CHAPTER 1

Orthogonal Expansions

1. The Fourier Method

Exercise 1. Form the linear combination

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos nct \sin nx$$

Then

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin nx$$

Using the exactly same calculation as in (3.5)–(3.7) in the text, we obtain

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

Observe that $u_t(x,0) = 0$ is automatically satisfied.

When the initial conditions are changed to $u(x,0) = 0$, $u_t(x,0) = g(x)$ then a linear combination of the fundamental solutions $u_n(x,t) = \cos nct \sin nx$ does not suffice. But, observe that $u_n(x,t) = \sin nct \sin nx$ now works and form the linear combination

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin nct \sin nx$$

Now $u(x,0) = 0$ is automatically satisfied and

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} ncb_n \sin nx$$

Again using the argument in (3.5)–(3.7), one easily shows the b_n are given by

$$b_n = \frac{2}{nc\pi} \int_0^{\pi} g(x) \sin nx \, dx$$

2. Orthogonal Expansions

Exercise 1. The requirement for orthogonality is

$$\int_0^{\pi} \cos mx \cos nx \, dx = 0, \quad m \neq n$$

For the next part make the substitution $y = \pi x/l$ to get

$$\int_0^l \cos(m\pi x/l)\cos(n\pi x/l); dx = \int_0^\pi \cos my \cos ny \; dy = 0, \quad m \neq n$$

We have

$$c_n = \frac{(f, \cos(n\pi x/l))}{||\cos(n\pi x/l)||^2}$$

Thus

$$c_0 = \frac{1}{l}\int_0^l f(x)dx, \quad c_n = \frac{2}{l}\int_0^l f(x)\cos(n\pi x/l)dx, \;\; n \geq 1$$

Exercise 3. Up to a constant factor, the Legendre polynomials are

$$P_0(x) = 1, \; P_1(x) = x, \; P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}, \; P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$$

The coefficients c_n in the expansion are given by the generalized Fourier coefficients

$$c_n = \frac{1}{||P_n||^2}(f, P_n) = \frac{\int_{-1}^1 e^x P_n(x)dx}{\int_{-1}^1 Pn(x)^2 dx}$$

The pointwise error is

$$E(x) = e^x - \sum_{n=0}^{3} c_n P_n(x)$$

The mean square error is

$$E = \left(\int_{-1}^1 [e^x - \sum_{n=0}^{3} c_n P_n(x)]^2 dx \right)^{1/2}$$

The maximum pointwise error is $\max_{-1 \leq x \leq 1} |E(x)|$.

Exercise 4. Use the calculus facts that

$$\int_0^b \frac{1}{x^p} dx < \infty, \;\; p < 1$$

and

$$\int_a^\infty \frac{1}{x^p} dx < \infty, \;\; p > 1 \;\; (a > 0)$$

Otherwise the improper integrals diverge. Thus $x^r \in L^2[0,1]$ if $r > -1/2$ and $x^r \in L^2[0,\infty]$ if $r < -1/2$ *and* $r > -1/2$, which is impossible.

Exercise 5. We have

$$\cos x = \sum_{n=1}^{\infty} b_n \sin 2nx, \quad b_n = \frac{4}{\pi}\int_0^{\pi/2} \cos x \sin 2nx \; dx$$

Also

$$\sin x = \sum_{n=1}^{\infty} b_n \sin nx$$

clearly forces $b_1 = 1$ and $b_n = 0$ for $n \geq 1$. Therefore the Fourier series of $\sin x$ on $[0, \pi]$ is just a single term, $\sin x$.

Exercise 6. (a) We find

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = 2(2x^2 - 1)$$
$$H_3(x) = 8x^3 - 12x, \quad H_4(x) = 16x^4 - 48x^2 + 12$$

(b) This is a straightforward verification. (c) To verify orthogonality, note that

$$-v_n'' + x^2 v_n = (2n+1)v_n, \qquad -v_m'' + x^2 v_m = (2m+1)v_m$$

Multiply the first equation by v_m and the second by v_n, subtract, and then integrate over R to get

$$\int_R (-v_n'' v_m + v_m'' v_n) dx = 2(n-m) \int_R v_m v_n dx$$

But integrating by parts twice gives

$$\int_R v_m v_n'' dx = \int_R v_n v_m'' dx$$

The boundary terms generated by the parts integration go to zero since the v_n and v_n' go to zero as $x \to \pm\infty$. Thus the left side of the equation above is zero, forcing

$$\int_R v_m v_n dx = 0 \quad \text{when} \;\; m \neq n$$

(d) If $f(x) = \sum_{n=0}^{\infty} c_n v_n(x)$, then

$$c_n = (f, v_n)/||v_n||^2 = \frac{\int_R f(x) v_n(x) dx}{\int_R v_n(x)^2 dx}$$

(e) Notice that

$$v_n(x)^2 = H_n(x)^2 e^{-x^2}$$

Thus

$$v_0(x)^2 = e^{-x^2}, \quad v_1(x)^2 = 4x^2 e^{-x^2}, \quad v_2(x)^2 = 4(2x^-1)^2 e^{-x^2}$$

The graphs are shown in the following figures (these plots are not normalized).

Exercise 7. A plot of $\psi_{mn}(x)$ is shown in the figure. The coefficients are given by

$$c_{mn} = \frac{\int_R f(x) \psi_{mn}(x) dx}{\int_R \psi_{mn}^2(x) dx}$$

Easily

$$\int_R \psi_{mn}^2(x) dx = 1$$

and

$$\int_R f(x) \psi_{mn}(x) dx = 2^{m/2} \left(\int_{n/2^m}^{(n+1/2)/2^m} f(x) dx - \int_{(n+1/2)/2^m}^{(n+1)/2^m} f(x) dx \right)$$

Exercise 8. Expanding

$$\begin{aligned} q(t) &= (f + tg, f + tg) \\ &= ||g||^2 t^2 + 2(f,g)t + ||f||^2 \end{aligned}$$

which is a quadratic in t. Because $q(t)$ is nonnegative (a scalar product of a function with itself is necessarily nonnegative because it is the norm-squared), the graph of the quadratic can never dip below the t axis. Thus it can have at most one real root.

FIGURE 1. Unnormalized wave functions for problem 6(e).

Thus the discriminant $b^2 - 4ac$ must be nonpositive. In this case the discriminant is

$$b^2 - 4ac = 4(f, g)^2 - 4||g||^2||f||^2 \leq 0$$

This gives the desired inequality.

FIGURE 2. Graph of the wavelet $\psi_{mn}(x)$.

3. Classical Fourier Series

Exercise 1. Since f is an even function, $b_n = 0$ for all n. We have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} dx = 1$$

and for $n = 1, 2, 3, \ldots,$

$$a_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \sin nx \; dx = \frac{2}{n\pi}\sin(n\pi/2)$$

Thus the Fourier series is

$$\begin{aligned} &\frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(n\pi/2)\cos nx \\ &= \frac{1}{2} + \frac{2}{\pi}\left(\cos x - \frac{1}{3}\cos 3x + \cdots\right) \end{aligned}$$

A plot of a two-term and a four-term approximation is shown in the figure.

Exercise 2. Because the function is even, $b_n = 0$. Then

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = 2\pi^2/3$$

and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx \; dx = \frac{4(-1)^n}{n^2}$$

FIGURE 3. Exercise 1.

So the Fourier series is

$$\frac{\pi^2}{3} + \sum_{n=0}^{\infty} \frac{4(-1)^n}{n^2} \cos nx$$

This series expansion of $f(x) = x^2$ must converge to $f(0) = 0$ at $x = 0$ since f is piecewise smooth and continuous there. This gives

$$0 = \frac{\pi^2}{3} + \sum_{n=0}^{\infty} \frac{4(-1)^n}{n^2}$$

which implies the result.

The frequency spectrum is

$$\gamma_0 = \frac{2\pi^2}{3\sqrt{2}}, \quad \gamma_n = \frac{4}{n^2}, \ n \geq 1$$

Exercise 3. This problem suits itself for a computer algebra program to calculate the integrals. We find $a_0 = 1$ and $a_n = 0$ for $n \geq 1$. Then we find

$$\begin{aligned} b_n &= \frac{1}{2\pi} \int_{-2\pi}^{0} (x+1) \sin(nx/2) dx + \frac{1}{2\pi} \int_{0}^{2\pi} x \sin(nx/2) dx \\ &= \frac{1}{\pi} \frac{-1 + (-1)^n + 4\pi(-1)^{n+1}}{n} \end{aligned}$$

Note that $b_n = -4/n\pi$ if n is even and $b_n = (-2 + 4\pi)/n\pi$ if n is odd. So the Fourier series is

$$f(x) = \frac{1}{2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{-1 + (-1)^n + 4\pi(-1)^{n+1}}{n} \sin(nx/2)$$

A five-term approximation is shown in the figure.

FIGURE 4. Five-term approximation in Exercise 3.

Exercise 4. Because $\cos ax$ is even we have $b_n = 0$ for all n. Next

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos ax \; dx = \frac{2\sin a\pi}{a\pi}$$

and, using a table of integrals or a software program, for $n \geq 1$,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos ax \cos nx \; dx \\ &= \frac{1}{\pi}\left(\frac{\sin(a-n)x}{2(a-n)} + \frac{\sin(a+n)x}{2(a+n)}\right)_{-\pi}^{\pi} \\ &= \frac{2a(-1)^n}{\pi(a^2-n^2)}\sin a\pi \end{aligned}$$

Therefore the Fourier series is

$$\cos ax = \frac{\sin a\pi}{a\pi} + \sum_{n=1}^{\infty} \frac{2a(-1)^n}{\pi(a^2-n^2)} \sin a\pi \cos nx$$

Substitute $x = 0$ to get the series for $\csc a\pi$.

Exercise 5. Here $f(x)$ is odd so $a_n = 0$ for all n. Then

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{0} -\frac{1}{2}\sin nx \; dx + \frac{1}{\pi}\int_{0}^{\pi} \frac{1}{2}\sin nx \; dx \\ &= \frac{1}{n\pi}(1-(-1)^n) \end{aligned}$$

Therefore

$$b_{=}\frac{2}{(2k-1)\pi}, \quad k = 1, 2, 3, \ldots$$

FIGURE 5. $S_3(x)$ and $S_7(x)$ in Exercise 5.

The Fourier series is

$$\sum_{k=1} \infty \frac{2}{(2k-1)\pi} \sin(2k-1)x$$

Graph of $S_3(x)$ and $S_7(x)$ are shown in the figure. In the accompanying figure a graph of $S_{10}(x)$ is presented; it still shows the overshoot near the discontinuity (Gibbs phenomenon). Here $S_N(x)$ is the sum of the first N terms.

4. Sturm-Liouville Problems

Exercise 1. Substituting $u(x,t) = g(t)y(x)$ into the PDE

$$u_t = (p(x)u_x)_x - q(x)u$$

gives

$$g'(t)y(x) = \frac{d}{dx}(p(x)g(t)y'(x)) - q(x)g(t)y(x)$$

Dividing by $g(t)y(x)$ gives

$$\frac{g'}{g} = \frac{(py')' - q}{y}$$

Setting these equal to $-\lambda$ gives the two differential equations for g and y.

Exercise 2. When $\lambda = 0$ the ODE is $-y'' = 0$ which gives $y(x) = ax+b$. Applying the boundary conditions forces $a = b = 0$ and so zero is not and eigenvalue. When $\lambda = -k^2 < 0$ then the ODE has general solution $y(x) = ae^{kx} + be^{-kx}$, which are exponentials. If $y = 0$ at $x = 0$ and $x = l$, then it is not difficult to show $a = b = 0$, which means that there are no negative eigenvalues (this is similar to the argument in the text). If $\lambda = k^2 > 0$ then $y(x) = a \sin kx + b \cos kx$. Then $y(0) = 0$ forces

FIGURE 6. $S_{10}(x)$ showing overshoot of the Fourier approximation in Exercise 5.

$b = 0$ and then $y(l) = a \sin kl = 0$. So $kl = n\pi$, $n = 1, 2, \ldots$ giving eigenvalues and eigenfunctions as stated.

Exercise 3. When $\lambda = 0$ the ODE is $-y'' = 0$ which gives $y(x) = ax + b$. But $y'(0) = a = 0$ and $y(l) = al + b = 0$, and so $a = b = 0$ and so zero is not an eigenvalue. When $\lambda = -k^2 < 0$ then the ODE has general solution $y(x) = ae^{kx} + be^{-kx}$, which are exponentials. If $y = 0$ at $x = 0$ and $x = l$, then it is not difficult to show $a = b = 0$, which means that there are no negative eigenvalues. If $\lambda = k^2 > 0$ then $y(x) = a \sin kx + b \cos kx$. Then $y'(0) = 0$ forces $a = 0$ and then $y(l) = b \cos kl = 0$. But the cosine function vanishes at $\pi/2$ plus a multiple of π, i.e.,

$$kl = \sqrt{\lambda} l = \pi/2 + n\pi$$

for $n = 0, 1, 2, \ldots$. This gives the desired eigenvalues and eigenfunctions as stated in the problem.

Exercise 4. When $\lambda = 0$ the ODE is $-y'' = 0$ which gives $y(x) = ax + b$. The boundary conditions force $a = 0$ but do not determine b. Thus $\lambda = 0$ is an eigenvalue with corresponding constant eigenfunctions. When $\lambda = -k^2 < 0$ then the ODE has general solution $y(x) = ae^{kx} + be^{-kx}$, which are exponentials. Easily, exponential functions cannot satisfy periodic boundary conditions, so there are no negative eigenvalues. If $\lambda = k^2 > 0$ then $y(x) = a \sin kx + b \cos kx$. Then $y'(x) = ak \cos kx - bk \sin kx$. Applying the boundary conditions

$$b = a \sin kl + b \cos kl, \quad a = a \cos kl - b \sin kl$$

We can rewrite this system as a homogeneous system

$$\begin{aligned} a\sin kl + b(\cos kl - 1) &= 0 \\ a(\cos kl - 1) - b\sin kl &= 0 \end{aligned}$$

A homogeneous system will have a nontrivial solution when the determinant of the coefficient matrix is zero, which is in this case reduces to the equation

$$\cos kl = 0$$

Therefore kl must be a multiple of 2π, or

$$\lambda_n = (2n\pi/l)^2, \quad n = 1, 2, 3, \ldots$$

The corresponding eigenfunctions are

$$y_n(x) = a_n \sin(2n\pi x/l) + b_n \cos(2n\pi x/l)$$

Exercise 6. The problem is

$$-y'' = \lambda y, \quad y(0) + y'(0) = 0,\ y(1) = 0$$

If $\lambda = 0$ then $y(x) = ax + b$ and the boundary conditions force $b = -a$. Thus eigenfunctions are

$$y(x) = a(1 - x)$$

If $\lambda < 0$ then $y(x) = a\cosh kx + b\sinh kx$ where $\lambda = -k^2$. The boundary conditions give

$$a + bk = 0, \quad a\cosh k + b\sinh k = 0$$

Thus $\sinh k - k\cosh k = 0$ or $k = \tanh k$ which has no nonzero roots. Thus there are no negative eigenvalues.

If $\lambda = k^2 > 0$ then $y(x) = a\cos kx + b\sin kx$. The boundary conditions imply

$$a + bk = 0, \quad a\cos k + b\sin k = 0$$

Thus $k = \tan k$ which has infinity many positive roots k_n (note that the graphs of k and $\tan k$ cross infinitely many times). So there are infinitely many positive eigenvalues given by $\lambda_n = k_n^2$.

Exercise 7. The SLP is

$$-y'' = \lambda y, \quad y(0) + 2y'(0) = 0,\ 3y(2) + 2y'(2) = 0$$

If $\lambda = 0$ then $y(x) = ax+b$. The boundary conditions give $b+2a = 0$ and $8a+3b = 0$ which imply $a = b = 0$. So zero is not an eigenvalue. Since this problem is a regular SLP we know by the fundamental theorem that there are infinitely many positive eigenvalues.

If $\lambda = -k^2 < 0$, then $y(x) = a\cosh kx + b\sinh kx$. The boundary conditions force the two equations

$$a + 2bk = 0, \quad (3\cosh 2k + 2k\sinh 2k)a + (3\sinh 2k + 2k\cosh 2k) = 0$$

This is a homogeneous linear system for a and b and it will have a nonzero solution when the determinant of the coefficient matrix is zero, i.e.,

$$\tan 2k = \frac{4k}{3 - 4k^2}$$

This equation has nonzero solutions at $k \approx \pm 0.42$. Therefore there is one negative eigenvalue $\lambda \approx -0.42^2 = -0.176$. (This nonlinear equation for k can be solved graphically using a calculator, or using a computer algebra package, or using the solver routine on a calculator).

Exercise 8. When $\lambda = 0$ the ODE is $y'' = 0$, giving $y(x) = Ax + B$. Now apply the boundary conditions to get

$$B - aA = 0, \quad Al + B + bA = 0$$

This homogeneous system has a nonzero solution for A and B if and only if $a + b = -abl$. (Note that the determinant of the coefficient matrix must be zero).

Exercise 9. Multiplying the differential equation by y and integrating from $x = 1$ to $x = \pi$ gives

$$-\int_1^\pi y(x^2y')'dx = \lambda \int_1^\pi y^2 dx$$

or, upon integrating the left side by parts,

$$-x^y y' \,|_1^\pi - \int_1^\pi x^2(y')^2 dx = \lambda \int_1^\pi y^2 dx$$

The boundary term vanishes because of the boundary conditions. Therefore, because both integrals are nonnegative we have $\lambda \geq 0$. If $\lambda = 0$ then y' =const= 0 (by the boundary conditions). So $\lambda \neq 0$ and the eigenvalues are therefore positive.

If $\lambda = k^2 > 0$, then the ODE becomes

$$x^2y'' + 2xy' + k^2y = 0$$

which is a Cauchy-Euler equation (see the Appendix on differential equations in the text). This can be solved to determine eigenvalues

$$\lambda_n = \left(\frac{n\pi}{\ln \pi}\right)^2 + \frac{1}{4}$$

with corresponding eigenfunctions

$$y_n(x) = \frac{1}{\sqrt{x}} \sin\left(\frac{n\pi}{\ln \pi} \ln x\right)$$

Exercise 10. The operator on the left side of the equation has variable coefficients and the ODE cannot be solved analytically in terms of simple functions.

Exercise 11. Multiply the equation by y and integrate from $x = 0$ to $x = l$ to get

$$-\int_0^l yy''dx + \int_0^l qy^2 dx = \lambda \int_0^l y^2 dx$$

Integrate the first integral by parts; the boundary term will be zero from the boundary conditions; then solve for λ to get

$$\lambda = \frac{\int_0^l (y')^2 dx + \int_0^l qy^2 dx}{||y||^2}$$

Clearly (note $y(x) \neq 0$) the second integral in the numerator and the integral in the denominator are positive ,and thus $\lambda > 0$. $y(x)$ cannot be constant because the boundary conditions would force that constant to be zero.

Exercise 12. Let y and u be two independent eigenfunctions corresponding to the single eigenvalue λ for the problem in Exercise 11. Then

$$-y'' + qy = \lambda y, \quad -u'' + qu = \lambda u$$

and both y and u satisfy the boundary conditions. Multiply the first by u and the second by y and subtract to get

$$u''y - uy'' = 0$$

or

$$(uy' - yu')' = 0$$

Thus $uy' - yu' = \text{const.} = 0$. The constant is zero by the boundary conditions. But, by the quotient rule for derivatives, this last equation is equivalent to $(y/u)' = 0$. Hence $y/u = C$ for some constant C and so $y = Cu$, which means y and u are not independent.

Observe that the *periodic* boundary value problem in Exercise 4 does have linear independent eigenfunctions $\cos(2n\pi x/l)$ and $\sin(2n\pi x/l)$ corresponding to the eigenvalue $\lambda = (2n\pi/l)^2$.

Exercise 13. By Exercise 6 in Section 3.2 the differential equation is identified as Hermite's equation, and it has solutions in $L^2(R)$ when $\lambda = \lambda_n = 2n + 1$ and the corresponding eigenfunctions are the Hermite functions

$$y_n(x) = H_n(x)e^{-x^2/2}$$

for $n = 0, 1, 2, \ldots$.