CHAPTER 2

Partial Differential Equations on Unbounded Domains

1. Cauchy Problem for the Heat Equation

Exercise 1a. Making the transformation $r = (x-y)/\sqrt{4kt}$ we have

$$\begin{aligned} u(x,t) &= \int_{-1}^{1} \frac{1}{\sqrt{4\pi kt}} e^{-(x-y)^2/4kt} dy \\ &= -\int_{(x+1)/\sqrt{4kt}}^{(x-1)/\sqrt{4kt}} \frac{1}{\sqrt{\pi}} e^{-r^2} dr \\ &= \frac{1}{2}\left(erf\ \left((x+1)/\sqrt{4kt}\right) - erf\ \left((x-1)/\sqrt{4kt}\right)\right) \end{aligned}$$

Exercise 1b. We have

$$u(x,t) = \int_0^\infty \frac{1}{\sqrt{4\pi kt}} e^{-(x-y)^2/4kt} e^{-y} dy$$

Now complete the square in the exponent of e and write it as

$$\begin{aligned} -\frac{(x-y)^2}{4kt} - y &= -\frac{x^2 - 2xy + y^2 + 4kty}{4kt} \\ &= -\frac{y + 2kt - x)^2}{4kt} + kt - x \end{aligned}$$

Then make the substitution in the integral

$$r = \frac{y + 2kt - x}{\sqrt{4kt}}$$

Then

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{\pi}} e^{kt-x} \int_{(2kt-x)/\sqrt{4kt}}^{\infty} e^{-r^2} dr \\ &= \frac{1}{2} e^{kt-x} \left(1 - erf\left((2kt-x)/\sqrt{4kt}\right)\right) \end{aligned}$$

Exercise 2. We have

$$|u(x,t)| \le \int_R |G(x-y,t)||\phi(y)|dy \le M \int_R G(x-y,t)dy = M$$

Exercise 3. Use

$$\begin{aligned} erf(z) &= \frac{2}{\sqrt{\pi}}\int_0^z e^{-r^2}dr \\ &= \frac{2}{\sqrt{\pi}}\int_0^z (1-r^2+\cdots)dr \\ &= \frac{2}{\sqrt{\pi}}(z-\frac{z^3}{3}+\cdots) \end{aligned}$$

This gives

$$w(x_0,t) = \frac{1}{2} + \frac{x_0}{\pi\sqrt{t}} + \cdots$$

Exercise 4. The verification is straightforward. We guess the Green's function in two dimensions to be

$$\begin{aligned} g(x,y,t) &= G(x,t)G(y,t) \\ &= \frac{1}{\sqrt{4\pi kt}}e^{-x^2/4kt}\frac{1}{\sqrt{4\pi kt}}e^{-y^2/4kt} \\ &= \frac{1}{4\pi kt}e^{-(x^2+y^2)/4kt} \end{aligned}$$

where G is the Green's function in one dimension. Thus g is the temperature distribution caused by a point source at $(x,y)=(0,0)$ at $t=0$. This guess gives the correct expression. Then, by superposition, we have the solution

$$u(x,y,t) = \int_{R^2}\frac{1}{4\pi kt}e^{-((x-\xi)^2+(y-\eta)^2)/4kt}\psi(\xi,\eta)d\xi d\eta$$

Exercise 6. Using the substitution $r = x/\sqrt{4kt}$ we get

$$\int_R G(x,t)dx = \frac{1}{\sqrt{\pi}}\int_R e^{-r^2}dr = 1$$

2. Cauchy Problem for the Wave Equation

Exercise 1. Applying the initial conditions to the general solution gives the two equations

$$F(x)+G(x)=f(x), \quad -cF'(x)+cG'(x)=g(x)$$

We must solve these to determine the arbitrary functions F and G. Integrate the second equation to get

$$-cF(x)+cG(x) = \int_0^x g(s)ds + C$$

Now we have two linear equations for F and G that we can solve simultaneously.

Exercise 2. Using d'Alembert's formula we obtain

$$\begin{aligned} u(x,t) &= \frac{1}{2c}\int_{x-ct}^{x+ct}\frac{ds}{1+0.25s^2} \\ &= \frac{1}{2c}2\arctan(s/2)\mid_{x-ct}^{x+ct} \\ &= \frac{1}{c}(\arctan((x+ct)/2)-\arctan((x-ct)/2)) \end{aligned}$$

Exercise 3. Let $u = F(x-ct)$. Then $u_x(0,t) = F'(-ct) = s(t)$. Then

$$F(t) = \int_0^t s(-r/c)dr + K$$

Then

$$u(x,t) = -\frac{1}{c}\int_0^{t-x/c} s(y)dy + K$$

Exercise 4. Letting $u = U/\rho$ we have

$$u_{tt} = U_{tt}/\rho, \quad u_\rho = U_\rho/\rho - U/\rho^2$$

and

$$u_{\rho\rho} = U_{\rho\rho}/\rho - 2U_\rho/\rho^2 + 2U/\rho^3$$

Substituting these quantities into the wave equation gives

$$U_{tt} = c^2 U_{\rho\rho}$$

which is the ordinary wave equation with general solution

$$U(\rho,t) = F(\rho-ct) + G(\rho+ct)$$

Then

$$u(\rho,t) = \frac{1}{r}(F(\rho-ct)+G(\rho+ct))$$

As a spherical wave propagates outward in space its energy is spread out over a larger volume, and therefore it seems reasonable that its amplitude decreases.

Exercise 5. The exact solution is, by d'Alembert's formula,

$$u(x,t) = \frac{1}{2}(e^{-|x-ct|}+e^{-|x+ct|}) + \frac{1}{2c}(\sin(x+ct)-\sin(x-ct))$$

Exercise 7. Write

$$v = \int_R H(s,t)u(x,s)ds$$

where

$$h(s,t) = \frac{1}{\sqrt{4\pi t(k/c^2)}}e^{-s^2/(4t(k/c^2))}$$

which is the heat kernel with k replaced by k/c^2. Thus H satisfies

$$H_t - \frac{k}{c^2}H_{xx} = 0$$

Then, we have

$$\begin{aligned} v_t - kv_{xx} &= \int_R (H_t(s,t)u(x,s) - kH(s,t)u_{xx}(x,s))ds \\ &= \int_R (H_t(s,t)u(x,s) - (k/c^2)H(s,t)u_{ss}(x,s))ds \end{aligned}$$

where, in the last step, we used the fact that u satisfies the wave equation. Now integrate the second term in the last expression by parts twice. The generated boundary terms will vanish since H and H_s go to zero as $|s| \to \infty$. Then we get

$$v_t - kv_{xx} = \int_R (H_t(s,t)u(x,s) - (k/c^2)H_{ss}(s,t)u(x,s))ds = 0$$

3. Ill-Posed Problems

Exercise 1. Consider the two problems

$$\begin{aligned} u_t + u_{xx} &= 0, \quad x \in R,\ t > 0 \\ u(x,0) &= f(x), \quad x \in R \end{aligned}$$

If $f(x) = 1$ the solution is $u(x,t) = 1$. If $f(x) = 1 + n^{-1}\sin nx$, which is a small change in initial data, then the solution is

$$u(x,t) = 1 + \frac{1}{n}e^{n^2 t}\sin nx$$

which is a large change in the solution. So the solution does not depend continuoulsy on the initial data.

Exercise 2. Integrating twice, the general solution to $u_{xy} = 0$ is

$$u(x,y) = F(x) + G(y)$$

where F and G are arbitrary functions. Note that the equation is hyperbolic and therefore we expect the problem to be an evolution problem where data is carried forward from one boundary to another; so a boundary value problem should not be well-posed since the boundary data may be incompatible. To observe this, note that

$$u(x,0) = F(x) + G(0) = f(x). \quad u(x,1) = F(x) + G(1) = g(x)$$

where f and g are data imposed along $y = 0$ and $y = 1$, respectively. But these last equations imply that f and g differ by a constant, which may not be true.

Exercise 3. We subtract the two solutions given by d'Alembert's formula, take the absolute value, and use the triangle inequality to get

$$\begin{aligned} |u^1 - u^2| &\le \frac{1}{2}|f^1(x-ct) - f^2(x-ct)| + \frac{1}{2}|f^1(x+ct) - f^2(x+ct)| \\ &\quad + \frac{1}{2c}\int_{x-ct}^{x+ct} |g^1(s) - g^2(s)|ds \\ &\le \frac{1}{2}\delta_1 + \frac{1}{2}\delta_1 + \frac{1}{2c}\int_{x-ct}^{x+ct} \delta_2 ds \\ &= \delta_1 + \frac{1}{2c}\delta_2(2ct) \\ &\le \delta_1 + T\delta_2 \end{aligned}$$

4. Semi-Infinite Domains

Exercise 2. We have

$$u(x,t) = \int_0^\infty (G(x-y,t) - G(x+y,t))dy = erf\ (x/\sqrt{4kt})$$

Exercise 3. For $x > ct$ we use d'Alembert's formula to get

$$u(x,t) = \frac{1}{2}((x-ct)e^{-(x-ct)} + (x+ct)e^{-(x+ct)})$$

For $0 < x < ct$ we have from (2.29) in the text

$$u(x,t) = \frac{1}{2}((x+ct)e^{-(x+ct)} - (ct-x)e^{-(ct-x)})$$

Exercise 4. Letting $w(x,t) = u(x,t) - 1$ we get the problem

$$w_t = kw_{xx}, \quad w(0,t) = 0, \ \ t > 0, \ \ ; w(x,0) = -1, \ \ x > 0$$

Now we can apply the result of the text to get

$$w(x,t) = \int_0^\infty (G(x-y,t) - G(x+y,t))(-1)dy = -erf\ (x/\sqrt{4kt})$$

Then

$$u(x,t) = 1 - erf\ (x/\sqrt{4kt})$$

Exercise 5. The problem is

$$\begin{aligned} u_t = ku_{xx}, \ \ x > 0, \ t > 0 \\ u(x,0) = 7000, \ \ x > 0 \\ u(0,t) = 0, \ \ t > 0 \end{aligned}$$

From Exercise 2 we know the temperature is

$$u(x,t) = 7000\ erf\ (x/\sqrt{4kt}) = 7000\frac{2}{\sqrt{\pi}}\int_0^{x/\sqrt{4kt}} e^{-r^2} dr$$

The geothermal gradient at the current time t_c is

$$u_x(0,t_c) = \frac{7000}{\sqrt{\pi k t_c}} = 3.7 \times 10^{-4}$$

Solving for t gives

$$t_c = 1.624 \times 10^{16} \text{ sec } = 5.15 \times 10^8 \text{ yrs}$$

This gives a very low estimate; the age of the earth is thought to be about 15 billion years.

There are many ways to estimate the amount of heat lost. One method is as follows. At $t = 0$ the total amount of heat was

$$\int_S \rho c u \, dV = 7000 \rho c \frac{4}{3}\pi R^3 = 29321 \rho c R^3$$

where S is the sphere of radius $R = 4000$ miles and density ρ and specific heat c. The amount of heat leaked out can be calculated by integrating the geothermal gradient up to the present day t_c. Thus, the amount leaked out is approximately

$$\begin{aligned}(4\pi R^2)\int_0^{t_c} -K u_x(0,t)dt &= -4\pi R^2 \rho c k (7000)\int_0^{t_c}\frac{1}{\sqrt{\pi k t}}dt \\ &= -\rho c R^2 (1.06 \times 10^{12})\end{aligned}$$

So the ratio of the heat lost to the total heat is

$$\frac{\rho c R^2 (1.06 \times 10^{12})}{29321 \rho c R^3} = \frac{3.62 \times 10^7}{R} = 5.6\%$$

Exercise 6. In preparation.

5. Sources and Duhamel's Principle

Exercise 1. By (2.45) the solution is

$$\begin{aligned}u(x,t) &= \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} \sin s ds \\ &= \frac{1}{c^2}\sin x - \frac{1}{2c^2}(\sin(x-ct) + \sin(x+ct))\end{aligned}$$

Exercise 2. The solution is

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} G(x-y, t-\tau)\sin y \, dy d\tau$$

where G is the heat kernel.

Exercise 3. The problem

$$w_t(x,t,\tau) + c w_x(x,t,\tau) = 0, \quad w(x,0,\tau) = f(x,\tau)$$

has solution (see Chapter 1)

$$w(x,t,\tau) = f(x-ct,\tau)$$

Therefore, by Duhamel's principle, the solution to the original problem is

$$u(x,t) = \int_0^t f(x - c(t-\tau), \tau) d\tau$$

Applying this formula when $f(x,t) = xe^{-t}$ and $c = 2$ gives

$$u(x,t) = \int_0^t (x - 2(t-\tau))e^{-\tau} d\tau$$

This integral can be calculated using integration by parts or a computer algebra program. We get

$$u(x,t) = -(x-2t)(e^{-t} - 1) - 2te^{-t} + 2(1 - e^{-t})$$

6. Laplace Transforms

Exercise 1. Taking the Laplace transform of the PDE gives, using the initial conditions,

$$U_{xx} - \frac{s^2}{c^2}U = -\frac{g}{sc^2}$$

The general solution is

$$U(x,s) = A(s)e^{-sx/c} + B(s)e^{sx/c} + \frac{g}{s^3}$$

To maintain boundedness, set $B(s) = 0$. Now $U(0,s) = 0$ gives $A(s) = -g/s^3$. Thus

$$U(x,s) = -\frac{g}{s^3}e^{-sx/c} + \frac{g}{s^3}$$

is the solution in the transform domain. Now, from a table or computer algebra program,

$$L^{-1}\left(\frac{1}{s^3}\right) = \frac{t^2}{2}, \quad L^{-1}(F(s)e^{-as}) = H(t-a)f(t-a)$$

Therefore

$$L^{-1}\left(\frac{1}{s^3}e^{-xs/c}\right) = H(t - x/c)\frac{(t-x/c)^2}{2}$$

Hence

$$u(x,t) = \frac{gt^2}{2} - gH(t-x/c)\frac{(t-x/c)^2}{2}$$

Exercise 2. Taking the Laplace transform of the PDE while using the initial condition gives, for $U = U(x,y,s)$,

$$U_{yy} - pU = 0$$

The bounded solution of this equation is

$$U = a(x,s)e^{-y\sqrt{s}}$$

The boundary condition at $y = 0$ gives $sU(x,o,s) = -U_x(x,0,s)$ or $a = -a_x$, or

$$a(x,s) = f(s)e^{-xs}$$

The boundary condition at $x = u = 0$ forces $f(s) = 1/s$. Therefore

$$U(x,y,s) = \frac{1}{s}e^{-xs}e^{-y\sqrt{s}}$$

From the table of transforms

$$u(x,y,t) = 1 - erf((y-x)/\sqrt{4t})$$

Exercise 3. Using integration by parts, we have

$$\begin{aligned} L\left(\int_0^t f(\tau)d\tau\right) &= \int_0^\infty \left(\int_0^t f(\tau)d\tau\right) e^{-st}dt \\ &= -\frac{1}{s}\int_0^\infty \left(\int_0^t f(\tau)d\tau\right) \frac{d}{ds}e^{-st}dt \\ &= -\frac{1}{s}\int_0^t f(\tau)d\tau \cdot e^{-st} \mid_0^\infty + \frac{1}{s}\int_0^\infty f(t)e^{-st}dt \\ &= \frac{1}{s}F(s) \end{aligned}$$

Exercise 4. Since $H = 0$ for $x < a$ we have

$$\begin{aligned} L\left(H(t-a)f(t-a)\right) &= \int_a^\infty f(t-a)e^{-st}dt \\ &= \int_0^\infty f(\tau)e^{-s(\tau+a)}d\tau = e^{-as}F(s) \end{aligned}$$

where we used the substitution $\tau = t - a,\ d\tau = dt$.

Exercise 5. The model is

$$\begin{aligned} u_t = u_{xx}, \quad x > 0,\ t > 0 \\ u(x,0) = u_0, \quad x > 0 \\ -u_x(0,t) = 0 - u(0,t) \end{aligned}$$

Taking the Laplace transform of the PDE we get

$$U_{xx} - sU = -u_0$$

The bounded solution is

$$U(x,s) = a(s)e^{-x\sqrt{s}} + \frac{u_0}{s}$$

The radiation boundary condition gives

$$-a(s)\sqrt{s} = a(s) + \frac{u_0}{s}$$

or

$$a(s) = -\frac{u_0}{s(1+\sqrt{s})}$$

Therefore, in the transform domain

$$U(x,s) = -\frac{u_0}{s(1+\sqrt{s})}e^{-x\sqrt{s}} + \frac{u_0}{s}$$

Using a table of Laplace transforms we find

$$u(x,t) = u_0 - u_0\left[erfc\left(\frac{x}{\sqrt{4t}}\right) - erfc\left(\sqrt{t} + \frac{x}{\sqrt{4t}}\right)e^{x+t}\right]$$

where $erfc(z) = 1 - erf(z)$.

Exercise 6. Taking the Laplace transform of the PDE gives, using the initial conditions,

$$U_{xx} - \frac{s^2}{c^2}U = 0$$

The general solution is

$$U(x,s) = A(s)e^{-sx/c} + B(s)e^{sx/c}$$

To maintain boundedness, set $B(s) = 0$. Now The boundary condition at $x = 0$ gives $U(0,s) = G(s)$ which forces $A(s) = G(s)$. Thus

$$U(x,s) = G(s)e^{-sx/c}$$

Therefore, using Exercise 4, we get

$$u(x,t) = H(t - x/c)g(t - x/c)$$

7. Fourier Transforms

Exercise 1. The convolution is calculated from

$$x \star e^{-x^2} = \int_{-\infty}^{\infty} (y-x)e^{-y^2}dy$$

Exercise 2. From the definition we have

$$\begin{aligned}
F^{-1}(e^{-a|\xi|})) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-a|\xi|}e^{-ix\xi}d\xi \\
&= \frac{1}{2\pi}\int_{-\infty}^{0} e^{a\xi}e^{-ix\xi}d\xi + \frac{1}{2\pi}\int_{0}^{\infty} e^{-a\xi}e^{-ix\xi}d\xi \\
&= \frac{1}{2\pi}\int_{-\infty}^{0} e^{a\xi-ix\xi}d\xi + \frac{1}{2\pi}\int_{0}^{\infty} e^{-a\xi-ix\xi}d\xi \\
&= \frac{1}{2\pi}\frac{1}{a-ix}e^{(a-ix)\xi}\;|_{-\infty}^{0} + \frac{1}{2\pi}\frac{1}{-a-ix}e^{(-a-ix)\xi}\;|_{0}^{\infty} \\
&= \frac{a}{\pi}\frac{1}{a^2+x^2}
\end{aligned}$$

Exercise 3a. Using the definition of the Fourier transform

$$2\pi F^{-1}(-\xi) = \int_{-\infty}^{\infty} u(x)e^{-i(-\xi)x}dx = F(u)(\xi)$$

Exercise 3b. From the definition,

$$\begin{aligned}
\hat{u}(\xi + a) &= \int_{-\infty}^{\infty} u(x)e^{i(\xi+a)x}dx \\
&= \int_{-\infty}^{\infty} u(x)e^{iax}e^{i\xi x}dx \\
&= F(e^{iax}u)(\xi)
\end{aligned}$$

Exercise 3c. Use 3(a) or, from the definition,

$$F(u(x+a)) = \int_{-\infty}^{\infty} u(x+a)e^{i\xi x}dx = \int_{-\infty}^{\infty} u(y)e^{i\xi(y-a)}dy = e^{-ia\xi}\hat{u}(\xi)$$

Exercise 4. From the definition

$$\begin{aligned}\hat{u}(\xi) &= \int_0^{\infty} e^{-ax}e^{i\xi x}dx \\ &= \int_0^{\infty} e^{(i\xi - a)x}dx \\ &= \frac{1}{i\xi - a}e^{(i\xi-a)x}\,|_0^{\infty} \\ &= \frac{1}{a - i\xi}\end{aligned}$$

Exercise 5. Observe that

$$xe^{-ax^2} = -\frac{1}{2a}\frac{d}{dx}e^{-ax^2}$$

Then

$$F(xe^{-ax^2}) = -\frac{1}{2a}(-i\xi)F(e^{-ax^2})$$

Now use (2.59).

Exercise 6. Take transforms of the PDE to get

$$\hat{u}_t = (-i\xi)^2\hat{u} + \hat{f}(\xi, t)$$

Solving this as a linear, first order ODE in t with ξ as a parameter, we get

$$\hat{u}(\xi, t) = \int_0^t e^{-x^2(t-\tau)}\hat{f}(\xi, \tau)d\tau$$

Taking the inverse Fourier transform, interchanging the order of integration, and applying the convolution theorem gives

$$\begin{aligned}u(x,t) &= \int_0^t F^{-1}\left[e^{-x^2(t-\tau)}\hat{f}(\xi,\tau)\right]d\tau \\ &= \int_0^t F^{-1}\left[e^{-x^2(t-\tau)}\right] \star f(x,\tau)d\tau \\ &= \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi(t-\tau)}}e^{-(x-y)^2/4(t-\tau)}f(y,\tau)dyd\tau\end{aligned}$$

Exercise 7. Proceeding exactly in the same way as in the derivation of (2.61) in the text, but with k replaced by I, we obtain the solution

$$u(x,t) = \frac{1}{\sqrt{4\pi it}}\int_{-\infty}^{\infty} e^{-(x-y)^2/4it}f(y)dy$$

where $u(x,0)=f(x)$. Thus

$$u(x,t)=\frac{1}{\sqrt{4\pi it}}\int_{-\infty}^{\infty}e^{i(x-y)^2/4t-y^2}dy$$

Here, in the denominator, $\sqrt{i}$ denotes the root with the positive real part, that is $\sqrt{i}=(1+i)/\sqrt{2}$.

Exercise 8. Letting $v=u_y$ we have

$$v_{xx}+v_{yy}=0,\quad x\in R,\ y>0;\quad v(x,0)=g(x)$$

Hence

$$v(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{yg(\tau)}{(x-\tau)^2+y^2}d\tau$$

Then

$$\begin{aligned}
u(x,y) &= \int_0^y v(x,\xi)d\xi \\
&= \int_0^y \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{yg(\tau)}{(x-\tau)^2+y^2}d\tau d\xi \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty}\int_0^y \frac{y}{(x-\tau)^2+y^2}d\xi d\tau \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}g(\tau)\left(\ln\left((x-\tau)^2+y^2\right)-\ln\left((x-\tau)^2\right)\right)d\tau \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}g(x-\xi)\ln\left(\xi^2+y^2\right)d\tau+C
\end{aligned}$$

Exercise 9. We have

$$\begin{aligned}
u(x,y) &= \frac{y}{\pi}\int_{-l}^{l}\frac{d\tau}{(x-\tau)^2+y^2} \\
&= \frac{1}{\pi}\left(\arctan\left(\frac{l-x}{y}\right)+\arctan\left(\frac{l+x}{y}\right)\right)
\end{aligned}$$

Exercise 10. From the definition of the Fourier transform

$$\begin{aligned}
F(u_x) &= \int_{-\infty}^{\infty}u_x(x,t)e^{i\xi x}dx \\
&= u(x,t)e^{i\xi x}\mid_{-\infty}^{\infty} - i\xi\int_{-\infty}^{\infty}u(\xi,t)e^{i\xi x}dx \\
&= -i\xi\hat{u}(\xi,t)
\end{aligned}$$

For the second derivative, integrate by parts twice and assume u and u_x tend to zero as $x\to\pm\infty$ to get rid of the boundary terms.

Exercise 11. In this case where f is a square wave signal,

$$F(f(x))=\int_{-\infty}^{\infty}f(x)e^{i\xi x}dx=\int_{-a}^{a}e^{i\xi x}dx=\frac{2\sin\xi x}{\xi}$$

Exercise 12. Taking the Fourier transform of the PDE

$$u_t = Du_{xx} - cu_x$$

gives

$$\hat{u}_t = -(D\xi^2 + i\xi c)\hat{u}$$

which has general solution

$$\hat{u}(\xi, t) = C(\xi)e^{-D\xi^2 t - i\xi ct}$$

The initial condition forces $C(\xi) = \hat{\phi}(\xi)$ which gives

$$\hat{u}(\xi, t) = \hat{\phi}(\xi)e^{-D\xi^2 t - i\xi ct}$$

Using

$$F^{-1}\left(e^{-D\xi^2 t}\right) = \frac{1}{\sqrt{4\pi Dt}}e^{-x^2/4Dt}$$

and

$$F^{-1}\left(\hat{u}(\xi, t)e^{-ia\xi}\right) = u(x + a)$$

we have

$$F^{-1}\left(e^{-i\xi ct}e^{-D\xi^2 t}\right) = \frac{1}{\sqrt{4\pi Dt}}e^{-(x+vt)^2/4Dt}$$

Then, by convolution,

$$u(x, t) = \phi \star \frac{1}{\sqrt{4\pi Dt}}e^{-(x+vt)^2/4Dt}$$

Exercise 13. (a) Substituting $u = \exp(i(kx - \omega t))$ into the PDE $u_t + u_{xxx} = 0$ gives $-i\omega + (ik)^3 = 0$ or $\omega = -k^3$. Thus we have solutions of the form

$$u(x, t) = e^{i(kx+k^3 t)} = e^{ik(x+k^2 t)}$$

The real part of a complex-valued solution is a real solution, so we have solutions of the form

$$u(x, t) = \cos[k(x + k^2 t)]$$

These are left traveling waves moving with speed k^2. So the temporal frequency ω as well as the wave speed $c = k^2$ depends on the spatial frequency, or wave number, k. Note that the wave length is proportional to $1/k$. Thus, higher frequency waves are propagated faster.

(b) Taking the Fourier transform of the PDE gives

$$\hat{u}_t = -(-i\xi)^3\hat{u}$$

This has solution

$$\hat{u}(\xi, t) = \hat{\phi}(\xi)e^{-i\xi^3 t}$$

where $\hat{\phi}$ is the transform of the initial data. By the convolution theorem,

$$u(x, t) = \phi(x) \star F^{-1}(e^{-i\xi^3 t})$$

Figure 1. Exercise 13c.

To invert this transform we go to the definition of the inverse. We have

$$\begin{aligned} F^{-1}(e^{-i\xi^3 t}) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi^3 t} e^{-i\xi x} d\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos(\xi^3 t + \xi x) d\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos\left(\frac{z^3}{3} + \frac{zx}{(3t)^{1/3}}\right)\frac{1}{(3t)^{1/3}} dz \\ &= \frac{1}{(3t)^{1/3}} \mathrm{Ai}\left(\frac{x}{(3t)^{1/3}}\right) \end{aligned}$$

where we made the substitution $\xi = z/(3t)^{1/3}$ to put the integrand in the form of that in the Airy function. Consequently we have

$$u(x,t) = \frac{1}{(3t)^{1/3}}\int_{-\infty}^{\infty} \phi(x-y)\mathrm{Ai}\left(\frac{y}{(3t)^{1/3}}\right) dy$$

(c) A graph of the function $e^{-x^2} \star \mathrm{Ai}(x)$ is shown in the figure. It was obtained with the Maple commands:

```
with(plots);
f:=Int(exp(-y*y)*Ai(x-y),y=-5..5);
plot (f,x=-10..10);
```

Exercise 14. The problem is

$$u_{tt} = c^2 u_{xx} = 0, \quad x \in R,\ t > 0$$
$$u(x,0) = f(x), \ \ u_t(x,0) = 0, \ \ x \in R$$

Taking Fourier transforms of the PDE yields

$$\hat{u}_{tt} + c^2\xi^2\hat{u} = 0$$

whose general solution is

$$\hat{u} = A(\xi)e^{i\xi ct} + B(\xi)e^{-i\xi ct}$$

From the initial conditions, $\hat{u}(\xi, 0) = \hat{f}(\xi)$ and $\hat{u}_t(\xi, 0) = 0$. Thus $A(\xi) = B(\xi) = 0.5\hat{f}(\xi)$. Therefore

$$\hat{u}(\xi, t) = 0.5\hat{f}(\xi)(e^{i\xi ct} + e^{-i\xi ct})$$

Now we use the fact that

$$F^{-1}\left(\hat{f}(\xi)e^{ia\xi}\right) = f(x - a)$$

to invert each term. Whence

$$u(x, t) = 0.5(f(x - ct) + f(x + ct))$$

8. Solving PDEs Using Computer Algebra Packages

Exercise 1. In preparation.

Exercise 2. In preparation.

Exercise 3. In preparation.