CHAPTER 5. Laplace Transforms

Sec. 5.1 Laplace Transform. Inverse Transform. Linearity. Shifting

Problem Set 5.1. Page 257

7. **Laplace transform.** Use the addition formula for the sine function.

13. **Use of the defining integral.** Problems 9-16 can be done by direct integration of the defining integral, with the proper lower and upper limits of integration. In some cases you will need integration by parts. In Prob. 13,

$$\int te^{-st}\,dt = \frac{te^{-st}}{-s} - \frac{1}{-s}\int e^{-st}\,dt$$
$$= \frac{te^{-st}}{-s} - \frac{1}{s^2}e^{-st}.$$

The lower limit of integration is 0, and the first term in the second line is 0 at $t = 0$. The upper limit of integration is k (see the figure in Problem Set 5.1). Hence by evaluating your result at the limits you obtain $ke^{-ks}/(-s)$ from the first term and $-(e^{-ks} - 1)/s^2$ from the second.

17. **Inverse transform.** The basic formulas are contained in Table 5.1 of Sec. 5.1. From the denominator of the given expression you see that formulas 7 and 8 of that table are needed. Since $s^2 + 3.24 = s^2 + 1.8^2$, you will obtain cos $1.8t$ and sin $1.8t$. From formula 7 you may conclude that the cosine term is 0.1 cos $1.8t$. From formula 8 you see that the sine term is 0.5 sin $1.8t$ because $0.9 = 1.8 \cdot 0.5$. The answer is the sum of these two expressions obtained.

23. **Inverse transform.** Factor out $1/L^2$.

27. **Partial fraction reduction** and formula 6 in Table 5.1 are needed. The two partial fractions will give you a sum of two exponential functions.

33. **First shifting theorem.** $\sinh t \cos t = (1/2)(e^t - e^{-t}) \cos t = (1/2)e^t \cos t - (1/2)e^{-t} \cos t$ and formula 7 in Table 5.1 together with the first shifting theorem give $(1/2)(s - 1)/[(s - 1)^2 + 1]$ as the transform of the first term and $-(1/2)(s + 1)/[(s + 1)^2 + 1]$ as the transform of the second term. The sum of the two transforms has the common denominator

$$(s^2 - 2s + 2)(s^2 + 2s + 2) = (s^2 + 2)^2 - 4s^2 = s^4 + 4$$

and the numerator

$$(1/2)(s - 1)(s^2 + 2s + 2) - (1/2)(s + 1)(s^2 - 2s + 2) = s^2 - 2.$$

This gives the answer shown in Appendix 2.

39. **First shifting theorem.** You can write $s/D = [(s + 1/2) - 1/2]/D$, where D is the given denominator. Now by the shifting theorem, $(s + 1/2)/D$ has the inverse $\exp(-t/2) \cos t$, and $(-1/2)/D$ has the inverse $(-1/2) \exp(-t/2) \sin t$.

Sec. 5.2 Transforms of Derivatives and Integrals. Differential Equations

Example 2. The question at the end. $f(t) = \sin \omega t, f''(t) = -\omega^2 f(t)$. Now take the transform and use (2) and $f'(0) = \omega \cos 0 = \omega$, obtaining

$$\mathcal{L}(f) = -\omega^2\mathcal{L}(f) = s^2\mathcal{L}(f) - \omega.$$

Now solve algebraically for $\mathcal{L}(f)$.

Problem Set 5.2. Page 264

1. **Initial value problem.** Denote the Laplace transform of the unknown function y (the solution sought) by Y. By (1) in Sec. 5.2 the transform of the derivative y' is $sY - y(0) = sY$; here the given initial condition is taken into account. From Table 5.1 in Sec. 5.1 you see that the transform of the right side of the given differential equation is $10/(s^2+1)$, Together this gives the subsidiary equation

$$sY + 3Y = 10/(s^2+1).$$

Algebraically solving for Y and writing the result in terms of partial fractions gives

$$Y = \frac{10}{(s+3)(s^2+1)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+1}.$$

For determining the constants A, B, C use your favorite method. For instance, taking the common denominator on the right, by equating the numerators on both sides, you have

$$10 = A(s^2+1) + (s+3)(Bs+C). \qquad \text{(a)}$$

Setting $s = -3$ gives

$$10 = A[(-3)^2+1], \quad \text{hence } A = 1.$$

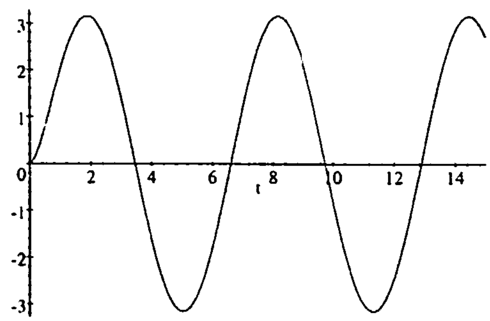
Equating the terms in s^2 on both sides of (a) and using $A = 1$, you obtain

$$0 = 1 + B, \quad \text{hence } B = -1.$$

Equating the constant terms on both sides of (a) and again using $A = 1$ gives

$$10 = 1 + 3C, \quad \text{hence } C = 3.$$

$1/(s+3)$ has the inverse transform e^{-3t} (see Table 5.1). The inverse transform of the next term $Bs/(s^2+1) = -s/(s^2+1)$ is $-\cos t$. The last term $C/(s^2+1) = 3/(s^2+1)$ has the inverse transform $3\sin t$. Together this gives the answer on p. A14 of Appendix 2. The figure shows that the solution is practically a harmonic oscillation; the influence of the transition term e^{-3t} is hardly visible.



Section 5.2. Problem 1. Solution of the initial value problem

11. **Different derivations.** (a) Use $\cos^2 t = 1 - \sin^2 t$. (b) Set $f = \cos^2 t$. Then $f(0) = 1$ and $f' = -2\cos t \sin t = -\sin 2t$. From (1) in Sec. 5.2 you now obtain

$$\mathcal{L}(-\sin 2t) = -2/(s^2+4) = s\mathcal{L}(f) - 1.$$

Solving for $\mathcal{L}(f)$ gives the answer

$$\mathcal{L}(f) = \frac{1}{s}\left(1 - \frac{2}{s^2+4}\right) = \frac{1}{s}\left(\frac{s^2+2}{s^2+4}\right).$$

19. Application of Theorem 3. The expression in the parentheses has the inverse transform $\cos 3t + \frac{1}{3}\sin 3t$. The factor s^2 in the denominator suggests two successive integrations. The first gives

$$\frac{1}{3}\sin 3t - \frac{1}{9}(\cos 3t - 1).$$

Integrate this again from 0 to t (see Theorem 3; don't forget the contribution from the lower limit of integration). This gives

$$-\frac{1}{9}(\cos 3t - 1) - \frac{1}{9}\left(\frac{1}{3}\sin 3t - t\right).$$

Multiply the result by the factor 9 shown in the enunciation of the problem. This gives the answer in Appendix 2.

Sec. 5.3 Unit Step Function. Second Shifting Theorem. Dirac's Delta Function

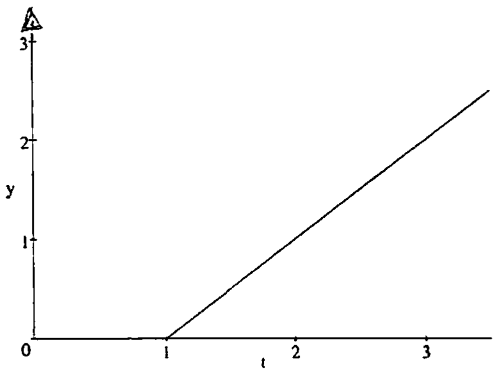
Example 1 involves the term $u(t - 2\pi)\sin t$. You must convert it to $u(t - 2\pi)\sin(t - 2\pi)$ before you can apply (3) because in (3) both f and u have the same argument $t - a$. Here these two expressions are equal because of periodicity. In most cases two such expressions will not be equal. For instance, if you have $tu(t - 4)$ (sketch it to see what it looks like!), before you apply (3), you have to write it as

$$tu(t-4) = (t - 4 + 4)u(t-4) = (t-4)u(t-4) + 4u(t-4).$$

Think this over before you go on.

Problem Set 5.3. Page 273

3. Unit step function. The given function $(t - 1)u(t - 1)$ is of the form (2) $f(t - a)u(t - a)$ with $f(t) = t$ and $a = 1$. The transform of $f(t)$ is $F(s) = 1/s^2$ (see Table 5.1 in Sec. 5.1). Hence (3) in the second shifting theorem shows that the given function has the transform $e^{-s}F(s) = e^{-s}/s^2$.



Section 5.3. Problem 3. Given function (which is zero for $t < 1$)

11. Laplace transform. Since $1 - u(t - 1)$ is 1 for t from 0 to 1 and 0 otherwise, you can represent the given function by $[1 - u(t-1)]e^{-t} = e^{-t} - u(t-1)e^{-t}$. The first term has the transform $1/(s - 1)$. Consider the second term and write it in the form (2) of this section,

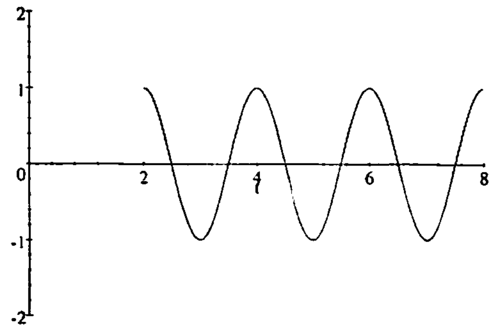
$$-u(t-1)e^{t} = -u(t-1)e^{t-1}e.$$

Keep this step in mind, which is always needed in using the second shifting theorem (Theorem 1). Formula (3) with $F(s) = 1/(s - 1)$ and $a = 1$ now gives the transform

$$-e \cdot e^{-s}/(s-1) = -e^{1-s}/(s-1).$$

Hence the answer is $(1 - e^{1-s})/(s - 1)$.

19. Inverse transform. $F(s) = s/(s^2 + \pi^2)$ has the inverse $f(t) = \cos \pi t$ and is multiplied by e^{-2s}, which corresponds to the multiplication of $f(t)$ by $u(t-2)$ and a shift of t in the cosine to $t-2$. But $\cos(\pi(t-2)) = \cos t$ by periodicity. Hence the answer is $u(t-2)\cos \pi t$, a cosine curve that begins at $t = 2$, the first part of it between 0 and 2 being cut off (see the figure).



Section 5.3. Problem 19. Inverse $u(t-2)\cos \pi t$

23. Initial value problem. Express the right side by a unit step function, obtaining

$$\begin{aligned} y'' + 9y &= 8[1 - u(t-\pi)] \sin t \\ &= 8 \sin t + 8u(t-\pi) \sin (t-\pi). \end{aligned}$$

Here $\sin t = -\sin(t-\pi)$ has been used and the form of the right side has been chosen in order to prepare for the use of (2) in this section. Now use the initial conditions $y(0) = 0$, $y'(0) = 4$ and (2) in Sec. 5.2 as well as (3) in this section. This gives the subsidiary equation

$$s^2 Y - s \cdot 0 - 4 + 9Y = 8/(s^2 + 1) + 8e^{-\pi s}/(s^2 + 1).$$

Solving algebraically for Y and using partial fractions gives

$$\begin{aligned} Y &= 4/(s^2 + 9) + 8/[(s^2 + 1)(s^2 + 9)] + 8e^{-\pi s}/[(s^2 + 1)(s^2 + 9)] \\ &= \frac{4}{s^2 + 9} + \frac{1}{s^2 + 1} - \frac{1}{s^2 + 9} + \left(\frac{1}{s^2 + 1} - \frac{1}{s^2 + 9}\right)e^{-\pi s}. \end{aligned}$$

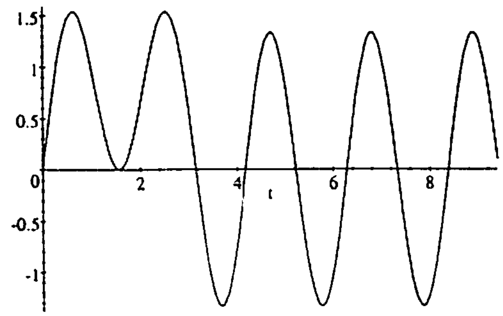
The inverse is

$$\begin{aligned} &\frac{4}{3} \sin 3t + \sin t - \frac{1}{3} \sin 3t \\ &\qquad + u(t-\pi)\left[\sin (t-\pi) - \frac{1}{3} \sin (3t - 3\pi)\right]. \end{aligned} \tag{a}$$

This gives $\sin 3t + \sin t$ if $0 < t < \pi$. In the second line,

$$\sin (t-\pi) - \frac{1}{3} \sin (3t - 3\pi) = -\sin t + \frac{1}{3} \sin 3t.$$

These two terms cancel two terms in the first line of (a), so that for $t > \pi$ the solution is $(4/3) \sin 3t$. The figure shows the solution thus obtained.



Section 5.3. Problem 23. Particular solution whose formula changes at π

37. ***RC*-circuit.** $100i + 10\int_0^t i(\tau)\,d\tau = 100[u(t-1) - u(t-2)]$. Now $q(0) = 0$ because of the limits of integration, and $i(0) = 0$ follows from $v(0) = 0$ and the equation. Using Theorem 3 in Sec. 5.2, obtain the subsidiary equation

$$100I + (10/s)I = 100(e^{-s}/s - e^{-2s}/s).$$

Division by 100, multiplication by s, and algebraic solution for I gives

$$I = e^{-s}/(s+0.1) - e^{-2s}/(s+0.1).$$

Now use the second shifting theorem to obtain the solution

$$i = e^{-0.1(t-1)}u(t-1) - e^{-0.1(t-2)}u(t-2).$$

Hence

$$i = \begin{cases} 0 & \text{if } t < 1, \\ e^{-0.1(t-1)} & \text{if } 1 < t < 2, \\ e^{-0.1(t-1)} - e^{-0.1(t-2)} & \text{if } t > 2. \end{cases}$$

Sec. 5.4 Differentiation and Integration of Transforms

Example 3. Proceeding as in Example 2 gives

$$-\frac{d}{ds}\left[\ln\left(1 - \frac{a^2}{s^2}\right)\right] = -\left(1 - \frac{a^2}{s^2}\right)^{-1}(-2)\left(-\frac{a^2}{s^3}\right)$$
$$= -2a^2/[s(s^2 - a^2)]$$
$$= 2/s - 2s/(s^2 - a^2).$$

The inverse is $f(t) = 2 - 2\cosh at$. Now divide by t.

Problem Set 5.4. Page 278

7. **Inverse transform.** $\sin t$ has the transform $F(s) = 1/(s^2+1)$; see Sec. 5.1 if necessary. The multiplication by e^{-t} corresponds to replacing s by $s + 1$, according to the first shifting theorem (Sec. 5.1), that is, $e^{-t}\sin t$ has the transform $F(s+1) = 1/[(s+1)^2 + 1] = 1/(s^2 + 2s + 2)$. Finally, the multiplication by t corresponds to the differentiation of the transform and multiplication by -1, so that you obtain the answer

$$\frac{1}{(s^2+2s+2)^2}(2s+2) = \frac{2s+2}{(s^2+2s+2)^2}.$$

9. **Inverse transform.** $1/s^3$ has the inverse transform $t^2/2$ (see Table 5.1 in Sec. 5.1). Hence by the first shifting theorem, the given function has the inverse transform $t^2e^{3t}/2$. Check the result by two successive applications of (1) in this section, as follows. $1/(s-3)$ has the inverse transform e^{3t}. Its derivative is $-1/(s-3)^2$ and has the inverse transform $-te^{3t}$, by (1). Similarly, its second derivative is $+2/(s-3)^3$ and has the inverse transform $-t(-te^{3t}) = +t^2e^{3t}$. Now divide by 2.

13. **Inverse transform.** Let $f(t)$ be the inverse transform of the given

$$\ln[(s^2+1)/(s-1)^2)] = \ln(s^2+1) - 2\ln(s-1). \qquad \text{(A)}$$

The derivative of (A) is

$$\frac{2s}{s^2+1} - \frac{2}{s-1}. \qquad \text{(B)}$$

The inverse transform of (B) is $2\cos t - 2e^t$. From (1*) you see that this is the inverse of $-tf(t)$. Hence the answer is

$$f(t) = -(2\cos t - 2e^t)/t = 2(e^t - \cos t)/t.$$

Sec. 5.5 Convolution. Integral Equations

Problem Set 5.5. Page 283

Write all the calculations in these problems, in particular, the occurring integrals and their evaluation, very orderly step by step on your worksheet, as explained in these solutions.

3. **Convolution by evaluating the integral (1).** In (1) you have $f(t) = e^t$, $g(t) = e^{-t}$, hence $f(\tau) = e^\tau$, $g(t-\tau) = e^{-(t-\tau)} = e^{-t+\tau}$. The function e^{-t} can be taken out from under the integral sign because you integrate with respect to τ. The remaining integrand is $e^\tau e^\tau = e^{2\tau}$. Integration gives $e^{2\tau}/2$, hence $e^{2t}/2$ at the upper limit of integration and $1/2$ at the lower limit of integration. This must now be subtracted and the result must be multiplied by the function e^{-t}, which was pulled out; thus,

$$e^{-t}(e^{2t}-1)/2 = (1/2)(e^t - e^{-t}) = \sinh t,$$

where the last equality is a consequence of the definition of the hyperbolic sine.

11. **Inverse transform.** $1/s^2$ has the inverse transform t, and $1/(s-1)$ has the inverse transform e^t. Hence the integrand of the corresponding convolution integral (1) is $\tau e^{t-\tau}$. (Of course, write this integral and all the following calculations on your worksheet!) Integration by parts gives $-\tau e^{t-\tau}$ plus the integral of $e^{t-\tau}$, the first minus sign being a consequence of the fact that you integrate with respect to τ, not t! The integrated part gives $-t$ from the upper limit and 0 from the lower. The remaining integral equals $-e^{t-\tau}$, so that its upper limit gives -1 and its lower contributes e^t. Hence the answer is $-t-1+e^t$.

19. **Initial value problem.** The subsidiary equation is

$$s^2Y + Y = 3s/(s^2+4);$$

here the initial conditions $y(0) = 0$, $y'(0) = 0$ have been used. Solving algebraically for Y gives

$$Y = \left(\frac{1}{s^2+1}\right)\left(\frac{3s}{s^2+4}\right).$$

The two functions on the right have the inverse transforms $\sin t$ and $3\cos 2t$, respectively. The integrand of the convolution of these two functions of t is

$$3\sin\tau\cos(2t-2\tau).$$

Convert this to a sum of sines by using (12) in Appendix A3.1. Choose $u = 2t-\tau$ and $v = 3\tau - 2t$. The result is

$$\frac{3}{2}\left[\sin(2t-\tau)+\sin(3\tau-2t)\right].$$

Integration with respect to τ gives

$$\frac{3}{2}\left[\cos(2t-\tau)-\frac{1}{3}\cos(3\tau-2t)\right].$$

Evaluating at the upper limit $\tau = t$ and subtracting the value at the lower limit $\tau = 0$ gives

$$\frac{3}{2}\left[\cos t-\frac{1}{3}\cos t-\left(\cos 2t-\frac{1}{3}\cos(-2t)\right)\right].$$

Here $\cos(-2t) = \cos 2t$. Simplification gives

$$\frac{3}{2}\left[\frac{2}{3}\cos t-\frac{2}{3}\cos 2t\right] = \cos t-\cos 2t.$$

Sec. 5.6 Partial Fractions. Differential Equations

Problem Set 5.6. Page 289

3. Unrepeated factors. The partial fraction representation of the given function is

$$1/s + (3/2)/(s-3) - (3/2)/(s+3).$$

You can find the inverse transform of each of these three terms in Table 5.1 (Sec. 5.1). Recalling the definition of the hyperbolic sine function, you obtain

$$1 + (3/2)e^{3t} - (3/2)e^{-3t} = 1 + 3\sinh 3t.$$

13. Verification by working backward. $\cosh at \cos at = (1/2)e^{at}\cos at + (1/2)e^{-at}\cos at$. Now $\cos at$ has the transform $s/(s^2+a^2)$. Hence, by the first shifting theorem, $e^{at}\cos at$ has the transform

$$(s-a)/[(s-a)^2+a^2] = (s-a)/(s^2-2as+2a^2) \qquad \text{(A)}$$

and $e^{-at}\cos at$ has the transform

$$(s+a)/[(s+a)^2+a^2] = (s+a)/(s^2+2as+2a^2). \qquad \text{(B)}$$

Now take the common denominator of the expressions in (A) and (B),

$$[(s^2+2a^2)-2as][(s^2+2a^2)+2as] = (s^2+2a^2)^2-4a^2s^2 = s^4+4a^4.$$

The corresponding numerator of the sum of (A) and (B) is $2s^3$ because all the other terms cancel in pairs. Multiply this by 1/2 (the factors at the very beginning of this calculation) to get the formula in the problem.

Sec. 5.7 Systems of Differential Equations

Example 1. The subsidiary equations are obtained by transforming the derivatives by means of (1) in Sec. 5.2, just as in the case of a single differential equation. For the first equation this gives

$$sY_1 - y_1(0) = -0.08Y_1 + 0.02Y_2 + 6/s.$$

For the second equation you obtain

$$sY_2 - y_2(0) = 0.08Y_1 + 0.08Y_2.$$

Inserting $y_1(0) = 0$ and $y_2(0) = 150$ and collecting terms in Y_1 and in Y_2 gives the subsidiary equations shown on p. 291.

Problem Set 5.7. Page 294

1. System of differential equations. The subsidiary equations are

$$sY_1 - 1 = -Y_1 + Y_2$$
$$sY_2 - 0 = -Y_1 - Y_2.$$

Note that the given initial conditions have been used, as in the example just explained, the notations being those used in the text. The next step is ordering these equations, taking all the Y-terms to the left. This gives

$$(s+1)Y_1 - \qquad Y_2 = -(-1)$$
$$Y_1 + (s+1)Y_2 = 0.$$

In the case of a single equation the subsidiary equation is solved algebraically. The same step must now be done with the system of subsidiary equations. Elimination or Cramer's rule (Sec. 6.6) gives

$$Y_1 = \frac{s+1}{(s+1)^2+1}$$
$$Y_2 = -\frac{1}{(s+1)^2+1}.$$

$s/(s^2+1)$ has the inverse transform $\cos t$, and $1/(s^2+1)$ has the inverse transform $\sin t$. This and the first shifting theorem give the answer

$$y_1 = e^{-t}\cos t, \qquad y_2 = -e^{-t}\sin t.$$

19. Electrical network. From Fig. 132 and Kirchhoff's laws you obtain the model

$$2i_1 + 4(i_1 - i_2) + i_1' = 195\sin t$$
$$4i_2 + 4(i_2 - i_1) + 2i_2' = 0$$

and $i_1(0) = 0$, $i_2(0) = 0$. The subsidiary equations are

$$2I_1 + 4(I_1 - I_2) + sI_1 = 195/(s^2+1)$$
$$4I_2 + 4(I_2 - I_1) + 2sI_2 = 0.$$

Collecting terms in I_1 and I_2 and dividing the second equation by 2 gives

$$(s+6)I_1 - \qquad 4I_2 = 195/(s^2+1)$$
$$-2I_1 + (s+4)I_2 = 0.$$

The solution by elimination or Cramer's rule is

$$I_1 = 195\frac{s+4}{(s^2+10s+16)(s^2+1)}$$
$$I_2 = 2\cdot 195\frac{1}{(s^2+10s+16)(s^2+1)}.$$

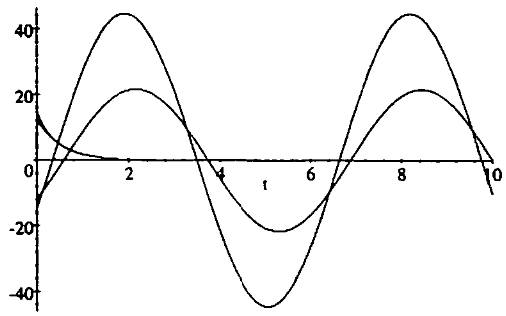
In terms of partial fractions, this becomes

$$I_1 = \frac{2}{s+8} + \frac{13}{s+2} - 3\frac{5s-14}{s^2+1}$$
$$I_2 = -\frac{1}{s+8} + \frac{13}{s+2} - 6\frac{2s-3}{s^2+1}.$$

All the terms obtained have inverse transforms that should by now be well known to you. The last term in each line is a mixture of transforms of cosines and sines that you can easily take apart and determine. This gives the answer

$$i_1 = 2e^{-8t} + 13e^{-2t} - 15\cos t + 42\sin t$$
$$i_2 = -e^{-8t} + 13e^{-2t} - 12\cos t + 18\sin t.$$

The figure shows the exponential terms in i_1 and i_2 and (separately) the steady-state solutions, which are harmonic oscillations. It shows that the transition period is rather short and of the same length of time for both currents, and that the initial conditions seem to be satisfied (as you can readily check from the formulas of the answer). Compare this figure with Fig. 132 in the text and comment. Why do the two curves cross the t-axis at different points?



Section 5.7. Problem 19. Exponential terms and steady-state solutions