CHAPTER 19. Numerical Methods for Differential Equations

Sec. 19.1 Methods for First-Order Differential Equations

Problem Set 19.1. Page 951

3. **Euler method.** This method is hardly used in practice because it is not accurate enough for most purposes, and there are other methods (Runge-Kutta methods, in particular) that give much more accurate values without too much more work. However, the Euler method explains the principle underlying this class of methods in the simplest possible form, and this is the purpose of the present problem. The latter has the advantage that it concerns a differential equation that can easily be solved exactly, so that you can observe the behavior of the error as the computation is progressing from step to step. The given inital value problem is

$$y' + 5x^4y^2 = 0, \qquad y(0) = 1.$$

For the Euler method you have to write the differential equation in the form

$$y' = f(x, y) = -5x^4y^2. \tag{A}$$

The required step size is $h = 0.2$, so that 10 steps will give approximate solution values from 0 to 2.0. Because of (A) the formula (3) for the Euler method takes the form

$$y_{n+1} = y_n + 0.2(-5x_n^4y_n^2) = y_n - x_n^4y_n^2. \tag{B}$$

Because of the initial condition $y(0) = 1$ your starting values are

$$x = x_0 = 0 \quad \text{and} \quad y = y_0 = 1.$$

The exact solution is obtained by separating variables. Dividing (A) by y^2 on both sides and integrating, you obtain

$$y'/y^2 = -5x^4, \quad -\frac{1}{y} = -x^5 + c.$$

Taking the reciprocal and multiplying by -1 gives

$$y = \frac{1}{x^5 + c^*} \qquad (c^* = -c).$$

From this and the initial condition $y(0) = 1$ you obtain $c^* = 1$. Hence the solution of the problem is

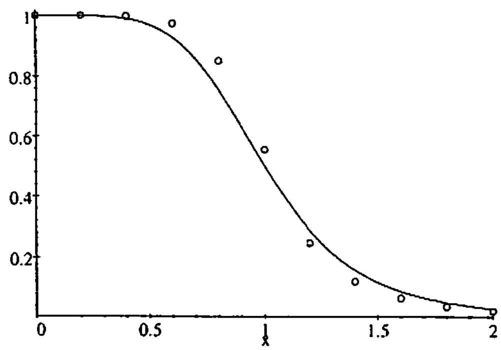
$$y = \frac{1}{x^5 + 1}. \tag{C}$$

Use (C) in computing the error of the approximations obtained from (B). The computations with 10S rounded to 6S give the values shown in the following table.

Table for Problem 3. Computations with Euler's Method

n	x_n	y_n	y_n^2	$x_n^4 y_n^2$	Exact	Error
0	0	1	1	0	1	0
1	0.2	1	1	0.001600	0.999680	−0.000320
2	0.4	0.998400	0.996803	0.025518	0.989864	−0.008536
3	0.6	0.972882	0.946499	0.122666	0.927850	−0.045031
4	0.8	0.850216	0.722867	0.296086	0.753194	−0.097022
5	1.0	0.554129	0.307059	0.307059	0.500000	−0.054129
6	1.2	0.247070	0.061044	0.126580	0.286671	+0.039601
7	1.4	0.120490	0.014518	0.055772	0.156783	0.036293
8	1.6	0.064718	0.004188	0.027449	0.087064	0.022346
9	1.8	0.037269	0.001389	0.014581	0.050262	0.012993
10	2.0	0.022688	0.000515	0.008236	0.030303	0.007615

It is interesting that the error is neither monotone increasing nor of constant sign, as you might have expected. Of course, this has to do with the particular form of the equation and its solution, which approaches zero as x approaches infinity. The figure shows the behavior of the solution and the approximate values marked as points.



Section 19.1. Problem 3. Solution curve and approximations by Euler's method

13. **Classical Runge-Kutta method.** This is perhaps the most popular method. The given initial value problem is (see Prob. 11)

$$y' = \frac{2}{x}\sqrt{y - \ln x} + \frac{1}{x}, \qquad y(1) = 0. \tag{A}$$

In Prob. 11 this was solved by Euler's method with $h = 0.1$ for 8 steps from 1.0 to 1.8. The error was determined from the exact solution and was found to increase from 0 to 0.05, approximately. The exact solution can be obtained as follows. The form of the differential equation suggests introducing the new unknown function

$$z = y - \ln x. \qquad \text{Then} \qquad z' = y' - \frac{1}{x} = \frac{2}{x}\sqrt{z}, \tag{B}$$

where the last equality sign follows by using (A). You can now separate the variables, obtaining

$$\frac{z'}{\sqrt{z}} = \frac{2}{x}.$$

By integration, $2\sqrt{z} = 2\ln x + c^*$, hence $\sqrt{z} = \ln x + c$. Squaring and then using (B), you have

$$z = (\ln x + c)^2, \qquad y = z + \ln x = (\ln x + c)^2 + \ln x.$$

Since $\ln 1 = 0$, you obtain from this and the initial condition $y(1) = c^2 = 0$. Hence the solution is $y = (\ln x)^2 + \ln x$, as shown in Prob. 11. The point of Prob. 13 is a comparison of the accuracy of Euler's method with that of the Runge-Kutta method. Now in the latter you have to compute four auxiliary quantities k_1, k_2, k_3, k_4 per step; hence in the required two steps this amounts to eight such computations, compared to eight steps in the Euler method in Prob. 11; in this sense, the comparison seems fair. The error will turn out to be about half of that of Euler's method. The results of the Runge-Kutta calculations (10S, rounded to 5S) are shown in the table.

Table for Problem 13. Computations with the Runge-Kutta method

x_n	y_n	k_1	k_2	k_3	k_4	Exact	Error
1.0	0	0.4	0.42197	0.44621	0.47501	0	0
1.4	0.43522	0.46529	0.47241	0.47440	0.47436	0.44969	0.01446
1.8	0.90744					0.93328	0.02584

Sec. 19.2 Multistep Methods

Problem Set 19.2. Page 955

1. **Adams-Moulton method.** The initial value problem to be solved is

$$y' = f(x,y) = x + y, \quad y(0) = 0. \tag{A}$$

The differential equation is linear. Hence you can solve it exactly, so that no numerical method would be needed. Indeed, write the equation in (A) in the standard form (1), Sec. 1.6,

$$y' - y = x,$$

and solve it by (4), Sec. 1.6, with $p = -1$, hence $h = -x$, obtaining

$$y(x) = e^x\left(\int e^{-x} x\,dx + c\right) = ce^x - x - 1.$$

The initial condition gives $y(0) = c - 0 - 1 = 0, c = 1$. Hence the solution of the initial value problem (A) is

$$y(x) = e^x - x - 1. \tag{B}$$

You can later use (B) for determining the errors of the approximate values obtained by the Adams-Moulton method. Now begin with the computation. From (A) you have

$$f_n = f(x_n, y_n) = x_n + y_n, \quad f_{n-1} = f(x_{n-1}, y_{n-1}) = x_{n-1} + y_{n-1}$$

and similarly for the other terms in (7a). Hence (7a) takes the form

$$y^*_{n+1} = y_n + \frac{0.1}{24}[55(x_n + y_n) - 59(x_{n-1} + y_{n-1}) + 37(x_{n-2} + y_{n-2}) - 9(x_{n-3} + y_{n-3})].$$

This gives the predictor. Similarly, the corrector (7b) takes the form

$$y_{n+1} = y_n + \frac{0.1}{24}[9(x_{n+1} + y^*_{n+1}) + 19(x_n + y_n) - 5(x_{n-1} + y_{n-1}) + (x_{n-2} + y_{n-2})].$$

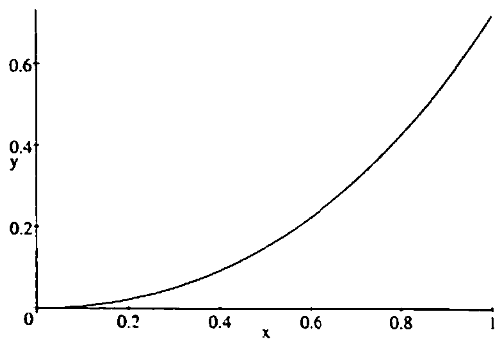
Arrange the numerical values obtained as in Table 19.9 on p. 955.

x_n	Starting y_n	Predicted y_n^*	Corrected y_n	Exact Values	Error $\times 10^{-8}$
0	0				
0.1	0.005 170 83				
0.2	0.021 402 6				
0.3	0.049 858 5				
0.4		0.091 820 10	0.091 824 54	0.091 824 70	16
0.5		0.148 716 45	0.148 721 31	0.148 721 27	−4
0.6		0.222 113 67	0.222 119 08	0.222 118 80	−28
0.7		0.313 747 30	0.313 753 27	0.313 752 71	−56
0.8		0.425 535 24	0.425 541 83	0.425 540 93	−91
0.9		0.559 597 13	0.559 604 42	0.559 603 11	−131
1.0		0.718 275 57	0.718 283 62	0.718 281 83	−180

You see that the differences between predictor and corrector are of the order 10^{-6} to 10^{-5}. These differences are monotone increasing, namely, in terms of the last three digits shown,

$$444,\ 486,\ 541,\ 597,\ 659,\ 729,\ 805.$$

The errors of the corrected values are much less, of the order 10^{-7} to 10^{-6}. This shows that the process of correcting the predicted values is definitely worthwhile. From $x = 0.5$ on, the error is negative and is monotone increasing in absolute value. Monotonicity is typical of many cases, but other behavior also appears, for instance, if the solution happens to be periodic. The solution of the present problem is monotone increasing (see the figure) because its derivative is nonnegative. In the figure the approximate values obtained lie practically on the curve; the errors are much too small for exhibiting them graphically.



Section 19.2. Problem 1. Solution curve

Sec. 19.3 Methods for Systems and Higher Order Equations

Problem Set 19.3. Page 961

1. Euler's method for systems. The given system is

$$y_1' = f_1(x, y) = 2y_1 - 4y_2, \qquad y_2' = f_2(x, y) = y_1 - 3y_2. \qquad \text{(A)}$$

Hence the recursion relation (5) with $h = 0.1$ takes the form (in components)

$$y_{1,n+1} = y_{1,n} + 0.1(2y_{1,n} - 4y_{2,n}), \qquad y_{2,n+1} = y_{2,n} + 0.1(y_{1,n} - 3y_{2,n}). \tag{B}$$

You see that this is not more complicated than Euler's method for a single equation, except for the fact that in each step you now have to work on two equations and each of them involves the result of both $y_{1,n}$ and $y_{2,n}$ of the previous step. Solve the system exactly, so that you can calculate the errors and judge the accuracy of the approximate values obtained by Euler's method. Proceed as in Sec. 3.3. The matrix of the system is

$$\mathbf{A} = \begin{bmatrix} 2 & -4 \\ 1 & -3 \end{bmatrix}.$$

Its characteristic equation is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (2-\lambda)(-3-\lambda) - (-4)1 = \lambda^2 + \lambda - 2 = (\lambda+2)(\lambda-1) = 0.$$

Hence the eigenvalues of **A** are -2 and 1. Corresponding eigenvectors are found by inserting the eigenvalues into the system of equations. Thus, for $\lambda = -2$ you have

$$(2+2)z_1 - 4z_2 = 0, \qquad \text{hence} \qquad z_2 = z_1,$$

and you can take $[1 \quad 1]^T$ as the first eigenvector. Note here that you have used z instead of x in Sec. 3.3 because x is supposed to be used as the independent variable. Similarly for $\lambda = 1$ you obtain

$$(2-1)z_1 - 4z_2 = 0, \quad \text{hence} \quad z_1 = 4z_2$$

and you can take $[4 \quad 1]^T$ as the second eigenvector. This gives the general solution

$$\mathbf{y}(x) = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-2x} + c_2\begin{bmatrix} 4 \\ 1 \end{bmatrix}e^{x}. \tag{C}$$

The given initial values are $y_1(0) = 3$, $y_2(0) = 0$, in vectorial form $\mathbf{y}(0) = [3 \quad 0]^T$. From this and (C) you obtain

$$\mathbf{y}(0) = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad \text{in components,} \quad \begin{aligned} c_1 + 4c_2 &= 3 \\ c_1 + c_2 &= 0 \end{aligned}.$$

The solution obtained by inspection, by elimination, or by Cramer's rule is $c_1 = -1$, $c_2 = 1$. Hence the given initial value problem has the solution

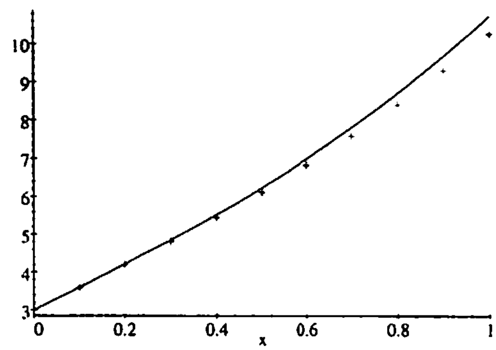
$$\mathbf{y}(x) = -\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-2x} + \begin{bmatrix} 4 \\ 1 \end{bmatrix}e^{x},$$

in components

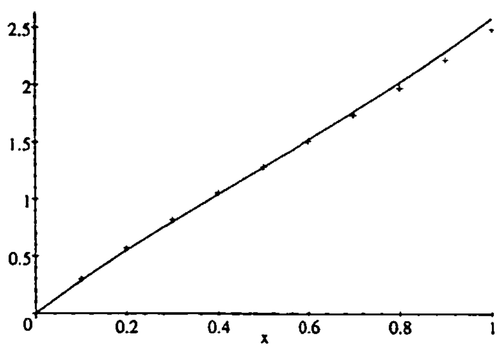
$$y_1(x) = -e^{-2x} + 4e^{x}$$
$$y_2(x) = -e^{-2x} + e^{x}.$$

The recursion formula (B) gives the approximate values for $x = 0, 0.1, 0.2, \ldots, 1.0$ shown in the table. The errors ϵ_1 and ϵ_2 of y_1 and y_2, respectively, are obtained by using (C). The first figure shows $y_1(x)$ as a curve in the xy_1-plane and the values obtained by Euler's method marked by crosses. The second figure shows $y_2(x)$ in the xy_2-plane and the approximate values as crosses. The third figure shows y_2 as a function of y_1. Hence this is the trajectory of the initial value problem plotted in the phase plane (the y_1y_2-plane). The values obtained by Euler's method are again marked by crosses. From the table and from the figures you see that the values are too inaccurate for practical purposes. This agrees with your experience in the application of Euler's method to a single differential equation.

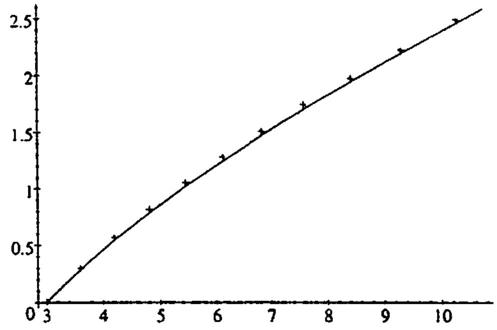
x	y_1	y_2	$\varepsilon(y_1)$	$\varepsilon(y_2)$
0	3	0	0	0
0.1	3.6	0.3	0.0	0.0
0.2	4.20	0.57	0.02	−0.02
0.3	4.812	0.819	0.039	−0.018
0.4	5.4468	1.0545	0.0712	−0.0120
0.5	6.11436	1.28283	0.11265	−0.00199
0.6	6.824100	1.509417	0.163181	0.011508
0.7	7.585153	1.739002	0.223261	0.028154
0.8	8.406583	1.975817	0.293684	0.047828
0.9	9.297573	2.223730	0.375541	0.070574
1.0	10.267596	2.486368	0.470964	0.096578



Section 19.3. Problem 1. Exact solution $y_1(x)$ and approximate values (crosses)



Section 19.3. Problem 1. Exact solution $y_2(x)$ and approximate values (crosses)



Section 19.3. Problem 1. Exact solution as a curve (trajectory) in the y_1y_2-plane and approximate values

5. Classical Runge-Kutta method. The initial value problem is the same as in Prob. 1. Two steps with the required (very large) $h = 0.5$ will give approximations for $x = 0.5$ and 1.0, which you can compare with the values in Prob. 1 obtained by Euler's method. The system is (in component form)

$$\begin{aligned} y_1' &= f_1(x,y) = 2y_1 - 4y_2 \qquad \text{(A)} \\ y_2' &= f_2(x,y) = y_1 - 3y_2. \end{aligned}$$

The initial values are $y_1(0) = 3$, $y_2(0) = 0$. Now use the formula (6) for the classical Runge-Kutta method in Sec. 19.3. Formula (6b) consists of four vector formulas. Since (A) consists of two equations, each of the vector functions in (6) has two components. This will give you $4 \cdot 2 = 8$ formulas for the components. You could write

$$\begin{aligned} \mathbf{k}_1 &= [k_{11}\ \ k_{12}]^T \qquad \text{(B)} \\ \mathbf{k}_2 &= [k_{21}\ \ k_{22}]^T \\ \mathbf{k}_3 &= [k_{31}\ \ k_{32}]^T \\ \mathbf{k}_4 &= [k_{41}\ \ k_{42}]^T. \end{aligned}$$

However, a simpler notation is that in Example 2 of Sec. 19.3 of the text, namely, instead of $\mathbf{k}_1, \dots, \mathbf{k}_4$ write

$$\begin{aligned} \mathbf{a} &= [a_1\ \ a_2]^T \qquad \text{(C)} \\ \mathbf{b} &= [b_1\ \ b_2]^T \\ \mathbf{c} &= [c_1\ \ c_2]^T \\ \mathbf{d} &= [d_1\ \ d_2]^T. \end{aligned}$$

(The text of the book uses the notation (B), which more distinctly shows the transition from the "scalar case" of a single differential equation to the "vector case" of a system of differential equations. (If you prefer (B), which forces you to carry along *two indices*, use it.) A further notational simplification giving the computation a simpler look is

$$y_1 = y, \quad y_2 = z \qquad \text{(D)}$$

because then instead of $y_{1,n}$ and $y_{2,n}$ (as in the text) you simply have $y_{1,n} = y_n$ and $y_{2,n} = z_n$. This will help in work by hand as well as in writing up a computer program. (Again, if you don't like (D), use y_1 and y_2.) From (6c), written in the notations (C) and (D), you obtain the following equations for calculating the auxiliary quantities occurring in the classical Runge-Kutta method

$$\begin{aligned} a_1 &= hf_1(x_n, y_n, z_n) \\ a_2 &= hf_2(x_n, y_n, z_n) \end{aligned}$$

and so on. Inserting

$$\begin{aligned} f_1 &= 2y_1 - 4y_2 = 2y - 4z \\ f_2 &= y_1 - 3y_2 = y - 3z \end{aligned}$$

from (A) and $h = 0.5$, as required, gives

$$a_1 = hf_1(x_n, y_n, z_n) = 0.5(2y_n - 4z_n)$$
$$a_2 = hf_2(x_n, y_n, z_n) = 0.5(y_n - 3z_n).$$

Note that in the system the independent variable x does not appear explicitly. This is an advantage because you need not pay attention to the place at which x is taken in each step (x_n or $x_n + h/2$ or $x_n + h$). Similarly,

$$b_1 = hf_1(x_n + h/2, y_n + a_1/2, z_n + a_2/2) = 0.5[2(y_n + a_1/2) - 4(z_n + a_2/2)]$$
$$b_2 = hf_2(x_n + h/2, y_n + a_1/2, z_n + a_2/2) = 0.5[(y_n + a_1/2) - 3(z_n + a_2/2)].$$

Note that in the general formula, b_1 and b_2 differ only by f_1 and f_2, so that from b_1 you can immediately see the form of b_2 by looking at the given system. Furthermore,

$$c_1 = hf_1(x_n + h/2, y_n + b_1/2, z_n + b_2/2) = 0.5[2(y_n + b_1/2) - 4(z_n + b_2/2)]$$
$$c_2 = hf_2(x_n + h/2, y_n + b_1/2, z_n + b_2/2) = 0.5[y_n + b_1/2 - 3(z_n + b_2/2)]$$

and finally

$$d_1 = hf_1(x_n + h, y_n + c_1, z_n + c_2) = 0.5[2(y_n + c_1) - 4(z_n + c_2)]$$
$$d_2 = hf_2(x_n + h, y_n + c_1, z_n + c_2) = 0.5[y_n + c_1 - 3(z_n + c_2)].$$

The recursion for x is

$$x_{n+1} = x_n + h.$$

The next values for $y_1 = y$ and $y_2 = z$ are given by (6c), that is,

$$y_{n+1} = y_n + \frac{1}{6}(a_1 + 2b_1 + 2c_1 + d_1)$$
$$z_{n+1} = z_n + \frac{1}{6}(a_2 + 2b_2 + 2c_2 + d_2).$$

The computation gives

x	y_1	y_2	Error of y_1	Error of y_2
0.5	6.218750	1.273438	0.008256	0.007404
1.0	10.728760	2.576721	0.009032	0.006225

The errors are much smaller than the corresponding errors in Prob. 1. Of course, further accuracy can be gained if we reduce $h = 0.5$ to, say, $h = 0.1$ (the value used in Prob. 1). The computation (10S rounded to 6D) gives the following values, exhibiting a very substantial reduction of the errors.

x	y_1	y_2	Error of y_1 $\times 10^6$	Error of y_2 $\times 10^6$
0.0	3	0	0	0
0.1	3.601 50	0.286 438	3	3
0.2	4.215 286	0.551 078	5	4
0.3	4.850 617	0.801 042	6	5
0.4	5.517 962	1.042 490	7	6
0.5	6.226 997	1.280 835	8	6
0.6	6.987 272	1.520 918	9	7
0.7	7.808 404	1.767 149	10	7
0.8	8.700 257	2.023 638	11	6
0.9	9.673 102	2.294 298	11	6
1.0	10.737 779	2.582 940	13	6

Sec. 19.4 Methods for Elliptic Partial Differential Equations

Problem Set 19.4. Page 969

1. **Derivation of (6c).** For this derivation you have to know Taylor's formula

$$u(x+h,y+k) = u + hu_x + ku_y + \frac{1}{2}(h^2 u_{xx} + 2hku_{xy} + k^2 u_{yy}) + \ldots \qquad \text{(A)}$$

where the function u and all the derivatives on the right are evaluated at (x, y). The further derivation is now automatic. If you replace h by $-h$ on the left, you get corresponding minus signs on the right, that is, a minus sign in the second and fifth term,

$$u(x-h,y+k) = u - hu_x + ku_y + \frac{1}{2}(h^2 u_{xx} - 2hku_{xy} + k^2 u_{yy}) + \ldots. \qquad \text{(B)}$$

The right side of (6c) tells you what further expressions you should consider, namely,

$$u(x+h,y-k) = u + hu_x - ku_y + \frac{1}{2}(h^2 u_{xx} - 2hku_{xy} + k^2 u_{yy}) + \ldots \qquad \text{(C)}$$

and

$$u(x-h,y-k) = u - hu_x - ku_y + \frac{1}{2}(h^2 u_{xx} + 2hku_{xy} + k^2 u_{yy}) + \ldots. \qquad \text{(D)}$$

The idea of the proof of (6c) now is to combine (A) ... (D) so that the derivative u_{xy} will remain, whereas all the other derivatives as well as the function u itself will drop out. In (A) minus (B) the function u and the derivatives u_y, u_{xx}, and u_{yy} drop out and you are left with

$$2hu_x + 2hku_{xy}. \qquad \text{(E)}$$

In (D) minus (C) the function u and the derivatives u_y, u_{xx}, and u_{yy} drop out and you are left with

$$-2hu_x + 2hku_{xy}. \qquad \text{(F)}$$

Addition (E) plus (F) gives $4hku_{xy}$. Division by $4hk$ gives u_{xy}, the left side of (6c). Now

$$\text{(E)} + \text{(F)} = \text{(A)} - \text{(B)} - \text{(C)} + \text{(D)}$$

and this is precisely the right side of (6c). This completes the derivation.

3. **Potential. Liebmann's method (Gauss-Seidel iteration).** Proceed as in Example 1. Sketch the square and the grid and indicate the boundary potential as well the notation for the four interior points at which

you have to find the potential. The linear system to be solved is (we indicate at each equation the point from which it results)

$$\begin{array}{llllll}
(P_{11}) & -4u_{11} & + u_{21} & + u_{12} & & = -330 \\
(P_{21}) & u_{11} & -4u_{21} & & + u_{22} & = -210 \\
(P_{12}) & u_{11} & & -4u_{12} & + u_{22} & = -330 \\
(P_{22}) & & u_{21} & + u_{12} & -4u_{22} & = -210.
\end{array}$$

The augmented matrix of this system is

$$\begin{bmatrix} -4 & 1 & 1 & 0 & -330 \\ 1 & -4 & 0 & 1 & -210 \\ 1 & 0 & -4 & 1 & -330 \\ 0 & 1 & 1 & -4 & -210 \end{bmatrix}.$$

Apply Gauss elimination. Row 1 is the pivot row. The next matrix is

$$\begin{bmatrix} -4 & 1 & 1 & 0 & -330 \\ 0 & -3.75 & 0.25 & 1 & -292.5 \\ 0 & 0.25 & -3.75 & 1 & -412.5 \\ 0 & 1 & 1 & -4 & -210 \end{bmatrix} \begin{array}{l} \\ \text{Row 2} + 0.25\,\text{Row 1} \\ \text{Row 3} + 0.25\,\text{Row 1} \\ \text{Row 4.} \end{array}$$

Row 2 is the pivot row. The next matrix is

$$\begin{bmatrix} -4 & 1 & 1 & 0 & -330 \\ 0 & -3.75 & 0.25 & 1 & -292.5 \\ 0 & 0 & -3.733333 & 1.066667 & -432 \\ 0 & 0 & 1.066667 & -3.733333 & -288 \end{bmatrix} \begin{array}{l} \\ \\ \text{Row 3} + (1/15)\,\text{Row 2} \\ \text{Row 4} + (4/15)\,\text{Row 2.} \end{array}$$

Row 3 is the next (and last) pivot row. Row 4 of the next matrix is Row 4 plus 2/7 times Row 3 of the previous matrix, namely,

$$[0 \quad 0 \quad 0 \quad -3.428571 \quad -411.428572].$$

Back substitution gives, in this order,

$$u_{22} = 120, \quad u_{12} = 150, \quad u_{21} = 120, \quad u_{11} = 150.$$

The result reflects a symmetry (can you see it now?), which you could have used to reduce your 4×4 system to a 2×2 system and solve the latter. We emphasize again that in practice, such systems are very large, due to much finer grids, and our problem serves to explain and illustrate the principle. For fine grids the matrix is sparse, so that the Liebmann method (Gauss-Seidel iteration) becomes advantageous. In our problem the iteration uses the system in the form (as in Example 1 of the text)

$$\begin{array}{lllll}
u_{11} = & & 0.25u_{21} + 0.25u_{12} & & + 82.5 \\
u_{21} = & 0.25u_{11} & & + 0.25u_{22} & + 52.5 \\
u_{12} = & 0.25u_{11} & & + 0.25u_{22} & + 82.5 \\
u_{22} = & & 0.25u_{21} + 0.25u_{12} & & + 52.5.
\end{array}$$

The iteration, starting with the suggested values 100, 100, 100, 100, gives the following values.

	Step 1	Step 2	Step 3	Step 4	Step 5	Step 10
100	132.5	145.3125	148.828125	149.707031	149.926758	149.999928
100	110.625	117.65625	119.414062	119.853516	119.963379	119.999964
100	140.625	147.65625	149.414062	149.853516	149.963379	149.999964
100	115.3125	118.828125	119.707031	119.926758	119.981690	119.999982

Fifteen steps give accurate 10S-values.

Sec 19.5 Neumann and Mixed Problems. Irregular Boundary

Problem Set 19.5. Page 975

3. **Mixed boundary conditions for the Laplace equation.** Proceed as in Example 1. The situation is simpler because you are dealing with the Laplace equation, whereas Example 1 in the text concerns a Poisson equation. Equation (1) in Example 1 is a list of boundary values, in which the values 0 are not included. There are 10 grid points on the boundary. For Prob. 3 the 10 boundary values, beginning at 0 and going around counterclockwise, are

$$1, \quad 1, \quad 1, \quad 1, \quad 1, \quad 1, \quad u_n(P_{22}) = 1, \quad u_n(P_{12}) = 1, \quad 1, \quad 1. \qquad \text{(A)}$$

Here $u_n(P_{22}) = u_y(P_{22})$ and $u_n(P_{12}) = u_y(P_{12})$, just as in Example 1; that is, the outer normal direction at the upper edge of the rectangle is the positive y-direction. For the two inner points you obtain two equations, corresponding to (2a) in Example 1, which you label (P_{11}) and (P_{21}); these are the inner points from which (and from whose neighbors) you get the two equations. The left sides of the equations are the same as in Example 1. The right sides differ; they are −2 (−1 from point P_{10} and −1 from point P_{01}) and −2 (−1 from point P_{20} and −1 from P_{31}), respectively. Hence these equations are

$$\begin{aligned} &(P_{11}) \qquad -4u_{11} + u_{21} + u_{12} = -2 \\ &(P_{21}) \qquad u_{11} - 4u_{21} + u_{22} = -2. \end{aligned} \qquad \text{(B)}$$

(Labeling the equations would not be absolutely necessary because the term with the coefficient −4 indicates whose stencil you are considering.) u_{12} and u_{22} are unknown because at P_{12} and P_{22} the normal derivative is given, not the function value u. As in Example 1 you extend the rectangle and the grid in the positive y-direction , introducing the points P_{13} and P_{23} as in Fig. 426b and assuming that the Laplace equation continues to hold in the extended rectangle. Then you can write down two more equations (the analog of (2b) in the example), namely,

$$\begin{aligned} &(P_{12}) \qquad u_{11} - 4u_{12} + u_{22} + u_{13} = -1 \\ &(P_{22}) \qquad u_{21} + u_{12} - 4u_{22} + u_{23} = -1. \end{aligned} \qquad \text{(C)}$$

In (P_{12}) the −1 on the right comes from P_{02}. In (P_{22}) the −1 comes from P_{32}. You now have these two additional equations, at the price of two new unknowns u_{13} and u_{23}, so it looks as if you have gained nothing. However, you have not yet used the condition on the upper edge of the original rectangle (the normal derivative being 1 there), and this is what you do next, just as in Example 1. This gives (since $h = 0.5$)

$$1 = \frac{\partial}{\partial y} u_{12} \approx \frac{u_{13} - u_{11}}{2h} = u_{13} - u_{11},$$

hence

$$u_{13} = u_{11} + 1$$

and

$$1 = \frac{\partial}{\partial y} u_{22} \approx \frac{u_{23} - u_{21}}{2h} = u_{23} - u_{21}$$

hence

$$u_{23} = u_{21} + 1.$$

Now substitute the expressions for u_{13} and u_{23} into (C) and simplify. In the first of these equations you have $u_{11} + u_{13} = 2u_{11} + 1$. In the second equation, $u_{21} + u_{21} = 2u_{21} + 1$. Taking the terms 1 to the right, the equations in (C) thus become

$$\begin{aligned} 2u_{11} \quad & - 4u_{12} + u_{22} = -2 \qquad \text{(D)} \\ 2u_{21} \quad & + u_{12} - 4u_{22} = -2. \end{aligned}$$

Your system to be solved consists of the four equations in (B) and (D). Its augmented matrix looks as follows. (For better orientation write the unknowns in a row above the matrix, where rs denotes the right side.)

$$\begin{array}{ccccc} u_{11} & u_{21} & u_{12} & u_{22} & rs \end{array}$$
$$\begin{bmatrix} -4 & 1 & 1 & 0 & -2 \\ 1 & -4 & 0 & 1 & -2 \\ 2 & 0 & -4 & 1 & -2 \\ 0 & 2 & 1 & -4 & -2 \end{bmatrix}$$

You can solve this by Gauss elimination. To eliminate u_{11}, use Row 1 as pivot row. Then compute a new matrix

$$\begin{bmatrix} -4 & 1 & 1 & 0 & -2 \\ 0 & -3.75 & 0.25 & 1 & -2.5 \\ 0 & 0.5 & -3.5 & 1 & -3 \\ 0 & 2 & 1 & -4 & -2 \end{bmatrix} \begin{array}{l} \\ \text{Row 2 + 0.25 Row 1} \\ \text{Row 3 + 0.5 Row 1} \\ \text{Row 4} \end{array}$$

Row 2 is the pivot row and is left unchanged. The next matrix is

$$\begin{bmatrix} -4 & 1 & 1 & 0 & -2 \\ 0 & -3.75 & 0.25 & 1 & -2.5 \\ 0 & 0 & -3.466667 & 1.133333 & -3.333333 \\ 0 & 0 & 1.133333 & -3.466667 & -3.333333 \end{bmatrix} \begin{array}{l} \\ \\ \text{Row 3 + \{0.5/3.75\}Row 2} \\ \text{Row 4 + \{2/3.75\}Row 2} \end{array}$$

Finally, Row 3 is the pivot row and Row 4 + (1.133333/3.466667) Row 3 is the new Row 4, which is of the form

$$[0 \quad 0 \quad 0 \quad -3.096154 \quad -4.423077].$$

Back substitution now gives, in this order,

$$u_{22} = 1.428571, \quad u_{12} = 1.428571, \quad u_{21} = 1.142857, \quad u_{11} = 1.142857.$$

The result indicates that you could have saved much of this work by using symmetry. In practice, this will not happen too often because it requires that the region shows symmetry and, in addition, the given boundary values exhibit the same symmetry. Our present problem satisfies these conditions; the rectangle and the boundary values are both symmetric with respect to the vertical line $x = 0.75$.

13. **Irregular boundary**. First make a sketch of your own, showing the region, the grid, and the numerical values of the given boundary data. (Use a red pencil for the latter, in order not to confuse them with notations for the boundary points.) On the x-axis the boundary values at the grid points are $u_{00} = 0$ (which will not be needed), $u_{10} = 3$, $u_{20} = 6$, $u_{30} = 9$ (not needed). On the y-axis you have $u_{01} = u_{02} = u_{03} = 0$. Furthermore, $u = 9 - 3y$ on the right vertical boundary gives $u_{31} = 9 - 3 \cdot 1 = 6$. On the upper horizontal portion of the boundary the potential is 0, hence $u_{13} = 0$. The sloping portion of the boundary is given by

$$y = 4.5 - x. \quad \text{Hence} \quad x = 4.5 - y.$$

For the lowest point of it you have $x = 3$; hence $y = 4.5 - 3 = 1.5$. For the next grid point on it you see that $y = 2$, hence $4.5 - 2 = 2.5$. The next grid point corresponds to $x = 2$, as you see from the figure; hence $y = 4.5 - 2 = 2.5$. For the highest point, $y = 3$, hence $x = 4.5 - 3 = 1.5$. Hence the four points just considered on the sloping portion of the boundary have the coordinates and potential $u(x, y) = x^2 - 1.5x = x(x - 1.5)$ as follows.

$$\begin{array}{lll} (3, 1.5) & \text{hence} & u = 3(3 - 1.5) = 4.5, \\ (2.5, 2) & \text{"} & u = 2.5(2.5 - 1.5) = 2.5 \\ (2, 2.5) & \text{"} & u = 2(2 - 1.5) = 1 \\ (1.5, 3) & \text{"} & u = 1.5(1.5 - 1.5) = 0. \end{array} \tag{A}$$

You will need only the second and third of these points and potentials. You are now ready to set up the linear system of equations. You have 4 inner points $P_{11}, P_{21}, P_{12}, P_{22}$. For the first three of these, you obtain equations of the usual form, namely (see the figure and the boundary values given or derived from the given formula referring to the portion of the boundary on the x-axis)

$$\begin{array}{ll} (\mathrm{P}_{11}:) & -4u_{11} + u_{21} + u_{12} \qquad\qquad = -u_{10} - u_{01} = -3 - 0 = -3 \\ (\mathrm{P}_{21}:) & u_{11} - 4u_{21} \qquad\qquad + u_{22} = -u_{20} - u_{31} = -6 - 6 = -12 \\ (\mathrm{P}_{12}:) & u_{11} \qquad\qquad - 4u_{12} + u_{22} = -u_{02} - u_{13} = -0 - 0 = 0. \end{array}$$

For P_{22} the situation is as in Fig. 427 in the text with $a = 1/2$ and $b = 1/2$. This case is given as a particular case of (5) on p. 973 at the bottom. From the stencil you see that the two points closer to P_{22} get a weight greater than 1, namely, 4/3, whereas the other two points (the ones that are at the usual distance h from the center P_{22} of the stencil) now each have the reduced weight 2/3 instead of 1; this is physically understandable. Accordingly, your fourth equation changes its usual form

$$u_{21} + u_{12} - 4u_{22} = \dots$$

to the form

$$(\mathrm{P}_{22}:) \qquad \frac{2}{3}u_{21} + \frac{2}{3}u_{12} - 4u_{22} = \frac{4}{3}(-2.5) + \frac{4}{3}(-1) = -4.666667.$$

If you had forgotten 4/3 on the right, you could have discovered it by checking whether the sum of the coefficients of all terms when taken to the left equals 0, that is,

$$\frac{2}{3} + \frac{2}{3} - 4 + \frac{4}{3} + \frac{4}{3} = 0.$$

Multiplication of the equation (P_{22}) by 3 gives the simpler form

$$(\mathrm{P}_{22}:) \qquad 2u_{21} + 2u_{12} - 12u_{22} = -14.$$

The augmented matrix of the linear system thus obtained is

$$\begin{bmatrix} -4 & 1 & 1 & 0 & -3 \\ 1 & -4 & 0 & 1 & -12 \\ 1 & 0 & -4 & 1 & 0 \\ 0 & 2 & 2 & -12 & -14 \end{bmatrix}.$$

By Gauss elimination you obtain the solution

$$u_{11} = 2, \qquad u_{21} = 4, \qquad u_{12} = 1, \qquad u_{22} = 2.$$

Take a look at the solution. Insert these four values in your sketch. Although you do not have too many valuės inside the region (just four), you can still obtain a qualitative picture of the equipotential lines (curves) in the region. The highest potential (9) is at the right lower corner. Now find $u = 8$ on the x-axis and on the vertical portion and draw the line $u = 8$ in the region; this curve looks like a quarter-circle. Do the same for $u = 7, 6, 5, 4$. For $u = 4$ you have the help that it must pass through P_{21}, that is, $(x, y) = (2, 1)$. Also, you can locate the endpoint of the curve on the sloping portion of the boundary. Next find $u = 3, 2, 1$ on the x-axis as well as on the sloping portion of the boundary. Draw the curves $u = 3, 2$ (passing through the points (1, 1) and (2, 2)), and $u = 1$ (passing through the point (1, 2)). This gives you a good qualitative picture of the potential as well as the impression that the values obtained by our calculations are reasonable approximate values of the potential at the four inner points.

Sec. 19.6 Methods for Parabolic Equations

Problem Set 19.6. Page 981

1. **Nondimensional form of the heat equation.** $\tilde{x}$ ranges from 0 to L. Hence $x = \tilde{x}/L$ ranges from 0 to 1. Now apply the chain rule, obtaining

$$\tilde{u}_{\tilde{t}} = u_t \frac{dt}{d\tilde{t}} = u_t \frac{c^2}{L^2}$$

and

$$\tilde{u}_{\tilde{x}\tilde{x}} = u_{xx}\left(\frac{dx}{d\tilde{x}}\right)^2 = u_{xx}\cdot\frac{1}{L^2}.$$

Now multiply the heat equation by L^2 and divide it by c^2.

3. **Explicit method for the heat equation.** $h = 1$ is the given step in x-direction. $k = 0.5$ is the given step in t-direction. Hence, to reach $t = 2$, you have to do 4 time steps. The initial temperature is $f(x) = x - 0.1x^2 = x(1 - 0.1x)$. It satisfies the conditions $f(0) = 0$ and $f(10) = 0$ at the ends of the bar. At the grid points $x = 0, 1, 2, ..., 10$ the initial temperature u is

$x = 0$	1	2	3	4	5	6	7	8	9	10
$u = 0$	0.9	1.6	2.1	2.4	2.5	2.4	2.1	1.6	0.9	0 .

In (5) you have $r = k/h^2 = 0.5$. Hence the first term of (5) on the right is 0, and (5) takes the form

$$u_{i,j+1} = 0.5(u_{i+1,j} + u_{i-1,j}). \tag{A}$$

i runs in the x-direction and j in the t-direction (the time direction). In each time row the first value and the last value are 0; this is the temperature at which the ends of the bar are kept at all times. From (A) you see that for obtaining the new value you have to take the arithmetic mean of two values in the preceding time row; one is one x-step to the left and the other one x-step to the right of the value which you want to calculate. Hence the simple computation looks as follows.

$x = 0$	0	1	2	3	4	5	6	7	8	9	10
$t = 0$	0	0.9	1.6	2.1	2.4	2.5	2.4	2.1	1.6	0.9	0
$t = 0.5$	0	0.8	1.5	2.0	2.3	2.4	2.3	2.0	1.5	0.8	0
$t = 1.0$	0	0.75	1.4	1.9	2.2	2.3	2.2	1.9	1.4	0.75	0
$t = 1.5$	0	0.7	1.325	1.8	2.1	2.2	2.1	1.8	1.325	0.7	0
$t = 2.0$	0	0.6625	1.25	1.7125	2.0	2.1	2.0	1.7125	1.25	0.6625	0

9. **Crank-Nicolson method.** Formula (9) was obtained by taking $r = k/h^2 = 1$. Since $h = 0.2$ is required, you must take

$$k = h^2 = 0.04.$$

This is the same value as in Example 1 in the text. Hence you must do 5 time steps to reach $t = 0.2$. The initial temperature is "triangular", $f(x) = x$ if $0 \leqq x \leqq 0.5$ and $f(x) = 1 - x$ if $0.5 \leqq x \leqq 1$. Since $h = 0.2$, you need the initial temperature at $x = 0, 0.2, 0.4, 0.6, 0.8, 1.0$. The values are

$$\begin{array}{lcccccc} i = & 0 & 1 & 2 & 3 & 4 & 5 \\ x = & 0 & 0.2 & 0.4 & 0.6 & 0.8 & 1.0 \\ u = & 0 & 0.2 & 0.4 & 0.4 & 0.2 & 0. \end{array} \tag{B}$$

Step 1 ($t = 0.04$). The use of (9) is required. For $j = 0$ this formula is

$$4u_{i,1} - u_{i+1,1} - u_{i-1,1} = u_{i+1,0} + u_{i-1,0}. \tag{C}$$

In each time row you have 6 values. Now 2 of them are 0 for all t. You have to determine the temperature at the remaining 4 inner points $x = 0.2, 0.4, 0.6, 0.8$, corresponding to $i = 1, 2, 3, 4$. For these values of i you obtain from (C) the system

$$\begin{array}{ll} (i = 1) & 4u_{1,1} - u_{2,1} - u_{0,1} = u_{2,0} + u_{0,0} \\ (i = 2) & 4u_{2,1} - u_{3,1} - u_{1,1} = u_{3,0} + u_{1,0} \\ (i = 3) & 4u_{3,1} - u_{4,1} - u_{2,1} = u_{4,0} + u_{2,0} \\ (i = 4) & 4u_{4,1} - u_{5,1} - u_{3,1} = u_{5,0} + u_{3,0}\,. \end{array}$$

(Perhaps you will find it helpful that we have retained the commas in the indices, although this would not have been absolutely necessary, as Example 1 in the book illustrates.) Since the initial temperature is symmetric with respect to $x = 0.5$ (and the temperature is 0 at both ends!), so is the temperature at the 4 inner points for all t. In formulas,

$$u_{3,j} = u_{2,j} \quad \text{and} \quad u_{4,j} = u_{1,j}. \tag{D}$$

If you insert (D) into your system of four equations and use that $u_{5,j} = u_{0,j} = 0$, you see that the third equation becomes identical with the second one, and the fourth one with the first one. Hence you can restrict yourself to the first and second equations. In these equations, $u_{0,1} = 0$, $u_{0,0} = 0$, and $u_{3,0} = u_{2,0}$ (see (B) or (C) with $j = 0$), so that these equations take the form

$$\begin{array}{ll} (i = 1) & 4u_{1,1} - u_{2,1} = u_{2,0} = 0.4 \\ (i = 2) & -u_{1,1} + 3u_{2,1} = u_{2,0} + u_{1,0} = 0.4 + 0.2 = 0.6, \end{array} \tag{E}$$

where $3u_{2,1}$ results from $4u_{2,1} - u_{3,1} = 4u_{2,1} - u_{2,1}$. The augmented matrix of this system is

$$\begin{bmatrix} 4 & -1 & 0.4 \\ -1 & 3 & 0.6 \end{bmatrix}.$$

By Gauss elimination you obtain the solution

$$u_{1,1} = 0.163636 = u_{4,1}, \qquad u_{2,1} = 0.254545 = u_{3,1}. \tag{F}$$

Step 2 ($t = 0.08$). The matrix of the system remains the same; only the right sides change. 0.4 was $u_{2,0}$. Hence you now have to take $u_{2,1}$ since you are now dealing with $j = 1$. The term 0.6 was the sum of $u_{2,0}$ and $u_{1,0}$. Hence you now have to take $u_{2,1} + u_{1,1}$ as the right side of the second equation. This gives the augmented matrix

$$\begin{bmatrix} 4 & -1 & 0.254545 \\ -1 & 3 & 0.418182 \end{bmatrix}.$$

The solution is

$$u_{1,2} = 0.107438 = u_{4,2}, \qquad u_{2,2} = 0.175207 = u_{3,2}.$$

Step 3 ($t = 0.12$). The augmented matrix is

$$\begin{bmatrix} 4 & -1 & 0.175207 \\ -1 & 3 & 0.282645 \end{bmatrix}.$$

The solution is

$$u_{1,3} = 0.0734786 = u_{4,3}, \qquad u_{2,3} = 0.118708 = u_{3,3}.$$

Step 4 ($t = 0.16$). The augmented matrix is

$$\begin{bmatrix} 4 & -1 & 0.118708 \\ -1 & 3 & 0.192186 \end{bmatrix}.$$

The solution is

$$u_{1,4} = 0.049846 = u_{4,4}, \qquad u_{2,4} = 0.080678 = u_{3,4}.$$

Step 5 ($t = 0.20$). The augmented matrix is

$$\begin{bmatrix} 4 & -1 & 0.080678 \\ -1 & 3 & 0.130524 \end{bmatrix}.$$

The solution is

$$u_{1,5} = 0.033869 = u_{4,5}, \qquad u_{2,5} = 0.054798 = u_{3,5}. \tag{G}$$

The series in Example 3 of Sec. 11.5 with $L = 1$ and $c = 1$ [the equation is assumed to be $u_t = u_{xx}$, hence $c = 1$; see (1)] is

$$u(x,t) = \frac{4}{\pi^2}\left(\sin \pi x\, e^{-\pi^2 t} - \frac{1}{9}\sin 3\pi x\, e^{-9\pi^2 t} + - \ldots\right).$$

Hence for $t = 0.2$ this becomes

$$u(x, 0.2) = \frac{4}{\pi^2}\left(\sin \pi x\, e^{-0.2\pi^2} - \frac{1}{9}\sin 3\pi x\, e^{-1.8\pi^2} + - \ldots\right). \tag{H}$$

The second term is already very small because π^2 is about 10 and e^{-18} is about 10^{-8}, and the exponential function in the next term would be e^{-49}, which is about 10^{-21}. Hence the values obtained from the sum of the first two terms of (G) are very accurate. The computation gives

	$x = 0.2$	$x = 0.4$
u in (G)	0.033869	0.054798
u in (H)	0.033091	0.053543
Error of u in G	−0.000778	−0.001255

Your calculation illustrates that, although our h is relatively large, the values obtained by the Crank-Nicolson method are rather accurate. Furthermore, you see that the series in Sec. 11.5 is very useful numerically.

Sec. 19.7 Methods for Hyperbolic Equations

Problem Set 19.7. Page 984

1. **Vibrating string problem (1)-(4).** From (4) you see that the ends of the string are at $x = 0$ and $x = 1$. The initial displacement is given by the parabola

$$f(x) = 0.1x(1 - x).$$

(The factor 0.1 has been included as a reminder that the wave equation (1) was derived under the assumptions that the string has a small displacement and makes small angles with the horizontal x-axis at all times.) The initial velocity of the string is assumed to be zero. Since $h = 0.2$, you need $f(x)$ at 0, 0.2, 0.4, 0.6, 0.8, 1.0, that is,

$$\begin{array}{lcccccc} x = & 0 & 0.2 & 0.4 & 0.6 & 0.8 & 1.0 \\ u(x,0) = f(x) = & 0 & 0.016 & 0.024 & 0.024 & 0.016 & 0\ . \end{array} \qquad \text{(A)}$$

From this and (8) with $g_i = 0$ (the initial velocity is zero!) and $k = 0.2$, as required in the problem, you obtain [also using (4)]

$$u(x,0.2) \quad 0 \quad 0.012 \quad 0.020 \quad 0.020 \quad 0.012 \quad 0\ . \qquad \text{(B)}$$

For the remaining calculations you have to use (6), whose right side in each step includes values from *two* preceding time rows. The numerical values obtained [including those in (A) and (B)] are as follows.

$x =$	0	0.2	0.4	0.6	0.8	1.0
$t = 0$	0	0.016	0.024	0.024	0.016	0
$t = 0.2$	0	0.012	0.020	0.020	0.012	0
$t = 0.4$	0	0.004	0.008	0.008	0.004	0
$t = 0.6$	0	−0.004	−0.008	−0.008	−0.004	0
$t = 0.8$	0	−0.012	−0.020	−0.020	−0.012	0
$t = 1.0$	0	−0.016	−0.024	−0.024	−0.016	0
$t = 1.2$	0	−0.012	−0.020	−0.020	−0.012	0
$t = 1.4$	0	−0.004	−0.008	−0.008	−0.004	0
$t = 1.6$	0	0.004	0.008	0.008	0.004	0
$t = 1.8$	0	0.012	0.020	0.020	0.012	0
$t = 2.0$	0	0.016	0.024	0.024	0.016	0

You see that these values correspond to one full cycle because the last line equals the first, so that for continuing t the string starts its next cycle. The reason can be seen from (11*) and (11) in Sec. 11.3 because for $c = 1$ and $L = 1$ you have $\lambda_n t = (cn\pi/L)t = n\pi t$ and for $t = 2$ this equals $2n\pi$, which is a period of the cosine and sine in (11).

7. Nonzero initial displacement and velocity. The initial displacement is

$$f(x) = 1 - \cos 2\pi x.$$

Since $h = 0.1$, you need its values at $x = 0, 0.1, ..., 1.0$. Now the curve of $f(x)$ is symmetric with respect to $x = 1/2$, as is clear by inspection; formally it is obtained from the addition formula for the cosine by calculating

$$\begin{aligned} \cos(2\pi(1-x)) &= \cos(2\pi - 2\pi x) = \cos 2\pi \cos 2\pi x + \sin 2\pi \sin 2\pi x \\ &= 1 \cdot \cos 2\pi x + 0. \end{aligned}$$

Hence you may calculate $f(0.1), ..., f(0.5)$ and then use $f(0.6) = f(0.4)$, etc. The values are (6D)

$$\begin{array}{lcccccc} x = & 0 & 0.1 & 0.2 & 0.3 & 0.4 & 0.5 \\ f(x) & 0 & 0.190983 & 0.690983 & 1.309017 & 1.809017 & 2.000000. \end{array}$$

The initial velocity is

$$g(x) = x - x^2 = x(1-x).$$

Its values for the same x will be needed in (8) to get started. $g(x)$ is also symmetric with respect to $x = 1/2$, so that it suffices to calculate $kg(x) = 0.1g(x)$ for $x = 0, 0.1, ..., 0.5$. We include $f(x)$ for convenience. $u(x, 0.1)$ is then calculated from (8) and $u(x, 0.2)$ from (6), with 10S and then rounded to 6D. Calculations of $u(x, 0.1)$ and $u(x, 0.2)$ are given after the table.

x	0	0.1	0.2	0.3	0.4	0.5
$f(x)$	0	0.190983	0.690983	1.309017	1.809017	2.000000
$0.1g(x)$	0	0.009	0.016	0.021	0.024	0.025
$u(x, 0.1)$	0	0.354492	0.766000	1.271000	1.678508	1.834017
$u(x, 0.2)$	0	0.575017	0.934509	1.135491	1.296000	1.357017

For $u(x, 0.1)$, formula (8) with $i = 1, 2, \ldots$ gives (we set again commas between the two indices)

$$u_{1,1} = \frac{1}{2}(u_{0,0} + u_{2,0}) + 0.1\,g_1 = \frac{1}{2}(0 + 0.690983) + 0.009 = 0.354492$$
$$u_{2,1} = \frac{1}{2}(u_{1,0} + u_{3,0}) + 0.1\,g_2 = \frac{1}{2}(0.190983 + 1.309017) + 0.016 = 0.766000$$
$$u_{3,1} = \frac{1}{2}(u_{2,0} + u_{4,0}) + 0.1\,g_3 = \frac{1}{2}(0.690983 + 1.809017) + 0.021 = 1.271000$$
$$u_{4,1} = \frac{1}{2}(u_{3,0} + u_{5,0}) + 0.1\,g_4 = \frac{1}{2}(1.309017 + 2.000000) + 0.024 = 1.678508$$
$$u_{5,1} = \frac{1}{2}(u_{4,0} + u_{6,0}) + 0.1\,g_5 = \frac{1}{2}(1.809017 + 1.809017) + 0.025 = 1.834017.$$

For the next time row ($t = 0.2$) you have to use (6), obtaining

$$u_{1,2} = u_{0,1} + u_{2,1} - u_{1,0} = 0 + 0.766000 - 0.190983 = 0.575017$$
$$u_{2,2} = u_{1,1} + u_{3,1} - u_{2,0} = 0.354492 + 1.271000 - 0.690983 = 0.934509$$
$$u_{3,2} = u_{2,1} + u_{4,1} - u_{3,0} = 0.766000 + 1.678508 - 1.309017 = 1.135491$$
$$u_{4,2} = u_{3,1} + u_{5,1} - u_{4,0} = 1.271000 + 1.834017 - 1.809017 = 1.296000$$
$$u_{5,2} = u_{4,1} + u_{6,1} - u_{5,0} = 1.678508 + 1.678508 - 2.000000 = 1.357016.$$