CHAPTER 8. Vector Differential Calculus. Grad, Div, Curl

Sec. 8.1 Vector Algebra in 2-Space and 3-Space

Problem Set 8.1. Page 407

1. Components. According to the definition of components you have to calculate the differences of the coordinates of the terminal point Q minus the corresponding coordinates of the initial point P of the vector. Thus, $v_1 = 4 - 1 = 3$, etc. Since the z-coordinates of P and Q are zero, the vector $\mathbf{v}$ is a vector in the xy-plane; it has no component in the z-direction. Sketch the vector, so that you see what it looks like as an arrow in the xyz-coordinate system in space.

15. Vector addition has properties quite similar to those of the addition of numbers because it is defined in terms of components and the latter are numbers. Prob. 15 illustrates the commutativity and Prob. 17 the associativity of vector addition.

19. Addition and scalar multiplication. It makes no difference whether you first multiply and then add, or whether you first add the given vectors and then multiply their sum by the scalar 4. This problem and Example 2 in the text illustrate formula (6b).

29. Forces were foremost among the applications that have suggested the concept of a vector, and forming the resultant of forces has motivated vector addition to a large extent. Thus, each of Problems 24-28 amounts to the addition of three vectors. "Equilibrium" means that the resultant of the given forces is the zero vector. Hence in Prob. 29 you must determine $\mathbf{p}$ such that

$$\mathbf{p} + \mathbf{q} + \mathbf{u} = \mathbf{0}.$$

Hence

$$\mathbf{p} = -\mathbf{q} - \mathbf{u} = -[3,\ 2,\ 0] - [-2,\ 4,\ 0] = [-3,\ -2,\ 0] + [2,\ -4,\ 0] = [-1,\ -6,\ 0].$$

33. Force in a rope. This is typical of many problems in mechanics. Choose an xy-coordinate system with the x-axis pointing horizontally to the right and the y-axis pointing vertically downward. Then the given weight is a force $\mathbf{w} = [w_1, w_2] = [0, w]$ pointing vertically downward. You have to determine the unknown force in the left rope, call it $\mathbf{u} = [u_1, u_2]$, and the unknown force $\mathbf{v} = [v_1, v_2]$ in the right rope. The three forces are in equilibrium (they have the resultant $\mathbf{0}$) because the system does not move. Thus,

$$\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}. \qquad \text{(A)}$$

You have two choices of giving $\mathbf{u}$ a direction, and similarly for $\mathbf{v}$. The choice is up to you. Suppose you choose the vectors to point from the point where the weight is attached upward to the points where the ropes are fixed. Then (see the figure on p. 406)

$$\begin{aligned} \mathbf{u} &= [u_1,\ u_2] = [-|\mathbf{u}|\cos\alpha,\ -|\mathbf{u}|\sin\alpha] \\ \mathbf{v} &= [v_1,\ v_2] = [|\mathbf{v}|\cos\alpha,\ -|\mathbf{v}|\sin\alpha] \\ \mathbf{w} &= [w_1,\ w_2] = [0,\ w], \end{aligned} \qquad \text{(B)}$$

where the components preceded by a minus sign are those that point in the negative direction of the corresponding coordinate axis (to the left or upward). From (A) and (B) you obtain for the horizontal components

$$-|\mathbf{u}|\cos\alpha + |\mathbf{v}|\cos\alpha = 0, \quad \text{hence} \quad |\mathbf{u}| = |\mathbf{v}|. \qquad \text{(C)}$$

Obviously, this could have been concluded from the symmetry of the figure. From (A) and (B) you obtain for the vertical components

$$-|\mathbf{u}|\sin\alpha - |\mathbf{v}|\sin\alpha + w = 0.$$

From this and (C) conclude

$$|\mathbf{u}|\sin\alpha = |\mathbf{v}|\sin\alpha = \frac{1}{2}w.$$

Hence **u** and **v** have equal vertical components. Also this is a consequence of the symmetry.

Sec. 8.2 Inner Product (Dot Product)

Problem Set 8.2. Page 413

5. Inner product. This problem illustrates the rule

$$(p\mathbf{b})\bullet(q\mathbf{c}) = (pq)(\mathbf{b}\bullet\mathbf{c}) \tag{A}$$

which can be used to simplify calculations.

For the given vectors you obtain from (2), which gives the inner product in terms of components, for the first expression (corresponding to the left side of (A))

$$(2\mathbf{b})\bullet(5\mathbf{c}) = (2[2, 0, -5])\bullet(5[4, -2, 1]) = [4, 0, -10]\bullet[20, -10, 5]$$
$$= 4\cdot 20 + 0 + (-10)\cdot 5 = 30$$

and for the second expression, again from (2), (corresponding to the right side of (A))

$$10(\mathbf{b}\bullet\mathbf{c}) = 10([2, 0, -5]\bullet[4, -2, 1]) = 10(8 + 0 + (-5)) = 30.$$

The general formula (A) can be proved as follows. Let $\mathbf{b} = [b_1, b_2, b_3]$ and $\mathbf{c} = [c_1, c_2, c_3]$. Then $p\mathbf{b} = [pb_1, pb_2, pb_3]$ by (5) in Sec. 8.1, and similarly for $q\mathbf{c}$. From this and (2) in this section,

$$(p\mathbf{b})\bullet(q\mathbf{c}) = [pb_1, pb_2, pb_3]\bullet[qc_1, qc_2, qc_3] = pb_1qc_1 + pb_2qc_2 + pb_3qc_3.$$

Hence the last expression contains a common factor pq, which you can pull out, obtaining

$$pq(b_1c_1 + b_2c_2 + b_3c_3).$$

From (2) (with **b** and **c** instead of **a** and **b**) you see that this equals $(pq)(\mathbf{b}\bullet\mathbf{c})$. But this is the right side of (A), and the proof is complete.

Looking back and comparing, you may find that this general proof was not much more difficult than the calculation in the case of the given special vectors. This is an interesting experience worth remembering because it may happen in other cases, too. And, as a general rule, if a general proof seems too difficult, start with a simple special case.

13. Work. This is a major motivation and application of inner products. In this problem, the force $\mathbf{p} = [2, 6, 6]$ acts in the displacement from the point A with coordinates $(3, 4, 0)$ to the point B with coordinates $(5, 8, 0)$. Both points lie in the xy-plane. Hence the same is true for the segment AB, which represents a "displacement vector" **d**, whose components are obtained as differences of corresponding coordinates of B minus A (endpoint minus initial point). Thus,

$$\mathbf{d} = [d_1, d_2, d_3] = [5-3, \ 8-4, \ 0-0] = [2, 4, 0].$$

This gives the work as an inner product (a dot product) as in Example 2 in the text,

$$W = \mathbf{p}\bullet\mathbf{d} = [2, 6, 6]\bullet[2, 4, 0] = 2\cdot 2 + 6\cdot 4 + 6\cdot 0 = 28.$$

19. Angle. Use (4).

25. Normal to a plane. Angle between planes. A normal to a plane is a straight line perpendicular to the plane. See Example 6 in the text. The angle γ between the given planes P_1 and P_2 is the angle between their normals. (Equivalently, if P is a plane perpendicular to the line of intersection between P_1 and P_2, then γ is the angle between the lines along which P_1 and P_2 intersect P.) Normal vectors of the given $P_1 : x + y + z = 1$ and $P_2 : x + 2y + 3z = 6$ are

$$\mathbf{n}_1 = [1, 1, 1] \quad \text{and} \quad \mathbf{n}_2 = [1, 2, 3],$$

respectively. Now use (4). You need $\mathbf{n}_1\bullet\mathbf{n}_2 = 1 + 2 + 3 = 6$ and in the denominator $|\mathbf{n}_1| = \sqrt{3}$ and

$|\mathbf{n}_2| = \sqrt{1+4+9} = \sqrt{14}$. This gives

$$\cos\gamma = 6/\sqrt{3\cdot 14} = 6/6.48074 = 0.92582, \quad \gamma = 22.2 \text{ degrees.}$$

Sec. 8.3 Vector Product (Cross Product)

Problem Set 8.3. Page 421

1. **Vector product.** Both the dot product and the cross product are concepts created for reasons of applications. The dot product is a scalar, as it occurs, for instance, in connection with work, as you have seen in Problem Set 8.2. The cross product is a vector perpendicular to the two vectors (or is the zero vector in the exceptional cases described in the definition). Examples 4-6 in the text illustrate important applications that helped to motivate this kind of product. Given $\mathbf{a} = [1, 2, 0]$ and $\mathbf{b} = [-3, 2, 0]$, you obtain the cross product $\mathbf{v} = \mathbf{a} \times \mathbf{b}$ from (2). The vectors **a** and **b** lie in the xy-plane (more precisely, they are parallel to this plane) since their components in the z-direction are zero. Hence their cross product must be perpendicular to the xy-plane, as follows directly from the definition. See Fig. 167. Much more easily to remember than (2) is (2**), which implies (2*), giving the components by second-order determinants. If you need help with determinants, look up the beginning of Sec. 6.6. From (2*) you obtain

$$v_1 = \begin{vmatrix} 2 & 0 \\ 2 & 0 \end{vmatrix} = 0, \quad v_2 = \begin{vmatrix} 0 & 1 \\ 0 & -3 \end{vmatrix} = 0, \quad v_3 = \begin{vmatrix} 1 & 2 \\ -3 & 2 \end{vmatrix} = 2 + 6 = 8.$$

Hence $\mathbf{v} = [0, 0, 8]$. From this you have $\mathbf{b} \times \mathbf{a} = -\mathbf{v} = [0, 0, -8]$. This illustrates that cross multiplication is not commutative but *anticommutative*; see (6) and Fig. 171. Hence always observe the order of the factors carefully.

17. **Scalar triple product.** This is the most useful of products with three or more factors. The reason is its geometric interpretation (see Figs. 175 and 176). Using (10) and developing the determinant by the third column gives

$$(\mathbf{b\,a\,d}) = \mathbf{b} \bullet (\mathbf{a} \times \mathbf{d}) = \begin{vmatrix} -3 & 2 & 0 \\ 1 & 2 & 0 \\ 6 & -7 & 2 \end{vmatrix} = 2\begin{vmatrix} -3 & 2 \\ 1 & 2 \end{vmatrix} = 2(-6-2) = -16.$$

The three vectors make up the three rows of the determinant, as you see. Interchanging two rows multiplies the determinant by -1; this gives the second result.

21. **Rotations** can be conveniently handled by vector products, as Example 5 of the text shows. For a rotation about the y-axis with $\omega = 9\,\text{sec}^{-1}$ the rotation vector **w**, which always lies in the axis of rotation (if you choose a point on the axis as the initial point of **w**), is

$$\mathbf{w} = [0, 9, 0]$$

or $\mathbf{w} = [0, -9, 0]$, depending on the sense of the rotation, which is not given in the problem. Also, $\mathbf{r} = [3, 4, 8]$, the position vector of the point P at which you want to find the velocity vector **v** (see Fig. 174). From these data the basic formula (9) gives

$$\mathbf{v} = \mathbf{w} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 9 & 0 \\ 3 & 4 & 8 \end{vmatrix} = (9\cdot 8)\mathbf{i} - 0\mathbf{j} + (-(9\cdot 3))\mathbf{k} = [72, 0, -27]$$

or $-[72, 0, -27] = [-72, 0, 27]$ if you take the other possible **w**. The speed is the length of the velocity vector **v**, that is,

$$|\mathbf{v}| = \sqrt{(-72)^2 + 0 + 27^2} = \sqrt{5913} = 76.896.$$

29. Area of a triangle. Given the vertices $A\,:\,(1,3,2)$, $B\,:\,(3,-4,2)$, $C\,:\,(5,0,-5)$, you can find the area from a vector product. (Try it without vectors.) All you have to do is to derive two vectors that form two sides of the triangle. Obviously, there are essentially three possibilities for doing this. For instance, derive **b** and **c** with common inital point A and terminal points B and C. This gives

$$\mathbf{b} = [3-1,\ -4-3,\ 2-2] = [2,-7,\ 0], \qquad \mathbf{c} = [5-1,\ 0-3,\ -5-2] = [4,\ -3,\ -7].$$

Their vector product is

$$\mathbf{v} = \mathbf{b}\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -7 & 0 \\ 4 & -3 & -7 \end{vmatrix} = (-7)\cdot(-7)\mathbf{i} - (2\cdot(-7))\mathbf{j} + (2\cdot(-3) - (-7)\cdot 4)\mathbf{k}$$

$$= 49\mathbf{i} + 14\mathbf{j} + 22\mathbf{k}.$$

By the definition of vector product, $|\mathbf{v}|$ gives the area of the parallelogram determined by these vectors,

$$|\mathbf{v}| = \sqrt{49^2 + 14^2 + 22^2} = \sqrt{3081} = 55.507.$$

Hence the answer is $(1/2)\sqrt{3081} = 27.753$.

Sec. 8.4 Vector and Scalar Functions and Fields. Derivatives

Problem Set 8.4. Page 427

1. Value at a point. $f(2,4) = 9\cdot 2^2 + 4\cdot 4^2 = 100$. Etc.

5. Level curves. arctan $(y/x) = k = const$ if and only if $y/x = c = const$, $y = cx$. Straight lines passing through the origin.

13. Level surfaces. $4x^2 + y^2 + 9z^2 = const.$ Division by 36 gives

$$\frac{x^2}{3^2} + \frac{y^2}{6^2} + \frac{z^2}{2^2} = c = const.$$

These are ellipsoids. For $c = 1$ the ellipsoid has the semi-axes 3, 6, 2 in the x-, y-, and z-directions, respectively. For other (positive) values you obtain an ellipsoid with semi-axes proportional to 3, 6, and 2, respectively.

19. Vector field. The vector function $\mathbf{v} = [y^2,\ 1]$ is constant on each horizontal straight line $y = const.$ Hence along each such line the arrows representing **v** have the same length and the same direction (they are parallel and their arrow heads point in the same direction).

23(e). Vector field. The vectors $\mathbf{v} = [\cos x,\ \sin x]$ are all unit vectors because $v_1^2 + v_2^2 = \cos^2 x + \sin^2 x = 1$. On each vertical straight line $x = const$ these vectors are constant (represented by parallel arrows with heads pointing in the same direction). For $x = 0$ they are horizontal (why?). As x increases from 0, their slope increases until x reaches the value $\pi/2$, at which $\cos x = 0$ and the arrows are vertical. Make a sketch corresponding to this discussion and continue it for values of $x > \pi/2$ until you reach 2π. From there on the vector field repeats itself, for reasons of periodicity.

27. Partial derivatives. Since vectors are differentiated componentwise, this is as in calculus, and you obtain the answer given on p. A21 of Appendix 2 of the book.

Sec. 8.5 Curves. Tangents. Arc Length

Problem Set 8.5. Page 433

5. Straight line. Eq. (4) gives the general parametric representation of a straight line

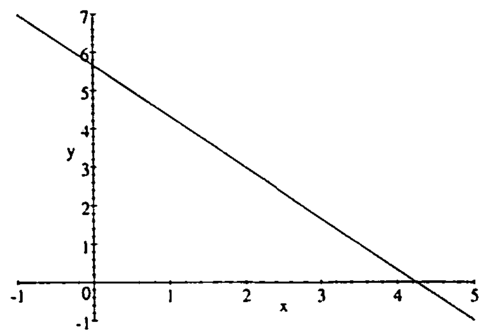
$$\mathbf{r}(t) = \mathbf{a} + t\mathbf{b} = [a_1 + tb_1,\ a_2 + tb_2,\ a_3 + tb_3].$$

Here, **a** is the position vector of a point P through which the line passes; hence this point corresponds to the parameter value $t = 0$. In this problem you can take $\mathbf{a} = [2,\ 3,\ 0]$, the position vector of A. The vector **b** gives the direction of the line. If **b** is a unit vector, then $|t|$ is the distance of the point with position vector $\mathbf{r}(t)$ from P. In the problem, **b** is not given, but must be determined, and you can take the vector with initial point A and terminal point B. Its components are the differences of corresponding coordinates, that is,

$$\mathbf{b} = [5-2,\ -1-3,\ 0-0] = [3,\ -4,\ 0].$$

Make sure that you understand this step. This gives the answer

$$\mathbf{r}(t) = [2,\ 3,\ 0] + t[3,\ -4,\ 0] = [2+3t,\ 3-4t,\ 0].$$



Section 8.5. Problem 5. Parametric representation of a straight line

11. Parametric representation of a circle. Circles occur often, and it is important that you fully understand this representation

$$\mathbf{r}(t) = [0,\ 5\cos t,\ 5\sin t].$$

In components,

$$x(t) = 0, \qquad y(t) = 5\cos t, \qquad z(t) = 5\sin t.$$

This gives

$$y^2 + z^2 = 25\cos^2 t + 25\sin^2 t = 25.$$

The circle lies in the yz-plane. Its center is the origin. Its radius is 5. A circle of radius c and center (a, b) in the xy-plane has the representation

$$\mathbf{r}(t) = [a + c\cos t,\ b + c\sin t,\ 0]. \tag{A}$$

From this you obtain

$$(x-a)^2 + (y-b)^2 = c^2\cos^2 t + c^2\sin^2 t = c^2, \tag{B}$$

the familiar representation often used in calculus. Before going on, make sure that you understand the relationship between (A) and (B).

23. Tangent to an ellipse. The ellipse is given by

$$\mathbf{r}(t) = [\cos t,\ 2\sin t]$$

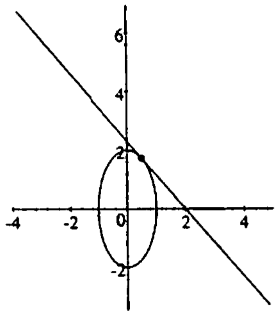
where t varies from 0 to 2π. You have to find the tangent at the point $P:\ (1/2, \sqrt{3})$. Now $\cos t = 1/2$

when $t = \pi/3 = 60$ degrees. Then $2 \sin t = 2 \sin(\pi/3) = 2(1/2)\sqrt{3} = \sqrt{3}$. Hence $\mathbf{r}(\pi/3) = [1/2, \ \sqrt{3}]$, so $t = \pi/3$ is in fact the parametric value of P. Now the slope of the tangent is $\mathbf{r}'(t) = [-\sin t, \ 2\cos t]$. At P this becomes

$$\mathbf{r}'(\pi/3) = [-\sin(\pi/3), \ 2\cos(\pi/3)] = [-(1/2)\sqrt{3}, \ 1].$$

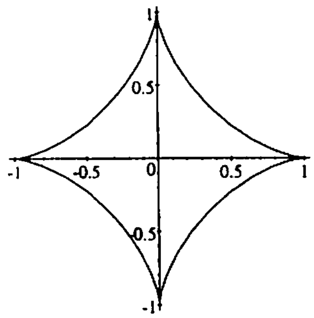
From these data and (9) you obtain a parametric representation of the tangent at P in the form

$$\mathbf{q}(w) = [1/2, \ \sqrt{3}] + w[-(1/2)\sqrt{3}, \ 1] = [0.500 - 0.866w, \ 1.732 + w].$$

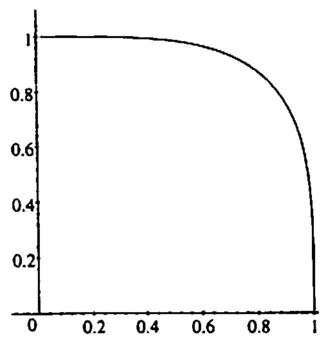


Section 8.5. Problem 23. Ellipse and tangent

33. Famous curves.



Section 8.5. CAS Project 33. Astroid



Section 8.5. CAS Project 33. Arc of Lamé's curve

Sec. 8.6 Curves in Mechanics. Velocity and Acceleration

Problem Set 8.6. Page 439

5. Acceleration. The path

$$\mathbf{r}(t) = [b \cos t, \quad b \sin t, \quad c] \qquad (b > 0)$$

is a circle of radius b in the horizontal plane $z = c$. The velocity is

$$\mathbf{v}(t) = \mathbf{r}'(t) = [-b \sin t, \quad b \cos t, \quad 0].$$

This vector is horizontal; there is no velocity in z-direction, as had to be expected for physical reasons. The speed is

$$|\mathbf{v}(t)| = \sqrt{(-b \sin t)^2 + (b \cos t)^2} = b.$$

Hence the speed is constant. This does not imply that the acceleration is zero because the velocity is changing direction. Indeed, the acceleration is

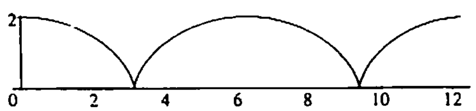
$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = [-b \cos t, \quad -b \sin t, \quad 0].$$

You see that $\mathbf{a}(t) = -\mathbf{r}(t)$. This is the centripetal acceleration. It points from the moving body to the center of the circle (the origin). All these results are in agreement with Example 1 of the text, with $R = b$ and $\omega = 1$. Note that the special $\omega = 1$ makes the centripetal acceleration *equal* to $-\mathbf{r}(t)$, whereas for general ω it will just be *proportional* to it.

9. Cycloid. This curve is shown in the figure. From the given formula

$$\mathbf{r}(t) = [R \sin \omega t + \omega R t, \quad R \cos \omega t + R]$$

you see that $t = 0$ gives $\mathbf{r}(0) = [0, \; 2R]$; this corresponds to the first maximum of the curve in the figure. For $t = \pi/\omega$ you obtain $\mathbf{r}(\pi/\omega) = [\pi R, \; 0]$; this corresponds to the first cusp of the curve in the figure. Differentiating $\mathbf{r}(t)$ twice and substituting those values of t, you obtain the answer on p. A21 in Appendix 2 of the book. If the wheel rolls without slipping on a circle (instead of the x-axis), one obtains an epicycloid or a hypocycloid. These curves are discussed in most engineering handbooks, for instance, on p. 290 of O. V. Eshbach, *Handbook of Engineering Fundamentals*, 3rd ed., New York: Wiley, 1975.



Section 8.6. **Problem 9.** Cycloid with $R = 1$.

Sec. 8.7 Curvature and Torsion of a Curve. *Optional*

Problem Set 8.7. Page 443

7. Curvature. The given curve

$$\mathbf{r}(t) = [a \cos t, \quad a \sin t, \quad ct]$$

is a helix, right-handed when $c > 0$ (Fig. 184 in Sec. 8.5), left-handed when $c < 0$ (Fig. 185), and degenerated to a circle when $c = 0$. In each case the curve lies on a circular cylinder of radius a and the z-axis as the axis of rotation. In Example 1 of the text it is shown how the curvature can be obtained from (1). This, however, presupposes that the curve is represented with the arc length s as parameter or the given representation can be easily converted to such a representation. This will hardly be the case in practice. For this reason, from a practical point of view, formula (1') is more important than formula (1) (which gives a better geometrical characterization of the curvature in principle). In the present case, an application of (1') proceeds as follows.

$$\begin{aligned}
\mathbf{r} &= [a\cos t, \quad a\sin t, \quad ct] \\
\mathbf{r}' &= [-a\sin t, \quad a\cos t, \quad c] \\
\mathbf{r}'' &= [-a\cos t, \ -a\sin t, \quad 0] \\
\mathbf{r}'\bullet\mathbf{r}' &= a^2\sin^2 t + a^2\cos^2 t + c^2 = a^2 + c^2 \\
\mathbf{r}'\bullet\mathbf{r}'' &= a^2\sin t\cos t - a^2\sin t\cos t + 0 = 0 \\
\mathbf{r}''\bullet\mathbf{r}'' &= a^2\cos^2 t + a^2\sin^2 t = a^2.
\end{aligned}$$

Hence the numerator in (1') is

$$\sqrt{(a^2+c^2)a^2} = a\sqrt{a^2+c^2}$$

and the denominator is $(a^2 + c^2)^{3/2}$. The quotient is the curvature

$$\kappa = a/(a^2+c^2).$$

For a circle of radius a you have $c = 0$ and get the curvature $\kappa = a/a^2 = 1/a$, the reciprocal of the radius, as expected. The amount of work was not much more than that in Example 1. In general the expression for s will be cumbersome or will even be given by a nonelementary integral, so that (1') must be used instead of (1).

13. Torsion. Use that for a plane curve, $\mathbf{u}$ and thus $\mathbf{u}'$ in (2) and (3) lie in the plane of the curve (when you choose a point of the curve as the initial point of these vectors). Use what this implies with respect to the cross product in (3), and then take a look at (4), from which you will see the result without any calculation.

15. Torsion. Use (4''') and straightforward calculation. By repeated differentiation you get for the determinant

$$\begin{vmatrix} 1 & 2t & 3t^2 \\ 0 & 2 & 6t \\ 0 & 0 & 6 \end{vmatrix} = 12.$$

The denominator requires more effort, but the expression

$$(1+4t^2+9t^4)(4+36t^2) - (4t+18t^3)^2$$

first obtained simplifies nicely by cancellation of terms.

Sec. 8.8 Review from Calculus in Several Variables. *Optional*

Problem Set 8.8. Page 446

7. Chain rule. The result is obtained by applying the chain rule and simplifying. From $w = xy$ and $x = e^u\cos v$, $y = e^u\sin v$ you obtain

$$\begin{aligned}
w_u = x_u y + x y_u &= e^u\cos v\, e^u\sin v + e^u\cos v\, e^u\sin v \\
&= 2e^{2u}\cos v\sin v = e^{2u}\sin 2v
\end{aligned}$$

and similarly for the partial derivative of w with respect to v. In the present case, substitution and differentiation is much simpler; you obtain

$$w = e^{2u}\cos v \sin v = \frac{1}{2}e^{2u}\sin 2v$$

and from this by differentiation and use of the chain rule

$$w_u = e^{2u}\sin 2v, \qquad w_v = e^{2u}\cos 2v.$$

Sec. 8.9 Gradient of a Scalar Field. Directional Derivative

Problem Set 8.9. Page 452

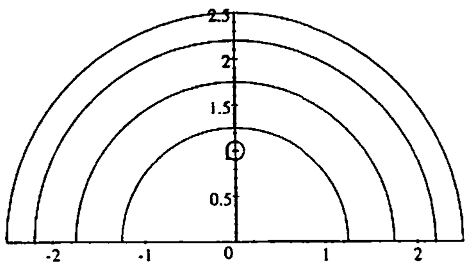
7. **Gradient.** Heat flows from higher to lower temperatures. If the temperature is given by $T(x,y,z)$, the isotherms are the surfaces $T = const$, and the direction of heat flow is the direction of $-\text{grad}\,T$. For $T = z/(x^2+y^2)$ you obtain, using the chain rule,

$$-\text{grad}\,T = -[-2xz/(x^2+y^2)^2, \quad -2yz/(x^2+y^2)^2, \quad 1/(x^2+y^2)]$$
$$= (1/(x^2+y^2)^2)[2xz, \quad 2yz, \quad -(x^2+y^2)].$$

The given point P has the coordinates $x = 0, y = 1, z = 2$. Hence at P,

$$-\text{grad}\,T(P) = [0, \quad 4, \quad -1].$$

(In the first printing of the textbook, the last minus sign is missing.) The figure shows the isotherms in the plane $z = 2$, that is, the circles of intersection of the paraboloids $T = c = const$ (thus $z = c(x^2+y^2)$) with the horizontal plane $z = 2$. The point P is marked by a small circle on the vertical y-axis.



Section 8.9. **Problem 9**. Isotherms in the horizontal plane $z = 2$

17. **Normal of a curve.** $f(x, y) = x^2 + y^2 = 25$ represents a circle of radius 5 with the center at the origin. A normal vector to the circle is

$$\mathbf{N} = \text{grad}\,f = [2x, \quad 2y].$$

At P : $(3, 4)$ this becomes

$$\mathbf{N}(P) = [6, \quad 8].$$

This normal vector has the length 10. Hence a unit normal vector of the circle at P is

$$\mathbf{n} = [0.6, \quad 0.8]$$

(and the other is $-\mathbf{n} = -[0.6, \quad 0.8]$).

23. **Potential.** Not every vector function (vector field) has a potential, that is, not every vector function $\mathbf{v}(x, y, z)$ can be obtained as the gradient of a scalar function $f(x, y, z)$.

In the present case, $\mathbf{v}(x,y) = [xy, \quad 2xy]$. If $\mathbf{v}$ had a potential, you should have $\mathbf{v} = \text{grad}\,f = [f_x, \quad f_y]$. Give it a trial. From the first component you obtain the condition

$$f_x = xy.$$

By integration with respect to x,

$$f = \frac{1}{2}x^2 y + g(y).$$

To understand this step, take the partial derivative of f with respect to x. Similarly, from the second component you obtain the condition

$$f_y = 2xy.$$

By integration with respect to y,

$$f = xy^2 + h(x).$$

Again, check this by differentiation with respect to y.

Now, if **v** had a potential f, you should be able to find a function $g(y)$ depending only on y and a function $h(x)$ depending only on x such that the two expressions for f agree. You see that this is not possible. From this it follows that **v** has no potential. (In Sec. 8.11 we shall discover a systematic method that will make you independent of trial and error.)

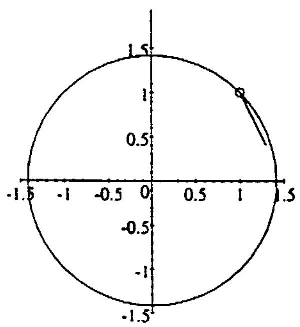
29. Directional derivative. The directional derivative measures the rate of change of a function f in a given direction in the plane or in space. The direction is given by a vector which we call **a**, and the derivative is given by (6'), for reasons explained in the text. In the present problem, $f = x^2 + y^2$, $\mathbf{a} = [2, \ -4]$, and $P : (1, \ 1)$. In (6') you need

$$|\mathbf{a}| = \sqrt{20}, \quad \text{grad} f = [2x, \ 2y], \quad \mathbf{a} \bullet \text{grad} f = 2 \cdot 2x - 4 \cdot 2y = 4x - 8y.$$

Hence at an arbitrary point the derivative of f in the direction of **a** is

$$(4x - 8y)/\sqrt{20} \quad \text{and at } P, \quad (4 - 8)/\sqrt{20} = -2/\sqrt{5} = -0.89.$$

The curves $f = const$ are concentric circles. The figure shows the circle through P, as well as P itself (the small circle) and the direction of **a** (the small segment inside the circle). At P the gradient is $[2, \ 2]$, pointing to the exterior of the circle. The angle between grad f and **a** is greater than 90 degrees; this gives the minus sign in -0.89. The value $|-0.89|$ is relatively small because **a** is not too much different from the tangent direction of the circle at P (for which the directional derivative is zero—explain!).



Section 8.9. Problem 29. Point P, contour line $f = const$ through P, and direction of the vector **a**

Sec. 8.10 Divergence of a Vector Field

Example 2. **v** is the velocity. ρ is the density. $\mathbf{u} = \rho\mathbf{v}$ has the components $u_1 = \rho v_1$, $u_2 = \rho v_2$, $u_3 = \rho v_3$. Hence ρv_2 on p. 455 is the mass flowing in the y-direction, and $(\rho v_2)_y$ is its value at some y, that is, the mass entering the box through the face $y = const$ (the left face in Fig. 199, which also shows the face $x + \Delta x = const$ as the right face and $z + \Delta z = const$ as the upper face.

Problem Set 8.10. Page 456

5. **Divergence.** The physical meaning and practical importance of the divergence of a vector function (a vector field) are explained in the text. div is applied to a vector function **v** and gives a scalar function div **v**, whereas grad is applied to a scalar function f and gives a vector function grad f. Of course, grad and div are not inverses of each other; they are entirely different operations, created because of their applicability in physics, geometry, and elsewhere. The calculation of div **v** by (1) is partial differentiation as in calculus. In the present problem, $\mathbf{v} = [v_1, \ v_2]$ has as its first component

$$v_1 = -y/(x^2+y^2), \quad \text{and you need} \quad \frac{\partial}{\partial x} v_1 = +2xy/(x^2+y^2)^2$$

with the factor $2x$ resulting from the chain rule. The second component is

$$v_2 = x/(x^2+y^2), \quad \text{and you need} \quad \frac{\partial}{\partial y} v_2 = -2xy/(x^2+y^2)^2.$$

The sum of the two expressions on the right is div **v**, and you see that div $\mathbf{v} = 0$.

11. **Incompressible flow.** The velocity vector is $\mathbf{v} = y\mathbf{i} = [y, \ 0, \ 0]$. Hence div $\mathbf{v} = 0$. This shows that the flow is incompressible; see (7) on p. 456. **v** is parallel to the x-axis. In the upper half-plane it points to the right and in the lower to the left. On the x-axis ($y = 0$) it is the zero vector. On each horizontal line $y = const$ it is constant. The speed is the larger the farther away from the x-axis you are. From $\mathbf{v} = y\mathbf{i}$ and the definition of a velocity vector you obtain

$$\mathbf{v} = [dx/dt, \ dy/dt, \ dz/dt] = [y, \ 0, \ 0].$$

This vector equation gives three equations for the corresponding components,

$$dx/dt = y, \quad dy/dt = 0, \quad dz/dt = 0.$$

Integration of $dz/dt = 0$ gives

$$z(t) = c_3 \quad \text{with} \quad c_3 = 0 \text{ for the face } z = 0, \quad c_3 = 1 \text{ for the face } z = 1$$

and $0 < c_3 < 1$ for particles inside the cube. Similarly, by integration of $dy/dt = 0$ you obtain $y(t) = c_2$ with $c_2 = 0$ for the face $y = 0$, $c_2 = 1$ for the face $y = 1$ and $0 < c_2 < 1$ for particles inside the cube.

Finally, $dx/dt = y$ with $y = c_2$ becomes $dx/dt = c_2$. By integration,

$$x(t) = c_2 t + c_1.$$

From this,

$$x(0) = c_1 \quad \text{with} \quad c_1 = 0 \text{ for the face } x = 0, \quad c_1 = 1 \text{ for the face } x = 1.$$

Also

$$x(1) = c_1 + c_2,$$

hence

$$x(1) = c_2 + 0 \text{ for the one face}, \quad x(1) = c_2 + 1 \text{ for the other face}.$$

This shows that the distance of these two parallel faces has remained the same, namely, 1. And since nothing happened in the y- and z-directions, this shows that the volume at time $t = 1$ is still 1, as it should be in the case of incompressibility.

13. **Formulas for the divergence.** These formulas help in simplifying calculations as well as in theoretical work. They follow by straightforward calculations directly from the definitions. For instance, by the definition of the divergence and by product differentiation you obtain

$$\begin{aligned} \operatorname{div}(f\mathbf{v}) &= (fv_1)_x + (fv_2)_y + (fv_3)_z \\ &= f_x v_1 + f_y v_2 + f_z v_3 + f[(v_1)_x + (v_2)_y + (v_3)_z] \\ &= (\operatorname{grad} f) \cdot \mathbf{v} + f \operatorname{div} \mathbf{v}. \end{aligned}$$

15. Laplacian. Problems 14-20 are calculational exercises proposed because of the importance of the Laplacian and the Laplace equation in several branches of physics (see the Index of the book). In general, the Laplacian is itself a function of x, y, z. In Prob. 15 it is constant,

$$\nabla^2 f = 8 + 18 + 2 = 28,$$

since f is so simple. The case of two variables x, y is best handled by complex analysis, as is explained in Chaps. 12 and 16.

Sec. 8.11 Curl of a Vector Field

Example 1. curl $\mathbf{v} = \mathbf{i}(z_y - (3xz)_z) - \mathbf{j}(z_x - (yz)_z) + \mathbf{k}((3xz)_x - (yz)_y) = -3x\mathbf{i} + y\mathbf{j} + (3z - z)\mathbf{k}$.

Problem Set 8.11. Page 459

7. Calculation of the curl. curl is the third of the three operators grad, div, curl designed to meet practical needs in connection with vector and scalar functions and fields. These operators increase the usefulness of vector calculus very significantly, notably in connection with integrals, as will be shown in Chap. 9. For the calculation of the curl use (1), where the "symbolic determinant" helps to memorize the actual formulas for the components given in (1) below the determinant. Given

$$v_1 = x^2yz, \quad v_2 = xy^2z, \quad v_3 = xyz^2.$$

With this you obtain from (1) for the components of $\mathbf{u}$ = curl $\mathbf{v}$ the expressions

$$u_1 = (v_3)_y - (v_2)_z = (xyz^2)_y - (xy^2z)_z = xz^2 - xy^2$$
$$u_2 = (v_1)_z - (v_3)_x = (x^2yz)_z - (xyz^2)_x = x^2y - yz^2$$
$$u_3 = (v_2)_x - (v_1)_y = (xy^2z)_x - (x^2yz)_y = y^2z - x^2z.$$

11. Fluid flow. Both div and curl characterize essential properties of flows, which are usually given in terms of the velocity vector field $\mathbf{v}(x, y, z)$. The present problem is two-dimensional, that is, in each plane $z = const$ the flow is the same. The given velocity is

$$\mathbf{v} = [y, \quad -x].$$

Hence div $\mathbf{v} = 0 + 0 = 0$. This shows that the flow is incompressible (see the previous section). Furthermore, from (1) in the present section you see that the first two components of the curl are zero because they consist of expressions involving v_3, which is zero, or involving the partial derivative with respect to z, which is zero because $\mathbf{v}$ does not depend on z. There remains

$$\text{curl}\,\mathbf{v} = ((v_2)_x - (v_1)_y)\mathbf{k} = (-1 - 1)\mathbf{k} = -2\mathbf{k}.$$

This shows that the flow is not irrotational. Now determine the paths of the particles of the fluid. From the definition of velocity you have

$$v_1 = dx/dt, \qquad v_2 = dy/dt.$$

From this and the given velocity vector $\mathbf{v}$ you see that

$$dx/dt = y \qquad \text{(A)}$$
$$dy/dt = -x. \qquad \text{(B)}$$

This system of differential equations can be solved by a trick worth remembering. The right side of (B) times the left side of (A) is $-x\,dx/dt$. This must equal the right side of (A) times the left side of (B), which is $y\,dy/dt$. Hence

$$-x\,dx/dt = y\,dy/dt.$$

You can now integrate with respect to t on both sides and multiply by -2, obtaining

$$x^2 = -y^2 + c.$$

This shows that the paths of the particles (the streamlines of the flow) are concentric circles

$$x^2 + y^2 = c.$$

This agrees with the fact that (2) on p.458 relates the curl with the rotation vector. In the present case,

$$\mathbf{w} = \frac{1}{2}\operatorname{curl}\mathbf{v} = \mathbf{k},$$

which shows that the z-axis is the axis of rotation.