in N}\{(x,\sigma)\mid\sigma(x)=x\}.$$

Now, for fixed σ, $|\{(x,\sigma) \mid \sigma(x)=x\}| = |\{x \mid x\in N, \sigma(x)=x\}| = |N_0(\sigma)|$, and for fixed x, $|\{(x,\sigma) \mid \sigma(x)=x\}| = |\{\sigma \mid \sigma\in\mathfrak{G}, \sigma(x)=x\}| = |\mathfrak{G}(x)|$. Hence, by passing to the cardinals in [6e], and with [5h] for (*):

$$|E| = \sum_{\sigma\in\mathfrak{G}} |N_0(\sigma)| = \sum_{x\in N} |\mathfrak{G}(x)| = \sum_{\varphi\in\Phi} \Big(\sum_{x\in\varphi} |\mathfrak{G}(x)|\Big)$$

$$\overset{(*)}{=} \sum_{\varphi\in\Phi} \Big(\sum_{x\in\varphi} \frac{|\mathfrak{G}|}{|\varphi|}\Big) = \sum_{\varphi\in\Phi} |\mathfrak{G}| = |\Phi|\,|\mathfrak{G}|. \quad ■$$

6.7. THEOREM OF PÓLYA

(I) *An example*

In order to clarify the aim of this section consider the following problem. *In how many ways can one paint the six faces of a cube in at most r colours, it being understood that two colourings will not be distinguished if they can be transformed into each other by a rotation of the cube.* In the last case the colourings are called *equivalent*. The class of colourings that are all equivalent to a given one, is called a *model* or a *configuration*. For example, in the case of two colours, white and blue, $r=2$, the colourings {blue: E, F; white: the rest} and {blue: A, C} are equivalent (Figure 44), but these two are not equivalent to the colouring {blue: A, B}. Direct counting shows that there are only 10 models for all possible $2^6=64$ possible colourings. Figure 45 shows the 6 models corresponding with at most 3 blue faces (blue = hatched), the 4 remaining models can be obtained from the set of models with at most 2 blue faces, by interchanging the colours white and blue.

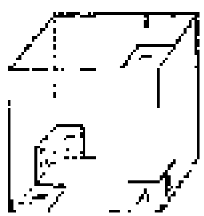


Fig. 44.

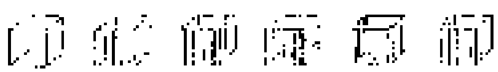


Fig. 45.

(II) *Statement of the problem*

Let D and R be two finite sets, $|D|=d$, $|R|=r$, and let $\mathfrak{G}$ be a group of permutations of D. $F=R^D$ is the set of maps of D into R, and $\mathfrak{F}$ is the partition of F consisting of the $\sim$ equivalence classes on F defined by:

[7a] $\qquad f\sim g \Leftrightarrow \exists\sigma\in\mathfrak{G}, \quad g=f(\sigma),$

which means: $\forall x\in D$, $g(x)=f(\sigma(x))$.

This is an equivalence indeed, because (I) $f=f(\iota)$, (II) $g=f(\sigma)\Rightarrow f=g(\sigma^{-1})$, (III) $g=f(\alpha)$, $h=g(\beta)\Rightarrow h=f(\alpha\beta)$. Each class $\mathfrak{f}\in\mathfrak{F}$ is called a *model*.

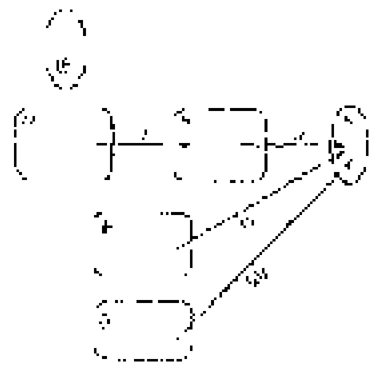


Fig. 46.

Let also A be a commutative ring, and w a map from R into A, called *weight*. We define the weight of $f\in F$ by:

[7b] $\qquad w(f) := \prod_{x\in D} w(f(x)),$

and the *inventory* of each subset $F'\subset F$, denoted by $W(F')$, by:

[7c] $\qquad W(F') := \sum_{f\in F'} w(f).$

It is easy to see, by [7a, b], that:

[7d] $\qquad f\sim g \Rightarrow w(f)=w(g).$

Thus we can define the *weight* $\mathfrak{W}(\mathfrak{f})$ of a *model* $\mathfrak{f}\in\mathfrak{F}$ by:

[7e] $\qquad \mathfrak{W}(\mathfrak{f}) := w(f), \quad$ where $\quad f\in\mathfrak{f}.$

(f is an arbitrary representative of the equivalence class $\mathfrak{f}$). Like in [7c], the *inventory* $\mathfrak{W}(\mathfrak{F}')$ of each $\mathfrak{F}' \subset \mathfrak{F}$ is defined as follows:

$$[7f] \qquad \mathfrak{W}(\mathfrak{F}') = \sum_{\mathfrak{f} \in \mathfrak{F}'} \mathfrak{W}(\mathfrak{f}).$$

The problem now is to compute $\mathfrak{W}(\mathfrak{F})$.

In the case of example (I), D is the set of the faces of the cube, R is the set with two elements, 'blue' and 'white'. The weight function w is defined by $w(\text{blue})=t$, $w(\text{white})=u$; A is the ring of polynomials in two variables t, u. $\mathfrak{G}$ is the group of permutations of the faces of the cube, which we studied already on p. 246, F is the set of colourings of the fixed cube, and $\mathfrak{F}$ is the set of models of colourings. If $W(f)=t^p u^q$, this means, by [7b], that the colouring f is of type (p, q) in the sense that f contains p blue faces and q white faces, $p+q=6$. Hence, we have:

$$[7g] \qquad \mathfrak{W}(\mathfrak{F}) = \sum_{\mathfrak{f} \in \mathfrak{F}} \mathfrak{W}(\mathfrak{f}) = \sum_{p,q} a(p,q)\, t^p u^q = P(t, u),$$

where $a(p,q)$ is the number of models of type (p,q). The *total* number of models is then equal to:

$$[7h] \qquad \sum_{p,q} a(p,q) = P(1,1).$$

(III) *Theorem of Pólya.* ([Pólya, 1937], and in other form [Redfield, 1927]. We follow the exposition of [De Bruijn, 1964].) *Let $Z(x_1, x_2, \ldots, x_d)$ be the cycle indicator polynomial of the group of permutations $\mathfrak{G}$ of D ([6c, d], p. 246), then we have for the value of the inventory of $\mathfrak{F}$:*

$$[7i] \qquad \mathfrak{W}(\mathfrak{F}) := \sum_{\mathfrak{f} \in \mathfrak{F}} \mathfrak{W}(\mathfrak{f}) = Z\Big(\sum_{y \in R} w(y), \sum_{y \in R} w^2(y), \ldots, \sum_{y \in R} w^d(y)\Big),$$

where $\mathfrak{W}, w, \mathfrak{F}, \mathfrak{f}, R$ are defined in the previous section.

■ *Let F_ξ be the set of the $f \in F$ for which $W(f)=\xi$. It appears that we can consider $\mathfrak{G}$ as a group of permutations of F_ξ (the verification is easy), when we define $\sigma(f)$, for $\sigma \in \mathfrak{G}$, $f \in F_\xi$, by:

$$[7j] \qquad \forall x \in D, \qquad \sigma(f)(x) = f(\sigma(x)).$$

It follows that the conditions of Theorem B (p. 249) are satisfied, if we take N instead of F_ξ, and if we change $N_1(\sigma)$ into:

$$[7k] \qquad F_\xi(\sigma) := \{f \mid f \in F_\xi,\ \sigma f = f\}$$

(here $\sigma f = f$ means that $\forall x \in D$, $f(\sigma(x)) = f(x)$). The number of models (= orbits) whose weight is ξ, $f \in F_\xi$, is hence equal to (using [6j], p. 249):

$$[7l] \qquad \frac{1}{|\mathfrak{G}|} \sum_{\sigma \in \mathfrak{G}} |F_\xi(\sigma)|.$$

Thus, by [7l] for (*), and by [7k] for (**),

$$[7m] \qquad \mathfrak{W}(\mathfrak{F}) = \sum_{\mathfrak{f} \in \mathfrak{F}} \mathfrak{W}(\mathfrak{f}) \overset{(*)}{=} \sum_{\xi \in A} \Big(\xi \cdot \frac{1}{|\mathfrak{G}|} \sum_{\sigma \in \mathfrak{G}} |F_\xi(\sigma)|\Big) =$$
$$= \frac{1}{|\mathfrak{G}|} \sum_{\sigma \in \mathfrak{G}} \Big(\sum_{\xi \in A} \xi |F_\xi(\sigma)|\Big) \overset{(**)}{=} \frac{1}{|\mathfrak{G}|} \sum_{\sigma \in \mathfrak{G}} \Big(\sum_{\sigma f = f} W(f)\Big).$$

In other words, if $\mathfrak{B} = (B_1, B_2, \ldots, B_k)$ is the partition of D consisting of the *orbits* of σ (in the sense of Definition D, p. 248), the last summation of [7m] can be taken over all f that are *constant* on each of these blocks $B \in \mathfrak{B}$. Giving such a function f is hence equivalent to giving a mapping φ of $\mathfrak{B}$ into R, $\varphi \in R^{\mathfrak{B}}$. Under these circumstances, choose $b_i \in B_i$, $i \in [k]$, and then apply Theorem A (p. 248) in (*) to obtain the expansion of a product of sums:

$$[7n] \qquad \sum_{\sigma f = f} W(f) = \sum_{\sigma f = f} \prod_{x \in D} w(f(x))$$
$$= \sum_{\sigma f = f} \prod_{1 \le i \le k} \{w(f(b_i))\}^{|B_i|}$$
$$= \sum_{\varphi \in R^{\mathfrak{B}}} \prod_{B \in \mathfrak{B}} \{w(\varphi(B))\}^{|B|}$$
$$\overset{(*)}{=} \prod_{B \in \mathfrak{B}} \sum_{y \in R} \{w(y)\}^{|B|}.$$

Thus we recognize the term of $Z(x_1, x_2, \ldots, x_d)$ corresponding with the permutation σ, [6c] (p. 247). In this term, x_1 should be replaced by $\sum_{y \in R} w(y)$, x_2 by $\sum_{y \in R} w^2(y)$, etc. Hence [7i] using [7m (**)]. ■

(IV) *Application to the cube*

We return to the cube of (I) with at most 2 colours. With the weight w

as defined on p. 252, we have $\sum_{y\in Y} w^k(y) = t^k + u^k$; hence by [6f] (p. 248) and [7g, i]:

$$\text{[7o]}\qquad P(t,u)=\tfrac{1}{24}\{(t+u)^6+3(t+u)^2(t^2+u^2)^2+6(t+u)^2\times(t^4+u^4)+6(t^2+u^2)^3+8(t^3+u^3)^2\}=$$
$$=t^6+t^5u+2t^4u^2+2t^3u^3+2t^2u^4+tu^5+u^6.$$

For instance, by [7g], the number of colourings with 4 blue faces and two white faces is equal to the coefficient of t^4u^2 in [7o], hence 2. More generally, if there are c colours, then we have in [7i]: $\sum_{y\in Y} w^k(y) = t_1^k + t_2^k + \cdots + t_c^k$, where $t_1, t_2, \ldots, t_c$ are c variables. Hence, by notation of Exercise 5 (p. 159), for the monomial symmetric functions:

$$\text{[7p]}\qquad P(t_1,t_2,\ldots,t_c)=\tfrac{1}{24}\{(t_1+t_2+\cdots+t_c)^6+3(t_1+t_2+\cdots+t_c)^2(t_1^2+t_2^2+\cdots+t_c^2)^2+\cdots\}=$$
$$=\sum^{(1)} t_1^6+\sum^{(2)} t_1t_2^5+2\sum^{(2)} t_1^2t_2^4+2\sum^{(2)} t_1^3t_2^3+$$
$$+2\sum^{(3)} t_1t_2t_3^4+3\sum^{(3)} t_1t_2^2t_3^3+5\sum^{(4)} t_1t_2t_3t_4^3+6\sum^{(3)} t_1^2t_2^2t_3^2+$$
$$+8\sum^{(4)} t_1t_2t_3^2t_4^2+15\sum^{(5)} t_1t_2t_3t_4t_5^2+30\sum^{(6)} t_1t_2t_3t_4t_5t_6.$$

For instance, there are 15 models of the cube that use 5 given colours for the faces (hence one colour is used twice). The total number v_c of models of cubes with at most c colours is obtained by putting $t_1=t_2=\cdots=t_c=1$ in [7p]. Then we obtain, after simplifications:

$$v_c=c+8\binom{c}{2}+30\binom{c}{3}+68\binom{c}{4}+75\binom{c}{5}+30\binom{c}{6},$$
$$v_2=10,\quad v_3=57,\quad v_4=240,\ \text{etc.}$$

For other applications of the theorem of Pólya, see Exercises 16-20 (pp. 262-265). (Some references to the theorem of Redfield-Pólya: [De Bruijn, 1959, 1963, 1964, 1967], [Foulkes, 1963, 1966], [Harary, 1967], [Read, 1968], [Riordan, 1957b], [Sheehan, 1967].)

SUPPLEMENT AND EXERCISES

1. *Cauchy identity.* Show that $\sum (c_1!\,c_2!\ldots 1^{c_1}2^{c_2}\ldots)^{-1}=1$, where the summation is taken over all sequences of integers $c_i\geq 0$ such that $c_1+2c_2+\cdots=n$.

2. *Return to the permutations with a given number of inversions.* Determine an explicit formula of minimal rank for the number $I(n,k)$ of permutations of $[n]$ with k inversions (cf. p. 237): $I(n,1)=n-1$, $I(n,2)=\binom{n}{2}-1$, $I(n,3)=(1/3!)n(n^2-7)$, $I(n,4)=(1/4!)n(n+1)(n^2+n-14), \ldots$ [Hint: [4i], p. 239, and the 'pentagonal' theorem of Euler, [5e], p. 104.]

3. *$\mathfrak{S}[n]$ and $\mathfrak{S}(N)$ as metric spaces.* (1) The expression $d(\alpha,\beta):=\max_{1\leq i\leq n}|\alpha(i)-\beta(i)|$, where α and β are permutations of $[n]:=\{1,2,\ldots,n\}$, defines a distance on the set $\mathfrak{S}[n]$ of all permutations of $[n]$. Let $\Phi(n,r)$ be the number of elements of an arbitrary ball of radius r, in other words, the number of permutations σ such that $d(\iota,\sigma)\leq r$, where ι stands for the identity permutation. Then, $\Phi(n,1)=F_n$, the *Fibonacci* number (p. 45). Moreover, $\Phi(n,2)=2\Phi(n-1,2)+2\Phi(n-3,2)-\Phi(n-5,2)$ ([Lagrange (R.), 1962a], [Mendelsohn, 1961]). More generally, the computation of $\Phi(n,r)$ is essentially the computation of a permanent (Exercise 15, p. 201). Between two elements α,β one can define also another distance function, namely the number of *inversions* of $\alpha\beta^{-1}$. (2) For each permutation $\alpha\in\mathfrak{S}(N)$, *N finite*, let $N(\alpha)$ be the set of the mobile points of α. Show that $d(\alpha,\beta):=|N(\alpha\beta^{-1})|$ defines a distance on $\mathfrak{S}(N)$. How many points are there in the ball $\{\sigma\mid d(\iota,\sigma)\leq k\}$? (cf. p. 180).

4. *Labeling $\mathfrak{S}[n]$ by inversions.* For every permutation $\sigma\in\mathfrak{S}[n]$ and every integer $k\in[n]$, let $x_k=x_k(\sigma)$ be the number of integers $j\leq k$ such that the pair $(j,k+1)$ is an inversion $(\sigma(j)>\sigma(k+1))$. Evidently $x_k\leq k$. So we can associate with σ the integer $x=x(\sigma)=x_1 1!+x_2 2!+x_3 3!+\cdots+x_{n-1}(n-1)!\leq n!-1$. Conversely, using the factorial representation of integers (Exercise 9, p. 117), show that each x, $0\leq x\leq n!-1$ is the label of a single permutation σ; how to determine this permutation? [*Example.* $\begin{pmatrix}1&2&3&4&5&6\\4&3&6&5&1&2\end{pmatrix}$ has for label $1.1!+1.3!+4.4!+4.5!=583$.]

***5.** *$\mathfrak{S}(N)$ as a lattice.* We associate with every permutation $\sigma\in\mathfrak{S}(N)$ the subset $E(\sigma)\subset\mathfrak{P}_2[N]$ consisting of the pairs $\{i,j\}$ which are not inverted: $i<j\Rightarrow\sigma(i)<\sigma(j)$. Show that $\sigma\leq\sigma'$ if $E(\sigma)\subset E(\sigma')$ endows $\mathfrak{S}[n]$ with a lattice structure ([Guilbaud, Rosenstiehl, 1963]).

6. *Conditioned permutations.* Let a_i be a sequence of integers $1\leqslant a_1<a_2<a_3<\cdots$ and let $\mathfrak{z}(n,k;a_1,a_2,\dots)$ be the number of permutations of N, $n=|N|$, with k orbits, such that each has a number of elements equal to one of the a_i. Then ([Gruder, 1953]):

$$\sum_{n,k}\mathfrak{z}(n,k;a_1,a_2,\dots)\frac{t^n}{n!}u^k=\exp\left\{u\left(\frac{t^{a_1}}{a_1}+\frac{t^{a_2}}{a_2}+\cdots\right)\right\}.$$

More generally, prove a theorem analogous to Theorem B (p. 98) for permutations.

7. *Derangements by number of orbits.* Let $d(n,k)$ be the number of derangements of N, $|N|=n$, with k orbits (p. 231), or permutations with k cycles of length $\geqslant 2$. (1) We have the following GF: $e^{-tu}(1-t)^{-u}=1+\sum_{2k\leqslant n}d(n,k)t^nu^k/n!$. [*Hint:* Use [3b], p. 233.] Hence, $\sum_k(-1)^{k-1}d(n,k)=n-1$. (2) The following recurrence relation holds: $d(n+1,k)=n\{d(n,k)+d(n-1,k-1)\}$, $d(0,0)=1$. ([Appell, 1880], [Carlitz, 1958a], [Tricomi, 1951] and Exercises 11 (p. 293) and 20 (p. 295) about the *associated* Stirling numbers of the first kind, $s_2(n,k)=(-1)^{n+k}d(n,k)$.) (3) For $k\geqslant 2$, and p prime, we have $d(p,k)\equiv 0 \pmod{p(p-1)}$. (4) For all integers l, $\sum_m(-1)^m d(l+m,m)=(-1)^l$. (5) Similarly, $\sum_m(-1)^m d(l+m,m)/(l+m-1)=0$. (6) We have $d(2k,k)=1.3.5\dots(2k-1)$; $d(2k+1,k)=\frac{2}{3}(2k+1)!\{(k-1)!2^k\}^{-1}$; $d(2k+2,k)=\{(4k+5)/18\}(2k+2)!\{(k-1)!2^k\}^{-1}$. A table of the $d(n,k)$ is given below:

$k\backslash n$	2	3	4	5	6	7	8	9	10
1	1	2	6	24	120	720	5040	40320	362880
2			3	20	130	924	7308	64224	623376
3					15	210	2380	26432	303660
4							105	2520	44100
5									945

$k\backslash n$	11	12	13	14	15
1	3628800	39916800	479001600	6227020800	87178291200
2	6636960	76998240	967524480	13096736640	190060335360
3	3678840	47324376	647536032	9418945536	145410580224
4	705320	11098780	177331440	2920525608	49952862960
5	34650	866250	18858840	389449060	7934927000
6		10395	540540	18288270	520059540
7				135135	8648640

(7) Show that the number $d_r(n,k)$ of permutations of N that have k orbits, all of length $\geqslant r$, satisfies the recurrence relation $d_r(n+1,k)=nd_r(n,k)+(n)_{r-1}d_r(n-r+1,k-1)$. N.B.: $d_1(n,k)=s(n,k)$, $d_2(n,k)=d(n,k)$. Cf. Exercise 7, p. 221 and Exercise 20, p. 295.)

8. *The $d(n,k)$ above are used in the asymptotic expansion of $Z_x(n)=1^{xn}+2^{xn}+\cdots+n^{xn}$.* Let $[r,q]=e^{-rx}(1-e^{-x})^{-q-1}$, $x\in\mathbb{C}$, $\operatorname{Re}x>0$. Then:

$$Z_x(n)\approx n^{xn}\sum_{k\geqslant 0}C_kn^{-k},$$

where $C_k=\sum d(q,q-k)A(q,r)(-x)^{q-k}[r,q]/q!$, a double finite summation where $k\leqslant q\leqslant 2k$, $r\leqslant q$, and where the $A(q,r)$ are the Eulerian numbers of p. 240. Thus, $C_0=[0,0]=(1-e^{-x})^{-1}$, $C_1=-(x/2)([1,2]+[2,2])$, ...

9. *The number of solutions of $\sigma^2=e$ in $\mathfrak{S}(N)$.* Let T_n be the number of permutations $\sigma\in\mathfrak{S}(N)$, $|N|=n$, such that $\sigma^2=e$ (= the identity permutation). Such a permutation, or *involution* (or *selfconjugate* permutation of Muir) has a cycle decomposition consisting of transpositions only. Deduce the following relations: $T_n=T_{n-1}+(n-1)T_{n-2}$, $T_0=T_1=1$, and $\sqrt{n}\leqslant T_n/T_{n-1}\leqslant\sqrt{n}+1$. Finally, $\sum_{n\geqslant 0}T_nt^n/n!=\exp(t+t^2/2)$. Show then that $T_n=n!\sum(i!\,j!\,2^j)^{-1}$ where the summation takes place over the pairs (i,j) such that $i+2j=n$. More generally, let $T(n,k)$ be the number of solutions of $\sigma^k=e$, $\sigma\in\mathfrak{S}(N)$ (hence $T_n=T(n,2)$); show that $\sum_{n\geqslant 0}T(n,k)t^n/n!=\exp\{\sum_{d|k}t^d/d\}$, where the last summation is taken over all divisors d of k. (See [Chowla, Herstein, Moore, 1952], [Chowla, Herstein, Scott, 1952], [Jacobsthal, 1949], [Moser, Wyman, 1955a], [Nicolas, 1969].) Use this to obtain the recurrence relation $T(n+1,k)=\sum_{d|k}(n)_{d-1}\times T(n-d+1,k)$ and the first values of $T(n,k)$:

$n\backslash k$	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
2	1	2	1	2	1	2	1
3	1	4	3	4	1	6	1
4	1	10	9	16	1	18	1
5	1	26	21	56	25	66	1
6	1	76	81	256	145	396	1
7	1	232	351	1072	505	2052	721

10. ***Permutations with ordered orbits, outstanding elements*** **([Sade, 1955]).** For each subset $A\subset[n]$, we denote by $\imath(A)$ the smallest integer $\in A$, called the *initial integer* of A. Let σ be a permutation of $[n]$, $\sigma\in\mathfrak{S}[n]$, whose orbits are numbered, say $\Omega_1(\sigma),\Omega_2(\sigma),\dots,\Omega_k(\sigma)$, $\sum_{1\le i\le k}\Omega_i(\sigma)=[n]$, such that $\imath(\Omega_1(\sigma))<\imath(\Omega_2(\sigma))<\dots<\imath(\Omega_k(\sigma))$, $k\le n$. (1) Let $F(n,k)$ be the number of $\sigma\in\mathfrak{S}[n]$ such that $n\in\Omega_k(\sigma)$. Show that

$$F(n,k)=(n-2)F(n-1,k)+F(n-1,k-1),\quad F(n,1)=\tfrac{1}{2}n!,\quad F(n,n)=1.$$

Make a complete study of this double sequence $F(n,k)$. (Find its GF, establish recurrence relations, etc.)

$n\backslash k$	1	2	3	4	5	6
1	1					
2	1	1				
3	3	2	1			
4	12	7	4	1		
5	60	33	19	7	1	
6	360	192	109	47	11	1

(2) Let $g(n,k,c)$ be the number of permutations of $[n]$ whose k-th orbit has c elements. Then $g(n,k,c)=(n-1)g(n-1,k,c)+g(n-1,k-1,c)$.
(3) An *outstanding element* $j(\in[n])$ of $\sigma\in\mathfrak{S}(n)$ is, by definition, an element such that $\sigma(j)>\sigma(i)$ for all $i<j$. We make the convention of calling 1 outstanding too. Show that the number of permutations of $[n]$ with k outstanding elements equals $s(n,k)$ ([Rényi, 1962]).

11. ***Alternating permutations of André, Euler numbers and tangent numbers.*** (For an exhaustive study of this problem, see [André, 1879a, 1881, 1883a, 1894, 1895], and [Entringer, 1966] for a reformulation. The expressions we find for $(\cos t)^{-1}$ and $\operatorname{tg} t$ give a *combinatorial interpretation* of the Euler and Bernoulli numbers, [14a,b], p. 48, and Exercise 36, p. 88.)
We will call a permutation $\sigma\in\mathfrak{S}[n]$ *alternating* if and only if the $(n-1)$ differences $\sigma(2)-\sigma(1)$, $\sigma(3)-\sigma(2),\dots,\sigma(n)-\sigma(n-1)$ have alternating signs. For example $\begin{pmatrix}1&2&3&4\\1&3&2&4\end{pmatrix}$ and $\begin{pmatrix}1&2&3&4\\3&1&4&2\end{pmatrix}$ are alternating, but $\begin{pmatrix}1&2&3&4\\3&2&1&4\end{pmatrix}$ and $\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}$ are not. We put $A_0=A_1=A_2=1$ and we let $2A_n$ be the number of alternating permutations of $[n]$, $n\ge3$. Show that $2A_{n+1}=\sum_{k=0}^{n}\binom{n}{k}A_kA_{n-k}$ and that $\sum_{n\ge0}A_nt^n/n!=\operatorname{tg}(\pi/4+t/2)$. Use this to

obtain:

$$\sum_{n\ge0}A_{2n}t^{2n}/(2n)!=(\cos t)^{-1}$$

and

$$\sum_{n\ge0}A_{2n+1}t^{2n+1}/(2n+1)!=\operatorname{tg} t.$$

Hence $A_{2n}=|E_{2n}|$, where E_{2n} is the Euler number (p. 48), and the A_{2n+1}, often called *tangent numbers*, have the following first values ([Knuth, Buckholtz, 1967], for $m\le120$; see also [Estanave, 1902], [Schlömilch, 1857], [Schwatt, 1931], [Toscano, 1936].):

m	1	3	5	7	9	11	13
A_m	1	2	16	272	7936	353792	22368256

m	15	17	19	21
A_m	1903757312	209865342976	29088885112832	4951498053124096

With Exercise 36, p. 88, and p. 49, $A_{2n-1}=(-1)^{n-1}B_{2n}4^n(4^n-1)/2n=4^{n-1}|G_{2n}|/n$. Also prove the following explicit values:

$$A_{2n}=\sum_{1\le j\le k\le 2n}(-1)^{j+n}\binom{2k}{k-j}j^{2n}/2^{k-1},$$

$$A_{2n+1}=\sum_{1\le j\le k\le 2n+1}(-1)^{j+n}\binom{2k}{k-j}(k-1)j^{2n}/2^{k-2}.$$

Moreover, as a function of the *Eulerian polynomials* $A_n(u)$ of p. 244, the tangent number A_{2n+1} equals $A_{2n+1}(-1)$.

Finally, it may be valuable to introduce *other tangent numbers* $T(n,k)$ such that $(\operatorname{tg} t)^k/k!=\sum_{n\ge k}T(n,k)t^n/n!$, in order to compute the $A_{2n+1}=T(2n+1,1)$. In fact, we have $T(n-1,k)=T(n,k-1)+k(k+1)\times T(n,k+1)$, hence the first values of $T(n,k)$:

$n\backslash k$	1	2	3	4	5	6	7	8	9	10	11
1	1										
2		1									
3	2		1								
4		8		1							
5	16		20		1						
6		136		40		1					
7	272		616		70		1				
8		3968		2016		112		1			
9	7936		28160		5376		168		1		
10		176896		135680		12432		240		1	
11	353792		1805056		508640		25872		330		1

Find a formula of rank 2 for $T(n, k)$. Of course, these numbers are *inverses* (p. 143) of the *arctangent numbers* $t(n, k)$ defined by $(\operatorname{arctg} t)^k/k! = \sum_{n\geqslant k} t(n, k)\, t^n/n!$, for which holds $t(n+1, k) = t(n, k-1) - n(n-1) \times t(n-1, k)$, the first values being:

n\k	1	2	3	4	5	6	7	8	9	10
1	1									
2		1								
3	−2		1							
4		−8		1						
5	24		−20		1					
6		184		−40		1				
7	−720		784		−70		1			
8		−8448		2464		−112		1		
9	40320		−52352		6384		−168		1	
10		648576		−229760		14448		−240		1
11	−3628800		5360256		−804320		29568		−330	

*12. *The number of terms of a symmetric determinant.* (1) Let be given two permutations $\alpha, \beta \in \mathfrak{S}(N)$, $|N| = n$. Show that the following relation is an equivalence relation: "If γ is a cycle of α (or β), then γ or γ^{-1} is a cycle of β (or α)". (2) The number of equivalence classes of type $[c_1, c_2, \ldots]$ equals $n!/(c_1!\,c_2!\ldots 1^{c_1} 2^{c_2} \ldots 2^{c_3+c_4+\cdots})$. (3) The total number a_n of classes satisfies $\sum_{n\geqslant 0} a_n t^n/n! = (1-t)^{-1/2}\exp(t/2 + t^2/4)$. (4) The difference between the numbers of 'even' classes and 'odd' classes, denoted by a'_n, satisfies $\sum_{n\geqslant 0} a'_n t^n/n! = (1+t)^{1/2}\exp(-t/2 + t^2/4)$ (cf. [3e], p. 2[illegible].) (5) It follows that $a_{n+1} = (n+1)\, a_n - \binom{n}{2} a_{n-2}$ and $a'_{n+1} = -(n-1)\, a'_n - \binom{n}{2} \times a'_{n-2}$. (6) Show that the numbers p_n and q_n of 'positive' and 'negative' terms of a *symmetric* determinant of order n satisfy $p_n + q_n = a_n$, $p_n - q_n = a'_n$. (7) Treat all the preceding questions for the case of 'derangements', in which case the determinant of (6) is supposed to have only 0 on the main diagonal. ([*Pólya, Szegö, II, 1925], p. 110, Exercises 45–46.)

*13. *Permutations by number of 'sequences'.* (For many other properties, see [André, 1898].) Let σ be a permutation of $[n]$, $\sigma \in \mathfrak{S}[n]$. A *sequence* of length $l\,(\geqslant 2)$ of σ is a maximal interval of integers $[i, i+l-1] = \{i, i+1, \ldots, i+l-1\}$ on which σ is monotonic. The sequence is called intermediary or left or right according to whether $1 < i$, $i+l-1 < n$ or $i = 1$ or $i+l-1 = n$. A *peak* of σ is a maximum with respect to σ. The peak (i, i) is called intermediary or left or right, when $1 < i < n$ or $\sigma(i-1) < \sigma(i) > \sigma(i+1)$ or $i = 1$, $\sigma(1) > \sigma(2)$ or $i = n$, $\sigma(n-1) < \sigma(n)$, respectively. Let $\mathbf{P}(n, s)$ be the set of permutations of $[n]$ with s sequences, and let $P_{n,s} = |\mathbf{P}(n, s)|$. Using the map φ, introduced in [5e] (p. 242) from $\mathbf{P}(n, s)$ into $\mathbf{P}(n-1, s) + \mathbf{P}(n-1, s-1) + \mathbf{P}(n-1, s-2)$, as well as the notations given above, show that $P_{n,s} = sP_{n-1,s} + 2P_{n-1,s-1} + (n-s)P_{n-1,s-2}$. For all k, $n \geqslant 2k+2$: $1^k P_{n,1} + 3^k P_{n,3} + 5^k P_{n,5} + \cdots = 2^k P_{n,2} + 4^k P_{n,4} + \cdots$. Finally, $\sum_{n,s} P_{n,s} u^s t^n/n! = (1+u)^{-1}\big((1-u)(1-\sin(v + t\cos v))^{-1} - 1\big)$, where $u = \sin v$.

n\s	1	2	3	4	5	6	7	8	9
2	2								
3	2	4							
4	2	12	10						
5	2	28	58	32					
6	2	60	236	300	122				
7	2	124	836	1852	1682	544			
8	2	252	2766	9576	14622	10332	2770		
9	2	508	8814	45096	104526	119964	59254	15872	
10	2	1020	27472	201030	650396	1106820	1037952	505909	101042

Evidently, $P_{n,n-1} = 2A_n$ (Exercise 14, p. 258.) For each sequence $Q = (q_1, q_2, \ldots, q_{n-1})$ of ± 1, let us denote the number of permutations $\sigma \in \mathfrak{S}[n]$ such that $q_j = \operatorname{sg}(\sigma(j+1) - \sigma(j))$, $j \in [n-1]$, by $[Q]$. Giving Q is evidently equivalent to giving the indices $k_1, k_2, \ldots, k_r$ of the q_k that are equal to -1 $(r \leqslant n-1)$. We use the convention $k_0 = 0$ and $k_{r+1} = n$. Show that:

$$[Q] = \det\begin{pmatrix} \binom{k_1}{k_0} & \binom{k_1}{k_1} & \cdots & \binom{k_1}{k_r} \\ \binom{k_2}{k_0} & \binom{k_2}{k_1} & \cdots & \binom{k_2}{k_r} \\ \cdots & \cdots & \cdots & \cdots \\ \binom{k_{r+1}}{k_0} & \binom{k_{r+1}}{k_1} & \cdots & \binom{k_{r+1}}{k_r} \end{pmatrix}.$$

([Niven, 1968], [De Bruijn, 1970].)

14. *Permutations of* $[n]$ *by number of components.* To every $\sigma \in \mathfrak{S}[n]$ we

associate the *division* $[n]=I_1+I_2+\cdots+I_k$, where the *components* I_h of σ are the *smallest intervals* such that $\sigma(I_h)=I_h$, $h=1,2,\ldots,k$. For example, $\sigma=\begin{pmatrix}1&2&3&4&5&6\\3&1&2&4&6&5\end{pmatrix}$ has the three components $\{1,2,3\}$, $\{4\}$, $\{5,6\}$, and the identity has n components. A permutation is said to be *indecomposable* if it has one component; so is σ, if $\sigma(k)=n-k+1$. We denote by $C(n,k)$ the number of permutations with k components. Introducing the Euler formal series $e(t):=\sum_{n\geq 0} n!t^n$ (see also Exercise 34, p. 171), we have the GF: $i(t):=\sum_{n\geq 1} C(n,1)\,t^n=1-(e(t))^{-1}$ and $\sum_{n\geq k} C(n,k)\,t^n=(i(t))^k$. Find a simple recurrence for $C(n,k)$. [*Hint.* Use $t^2e'=(1-t)e-1$ (Exercise 16, p. 294).] Here are the first values of $C(n,k)$.

$n\backslash k$	1	2	3	4	5	6	7	8	9	10
1	1									
2	1	1								
3	3	2	1							
4	13	7	3	1						
5	71	32	12	4	1					
6	461	177	58	18	5	1				
7	3447	1142	327	92	25	6	1			
8	29093	8411	2109	531	135	33	7	1		
9	273343	69692	15366	3441	804	198	42	8	1	
10	2829325	642581	125316	24892	5226	1145	252	52	9	1

15. *Cayley representation of a finite group.* Let N be a finite multiplicatively written group, $n=|N|$. With every $a\in N$ we associate the permutation σ_a of N defined by $\sigma_a x:=ax$, $x\in N$. Let $\mathfrak{G}$ be the group of these permutations, called the Cayley representation of the group N. Show that $\mathfrak{G}$ is isomorphic to N ($\Leftrightarrow \sigma_a\sigma_b=\sigma_{ab}$) and that $Z(\mathfrak{G};x_1,x_2,\ldots)=(1/n)\sum' \nu(d)\,(x_d)^{n/d}$, where d runs through the set of divisors ≥ 1 of n, and where $\nu(d)$ is the number of elements $a\,(\in N)$ with *order* d.

16. *Cube and octahedron.* (1) Let N be the set of the 8 *vertices* of a cube, and let $\mathfrak{G}$ be the group of permutations of N induced by the rotations of this cube. Then the cycle indicator polynomial $Z(x)$ equals $\frac{1}{24}(x_1^8+9x_2^4+6x_4^2+8x_1^2x_3^2)$. Prove that if N is the set of the 12 *edges*, we have $Z(x)=\frac{1}{24}(x_1^{12}+3x_2^6+6x_4^3+6x_1^2x_2^5+8x_3^4)$. (2) Show that there are only three different ways to distribute three red balls, two black balls and one white ball over the vertices of a regular octahedron in euclidean three-dimensional space. The octahedron is supposed to be freely movable. Generalize to c colours, as on p. 251.

17. *Colourings of a roulette.* (1) Let $\mathfrak{G}$ be the cyclic group of order n. Show that $Z(\mathfrak{G};x_1,x_2,\ldots)=(1/n)\sum_{k=0}^{n-1} x_{n/(k,n)}^{(k,n)}$, where (k,n) is the GCD of k and n. (2) Use this to obtain

$$Z(\mathfrak{G};x_1,x_2,\ldots)=(1/n)\sum_{d|n}\varphi(d)\,(x_d)^{n/d},$$

where $\varphi(d)$ is the Euler function (p. 193), and $d|n$ means 'd divides n'. (3) Now consider a roulette. This is a disc freely rotating around its axis, and divided into n equal sectors. Show that the number of ways to paint the sectors of the roulette into k given colours equals $(1/n)\sum_{d|n}\varphi(d)k^{n/d}$. (Two ways which can be transformed into each other by a rotation are considered equal. [Jablonski, 1892].)

18. *Necklaces with two colours.* Let N be the set of n vertices of a regular polygon, $n=|N|$. Let be given α blue beads and $(n-\alpha)$ red beads, $0\leq\alpha\leq n$. On each vertex a bead is placed, thus obtaining a *necklace*. Let P_n^α be the number of different necklaces. Two necklaces that can be transformed into each other by rotation, or reflection with respect to a diameter, or both, are not distinguished from each other. Then we have $P_n^1=1$, $P_n^2=[n/2]$, $P_n^3=n^2/12$ if $n\equiv 0 \pmod 6$ or $(n^2-1)/12$ if $n\equiv\pm 1 \pmod 6$ or $(n^2-4)/12$ if $n\equiv\pm 2 \pmod 6$ or $(n^2+3)/12$ if $n\equiv 3 \pmod 6$. Compute P_n^4 and generalize. ([Demonde, 1816], [Gilbert, Riordan, 1961], [Lagrange, R., 1962b], [Moreau, 1872], *[Riordan, 1958], p. 162, [Titsworth, 1964].)

*19. *The number of unlabeled graphs.* Two graphs $\mathscr{G}$ and $\mathscr{G}'$ over N are called *equivalent*, or *isomorphic* if there exists a permutation σ of N, which induces a map from the set of edges of $\mathscr{G}$ onto the set of edges of $\mathscr{G}'$. In other words, $\exists\sigma\in\mathfrak{S}(N)$, $\{x,y\}\in\mathscr{G}\Leftrightarrow\{\sigma(x),\sigma(y)\}\in\mathscr{G}'$. Each equivalence class, thus obtained, is called an *unlabeled graph*, abbreviated UG (graphs as we have seen on p. 61 are called *labeled graphs*, to distinguish them from the UG; their vertices are distinguishable). For instance, there are three UG's with 4 nodes and 3 edges: $\mathscr{G}_1$, $\mathscr{G}_2$, $\mathscr{G}_3$ ($\mathscr{G}_4$ is equivalent to $\mathscr{G}_1$) (see Fig. 36a).

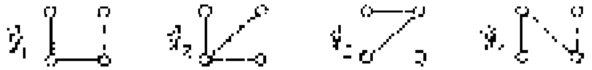


Fig. 47.

From now on, $N=[n]=\{1,2,\ldots,n\}$. With each $\sigma\in\mathfrak{S}(n)$ we associate the permutation $\tilde\sigma$ of $\mathfrak{P}_2[n]$ defined for each pair $\{x,y\}$ by $\tilde\sigma\{x,y\}:=\{\sigma(x),\sigma(y)\}$. The set of the $\tilde\sigma$ forms together the 'group of pairs', denoted by $\mathfrak{S}^{(2)}[n]$ ($\leqslant\mathfrak{S}(\mathfrak{P}_2[n])$), which has a cycle indicator polynomial $Z(\mathfrak{S}^{(2)}[n],x_1,x_2,\ldots)$, denoted by $Z_n(x_1,x_2,\ldots)$. (1) Show that the number $g_{n,k}$ of UG satisfies $\sum_k g_{n,k}x^k=Z_n(1+x,1+x^2,1+x^3,\ldots)$. (2) For $\sigma\in\mathfrak{S}[n]$ of type $[c_1,c_2,\ldots]$, let $I_k[c_1,c_2,\ldots]$ be the number of k-orbits (in $\mathfrak{P}_2[n]$) of $\tilde\sigma$. Then, $Z_n(x_1,x_2,\ldots)$ equals:

$$\frac{1}{n!}\sum_{c_1+2c_2+\cdots=n}\frac{n!}{c_1!\,c_2!\ldots 1^{c_1}2^{c_2}\ldots}x_1^{I_1[c_1,c_2,\ldots]}x_2^{I_2[c_1,c_2,\ldots]}\ldots.$$

(3) Show that $I_k[c_1,c_2,\ldots,c_n]=c_{2k}\binom{c_k}{2}(k-1+\langle k/2\rangle)+(1/2k)\times\times\sum ij\,c_i(c_j-\delta_{i,j})$, where $[i,j]$ is the LCM of i and j, $\delta_{i,j}$ the Kronecker symbol, and $\langle x\rangle=x$ if x is an integer, and $=0$ otherwise, the summation being taken over all (i,j) such that $1\leqslant i\leqslant j\leqslant n$ and $[i,j]=k$. (This theorem, in this form, is due to [Oberschelp, 1967]. Counting unlabeled graphs and digraphs is done in the fundamental paper by [Pólya, 1937], and also in [Harary] and [Read], among others.) Thus, $Z_2=x_1$, $Z_3=(1/3!)(x_1^3+3x_1x_2+2x_3)$, $Z_4=(1/4!)(x_1^6+9x_1^2x_2^2+8x_3^2+6x_2x_4)\ldots$. The first values of $g_{n,k}$ are:

$n\backslash k$	1	2	3	4	5	6	7	8	9	10
2	1									
3	1	1	1							
4	1	2	3	2	1	1				
5	1	2	4	6	6	6	4	2	1	1
6	1	2	5	9	15	21	24	24	21	15
7	1	2	5	10	21	41	65	97	131	148
8	1	2	5	11	24	56	115	221	402	663

*20. *The number of unlabeled m-graphs.* Let us call any system of m-blocks (p. 7) of N an m-graph of N. In particular, an ordinary graph is a 2-graph. Let $g_n^{(m)}$ be the *total* number of *unlabeled* m-graphs (in the sense of the previous exercise). Then, for fixed m, when $n\to\infty$

$$g_n^{(m)}\sim\frac{2^{\binom{n}{m}}}{n!}\left\{1+\frac{\binom{n}{2}}{2^{\binom{n-1}{m-1}}}(1+o(1))\right\}.$$

([Oberschelp, 1968]; see also [*Cattrup, 1950], [Davis, 1953], [Mišek, 1963, 1964], Pólya, 1940]).

*21. *Rearrangements.* This is a generalization of as well a *permutation* and a *minimal path* (p. 20). Let $\mathfrak{X}:=\{x_1,x_2,\ldots,x_n\}$ be a finite set with n elements. A *rearrangement* of $\mathfrak{X}$, (abbreviated RA) is a word of $\mathfrak{X}$ (p. 18). More precisely, a $(c_1,c_2,\ldots,c_n)$-RA of $\mathfrak{X}$, say f, is a word in which the letter x_i occurs c_i times, $c_i\geqslant 0$, $i\in[n]$. We say also 'RA of $x_1^{c_1}x_2^{c_2}\ldots x_n^{c_n}$', or 'word of *specification* $(c_1,c_2,\ldots,c_n)$', and we denote $f\in\mathfrak{R}(c_1,c_2,\ldots,c_n)$. For instance, for $\mathfrak{X}:=\{a,b,c\}$ the RA $f_1=b\,a\,c\,b\,c\,b\,c\,c\,b$ and $f_2=c\,c\,a\,a\,a\,c\,a$ are of specification $(2,4,4)$ and $(4,0,3)$, respectively. For $c_1=c_2=\cdots=c_n=1$, we get back the *permutations* of $\mathfrak{X}$. A RA can be represented as a *minimal path* in the euclidean R^n, which describes a process of counting ballots for an election with n candidates. The word f_2 is shown in Figure 48. (1) The number of $(c_1,c_2,\ldots,c_n)$-RA equals $(c_1,c_2,\ldots,c_n)$ (p. 27). (2) A *sequence* of $f\in\mathfrak{R}(c_1,c_2,\ldots,c_n)$ is a maximal row of consecutive x_i in f, $i\in[n]$. For instance, f_1 has 7 sequences. What is the

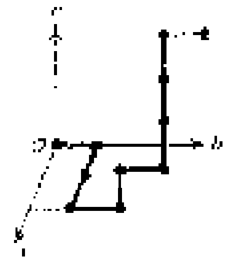


Fig. 48.

number of the $f\in\mathfrak{R}(c_1,c_2,\ldots)$ having s sequences. ([*David, Barton, 1962], p. 1195) (3) Compute $f_{r_1,r_2,\ldots,r_n}(c_1,c_2,\ldots,c_n)$, which is the number of the $(c_1,c_2,\ldots,c_n)$ RA such that between two letters x_i there are at least r_i other letters. (A generalization of [3d], p. 21, and Exercise 1, p. 198.) (4) If $\mathfrak{X}=[n]$, then we can consider f as a map from $[p]$, $p=c_1+c_2$

$=e_1+e_2+\cdots+e_n$, into $[n]$ such that for all $i\in[n]$, $|f^{-1}(i)|=e_i$. (Figure 45 shows $f=2\,1\,1\,2\,3\,2\,3\,3\,3\,2$.) An *inversion* of f is a pair (i,j) such that $1\le i<j\le r$ and $f(i)>f(j)$ (f_1 has 7 inversions). Show that the number $b(e_1,e_2,\ldots,e_n;k)$ of $(e_1,e_2,\ldots,e_n)$-RA of $[n]$ with k inversions, $e_1, e_2,\ldots\ge 1$,

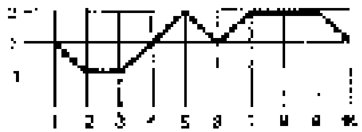


Fig. 45.

has for GF $\sum_k b(e_1,e_2,\ldots;k)u^k$ the following rational fraction:

$$\frac{(1-u)(1-u^2)\cdots(1-u^r)}{\prod_{i=1}^{e_1}(1-u^i)\prod_{i=1}^{e_2}(1-u^i)\cdots\prod_{i=1}^{e_n}(1-u^i)}.$$

(For $e_1=e_2=\cdots=1$, we recover [7a], p. 239.) (5) We call the sum $I(f)$ of the indices $j\in[r-1]$ such that $f(j)>f(j+1)$ (f is a $(e_1,e_2,\ldots,e_n)$ RA of $[n]$) the *index* of f. So the index is the sum of the j where there is a *descent (or fall)*. Show that the number of RA for which $I(f)=k$ equals $b(e_1,e_2,\ldots;k)$. ([MacMahon, 1913, 1916] gives a proof using the GF; [Foata, 1968] and [Cartier, Foata, 1969] give a 'bijective' proof.) (6) An *ascent* (or rise) of a $(e_1,e_2,\ldots)$-RA of $[n]$, f, is an index i such that $f(i)<f(i+1)$. Compute the number $A(e_1,e_2,\ldots,e_n;k)$ of the RA with $(k-1)$ ascents. (These numbers are a generalization of the Eulerian numbers [5e], p. 242. They give the solution to the problem of Simon Newcomb (p. 246).)

***22.** *Folding a strip of stamps.* Given a strip of n stamps labelled $1, 2, \ldots, n$ from left to right, the problem is to determine the number $A(n)$ of ways this strip can be folded along the perforations so that the stamps are piled one on top of each other without destroying the continuity of the strip. It is supposed that stamp labelled 1 has its front side facing the top of the pile and its left edge on the left as we look down on the pile. So $A(1)=1$, $A(2)=2$, and $A(3)=6$ as it is shown by the following figures:

If $n\ge 2$, prove that $A(n)=2n\,a(n)$, where $a(n)$ is a positive integer. Here are the known values of $a(n)$:

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$a(n)$	1	1	2	5	12	33	87	252	703	2105	6099	18689	55639	173423	526937

n	17	18	19	20	21	22	23
$a(n)$	1561047	5137252	16391515	51255709	164951529	538138861	1688329630

n	24	25	26	27	28
$a(n)$	5382512216	17547919924	56335234064	184596351277	596362337295

(Up to 10: [Touchard, 1950, 1952]; up to 12: [Sade, 1949a]; up to 16: [Koehler, 1968]; up to 28: [Lunnon, 1971].)

***23.** *An explicit and combinatorial Stirling expansion for the gamma function of large argument.* Using the Watson lemma for Laplace transforms, show that

$$\Gamma(x)\approx\left(\frac{x}{e}\right)^x\sqrt{\frac{2\pi}{x}}\left(1+\sum_{q\ge1}\frac{c_q}{x^q}\right),\qquad x\to\infty,$$

where the coefficients

$$c_q=\sum_{k=1}^{2q}(-1)^k\frac{d_3(2q+2k,k)}{2^{q+k}(q+k)!}$$

use the number $d_3(n,k)$ of permutations of $[n]$ with k orbits all ≥ 3 (See Exercise 7 p. 256). The first values of c_q are (for $q\le 20$, see [Wrench, 1968]):

q	1	2	3	4	5	6	7
c_q	$\frac{1}{12}$	$\frac{1}{288}$	$\frac{139}{51840}$	$\frac{571}{2488320}$	$\frac{163879}{209018880}$	$\frac{5246819}{75246796800}$	$\frac{534703531}{902961561600}$

EXAMPLES OF INEQUALITIES AND ESTIMATES

In the preceding chapters we have established explicit formulas for counting sets. The sets we wanted to count were of the following type: a finite set N with n elements is given, and then we studied sets of combinatorial objects bound to N that satisfied some additional conditions. If these conditions are not simple, then the explicit formula is usually not simple either, difficult to obtain, and little efficient. It can often be replaced advantageously by upper and lower bounds. Evidently, the closer these bounds fit, the better.

In most of the cases we want to determine conditions in the form of inequalities between certain parameters (integers) that guarantee the existence or non-existence of configurations between these parameters. The search for such inequalities has the charm of challenging problems, since there is no general rule for obtaining this kind of results.

In this chapter we give also an example of the use of probabilistic language, and, moreover, an asymptotic expansion of the most easy kind.

7.1. CONVEXITY AND UNIMODALITY OF COMBINATORIAL SEQUENCES

Just as in the case of functions of a real variable, it is interesting to know the global behaviour of combinatorial sequences of integers: monotony, convexity, extrema; this is a fertile source of inequalities, which are particularly useful in estimates.

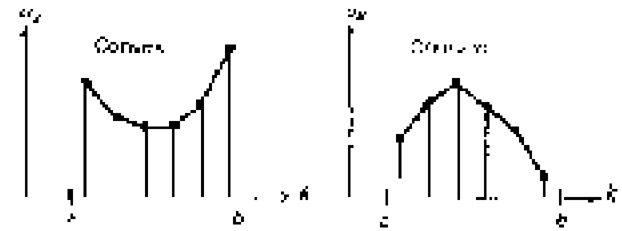
In this respect we recall some definitions.

I. A real sequence v_k, $k=0,1,2,\ldots$, is called *convex* on an interval $[a,b]$ (containing at least 3 consecutive integers) when:

$$[1a] \qquad v_k \leqslant \tfrac{1}{2}(v_{k-1}+v_{k+1}), \quad k\in[a+1,b-1].$$

It is called *concave* on $[a,b]$ if, in [1a], $\leqslant$ is replaced by $\geqslant$. In the case where the inequalities are *strict* for *all* k, v_k is called *strictly convex* or *strictly concave*. [1a] is equivalent to $\Delta^2 v_k := v_{k+2}-2v_{k+1}+v_k \geqslant 0$ for all

$k\in[a,b-2]$ (p. 13). The polygonal representation of v_k has hence the form of Figures 50a or b. For instance, $v_n=\binom{n}{m}$, m fixed $\geqslant 2$, is strictly convex on $[m,\infty]$, because $\Delta^2 v_n=\binom{n+2}{m}-2\binom{n+1}{m}+\binom{n}{m}=\binom{n}{m-2}>0$.



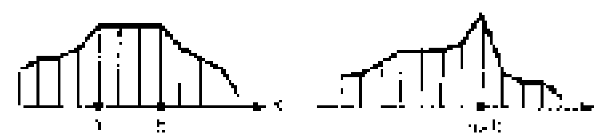
(a) (b)

Fig. 50.

II. A real sequence v_k, $k=0,1,2,\ldots$, is called *unimodal* if there exist two integers a and b such that:

$$[1b] \qquad k\leqslant a-2 \Rightarrow v_k\leqslant v_{k+1};\quad v_{a-1}<v_a=v_{a+1}=\cdots=v_b>v_{b+1};\quad k\geqslant b+1 \Rightarrow v_k\geqslant v_{k+1}.$$

Figure 51a represents the polygon of a unimodal sequence in the case of a *plateau* ($\Leftrightarrow a<b$) with 4 points, and Figure 51b shows the case of a *peak* ($\Leftrightarrow a=b$).



(a) (b)

Fig. 51.

For instance, $v_k=\binom{n}{k}$, n fixed $\geqslant 2$, is unimodal on $[0,n]$ with a peak in $k=\frac{1}{2}n$ if n is even, and with a plateau in $k=(n\pm 1)/2$ if n is odd.

III. A real sequence $v_k>0$, $k=0,1,2,\ldots$, is called *logarithmically convex* in $[a,b]$ if:

$$[1c] \qquad v_k^2 \leqslant v_{k-1}v_{k+1}, \quad k\in[a+1,b-1].$$

It is called *logarithmically concave* if, in [1c], $\leqslant$ is everywhere replaced by $\geqslant$. In the case that the inequalities are *strict* for all k, v_k is called *strictly logarithmically convex* (or *concave*).

The terminology adopted here originates from the fact that [1c] is equivalent to saying that $w_k := \log v_k$ is convex.

THEOREM A. *Each sequence* $v_k(>0)$ *which is logarithmically concave on its interval of definition, say* $[a, b]$, *is there either nondecreasing or nonincreasing or unimodal. Moreover, in the last case, if* v_k *is strictly logarithmically concave, then* v_k *has either a peak or a plateau with 2 points.*

■ $v_k^2 \geqslant v_{k-1}v_{k+1}$ can be written as $v_k/v_{k-1} \geqslant v_{k+1}/v_k$, which proves that $z_k := v_k/v_{k-1}$ is decreasing on $[a+1, b]$, where a and b are supposed to be integers without loss of generality. If $z_b \geqslant 1$ (or $z_{a+1} \leqslant 1$), v_k is increasing (or decreasing) on $[a, b]$. If $z_{a+1} > 1$ and $z_b < 1$, v_k is evidently unimodal. In the last case, if z_k decreases strictly, then there is at most one value of k such that $z_k = 1$, which gives then a plateau of 2 points. ■

THEOREM B. *If the generating polynomial:*

$$[1d] \qquad P(x) := \sum_{0 \leqslant k \leqslant p} v_k x^k, \quad v_p \neq 0,$$

of a finite sequence $v_k(\geqslant 0)$, $0 \leqslant k \leqslant p$, *has only real roots* $(\leqslant 0)$, *then:*

$$[1e] \qquad v_k^2 \geqslant v_{k-1}v_{k+1}\frac{k}{k-1}\cdot\frac{p-k+1}{p-k}, \quad k \in [2, p-1]$$

(*this is one form of the Newton inequalities*, [*Hardy, Pólya, Littlewood, 1952], p. 104); *hence* v_k *is unimodal, either with a peak or with a plateau of 2 points.*

■ Let us first suppose that all the $v_k > 0$. Applying the theorem of Rolle, the polynomial $Q(x, y) := \sum_{k=0}^{p} v_k x^k y^{p-k}$ has only roots with real y/x, so the polynomials $\partial Q/\partial x$ and $\partial Q/\partial y$ also have this property; inductively we find then that this is true for all $\partial^{s+t}Q/\partial x^s \partial y^t$, $s+t \leqslant p-1$. This holds particularly for the second-degree polynomials $\partial^{p-2}Q/\partial x^{k-1}\partial y^{p-k-1}$, whose discriminant is consequently $\geqslant 0$, hence [1e]. Now, if there does exist an l such that $v_l = 0$, $0 < l < p$, then all the roots of $P(x) = 0$ are zero, since these are numbers $\leqslant 0$ whose $(p-l)$-th elementary symmetric function is zero; so finally, $v_k = 0$, $0 \leqslant k \leqslant p-1$, hence [1e] follows again. ■

Now we have a powerful tool for proving unimodality of certain combinatorial sequences.

THEOREM C. *The sequence of the absolute values of the Stirling numbers of the first kind* $s(n, k)$, n *fixed* $(\geqslant 1)$, k *variable* $(\leqslant n)$ *is unimodal, with a peak or plateau of 2 points.*

(In fact, only the peak exists, [Erdős, 1953]; for estimates of its abscissa, see [Hammersley, 1951], [Moser, Wyman, 1958b].)

■ In fact, the "horizontal" polynomial ([5f], p. 213) $\sum_k |s(n, k)| x^k = x(x+1)\cdots(x+n-1)$ has only real roots, and we can apply Theorem B. ■

THEOREM D. *The sequence* $S(n, k)$ *of the Stirling numbers of the second kind,* n *fixed* $(\geqslant 1)$, k *variable* $(\leqslant n)$, *is unimodal with a peak or plateau of 2 points* ([Harper, 1967], [Lieb, 1968]. See also [Baron, 1968], [Dobson, 1968], [Dobson, Rennie, 1969], [Harborth, 1968], [Kanold, 1968a, b], [Wegner, 1970], and Exercise 20, p. 295.)

■ We know ([2b], p. 206) that the $P_n = P_n(x) := \sum_{k \geqslant 0} S(n, k)x^k$ satisfy:

$$\Phi = \Phi(t, x) := \sum_{n \geqslant 0} P_n(x)\frac{t^n}{n!} = \exp(x(e^t - 1)).$$

Now $x\Phi + x\,\partial\Phi/\partial x - \partial\Phi/\partial t = 0$. Hence:

$$[1f] \qquad P_n = x\left(P_{n-1} + \frac{dP_{n-1}}{dx}\right), \quad n \geqslant 1.$$

For $H_n := e^x P_n$, [1f] gives then $H_n = x\,dH_{n-1}/dx$. Applying the theorem of Rolle repeatedly shows the roots of H_n to be all $\leqslant 0$, hence also the roots of P_n are $\leqslant 0$, as they are the same. Then apply Theorem B again. ■

7.5. SPERNER SYSTEMS

DEFINITION. *A system* $\mathscr{S}$ *of distinct blocks of a finite set* N, $\mathscr{S} \subset \mathfrak{P}(N)$, is called a *Sperner system*, if for any two blocks, one is not contained in the other. In other words, if $\mathfrak{s}(N)$ is the family of these systems:

$$(\mathscr{S} \in \mathfrak{s}(N)) \Leftrightarrow (B, B' \in \mathscr{S}) \Rightarrow (B \not\subset B' \quad \text{and} \quad B' \not\subset B)).$$

THEOREM [Sperner, 1928]. *The maximum number of blocks of a Sperner system equals* $\binom{n}{[n/2]}$, *where* $[x]$ *is the largest integer* $\leqslant x$.

■ For all $\mathscr{S} \in \mathfrak{s}(N)$, we will prove with [Lubell, 1966]:

$$[2a] \qquad \sum_{B \in \mathscr{S}} \frac{1}{\binom{n}{|B|}} \leqslant 1.$$

This will imply the theorem, because $\binom{n}{k} \leqslant \binom{n}{[n/2]}$ for all $k \in [0, n]$, hence:

$$\sum_{B \in \mathscr{S}} \frac{1}{\binom{n}{|B|}} \geqslant \sum_{B \in \mathscr{S}} \frac{1}{\binom{n}{[n/2]}} = \frac{|\mathscr{S}|}{\binom{n}{[n/2]}}.$$

From this we get, using [2a], $|\mathscr{S}| \leqslant \binom{n}{[n/2]}$. This maximum value is reached by the Sperner system $\mathfrak{P}_{[n/2]}(N)$. We now prove [2a]. We introduce the name *chain* for a system $\mathscr{C} = \{C_1, C_2, \ldots, C_l\}$ of N, $\mathscr{C} \subset \mathfrak{P}(N)$ such that $C_1 \subset C_2 \subset \cdots \subset C_l$, with *strict* inclusions. A chain is called *maximal* if it has a maximum number of blocks, namely n. Let $\mathfrak{c}(N)$ be the family of maximal chains of N. A maximal chain is evidently completely determined by the permutation $(x_1, x_2, \ldots, x_n)$ of N, given by: $x_1 = C_1$, $x_2 = C_2 - C_1, \ldots, x_n = C_n - C_{n-1}$. Hence $|\mathfrak{c}(N)| = n!$. Now we observe that a given system $\mathscr{S}$ is a Sperner system if and only if *each* chain $\mathscr{C} \in \mathfrak{c}(N)$ satisfies $|\mathscr{C} \cap \mathscr{S}| = 0$ or 1. Let $\mathfrak{c}_{\mathscr{S}}$ be the family of chains $\mathscr{C} \in \mathfrak{c}(N)$ such that $|\mathscr{C} \cap \mathscr{S}| = 1$. We define the map φ from $\mathfrak{c}_{\mathscr{S}}$ into $\mathscr{S}$ by $\varphi(\mathscr{C}) :=$ the unique block $B \in \mathscr{C} \cap \mathscr{S}$. Of course φ is surjective, and for all $B \in \mathscr{S}$, $|\varphi^{-1}(B)| = |B|!(n - |B|)!$. It follows that:

$$[2b] \qquad |\mathfrak{c}_{\mathscr{S}}| = \sum_{B \in \mathscr{S}} |\varphi^{-1}(B)| = \sum_{B \in \mathscr{S}} |B|!\,(n - |B|)!.$$

It now suffices to combine $|\mathfrak{c}_{\mathscr{S}}| \leqslant |\mathfrak{c}(N)| = n!$ with [2b] to obtain [2a]. ■

The number $s(n) := |\mathfrak{s}(N)|$ of Sperner systems (considered systems without repetition, in the sense of p. 3) is just, up to 2 units, the number of elements of a free distributive lattice with n generators, or, the number of monotone increasing Boolean functions with n variables. Since [Dedekind, 1897] numerous efforts have been made to compute or estimate this number [Agnew, 1961], [Gilbert, 1954], [Rivière, 1968], [Yamamoto, 1954]. Actually, the known values are:

n	1	2	3	4	5	6	7
$s(n)$	1	4	18	166	7579	7828352	2414682040996

($s(5)$ due to [Church, 1940], $s(6)$ due to [Ward, 1946], $s(7)$ due to [Church, 1965].) The following upper and lower bounds hold:

$$2^{\binom{n}{[n/2]}} \leqslant s(n) \leqslant 3^{\binom{n}{[n/2]}}$$

([Hansel, 1966]) and also the asymptotic equivalent $\log_2 s(n) \sim \binom{n}{[n/2]}$ ([Kleitman, 1969], [Shapiro, 1970]). Various extensions of the Sperner theorem have been suggested ([Chao-Ko, Erdös, Rado, 1961], [Hilton, Milner, 1967], [Katona, 1966, 1968], [Kleitman, 1968b], [Meshalkin, 1963], [Milner, 1968]).

7.3 ASYMPTOTIC STUDY OF THE NUMBER OF REGULAR GRAPHS OF ORDER TWO ON N

(I) *Graphical and geometrical formulation of the problem*

A regular graph of order r (integer $\geqslant 0$) is a graph on N, $|N| = n$, such that there are r edges adjacent to every node $x \in N$. Let $G(n, r)$ be the number of these graphs. Evidently $G(n, 0) = 1$. For computing $G(n, 1)$, observe that giving a regular graph of order 1 is equivalent to giving a partition of N into disjoint pairs (the edges). Hence $G(2m+1, 1) = 0$ and $G(2m, 1) = (2m)!/(2^m m!)$. We investigate now $G(n, 2) = a_n$. First, we give a geometric interpretation to these numbers ([Whitworth, 1901], p. 269, Exercise 162).

Let be given a set Δ of n straight lines in the plane, $\delta_1, \delta_2, \ldots, \delta_n$, lying in *general position* (no two among them are parallel, and no three among them are concurrent). Let P be the set of their points of intersection, $|P|=\binom{n}{2}$. We call any set of n points from P such that any three different points are not collinear, a *cloud*. An example is shown in Figure 52, in

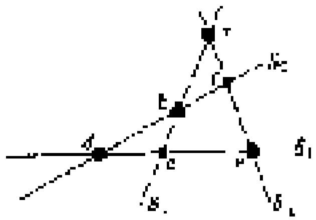


Fig. 52.

the case $n=4$, $\{a, b, d, e\}$. Let $\mathcal{N}(\Delta)$ stand for the set of *clouds* of Δ; then we have:

$$[3a] \qquad N\in\mathcal{N}(\Delta) \Leftrightarrow N\subset P; \quad |N|=n; \quad (\{a, b, c\}\subset P, \; \delta\in\Delta) \Rightarrow \{a, b, c\}\not\subset\delta.$$

Giving a cloud is hence equivalent to giving a regular graph of order 2: it suffices to identify the lines $\delta_1, \delta_2, \ldots, \delta_n$ with the nodes $x_1, x_2, \ldots, x_n$ of N, and each *point of intersection* $\delta_i\cap\delta_j$ with the *edge* $\{x_i, x_j\}$.

For example, with 3 points, we can get only 1 cloud; with 4 points, we have 3 clouds, since the clouds in $\{\delta_1, \delta_2, \delta_3, \delta_4\}$ (Figure 52) are the sets $\{a, b, d, e\}$, $\{a, c, d, f\}$, $\{b, c, e, f\}$. *The problem is to determine the number* $\gamma_n=|\mathcal{N}(\Delta)|$ *of clouds of* Δ.

(II) *A recurrence relation* ([Robinson, 1951, 1952], [Carlitz, 1954b, 1960b]).

Let now $M:=\{a_1, a_2, \ldots, a_{n-1}\}$ be a *cloud* of $\Gamma:=\{\delta_1, \delta_2, \ldots, \delta_{n-1}\}$. It is clear, by [3a], that every straight line δ_i, $i\in[n-1]$, contains exactly two points of M. Now we add an n-th line δ_n, so we obtain $\Delta:=\{\delta_1, \delta_2, \ldots, \delta_{n-1}, \delta_n\}$. We consider then an arbitrary point a_i of M, which belongs to 2 lines, say δ_k and δ_l (of Γ), that intersect δ_n in the points u and v (Figure 53). It is easily seen that $N_i=\{a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{n-1}, u, v\}$ is a *cloud* of Δ. Thus, if we let a_i run through the set $a_1, a_2, \ldots, a_{n-1}$, we

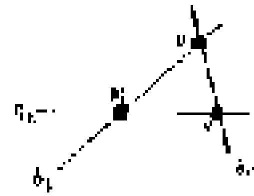


Fig. 53.

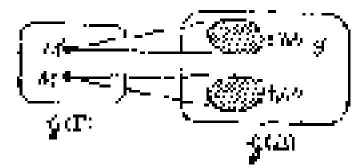


Fig. 54.

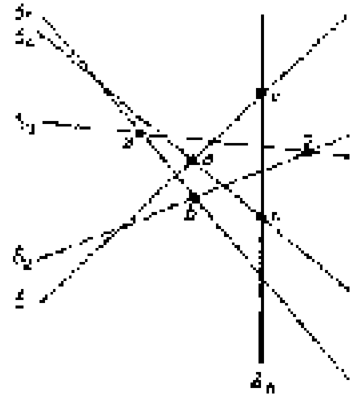


Fig. 55.

associate with every cloud $M\in\mathcal{N}(\Gamma)$ a set $\Phi(M)$ of $(n-1)$ clouds of Δ:

$$[3b] \qquad \Phi(M)\subset\mathcal{N}(\Delta), \qquad |\Phi(M)|=n-1.$$

On the other hand, each cloud $N\in\mathcal{N}(\Delta)$ obtained in the preceding way (Figure 54) is obtained in one way only:

$$[3c] \qquad M, M'\in\mathcal{N}(\Gamma), \qquad M\neq M' \Rightarrow \Phi(M)\cap\Phi(M')=\emptyset.$$

But in this way $\mathcal{N}(\Delta)$ is not completely obtained, because there exist *singular* clouds N of Δ that do not belong to any $\Phi(M)$, for instance, the cloud shown in Figure 55. Let $\mathcal{S}$ be the set of singular clouds of Δ. Giving a cloud $\in\mathcal{S}$ is evidently equivalent to giving a pair $\{u, v\}$ among the $(n-1)$ points of δ_n, and to giving a cloud on the $(n-3)$ lines δ_i *that do not pass through* $\{u, v\}$. Hence

$$[3d] \qquad |\mathcal{S}|=\gamma_{n-3}\binom{n-1}{2}.$$

Now, according to [3c] we have the division:

$$\mathscr{G}(A) = \Big(\sum_{M \in \mathscr{M}(A)} \mathscr{G}(M)\Big) + \mathscr{S};$$

this gives, after passing to the cardinalities (using [3b] for (*)):

$$g_n = |\mathscr{G}(A)| = \sum_{M \in \mathscr{M}(A)} |\mathscr{G}(M)| + |\mathscr{S}| \overset{(*)}{=} (n-1)\,|\mathscr{G}(\Gamma)| + |\mathscr{S}|.$$

Finally, by [3d]:

[3e] $$g_n = (n-1)\,g_{n-1} + \binom{n-1}{2} g_{n-3},$$
$$n \geq 3; \quad g_0 = 1, \quad g_1 = g_2 = 0.$$

n	0	1	2	3	4	5	6	7	8	9	10	11
g_n	1	0	0	1	3	12	70	465	3507	30016	286884	3026655

n	12	13	14	15	16
g_n	34944085	438263364	5933502822	86248951243	1339751921865

(III) *A generating function*

Using [3e] for (*), we get:

[3f] $$g(t) := \sum_{n \geq 0} g_n \frac{t^n}{n!} = 1 + \sum_{n \geq 3} g_n \frac{t^n}{n!}$$
$$\overset{(*)}{=} 1 + \sum_{n \geq 3} (n-1)\, g_{n-1} \frac{t^n}{n!} + \sum_{n \geq 3} \binom{n-1}{2} g_{n-3} \frac{t^n}{n!}.$$

Taking the derivative of [3f] with respect to t:

$$g'(t) = t \sum_{n \geq 2} g_{n-1} \frac{t^{n-2}}{(n-2)!} + \frac{t^2}{2} \sum_{n \geq 3} g_{n-3} \frac{t^{n-3}}{(n-3)!}$$
$$= t g'(t) + \frac{t^2}{2} g(t).$$

Thus, considering $g(t)$ as a function defined in a certain interval (to be specified later), we obtain the differential equation $g'(t)/g(t) = t^2/2(1-t)$, which gives, by integration on $(-1, +1)$ and exponentiation, and observing that $g(0) = g_0 = 1$:

[3g] $$g(t) = \sum_{n \geq 0} g_n \frac{t^n}{n!} = \frac{1}{\sqrt{1-t}} \exp\left(-\frac{t^2 + 2t}{4}\right).$$

(IV) *The asymptotic expansion*

We will use the 'method of *Darboux*' ([Darboux, 1878]) which is stated below. No proof will be given.

THEOREM. *Let $g(z) = \sum_{n \geq 0} g_n z^n/n!$ be a function of the complex variable z, regular for $|z| < 1$, and with a finite number l of singularities on the unit circle $|z| = 1$: $e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_l}$. We suppose that in a neighbourhood of each of these singularities $e^{i\theta_k}$, $g(z)$ has an expansion of the following form:*

[3h] $$g(z) = \sum_{p \geq 0} c_p^{(k)} (1 - z e^{-i\theta_k})^{\alpha_k + p\beta_k}, \quad k \in [l],$$

where the α_k are complex numbers, and all $\beta_k > 0$. The branch chosen for each bracketed expression is that which is equal to 1 for $z = 0$. Under these circumstances, g_n has the following asymptotic expansion $(n \to \infty)$:

[3i] $$g_n = \sum_{0 \leq p \leq N(q)} \; \sum_{1 \leq k \leq l} c_p^{(k)} (\alpha_k + p\beta_k)_n (-e^{-i\theta_k})^n + O(n^{-q} n!).$$

In [3i], $N(q)$ is the smallest integer $\geq \max_{1 \leq k \leq l} \beta_k^{-1}(q - \mathrm{Re}(\alpha_k) - 1)$, and $O(n^{-q} n!)$ means a sequence u_n such that $u_n/(n^{-q} n!)$ is bounded for $n \to \infty$.

It is important to observe that *formally* the asymptotic expansion [3i] of g_n, up to the O term, can be obtained by gathering *for each singularity* $e^{i\theta_k}$ the coefficient of $z^n/n!$ in [3h].

We apply this theorem to the function $g(z)$, defined by [3g]: the only singularity is in $z = 1$. The expansion [3h] can be obtained using the *Hermite* polynomials $H_n(x)$, [14n] (p. 50). Thus, if we put $u = 1 - z$:

$$g(z) = e^{-3/4} u^{-1/2} \exp\left(u - \frac{u^2}{4}\right) = e^{-3/4} u^{-1/2} \sum_{p \geq 0} \frac{H_p(1)}{2^p p!} u^p$$
$$= e^{-3/4}\left(u^{-1/2} + u^{1/2} + \tfrac{1}{4} u^{3/2} - \tfrac{1}{12} u^{5/2} - \cdots\right).$$

Hence, by [3i], where $l = 1$, $e^{i\theta} = 1$, $c_p^{(1)} = c_p = H_p(1)/2^p p!$, $\alpha = -\frac{1}{2}$, $\beta = 1$.

$\zeta(q)\sim q-1$ for all integers $q\geq 1$, we get the asymptotic expansion of g_n:

$$[3j]\qquad g_n=e^{-3/4}\sum_{q=0}^{r-1}\left\{\frac{\varphi_q(1)}{2^q q!}(-1)^q(\ldots)\right\}+O(n^{\ldots}),\ n\to\infty.$$

Taking into account the Stirling formula $n!=n^n e^{-n}\sqrt{2\pi n}\,(1+O(n^{-1}))$, [3j] gives us, if we take only the first term ($q=0$):

$$g_n=e^{-3/4}\sqrt{2}\,n^n e^{-n}(1+O(n^{-1/2}))\sim e^{-3/4}\sqrt{2}\,n^n e^{-n}.$$

(V) *A direct computation*

We could have determined g_n directly, by an argument analogous to that on p. 235. It is the number of symmetric and antireflexive relations on $[n]$ such that each section has 2 elements. Hence:

$$g_n=G(n,2)=\mathbf{C}_{u_1^2u_2^2\ldots u_n^2}\prod_{1\leq i<j\leq n}(1+u_iu_j)$$

from which follows, after some computations:

$$g_n=G(n,2)=\frac{1}{4^n}\sum_{\alpha_1+2\alpha_2+\beta_1=n}\frac{(-1)^{\alpha_1+\beta_1}}{\alpha_1!\,\alpha_2!\,\beta_1!}\times$$
$$\times(2\alpha_1)!\,(2\alpha_2-\beta_1)!\,\frac{1}{2^{\alpha_1}}\binom{\cdot}{\cdot},$$

(which leads to the GF [3g] and conversely).

(VI) *The general case*

The *explicit* computation of $G(n,r)$ (p. 273) can also be done by:

$$G(n,r)=\mathbf{C}_{u_1^ru_2^r\ldots u_n^r}\prod_{1\leq i<j\leq n}(1+u_iu_j),$$

but the formulas become very quickly extremely complicated. Thus, $G(2m+1,3)=0$ and

$$G(2m,3)=\sum_{\ldots}\frac{(-1)^{\alpha_2+\beta_1}}{\ldots}\times$$
$$\times\frac{(2m)!\,(2\gamma_1)!}{\alpha_1!\,\alpha_2!\,\alpha_3!\,\beta_1!\,(\gamma_1+\gamma_2-m)!},$$

where $\alpha_1+2\alpha_2+3\alpha_3+\beta_1=3m$ and $\alpha_1+\alpha_2\geq m$. The first values of $G(n,r)$ are:

$n\backslash k$	0	1	2	3	4	5	6	7
1	1							
2	[illegible]	1						
3	1	0	1					
4	1	3	3	1				
5	1	0	12	0	1			
6	1	15	70	70	15	1		
7	1	0	[illegible]	0	465	0	1	
8	1	105	3507	19355	19355	3507	105	1

7.4. RANDOM PERMUTATIONS

We take for probability space $(\Omega,\mathscr{A},\mathbf{P})$ the following: $\Omega=\mathfrak{S}[n]$ (the set of all permutations of $[n]=\{1,2,\ldots,n\}$), $\mathscr{A}=\mathfrak{P}(\mathfrak{S}[n])$ (the set of all subsets of $\mathfrak{S}[n]$), and the probability measure $\mathbf{P}$ that for which all permutations have equal probability:

$$[4a]\qquad \omega\in\mathfrak{S}[n]\Rightarrow\mathbf{P}(\omega)=\frac{1}{n!};\qquad A\subset\mathfrak{S}[n]\Rightarrow\mathbf{P}(A)=\frac{|A|}{n!}.$$

(Definitions A and B, p. 189; we observe that the probabilistic terminology used in this section is defined in Exercise 11, p. 160).

We are now interested in the sequence of RV (random variables) C_n: $\Omega\to N$ defined by:

$$[4b]\qquad C_n=C_n(\omega)=\text{the number of cycles of }\omega.$$

According to Theorem B (p. 234) and to [4a] above for $(*)$, we obtain the following distribution for the C_n:

$$[4c]\qquad p_n(k):=\mathbf{P}(C_n=k)\overset{(*)}{=}\frac{s(n,k)}{n!},$$

where the $s(n,k)$ are the unsigned Stirling numbers of the first kind. Consequently, the GF of the *probabilities* of C_n becomes, using [5f], p. 213, for $(**)$:

$$[4d]\qquad \varphi(u)=\varphi_{C_n}(u)=\sum_k p_n(k)u^k=\sum_k\frac{s(n,k)}{n!}u^k=$$
$$\overset{(**)}{=}\frac{1}{n!}u(u+1)\cdots(u+n-1),$$

from which we obtain the GF of the *cumulants* of C_n:

[4e] $$\gamma(t) = \log\{g(e^t)\} = \sum_{1\le i\le n} \log\left(1 + \frac{e^t - 1}{i}\right).$$

We expand [4e] using [2a] (p. 206) for (*); then we obtain:

[4f] $$\sum_{m\ge 0} \kappa_m \frac{t^m}{m!} = \gamma(t) = \sum_{1\le i\le n}\left\{\sum_{l\ge 1} \frac{(-1)^{l-1}}{l}\left(\frac{e^t-1}{i}\right)^l\right\}$$

$$\overset{(*)}{=} \sum_{1\le i\le n}\left\{\sum_{l\ge 1}\frac{(-1)^{l-1}}{i^l}(l-1)!\sum_{m\ge l} S(m,l)\frac{t^m}{m!}\right\},$$

and by identifying the coefficients of $t^m/m!$ in [4f].

THEOREM A. *The cumulants of the RV C_n defined by* [4b] *equal*

[4g] $$\kappa_m = \kappa_m(C_n) = \sum_{1\le l\le m}\{(-1)^{l-1}(l-1)!\, S(m,l)\,\zeta_n(l)\},$$

where $S(m,l)$ is the Stirling number of the second kind and

[4h] $$\zeta_n(l) := \sum_{1\le i\le n}\frac{1}{i^l} = 1 + \frac{1}{2^l} + \cdots + \frac{1}{n^l}.$$

Thus, by passing to the moments:

$$\mu'_1 = \mathrm{E}(C_n) = \kappa_1 = \zeta_n(1),$$

[4i] $$\mu_2 = \operatorname{var} C_n = \mathrm{D}^2(C_n) = \kappa_2 = \zeta_n(1) - \zeta_n(2).$$

For studying the behaviour of the limit of C_n, we state the *central limit theorem* (in very general form due to [Lindeberg, 1922], see, for instance, [*Renyi, 1966], p. 412–21, for a proof):

THEOREM B. *Let $X_{n,i}$ be a double sequence of RV, defined for $n\in\mathbb{N}$ and $1\le i\le k_n$, where k_n are given integers >0. We suppose that the variables $X_{n,i}$, n fixed, i variable in $[k_n]$, are independent, which is formulated by saying 'the $X_{n,i}$ are row-independent'. If we define now RV S_n and $Y_{n,i}$ by:*

[4j] $$S_n := \sum_{1\le i\le k_n} X_{n,i}, \qquad Y_{n,i} := \frac{X_{n,i} - \mathrm{E}(X_{n,i})}{\mathrm{D}(S_n)},$$

with, for distribution function of $Y_{n,i}$:

[4k] $$G_{n,i}(y) := \mathrm{P}(Y_{n,i} < y),$$

then the condition [4l] (*of Lindeberg*):

[4l] $$\forall \varepsilon > 0, \quad \lim_{n\to\infty} \sum_{1\le i\le k_n} \int_{|y|>\varepsilon} y^2\, dG_{n,i}(y) = 0$$

implies [4m] (*central limit theorem*):

[4m] $$\lim_{n\to\infty} \mathrm{P}\left\{\frac{S_n - \mathrm{E}(S_n)}{\mathrm{D}(S_n)} < x\right\} = \Phi(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\, dt.$$

The conclusion [4m] *still holds when $\mathrm{E}(S_n)$ and $\mathrm{D}(S_n)$ are replaced by equivalent values, when $n\to\infty$.*

The role of the RV S_n will be played by C_n [4b], for our application. Thus, we have to interpret C_n as a sum [4j]. To do this, we define the sequence $X_{n,i}$ of *row-independent* RV, $1\le i\le n$, by:

[4n] $$\mathrm{P}(X_{n,i} = 1) = 1/i, \qquad \mathrm{P}(X_{n,i} = 0) = 1 - 1/i.$$

The GF of the probabilities of the $X_{n,i}$ equal $g_{X_{n,i}}(u) = (i-1+u)/i$. Thus we get, by [4a] for (*), and by the row-independence for (**):

$$g_{C_n}(u) \overset{(*)}{=} \prod_{1\le i\le n} g_{X_{n,i}}(u) \overset{(**)}{=} g_{\Sigma X_{n,i}}(u),$$

from which follows:

[4o] $$C_n = \sum_{1\le i\le n} X_{n,i}.$$

Furthermore, we show that condition [4l] is satisfied by the $X_{n,i}$. Because of [4i]:

$$\mathrm{D}^2(C_n) = \sum_{1\le i\le n}\frac{i-1}{i^2} \ge \sum_{2\le i\le n}\frac{1}{i+2} \ge \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n} >$$

$$> \log n - 1 - \frac{1}{2} - \frac{1}{3} > \log n - 2.$$

Hence:

$$|Y_{i,n}| = \left|\frac{X_{n,i}-\mathbf{E}(X_{n,i})}{\mathbf{D}(C_n)}\right| < \frac{1-\frac{1}{i}}{\sqrt{\log n - 3}} < \frac{1}{\sqrt{\log n - 2}},$$

which, for n sufficiently large, implies $|Y_{i,n}|<\varepsilon$, in other words, $\int_{|y|>\varepsilon} y^2\, dG_{i,n}(y)=0$, for all $i\in[n]$, hence [4l] follows. Finally, we use $\mathbf{E}(S_n)\sim\log n$ and $\mathbf{D}(S_n)\sim\sqrt{\log n}$ to obtain by [4m]:

$$\lim_{n\to\infty}\mathbf{P}\left\{\frac{C_n-\log n}{\sqrt{\log n}}<x\right\}=\Phi(x).$$

In other words ([*Feller, I, 1968], p. 258): "The number of permutations with a number of orbits between $\log n+\alpha\sqrt{\log n}$ and $\log n+\beta\sqrt{\log n}$, $\alpha<\beta$, equals approximately $n!\{\Phi(\beta)-\Phi(\alpha)\}$."

We give, rapidly, another example of RV associated with random permutations. We will deal with $I_n=I_n(\sigma)$, the number of *inversions* of the permutation σ (p. 237). The GF of the probabilities is ([4b], p. 239):

[4p]
$$G_n(u)=\frac{1}{n!}\prod_{1\le j\le n}\frac{1-u^j}{1-u}$$
$$=\frac{1+u}{2}\cdot\frac{1+u+u^2}{3}\cdots\cdots\frac{1+u+u^2+\cdots+u^{n-1}}{n},$$

hence we get for the GF of the cumulants:

$$\chi(t)=\sum_{m\ge 1}\kappa_m\frac{t^m}{m!}=\sum_{1\le j\le n}\log\frac{e^{jt}-1}{j(e^t-1)}$$
$$=\sum_{1\le j\le n}\log\left(1+\frac{j}{2}t+\frac{j^2}{3}\frac{t^2}{2!}+\cdots\right)-\sum_{1\le j\le n}\log\left(1+\frac{1}{2}t+\frac{1}{3}\frac{t^2}{2!}+\cdots\right).$$

By [5a] (p. 160) follows: $\kappa_m=\sum_{j=1}^{n}L_m(j/2, j^2/3,\ldots)-nL_m(1/2,1/3,\ldots)$. Hence $\mu_1=\mathbf{E}(I_n)=\kappa_1=n(n-1)/4$ (cf. p. 160), $\mu_2=\mathbf{D}^2(I_n)=\kappa_2=n(n-1)(2n+5)/72$; in other words $\mathbf{E}(I_n)\sim n^2/4$, $\mathbf{D}(I_n)\sim n^{3/2}/6$.

The factorization [4p] suggests that we define the *row-independent* RV $X_{n,j}$ by $\mathbf{P}(X_{n,j}=k)=1/j$, where $(k+1)\in[j]$, and then we prove easily that the Lindeberg condition [4l] is satisfied. Thus:

$$\lim_{n\to\infty}\mathbf{P}\left\{\frac{I_n-n^2/4}{n^{3/2}/6}<x\right\}=\Phi(x).$$

In other words: "The number of permutations whose number of inversions lies between $n^2/4+\alpha n^{3/2}/6$ and $n^2/4+\beta n^{3/2}/6$, $\alpha<\beta$, equals approximately $n!\{\Phi(\beta)-\Phi(\alpha)\}$." ([*Feller, I, 1968], p. 257. For many other problems of random permutations see [Gontcharoff, 1944] and [Shepp, Lloyd, 1966].

7.5. THEOREM OF RAMSEY

The Ramsey theorem generalizes the 'Dirichlet pigeon-hole principle': If $n+1$ objects are distributed over n pigeon-holes, at least one pigeon-hole contains at least two objects. It introduces a sequence of numbers whose computation and estimation are still among the most fascinating problems of combinatorial analysis.

(I) *Statement of the 'bicolour' theorem, and definition of the Ramsey numbers* $r(b;p,q)$

DEFINITION. *Let three integers be given, b, p, q, $1\le b\le p, q$. A finite set N is called Ramsey-(b,p,q) if for all divisions $(\mathfrak{A},\mathfrak{B})$ of $\mathfrak{P}_b(N)$ into two subsets, $\mathfrak{A}+\mathfrak{B}=\mathfrak{P}_b(N)$, (p. [illegible]) at least one of the following two statements is true:*

[5a] *There exists a P such that $P\in\mathfrak{P}_p(N)$, $\mathfrak{P}_b(P)\subset\mathfrak{A}$.*

[5b] *There exists a Q such that $Q\in\mathfrak{P}_q(N)$, $\mathfrak{P}_b(Q)\subset\mathfrak{B}$.*

Now we can state the 'bicolour' theorem of Ramsey. It is called the 'bicolour' theorem, because a division into two subsets $\mathfrak{A}+\mathfrak{B}$ is equivalent to colouring each block $B\in\mathfrak{P}_b(N)$ in one of two given colours, say, crimson and dove-gray.

THEOREM. *There exists a triple sequence $r(b;p,q)$ of integers >0, called bicolour b-ary Ramsey numbers (multicolour numbers will be investigated in Exercise 26, p. 298), which is characterized by the following property*

[5c] *concerning an arbitrary finite set N:*

[5c] N is Ramsey-$(b; p, q) \Leftrightarrow |N| \geqslant \rho(b; p, q)$.

Moreover:

[5d] $\rho(b; p, q) \leqslant 1 + \rho(b-1; \rho(b; p-1, q), \rho(b; p, q-1))$.

([Ramsey, 1930]. Our exposition is an adaptation of [*Ryser, 1963], pp. 38–43.)

(II) *Some special values of $\rho(b; p, q)$*

First, it is clear that the roles of p, $\mathscr{C}$ and q, $\mathscr{D}$ are symmetric; so:

[5e] $\rho(b; p, q) = \rho(b; q, p)$.

We also show:

[5f] $\rho(1; p, q) = p + q - 1, \quad 1 \leqslant p, q$.

■ Let N be a finite set, such that $|N| = n \geqslant p + q - 1$. Suppose a division of $\mathfrak{P}_1(N) = N$ into two subsets $\mathscr{C} + \mathscr{D} = N$ is given. Then we have $|\mathscr{C}| + |\mathscr{D}| = n \geqslant p + q - 1$, hence $|\mathscr{C}| \geqslant p$ or $|\mathscr{D}| \geqslant q$. If $|\mathscr{C}| \geqslant p$, there exists a $P \in \mathfrak{P}_p(N)$ such that $\mathfrak{P}_1(P)(= P) \subset \mathscr{C}$; if $|\mathscr{D}| \geqslant q$, there exists likewise a $Q \in \mathfrak{P}_q(N)$ such that $\mathfrak{P}_1(Q)(= Q) \subset \mathscr{D}$. Thus N is Ramsey-$(1; p, q)$ if $n \geqslant p + q - 1$.

Conversely, if $|N| < p + q - 1$, in other words, if $|N| = n \leqslant p + q - 2$ we only have to choose a division into two subsets $\mathscr{C} + \mathscr{D} = N$ such that $|\mathscr{C}| = p - 1$, $|\mathscr{D}| = q - 1$ to see that N cannot be Ramsey-$(1; p, q)$.

Finally, we prove:

[5g] $\rho(b; b, q) = q \; (= \rho(b; q, b)), \quad b \leqslant q$.

■ We first prove that each finite set N such that $n = |N| \geqslant q$ is Ramsey-$(b; b, q)$. For a division into two subsets $\mathscr{C} + \mathscr{D} = \mathfrak{P}_b(N)$ there are two cases:

(I) $\mathscr{C} \neq \emptyset$. Then choose $P \in \mathscr{C}$; hence $|P| = p = b$ which implies hence evidently $\mathfrak{P}_b(P) = \{P\} \subset \mathscr{C}$.

(II) $\mathscr{C} = \emptyset$. Then $\mathscr{D} = \mathfrak{P}_b(N)$. Now $n = |N| \geqslant q$. Hence $\mathfrak{P}_q(N)$ is not empty, and we can choose Q there. Necessarily $|Q| = q$ and $\mathfrak{P}_b(Q) \subset \mathfrak{P}_b(N) = \mathscr{D}$.

Conversely, if $|N| < q$, in other words, if $|N| = n \leqslant q - 1$, it suffices to choose the division into two subsets $\mathscr{C} + \mathscr{D} = \mathfrak{P}_b(N)$ such that $\mathscr{C} = \emptyset$ to see (by $\mathfrak{P}_q(N) = \emptyset$) that N cannot be Ramsey-$(b; b, q)$. ■

Taking into account [5e, f, g] we suppose from now on that:

[5h] $1 < b < p, q$.

(III) *Choice of the induction for $\rho(b; p, q)$*

Let $\mathbf{R}(b)$ be the table of the values of the double sequence $\rho(b; p, q)$, $p, q \geqslant 1$, b fixed $\geqslant 1$, extended by $\rho(b; p, q) = 0$ if not $1 \leqslant b \leqslant p, q$. We know already $\mathbf{R}(1)$, according to [5f]. To prove the existence of $\rho(b; p, q)$, $1 \leqslant c \leqslant b-1$, we suppose the existence of all the tables $\mathbf{R}(c)$ where b is fixed $\geqslant 2$ (= existence of all the $\rho(c; p, q)$, with $c \leqslant b-1$, p, q), as well as the existence of:

[5i] $p' := \rho(b; p-1, q)$ and $q' := \rho(b; p, q-1)$

in the table $\mathbf{R}(b)$. From these induction hypotheses we will deduce now the existence of $\rho(b; p, q)$, and simultaneously also:

[5j] $\rho(b; p, q) \leqslant 1 + \rho(b-1; p', q')$,

in other words [5d], because of [5i].

(IV) *Proof of the theorem of Ramsey*

We observe that [5j] is equivalent to proving that every finite set N that satisfies:

[5k] $n = |N| \geqslant 1 + \rho(b-1; p', q')$

is Ramsey-$(b; p, q)$ (p', q' defined in [5i]).

Let N be such that [5k] holds, and choose $x \in N$, and let $M := N - \{x\}$; then, by [5k]:

[5l] $|M| = n - 1 \geqslant \rho(b-1; p', q')$.

Now we associate with the division $\mathscr{C} + \mathscr{D} = \mathfrak{P}_b(N)$ the division $\mathscr{C}' + \mathscr{D}' = \mathfrak{P}_{b-1}(M)$, defined by:

[5m] $\mathscr{C}' := \{C \setminus \{x\} \mid C \in \mathscr{C}\}, \qquad \mathscr{D}' := \{D \setminus \{x\} \mid D \in \mathscr{D}\}$.

According to [5l], M is Ramsey-$(b-1; p', q')$, which implies for $\mathscr{C}'$ and

$\mathscr{D}'$ that at least one of the following two statements is true:

[5n] There exists an X such that $X \in \mathfrak{P}_{p'}(M)$, $\mathfrak{P}_{k-1}(X) \subset \mathscr{C}'$.

[5o] There exists an Y such that $Y \in \mathfrak{P}_{q'}(M)$, $\mathfrak{P}_{k-1}(Y) \subset \mathscr{D}'$.

We suppose now that we are in the case [5n]. Because $|X| = p' = \rho(k; p-1, q)$, the set X is *Ramsey*-$(k; p-1, q)$; hence, we have for the division $\mathscr{C}''+\mathscr{D}''=\mathfrak{P}_k(X)$, defined by

[5p] $\mathscr{C}'' := \mathscr{C} \cap \mathfrak{P}_k(X), \qquad \mathscr{D}'' := \mathscr{D} \cap \mathfrak{P}_k(X),$

at least one of the following two possibilities:

[5q] There exists a P' such that $P' \in \mathfrak{P}_{p-1}(X)$, $\mathfrak{P}_k(P') \subset \mathscr{C}''$.

[5r] There exists a Q such that $Q \in \mathfrak{P}_q(X)$, $\mathfrak{P}_k(Q) \subset \mathscr{D}''$.

In the case [5r], evidently $Q \in \mathfrak{P}_q(N)$, because $X \subset N$; hence $\mathfrak{P}_k(Q) \subset \mathscr{D}$, since $\mathscr{D}'' \subset \mathscr{D}$, [5p]. *So we have proved* [5e].

In the case [5q], we will show that the set $P := P' \cup \{x\}$ satisfies [5d] indeed, in other words, that $\mathfrak{P}_k(P) \subset \mathscr{C}$. We put:

[5s] $\mathfrak{X}_0 := \{B \mid B \in \mathfrak{P}_k(P), x \notin B\},$
$\mathfrak{X}_1 := \{B \mid B \in \mathfrak{P}_k(P), x \in B\}.$

Hence:

[5t] $\mathfrak{P}_k(P) = \mathfrak{X}_0 + \mathfrak{X}_1.$

We have $\mathfrak{X}_0 \subset \mathscr{C}$; this follows from: (1) $\mathfrak{X}_0 \subset \mathfrak{P}_k(P')$ by definition [5s] of $\mathfrak{X}_0$; (2) $\mathfrak{P}_k(P') \subset \mathscr{C}''$, [5q]; (3) $\mathscr{C}'' \subset \mathscr{C}$, [5p]. Similarly $\mathfrak{X}_1 \subset \mathscr{C}$, because all $K \in \mathfrak{X}_1$ are of the form $K = H + \{x\}$, where $H \in \mathfrak{P}_{k-1}(P')$, [5s]; now, because of [5q], $\mathfrak{P}_{k-1}(P') \subset \mathfrak{P}_{k-1}(X)$; hence, by [5n], $\mathfrak{P}_{k-1}(P') \subset \mathscr{C}'$; consequently, by [5m, i], $K \in \mathscr{C}$. Finally, [5t] implies $\mathfrak{P}_k(P) \subset \mathscr{C}$, in other words [5d].

A similar argument, mutatis mutandis, is carried out in the case [5o].

For the computation and the properties of the Ramsey numbers, we refer to several authors who have worked on this problem ([Erdös, 1947, 1957–58, 1964], [Giraud, 1968a, b, 1969a, b], [Graver, Yackel, 1966, 1968], [Greenwood, Gleason, 1955], [Kalbfleisch, 1965, 1966, 1967a, b, 1968], [Krieger, 1968], [Walker, 1968], [Yackel, 1972], [Znám, 1967]).

7.6. BINARY (BICOLOUR) RAMSEY NUMBERS

In this section we deal with the numbers $\rho(2; p, q)$, [5c] (p. 284), which we will denote in the sequel by $\rho(p, q)$, $2 \leq p, q$. We give a new definition of these numbers in terms of graph theory (p. 61).

Giving a division into two sets $\mathscr{C}+\mathscr{D}=\mathfrak{P}_2(N)$ is equivalent to giving a graph $\mathscr{G}$ on N, if we make the convention that $\mathscr{C}=\mathscr{G}$ and $\mathscr{D}=\bar{\mathscr{G}}=\mathfrak{P}_2(N)-\mathscr{G}$. This is also equivalent to painting the edges of the complete graph $\mathfrak{P}_2(N)$ in blue and white colours, that is, painting blue the edges in $\mathscr{C}$, and white the edges in $\mathscr{D}$. This explains why the numbers $\rho(2; p, q) = \rho(p, q)$ are called bicolour numbers.

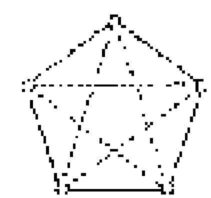


Fig. 56

With every graph $\mathscr{G}$ on N, we associate the following two numbers: (1) The number $c(\mathscr{G})$, which is equal to the maximum number of elements of a *complete subgraph* of $\mathscr{G}$; (2) the number $i(\mathscr{G})$, equal to the maximum number of elements of *independent* sets $\mathscr{G}$ (i.e. complete subgraphs of $\bar{\mathscr{G}}$). Let now be given two integers $p, q > 0$. We say that $\mathscr{G}$ is a (p, q)-graph if $c(\mathscr{G}) < p$ and $i(\mathscr{G}) < q$. This means that $\mathscr{G}$ [or $\bar{\mathscr{G}}$] does not contain a complete subgraph of p elements [or q elements]. Hence, the negation of [5e] (p. 284) can be written:

[6a] there exists a (p, q)-graph $\mathscr{G} \subset \mathfrak{P}_2(N) \Leftrightarrow |N|+1 \leq \rho(p, q)$.

and the problem becomes that of constructing (p, q)-graphs with the largest number of vertices, thus providing a constructive procedure for obtaining lower bounds for the Ramsey numbers $\rho(p, q)$.

We will illustrate this with the computation of $\rho(3, 3)$. Inequality [5d] (p. 284) combined with [5f] $(k=2)$ gives:

[6b] $\rho(p, q) \leq \rho(p, q-1) + \rho(p-1, q).$

This gives, together with $p(2,3)=p(3,2)=3$ and [5g]:

$$[6c] \qquad p(3,3)\leqslant 6.$$

On the other hand, the graph $\mathscr{G}$ of Figure 56 over N_5 ($|N_5|=5$), whose edges are indicated by full lines, does not contain any *triangle* (=complete subgraph of 3 elements); the complementary graph neither does ($\bar{\mathscr{G}}$ is indicated by dotted lines). Hence, by [6a]:

$$[6d] \qquad p(3,3)\geqslant 6.$$

Together, [6c, d] imply $p(3,3)=6$.

Below the first values of $p(p,q)$ that are either known or for which bounds are known. The table should be completed by symmetry (cf. [5e], p. 281). For $p(3,8)$, for instance, 27–30 means $27\leqslant p(3,8)\leqslant 30$.

$p\backslash q$	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10
3		6	9	14	18	23	27–30	36–37	39–44
4			18	25–28	34–44	?–66	?–94	?–129	?–170
5				38–55	51–94	?–156	?–242	?–364	?–521
6					102–169	?–322	?–541	?–886	?–1371
7						110–586	?–1131	?–1974	?–3255

7. SQUARES IN RELATIONS

Let M be a finite set and a an integer, $1\leqslant a\leqslant m=|M|$. Determine the smallest integer $\mathfrak{f}=\mathfrak{f}(m,a)$, such that each relation $\mathfrak{R}$ on M, $\mathfrak{R}\subset M^2$ $(=M\times M)$, $|\mathfrak{R}|=k\geqslant\mathfrak{f}$, contains at least one a^2-square (Zarankiewicz, 1951). This is a product set of the form $AA'=A\times A'$, where $A,A'\subset M$, $|A|=|A'|=a$. In other words, when $\mathscr{C}=\mathscr{C}(a)$ is the set of a^2-squares of M^2:

$$[7a] \qquad k\geqslant\mathfrak{f}(m,a)\Leftrightarrow\forall\mathfrak{R}\in\mathfrak{P}_k(M^2),\quad \exists(A,A')\in\mathfrak{P}_a(M)^2,\quad AA'\subset\mathfrak{R},$$

where $\mathfrak{P}_a(M)^2:=\mathfrak{P}_a(M)\times\mathfrak{P}_a(M)$. Evidently $a^2\leqslant\mathfrak{f}\leqslant m^2$.

We transform [7a] by introducing for each a^2-square $AA'\in\mathscr{C}$ the set of $\mathfrak{r}(A,A')$ of the k-relations on M that contain AA'. Hence [7a] is equivalent with:

$$[7b] \qquad k\geqslant\mathfrak{f}(m,a)\Leftrightarrow\mathfrak{P}_k(M^2)=\bigcup_{(A,A')\in\mathfrak{P}_a(M)^2}\mathfrak{r}(A,A').$$

This will provide us with a lower bound for $\mathfrak{f}$.

THEOREM A. *There exists a constant $c_1=c_1(a)>0$ independent of m such that:*

$$[7c] \qquad \mathfrak{f}(m,a)>c_1 m^2\, m^{-2/a}.$$

■ In fact, [7b] implies, by [7d] (p. 194):

$$[7d] \qquad |\mathfrak{P}_k(M^2)|\leqslant\sum_{(A,A')\in\mathfrak{P}_a(M)^2}|\mathfrak{r}(A,A')|.$$

Now

$$[7e] \qquad |\mathfrak{P}_k(M^2)|=\binom{m^2}{k},\qquad |\mathfrak{P}_a(M)^2|=\binom{m}{a}^2,$$
$$|\mathfrak{r}(A,A')|=\binom{m^2-a^2}{k-a^2}.$$

Hence [7b] becomes, by [7d, e]:

$$[7f] \qquad k\geqslant\mathfrak{f}(m,a)\Rightarrow\binom{m^2}{k}\overset{(*)}{\leqslant}\binom{m}{a}^2\binom{m^2-a^2}{k-a^2}.$$

We weaken (*) by using: (1) (*)⇒(**), (2) $(m)_a\leqslant m^a$, for (***), (3) $m^2/k\leqslant(m^2-l)/(k-l)$ for (****):

$$[7g] \qquad \left(\frac{m^2}{k}\right)^{a^2}\overset{(****)}{\leqslant}\frac{m^2(m^2-1)\cdots(m^2-a^2+1)}{k(k-1)\cdots(k-a^2+1)}\overset{(**)}{\leqslant}\binom{m}{a}^2\overset{(***)}{<}\frac{m^{2a}}{(a!)^2}.$$

Hence, by [7f, g]:

$$k\geqslant\mathfrak{f}(m,a)\Rightarrow\left(\frac{m^2}{k}\right)^{a^2}<\frac{m^{2a}}{(a!)^2}\Leftrightarrow k>(a!)^{2/a^2}\,m^2\,m^{-2/a},$$

which is [7c]. ■

THEOREM B. *There exists a constant $c_2=c_2(a)>0$ independent of m such that:*

$$[7h] \qquad \mathfrak{f}(m,a)\leqslant c_2 m^2\, m^{-1/a}.$$

■ Let $\mathfrak{N} \in \mathfrak{P}_k(M^2)$. We put $M = [m] := \{1, 2, \ldots, m\}$ and

$$[7i] \qquad r_j := |\langle \mathfrak{N} | j \rangle| \quad \left(\text{hence } \sum_{j=1}^{m} r_j = k\right),$$

where $\langle \mathfrak{N} | j \rangle$ means the second section of $\mathfrak{N}$ by j (see Figure 57). Clearly $\mathfrak{N}$ contains an a^2-square, if there exists an a-block $A (\subset M)$ which is contained in at least a of the subsets $\langle \mathfrak{N} | j \rangle$ of M, $j \in [m]$. Now, according to an argument analogous to the 'pigeon-hole principle' (p. 91), this happens as soon as:

$$[7j] \qquad \sum_{j=1}^{m} \binom{r_j}{a} > (a-1)\binom{m}{a} \qquad (\Rightarrow k \geq f(m, a)).$$

We now must majorize k as good as possible, using [7i, j]. (For a more precise statement, see [Znám, 1963, 1965], [Guy, Znám, 1968].) By convexity of the function $\binom{x}{a}$ for $x \geq a$ (its second derivative is always positive: $d^2(x)_a/dx^2 = 2(x)_a \sum_{0 \leq i < j \leq a-1} ((x-i)(x-j))^{-1}$) and the related Jensen inequality, we obtain, using [7i] for (∗):

$$[7k] \qquad \sum_{1 \leq j \leq m} \binom{r_j}{a} \geq m \binom{\dfrac{r_1 + r_2 + \cdots + r_m}{m}}{a} \overset{(*)}{=} m \binom{k/m}{a} > m \frac{((k/m) - a)^a}{a!}.$$

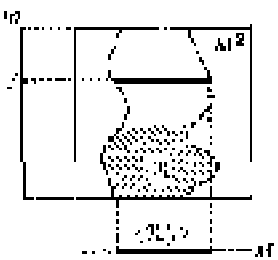


Fig. 57.

Consequently, by $\binom{m}{a} < m^a/a!$ for (∗∗):

$$[7l] \qquad m \frac{((k/m) - a)^a}{a!} \overset{(**)}{\geq} (a-1)\frac{m^a}{a!} \Rightarrow$$
$$\Rightarrow k \geq am + (a-1)^{1/a} m^2 . m^{-1/a} \Rightarrow k \geq f(m, a).$$

Hence $f(m, a) < am + (a-1)^{1/a} m^2 . m^{-1/a}$, which implies [7h]. ■

The following is a table of the known values of $f(m, a)$. (See all the quoted papers by Guy and Znám, and Exercise 29, p. 300.)

$a \backslash m$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	
2	4	7	10	13	17	22	25	30	35	40	46	53	57	61	→
3		9	14	21	27	34	43	50	61	?	?	…			
4			16	23	?	?	?	…							
5				25	34	?	?								
⋮				⋱	⋱										

It has been proved that $f(m, 2) \sim m^{3/2}$, $m \to \infty$ ([Čulík, 1956], [Hyltén-Cavallius, 1958], [Kövári, Sós, Turán, 1954], [Reiman, 1958], [W. G. Brown, 1966]), but no asymptotic expression is known for $f(m, a)$, $a \geq 3$, fixed, $m \to \infty$. A conjecture is that there exist constants $c(a) > 0$ such that

$$f(m, a) \sim c(a)\, m^2 m^{-1/a}.$$

SUPPLEMENT AND EXERCISES

1. *Vertical convexity of Stirling numbers and Bell numbers* (for a generalization of these properties see [Comtet, 1972]). (1) Show that for fixed k, the sequence $S(n, k)$ is convex, $n \geq k$. Same question for $s(n, k)$. (2) The sequence of numbers $\varpi(n)$ of partitions of a set with n elements (p. 210) is convex.

2. *Subsequences of the Pascal triangle.* The sequence $u_n := \binom{2n}{n}$ is convex. Does $\Delta^k u_n \geq 0$ for $k \geq 3$ also hold? Analogous questions for $\binom{2n-1}{n}$ and $\binom{bn}{an}$, a, b, c integer, $1 \leq a \leq b$. For a and b integers ≥ 1, and $n \to \infty$, we have:

$$\binom{(a+b)n}{an} \sim \frac{(a+b)^{(a+b)n+1/2}}{a^{an+1/2}b^{bn+1/2}} \frac{1}{\sqrt{2\pi n}}.$$

[Use the *Stirling formula* $n! \sim (n/e)^n\sqrt{2\pi n}$.]

3. *Unimodality of the Eulerian numbers.* Show that the Eulerian polynomials $A_n(u)$ (p. 244) form a *Sturm sequence*, that is, $A_n(u)$ has n real roots (≤ 0), *separated* by the roots of $A_{n-1}(u)$. [*Hint*: Use the recurrence relation $A_n(u) = (u-u^2)A'_{n-1}(u) + nuA_{n-1}(u)$.] Use this to prove that the sequence $A(n,k)$, for fixed n, is unimodal.

4. *Minimum of a partition of integers function.* With every partition $(y) = (y_1, y_2, \ldots, y_m)$ of n into m summands $y_1+y_2+\cdots+y_m = n$, $y_1 \geq y_2 \geq \cdots \geq y_m \geq 1$, we associate $W(y) := \sum_{i=1}^m \binom{y_i}{k}$. Then, for m, n, k fixed, the minimum of $W(y)$ occurs for a partition (y) that satisfies $|y_i - y_j| \leq 1$ for all (i, j) such that $1 \leq i < j \leq m$.

5. *The most agglomerated system.* Let N be a set, and $\mathscr{S}$ a system of N consisting of k (distinct) blocks all with $b (\geq 1)$ elements, $\mathscr{S} \in \mathfrak{P}_k(\mathfrak{P}_b(N))$. Then $M := \bigcup_{B \in \mathscr{S}} B$ has for minimal number of elements the smallest integer m such that $k \leq \binom{m}{b}$. (b, k fixed).

6. *Partition into unequal blocks.* The maximum number of blocks of a partition of N, $|N| = n$, into blocks with *all* different numbers of elements equals the largest integer $\leq (1/2)(-1+\sqrt{8n+1})$.

7. *Bounds for* $S(n,k)$. The inequalities $k^{n-k} \leq S(n,k) \leq \binom{n-1}{k-1} k^{n-k}$ follow from [2e] (p. 207). Improve these bounds for the Stirling numbers of the second kind.

8. *The number of k-Sperner systems.* The number $s(n,k)$ of Sperner systems with k blocks of N, $|N| = n$, satisfies $s(n,2) = (1/2!)(4^n - 2\cdot 3^n + 2^n)$, $s(n,3) = (1/3!)(8^n - 6\cdot 6^n + 6\cdot 5^n + 3\cdot 4^n - 6\cdot 3^n + 2\cdot 2^n)$, $s(n,4) = (1/4!)(16^n - 12\cdot 12^n + 24\cdot 10^n + 4\cdot 9^n - 18\cdot 8^n + 6\cdot 7^n - 36\cdot 6^n + 36\cdot 5^n + 11\cdot 4^n - 22\cdot 3^n + 6\cdot 2^n)$ ([Ullman, 1975]). *Determine for $s(n,k)$ an explicit formula of minimal rank.

9. *Asymptotic expansion of the Stirling numbers.* (For a detailed study of this matter, see [Moser, Wyman, 1958b, c].) We suppose k fixed, and $n \to \infty$. (1) $S(n,k) \sim k^n/k!$. [*Hint*: [7d], p. 196, and [7b], p. 204.] (2) $s(n+1, k+1) \sim (n!/k!)\log^k n$. Moreover, [75] (p. 217) gives a complete asymptotic expansion.

***10.** *Alike binomial coefficients (ABC).* These are *integers* of the form $n!(a!\,b!)^{-1}$, where n, a, b are integers too, such that $a+b \geq n$. Every binomial is evidently ABC. Show the existence of a universal constant $c>0$ such that $a+b < n + c\log n$ for each ABC ([Erdős, 1968]).

***11.** *Around a definition of e.* It is well known that $\varphi(t) := (1+t)^{1/t}$ approaches e if t tends to 0. More precisely, $\varphi(t) = e(1 - \sum_{n\geq 1} A(n)t^n) = \sum_{n \geq 0} a(n)t^n$, where the *rational* numbers $A(1), A(2), \ldots$ equal $1/2$, $11/24$, $7/16$, $2447/5760$, $959/2304$, $238043/580608$, $67223/165888, \ldots$, and where $a(n) = eA(n)$ has an asymptotic expansion:

$$a_n \approx 1 + \frac{1}{n} + \sum_{\nu \geq 1} \frac{P_\nu(\log n)}{n^{\nu+1}},$$

where $n \to \infty$ and $P_\nu(x)$ are polynomials of degree ν, $P_1(x) = \zeta(2) - 1 - x, \ldots$

***12.** *Inverting the harmonic numbers.* Let us consider a strictly increasing real *sequence* $f(u)$, $u \geq a$, $b = f(a)$, $f(\infty) = \infty$. For any real number $x \geq b$, we write $f^{\langle -1 \rangle}(x)$ for the largest integer $u \leq x$. For example, if $f(u) = u$, we find $f^{\langle -1\rangle}(x) = [x]$, the integral part of x. (1) For the *harmonic* sequence $f(n) = 1 + 2^{-1} + 3^{-1} + \cdots + n^{-1}$ and for any $x \geq 2$, we have $f^{\langle -1 \rangle}(x) = [e^{x-\gamma} - (1/2) + (3/2)(e^{x-\gamma}+1)^{-1}]$ or the same integer plus one ([Comtet, 1967], [Boas, Wrench, 1971]; $\gamma = 0.5772\ldots$ is the Euler constant). (2) More generally, calculate $f^{\langle -1 \rangle}(x)$, where $f(n) = 1 + 2^{-s} + 3^{-s} + \cdots + n^{-s}$, $s < 1$.

***13.** *Cauchy numbers (or integral of the rising and falling factorials).* (See [Lienard, 1946], [Nyström, 1930], [Wachs, 1947]). Let us consider the

Cauchy numbers of the first type $a_n := \int_0^1 (x)_n\,dx$ and of the second type $b_n := \int_0^1 \langle x\rangle_n\,dx$. (1) $\sum a_n t^n/n! = t(\log(1+t))^{-1}$, $\sum b_n t^n/n! = (-t)\times((1-t)\log(1-t))^{-1}$. (2) $a_n = \sum_k s(n,k)/(k+1)$, $b_n = \sum_k |s(n,k)|/(k+1)$. (3) $a_n = \sum_{k=1}^{n}(-1)^{k-1}(n)_k a_{n-k}/(k+1)$, $b_n = \sum_{k=1}^{n}(n)_k b_{n-k}/(k+1)$.

n	0	1	2	3	4	5	6	7	8	9	10
a_n	1	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{19}{30}$	$\frac{9}{4}$	$\frac{863}{84}$	$\frac{1375}{24}$	$\frac{33953}{90}$	$\frac{57281}{20}$	$\frac{3250433}{132}$
b_n	1	$\frac{1}{2}$	$\frac{5}{6}$	$\frac{9}{4}$	$\frac{251}{30}$	$\frac{475}{12}$	$\frac{19087}{84}$	$\frac{36799}{24}$	$\frac{1070017}{90}$	$\frac{2082753}{20}$	$\frac{134211265}{132}$

(4) When $n\to\infty$, we have $a_n/n! \sim (-1)^{n+1}n^{-1}(\log n)^{-2}$ and $b_n/n! \sim (\log n)^{-1}$.

14. *Representations of zero as a sum of different summands between $-n$ and $+n$.* Let $A(n)$ be the number of solutions of $\sum_{k=-n}^{n} kx_k = 0$, where x_k equals 0 or 1. Then, when $n\to\infty$, $A(n)\sim 3^{1/2}\pi^{-1/2}2^{2n+1}n^{-3/2}$ ([Van Lint, 1967], [Entringer, 1968]).

15. *Sum of the inverses of binomial and multinomial coefficients.* The sequence $I_n := \sum_{k=0}^{n}\binom{n}{k}^{-1}$ equals $(n+1)2^{-n-1}\sum_{k=1}^{n+1}2^k/k$. (For a probabilistic remark, see [*Letac, 1970], p. 14). It satisfies the recurrence $2I_n = ((n+1)/n)I_{n-1}+2$ and has the following (divergent) asymptotic expansion: $I_n/2 \approx 1+\sum_{p\geq 0} b_p n^{-p-1}$, where the integers b_p have as GF: $\sum_{p\geq 0} b_p t^p/p! = (2-e^t)^{-2}$.

p	0	1	2	3	4	5	6	7	8	9	10
b_p	1	2	8	44	308	2612	25988	296564	3816548	54667412	862440068

In the same way, $I_n(x) := \sum_{k=0}^{n}\binom{n}{k}^{-1}x^k = (n+1)(x/(1+x))^{n+1}\sum_{k=1}^{n+1}(1+x^k)(1-x)^k(k(1+x))^{-1}x^{-k}$ and $(1+(1/x))I_n(x)-(1+1/n)I_{n-1}(x) = x^n+x^{-1}$.

***16.** *The coefficients of $(\sum n!t^n)^{-1}$* ([Comtet, 1972]). Let $\alpha(t)$ be the purely formal series $\sum_{n\geq 0} n!t^n$. We define the coefficients $f(n)$ by $(\alpha(t))^{-1} = 1-\sum_{n\geq 1} f(n)t^n$. (1) The $f(n)$ are positive integers such that $f(p+1)\equiv 1 \pmod p$ for p prime. (2) We have the following asymptotic expansion $f(n)/n! \approx 1-\sum_{k\geq 1} A_k/(n)_k = 1-2/n-1/(n)_2-4/(n)_3-\dots$, where $A_k = f(k)-(k-1)f(k-1)$, $k\geq 2$. (3) The sequence $f(n)/n!$ (which tends to 1) is increasing for $n\geq 2$ and concave for $n\geq 4$.

n	1	2	3	4	5	6	7	8	9	10
$f(n)$	1	1	3	13	71	461	3447	29093	273343	2829325

(Cf. Exercise 16, p. 261 and Exercise 15, p. 294.)

***17.** *Sum of the logarithms of the binomial coefficients.* (1) Show that $\lim_{n\to\infty}\{n^{-2}\sum_{k=0}^{n}\log\binom{n}{k}\} = 1/2$. (2) More generally, for all integers $p\geq 1$, we have $\lim_{n\to\infty}\{n^{-2}\sum_{k=0}^{n}\log\binom{pn}{pk}\} = p/2$. ([Gould, 1964b], and for a generalization, [Carlitz, 1966c], [Hayes, 1956].)

***18.** *Examples of applications of the method of Darboux* (p. 277). Determine the asymptotic expansions for the Bernoulli and Euler numbers (p. 48), the e_n (p. 52).

***19.** *Cycles of a random permutation.* In the probability space defined on p. 279, for each integer $r\geq 1$, we introduce the RV $C_{n,r}$ equal to the number of r-orbits of ω. Show that the GF for its probabilities equals $\sum_{0\leq j\leq n/r}(u-1)^j r^{-j}/j!$. Deduce that, for r fixed, and n tending to ∞, $C_{n,r}$ tends to a Poisson RV with parameter $1/r$.

***20.** *The number of orbits in a random derangement.* We define the *associated Stirling numbers of the first kind* $s_2(n,k)$ by $\sum_{n,k} s_2(n,k)t^n u^k/n! = e^{-tu}(1+t)^u$. (1) The number $d(n,k)$ of derangements of N, $|N|=n$, with k orbits (p. 231, and Exercise 7, p. 256) equals $|s_2(n,k)|$. (2) The polynomials $P_n(u) := \sum_k d(n,k)u^k$ have all different and negative roots ([Tricomi, 1951], [Carlitz, 1958a]). (3) We consider the 'random' derangements ω of N (for which we must specify the probability space!), and the RV $A_n = A_n(\omega)$ = the number of orbits of ω. Study the asymptotic properties of the $d(n,k)$, analogous to those of $s(n,k)$ (pp. 279–283).

***21.** *Random partitions of integers.* We consider all partitions ϖ of n equally probable, $\mathbf{P}(\varpi) := (p(n))^{-1}$. We let S_n be the RV equal to the number of summands of ϖ. Hence $\mathbf{P}(S_n = m) = P(n,m)/p(n)$ (p. 94). Show that $\mathbf{E}(S_n) = (p(n))^{-1}\sum_{1\le r\le n} d(r)\,p(n-r)$, where $d(r)$ is the number of divisors of r. [*Hint*: Take the derivative with respect to u, [2c] (p. 97), put $u=1$, and use Exercise 16 (4), p. 161.] (For estimates of the first three moments, see [Luthra, 1958], and for the abscissa of the 'peak', [Szekeres, 1953].)

***22.** *Random tournaments.* We define a random tournament (cf. p. 68) $\mathcal{T} = \mathcal{T}(\omega)$ over N, $|N| = n$, by making random choices for each pair $\{x_i, x_j\} \in \mathfrak{P}_2(N)$, the arcs $\overrightarrow{x_i x_j}$ and $\overrightarrow{x_j x_i}$ being equiprobable, and the $\binom{n}{2}$ choices independent. (1) Let $C_n = C_n(\omega)$ be the number of 3-cycles of $\mathcal{T}(\omega)$ (for example, $\{x_1, x_2, x_4\}$, Figure 18, p. 68, is a 3-cycle). Show that $\mathbf{E}(C_n) = (1/4)\binom{n}{3}$, $\operatorname{var} C_n = (3/16)\binom{n}{3}$. [*Hint*: Define $\binom{n}{3}$ random variables $X_{i,j,k}$, $\{i,j,k\} \in \mathfrak{P}_3[n]$, by $X_{i,j,k} = 1$ if $\{x_i, x_j, x_k\}$ is the support of a 3-cycle, and $= 0$ otherwise; observe then that $\mathbf{E}(X_{i,j,k}) = 1/4$.] (2) More generally, let C_k be the number of k-cycles of $\mathcal{T}$, then we have $\mathbf{E}(C_k) = \binom{n}{k}(k-1)!\,2^{-k}$ and $\operatorname{var} C_k = O(n^{2k-3})$ when $n \to \infty$. (A deep study and a vast bibliography on random tournaments are found in [*Moon, 1968].)

***23.** *Random partitions of a set, mode of $S(n,k)$.* With every finite set N, $|N| = n$, we associate the probability space $(\Omega, \mathscr{A}, \mathbf{P})$, where Ω is the set of *partitions* of N, $\mathscr{A} := \mathfrak{P}(\Omega)$, and $\mathbf{P}(\omega) = 1/|\Omega| = 1/\varpi(n)$ (p. 210) for each partition $\omega \in \Omega$. We are now interested in the study of the RV $B_n = B_n(\omega)$, the *number of blocks* of ω. (1) $\mathbf{P}(B_n = k) = S(n,k)/\varpi(n)$, where $S(n,k)$ is the Stirling number of the second kind (p. 204). The GF of the probabilities is hence equal to $P_n(u)/\varpi(n)$, where $P_n(u) := \sum_k S(n,k)u^k$. (2) The moments μ'_m (not central) of B_n equal:

$$\sum_{i=0}^{m}\left\{\frac{\varpi(n+i)}{\varpi(n)}\sum_{i\le j\le k\le m}(-1)^{k-j}\binom{k}{j}S(m,k)\,s(j,i)\right\}.$$

(3) Using Theorem D (p. 271) we have $P_n(u) = \prod_{1\le i\le n}(u+\alpha_i)$, where $0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_n$, and defining the n *independent* RV $X_{n,i}$ by $\mathbf{P}(X_{n,i} = 0) = \alpha_i(1+\alpha_i)^{-1}$, $\mathbf{P}(X_{n,i} = 1) = (1+\alpha_i)^{-1}$, show that $B_n = \sum_i X_{n,i}$. (4) We have the following asymptotic result ([Moser, Wyman, 1955b], [Rényi, Szekeres, 1957], [*De Bruijn, 1961], p. 102 [saddlepoint method applied to [4b], p. 210]; cf. Exercise 22, p. 228; see also [Haigh, 1972]) for $Re^R = n \to \infty$:

$$\varpi(n) \sim (R+1)^{-1/2}\exp\{n(R + R^{-1} - 1) - 1\}.$$

This allows us to verify condition ([41], p. 286) of *Lindeberg* and to apply the central limit theorem. Deduce from this an estimate for $\max_k S(n,k)$ and for the corresponding 'abscissa' (the 'mode') $k = K(n) \sim n/\log n$ ([Harper, 1967], [Kanold, 1968a, b], and especially [Wegner, 1972]). (5) Determine a *complete asymptotic expansion* for $K(n)$, $n \to \infty$. [*Hint*: Start from [1b], p. 254.]

***24.** *Random words.* Let $\mathfrak{X} := \{x_1, x_2, \ldots, x_n\}$ be a finite set, or *alphabet*, $|\mathfrak{X}| = n$. At every epoch $t = 1, 2, 3, \ldots$ we choose at random a *letter* from $\mathfrak{X}$, each letter having the probability $1/n$, and the choices at different moments are independent. In this way we obtain an infinite random word f, and the section consisting of the first t letters is called $f(t)$. In the sequel of this text, $T = T(\omega)$ is the RV which equals the *first epoch* that a certain event E concerning f occurs. (1) **Birthdays.** E is the event "one of the letters of $f(t)$ has appeared k times". Put $\exp_k u = 1 + u + u^2/2! + \cdots + u^k/k!$, and show that:

$$\mathbf{E}(T) = \int_0^\infty \{\exp_{k-1}(t/n)\}^n e^{-t}\,dt.$$

Use this to obtain, for fixed k, $\mathbf{E}(T) \sim (k!)^{1/k}\Gamma(1+1/k)\,n^{1-1/k}$ for $n \to \infty$ ([Klamkin, Newman, 1967]). (So, for $n = 365$, $k = 2$, one needs in average 25 guests to a party, to find that two of them have the same birthday, which may be surprising.) (2) **The matchboxes of Banach.** A certain mathematician always carries two matchboxes with him. Both contain initially N matches. Each time he wants a match, he draws a box at random. Certainly a moment will come that he draws an empty box. Let R be the RV equal to the number of matches left in the other box. Show that $\mathbf{P}(R=r)=\binom{2k-r}{k}2^{-2k+r}$ and that $\mathbf{E}(R)=(2k+1)2^{-2k}\binom{2k}{k}-1 \approx 2\sqrt{k/\pi}-1$ ([*Feller, 1968], I, p. 238, [Konecky, 1961]). Also compute the moments $\mathbf{E}(R^m)$, $m=2, 3, \ldots$ (3) **Picture collector.** T is now the event "each letter has appeared k times" in $f(t)$. Then $\mathbf{E}(T)=n\log n+(k-1)n\log\log n+n(\gamma-\log[(k-1)!])+o(n)$ when $n\to\infty$; γ is the Euler constant. ([Newman, Shepp, 1960], [Erdős, Rényi, 1961]). (Thus, when every bar of chocolate goes together with a picture, one must buy in average $n\log n$ of these bars in order to obtain the complete collection of different pictures used by the manufacturer.) (4) **The monkey typist.** Let g be a word of length l and E the event "the last l letters of $f(t)$ form the word g". If the l letters of g are different, $1\leqslant l\leqslant n$, then $\mathbf{E}(T)=n^l$. In the general case the 'periods' of g play a role ([Solov'ev, 1966]).

***25.** *Similarly loaded dice.* (1) Show that it is not possible to load similarly two dice in such a way that the total score will be an equidistributed RV (on the values 2, 3, ..., 11, 12). [*Hint:* In the contrary case, use the GF of the probabilities to show the existence of $x_0, x_1, \ldots, x_5$ such that $(x_0t+x_1t^2+\cdots+x_5t^6)^2=\frac{1}{11}(t^2+t^3+\cdots+t^{11}+t^{12})$.] (2) The following is a more difficult question (see [Clements, 1968]) suggested by the preceding. Let $x=(x_0, x_1, \ldots, x_m)\in R^{m+1}$ and r an integer $\geqslant 1$. We define the $c_q(x)$ by: $(x_0+x_1t+\cdots+x_mt^m)^r=\sum_{q=0}^{rm}c_q(x)t^q$ and put $M(x):=\max c_q(x)$ for $0\leqslant q\leqslant rm$. Compute $\min M(x)$ on the set of all x such that $x_0, x_1, \ldots, x_m\geqslant 0$ and $x_0+x_1+\cdots+x_m=1$. (3) Answer these two questions when the two dice may be *independently* loaded.

***26.** *Multicolour Ramsey numbers.* Let be given integers $b, p_1, p_2, \ldots, p_k$ such that $1\leqslant b\leqslant p_1, p_2, \ldots, p_k$. A finite set N is called *Ramsey-$(b; p_1, p_2, \ldots, p_k)$* if and only if, for all k-divisions $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_k$ of $\mathfrak{P}_b(N)$, $\mathfrak{P}_b(N)=\mathscr{C}_1+\mathscr{C}_2+\cdots+\mathscr{C}_k$, there exists an integer $i\in[k]$ and a block $P\in\mathfrak{P}_{p_i}(N)$ such that $\mathfrak{P}_b(P)\subset\mathscr{C}_i$. (1) Show by induction on k, the existence of *k-color b-ary Ramsey numbers*, denoted by $\rho(b; p_1, p_2, \ldots, p_k)$ and satisfying:

$$N \text{ is Ramsey-}(b; p_1, p_2, \ldots, p_k) \Leftrightarrow |N|\geqslant \rho(b; p_1, p_2, \ldots, p_k)$$

(2) Moreover, show that $\rho(1; p_1, p_2, \ldots, p_k)=p_1+p_2+\cdots+p_k-k+1$ ([*Ryser, 1963], p. 38, and [*Dembowski, 1968], p. 29). We note:

$\rho(2; 3, 3, 3)=17$, $\rho(2; 3, 3, 4)\geqslant 30$, $\rho(2; 4, 4, 4)\geqslant 80$, $\rho(2; 5, 5, 5)\geqslant 200$, $\rho(2; 3, 3, 3, 3)\geqslant 62$, $\rho(2; 3, 3, 3, 3, 3)\geqslant 278$. (3) As an application of (1) show that for every integer $k\geqslant 1$ there exists an integer $B(k)$ with the following property: when $n\geqslant B(k)$, each k-division $(A_1, A_2, \ldots, A_k)$ of $[n]$, $A_1+A_2+\cdots+A_k=[n]$ is such that one of the subsets A_i contains three numbers of the form $x, y, x+y$. [*Hint:* For $n\geqslant\rho(2; 3, 3, \ldots, 3)$, where the number 3 occurs k times, apply (1) to the division $\mathscr{C}_1+\mathscr{C}_2+\cdots+\mathscr{C}_k=\mathfrak{P}_2[n]$ defined by: $\{a, b\}\in\mathscr{C}_i \Leftrightarrow |a-b|\in A_i$.]

***27.** *Convex polygons whose vertices form a subset of a given point set of the plane* ([Erdős, Szekeres, 1935], explained in [*Ryser, 1963], p. 43, and [*Dembowski, 1968], p. 30). Let N be a finite set of points in the plane such that no three among them are collinear, N is *general*, for short. An m-gon extracted from N will be the following: a closed polygonal line $\mathscr{P}$, not necessarily convex, whose vertices are different and belong to N. Such a polygon $\mathscr{P}$ is considered as a set of pairs of N (its sides), $\mathscr{P}\subset\mathfrak{P}_2(N)$. (1) From every *general* set A, $|A|=5$, we can extract a convex quadrilateral. (2) Let M be a *general* set, $|M|\geqslant 4$, such that for each $Q\subset M$, $|Q|=4$, one of the three quadrilaterals whose vertex set is Q, is convex. Then, there exists a convex m-gon extracted from M, $|M|=m$. [*Hint:* If not, the convex hull of M would be spanned by less than m points, consequently there would exist a Q whose three quadrilaterals are not convex.] (3) Deduce from (1, 2) the following theorem: For every integer $m\geqslant 3$ there exists an integer $f(m)$ such that from every finite *general* set containing at least $f(m)$ points of the plane, a convex m-gon can be extracted. [*Hint:* We have $f(3)=3$, $f(4)=5$; for $m\geqslant 5$, apply the theorem on p. 283, $p=m$, $q=5$, $\mathscr{C}+\mathscr{D}=\mathfrak{P}_4(N)$, where $\mathscr{C}$ is the set of the Q, $|Q|=4$, such that one of 3 extracted quadrilaterals is convex.]

28. *Monotonic subsets of a sequence.* Let X be a set of real numbers >0, $X:=\{x_1, x_2, x_3, \ldots\}$, $0<x_1<x_2<x_3<\cdots$. For all integers $h, k\geqslant 1$, we put $r(h, k):=(h-1)(k-1)+1$. Let N be a subset with n elements of X, $N\subset X$, $|N|=n$, and let φ be a map of N into $\mathbf{R}$. We first suppose that $n=r(h, k)$. Show that there exists either a subset $H\subset N$, $|H|=h$, on which the restriction of φ is increasing (not necessarily strictly), or a subset $K\subset N$, $|K|=k$, on which φ is decreasing (not necessarily strictly). [*Hint:* Argue by induction on $n\geqslant 2$, and fixed h. For $A\subset N$, $|A|=r(h, k)-1$,

$|M|=r(n, k+1)$, apply the induction hypothesis to each of the sets $M_z:=A\cup\{z\}$, where z runs through $N-A$. If $m<r(n,k)$, the property does not hold.

29. *Zarankiewicz numbers.* The numbers $t(n,a)$ defined on p. 248 satisfy $t(a,a)=a^2$ and $t(a+1,a)=(a+1)^2-2$.

***30.** *Complete subgraphs in graphs with sufficiently high degrees.* A necessary and sufficient condition that every graph $\mathscr{G}$ of N, $|N|=n$, all of its degrees exceeding or equalling k ($\forall x\in N$, $\delta(x)\geqslant k$, p. 61), contains a complete subgraph with p nodes is $k>n(p-2)/(p-1)$ ([Turán, 1941], [Zarankiewicz, 1947]).

***31.** *Maximum of a certain quadratic form* ([Motzkin, Straus, 1965]). Let E be the set of vectors $x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ whose real coordinates x_i satisfy $x_i\geqslant 0$, $i\in[n]$, and $x_1+x_2+\cdots+x_n=1$. (1) Let $F(x)$ denote the quadratic form $\sum_{1\leqslant i<j\leqslant n} x_i x_j$ (for instance $F_3=x_1x_2+x_1x_3+x_2x_3$). Show that $\max_{x\in E}F(x)=(1-1/n)/2$. (2) More generally, let $\mathfrak{G}$ be a graph over $[n]=\{1,2,\ldots,n\}$ (p. 61) and $F_{\mathfrak{G}}(x)=\sum_{\{i,j\}\in\mathfrak{G}} x_ix_j$. Show that $\max_{x\in E}F_{\mathfrak{G}}(E)$ equals $(1-1/k)/2$, where k is the *maximum* number of nodes of complete subgraphs contained in $\mathfrak{G}$ (p. 62); in other words, the maximum value of the number of elements of sets $H\subset[n]$ such that $\mathfrak{P}_2(H)\subset\mathfrak{G}$. [*Hint*: If $K:=\{i_1,i_2,\ldots,i_k\}$ is the set of nodes of a complete subgraph of $\mathfrak{G}$, then a lower bound for $\max F_{\mathfrak{G}}(x)$ is given by the value of $F_{\mathfrak{G}}(x)$ for $x_j=1/k$ if $j\in K$ and $=0$ otherwise. For the other inequality, use induction; first observe that the maximum occurs in an interior point of E.]

***32.** *Systems of distinct representatives.* Let $\mathscr{S}:=\{B_1,B_2,\ldots,B_n\}$ be a system of blocks, not necessarily different from N, $B_i\subset N$, $i\in[n]$. A block $M=\{x_1,x_2,\ldots,x_n\}\subset N$ is called a *system of distinct representatives*, abbreviated SDR, if and only if $x_i\in B_i$ for all $i\in[n]$. A necessary and sufficient condition that $\mathscr{S}$ admits a SDR is that for every subsystem $\mathscr{S}'\subset\mathscr{S}$ we have $|\bigcup_{B\in\mathscr{S}'}B|\geqslant|\mathscr{S}'|$. (The preceding statement, due to [Hall (P.), 1935], answers in particular the '*marriage problem*': n boys know a certain number of girls; under what conditions can each boy marry a girl he knows already? (One girl may be acquainted with several boys!...) See also [Halmos, Vaughan, 1950], [Mirsky, Perfect, 1966], [Mirsky, 1967], [Rado, 1967].) [*Hint*: Argue by complete induction on n, using *critical* subsystems $\mathscr{S}'\subset\mathscr{S}$ in the sense that $|\bigcup_{B\in\mathscr{S}'}B|=|\mathscr{S}'|$. If no subsystem is critical, take one point $x\in B_1$ and remove it from each of the blocks $B_2,B_3,\ldots$ (if it occurs there). Thus we obtain a new system $B_2',B_3',\ldots,B_n'$, which can be handled by the induction hypothesis. If there exists a critical system, then there exists a largest integer k such that (after changing the indices) $|B_1\cup B_2\cup\cdots\cup B_k|=k$ $(<n)$ and then we can choose a SDR for $B_1,B_2,\ldots,B_k$, say A_0. Then we show next that the system $C_{k+1},C_{k+2},\ldots$, where $C_j:=B_j\setminus A_0$, also satisfies the induction hypothesis, so has also a SDR, say A_1. Hence the required SDR is $A_0\cup A_1$.] Deduce from this that every latin $k\times n$ rectangle (p. 182), $1\leqslant k\leqslant n-1$, can be extended into a $(k+1)\times n$ rectangle by adding one row.

33. *Agglutinating systems.* A system $\mathscr{S}$ of blocks of N, $\mathscr{S}\subset\mathfrak{P}'(N)$, $|N|=n$, is called *agglutinating* if any two of them are not disjoint. Show that the number $|\mathscr{S}|$ of blocks of such a system is less than or equal to 2^{n-1}, and that this number is a least upper bound. [*Hint*: Let $\mathscr{S}^*$ be the system of complements of the blocks of $\mathscr{S}$, then we have, in the sense of [10e] (p. 38), $\mathscr{S}+\mathscr{S}^*\subset\mathfrak{P}(N)$.] *Let, more generally, $\mathscr{F}_1,\mathscr{F}_2,\ldots,\mathscr{F}_k$ be k agglutinating systems of N, then $|\bigcup_{j=1}^{k}\mathscr{F}_j|\leqslant 2^n-2^{n-k}$ ([Katona, 1964], [Kleitman, 1966, 1968a]).

34. *A weighing problem.* Let be given n $(\geqslant 2)$ coins, all of the same weight, except one, which is a little lighter. Show that the minimum number of weighings which must be performed to discover the counterfeit coin equals the smallest integer $\geqslant\log_3 n$, where $x=\log_3 y\Leftrightarrow y=3^x$ (the scale used only allows the comparison of weights). (For this subject see [Cairns, 1963], [Erdős, Rényi, 1963].)

35. *The number of groups of order n.* Let $g(n)$ be the number of finite not isomorphic groups G of order n, $|G|=n$. (1) Use the Cayley table (=the multiplication table) of G to show that $g(n)\leqslant n^{n^2}$. (2) The Cayley table of G is completely known if we know it for $S\times G$ only, where S is a *system of generators* of G. (3) Let S be a *minimal* system of generators (so there does not exist a system of generators with a smaller number of elements). Show that $2^{|S|}\leqslant n$. Deduce that $g(n)\leqslant n^{n\log_2 n}$, where $\log_2 n$ means the logarithm with base 2 of n ([Gallagher, 1967]). The following table of

$g(n)$ is taken from [*Coxeter, Moser, 1965], p. 134. See also [Newman, 1967], [James, Connor, 1969].)

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$g(n)$	1	1	1	2	[illegible]	2	1	5	2	2	1	5	1	2	1	14	1

n	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
$g(n)$	5	1	5	2	[illegible]	1	15	2	2	5	4	1	4	1	51

36. *A minimax inequality.* Let $a_{i,j}$, $i\in[m]$, $j\in[n]$ be mn real numbers. Then $\min_i \max_j a_{i,j} \geq \max_j \min_i a_{i,j}$. (For extensions, see [Schützenberger, 1957].)

*37. *Two examples of extremal problems in* $[n]$ ([Klamkin, Newman, 1969]). (1) Let $\mathscr{S}$ be a system of k pairs of $[n]$, $\mathscr{S}=\{A_1, A_2, \ldots, A_k\}$, $A_i\subset[n]$, $|A_i|=2$, all disjoint, and such that the k integers $\sum_{x\in A_i} x$, $i\in[k]$, are all different and smaller than n. Then the largest possible value for k, denoted by $\varphi(n)$, satisfies $(2n/5)-3\leq\varphi(n)\leq(2n-3)/5$. (2) Let $\mathscr{S}$ be a system of k triples of $[n]$, $\mathscr{S}=\{A_1, A_2, \ldots, A_k\}$, $A_i\subset[n]$, $|A_i|=3$, all disjoint, and such that for all $i\in[k]$, $\sum_{x\in A_i} x=n$. Then the largest possible value for k, denoted by $\psi(n)$, satisfies $\psi(n)\sim(2/9)n$, for $n\to\infty$. (The reader will find in [*Erdös, 1963] a large number of difficult and extremely interesting combinatory problems concerning arithmetical extremal problems.)

*38. *Multiple points on a polygonal contour.* Let $A_1, A_2, \ldots, A_n$ be points in the plane, $n>2$, and let Γ be the polygonal contour whose sides are $A_1A_2, A_2A_3, \ldots, A_{n-1}A_n, A_nA_1$. A *multiple point* of Γ is any point, different from the A_i, through which pass at least two sides of Γ. Show that the number s_n of these multiple points satisfies $s_n\leq(1/2)n(n-4)+1$ for n even, and $s_n\leq(1/2)n(n-3)$ for n odd. These inequalities cannot be improved ([Bergmann, 1959]; see also [Jordan (Camille), 1920]).

*39. *Separating systems.* A *separating system* (or *Kolmogoroff system* or T_0 system) of N is any system $\mathscr{S}\subset\mathfrak{P}(N)$ such that for all x and $y\in N$, $x\neq y$, there exists either an A such that $x\in A\in\mathscr{S}$, $y\notin A$, or a B such that $y\in B\in\mathscr{S}$, $x\notin B$ (not exclusive or). Compute or estimate the number of separating systems of N, $|N|=n$.

*40. *Multicoverings.* An *l-multicovering* of N is any system $\mathscr{S}\subset\mathfrak{P}(N)$ such that each $x\in N$ is contained in exactly l blocks of $\mathscr{S}$; (the blocks are distinct). Compute and estimate the number of l-coverings of N, $|N|=n$ ([Comtet, 1968b], [Baroti, 1970], [*[illegible], 19[illegible], p. [illegible]]). Here are the first values of $C_2(n,k)$, the number of 2-coverings of N with k blocks, and $C_2(n)=\sum_k C_2(n,k)$, the total number of 2-coverings:

$n\backslash k$	3	4	5	6	7	8	9	10	$C_2(n)$
2	1								1
3	[illegible]	1							8
4	15	39	25	3					80
5	90	280	472	256	40				1368
6	621	1975	6135	7255	3006	535	15		19232
7	35[illegible]	11284	70700	149600	131876	51540	11455	421	424000

41. *At most 1-overlapping systems.* These are systems $\mathscr{S}$ of N, consisting of k blocks, $\mathscr{S}\subset\mathfrak{P}_k(N)$, such that for any A and B, $A\neq B$, we have $|A\cap B|\leq 1$. If $\varphi(n,k)$ is the largest possible number of blocks of such a system $\mathscr{S}$, show that $\varphi(n,k)\sim n^2/k(k-1)$, for $n\to\infty$ ([Erdös, Hanani, 1963], [Schönheim, 1966]).

*42. *Geometries.* A *geometry* (or *linear system*) of N is a system $\mathscr{S}\subset\mathfrak{P}(N)$ whose blocks, called *lines*, satisfy the following two conditions: (1) Each pair $A\subset N$, $|A|=2$, is contained in precisely one line; (2) each line contains at least two points. The following are the known values of $g(n)$, which is the number of geometries of N, $|N|=n$, and the numbers $g^*(n)$, which are the number of nonisomorphic ones:

n	1	2	3	4	5	6	7	8	9
$g(n)$	1	[illegible]	2	6	52	[illegible]	8390	1[illegible]6399	50[illegible]65[illegible]51
$g^*(n)$	1	[illegible]	3	3	5	10	2[illegible]	65	381

Compute and estimate $g(n)$ and $g^*(n)$. (For $n\geq 10$, we have $2^n<g^*(n)<g(n)<2^{\binom{n}{2}}$, these inequalities and their numerical values being due to [Doyen, 1967].)

*43. *Steiner triple systems.* A Steiner triple system over N or simply a "triple system", is a set $\mathscr{S}$ of triples of N, $\mathscr{S}\subset\mathfrak{P}_3(N)$, such that every pair of elements of N is contained in exactly one triple. In the sense of the previous exercise, this is a 'geometry' in which every line has three points. We suppose N finite, $|N|=n$. (1) A necessary and sufficient condition for the existence of a triple system is that n is of the form $6k+1$ or $6k+3$. (2) Let $s(n)$ denote the number of triple systems (of N), and $s^*(n)$ the number of nonisomorphic ones. The known values are:

n	1	3	7	9	13	15
$s(n)$	1	1	30	840	1197504000	60281712691200
$s^*(n)$	1	1	1	1	2	80

Compute and estimate $s(n)$ and $s^*(n)$, where $n \equiv 1$ or $3 \pmod 6$. (See [Doyen, Valette, 1971].)

FUNDAMENTAL NUMERICAL TABLES

Factorials with their prime factor decomposition

1	= 1!	= 1
2	= 2!	= 2
6	= 3!	= 2.3
24	= 4!	= $2^3.3$
120	= 5!	= $2^3.3.5$
720	= 6!	= $2^4.3^2.5$
5040	= 7!	= $2^4.3^2.5.7$
40320	= 8!	= $2^7.3^2.5.7$
3 62880	= 9!	= $2^7.3^4.5.7$
36 28800	= 10!	= $2^8.3^4.5^2.7$
399 16800	= 11!	= $2^8.3^4.5^2.7.11$
4790 01600	= 12!	= $2^{10}.3^5.5^2.7.11$
62270 20800	= 13!	= $2^{10}.3^5.5^2.7.11.13$
8 71782 91200	= 14!	= $2^{11}.3^5.5^2.7^2.11.13$
130 76743 68000	= 15!	= $2^{11}.3^6.5^3.7^2.11.13$
2092 27898 88000	= 16!	= $2^{15}.3^6.5^3.7^2.11.13$
35568 74280 96000	= 17!	= $2^{15}.3^6.5^3.7^2.11.13.17$
6 40237 37057 28000	= 18!	= $2^{16}.3^8.5^3.7^2.11.13.17$
121 64510 04088 32000	= 19!	= $2^{16}.3^8.5^3.7^2.11.13.17.19$
2432 90200 81766 40000	= 20!	= $2^{18}.3^8.5^4.7^2.11.13.17.19$
51090 94217 17094 40000	= 21!	= $2^{18}.3^9.5^4.7^3.11.13.17.19$
11 24000 72777 76076 80000	= 22!	= $2^{19}.3^9.5^4.7^3.11^2.13.17.19$
258 52016 73888 49766 40000	= 23!	= $2^{19}.3^9.5^4.7^3.11^2.13.17.19.23$
6204 48401 73323 94393 60000	= 24!	= $2^{22}.3^{10}.5^4.7^3.11^2.13.17.19.23$
1 55112 10043 33098 59840 00000	= 25!	= $2^{22}.3^{10}.5^6.7^3.11^2.13.17.19.23$
40 32914 61126 60563 55840 00000	= 26!	= $2^{23}.3^{10}.5^6.7^3.11^2.13^2.17.19.23$
1088 88694 50418 35216 07680 00000	= 27!	= $2^{23}.3^{13}.5^6.7^3.11^2.13^2.17.19.23$
30488 83446 11713 86050 15040 00000	= 28!	= $2^{25}.3^{13}.5^6.7^4.11^2.13^2.17.19.23$
8 84176 19937 39701 95454 36160 00000	= 29!	= $2^{25}.3^{13}.5^6.7^4.11^2.13^2.17.19.23.29$
265 25285 98121 91058 63630 84800 00000	= 30!	= $2^{26}.3^{14}.5^7.7^4.11^2.13^2.17.19.23.29$

Binomial coefficients $\binom{n}{k}$

n\k	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	1													
1	1	1												
2	1	2	1											
3	1	3	3	1										
4	1	4	6	4	1									
5	1	5	10	10	5	1								
6	1	6	15	20	15	6	1							
7	1	7	21	35	35	21	7	1						
8	1	8	28	56	70	56	28	8	1					
9	1	9	36	84	126	126	84	36	9	1				
10	1	10	45	120	210	252	210	120	45	10	1			
11	1	11	55	165	330	462	462	330	165	55	11	1		
12	1	12	66	220	495	792	924	792	495	220	66	12	1	
13	1	13	78	286	715	1287	1716	1716	1287	715	286	78	13	1
14	1	14	91	364	1001	2002	3003	3432	3003	2002	1001	364	91	14
15	1	15	105	455	1365	3003	5005	6435	6435	5005	3003	1365	455	105
16	1	16	120	560	1820	4368	8008	11440	12870	11440	8008	4368	1820	560
17	1	17	136	680	2380	6188	12376	19448	24310	24310	19448	12376	6188	2380
18	1	18	153	816	3060	8568	18564	31824	43758	48620	43758	31824	18564	8568
19	1	19	171	969	3876	11628	27132	50388	75582	92378	92378	75582	50388	27132
20	1	20	190	1140	4845	15504	38760	77520	125970	167960	184756	167960	125970	77520
21	1	21	210	1330	5985	20349	54264	116280	203490	293930	352716	352716	293930	203490
22	1	22	231	1540	7315	26334	74613	170544	319770	497420	646646	705432	646646	497420
23	1	23	253	1771	8855	33649	100947	245157	490314	817190	1144066	1352078	1352078	1144066
24	1	24	276	2024	10626	42504	134596	346104	735471	1307504	1961256	2496144	2704156	2496144
25	1	25	300	2300	12650	53130	177100	480700	1081575	2042975	3268760	4457400	5200300	5200300
26	1	26	325	2600	14950	65780	230230	657800	1562275	3124550	5311735	7726160	9657700	10400600

The most extended existing table is that of [MILLER, 1954], $n \leqslant 200$.

The number $P(n, m)$ of partitions of n into m summands and the number $p(n) = \sum_m P(n, m)$ of partitions of n

$p(n)$	$n\backslash m$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1											
2	2	1	1										
3	3	1	1	1									
5	4	1	2	1	1								
7	5	1	2	2	1	1							
11	6	1	3	3	2	1	1						
15	7	1	3	4	3	2	1	1					
22	8	1	4	5	5	3	2	1	1				
30	9	1	4	7	6	5	3	2	1	1			
42	10	1	5	8	9	7	5	3	2	1	1		
56	11	1	5	10	11	10	7	5	3	2	1	1	
77	12	1	6	12	15	13	11	7	5	3	2	1	1
101	13	1	6	14	18	18	14	11	7	5	3	2	1
135	14	1	7	16	23	23	20	15	11	7	5	3	2
176	15	1	7	19	27	30	26	21	15	11	7	5	3
231	16	1	8	21	34	37	35	28	22	15	11	7	5
297	17	1	8	24	39	47	44	38	29	22	15	11	7
385	18	1	9	27	47	57	58	49	40	30	22	15	11
490	19	1	9	30	54	70	71	65	52	41	30	22	15
627	20	1	10	33	64	84	90	82	70	54	42	30	22
792	21	1	10	37	72	101	110	105	89	73	55	42	30
1002	22	1	11	40	84	119	136	131	116	94	75	56	42
1255	23	1	11	44	94	141	163	164	146	123	97	76	56
1575	24	1	12	48	108	164	199	201	186	157	128	99	77
1958	25	1	12	52	120	192	235	248	230	201	164	131	100

For $m \geqslant n/2$ (right of the bold-face figures), the table is completed by $P(n, m) = p(n-m)$. A table of $P(n, m)$, for $n \leqslant 100$, is found in [TODD, 1949].

Partial exponential Bell polynomials $B_{n,k}(x_1, x_2, \ldots)$

$B_{1,1} = 1$, $B_{2,1} = 2$, $B_{2,2} = 1^2$, $B_{3,1} = 3$, $B_{3,2} = 3.12$, $B_{3,3} = 1^3$, $B_{4,1} = 4$, $B_{4,2} = 4.13 + 3.2^2$, $B_{4,3} = 6.1^2 2$, $B_{4,4} = 1^4$, $B_{5,1} = 5$, $B_{5,2} = 5.14 + 10.23$, $B_{5,3} = 10.1^2 3 + 15.12^2$, $B_{5,4} = 10.1^3 2$, $B_{5,5} = 1^5$, $B_{6,1} = 6$, $B_{6,2} = 6.15 + 15.24 + 10.3^2$, $B_{6,3} = 15.1^2 4 + 60.123 + 15.2^3$, $B_{6,4} = 20.1^3 3 + 45.1^2 2^2$, $B_{6,5} = 15.1^4 2$, $B_{6,6} = 1^6$, $B_{7,1} = 7$, $B_{7,2} = 7.16 + 21.25 + 35.34$, $B_{7,3} = 21.1^2 5 + 105.124 + 70.13^2 + 105.2^2 3$, $B_{7,4} = 35.1^3 4 + 210.1^2 23 + 105.12^3$, $B_{7,5} = 35.1^4 3 + 105.1^3 2^2$, $B_{7,6} = 21.1^5 2$, $B_{7,7} = 1^7$, $B_{8,1} = 8$, $B_{8,2} = 8.17 + 28.26 + 56.35 + 35.4^2$, $B_{8,3} = 28.1^2 6 + 168.125 + 280.134 + 210.2^2 4 + 280.23^2$, $B_{8,4} = 56.1^3 5 + 420.1^2 24 + 280.1^2 3^2 + 840.12^2 3 + 105.2^4$, $B_{8,5} = 70.1^4 4 + 560.1^3 23 + 420.1^2 2^3$, $B_{8,6} = 56.1^5 3 + 210.1^4 2^2$, $B_{8,7} = 28.1^6 2$, $B_{8,8} = 1^8$, $B_{9,1} = 9$, $B_{9,2} = 9.18 + 36.27 + 84.36 + 126.45$, $B_{9,3} = 36.1^2 7 + 252.126 + 504.135 + 378.2^2 5 + 315.14^2 + 1260.234 + 280.3^3$, $B_{9,4} = 84.1^3 6 + 756.1^2 25 + 1260.1^2 34 + 1890.12^2 4 + 2520.123^2 + 1260.2^3 3$, $B_{9,5} = 126.1^4 5 + 1260.1^3 24 + 840.1^3 3^2 + 3780.1^2 2^2 3 + 945.12^4$, $B_{9,6} = 126.1^5 4 + 1260.1^4 23 + 1260.1^3 2^3$, $B_{9,7} = 84.1^6 3$

[illegible]

* The letter x occurring in [3d] (p. 134) has not been written here to save space. Thus, [illegible] should read [illegible].

Logarithmic polynomials

$L_1 = 1^1$; $L_2 = 2^1 - 1^2$; $L_3 = 3^1 - 3.2^11^1 + 2.1^3$; $L_4 = 4^1 - 4.3^11^1 - 3.2^2 + 12.2^11^2 - 6.1^4$; $L_5 = 5^1 - 5.4^11^1 - 10.3^12^1 + 20.3^11^2 + 30.2^21^1 - 60.2^11^3 + 24.1^5$; $L_6 = 6^1 - 6.5^11^1 - 15.4^12^1 + 30.4^11^2 - 10.3^2 + 120.3^12^11^1 - 120.3^11^3 + 30.2^3 - 270.2^21^2 + 360.2^11^4 - 120.1^6$; $L_7 =$ [illegible]; $L_8 =$ [illegible]

Partial ordinary Bell polynomials $\hat{B}_{n,k}(x_1, x_2, \ldots)$

[illegible]

Multinomial coefficients $(a_1, a_2, \ldots, a_m) = \frac{(a_1 + a_2 + \cdots + a_m)!}{a_1! a_2! \cdots a_m!}$

The bold-face numbers indicate the values of $n = a_1 + a_2 + \cdots + a_m$. For saving place, we write (1^3) instead of $(1, 1, 1)$, $(3^2 2 1^2)$ instead of $(3, 3, 2, 1, 1)$, etc. ...

2: $(2) = 1$, $(1^2) = 2$ **3**: $(3) = 1$; $(21) = 3$; $(1^3) = 6$ **4**: $(4) = 1$; $(31) = 4$, $(2^2) = 6$; $(21^2) = 12$; $(1^4) = 24$ **5**: $(5) = 1$; $(41) = 5$, $(32) = 10$; $(31^2) = 20$, $(2^21) = 30$; $(21^3) = 60$; $(1^5) = 120$ **6**: $(6) = 1$; $(51) = 6$, $(42) = 15$, $(3^2) = 20$; $(41^2) = 30$, $(321) = 60$, $(2^3) = 90$; $(31^3) = 120$, $(2^21^2) = 180$; $(21^4) = 360$; $(1^6) = 720$ **7**: $(7) = 1$; $(61) = 7$, $(52) = 21$, $(43) = 35$; $(51^2) = 42$, $(421) = 105$, $(3^21) = 140$, $(32^2) = 210$; $(41^3) = 210$, $(321^2) = 420$, $(2^31) = 630$; $(31^4) = 840$, $(2^21^3) = 1260$; $(21^5) = 2520$; $(1^7) = 5040$ **8**: $(8) = 1$; $(71) = 8$, $(62) = 28$, $(53) = 56$, $(4^2) = 70$; $(61^2) = 56$, $(521) = 168$, $(431) = 280$, $(42^2) = 420$, $(3^22) = 560$; $(51^3) = 336$, $(421^2) = 840$, $(3^21^2) = 1120$, $(32^21) = 1680$, $(2^4) = 2520$; $(41^4) = 1680$, $(321^3) = 3360$, $(2^31^2) = 5040$; $(31^5) = 6720$, $(2^21^4) = 10080$; $(21^6) = 20160$; $(1^8) = 40320$ **9**: $(9) = 1$; $(81) = 9$, $(72) = 36$, $(63) = 84$, $(54) = 126$; $(71^2) = 72$, $(621) = 252$, $(531) = 504$, $(4^21) = 630$, $(52^2) = 756$, $(432) = 1260$, $(3^3) = 1680$; $(61^3) = 504$, $(521^2) = 1512$, $(431^2) = 2520$, $(42^21) = 3780$, $(3^221) = 5040$, $(32^3) = 7560$; $(51^4) = 3024$, $(421^3) = 7560$, $(3^21^3) = 10080$, $(32^21^2) = 15120$, $(2^41) = 22680$; $(41^5) = 15120$, $(321^4) = 30240$, $(2^31^3) = 45360$; $(31^6) = 60480$, $(2^21^5) = 90720$; $(21^7) = 181440$; $(1^9) = 362880$ **10**: $(10) = 1$; $(91) = 10$, $(82) = 45$, $(73) = 120$, $(64) = 210$, $(5^2) = 252$; $(81^2) = 90$, $(721) = 360$, $(631) = 840$, $(541) = 1260$, $(62^2) = 1260$, $(532) = 2520$, $(4^22) = 3150$, $(43^2) = 4200$; $(71^3) = 720$, $(621^2) = 2520$, $(531^2) = 5040$, $(4^21^2) = 6300$, $(52^21) = 7560$, $(4321) = 12600$, $(3^31) = 16800$, $(42^3) = 18900$, $(3^22^2) = 25200$; $(61^4) = 5040$, $(521^3) = 15120$, $(431^3) = 25200$, $(42^21^2) = 37800$, $(3^221^2) = 50400$, $(32^31) = 75600$, $(2^5) = 113400$; $(51^5) = 30240$, $(421^4) = 75600$, $(3^21^4) = 100800$, $(32^21^3) = 151200$, $(2^41^2) = 226800$; $(41^6) = 151200$, $(321^5) = 302400$, $(2^31^4) = 453600$; $(31^7) = 604800$, $(2^21^6) = 907200$; $(21^8) = 1814400$; $(1^{10}) = 3628800$

Stirling numbers of the first kind $s(n, k)$

n\k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1														
2	−1	1													
3	2	−3	1												
4	−6	11	−6	1											
5	24	−50	35	−10	1										
6	−120	274	−225	85	−15	1									
7	720	−1764	1624	−735	175	−21	1								
8	−5040	13068	−13132	6769	−1960	322	−28	1							
9	40320	−109584	118124	−67284	22449	−4536	546	−36	1						
10	−362880	1026576	−1172700	723680	−269325	63273	−9450	870	−45	1					
11	3628800	−10628640	12753576	−8409500	3416930	−902055	157773	−18150	1320	−55	1				
12	−39916800	120543840	−150917976	105258076	−45995730	13339535	−2637558	357423	−32670	1925	−66	1			
13	479001600	−1486442880	1931559552	−1414014888	657206836	−206070150	44990231	−6926634	749463	−55770	2717	−78	1		
14	−6227020800	19802759040	−26596717056	20313753096	−9957703756	3336118786	−790943153	135036473	−16669653	1474473	−91091	3731	−91	1	
15	87178291200	−283465647360	392156797824	−310989260400	159721605680	−56663366760	14409322928	−2681453775	368411615	−37312275	2749747	−143325	5005	−105	1

For a table of the $s(n, k)$, $n \leq 50$, see [Mitrinović (D. S. and R. S.), 1960a, b, 1961] and for several extensions [Mitrinović (D. S. and R. S.), 1963, 1963a, b, 1964, 1965, 1966].

Stirling numbers of the second kind $S(n, k)$ and exponential numbers $\varpi(n) = \sum_k S(n, k)$

$\varpi(n)$	n\k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1														
2	2	1	1													
5	3	1	3	1												
15	4	1	7	6	1											
52	5	1	15	25	10	1										
203	6	1	31	90	65	15	1									
877	7	1	63	301	350	140	21	1								
4140	8	1	127	966	1701	1050	266	28	1							
21147	9	1	255	3025	7770	6951	2646	462	36	1						
115975	10	1	511	9330	34105	42525	22827	5880	750	45	1					
678570	11	1	1023	28501	145750	246730	179487	63987	11880	1155	55	1				
4213597	12	1	2047	86526	611501	1379400	1323652	627396	159027	22275	1705	66	1			
27644437	13	1	4095	261625	2532530	7508501	9321312	5715424	1899612	359502	39325	2431	78	1		
190899322	14	1	8191	788970	10391745	40075035	63436373	49329280	27971800	5135130	752752	66066	3367	91	1	
1382958545	15	1	16383	2375101	42355950	210766920	420693273	408741333	216627840	67128490	12662650	1479478	106470	4550	105	1

For a table of $S(n, k)$, $n \leq 25$, see [Miksa, 1950], and for $\varpi(n)$, $n \leq 74$ [Levine, Dalton, 1962].

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The bibliographical references in the text only indicate the name of the author and the year of publication; *italic* indicates a book. Suffixes a, b, c distinguish between different papers by the same author, published in the same year. We use chiefly the abbreviations of *Mathematical Reviews*, except the following: *A.* : American ; *A.J.M.* : American Journal of Mathematics ; *A.M.M.* : The American Mathematical Monthly ; *A.M.S.* : American Mathematical Society ; *C.J.M.* : Canadian Journal of Mathematics ; *C.M.B.* : Canadian Mathematical Bulletin ; *C.R.* : Comptes rendus hebdomadaires des séances de l'Académie des Sciences (Paris) ; *Crelle* : Journal für die reine und angewandte Mathematik ; *I.* : Institut ; *J.* : Journal ; *J.C.T.* : Journal of Combinatorial Theory ; *M.* : Mathematic(s, al), Mathématique(s), Mathematik, etc. ; *M.J.* : Mathematical Journal ; *N.* : National ; repr. : reprinted by ; *S.* : Society, Société, etc. ; *U.* : University, etc. ; *Z.* : Zeitschrift.

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INDEX

(The letter f indicates a numerical table)