CHAPTER 14. Power Series, Taylor Series

Sec. 14.1 Sequences, Series, Convergence Tests

Problem Set 14.1. Page 740

1. **Uniqueness of limit.** A formal proof is given on p. A34 in Appendix 2. A standard idea for many uniqueness proofs is to proceed indirectly, that is, one assumes that there are two objects of the kind considered and shows that they are identical. In the present problem one assumes the existence of two limits and shows that they are identical. The idea of doing this is that one draws two circles, one around each of the two limits and so small that they do not intersect. Then, by the definition of a limit. the first of these circles must contain all the terms of the sequence in its interior, except for at most finitely many of them. But the same must also be true for the second circle, again because of the definition of a limit; that is, it must also contain all the terms in its interior, except for finitely many of them, But this is impossible because the two circles lie outside of each other, their interiors have no points in common.

   How comes that nothing can happen if the two limits are "very close to each other"? Well, they are distinct points and they are kept fixed. Hence they have a positive distance d from each other (which may be extremely small but not zero—otherwise the two points would be identical). And if you choose circles of radius, say, $d/3$ or $d/4$, you obtain disjoint circular disks, as needed.

11. **Boundedness.** Let $\{z_n\}$ be bounded, say, $|z_n| < K$ for some K and all n. Set $z_n = x_n + iy_n$ as in the text. Then boundedness of the sequences $\{x_n\}$ and $\{y_n\}$ can be seen from

$$|x_n| \leqq |z_n| < K, \qquad |y_n| \leqq |z_n| < K.$$

Here it was used that for $z = x + iy$ you always have

$$x^2 \leqq x^2 + y^2 = |z|^2 \quad \text{hence} \quad |x| \leqq |z|$$

and similarly for the imaginary part y, namely, $|y| \leqq |z|$.

Conversely, let $\{x_n\}$ and $\{y_n\}$ be bounded, say,

$$|x_n| < K, \qquad |y_n| < K.$$

Then $x_n^2 < K^2$, $y_n^2 < K^2$, so that

$$|z_n|^2 = x_n^2 + y_n^2 < 2K^2$$

By taking square roots this gives

$$|z_n| < k \qquad (k = K\sqrt{2}).$$

Hence $\{z_n\}$ is bounded.

Can you see that this proof is very similar to that of Theorem 1? Just set $c = a + ib = 0$, and write K for ϵ. Then you see that the idea is practically the same in both proofs.

13. **Convergence test.** Apply the ratio test. For this you need

$$a_n = n^2 i^n/2^n \quad \text{hence} \quad |a_n| = n^2/2^n$$

where it was used that $|i^n| = 1$ for all n, and

$$a_{n+1} = (n+1)^2 i^{n+1}/2^{n+1}$$

hence

$$|a_{n+1}| = (n+1)^2/2^{n+1}.$$

The quotient is

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^2/2^{n+1}}{n^2/2^n} = \frac{1}{2}\left(\frac{n+1}{n}\right)^2.$$

Obviously, it approaches 1/2 as n approaches infinity. This shows that the series converges.

The intuitive qualitative reason for this result is the fact that the exponential factor 2^n in the denominator increases eventually much more rapidly than the factor n^2 in the numerator.

15. Failure of the ratio test. Divergence by comparison. The ratio

$$\frac{a_{n+1}}{a_n} = \frac{1/\sqrt{n+1}}{1/\sqrt{n}} = \sqrt{\frac{n}{n+1}}$$

approaches 1 as n approaches infinity. Hence no conclusion can be drawn from the ratio test. However, the answer is obtained by comparing with the harmonic series. You have

$$\sqrt{2} < 2 \quad \text{hence} \quad 1/\sqrt{2} > 1/2 \tag{A}$$

$$\sqrt{3} < 3 \quad \text{hence} \quad 1/\sqrt{3} > 1/3, \quad \text{etc.}$$

Now the harmonic series diverges. Hence its partial sums must eventually become greater than any (fixed) bound, no matter how large. But because of the infinitely many inequalities (A) each partial sum of the given series (except for the first, which equals 1) must be larger than the corresponding partial sum of the harmonic series; hence these partial sums must also eventually become larger than any given bound. This means that the given series also diverges.

Sec. 14.2 Power Series

Problem Set 14.2. Page 745

1. Radius of convergence. You can immediately see that the center is $-i\sqrt{2}$. The radius of convergence equals 1 because $a_n = n$, $a_{n+1} = n + 1$, and the Cauchy-Hadamard formula (6) gives

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{n}{n+1} = 1.$$

11. Cauchy-Hadamard formula. The center is 0. The radius of convergence can be determined by the Cauchy-Hadamard formula (6). For this you need

$$a_n = \frac{(3n)!}{2^n (n!)^3} \quad \text{and} \quad a_{n+1} = \frac{(3(n+1))!}{2^{n+1}((n+1)!)^3}.$$

Now

$$(3(n+1))! = (3n+3)! = (3n+3)(3n+2)(3n+1)(3n)!$$

and $(3n)!$ will cancel when you form the quotient a_n/a_{n+1}. Similarly,

$$((n+1)!)^3 = (n+1)^3 (n!)^3$$

and $(n!)^3$ will cancel. Finally, $2^{n+1}/2^n = 2$. Together,

$$\frac{a_n}{a_{n+1}} = \frac{2(n+1)^3}{(3n+3)(3n+2)(3n+1)}.$$

The limit of this quotient as n approaches infinity equals the quotient of the highest power of n, which is n^3 in both the numerator and the denominator; thus,

$$2n^3/(27n^3) = 2/27.$$

This is the radius of convergence. It is relatively small. The reason is that $(3n)!$ in the numerator of the general coefficient grows much faster than $(n!)^3$ in the denominator, about 27 times as fast, the first few values of the quotient being

$$1,\ 6,\ 90,\ 1680,\ 34650,\ 756756,\ 17153136,\ 399072960.$$

17. **Extension of Theorem 2.** The given series

$$3^2 z^2 + z^3 + 3^4 z^4 + z^5 + 3^6 z^6 \tag{A}$$

consists of the geometric series $z^3 + z^5 + z^7 + \ldots$, which has radius of convergence 1, and the geometric series $(3z)^2 + (3z)^4 + (3z)^6 + \ldots$, which converges for $|3z|^2 < 1$, hence $|3z| < 1$, thus $|z| < 1/3$. It follows that the given series has radius of convergence 1/3.

In principle, the series (A) is similar to that in Example 6. It can be written

$$\sum_{n=2}^{\infty} a_n z^n$$

where

$$a_n = \frac{1}{2}(1 + (-1)^n)3^n + \frac{1}{2}(1 + (-1)^{n+1});$$

indeed, the first summand in a_n equals 3^n if n is even and 0 if n is odd; the second summand in a_n equals 0 if n is even and 1 if n is odd. The sequence of the n^{th} roots $|a_n|^{1/n}$ of $|a_n|$ has the two limit points 3 and 1, and the reciprocal 1/3 of the greatest limit point is the radius of convergence, as in Example 6 in the text.

Sec. 14.3 Functions Given by Power Series

Problem Set 14.3. Page 750

3. **Radius of convergence by differentiation (Theorem 3).** The geometric series

$$\sum_{n=0}^{\infty} \left(\frac{z}{5}\right)^n$$

converges for $|z/5| < 1$, thus for $|z| < 5$. By Theorem 3, the same holds for the derived series

$$\sum_{n=1}^{\infty} \frac{n z^{n-1}}{5^n} \tag{A}$$

(where you can sum from $n = 1$ because the term for $n = 0$ is 0) and for the derived series of (A)

$$\sum_{n=2}^{\infty} \frac{n(n-1)z^{n-2}}{5^n}.$$

Hence the same is true for

$$z^2 f''(z) = \sum_{n=2}^{\infty} n(n-1)\left(\frac{z}{5}\right)^n.$$

This is the given series and proves that it has the radius of convergence 5.

7. **Radius of convergence by integration (Theorem 4).** The factors $n + 2$ and $n + 1$ in the denominator of the coefficients suggests determining the radius of convergence by using two successive integrations. Since $(-4)^n/2^n = (-2)^n$, you may start from

$$\sum_{n=0}^{\infty} (-1)^n 2^n z^{2n} = \sum_{n=0}^{\infty} (-1)^n (2z^2)^n.$$

This geometric series converges for $|2z^2| < 1$, hence $|z^2| < 1/2$ or $|z| < 1/\sqrt{2}$. The same is true for this series multiplied by z, that is,

$$\sum_{n=0}^{\infty} (-1)^n 2^n z^{2n+1}. \tag{B}$$

Integration and cancellation of a factor 2 in the numerator and denominator give

$$\sum_{n=0}^{\infty} \frac{(-1)^n 2^n z^{2n+2}}{2n+2} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{n-1} z^{2n+2}}{n+1}.$$

This series has the same radius of convergence $1/\sqrt{2}$ as (B). The same is true for this series multiplied by z, that is,

$$\sum_{n=0}^{\infty} \frac{(-1)^n 2^{n-1} z^{2n+3}}{n+1}.$$

Another integration and cancellation of 2 give

$$\sum_{n=0}^{\infty} \frac{(-1)^n 2^{n-1} z^{2n+4}}{(2n+4)(n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{n-2} z^{2n+4}}{(n+2)(n+1)}.$$

By Theorem 4 this series also has the radius of convergence $1/\sqrt{2}$. Multiplication by $4/z^4$ yields the given series, which thus has the radius of convergence $1/\sqrt{2}$.

15. Cauchy product. The observation that

$$\frac{1}{(1-z)^2} = \frac{1}{1-z} \cdot \frac{1}{1-z}$$

suggests trying the geometric series

$$(1 + z + z^2 + z^3 + \ldots)(1 + z + z^2 + z^3 + \ldots) = \sum_{n=0}^{\infty} a_n z^n.$$

Now you obtain the power z^n on the left as the sum of the products

$$1 \cdot z^n + z \cdot z^{n-1} + z^2 \cdot z^{n-2} + \ldots + z^{n-1} \cdot z + z^n \cdot 1.$$

These are $n + 1$ terms. Hence $a_n = n + 1$, as claimed.

A more natural approach seems differentiation of the geometric series and of its sum, obtaining

$$\frac{1}{(1-z)^2} = \left(\frac{1}{1-z}\right)' = \sum_{n=1}^{\infty} n z^{(n-1)} = \sum_{s=0}^{\infty} (s+1) z^s,$$

where $n = s + 1$, hence $s = n - 1$, so that the summation over s starts with 0.

Sec. 14.4 Taylor Series and Maclaurin Series

Example 2 shows the Maclaurin series of the exponential function. Using it for defining e^z would have forced us to introduce series rather early. I tried this out several times, but found the approach chosen in this book didactically superior.

Problem Set 14.4. Page 757

1. Cosine. Use the familiar series for $\cos s$ and set $s = 2z^2$.

5. Geometric series. The denominator of

$$f(z) = (z+2)/(1-z^2)$$

suggests starting from the geometric series (with z^2 instead of z), that is,

$$1/(1-z^2) = 1 + z^2 + z^4 + z^6 + \ldots.$$

Multiplication by $z + 2$ gives the result

$$f(z) = (2+z)(1 + z^2 + z^4 + z^6 + \ldots) = 2 + z + 2z^2 + z^3 + 2z^4 + z^5 + 2z^6 + \ldots. \qquad \text{(A)}$$

The radius of convergence is 1 because the multiplication by $2 + z$ does not change it. In terms of summation signs the calculation is

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The radius of convergence is 1 because the multiplication by $2 + z$ does not change it. In terms of summation signs the calculation is

$$f(z) = (2+z)\sum_{n=0}^{\infty} z^{2n} = \sum_{n=0}^{\infty} (2z^{2n} + z^{2n+1}) = \sum_{n=0}^{\infty} \frac{(3+(-1)^n)z^n}{2}.$$

Indeed, $(-1)^n = +1$ if n is even, so that $(3+(-1)^n)/2 = 4/2 = 2$; this is the coefficient of every even power of z in (A). And $(-1)^n = -1$ if n is odd, so that then $(3+(-1)^n)/2 = 2/2 = 1$; this is the coefficient of every odd power in (A).

11. Fresnel integral. Start from the Maclaurin series of $\sin x$. Set $x = t^2$. Perform termwise integration, obtaining

$$\int_0^z \sum_{n=0}^{\infty} \frac{(-1)^n t^{(4n+2)}}{(2n+1)!} dt = \sum_{n=0}^{\infty} \frac{(-1)^n t^{(4n+3)}}{(2n+1)!(4n+3)} \Bigg|_{t=0}^{z}.$$

Now set $t = z$; this is the contribution from the upper limit of integration. The lower limit of integration gives 0, so that you obtain the answer by setting $t = z$ in the series on the right.

17. Taylor series. Use the method explained in Example 7, based on the geometric series, as follows.

$$\frac{1}{z} = \frac{1}{[z-2]+2} = \frac{1}{2[1+\frac{z-2}{2}]} = \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n \left(\frac{z-2}{2}\right)^n = \sum_{n=0}^{\infty} (-1)^n 2^{-n-1} (z-2)^n. \qquad \text{(B)}$$

This series converges for $|(z-2)/2| < 1$, thus $|z-2| < 2$. Hence the radius of convergence is $R = 2$.

In the present case the use of the coefficient formula in (1) would also be quite simple and straightforward. Indeed, by differentiation,

$$f(z) = 1/z, \quad f'(z) = -1/z^2, \quad f''(z) = +2/z^3, \quad f'''(z) = -3!/z^3$$

and in general,

$$f^{(n)}(z) = (-1)^n n!/z^{n+1}.$$

This implies for the center $z = 2$

$$f^{(n)}(2) = (-1)^n n!/2^{n+1}.$$

Division by $n!$ gives the coefficient

$$a_n = f^{(n)}(2)/n! = (-1)^n/2^{n+1},$$

in agreement with (B).

Sec. 14.5 Uniform Convergence. *Optional*

Problem Set 14.5. Page 766

1. Power series. This follows from Theorem 1 because the series has radius of convergence $R = 1$, so that it converges for $|z - i| < 1$.

7. Power series. By Theorem 1, a power series in powers of $z - z_0$ converges uniformly in the closed disk $|z - z_0| \leqq r$, where $r < R$ and R is the radius of convergence of the series. Hence solving Probs. 7 – 12 amounts to determining the radius of convergence.

In Prob. 7 you have a power series in powers of

$$Z = (z+i)^2 \qquad \text{(A)}$$

of the form

$$\sum_{n=0}^{\infty} a_n Z^n \qquad \text{(B)}$$

with coefficients $a_n = 1/5^n$. Hence the Cauchy-Hadamard formula in Sec. 14.2 gives the radius of convergence R^* of this series in Z in the form

$$\frac{a_n}{a_{n+1}} = \frac{5^{-n}}{5^{-(n+1)}} = 5.$$

Hence the series (B) converges uniformly in every closed disk $|Z| \leqq r^* < R^* = 5$. Substituting (A) and taking square roots, you see that this means uniform convergence of the given power series in powers of $z + i$ in every closed disk

$$|z+i| \leqq r < R = \sqrt{5}. \qquad \text{(C)}$$

You can also write this differently by setting

$$\delta = R - r. \qquad \text{(D)}$$

Then from $R > r$ by subtracting r on both sides you have $\delta = R - r > r - r = 0$, thus $\delta > 0$. Furthermore, from (D) you have $r = R - \delta$. Together,

$$|z+i| \leqq R - \delta = \sqrt{5} - \delta \qquad (\delta > 0).$$

This is the form in which the answer is given in Appendix 2 of the book.