CHAPTER 3. Systems of Differential Equations, Phase Plane, Qualitative Methods

Sec. 3.1 Introductory Examples

Example 2. Spend time on Fig. 76 until you feel that you fully understand the difference between (b) (the usual representation in calculus) and (c), because trajectories will play an important role throughout this chapter. Try to understand the reasons for the following. The trajectory starts at the origin. It reaches its highest point where y_2 has a maximum (before $t = 1$). It has a vertical tangent where I_1 has a maximum, short after $t = 1$. As t increases from there to $t = 5$, the trajectory goes downward until it almost reaches the I_1-axis at 3; this point is a limit as $t \to \infty$. In terms of t the trajectory goes up faster than it comes down.

Problem Set 3.1. Page 158

5. **Electrical network.** The problem amounts to the determination of the two arbitrary constants in a general solution of a system of two differential equations in two unknown functions I_1 and I_2, representing the currents in an electrical network shown in Fig. 76 in Sec. 3.1. You will see that this is quite similar to the corresponding task for a single second-order differential equation. That solution is given by (6), in components

$$I_1(t) = 2c_1e^{-2t} + c_2e^{-0.8t} + 3, \qquad I_2(t) = c_1e^{-2t} + 0.8c_2e^{-0.8t}.$$

Setting $t = 0$ and using the given initial conditions $I_1(0) = 9$, $I_2(0) = 0$ gives

$$I_1(0) = 2c_1 + c_2 + 3 = 9 \qquad \text{(a)}$$

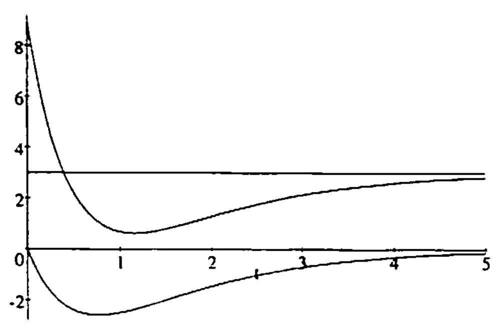
$$I_2(0) = c_1 + 0.8c_2 = 0. \qquad \text{(b)}$$

From (b) you have $c_1 = -0.8c_2$. Substituting this into (a) and simplifying gives $2(-0.8c_2) + c_2 = 6$, hence $-0.6c_2 = 6$ or $c_2 = -10$, and $c_1 = 8$. The answer is (note that $2c_1 = 16$)

$$I_1(t) = 16e^{-2t} - 10e^{-0.8t} + 3$$

$$I_2(t) = 8e^{-2t} - 8e^{-0.8t}.$$

These currents are shown in the figure. $I_1(t)$ has the limit 3, as expected. $I_2(t)$ comes out negative; this means it is directed opposite to the arrows shown in Fig. 76(a), which had been assumed arbitrarily at the beginning of the process of modeling; this had to be done because at the beginning, one does not know in what directions the currents will actually flow.



Section 3.1. Problem 5. Currents I_1 (upper curve) and its limit 3 and I_2

9. **Conversion of single differential equations to a system** is an important process, which always follows the pattern shown in formulas (9) and (10) of Sec. 3.1. The present equation $y'' - 9y = 0$ can be readily

solved. A general solution is $y = c_1 e^{3t} + c_2 e^{-3t}$. The point of the problem is not to explain a (complicated) solution method for a simple problem, but to explain the relation between systems and single equations and their solutions. In the present case the formulas (9) and (10) give $y_1 = y, y_2 = y'$ and

$$y_1' = y_2$$
$$y_2' = 9y_1$$

(because the given equation can be written $y'' = 9y$, hence $y_1'' = y_1$, but $y_1'' = y_2'$). In matrix form (as in Example 3 of the text) this is

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix} \mathbf{y}.$$

The characteristic equation is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ 9 & -\lambda \end{vmatrix} = \lambda^2 - 9 = 0.$$

The eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -3$. For λ_1 you obtain an eigenvector from (13) in Sec. 3.0 with $\lambda = \lambda_1$, that is,

$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{x} = \begin{bmatrix} -3 & 1 \\ 9 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3x_1 + x_2 \\ 9x_1 - 3x_2 \end{bmatrix} = \mathbf{0}.$$

From the first equation $-3x_1 + x_2 = 0$ you have $x_2 = 3x_1$. An eigenvector is determined only up to a nonzero constant. Hence, in the present case, a convenient choice is $x_1 = 1, x_2 = 3$. The second equation gives the same result and is not needed. For the second eigenvalue, $\lambda_2 = -3$, the procedure is the same, namely,

$$(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{x} = \begin{bmatrix} 3 & 1 \\ 9 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 + x_2 \\ 9x_1 + 3x_2 \end{bmatrix} = \mathbf{0}.$$

You now have $3x_1 + x_2 = 0$, hence $x_2 = -3x_1$, and can choose $x_1 = 1, x_2 = -3$. The eigenvectors obtained are

$$\mathbf{x}^{(1)} = [1 \quad 3]^T \quad \text{and} \quad \mathbf{x}^{(2)} = [1 \quad -3]^T.$$

Multiplying these by e^{3t} and e^{-3t}, respectively, and taking a linear combination involving two arbitrary constants c_1 and c_2 gives a general solution of the present system in the form

$$\mathbf{y} = c_1[1 \quad 3]^T e^{3t} + c_2[1 \quad -3]^T e^{-3t}.$$

In components, this is

$$y_1 = c_1 e^{3t} + c_2 e^{-3t}$$
$$y_2 = 3c_1 e^{3t} - 3c_2 e^{-3t}.$$

Here you see that $y_1 = y$ is a general solution of the given equation, and $y_2 = y_1' = y'$ is the derivative of this solution, as had to be expected because of the definition of y_2 at the beginning of the process. Incidentally, you can use $y_2 = y_1'$ for checking your result.

Sec. 3.3 Homogeneous Systems with Constant Coefficients. Phase Plane, Critical Points

Example 2. The characteristic equation is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = (\lambda - 1)^2 = 0.$$

Thus $\lambda = 1$ is an eigenvalue. Any nonzero vector with two components is an eigenvector because $\mathbf{A}\mathbf{x} = \mathbf{x}$ for any $\mathbf{x}$; indeed, $\mathbf{A}$ is the 2×2 unit matrix! Hence you can take $\mathbf{x}^{(1)} = [1 \quad 0]^T$ and $\mathbf{x}^{(2)} = [0 \quad 1]^T$ or any other two linearly independent vectors with two components. This gives the solution on p. 165.

Example 3. $(1-\lambda)(-1-\lambda) = (\lambda-1)(\lambda+1) = 0$, and so on.

Problem Set 3.3. Page 169

3. **General solution.** The matrix of the system is

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}.$$

The characteristic equation is

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) - 1\cdot 3 = \lambda^2 - 4 = 0.$$

Hence the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -2$. An eigenvector corresponding to λ_1 is obtained by solving $(\mathbf{A}-\lambda_1\mathbf{I})\mathbf{x} = (\mathbf{A}-2\mathbf{I})\mathbf{x} = \mathbf{0}$, in components,

$$\begin{aligned} (1-2)x_1 + \quad\quad x_2 &= 0 \\ 3x_1 + (-1-2)x_2 &= 0. \end{aligned}$$

Each of the equations gives $x_1 = x_2$ (and you need only one of them). Hence you can take $\mathbf{x}^{(1)} = [1 \quad 1]^T$ as an eigenvector corresponding to $\lambda_1 = 2$. For $\lambda_2 = -2$ those component equations are $(1+2)x_1 + x_2 = 0$ (both of the same form) and you can take $x_1 = 1, x_2 = -3$, so that an eigenvector corresponding to $\lambda_2 = -2$ is $\mathbf{x}^{(2)} = [1 \quad -3]^T$. This gives as a general solution of the system

$$\mathbf{y} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{2t} + c_2\begin{bmatrix} 1 \\ -3 \end{bmatrix}e^{-2t},$$

in components,

$$\begin{aligned} y_1 &= c_1e^{2t} + c_2e^{-2t} \\ y_2 &= c_1e^{2t} - 3c_2e^{-2t}. \end{aligned}$$

This agrees with the answer in Appendix 2, with c_1 and c_2 interchanged. (Of course, the notation for arbitrary constants is up to us.)

5. **General solution.** In this problem you will see the typical calculations in the case of complex eigenvalues. The matrix of the system is

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

This gives the characteristic equation

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 2 = 0.$$

The eigenvalues are complex conjugates, $\lambda_1 = 1+i$ and $\lambda_2 = 1-i$. Eigenvectors $\mathbf{x}$ are obtained from $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, as before. This is a vector equation, and you need only the first of the two corresponding scalar equations. For $\lambda = \lambda_1 = 1+i$ it is

$$(1-(1+i))x_1 - x_2 = 0,$$

thus

$$-ix_1 = x_2; \qquad \text{say,} \qquad x_1 = 1 \quad \text{and} \quad x_2 = -i.$$

So the new aspect is that this eigenvector $\mathbf{x}^{(1)} = [1 \quad -i]^T$ is no longer real but is complex. Similarly, for $\lambda = \lambda_2 = 1-i$ you get an eigenvector from

$$(1 - (1 - i))x_1 - x_2 = 0,$$

thus

$$ix_1 = x_2; \quad \text{say,} \quad x_1 = 1 \quad \text{and} \quad x_2 = i.$$

This gives the eigenvector $\mathbf{x}^{(2)} = [1 \quad i]^T$. Using the Euler formula

$$e^{it} = \cos t + i \sin t, \qquad e^{-it} = \cos t - i \sin t$$

(see Sec. 2.3) you can write a complex general solution

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^t(\cos t + i \sin t) + c_2 \begin{bmatrix} 1 \\ i \end{bmatrix} e^t(\cos t - i \sin t).$$

Writing this in terms of components and collecting cosine and sine terms, you obtain

$$\begin{aligned} y_1 &= c_1 e^t(\cos t + i \sin t) + c_2 e^t(\cos t - i \sin t) \\ &= e^t(A \cos t + B \sin t), \qquad A = c_1 + c_2, \quad B = ic_1 - ic_2 \\ y_2 &= -ic_1 e^t(\cos t + i \sin t) + ic_2 e^t(\cos t - i \sin t) \\ &= e^t(C \cos t + D \sin t), \qquad C = -ic_1 + ic_2, \quad D = c_1 + c_2. \end{aligned}$$

You see that $C = -B$ and $D = A$. You can now write a real general solution in vector form, namely,

$$\mathbf{y} = e^t\left(\begin{bmatrix} A \\ -B \end{bmatrix} \cos t + \begin{bmatrix} B \\ A \end{bmatrix} \sin t\right).$$

15. Initial value problem. The matrix of the given system is

$$\mathbf{A} = \begin{bmatrix} -14 & 10 \\ -5 & 1 \end{bmatrix}.$$

From $\mathbf{A}$ you obtain the characteristic equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -14 - \lambda & 10 \\ -5 & 1 - \lambda \end{vmatrix} = (-14 - \lambda)(1 - \lambda) + 50 = 0.$$

Simplification gives $\lambda^2 + 13\lambda + 36 = 0$. Hence the eigenvalues are $\lambda_1 = -9$ and $\lambda_2 = -4$. Corresponding eigenvectors are obtained from

$$[-14 - (-9)]x_1 + 10x_2 = 0, \quad \text{thus} \quad -5x_1 + 10x_2 = 0, \quad \text{say,} \quad x_1 = 2, \quad x_2 = 1$$

and

$$[-14 - (-4)]x_1 + 10x_2 = 0, \quad \text{thus} \quad -10x_1 + 10x_2 = 0, \quad \text{say,} \quad x_1 = 1, \quad x_2 = 1.$$

Hence the corresponding general solution is

$$\mathbf{y} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-9t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-4t}.$$

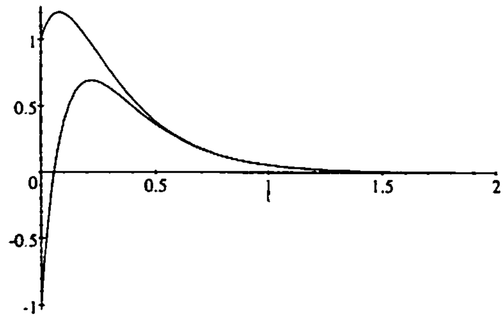
From this and the initial conditions $y_1(0) = -1$, $y_2(0) = 1$, written in vector form, you obtain

$$\mathbf{y}(0) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \text{thus} \quad \begin{aligned} 2c_1 + c_2 &= -1 \\ c_1 + c_2 &= 1 \end{aligned}.$$

The solution is $c_1 = -2$, $c_2 = 3$, so that you obtain the particular solution

$$\mathbf{y} = -2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-9t} + 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-4t}, \quad \text{thus} \quad \begin{aligned} y_1 &= -4e^{-9t} + 3e^{-4t} \\ y_2 &= -2e^{-9t} + 3e^{-4t} \end{aligned}.$$

The figure shows that both y_1 (the lower curve) and y_2 have a maximum and then approach zero in a monotone fashion.



Section 3.3. Problem 15. Particular solutions y_1 (lower curve) and y_2

Sec. 3.4 Criteria for Critical Points. Stability

Problem Set 3.4. Page 174

3. **Saddle.** The type of a critical point is determined by quantities closely related to the eigenvalues of the matrix of a system, namely, the trace p, which is the sum of the eigenvalues, the determinant q, which is the product of the eigenvalues, and the discriminant Δ, which equals $p^2 - 4q$; see (9) in Sec. 3.4. In Prob. 3 the matrix is

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

Hence $p = 1 + 1 = 2$, $q = 1 - 4 = -3$, and $\Delta = 4 - 4\cdot(-3) = 16$. Since $q < 0$, the system has a saddle point at 0, which is always unstable, as follows from (10c) in Sec. 3.4, and is plausible from Fig. 80 in Sec. 3.3. To solve the system, you need the eigenvalues, which you obtain as solutions of the characteristic equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0.$$

Hence the eigenvalues are -1 and 3. Their signs differ, which makes q negative and causes a saddle point. An eigenvector $\mathbf{x}^{(1)}$ for -1 is obtained from

$$(1 - (-1))x_1 + 2x_2 = 0, \quad \text{thus} \quad x_2 = -x_1, \quad \text{say,} \quad x_1 = 1, \quad x_2 = -1.$$

Hence $\mathbf{x}^{(1)} = [1 \quad -1]^T$. Similarly, for the eigenvalue 3 you obtain an eigenvector from

$$(1 - 3)x_1 + 2x_2 = 0, \quad \text{thus} \quad x_2 = x_1, \quad \text{say,} \quad x_1 = 1, \quad x_2 = 1.$$

This gives the eigenvector $\mathbf{x}^{(2)} = [1 \quad 1]^T$. Hence a general solution is

$$y = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}.$$

17. **Perturbation of center.** Example 4 in Sec. 3.3, to which the problem refers, shows two methods of solution, a systematic method and the shortcut. The first of them is similar to the procedure explained in this Manual in Prob. 5 of Problem Set 3.3 and can be completed following that method. The purpose of this Prob. 17 is to become aware of the fact that inaccuracies in the coefficients of a system (errors caused by rounding or in the process of physical measurements, etc.) can change the type of a critical point. In this problem it is suggested to go from $\mathbf{A}$ to $\mathbf{B} = \mathbf{A} + 0.1\mathbf{I}$, but it will be obvious from the analysis that smaller deviations would have a similar effect. Given

$$A = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}, \qquad \text{hence} \qquad B = A + 0.1I = \begin{bmatrix} 0.1 & 1 \\ -4 & 0.1 \end{bmatrix}.$$

For **A** you have $p = 0 + 0 = 0$, $q = -1\cdot(-4) = 4$. This confirms that the system in Example 4 has a center as its critical point; see (9c) in Sec. 3.4. For **B** you have $p = 0.1 + 0.1 = 0.2 \neq 0$, $q = 0.01 + 4 = 4.01$, and $\Delta = p^2 - 4q = 0.04 - 16.04 = -16 < 0$, which gives a spiral point by (9d) in Sec. 3.4. The eigenvalues of **A** are pure imaginary, $2i$ and $-2i$ (see Example 4 in Sec. 3.3), and it is interesting that the eigenvalues of **B** are $0.1 + 2i$ and $0.1 - 2i$, that is, they were changed by the same amount by which the main diagonal entries were changed (this reflects a general "shifting property"). Indeed, the characteristic equation of **B** is

$$\det(B - \lambda I) = \begin{vmatrix} 0.1 - \lambda & 1 \\ -4 & 0.1 - \lambda \end{vmatrix} = (0.1 - \lambda)^2 + 4 = \lambda^2 - 0.2\lambda + 4.01 = 0.$$

The roots (the eigenvalues of **B**) are $0.1 + 2i$ and $0.1 - 2i$.

Sec. 3.5 Qualitative Methods for Nonlinear Systems

Example 1. The critical point at (0, 0) turns out to be a center. This follows from the general criteria in Sec. 3.4. This is the first result. The next result follows from this and the periodicity of $\sin\theta = \sin y_1$ with 2π. Namely, the points $\pm 2\pi, \pm 4\pi, \ldots$ must also be centers. (Keep in mind that y_1 is just another notation for θ, introduced to fit the notation of our general discussions in this chapter.) The third result concerns the critical point $(\pi, 0)$ at $\theta = \pi$ of the θ-axis. The trick now is to move the origin to this point because our criteria were derived under the assumption that the critical point to be discussed is at the origin. This is the idea of the transformation (a translation)

$$\theta - \pi = y_1, \qquad \text{thus} \qquad \theta = \pi + y_1. \tag{A}$$

You see that $\theta = \pi$ now corresponds to our new $y_1 = 0$; we are at the new origin. Think about this before going on. From (A), $\sin\pi = 0$, and $\cos\pi = -1$ it follows that

$$\sin\theta = \sin(\pi + y_1) = \sin\pi\cos y_1 + \cos\pi\sin y_1 = -\sin y_1 = -y_1 + \frac{y_1^{\,3}}{6} - + \ldots,$$

as indicated in the example.

Problem Set 3.5. Page 183

5. **Linearization** begins with the determination of the positions of the critical points. As a system the given equation $y'' - y + y^2 = 0$ becomes (see Sec. 3.1 for the general formula)

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_1 - y_1^2. \end{aligned} \tag{a}$$

The critical points are obtained from $y_2 = 0$ (then $y_1' = 0$), $y_1 - y_1^2 = y_1(1 - y_1) = 0$ (then $y_2' = 0$). Hence they are at 0 and 1 on the y_1-axis. Linearization is then done for each critical point separately. and in each case the point is first shifted to the origin by a suitable change of coordinates, as explained in somewhat more detail just above (in connection with Example 1). Accordingly, begin with the critical point (0, 0). No transformation of y_1 or y_2 is necessary because the point already has the required position. From (a) you obtain the system linearized at the origin simply by dropping the quadratic term. This linearized system is

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_1. \end{aligned}$$

Its matrix is

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Calculate $p = 0 + 0 = 0$, $q = -1 \cdot 1 = -1$, and $\Delta = p^2 - 4q = 4$. Since $q < 0$, this is a saddle by (9b) in Sec. 3.4. (You do not need p and Δ.) The second critical point is at $y_1 = 1$, $y_2 = 0$. Hence make a shift by setting $y_1 = 1 + \tilde{y}_1$, $y_2 = \tilde{y}_2$. Then $y_1' = \tilde{y}_1'$,

$$y_1 - y_1{}^2 = y_1(1 - y_1) = (1 + \tilde{y}_1)(-\tilde{y}_1) = -\tilde{y}_1 - \tilde{y}_1^2$$

and (a) takes the form

$$\tilde{y}_1' = \tilde{y}_2$$
$$\tilde{y}_2' = -\tilde{y}_1 - \tilde{y}_1^2.$$

Linearize this by dropping the nonlinear term (the last term in the second differential equation). This gives the linearized system

$$\begin{aligned} \tilde{y}_1' &= \tilde{y}_2 \\ \tilde{y}_2' &= -\tilde{y}_1 \end{aligned} \qquad \text{whose matrix is} \qquad \mathbf{A}_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Calculate $p = 0$, $q = 1$, and conclude from (9c) in Sec. 3.4 that the critical point at $(1, 0)$ is a center.

13. **Trajectories.** $yy'' + y'^2 = (yy')' = 0$. By integration, $yy' = const$ or $y_1 y_2 = const$. These are the familiar hyperbolas with the coordinate axes as asymptotes.

Sec. 3.6 Nonhomogeneous Linear Systems

Example 1. The solution of the homogeneous system (not shown in the text) proceeds as before. That is, the characteristic equation of the matrix **A** is

$$(2 - \lambda)(-3 - \lambda) - (-4) \cdot 1 = \lambda^2 + \lambda - 2 = (\lambda - 1)(\lambda + 2) = 0.$$

Hence the eigenvalues are 1 and –2. Eigenvectors are obtained for $\lambda = 1$ from

$$(2 - 1)x_1 - 4x_2 = 0, \qquad \text{say,} \qquad x_1 = 4, \qquad x_2 = 1$$

and for $\lambda = -2$ from

$$(2 - (-2))x_1 - 4x_2 = 0, \qquad \text{say,} \qquad x_1 = 1, \qquad x_2 = 1.$$

This gives the solution of the homogeneous equation shown in the answer on p. 185.

Problem Set 3.6. Page 189

3. **General solution.** e^{3t} and $-3e^{3t}$ are such that you can apply the method of undetermined coefficients for determining a particular solution of the nonhomogeneous system. For this purpose you must first determine a general solution of the homogeneous system. The matrix of the latter is

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

It has the characteristic equation $\lambda^2 - 1 = 0$. Hence the eigenvalues of **A** are $\lambda_1 = -1$ and $\lambda_2 = 1$. Eigenvectors $\mathbf{x} = \mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are obtained from $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ with $\lambda = \lambda_1 = -1$ and $\lambda = \lambda_2 = 1$, respectively. For $\lambda_1 = -1$ you obtain

$$x_1 + x_2 = 0, \qquad \text{thus} \qquad x_2 = -x_1, \qquad \text{say,} \qquad x_1 = 1, \qquad x_2 = -1.$$

Similarly, for $\lambda_2 = 1$ you obtain

$$-x_1 + x_2 = 0, \quad \text{thus} \quad x_2 = x_1, \quad \text{say,} \quad x_1 = 1, \quad x_2 = 1.$$

Hence eigenvectors are $\mathbf{x}^{(1)} = [1 \quad -1]^T$ and $\mathbf{x}^{(2)} = [1 \quad 1]^T$. This gives the general solution of the homogeneous system

$$\mathbf{y}^{(h)} = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t}.$$

Now determine a particular solution of the nonhomogeneous system. Using the notation in the text (Sec. 3.6) you have on the right $\mathbf{g} = [1 \quad -3]^T e^{3t}$. This suggests the choice

$$\mathbf{y}^{(p)} = \mathbf{u}e^{3t} = [u_1 \quad u_2]^T e^{3t}. \tag{a}$$

Here $\mathbf{u}$ is a constant vector to be determined. The Modification Rule is not needed because 3 is not an eigenvalue of $\mathbf{A}$. Substitution of (a) into the given system $\mathbf{y}' = \mathbf{Ay} + \mathbf{g}$ yields

$$\mathbf{y}^{(p)\prime} = 3\mathbf{u}e^{3t} = \mathbf{Ay}^{(p)} + \mathbf{g} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} e^{3t} + \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{3t}.$$

Omitting the common factor e^{3t}, you obtain in terms of components

$$3u_1 = u_2 + 1 \qquad \text{ordered} \qquad 3u_1 - u_2 = 1$$

$$3u_2 = u_1 - 3 \qquad\qquad\qquad -u_1 + 3u_2 = -3.$$

Solution by elimination or by Cramer's rule (Sec. 6.6) gives $u_1 = 0$ and $u_2 = -1$. Hence the answer is

$$\mathbf{y} = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{3t}.$$

17. **Network.** First derive the model. For the left loop of the electrical network you obtain from Kirchhoff's voltage law

$$LI_1' + R_1(I_1 - I_2) = E \tag{a}$$

because both currents flow through R_1, but in opposite directions, so that you have to take their difference. For the right loop you similarly obtain

$$R_1(I_2 - I_1) + R_2I_2 + \frac{1}{C}\int I_2\,dt = 0. \tag{b}$$

Insert the given numerical values in (a). Do the same in (b) and differentiate (b) in order to get rid of the integral. This gives

$$I_1' + 2(I_1 - I_2) = 200$$

$$2(I_2' - I_1') + 8I_2' + 2I_2 = 0.$$

Write the terms in the first of these two equations in the usual order, obtaining

$$I_1' = -2I_1 + 2I_2 + 200. \tag{a1}$$

Do the same in the second equation as follows. Collecting terms and then dividing by 10, you first have

$$10I_2' - 2I_1' + 2I_2 = 0 \quad \text{or} \quad I_2' - 0.2I_1' + 0.2I_2 = 0.$$

To obtain the usual form, you have to get rid of the term in I_1', which you replace by using (a1). This gives

$$I_2' - 0.2(-2I_1 + 2I_2 + 200) + 0.2I_2 = 0.$$

Collecting terms and ordering them as usual, you obtain

$$I_2' = -0.4I_1 + 0.2I_2 + 40. \tag{b1}$$

(a1) and (b1) are the two equations of the system that you use in your further work. The matrix of the corresponding homogeneous system is

$$\mathbf{A} = \begin{bmatrix} -2 & 2 \\ -0.4 & 0.2 \end{bmatrix}.$$

Its characteristic equation is ($\mathbf{I}$ is the unit matrix)

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (-2 - \lambda)(0.2 - \lambda) - (-0.4)\cdot 2 = \lambda^2 + 1.8\lambda + 0.4 = 0.$$

This gives the eigenvalues

$$\lambda_1 = -0.9 + \sqrt{0.41} = -0.259688$$

and

$$\lambda_2 = -0.9 - \sqrt{0.41} = -1.540312.$$

Eigenvectors are obtained from $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ with $\lambda = \lambda_1$ and $\lambda = \lambda_2$. For λ_1 this gives

$$(-2 - \lambda_1)x_1 + 2x_2 = 0, \quad \text{say,} \quad x_1 = 2 \quad \text{and} \quad x_2 = 2 + \lambda_1.$$

Similarly, for λ_2 you obtain

$$(-2 - \lambda_2)x_1 + 2x_2 = 0, \quad \text{say,} \quad x_1 = 2 \quad \text{and} \quad x_2 = 2 + \lambda_2.$$

The eigenvectors thus obtained are

$$\mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ 2 + \lambda_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1.1 + \sqrt{0.41} \end{bmatrix}$$

and

$$\mathbf{x}^{(2)} = \begin{bmatrix} 2 \\ 2 + \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1.1 - \sqrt{0.41} \end{bmatrix}.$$

This gives as a general solution of the homogeneous system

$$\mathbf{I}^{(h)} = c_1 \mathbf{x}^{(1)} e^{\lambda_1 t} + c_2 \mathbf{x}^{(2)} e^{\lambda_2 t}.$$

You finally need a particular solution $\mathbf{I}^{(p)}$ of the given nonhomogeneous system $\mathbf{J}' = \mathbf{AJ} + \mathbf{g}$, where $\mathbf{g} = [200 \quad 40]^T$ is constant, and $\mathbf{J} = [I_1 \quad I_2]^T$ is the vector of the currents. The method of undetermined coefficients applies. Since $\mathbf{g}$ is constant, you can choose a constant $\mathbf{I}^{(p)} = \mathbf{u} = [u_1 \quad u_2]^T = \mathbf{const}$ and substitute it into the the system, obtaining, since $\mathbf{u}' = \mathbf{0}$,

$$\mathbf{I}^{(p)\prime} = \mathbf{0} = \mathbf{Au} + \mathbf{g} = \begin{bmatrix} -2 & 2 \\ -0.4 & 0.2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 200 \\ 40 \end{bmatrix} = \begin{bmatrix} -2u_1 + 2u_2 + 200 \\ -0.4u_1 + 0.2u_2 + 40 \end{bmatrix}.$$

Hence you can determine u_1 and u_2 from the system

$$-2u_1 + 2u_2 = -200$$
$$-0.4u_1 + 0.2u_2 = -40.$$

The solution is $u_1 = 100$, $u_2 = 0$. The answer is

$$\mathbf{J} = \mathbf{I}^{(h)} + \mathbf{I}^{(p)}.$$