CHAPTER 1

The Physical Origins of Partial Differential Equations

1. Mathematical Models

Exercise 1. The verification that $u = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/4kt}$ satisfies the heat equation $u_t = ku_{xx}$ is straightforward differentiation. For larger k, the profiles flatten out much faster.

Exercise 2. The problem is straightforward differentiation. Taking the derivatives is easier if we write the function as $u = \frac{1}{2}\ln(x^2 + y^2)$.

Exercise 3. Integrating $u_{xx} = 0$ with respect to x gives $u_x = \phi(t)$ where ϕ is an arbitrary function. Integrating again gives $u = \phi(t)x + \psi(t)$. But $u(0,t) = \psi(t) = t^2$ and $u(1,t) = \phi(t) \cdot 1 + t^2$, giving $\phi(t) = 1 - t^2$. Thus $u(x,t) = (1 - t^2)x + t^2$.

Exercise 4. Leibniz's rule gives

$$u_t = \frac{1}{2}(g(x + ct) + g(x - ct))$$

Thus

$$u_{tt} = \frac{c}{2}(g'(x + ct) - g'(x - ct))$$

In a similar manner

$$u_{xx} = \frac{1}{2c}(g'(x + ct) - g'(x - ct))$$

Thus $u_{tt} = c^2 u_{xx}$.

Exercise 5. If $u = e^{at}\sin bx$ then $u_t = ae^{at}\sin bx$ and $u_{xx} = -b^2 e^{at}\sin bx$. Equating gives $a = -b^2$.

Exercise 6. Letting $v = u_x$ the equation becomes $v_t + 3v = 1$. Multiply by the integrating factor e^{3t} to get

$$\frac{\partial}{\partial t}(ve^{3t}) = e^{3t}$$

Integrate with respect to t to get

$$v = \frac{1}{3} + \phi(x)e^{-3t}$$

where ϕ is an arbitrary function. Thus

$$u = \int v dx = \frac{1}{3}x + \Phi(x)e^{-3t} + \Psi(t)$$

Exercise 7. Let $w = e^u$ or $u = \ln w$ Then $u_t = w_t/w$ and $u_x = w_x/w$, giving $w_{xx} = w_{xx}/w - w_x^2/w^2$. Substituting into the PDE for u gives, upon cancellation, $w_t = w_{xx}$.

Exercise 8. It is straightforward to verify that $u = \arctan(y/x)$ satisfies the Laplace equation. We want $u \to 1$ as $y \to 0$ $(x > 0)$, and $u \to -1$ as $y \to 0$ $(x < 0)$. So try

$$u = 1 - \frac{2}{\pi} \arctan \frac{y}{x}$$

We want the branch of $\arctan z$ with $0 < \arctan z < \pi/2$ for $z > 0$ and $\pi/2 < \arctan z < \pi$ for $z < 0$.

Exercise 9. Differentiate under the integral sign to obtain

$$u_{xx} = \int_0^\infty -\xi^2 c(\xi) e^{-\xi y} \sin(\xi x) d\xi$$

and

$$u_{yy} = \int_0^\infty \xi^2 c(\xi) e^{-\xi y} \sin(\xi x) d\xi$$

Thus

$$u_{xx} + u_{yy} = 0$$

.

Exercise 10. In preparation.

2. Conservation Laws

Exercise 1. Since $A = A(x)$ depends on x, it cannot cancel from the conservation law and we obtain

$$A(x)u_t = -(A(x)\phi)_x + A(x)f$$

Exercise 2. The solution to the initial value problem is $u(x,t) = e^{-(x-ct)^2}$. When $c = 2$ the wave forms are bell-shaped curves moving to the right at speed two.

Exercise 3. Letting $\xi = x - ct$ and $\tau = t$, the PDE $u_t + cu_x = -\lambda u$ becomes $U_\tau = -\lambda U$ or $U = \phi(\xi)e^{-\lambda t}$. Thus

$$u(x,t) = \phi(x - ct)e^{-\lambda t}$$

Exercise 4. In the new dependent variable w the equation becomes $w_t + cw_x = 0$.

Exercise 5. In preparation.

Exercise 6. From Exercise 3 we have the general solution $u(x,t) = \phi(x-ct)e^{-\lambda t}$. For $x > ct$ we apply the initial condition $u(x,0) = 0$ to get $\phi \equiv 0$. Therefore $u(x,t) = 0$ in $x > ct$. For $x < ct$ we apply the boundary condition $u(0,t) = g(t)$ to get $\phi(-ct)e^{-\lambda t} = g(t)$ or $\phi(t) = e^{\lambda t/c}g(-t/c)$. Therefore $u(x,t) = g(t-x/c)e^{-\lambda x/c}$ in $0 \le x < ct$.

Exercise 7. Making the transformation of variables $\xi = x - t$, $\tau = t$, the PDE becomes $U_\tau - 3U = \tau$, where $U = U(\xi,\tau)$. Multiplying through by the integrating factor $\exp(-3\tau)$ and then integrating with respect to τ gives

$$U = -\left(\frac{\tau}{3} + \frac{1}{9}\right) + \phi(\xi)e^{3\tau}$$

or

$$u = -\left(\frac{t}{3} + \frac{1}{9}\right) + \phi(x-t)e^{3t}$$

Setting $t = 0$ gives $\phi(x) = x^2 + 1/9$. Therefore

$$u = -\left(\frac{t}{3} + \frac{1}{9}\right) + ((x-t)^2 + \frac{1}{9})e^{3t}$$

Exercise 8. Letting $n = n(x,t)$ denote the concentration in mass per unit volume, we have the flux $\phi = cn$ and so we get the conservation law

$$n_t + cn_x = -r\sqrt{n} \quad 0 < x < l,\ t > 0$$

The initial condition is $u(x,0) = 0$ and the boundary condition is $u(0,t) = n_0$. To solve the equation go to characteristic coordinates $\xi = x - ct$ and $\tau = t$. Then the PDE for $N = N(\xi,\tau)$ is $N_\tau = -r\sqrt{N}$. Separate variables and integrate to get

$$2\sqrt{N} = -r\tau + \Phi(\xi)$$

Thus

$$2\sqrt{n} = -rt + \Phi(x-ct)$$

Because the state ahead of the leading signal $x = ct$ is zero (no nutrients have arrived) we have $u(x,t) \equiv 0$ for $x > ct$. For $x < ct$, behind the leading signal, we compute Φ from the boundary condition to be $\Phi(t) = 2\sqrt{n_o} - rt/c$. Thus, for $0 < x < ct$ we have

$$2\sqrt{n} = -rt + 2\sqrt{n_0} - \frac{r}{c}(x-ct)$$

Along $x = l$ we have $n = 0$ up until the signal arrives, i.e., for $0 < t < l/c$. For $t > l/c$ we have

$$n(l,t) = (\sqrt{n_0} - \frac{rl}{2c})^2$$

Exercise 9. The graph of the function $u = G(x+ct)$ is the graph of the function $y = G(x)$ shifted to the *left* ct distance units. Thus, as t increases the profile $G(x+ct)$ moves to the left at speed c. To solve the equation $u_t - cu_x = F(x,t,u)$ on would transform the independent variables via $x = x + ct$, $\tau = t$.

Exercise 10. The conservation law for traffic flow is

$$u_t + \phi_x = 0$$

If $\phi(u) = \alpha u(\beta - u)$ is chosen as the flux law, then the cars are jammed at the density $u = \beta$, giving no movement or flux; if $u = 0$ there is no flux because there are no cars. The nonlinear PDE is

$$u_t + (\alpha u(\beta - u))_x = 0$$

or

$$u_t + \alpha(\beta - 2u)u_x = 0$$

Exercise 11. Transform to characteristic coordinates $\xi = x - vt, \quad \tau = t$ to get

$$U_\tau = -\frac{\alpha U}{\beta + U}, \quad U = U(\xi, \tau)$$

Separating variables and integrating yields, upon applying the initial condition and simplifying, the implicit equation

$$u - \alpha t - f(x) = \beta \ln(u/f(x))$$

Graphing the right and left sides of this equation versus u (treating x and $t > 0$ as parameters) shows that there are two crossings, or two roots u; the solution is the smaller of the two.

Exercise 12. In preparation.

3. Diffusion

Exercise 1. We have$u_{xx}(6, T) \approx (58 - 2(64) + 72)/2^2 = 0.5$. Since $u_t = ku_{xx} > 0$, the temperatue will increase. We have

$$u_t(T, 6) \approx \frac{u(T + 0.5, 6) - u(T, 6)}{0.5} \approx ku_{xx}(T, 6) \approx 0.02(0.5)$$

This gives $u(T + 0.5, 6) \approx 64.005$.

Exercise 2. Taking the time derivative

$$\begin{aligned} E'(t) &= \frac{d}{dt}\int_0^l u^2 dx = \int_0^l 2uu_t dx = 2k\int_0^l uu_{xx} dx \\ &= 2kuu_x \mid_0^l - 2k\int u_x^2 \leq 0 \end{aligned}$$

Thus E in nonincreasing, so $E(t) \leq E(0) = \int_0^l u_0(x)dx$. Next, if $u_0 \equiv 0$ then $E(0) = 0$. Therefore $E(t) \geq 0, \; E'(t) \leq 0, E(0) = 0$. It follows that $E(t) = 0$. Consequently $u(x, t) = 0$.

Exercise 3. Take

$$w(x, t) = u(x, t) - \frac{h(t) - g(t)}{l}(x - l) - g(t)$$

Then w will satisfy homogeneous boundary conditions. We get the problem

$$\begin{aligned} w_t &= kw_{xx} - F(x,t), \quad 0 < x < l,\ t > 0 \\ w(0,t) &= w(l,t) = 0, \quad t > 0 \\ w(x,0) &= G(x), \quad 0 < x < l \end{aligned}$$

where

$$F(x,t) = \frac{l-x}{l}(h'(t) - g'(t)) - g'(t), \quad G(x) = u(x,0) - \frac{h(0)-g(0)}{l}(x-l) - g(0)$$

Exercise 4. This is a straightforward calculation.

Exercise 5. The steady state problem for $u = u(x)$ is

$$ku'' + 1 = 0, \quad u(0) = 0,\ \ u(1) = 1$$

Solving this boundary value problem by direct integration gives the steady state solution

$$u(x) = -\frac{1}{2k}x^2 = (1 + \frac{1}{2k})x$$

which is a concave down parabolic temperature distribution.

Exercise 6. The steady-state heat distribution $u = u(x)$ satisfies

$$ku'' - au = 0, \quad u(0) = 1,\ u(1) = 1$$

The general solution is $u = c_1 \cosh \sqrt{a/k}x + c_2 \sinh \sqrt{a/k}x$. The constants c_1 and c_2 can be determined by the boundary conditions.

Exercise 7. The boundary value problem is

$$\begin{aligned} u_t &= Du_{xx} + ru(1 - u/K), \quad 0 < x < l,\ t > 0 \\ u_x(0,t) &= u_x(l,t) = 0, \quad t > 0 \\ u(x,0) &= ax(l-x), \quad 0 < x < l \end{aligned}$$

For long times we expect a steady state density $u = u(x)$ to satisfy $-Du'' + ru(1 - u/K) = 0$ with insulated boundary conditions $u'(0) = u'(l) = 0$. There are two obvious solutions to this problem, $u = 0$ and $u = K$. From what we know about the logistics equation

$$\frac{du}{dt} = ru(1 - u/K)$$

(where there is no spatial dependence and no diffusion, and $u = u(t)$), we might expect the the solution to the problem to approach the stable equilibrium $u = K$. In drawing profiles, note that the maximum of the initial condition is $al^2/4$. So the two cases depend on whether this maximum is below the carrying capacity or above it. For example, in the case $al^2/4 < K$ we expect the profiles to approach $u = K$ from below.

Exercise 8 These facts are directly verified.

4. PDE Models in Biology

Exercise 1. We have

$$\begin{aligned} u_t &= (D(u)u_x)_x = D(u)u_{xx} + D(u)_x u_x \\ &= D(u)u_{xx} + D'(u)u_x u_x. \end{aligned}$$

Exercise 2. The steady state equation is $(Du')' = 0$, where $u = u(x)$. If $D =$ constant then $u'' = 0$ which has a linear solution $u(x) = ax + b$. Applying the two end conditions ($u(0) = 4$ and $u'(2) = 1$) gives $b = 4$ and $a = 1$. Thus $u(x) = x + 4$. The left boundary condition means the concentration is held at the value $u = 4$, and the right boundary condition means $-Du'(2) = -D$, meaning that the flux is $-D$. So matter is entering the system at $L = 2$ (moving left). In the second case we have

$$\left(\frac{1}{1+x}u'\right)' = 0.$$

Therefore

$$\frac{1}{1+x}u' = a$$

or

$$u' = a(1+x).$$

The right boundary condition gives $a = 1/3$. Integrating again and applying the left boundary condition gives

$$u(x) = \frac{1}{3}x + \frac{1}{6}x^2 + 4.$$

In the third case the equation is

$$(uu')' = 0,$$

or $uu' = a$.This is the same as

$$\frac{1}{2}(u^2)' = a,$$

which gives

$$\frac{1}{2}u^2 = ax + b.$$

¿From the left boundary condition $b = 8$. Hence

$$u(x) = \sqrt{2ax + 16}.$$

Now the right boundary condition can be used to obtain the other constant a . Proceeding,

$$u'(2) = \frac{a}{2\sqrt{a+4}} = 1.$$

Thus $a = 2 + \sqrt{20}$.

Exercise 3. The general solution of $Du'' - cu' = 0$ is $u(x) = a + be^{cx/D}$. In the second case the equation is $Du'' - cu' + ru = 0$. The roots of the characteristic polynomial are

$$\lambda_{\pm} = \frac{c}{2D} \pm \frac{\sqrt{c^2 - 4Dr}}{2D}.$$

There are three cases, depending upon upon the discriminant c^2-4Dr. If $c^2-4Dr = 0$ then the roots are equal ($\frac{c}{2D}$) and the general solution has the form

$$u(x) = ae^{cx/2D} + bxe^{cx/2D}.$$

If $c^2 - 4Dr > 0$ then there are two real roots and the general solution is

$$u(x) = ae^{\lambda_+ x} + be^{\lambda_- x}.$$

If $c^2 - 4Dr < 0$ then the roots are complex and the general solution is given by

$$u(x) = ae^{cx/2D}\left(a\cos\frac{\sqrt{4Dr-c^2}}{2D}x + b\sin\frac{\sqrt{4Dr-c^2}}{2D}x\right).$$

Exercise 4. If u is the concentration, use the notation $u = v$ for $0 < x < L/2$, and $u = w$ for $L/2 < x < L$.The PDE model is then

$$\begin{aligned} v_t &= v_{xx} - \lambda v, \quad 0 < x < L/2, \\ w_t &= w_{xx} - \lambda w, \quad L/2 < x < L. \end{aligned}$$

The boundary conditions are clearly $v(0,t) = w(L,t) = 0$, and continuity at the midpoint forces $v(L/2) = w(L/2)$. To get a condition for the flux at the midpoint we take a small interval $[L/2-\epsilon, L/2+\epsilon]$. The flux in at the left minus the flux out at the right must equal 1, the amount of the source. In symbols,

$$-v_x(L/2-\epsilon, t) + w_x(L/2+\epsilon) = 1.$$

Taking the limit as $\epsilon \to 0$ gives

$$-v_x(L/2, t) + w_x(L/2) = 1.$$

So, there is a jump in the derivative of the concentration at the point of the source. The steady state system is

$$\begin{aligned} v'' - \lambda v &= 0, \quad 0 < x < L/2, \\ w'' - \lambda w &= 0, \quad L/2 < x < L, \end{aligned}$$

with conditions

$$\begin{aligned} v(0) &= w(L) = 0, \\ v(L/2) &= w(L/2), \\ -v'(L/2) + w'(L/2) &= 1. \end{aligned}$$

Let $r = \sqrt{\lambda}$. The general solutions to the DEs are

$$v = ae^{rx} + be^{-rx}, \quad w = ce^{rx} + de^{-rx}.$$

The four constants a, b, c, d may be determined by the four subsiduary conditions.

Exercise 5. The steady state equations are

$$\begin{aligned} v'' &= 0, \quad 0 < x < \xi, \\ w'' &= 0, \quad \xi < x < L, \end{aligned}$$

The conditions are

$$\begin{aligned} v(0) &= w(L) = 0, \\ v(\xi) &= w(\xi), \\ -v'(\xi) + w'(\xi) &= 1. \end{aligned}$$

Use these four conditions to determine the four constants in the general solution to the DEs. We finally obtain the solution

$$v(x) = \frac{\xi - L}{L}x, \quad w(x) = \frac{x - L}{L}\xi.$$

Exercise 6. The equation is

$$u_t = u_{xx} - u_x, \quad 0 < x < L$$

(With no loss of generality we have taken the constants to be equal to one). Integrating from $x = 0$ to $x = L$ gives

$$\int_0^L u_t dx = \int_0^L u_{xx} dx - \int_0^L u_x dx.$$

Using the fundamental theorem of calculus and bringing out the time derivative gives

$$\frac{\partial}{\partial t}\int_0^L u dx = u_x(L,t) - u_x(0,t) - u(L,t) + u(0,t) = -flux(L,t) + flux(0,t) = 0.$$

Exercise 7. The model is

$$\begin{aligned} u_t &= Du_{xx} + agu_x, \\ u(\infty, t) &= 0, \quad -Du_x(0,t) - agu(0,t) = 0. \end{aligned}$$

The first boundary condition states the concentration is zero at the bottom (a great depth), and the second condition states that the flux through the surface is zero, i.e., no plankton pass through the surface. The steady state equation is

$$Du'' + agu' = 0,$$

which has general solution

$$u(x) = A + Be^{-agx/D}.$$

The condition $u(\infty) = 0$ forces A=0. The boundary condition $-Du'(0) - agu(0) = 0$ is satisfied identically. So we have

$$u(x) = u(0)e^{-agx/D}.$$

Exercise 8. Notice that the dimensions of D are length-squared per unit time, so we use $D = L^2/T$, where L and T are the characteristic length and time, respectively. For sucrose,

$$L = \sqrt{DT} = \sqrt{(4.6 \times 10^{-6})(60 \times 60 \times 24)} = 0.63 \text{ cm}.$$

For the insect,

$$T = \frac{L^2}{D} = \frac{10000^2}{2.0 \times 10^{-1}} = 5 \times 10^8 \text{ sec} \;=\; 6000 \text{ days}.$$

Exercise 9. Solve each of the DEs, in linear, cylindrical, and spherical coordinates, respectively:

$$\begin{aligned} Du'' &= 0, \\ \frac{D}{r}(ru')' &= 0, \\ \frac{D}{\rho^2}(\rho^2 u')' &= 0. \end{aligned}$$

Exercise 10. Let $q = 1 - p$ and begin with the equation

$$u(x, t+\tau) - u(x,t) = pu(x-h,t) + qu(x+h,t) - pu(x,t) - qu(x,t),$$

or

$$u(x,t+\tau) = pu(x-h,t) + qu(x+h,t).$$

Expanding in Taylor series (u and its derivatives are evaluated at (x,t)),

$$u + u_t\tau + \cdots = pu - pu_x h + \frac{1}{2}pu_{xx}h^2 + qu + qu_x h + \frac{1}{2}qu_{xx}h^2 + \cdots,$$

or

$$\tau u_t = (1-2p)u_x h + \frac{1}{2}u_{xx}h^2 + \cdots.$$

Then

$$u_t = (1-2p)u_x\frac{h}{\tau} + \frac{h^2}{2\tau}u_{xx} + \cdots.$$

Taking the limit as $h, \tau \to 0$ gives

$$u_t = cu_x + Du_{xx},$$

with appropriately defined special limits.

Exercise 11. Similar to the example on page 30.

Exercise 12. Draw two concentric circles of radius $r = a$ and $r = b$. The total amount of material in between is

$$2\pi\int_a^b u(r,t)rdr.$$

The flux through the circle $r = a$ is $-2\pi aDu_r(a,t)$ and the flux through r=b is $-2\pi bDu_r(b,t)$. The time rate of change of the total amount of material in between equals the flux in minus the flux out, or

$$2\pi\frac{\partial}{\partial t}\int_a^b u(r,t)rdr. = -2\pi aDu_r(a,t) + 2\pi bDu_r(b,t),$$

or

$$\int_a^b u_t(r,t)rdr = \int_a^b D\frac{\partial}{\partial r}(ru_r(r,t))dr.$$

Since a and b are arbitrary,

$$u_t(r,t)r = D\frac{\partial}{\partial r}(ru_r(r,t)).$$

5. Vibrations and Acoustics

Exercise 1. In the vertical force balance the term $-\int_0^l g\rho_0(x)dx$ should be added to the right side to account for gravity acting downward.

Exercise 2. In the vertical force balance the term $-int_0^l \rho_0(x) k u_t dx$ should be added to the right side to account for damping.

Exercise 4. The initial conditions are found by setting $t = 0$ to obtain

$$u_n(x, 0) = \sin \frac{n\pi x}{l}$$

The temporal frequency of the oscillation is $\omega \equiv n\pi c/l$ with period $2\pi/\omega$. As the length l increases, the frequency decreases, making the period of oscillation longer. The tension is τ satisfies $\rho_0 c^2 = \tau$. As τ increases the frequency increases so the oscillations are faster. Thus, tighter strings produce higher frequencies; longer string produce lower frequencies.

Exercise 5. The calculation follows directly by applying the hint.

Exercise 6. We have

$$c^2 = \frac{dp}{d\rho} = k\gamma\rho^{\gamma-1} = \frac{\gamma p}{\rho}$$

Exercise 8. Assume $\tilde{\rho}(x, t) = F(x - ct)$, a right traveling wave, where F is to be determined. Then this satisfies the wave equation automatically and we have $\tilde{\rho}(0, t) = F(-ct) = 1 - 2\cos t$, which gives $F(t) = 1 - 2\cos(-t/c)$. Then

$$\tilde{\rho}(x, t) = 1 - 2\cos(t - x/c)$$

6. Quantum Mechanics

Exercise 1. This is a straightforward verification using rules of differentiation.

Exercise 2. Observe that

$$|\Psi(x, t)| = |y(x)||Ce^{-iEt/\hbar}| = C|y(x)|$$

Exercise 3. Substitute $y = e^{-ax^2}$ into the Schrödinger equation to get

$$E = \hbar^2 a/m, \quad a^2 = \frac{1}{4} mk/\hbar^2$$

This gives

$$y(x) = Ce^{-0.5\sqrt{mk}x^2/\hbar}$$

To find C impose the normality condition $\int_R y(x)^2 dx = 1$ and obtain

$$C = \left(\frac{mk}{2\pi\hbar}\right)^{1/4}$$

Exercise 5. Let $b^2 \equiv 2mE/\hbar^2$. Then the ODE

$$y'' + b^y = 0$$

has general solution

$$y(x) = A\sin bx + B\cos bx$$

The condition $y(0) = 0$ forces $B = 0$. The condition $y(\pi) = 0$ forces $\sin b\pi = 0$, and so (assuming $B \neq 0$) b must be an integer, i.e., $n^2 \equiv 2mE/\hbar^2$. The probability density functions are

$$y_n^2(x) = B^2 \sin^2 nx$$

with the constants B chosen such that $\int_0^\pi y^2 dx = 1$. One obtains $B = \sqrt{2/\pi}$. The probabilities are

$$\int_0^0 .25 y_n^2(x)dx$$

7. Heat Flow in Three Dimensions

In these exercises we use the notation ∇ for the gradient operation grad.

Exercise 1. We have

$$div(\nabla u) = div(u_x, u_y, u_z) = u_{xx} + u_{yy} + u_{zz}$$

Exercise 2. For nonhomogeneous media the conservation law (1.40) becomes

$$c\rho u_t - div(K(x, y, z)\nabla u) = f$$

So the conductivity K cannot be brought out of the divergence.

Exercise 3. Integrate both sides of the PDE over Ω to get

$$\begin{aligned}\int_\Omega f dV &= \int_\Omega -K\Delta u dV = \int_\Omega -K div(\nabla u)dV \\ &= \int_{\partial\Omega} -K\ \nabla u \cdot n dA = \int_{\partial\Omega} g dA\end{aligned}$$

The left side is the net heat generated inside Ω from sources; the right side is the net heat passing through the boundary. For steady-state conditions, these must balance.

Exercise 4. Follow the suggestion and use the divergence theorem.

Exercise 5. Follow the suggestion in the hint to obtain

$$-\int_\Omega \nabla u \cdot \nabla u\, dV = \lambda \int_\Omega u^2 dV$$

Both integrals are nonnegative, and so λ must be nonpositive. Note that $\lambda \neq 0$; otherwise $u = 0$.

Exercise 6. This calculation is on page 137 of the text.

Exercise 7. In preparation.

8. Laplace's Equation

Exercise 1. The temperature at the origin is the average value of the temperature around the boundary, or

$$u(0,0) = \frac{1}{2\pi}\int_0^{2\pi}(3\sin 2\theta + 1)d\theta$$

The maximum and minimum must occur on the boundary. The function $f(\theta) = 3\sin 2\theta + 1$ has an extremum when $f'(\theta) = 0$ or $6\cos 2\theta = 0$. The maxima then occur at $\theta = \pi/4, 5\pi/4$ and the minima occur at $\theta = 3\pi/4, 7\pi/4$.

Exercise 2. We have $u(x) = ax + b$. But $u(0) = b = T_0$ and $u(l) = al + T_0 = T_1$, giving $a = (T_1 - T_0)/l$. Thus

$$u(x) = \frac{T_1 - T_0}{l}x + T_0$$

which is a straight line connecting the endpoint temperatures. When the right end is insulated the boundary condition becomes $u'(l) = 0$. Now we have $a = 0$ and $b = T_0$ which gives the constant distribution

$$u(x) = T_0$$

Exercise 3. The boundary value problem is

$$-((1 + x^2)u')' = 0, \quad u(0) = 1,\ u(1) = 4$$

Integrating gives $(1 + x^2)u' = c_1$, or $u' = c_1/(1 + x^2)$. Integrating again gives

$$u(x) = c_1 \arctan x + c_2$$

But $u(0) = c_2 = 1$, and $u(1) = c_1 \arctan 1 + 1 = 4$. Then $c_1 = 6/\pi$.

Exercise 4. Assume $u = u(r)$ where $r = \sqrt{x^2 + y^2}$. The chain rule gives

$$u_x = u'(r)r_x = u'(r)\frac{x}{\sqrt{x^2 + y^2}}$$

Then, differentiating again using the product rule and the chain rule gives

$$u_{xx} = \frac{x^2}{r^2}u''(r) + \frac{y^2}{r^3}$$

Similarly

$$u_{yy} = \frac{y^2}{r^2}u''(r) + \frac{x^2}{r^3}$$

Then

$$\Delta u = u'' + \frac{1}{r}u' = 0$$

This last equation can be written

$$(ru')' = 0$$

which gives the radial solutions

$$u = a \ln r + b$$

which are logrithmic. The one dimensional Laplace equation $u'' = 0$ has linear solutions $u = ax + b$, and the three dimensional Laplace equation has algebraic power solutions $u = a\rho^{-1} + b$. In the two dimensional problem we have $u(r) = a \ln r + b$ with $u(1) = 0$ and $u(2) = 10$. Then $b = 0$ and $a = 10/\ln 2$. Thus

$$u(r) = 10\frac{\ln r}{\ln 2}$$

Exercise 5. We have $\nabla V = E$. Taking the divergence of both sides gives $\Delta V = div \nabla V = div\ E = 0$.

9. Classification of PDEs

Exercise 1. The equation

$$u_{xx} + 2ku_{xt} + k^2 u_{tt} = 0$$

is parabolic because $B^2 - 4AC = 4k^2 - 4k^2 = 0$. Make the transformation

$$x = \xi, \quad \tau = x - (B/2C)t = x - t/k$$

Then the PDE reduces to the canonical form $U_{\xi\xi} = 0$. Solve by direct integration. Then $U_\xi = f(\tau)$ and $U = \xi f(\tau) + g(\tau)$. Therefore

$$u = xf(x - t/k) + g(x - t/k)$$

where f, g are arbitrary functions.

Exercise 2. The equation $2u_{xx} - 4u_{xt} + u_x = 0$ is hyperbolic. Make the transformation $\xi = 2x + t, \tau = t$ and the PDE reduces to the canonical form

$$U_{\xi\tau} - \frac{1}{4}U_\xi = 0$$

Make the substitution $V = U_\xi$ to get $V_\tau = 0.25V$, or $V = F(\xi)e^{\tau/4}$. Then $U = f(\xi)e^{\tau/4} + g(\tau)$, giving

$$u = f(2x + t)e^{t/4} + g(t)$$

Exercise 3. The equation $xu_{xx} - 4u_{xt} = 0$ is hyperbolic. Under the transformation $\xi = t$, $\tau = t + 4\ln x$ the equation reduces to

$$U_{\xi\tau} + \frac{1}{4}U = 0$$

Proceeding as in Exercise 2 we obtain

$$u = f(t + 4\ln x)e^{t/4} + g(t)$$

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Exercise 5. The discriminant for the PDE

$$u_{xx} - 6u_{xy} + 12u_{yy} = 0$$

is $D = -12$ is negative and therefore it is elliptic. Take $b = 1/4 + \sqrt{3}i/12$ and define the complex transformation $\xi = x + by$, $\tau = x + \bar{b}y$. Then take

$$\alpha = \frac{1}{2}(\xi + \tau) = x + \frac{1}{4}y$$

and

$$\beta = \frac{1}{2i}(\xi - \tau) = \frac{\sqrt{3}}{12}y$$

Then the PDE reduces to Laplace's equation $u_{\alpha\alpha} + u_{\beta\beta} = 0$.

Exercise 6. In preparation.

Exercise 7. In preparation.