CHAPTER 2. Linear Differential Equations of Second and Higher Order

Sec. 2.1 Homogeneous Linear Equations of Second Order

Problem Set 2.1. Page 71

7. **Reduction to first order.** $y'' + e^y y'^3 = 0$ is of the form $F(y, y', y'') = 0$, so that you can set $z = y'$ and $y'' = (dz/dy)\,z$ (see Prob. 2). Substitution of this and division by z gives $dz/dy + e^y z^2 = 0$. By separation of variables, $dz/z^2 = -e^y\,dy$. Integration on both sides and multiplication by -1 gives $1/z = e^y + c_1$. Now by calculus, $z = dy/dx$ implies $dx/dy = 1/z$. Hence you can separate again and then integrate,

$$dx = (e^y + c_1)\,dy$$

$$x = e^y + c_1 y + c_2.$$

13. **Motion.** Expressing the given data in formulas gives $y'y'' = 1$, $y(0) = 2$, $y'(0) = 2$. By integration, $y'^2/2 = t + C$, hence $y' = \sqrt{2t + c_1}$, where $c_1 = 2C$. If you wish, you can now use the second initial condition to get $y'(0) = \sqrt{c_1} = 2$, hence $c_1 = 4$, so that $y' = \sqrt{2t+4}$. By another integration and the use of the first initial condition you obtain

$$y = \frac{1}{3}(2t+4)^{3/2} + c_2, \quad y(0) = \frac{1}{3}4^{3/2} + c_2 = \frac{8}{3} + c_2 = 2, \quad c_2 = -\frac{2}{3}.$$

This gives the answer

$$y = \frac{1}{3}(2t+4)^{3/2} - \frac{2}{3}.$$

Sec. 2.2 Second-Order Homogeneous Equations with Constant Coefficients

Problem Set 2.2. Page 75

7. **General solution.** Problems 1-9 amount to solving a quadratic equation (3), the characteristic equation. Observe that the solutions (4) refer to the case that y'' has the coefficient 1. For the present equation you can write $y'' - (30/9)\,y' + (25/9)\,y = 0$. Then the radicand in (4) is $225/81 - 25/9 = 0$, so that you have a double root $15/9 = 5/3$. The corresponding general solution is $y = (c_1 + c_2 x)\exp(5x/3)$.

15. **Initial value problem.** To solve an initial value problem, first determine a general solution by solving the characteristic equation $\lambda^2 + 2.2\lambda + 1.17 = 0$. The roots (4) are -1.3 and -0.9. The corresponding general solution is

$$y = c_1 e^{-1.3x} + c_2 e^{-0.9x}. \tag{a}$$

Because of the second initial condition you also need the derivative

$$y' = -1.3c_1 e^{-1.3x} - 0.9c_2 e^{-0.9x}. \tag{b}$$

In (a) and (b) you now put $x = 0$ and equate the result to 2 and -2.6, respectively (the given initial values), that is,

$$c_1 + c_2 = 2, \quad -1.3c_1 - 0.9c_2 = -2.6.$$

The solution is $c_1 = 2$, $c_2 = 0$, so that you get the answer $y = 2e^{-1.3x}$. Note that, in general, both solutions of a basis of solutions would appear; in that sense our present initial conditions are special.

21. **Linear independence and dependence.** This problem is typical of cases where one must use functional relations to prove linear dependence. Namely, $\ln x$ and $\ln(x^4) = 4\ln x$ are linearly dependent on any

interval of the positive semi-axis. Graphs may help when the functions are very complicated and transformations are not so obvious as in this problem; then you may find out whether the curves of the functions look "proportional".

Sec. 2.3 Case of Complex Roots. Complex Exponential Function

Problem Set 2.3. Page 80

5. **General solution.** $y'' + 1.6\,y' + 0.64\,y = 0$ (the given equation divided by 2.5) has the characteristic equation $\lambda^2 + 1.6\,\lambda + 0.64 = (\lambda + 0.8)^2 = 0$ with the double root -0.8. This is Case II, with the general solution as given in Appendix 2.

7. **General solution.** Division by 16 gives $y'' - 0.5y' + 0.3125y = 0$. From (3) you thus obtain the roots
$$\lambda_1 = 0.25 + 0.5\sqrt{0.25 - 1.25} = 0.25 + 0.5\,i \quad \text{and} \quad \lambda_2 = 0.25 - 0.5\,i.$$
Note that if an equation (with real coefficients) has a complex root, the conjugate of the root must also be a root. The real part is 0.25 and gives the exponential function exp $(0.25x)$. The imaginary parts are 0.5 and -0.5 and give the cosine and sine terms. Together,
$$y = e^{0.25x}(A \cos 0.5x + B \sin 0.5x),$$
which is oscillating with an increasing maximum amplitude.

21. **Boundary value problems** will be less important to us than initial value problems. The determination of a particular solution by using given boundary conditions is similar to that for an initial value problem. In the present problem the characteristic equation is $\lambda^2 + 2\,\lambda + 2 = 0$. Its roots are
$$\lambda_1 = -1 + \sqrt{1-2} = -1 + i \text{ and } \lambda_2 = -1 - i.$$
This gives the real general solution
$$y = e^{-x}(A \cos x + B \sin x).$$
On the left boundary, $y(0) = A = 1$. On the right boundary, $y(\pi/2) = B \exp(-\pi/2) = 0$, hence $B = 0$. Hence the answer is $y = e^{-x} \cos x$.

Sec. 2.4 Differential Operators. *Optional*

Problem Set 2.4. Page 83

3. **Differential operators.** $(D-2)(D+1)e^{2x} = 0$ because
$$(D-2)e^{2x} = 2e^{2x} - 2e^{2x} = 0.$$
For the second of the four given functions you first have
$$(D-2)xe^{2x} = e^{2x} + 2xe^{2x} - 2xe^{2x} = e^{2x}$$
and then
$$(D+1)e^{2x} = 2e^{2x} + e^{2x} = 3e^{2x}.$$
Similarly for the other functions.

13. **General solution.** The optional Sec. 2.4 introduces to the operator notation and shows how it can be applied to linear differential equations with constant coefficients. The facts considered are essentially as before, merely the notation changes. The given equation, divided by 10, is
$$(D^2 + 1.2D + 0.36)y = (D + 0.6)^2 y = 0.$$
It shows that the characteristic equation has the double root -0.6, so that the corresponding general solution is

$$y = (c_1 + c_2 x)e^{-0.6x}.$$

Sec. 2.5 Modeling: Free Oscillations (Mass-Spring Systems)

Problem Set 2.5. Page 90

1. **Harmonic oscillations.** Formula (4*) gives a better impression than a sum of cosine and sine terms because the maximum amplitude C and phase shift δ readily characterize the harmonic oscillation. The result follows by direct calculation, starting from the general solution

$$y = A \cos \omega_0 t + B \sin \omega_0 t$$

and using the initial conditions, first $y(0) = A = y_0$ and then

$$y' = \text{a sine term} + \omega_0 B \cos \omega_0 t, \qquad y'(0) = \omega_0 B = v_0,$$

where v suggests 'velocity'. This gives the particular solution

$$y = y_0 \cos \omega_0 t + (v_0/\omega_0) \sin \omega_0 t.$$

Accordingly, in (4*),

$$C = \sqrt{y_0{}^2 + \left(\frac{v_0}{\omega_0}\right)^2}, \qquad \tan \delta = \frac{v_0/\omega_0}{y_0}.$$

The derivation of (4*) suggested in the text begins with

$$\begin{aligned} y(t) &= C \cos (\omega_0 t - \delta) = C(\cos \omega_0 t \cos \delta + \sin \omega_0 t \sin \delta) \\ &= C \cos \delta \cos \omega_0 t + C \sin \delta \sin \omega_0 t = A \cos \omega_0 t + B \sin \omega_0 t. \end{aligned}$$

By comparing you see that

$$A^2 + B^2 = C^2 \cos^2 \delta + C^2 \sin^2 \delta = C^2$$

and

$$\tan \delta = \frac{\sin \delta}{\cos \delta} = \frac{C \sin \delta}{C \cos \delta} = \frac{B}{A}.$$

7. **Determination of frequencies.** $\omega_0 = \sqrt{k/m}$; see (4). Hence the frequencies are

$$\frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k_1}{m}} \quad \text{and} \quad \frac{1}{2\pi}\sqrt{\frac{k_2}{m}},$$

respectively. To prove $k = k_1 + k_2$, fix $s = s_0$ (for instance, $s_0 = 1$), choose $W_1 = k_1 s_0$ and $W_2 = k_2 s_0$, and add (couple the two systems), where k is the spring constant of the two systems

$$W = W_1 + W_2 = (k_1 + k_2)s_0 = ks_0$$

combined.

15. **Underdamping.** Equate the derivative to zero.

Sec. 2.6 Euler-Cauchy Equation

Problem Set 2.6. Page 96

3. **General solution.** Problems 2-13 are solved as explained in the text by determining the roots of the auxiliary equation (3). This is similar to the method for constant-coefficient equations in Secs. 2.2 and 2.3, but note well that the linear term in (3) is $(a - 1)\,m$, not am. Thus in Prob. 3 you have

$$m(m-1) - 20 = m^2 - m - 20 = 0.$$

The roots are -4 and 5. Hence a general solution is $y = c_1 x^{-4} + c_2 x^5$. The value $x = 0$ is excluded.

Similarly, the case of a double root of (3) gives a logarithmic term [see (7) in Sec. 2.6] and $x = 0$ and all negative x must be excluded.

7. **Pure imaginary roots.** The auxiliary equation is $m^2 + 1 = 0$. It has the roots $i = \sqrt{-1}$ and $-i$. Hence in (8) of Sec, 2.6 you have $\mu = 0$ (the real part of the roots is zero) and $\nu = 1$, so that (8) becomes simply $y = A\cos(\ln x) + B\sin(\ln x)$.

15. **Initial value problems** for Euler-Cauchy equations are solved as for constant-coefficient equations by first determining a general solution. The initial values must not be given at 0, where the coefficients of (1), written in standard form

$$y'' + \frac{a}{x}y' + \frac{b}{x^2}y = 0,$$

become infinite, but must refer to some other point, for instance, to $x = 1$. In Prob. 15 the auxiliary equation is

$$4m(m-1) + 24m + 25 = 0 \quad \text{or} \quad m^2 + 5m + 6.25 = 0.$$

It has the double root -2.5. The corresponding general solution (7). Sec. 2.6, is

$$y = (c_1 + c_2 \ln x)x^{-2.5}.$$

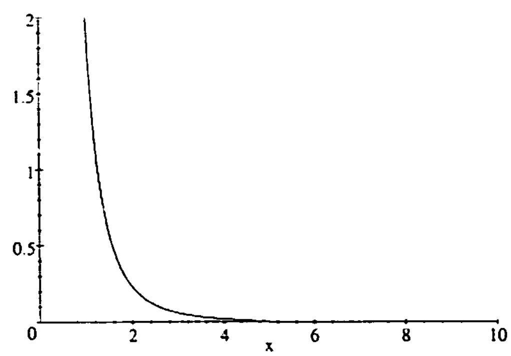
The first initial condition gives $y(1) = c_1 = 2$. For the second initial condition $y'(1) = -6$ you need the derivative. With $c_1 = 2$ the latter is

$$y' = \frac{c_2}{x}x^{-2.5} - 2.5(2 + c_2 \ln x)x^{-3.5}.$$

Setting $x = 1$, you thus obtain (since $\ln 1 = 0$)

$$y'(1) = c_2 - 5 = -6, \qquad \text{hence} \quad c_2 = -1.$$

The figure shows the particular solution obtained, $y = (2 - \ln x)x^{-2.5}$. For $x > 7.4$ the logarithm is greater than 2, so that for these x the solution becomes negative, but this can hardly be seen from the figure because the x-factor is very small in absolute value when x is large.



Section 2.6. Problem 15. Particular solution satisfying $y(1) = 2, y'(1) = -6$

Sec. 2.7 Existence and Uniqueness Theory. Wronskian

The **Wronskian** $W(y_1, y_2)$ of two solutions y_1 and y_2 of a differential equation is defined by (5), Sec. 2.7. It is conveniently written as a second-order determinant (but this is not essential for using it; you need not be familiar with determinants here). It serves for checking linear independence or dependence, which is important in obtaining bases of solutions. The latter are needed, for instance, in connection with initial value problems, where a single solution will generally not be sufficient for satisfying two given initial conditions. Of course, two functions are linearly independent if and only if their quotient is not constant. To check this, you would not need Wronskians, but we discuss them here in the simple case of second-order differential equations as a preparation for Secs. 2.13-2.15 on higher order equations, where Wronskians will show their power and are extremely useful.

Problem Set 2.7. Page 100

3. **Basis, Wronskian.** For $a > 0$ these solutions

$$y_1 = e^{-ax/2} \cos 3x \quad \text{and} \quad y_2 = e^{-ax/2} \sin 3x$$

represent damped vibrations, x being time. Their Wronskian is obtained by straightforward differentiation or by the following trick. From the quotient rule and (5), Sec. 2.7, it follows that

$$W = (y_2/y_1)' y_1^2, \tag{A}$$

where the prime denotes the derivative. In the present problem, $y_2/y_1 = \tan 3x$ has the derivative $3/\cos^2 3x$ (chain rule!). Furthermore, $y_1^2 = e^{-ax} \cos^2 3x$. The product of the two expressions is the Wronskian $W = 3e^{-ax}$.

5. **Wronskian.** Formula (A) in Prob. 3 gives $(x^4(\ln x)/x^4)'x^8 = (\ln x)'x^8 = x^7$.

7. **Wronskian.** Formula (A) in Prob. 3 contains

$$(\tan(2\ln x))' = [1/\cos^2(2\ln x)]\cdot\frac{2}{x},$$

the last factor resulting from the chain rule. Now $y_1^2 = x^{2\mu}\cos^2(2\ln x)$, and the product is $W = 2x^{2\mu-1}$.

11. **Equation for a given basis.** Problems 9-17 survey the most important types of equations discussed so far. The form in Prob. 11 suggests an Euler-Cauchy equation with a double root (because of the logarithmic term). Now

$$(m-2)^2 = m(m-1) - 3m + 4 \quad \text{shows that} \quad x^2y'' - 3xy' + 4y = 0.$$

From (5) you obtain the Wronskian

$$W = x^2(2x\ln x + x) - (x^2 \ln x)2x = x^3.$$

Check this by (A) in Prob. 3, obtaining $(\ln x)'x^4 = x^3$.

Sec. 2.8 Nonhomogeneous Equations

Verification of solutions proceeds for nonhomogeneous equations as it does for homogeneous equations, namely, by the calculation of y' and y'' and substitution of y, y', and y''. It is interesting that in Probs. 1-8 most solutions to some extent resemble the form of the functions on the right side of the equation. This observation gives the idea of a method for determining particular solutions to be discussed in the next section.

Problem Set 2.8. Page 103

7. **General solution.** $y_p = \ln \pi x = \ln x + \ln \pi$ has the derivatives $1/x$ and $-1/x^2$. Substitution gives $y'' + y = -1/x^2 + \ln \pi x$. A general solution of the homogeneous equation is $A\cos x + B\sin x$. Hence the answer (a general solution of the nonhomogeneous equation) is $y = A\cos x + B\sin x + \ln \pi x$.

11. **Initial value problem.** To solve an initial value problem, you must first determine a general solution of the nonhomogeneous equation (because if you first determine a particular solution of the homogeneous equation satisfying the initial conditions, the addition of a solution y_p will generally change the value of the entire solution and its derivative at the point at which the initial conditions are given). Now a general solution of the homogeneous equation $y'' - y = 0$ is $c_1 e^x + c_2 e^{-x}$. The particular solution $y_p = xe^x$ may

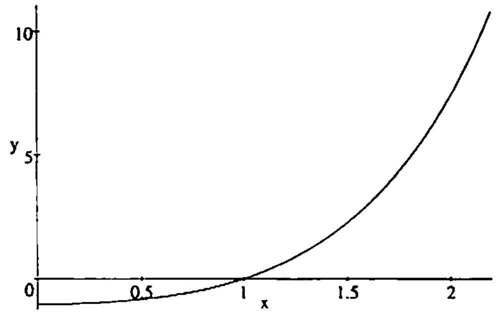
come as a surprise because $2e^x$ on the right might have suggested $y_p = ke^x$ with a suitable constant k, but if you substitute this, you get $0 = 2e^x$. Choosing $y_p = xe^x$ gives $y_p' = (x+1)e^x$ and $y_p'' = (x+2)e^x$, so that substitution yields $(x+2)e^x - xe^x = 2e^x$ and verifies that y_p is indeed a solution. Hence a general solution of the given equation is

$$y = c_1 e^x + c_2 e^{-x} + xe^x.$$

From the first initial condition, $y(0) = c_1 + c_2 = -1$. From the derivative and the second initial condition,

$$y' = c_1 e^x - c_2 e^{-x} + (x+1)e^x, \quad y'(0) = c_1 - c_2 + 1 = 0.$$

The solution of this system of two equations is $c_1 = -1$, $c_2 = 0$. This gives the answer $y = -e^x + xe^x = (x-1)e^x$ shown in the figure.



Section 2.8. Problem 11. Solution of the initial value problem

Sec. 2.9 Solution by Undetermined Coefficients

New in this section and problem set is the determination of a particular solution y_p by the method of undetermined coefficients. Because of the Modification Rule it is necessary to first determine a general solution of the homogeneous equation since the form of y_p differs depending on whether or not the function (or a term of it) on the right side of the differential equation is a solution of the homogeneous equation. If you forget to take this into account, you will not be able to determine the coefficients; in this sense the method will warn you that you made a mistake.

Problem Set 2.9. Page 107

1. **General solution.** A general solution of the homogeneous equation (1) $y'' + 4y = 0$ is $y_h = A\cos 2x + B\sin 2x$. The function $\sin 3x$ on the right is not a solution of (1). Hence the Modification Rule does not apply. Table 2.1 requires that you start from $y_p = K\cos 3x + M\sin 3x$. Two differentiations give $y_p'' = -9K\cos 3x - 9M\sin 3x$. Substituting this and y_p into the given equation yields

$$-9K\cos 3x - 9M\sin 3x + 4(K\cos 3x + M\sin 3x) = \sin 3x.$$

Since there is no cosine term on the right, this implies $-9K + 4K = 0$, hence $K = 0$. For the sine terms, $-9M + 4M = 1$, hence $M = -0.2$. This gives the answer $y = A\cos 2x + B\sin 2x - 0.2\sin 3x$.

11. **Modification rule.** The characteristic equation of the homogeneous equation is $\lambda^2 + 10\lambda + 25 = (\lambda+5)^2 = 0$. Hence it has the double root -5, so that a general solution of the homogeneous equation is $y_h = (c_1 + c_2 x)e^{-5x}$. This shows that e^{-5x} is a solution of the homogeneous equation. Hence you must apply the Modification Rule. More precisely, since you are dealing with a *double root*, you must multiply the usual choice e^{-5x} by x^2, (In the case of a simple root you would have to multiply by x.) Accordingly, choose $y_p = kx^2e^{-5x}$. By differentiation,

$$y_p' = k(2x - 5x^2)e^{-5x}, \qquad y_p'' = k(2 - 10x - 10x + 25x^2)e^{-5x}.$$

Substitution of these expressions into the differential equation $y'' + 10y' + 25y = e^{-5x}$ and omission of the common factor e^{-5x} on both sides of the equation gives

$$k(2 - 20x + 25x^2) + 10k(2x - 5x^2) + 25kx^2 = 1.$$

In this equation, x^2 has the coefficient $25k + 10k(-5) + 25k = 0$. Similarly, x has the coefficient $-20k + 10k \cdot 2 = 0$. Finally, the constant terms give $2k = 1, k = 0.5$. Hence the answer (a general solution of the given nonhomogeneous equation) is

$$y = (c_1 + c_2 x)e^{-5x} + 0.5x^2 e^{-5x}.$$

17. **Initial value problem.** A general solution of the homogeneous equation $y'' - 4y = 0$ is $y_h = c_1 e^{2x} + c_2 e^{-2x}$. The right side $e^{-2x} - 2x$ has two terms. The first is a solution of the homogeneous equation, the corresponding root of the characteristic equation being simple. Hence the Modification rule calls for kxe^{-2x} instead of the usual ke^{-2x}. By Table 2.1 in Sec. 2.9 the second term $-2x$ calls for the choice $ax + b$ (line 2 of the table, with a more convenient notation). Together, $y_p = kxe^{-2x} + ax + b$. By differentiation,

$$y_p' = k(1 - 2x)e^{-2x} + a, \qquad y_p'' = k(-2 - 2 + 4x)e^{-2x}.$$

Substitution into the nonhomogeneous equation gives

$$k(-4 + 4x)e^{-2x} - 4kxe^{-2x} - 4(ax + b) = e^{-2x} - 2x.$$

The terms in xe^{-2x} drop out. The e^{-2x}-terms give $-4k = 1, k = -1/4$. The x-terms give $-4a = -2, a = 1/2$. The constant terms give $b = 0$. Hence a general solution of the given equation is

$$y = c_1 e^{2x} + c_2 e^{-2x} + 0.5x - 0.25xe^{-2x}.$$

$y(0) = 0$ gives $y(0) = c_1 + c_2 = 0$. By differentiation of y,

$$y' = 2c_1 e^{2x} - 2c_2 e^{-2x} + 0.5 - 0.25(1 - 2x)e^{-2x}.$$

$y'(0) = 0$ thus gives $y'(0) = 2c_1 - 2c_2 + 0.5 - 0.25 = 0$. The solution of these two equations is $c_1 = -1/16, c_2 = 1/16$. Hence the answer is

$$y = -\frac{1}{16}(e^{2x} - e^{-2x}) + \frac{1}{2}x - \frac{1}{4}xe^{-2x}.$$

The exponential terms combine into $-(\sinh 2x)/8$, as given in Appendix 2.

Sec. 2.10 Solution by Variation of Parameters

Problem Set 2.10. Page 111

1. **General solution.** The right side e^{2x}/x does not permit the method of undetermined coefficients (which would be simpler than the present method). The homogeneous equation $y'' - 4y' + 4y = 0$ has the characteristic equation $\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0$. It has the double root 2. Hence a basis of solutions is $y_1 = e^{2x}$ and $y_2 = xe^{2x}$. Now determine a particular solution of the given equation by (2), Sec. 2.10. In (2) you need the Wronskian

$$W = e^{2x}(xe^{2x})' - xe^{2x}(e^{2x})' = e^{4x}(1 + 2x) - e^{4x}2x = e^{4x}.$$

The integrands of the integrals in (2) are

$$xe^{2x}(e^{2x}/x)/e^{4x} = 1 \quad \text{and} \quad e^{2x}(e^{2x}/x)/e^{4x} = 1/x.$$

Integration gives x and $\ln |x|$, respectively. From (2) you thus obtain the particular solution

$$y_p = -xe^{2x} + xe^{2x}\ln|x|.$$

Hence the corresponding general solution of the given nonhomogeneous equation is

$$y = (c_1 + c_2 x)e^{2x} + (-x + x\ln|x|)e^{2x}.$$

3. General solution. This equation can also be solved by undetermined coefficients, starting from $y_p = e^{-x}(K \cos x + M \sin x)$. Try it.

11. Euler-Cauchy equation. The homogeneous equation $x^2y'' - 4xy' + 6y = 0$ has the auxiliary equation $m^2 - 5m + 6 = 0$. The roots are $m_1 = 2$ and $m_2 = 3$. Hence a general solution of the homogeneous equation is $y_h = c_1x^2 + c_2x^3$. Try to find a particular solution of the nonhomogeneous equation by undetermined coefficients, setting $y_p = Cx^{-4}$. Then $y_p' = -4Cx^{-5}$, $y_p'' = 20Cx^{-6}$, and substitution into the given equation yields

$$20Cx^{-4} + 16Cx^{-4} + 6Cx^{-4} = 21x^{-4}, \qquad 42C = 21, \qquad C = 1/2.$$

Hence a general solution of the given nonhomogeneous equation is

$$y = c_1x^2 + c_2x^3 + \frac{1}{2}x^{-4}.$$

15. Euler-Cauchy equation. Determine y_p by (2) in Sec. 2.10. It is quite important that you first write the given equation in standard form

$$y'' - 2y'/x + 2y/x^2 = x \cos x. \qquad \text{Hence } r = x \cos x \qquad (\text{not } x^3 \cos x!).$$

The auxiliary equation of the homogeneous differential equation is $m^2 - 3m + 2 = 0$ and has the roots $m_1 = 1$, $m_2 = 2$. This gives the basis of solutions $y_1 = x$, $y_2 = x^2$. In (2) you need the Wronskian $W = x(2x) - 1 \cdot x^2 = x^2$. Hence the first integral in (2) has the integrand $x^2(x \cos x)/x^2 = x \cos x$. Integration by parts gives $x \sin x$ minus the integral of $\sin x$, which is $+\cos x$. Together, $x \sin x + \cos x$. The second integral in (2) has the integrand $x(x \cos x)/x^2 = \cos x$. Integration gives $\sin x$. From this and (2) you obtain

$$y_p = -x(x \sin x + \cos x) + x^2 \sin x = -x \cos x.$$

The answer (a general solution of the nonhomogeneous equation) is

$$y = c_1x + c_2x^2 - x \cos x.$$

Sec. 2.11 Modeling: Forced Oscillations. Resonance

In the solution a, b of (4) (the formula after (4) without number) the denominator is the coefficient determinant. The numerator of a is the determinant

$$\begin{vmatrix} F_0 & \omega c \\ 0 & k - m\omega^2 \end{vmatrix} = F_0(k - m\omega^2).$$

Similarly for b.

Problem Set 2.11. Page 117

Problems 1-17 involve driving forces such that the method of undetermined coefficients (Sec. 2.9) can be applied.

3. Steady-state solution. Because of the function $\sin 0.2t$ on the right you have to choose $y_p = K \cos 0.2t + M \sin 0.2t$. By differentiation,

$$y_p' = -0.2K \sin 0.2t + 0.2M \cos 0.2t,$$

$$y_p'' = -0.04K \cos 0.2t - 0.04M \sin 0.2t.$$

Substitute this into the equation $y'' + 2y' + 4y = \sin 0.2t$. To get a simple formula, use the abbreviations $C = \cos 0.2t$ and $S = \sin 0.2t$. Then

$$-0.04KC - 0.04MS + 2(-0.2KS + 0.2MC) + 4(KC + MS) = S.$$

Now collect the C-terms and the S-terms on the left and equate their sums to 0 (there is no C-term on the right) and 1, respectively,

$$-0.04K + 0.4M + 4K = 3.96K + 0.4M = 0$$

$$-0.04M - 0.4K + 4M = -0.4K + 3.96M = 1.$$

Elimination or Cramer's rule (Sec. 6.6) gives the solution $K = -0.02525, M = 0.2500$ (more exactly, $K = -0.025249975$, $M = 0.249974750$). Hence the steady-state solution is

$$y = -0.02525 \cos 0.2t + 0.2500 \sin 0.2t.$$

15. Initial value problem. Divide by 4 to have the standard form $y'' + 2y' + 0.75y = 106.25 \sin 2t$. (This is convenient, although not absolutely necessary.). The characteristic equation of the homogeneous equation is $\lambda^2 + 2\lambda + 0.75 = 0$. The roots are $-1/2$ and $-3/2$. Hence a general solution of the homogeneous equation is $y_h = c_1 e^{-0.5t} + c_2 e^{-1.5t}$. Now determine a particular solution y_p. The right side calls for the choice $y_p = K \cos 2t + M \sin 2t$. Both terms will be needed because the equation has a damping term, which causes a phase shift (in contrast to Prob. 13, where there is no damping and y_p is a sine term). By differentiation,

$$y_p' = -2K \sin 2t + 2M \cos 2t, \qquad y_p'' = -4K \cos 2t - 4M \sin 2t.$$

Using the abbreviations $C = \cos 2t$, $S = \sin 2t$ and substituting y_p and its derivatives into the given equation yields

$$-4KC - 4MS + 2(-2KS + 2MC) + 0.75(KC + MS) = 106.25S.$$

The sum of the cosine terms on the left must equal 0 since there is no cosine term on the right. Similarly, the sum of the sine terms on the left must equal 106.25. This gives the linear system of two equations

$$-4K + 4M + 0.75K = -3.25K + 4.00M = 0$$

$$-4M - 4K + 0.75M = -4.00K - 3.25M = 106.25.$$

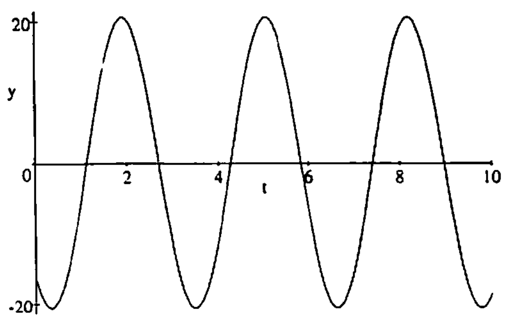
By elimination or by Cramer's rule (Sec. 6.6) you obtain the solution $K = -16$, $M = -13$. This gives the general solution

$$y(t) = c_1 e^{-0.5t} + c_2 e^{-1.5t} - 16 \cos 2t - 13 \sin 2t.$$

From this and the first initial condition, $y(0) = c_1 + c_2 - 16 = -16$. The derivative is

$$y'(t) = -0.5c_1 e^{-0.5t} - 1.5c_2 e^{-1.5t} + 32 \sin 2t - 26 \cos 2t,$$

and the second initial condition gives $y'(0) = -0.5c_1 - 1.5c_2 - 26 = -26$. By inspection or by elimination, $c_1 = 0$, $c_2 = 0$. This gives the answer $y = -16 \cos 2t - 13 \sin 2t$. This is a harmonic oscillation of period π and maximum amplitude $\sqrt{16^2 + 13^2} = 20.62$ (see the figure). It is interesting that because of the initial conditions the solution of the homogeneous equation does not contribute to the answer, so that there is no transition period.



Section 2.11. Problem 15. Solution without transition period

Sec. 2.12 Modeling of Electric Circuits

Example 1. The linear system of equations near the end of the example consists of (7) and an unnumbered equation, namely,

$$c_1 + \quad c_2 = 0.484 \qquad \text{(from } I(0) = 0)$$

$$-10c_1 - 990c_2 = -1.380 \cdot 377 = -520.26 \qquad \text{(from } I'(0) = 0).$$

It can be solved by elimination, namely, $c_2 = 0.484 - c_1$, hence

$$-10c_1 - 990(0.484 - c_1) = -520.26$$

and from this,

$$c_1 = \frac{1}{980}(479.16 - 520.26) = -0.041939,$$

so that $c_2 = 0.484000 + 0.041939 = 0.525939$.

Problem Set 2.12. Page 122

7. **Transient current.** In the model (1) the right side is the derivative of the elctromotive force $E = 25\cos 100t$, that is, $E' = -2500 \sin 100t$. Hence (1), divided by $L = 0.5$, is

$$I'' + 80I' + 1500I = -5000 \sin 100t. \tag{a}$$

The characteristic equation $\lambda^2 + 80\lambda + 1500 = 0$ has the roots -30 and -50. Hence a general solution of the homogeneous equation is

$$I_h = c_1 e^{-30t} + c_2 e^{-50t}.$$

This solution approaches zero as t goes to infinity, regardless of initial conditions. A particular solution I_p of the nonhomogeneous equation is obtained by substituting $I_p = K\cos 100t + M \sin 100t$ and its derivatives

$$I_p' = -100K \sin 100t + 100M \cos 100t,$$

$$I_p'' = -10000K \cos 100t - 10000M \sin 100t$$

into (a). Writing $C = \cos 100t$, $S = \sin 100t$, you obtain

$$-1000KC - 10000MS + 80(-100KS + 100MC) + 1500(KC + MS) = -5000S.$$

The sum of the cosine coefficients must be zero since there is no cosine term on the right side of (a). Similarly, the sum of the sine coefficients must equal -5000. This gives a system for determining K and M, namely,

$$-10000K + 8000M + 1500K = -8500K + 8000M = 0$$

$$-10000M - 8000K + 1500M = -8000K - 8500M = -5000.$$

Solving the first equation for M gives $M = 1.0625K$. Substituting this into the second equation, you find

$$-8000K - 8500M = -17031.25K = -5000, \qquad K = 0.293578.$$

From this, $M = 1.0625K = 0.311927$. Hence the answer is

$$I = c_1 e^{-30t} + c_2 e^{-50t} + 0.293578 \cos 100t + 0.311927 \sin 100t.$$

The exponential terms go to zero and the steady-state solution is a harmonic oscillation whose frequency equals that of the electromotive force. (The decimal fractions are approximations of the exact coefficients 32/109 and 34/109 given in the answer in Appendix 2.)

15. ***LC*-circuit.** Differentiating $E = 220 \sin 4t$ gives $E' = 880 \cos 4t$. Hence the model of the circuit is $2I'' + 200I = 880 \cos 4t$. Division by 2 gives

$$I'' + 100I = 440 \cos 4t.$$

A general solution of the homogeneous equation is $I_h = A\cos 10t + B\sin 10t$. You can find a particular solution of the form $I_p = K\cos 4t$. It is not necessary to add a term $M\sin 4t$ because there is no term in I'

(physically: no damping, no phase shift). Substitution gives $K(-16 + 100)\cos 4t = 440\cos 4t$. Hence $K = 110/21 = 5.238$. Consequently, a general solution of the model is

$$I(t) = A \cos 10t + B \sin 10t + 5.238 \cos 4t. \qquad \text{(b)}$$

This is a superposition of two harmonic oscillations.

Now use the initial conditions. For the first condition this is simple:

$$I(0) = A + 5.238 = 0, \qquad \text{hence} \qquad A = -5.238.$$

The second initial condition is $Q(0) = 0$, meaning that at $t = 0$ the capacitor is uncharged. To use this condition, proceed as in Example 1 on p. 121. From (1′) on p. 119 with $R = 0$ (an LC-circuit has $R = 0$!) and $\int I\,dt = Q$ you have

$$LI' + Q/C = 220 \sin 4t.$$

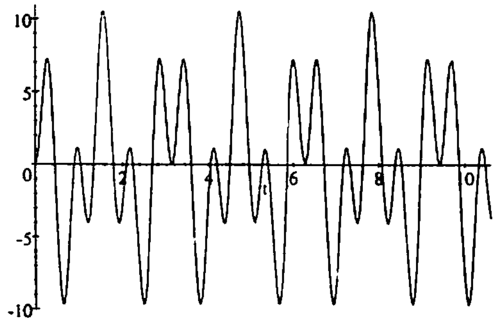
For $t = 0$ and $Q(0) = 0$ this gives $LI'(0) + 0 = 0$, hence $I'(0) = 0$. Now by differentiating (b) you obtain two sine terms, which are 0 when $t = 0$, and the cosine term $10B \cos 10t$, which equals $10B$ when $t = 0$. From this and $I'(0) = 0$ you have $B = 0$. You thus obtain the answer

$$I = 5.238(\cos 4t - \cos 10t).$$

Note that the term in $\cos 4t$ appears regardless of the initial conditions, whereas the other term is present because of these conditions. The figure shows that the oscillation is periodic (what is the shortest period?) and looks rather complicated. Additional insight into its character is obtained from (12) in Appendix A3.1, which gives

$$\cos 4t - \cos 10t = 2 \sin 7t \sin 3t.$$

This is similar to Fig. 59 in Sec. 2.11, but less distinct because $3t$ is not small enough compared to $7t$.



Section 2.12. Problem 15. Superposition of two harmonic oscillations in an LC-circuit

Sec. 2.13 Higher Order Linear Differential Equations

Example 2. The determinant of this homogeneous system is not zero (add Row 1 to Row 2, then develop by Row 2); hence the system has only the trivial solution (all unknowns zero). Or, calling the equations (a), (b), (c), you obtain $k_2 = k_1 + k_3$ from (a), then $k_2 = 0$ from this and (b), then $k_3 = -k_1$ from (a), then $-6k_1 = 0$ from (c), then $k_3 = 0$ from (b).

Problem Set 2.13. Page 131

3. **Wronskian. Initial value problem.** Calculate the Wronskian W. Since $f = e^{-3x}$ has the derivatives $f' = -3f$ and $f'' = 9f$, you obtain a factor f in each column, so that you can factor out $f^3 = e^{-9x}$ from the determinant, the remaining determinant being

$$\begin{vmatrix} 1 & x & x^2 \\ -3 & 1-3x & 2x-3x^2 \\ 9 & -6+9x & 2-12x+9x^2 \end{vmatrix}.$$

To simplify this, add 3 times Row 1 to Row 2 and subtract 9 times Row 1 from Row 3. The result is a determinant that can readily be developed by the first column, giving the value 2, that is,

$$\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & -6 & 2-12x \end{vmatrix} = 1\,(2-12x)+12x = 2.$$

Hence the Wronskian is $W = 2e^{-9x}$. The corresponding differential equation is obtained by first noting that because of the form of these solutions the characteristic equation must have the triple root -3, that is, it must be of the form (use the binomial formula)

$$(\lambda+3)^3 = \lambda^3+9\lambda^2+27\lambda+27 = 0.$$

Hence these functions are solutions of the differential equation

$$y''' + 9y'' + 27y' + 27y = 0,$$

as claimed, and they are linearly independent because their Wronskian is not zero. Now write down the corresponding general solution and its first and second derivatives for general x as well as for $x = 0$, and use the initial conditions to determine the three arbitrary constants. This looks as follows.

$$\begin{aligned} y(x) &= (c_1 + c_2x + c_3x^2)e^{-3x}, \\ y(0) &= c_1 = 4 \\ y'(x) &= (c_2 + 2c_3x - 3c_1 - 3c_2x - 3c_3x^2)e^{-3x} \\ y'(0) &= c_2 - 3\cdot 4 = -13, \qquad c_2 = -1 \\ y''(x) &= (2c_3 - 3c_2 - 3(c_2 - 3c_1) + \textit{further terms})e^{-3x} \\ y''(0) &= 2c_3 - 3(-1) - 3(-1-3\cdot 4) = 2c_3 + 42 = 46, \quad c_3 = 2 \end{aligned}$$

where "*further terms*" are those that give zero when $x = 0$, so you do not need to write them down. This gives the answer

$$y = (4 - x + 2x^2)e^{-3x}.$$

11. Linear dependence. Use $\cos^2 x + \sin^2 x = 1$.

15. Linear dependence. Consider the difference of the first two functions.

Sec. 2.14 Higher Order Homogeneous Equations with Constant Coefficients

Example 1. In the Wronskian, pull out a factor e^{-x} from the first column, e^{x} from the second, and e^{2x} from the third. Hence the Wronskian equals e^{2x} times the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \\ 1 & 1 & 4 \end{vmatrix}.$$

The latter is not zero (subtract Column 1 from Column 2 and develop by Column 2, to get $2(4-1) = 6$).

Problem Set 2.14. Page 137

3. **General solution.** Use that the characteristic equation is a quadratic equation in λ^2. Thus,

$$\lambda^4 - 2\lambda^2 + 1 = (\lambda^2 - 1)^2 = ((\lambda + 1)(\lambda - 1))^2 = (\lambda + 1)^2(\lambda - 1)^2 = 0.$$

In some of the other problems in this set it may be necessary to overcome the practical difficulty of determining the roots by using a numerical method, such as Newton's method (Sec. 17.2) (although we have chosen values such that one root, λ_1, may often be found by inspection and the remaining roots then by dividing the characteristic equation by $\lambda - \lambda_1$).

13. **Initial value problem.** $\lambda_1 = 1$ by inspection. Dividing by $\lambda - 1$ now gives

$$(\lambda^3 - \lambda^2 - \lambda + 1) \div (\lambda - 1) = \lambda^2 - 1 = (\lambda + 1)(\lambda - 1).$$

From this and the given initial values you get the corresponding general solution, its derivatives, the values at zero, and conditions (a)-(c) for the arbitrary constants in the general solution, as follows.

$$y(x) = (c_1 + c_2 x)e^x + c_3 e^{-x}$$

$$y(0) = c_1 + c_3 = 2 \quad \text{(from the first initial condition)} \qquad \text{(a)}$$

$$y'(x) = (c_2 + c_1 + c_2 x)e^x - c_3 e^{-x}$$

$$y'(0) = c_2 + c_1 - c_3 = 1 \quad \text{(from the second)} \qquad \text{(b)}$$

$$y''(x) = (c_2 + c_2 + c_1 + c_2 x)e^x + c_3 e^{-x}$$

$$y''(0) = c_1 + 2c_2 + c_3 = 0 \quad \text{(from the third).} \qquad \text{(c)}$$

Write the system (a)-(c) more orderly,

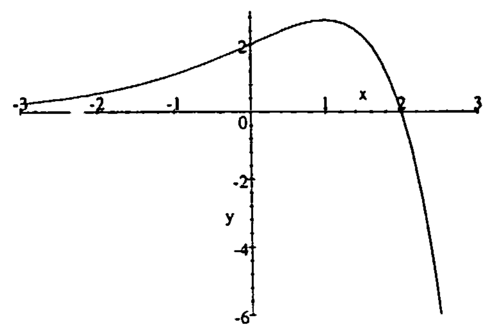
$$c_1 \qquad + c_3 = 2 \qquad \text{(a)}$$

$$c_1 + \quad c_2 - c_3 = 1 \qquad \text{(b)}$$

$$c_1 + 2c_2 + c_3 = 0. \qquad \text{(c)}$$

To solve this, apply the Gauss elimination or Cramer's rule or simply form (c) minus (a) to get $2c_2 = -2$, hence $c_2 = -1$, then form (a) plus (b), obtaining $2c_1 + c_2 = 2c_1 - 1 = 3$, $c_1 = 2$, and finally use (a), obtaining $c_3 = 2 - c_1 = 0$. Together, this gives the answer $y = (2 - x)e^x$.

The figure shows that y has a maximum at $x = 1$; this can be confirmed by using the derivative. From the change of the tangent direction with increasing x conclude that for positive x the second derivative must always be negative. Indeed, the previous formula for $y''(x)$ with the constants as just determined shows that $y'' = -xe^x$.



Section 2.14. Problem 13. Solution of the initial value problem

19. CAS Project. (c) Without a computer, the equation can be solved as follows. The auxiliary equation is

$$m(m-1)(m-2)+m(m-1)-2m+2 = 0.$$

The sum of the last two terms is $-2(m-1)$. Hence you now have a common factor $m-1$ and can write the auxiliary equation as

$$(m-1)(m(m-2)+m-2) = (m-1)(m^2-m-2) = (m-1)(m-2)(m+1).$$

The corresponding general solution is $y = c_1 x + c_2 x^2 + c_3/x$.

Sec. 2.15 Higher Order Nonhomogeneous Equations

Problem Set 2.15. Page 141

1. General solution. The characteristic equation of the homogeneous equation is

$$\lambda^3+3\lambda^2+3\lambda+1 = (\lambda+1)^3.$$

It has the triple root -1, so that a basis of solutions is

$$y_1 = e^{-x}, \quad y_2 = xe^{-x}, \quad y_3 = x^2 e^{-x}$$

and the corresponding general solution of the homogeneous equation is

$$y_h = (c_1+c_2 x+c_3 x^2)e^{-x}.$$

The right side is such that you can use the method of undetermined coefficients. The Modification Rule is not needed since none of the terms on the right is a solution of the homogeneous equation. Start from (see Table 2.1 in Sec, 2.9 if necessary)

$$y_p = Ce^x + K_1 x + K_0.$$

Substitution of this and the derivatives

$$y_p' = Ce^x + K_1, \quad y_p'' = Ce^x, \quad y_p''' = Ce^x$$

into the given equation

$$y''' + 3y'' + 3y' + y = 8e^x + x + 3$$

gives

$$C(1+3+3+1)e^x + 3K_1 + K_1 x + K_0 = 8e^x + x + 3.$$

From this you see that $C = 1$, $K_1 = 1$, $3K_1 + K_0 = 3$, $K_0 = 0$, and the answer is

$$y = y_h + y_p = (c_1+c_2 x+c_3 x^2)e^{-x} + e^x + x.$$

11. Initial value problem. The auxiliary equation of this Euler-Cauchy equation is

$$m(m-1)(m-2)-3m(m-1)+6m-6 = 0. \qquad \text{(A)}$$

Ordering terms gives

$$m^3-6m^2+11m-6 = 0. \qquad \text{(B)}$$

$m = 1$ is a root. This can be seen from (B) by inspection or from (A) by noting that $6m-6 = 6(m-1)$, so that (A) has a common factor $m-1$. Division of (B) by $m-1$ gives

$$(m^3-6m^2+11m-6) \div (m-1) = m^2-5m+6 = (m-2)(m-3).$$

Hence 2 and 3 are roots, and a general solution of the homogeneous equation is

$$y_h = c_1 x + c_2 x^2 + c_3 x^3.$$

Now determine a particular solution of the nonhomogeneous equation. Try the method of undetermined coefficients, setting

$$y_p = Kx^5. \quad \text{Then } y_p' = 5Kx^4, \ y_p'' = 20Kx^3, \ y_p''' = 60Kx^2.$$

Differentiation has reduced the exponent, but this will be compensated by the increasing power in successive coefficients, so that you obtain a common factor x^5, namely, since the equation is

$$x^3y''' - 3x^2y'' + 6xy' - 6y = 24x^5,$$

you obtain

$$(60 - 3 \cdot 20 + 6 \cdot 5 - 6)Kx^5 = 24Kx^5 = 24x^5, \qquad \text{hence} \quad K = 1.$$

This gives the general solution of the nonhomogeneous equation, its derivatives, their values at $x = 1$, and three equations (a), (b), (c) for determining the arbitrary constants by using the initial conditions, as follows.

$$y(x) = c_1x + c_2x^2 + c_3x^3 + x^5$$

$$y(1) = c_1 + c_2 + c_3 + 1 = 1, \qquad c_1 + c_2 + c_3 = 0 \qquad \text{(a)}$$

$$y'(x) = c_1 + 2c_2x + 3c_3x^2 + 5x^4$$

$$y'(1) = c_1 + 2c_2 + 3c_3 + 5 = 3, \qquad c_1 + 2c_2 + 3c_3 = -2 \qquad \text{(b)}$$

$$y''(x) = 2c_2 + 6c_3x + 20x^3$$

$$y''(1) = 2c_2 + 6c_3 + 20 = 14, \qquad 2c_2 + 6c_3 = -6\ . \qquad \text{(c)}$$

You can solve (a), (b), (c) by elimination. (b) minus (a) gives

$$c_2 + 2c_3 = -2. \qquad \text{(d)}$$

(c) minus 2(d) gives $2c_3 = -2$, hence $c_3 = -1$. From this and (d) there follows $c_2 = -2 - 2c_3 = 0$. From this and (a) you finally have $c_1 = -c_2 - c_3 = 1$. This gives the answer $y = x - x^3 + x^5$.