CHAPTER 1

# Partial Differential Equations on Bounded Domains

## 1. Separation of Variables

**Exercise 1.** The solution is

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-n^2 t} \sin nx$$

where

$$a_n = \frac{2}{\pi}\int_{\pi/2}^{\pi} \sin nx dx = -\frac{2}{n\pi}((-1)^n - \cos(n\pi/2))$$

Thus

$$u(x,t) = \frac{2}{\pi}e^{-t}\sin x - \frac{2}{\pi}e^{-4t}\sin 2x + \frac{2}{3\pi}e^{-9t}\sin 3x + \frac{2}{5\pi}e^{-25t}\sin 5x + \cdots$$

**Exercise 2.** The solution is given by formula (4.14) in the text, where the coefficients are given by (4.15) and (4.16). Since $G(x) = 0$ we have $c_n = 0$. Then

$$d_n = \frac{2}{\pi}\int_0^{\pi/2} x \sin nx \; dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} (\pi - x)\sin nx \; dx$$

Using the antiderivative formula $\int x \sin nx \; dx = (1/n^2)\sin nx - (x/n)\cos nx$ we integrate to get

$$d_n = \frac{4}{\pi n^2}\sin\frac{n\pi}{2}$$

**Exercise 3.** Substituting $u(x,y) = \phi(x)\psi(y)$ we obtain the Sturm-Liouville problem

$$-\phi'' = \lambda\phi, \quad x \in (0,l); \quad \phi(0) = \phi(l) = 0$$

and the differential equation

$$\psi'' - \lambda\psi = 0$$

The SLP has eigenvalues and eigenfunctions

$$\lambda_n = n^2\pi^2/l^2, \quad \phi_n(x) = \sin(n\pi x/l)$$

and the solution to the $\psi$–equation is

$$\psi_n(y) = a_n \cosh(n\pi y/l) + b_n \sinh(n\pi y/l)$$

Therefore

$$u(x,y) = \sum_{n=1}^{\infty} (a_n \cosh(n\pi y/l) + b_n \sinh(n\pi y/l)) \sin(n\pi x/l)$$

Now we apply the boundary conditions:

$$u(x,0) = F(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/l)$$

and

$$u(x,1) = G(x) = \sum_{n=1}^{\infty} (a_n \cosh(n\pi/l) + b_n \sinh(n\pi/l)) \sin(n\pi x/l)$$

Thus

$$a_n = \frac{2}{l} \int_0^{\pi} F(x) \sin(n\pi x/l) dx$$

and

$$a_n \cosh(n\pi/l) + b_n \sinh(n\pi/l) = \frac{2}{l} \int_0^{\pi} G(x) \sin(n\pi x/l) dx$$

which gives the coefficients $a_n$ and $b_n$.

**Exercise 4.** Substituting $u = y(x)g(t)$ into the PDE and boundary conditions gives the SLP

$$-y'' = \lambda y, \quad y(0) = y(1) = 0$$

and, for $g$, the equation

$$g'' + kg' + c^2 \lambda g = 0$$

The SLP has eigenvalues and eigenfunctions

$$\lambda_n = n^2\pi^2, \quad y_n(x) = \sin n\pi x, \;\; n = 1, 2, \ldots$$

The $g$ equation is a linear equation with constant coefficients; the characteristic equations is

$$m^2 + km + c^2\lambda = 0$$

which has roots

$$m = \frac{1}{2}(-k \pm \sqrt{k^2 - 4c^2n^2\pi^2})$$

By assumption $k < 2\pi c$, and therefore the roots are complex for all $n$. Thus the solution to the equation is (see the Appendix in the text on ordinary differential equations)

$$g_n(t) = e^{-kt}(a_n \cos(m_n t) + b_n \sin(m_n t))$$

where

$$m_n = \frac{1}{2}\sqrt{4c^2n^2\pi^2 - k^2})$$

Then we form the linear combination

$$u(x,t) = \sum_{n=1}^{\infty} e^{-kt}(a_n \cos(m_n t) + b_n \sin(m_n t)) \sin(n\pi x)$$

Now apply the initial conditions. We have

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$$

and thus

$$a_n = \int_0^1 f(x) \sin n\pi x \, dx$$

The initial condition $u_t = 0$ at $t = 0$ yields

$$u_t(x, 0) = 0 = \sum_{n=1}^{\infty} (b_n m_n - k a_n) \sin(n\pi x)$$

Therefore

$$b_n m_n - k a_n = 0$$

or

$$b_n = \frac{k a_n}{m_n} = \frac{k}{m_n} \int_0^1 f(x) \sin n\pi x \, dx$$

## 2. Flux and Radiation Conditions

**Exercise 1.** This problem models the transverse vibrations of a string of length $l$ when the left end is fixed (attached) and the right end experience no force; however, the right end can move vertically. Initially the string is displaced by $f(x)$ and it is not given an initial velocity.

Substituting $u = y(x)g(t)$ into the PDE and boundary conditions gives the SLP

$$-y'' = \lambda y, \quad y(0) = y'(l) = 0$$

and, for $g$, the equation

$$g'' + c^2 \lambda g = 0$$

The SLP has eigenvalues and eigenfunctions

$$\lambda_n = ((2n+1)\pi/l)^2, \quad y_n(x) = \sin((2n+1)\pi x/l), \quad n = 0, 1, 2, \ldots$$

and the equation for $g$ has general solution

$$g_n(t) = a_n \sin((2n+1)\pi ct/l) + b_n \cos((2n+1)\pi ct/l)$$

Then we form

$$u(x,t) = \sum_{n=0}^{\infty} (a_n \sin((2n+1)\pi ct/l) + b_n \cos((2n+1)\pi ct/l)) \sin((2n+1)\pi x/l)$$

Applying the initial conditions,

$$u(x,t) = f(x) = \sum_{n=0}^{\infty} b_n \sin((2n+1)\pi x/l)$$

which yields

$$b_n = \frac{1}{||\sin((2n+1)\pi x/l)||^2} \int_0^l f(x) \sin((2n+1)\pi x/l)$$

And

$$u_t(x, 0) = 0 = \sum_{n=0}^{\infty} a_n c \lambda_n \sin((2n+1)\pi x/l)$$

which gives $a_n = 0$. Therefore the solution is

$$u(x,t) = \sum_{n=0}^{\infty} b_n \cos((2n+1)\pi ct/l)) \sin((2n+1)\pi x/l)$$

**Exercise 2.** This problem models the steady state temperatures in a rectangular plate that is insulated on both sides, whose temperature is zero on the top, and whose temperature is $f(x)$ along the bottom. Letting $u = g(y)\phi(x)$ and substituting into the PDE and boundary conditions gives the Sturm-Liouville problem

$$-\phi'' = \lambda\phi, \quad \phi'(0) = \phi'(a) = 0$$

and the differential equation

$$g'' - \lambda g = 0$$

The eigenvalues and eigenfunctions are $\lambda_0 = 0,\ \phi(x) = 1$ and

$$\lambda_n = n^2\pi^2/a^2, \quad \phi_n(x) = \cos(n\pi x/a), \quad n = 1, 2, 3, \ldots$$

The solution to the $g$ equation is, corresponding to the zero eigenvalue, $g_0(y) = c_0 y + d_0$, and corresponding to the positive eigenvalues,

$$g_n(y) = c_n \sinh(n\pi y/a) + d_n \cosh(n\pi y/a)$$

Thus we form the linear combination

$$u(x,y) = c_0 y + d_0 + \sum_{n=1}^{\infty} (c_n \sinh(n\pi y/a) + d_n \cosh(n\pi y/a)) \cos(n\pi x/a)$$

Now apply the boundary conditions on $y$ to compute the coefficients:

$$u(x,0) = f(x) = d_0 + \sum_{n=1}^{\infty} d_n \cos(n\pi x/a)$$

which gives

$$d_0 = \frac{1}{a}\int_0^a f(x)dx, \;\; d_n = \frac{2}{a}\int_0^a f(x)\cos(n\pi x/a)dx$$

Next

$$u(x,b) = 0 = c_0 b + d_0 + \sum_{n=1}^{\infty} (c_n \sinh(n\pi b/a) + d_n \cosh(n\pi b/a)) \cos(n\pi x/a)$$

Therefore

$$c_0 = -d_0/b, \quad c_n = -\frac{\cosh(n\pi b/a)}{\sinh(n\pi b/a)} d_n$$

**Exercise 3.** The flux at $x = 0$ is $\phi(0,t) \equiv -u_x(0,t) = -a_0 u(0,t) > 0$, so there is heat flow into the bar and therefore adsorption. At $x = 1$ we have $\phi(1,t) = -u_x(1,t) = a_1 u(1,t) > 0$, and therefore heat is flowing out of the bar, which is radiation. The right side of the inequality $a_0 + a_1 > -a_0 a_1$ is positive, so the positive constant $a_1$, which measures radiation, must greatly exceed the negative constant $a_0$, which measures adsorption.

In this problem substituting $u = y(x)g(t)$ leads to the Sturm-Liouville problem

$$-y'' = \lambda y, \quad y'(0) - a_0 y(0) = 0, \; y'(1) + a_1 y(1) = 0$$

and the differential equation

$$g' = \lambda g$$

There are no nonpositive eigenvalues. If we take $\lambda = k^2 > 0$ then the solutions are

$$y(x) = a\cos kx + b\sin kx$$

Applying the two boundary conditions leads to the nonlinear equation

$$\tan k = \frac{(a_0 + a_1)k}{k^2 - a_0 a_1}$$

To determine the roots $k$, and thus the eigenvalues $\lambda = k^2$, we can graph both sides of this equation to observe that there are infinitely many intersections occuring at $k_n$, and thus there are infinitely many eigenvalues $\lambda_n = k_n^2$. The eigenfunctions are

$$y_n(x) = \cos k_n x + \frac{a_0}{k_n}\sin k_n x$$

So the solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n^2 t}(\cos k_n x + \frac{a_0}{k_n}\sin k_n x)$$

The $c_n$ are then the Fourier coefficients

$$c_n = (f, y_n)/||y_n||^2$$

If $a_0 = -1/4$ and $a_1 = 4$ then

$$\tan k = \frac{3.75k}{k^2 + 1}$$

From a graphing calculator, the first four roots are approximately $k_1 = 1.08,\ k_2 = 3.85,\ k_3 = 6.81,\ k_4 = 9.82$.

**Exercise 4.** Letting $c(x,t) = y(x)g(t)$ leads to the periodic boundary value problem

$$-y'' = \lambda y, \quad y(0) = y(2l),\ y'(0) = y'(2l)$$

and the differential equation

$$g' = \lambda D g$$

which has solution

$$g(t) = e^{-D\lambda t}$$

The eigenvalues and eigenfunctions are found exactly as in the solution of Exercise 4, Section 3.4, with $l$ replaced by $2l$. They are

$$\lambda_0 = 0, \lambda_n = n^2\pi^2/l^2, \quad n = 1, 2, \ldots$$

and

$$y_0(x) = 1, \quad y_n(x) = a_n\cos(n\pi x/l) + b_n\sin(n\pi x/l), \quad n = 1, 2, \ldots$$

Thus we form

$$c(x,t) = a_0/2 + \sum_{n=1}^{\infty} e^{-n^2\pi^2 Dt/l^2}(a_n\cos(n\pi x/l) + b_n\sin(n\pi x/l)$$

Now the initial condition gives

$$c(x,0) = f(x) = a_0/2 + \sum_{n=1}^{\infty}(a_n\cos(n\pi x/l) + b_n\sin(n\pi x/l)$$

which is the Fourier series for $f$. Thus the coefficients are given by

$$a_n = \frac{1}{l}\int_0^2 lf(x)\cos(n\pi x/l)dx, \quad n = 0, 1, 2, /ldots$$

and

$$b_n = \frac{1}{l}\int_0^2 lf(x)\sin(n\pi x/l)dx, \quad n = 1, 2, /ldots$$

## 3. Laplace's Equation

**Exercise 2.** Substituting $u(r,\theta) = g(\theta)y(r)$ into the PDE and boundary conditions gives the Sturm-Liouville problem

$$-g'' = \lambda g, \quad g(0) = g(\pi/2) = 0$$

and the differential equation

$$r^2y'' + ry' + \lambda y = 0$$

This SLP has been solved many times in the text and in the problems. The eigenvalues and eigenfunctions are

$$\lambda_n = 4n^2, \quad g_n(\theta) = \sin(2n\theta), \quad n = 1, 2, \dots$$

The $y$ equation is a Cauchy-Euler equation and has bounded solution

$$y_n(r) = r^{2n}$$

Form

$$u(r,\theta) = \sum_{n=1}^{\infty} b_n r^{2n}\sin(2n\theta)$$

Then the boundary condition at $r = R$ gives

$$u(R,\theta) = f(\theta) = \sum_{n=1}^{\infty} b_n R^{2n}\sin(2n\theta)$$

Hence the coefficients are

$$b_n = \frac{1}{\pi R^{2n}}\int_0^{\pi/2} f(\theta)\sin(2n\theta)d\theta$$

**Exercise 3.** Substituting $u(r,\theta) = g(\theta)y(r)$ into the PDE and boundary conditions gives the Sturm-Liouville problem

$$-g'' = \lambda g, \quad g(0) = g'(\pi/2) = 0$$

and the differential equation

$$r^2y'' + ry' + \lambda y = 0$$

The eigenvalues and eigenfunctions are

$$\lambda_n = (2n+1)^2, \quad g_n(\theta) = \sin((2n+1)\theta), \quad n = 0, 1, 2, \dots$$

The $y$ equation is a Cauchy-Euler equation and has bounded solution

$$y_n(r) = r^{2n+1}$$

Form

$$u(r,\theta) = \sum_{n=1}^{\infty} b_n r^{2n+1} \sin((2n+1)\theta)$$

Then the boundary condition at $r = R$ gives

$$u(R,\theta) = f(\theta) = \sum_{n=1}^{\infty} b_n R^{2n+1} \sin((2n+1)\theta)$$

Hence the coefficients are

$$b_n = \frac{1}{\pi R^{2n+1}} \int_0^{\pi/2} f(\theta) \sin((2n+1)\theta) d\theta$$

**Exercise 4.** Substituting $r = 0$ into Poisson's integral formula (4.34) of the text we instantly get the temperature at the origin as

$$u(0,0) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta$$

The right side is the average of the function $f(\theta)$ over the interval $[0, 2\pi]$.

**Exercise 5.** Let $w = u + v$ where $u$ satisfies the Neumann problem and $v$ satisfies the boundary condition $n \cdot \nabla v = 0$. Then

$$\begin{aligned}
E(w) &= E(u+v) \\
&= \frac{1}{2} \int_{\Omega} (\nabla u \cdot \nabla u + 2\nabla u \cdot \nabla v + \nabla v \cdot \nabla v) dV - \int_{\partial\Omega} (hu - hv) dA \\
&= E(u) + \int_{\Omega} \nabla u \cdot \nabla v \; dV + \frac{1}{2} \int_{\Omega} \nabla v \cdot \nabla v dV - \int_{\partial\Omega} hv \; dA \\
&= E(u) + \int_{\partial\Omega} v \nabla u \cdot n \; dA - \int_{\Omega} v \Delta u \; dV + \frac{1}{2} \int_{\Omega} \nabla v \cdot \nabla v dV - \int_{\partial\Omega} hv \; dA \\
&= E(u) + \int_{\partial\Omega} vh \; dA + \frac{1}{2} \int_{\Omega} \nabla v \cdot \nabla v dV - \int_{\partial\Omega} hv \; dA \\
&= E(u) + \frac{1}{2} \int_{\Omega} \nabla v \cdot \nabla v dV
\end{aligned}$$

So $E(u) \leq E(w)$.

**Exercise 6.** Multiply both sides of the PDE by $u$ and integrate over $\Omega$. We obtain

$$\int_{\Omega} u \Delta u \; dV = c \int_{\Omega} u^2 dV$$

Now use Green's first identity to obtain

$$\int_{\partial\Omega} u \nabla u \cdot n dA - \int_{\Omega} \nabla u \cdot \nabla u \; dV = c \int_{\Omega} u^2 dV$$

or

$$-\int_{\partial\Omega} a u^2 dA - \int_{\Omega} \nabla u \cdot \nabla u \; dV = c \int_{\Omega} u^2 dV$$

The left side is negative and the right side is positive. Then both must be zero, or

$$\int_\Omega u^2 dV = 0$$

Hence $u = 0$ in $\Omega$.

The uniqueness is standard. Let $u$ and $v$ be two solutions to the boundary value problem

$$\Delta u - cu = f, \ \ x \in \Omega \qquad n \cdot \nabla u + au = g, \quad x \in \partial\Omega$$

Then the difference $w = u - v$ satisfies the homogeneous problem

$$\Delta w - cw = 0, \ \ x \in \Omega \qquad n \cdot \nabla w + aw = 0, \quad x \in \partial\Omega$$

By the first part of the problem we know $w = 0$ and therefore $u = v$.

**Exercise 7.** The solution is given by equation (4.31) in the text. Here

$$f(\theta) = 4 + 3\sin\theta$$

The right side is its Fourier series, so the Fourier coefficients are given by

$$\frac{a_0}{2} = 4, \ \ Rb_1 = 3$$

with all the other Fourier coefficients identically zero. So the solution is

$$u(r,\theta) = 4 + \frac{3r}{R}\sin\theta$$

**Exercise 8.** Multiplying the equation $\Delta u = 0$ by $u$, integrating over $\Omega$, and then using Green's identity gives

$$\int_\Omega u\Delta u \, dV = \int_{\partial\Omega} u\nabla u \cdot n dA - \int_\Omega \nabla u \cdot \nabla u dV = 0$$

Thus

$$\int_\Omega \nabla u \cdot \nabla u dV = 0$$

which implies

$$\nabla u = 0$$

Thus $u$ =constant.

## 4. Cooling of a Sphere

**Exercise 1.** The problem is

$$-y'' - \frac{2}{\rho}y' = \lambda y, \quad y(0) \text{ bounded}, \ y(\pi) = 0$$

Making the transformation $Y = \rho y$ we get $-Y'' = \lambda Y$. If $\lambda = -k^2 < 0$ then

$$Y = a\sinh k\rho + b\cosh k\rho$$

or

$$y = \rho^{-1}(a\sinh k\rho + b\cosh k\rho)$$

For boundedness at $\rho = 0$ we set $b = 0$. Then $y(\pi) = 0)$ forces $\sinh k\pi = 0$. Thus $k = 0$. Consequently, there are no negative eigenvalues.

FIGURE 1. Temperature at the center of the sphere in Exercise 2.

**Exercise 2.** From the formula developed in the text the temperature at $\rho = 0$ is

$$u(0,t) = 74 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} e^{-n^2 kt}$$

where $k = 5.58$ inches-squared per hour. A graph of an approximation using fifty terms is shown in the figure.

**Exercise 3.** The boundary value problem is

$$\begin{aligned} u_t &= k(u_{\rho\rho} + \frac{2}{\rho} u_\rho) \\ u_\rho(R,t) &= -hu(R,t), \quad t > 0 \\ u(\rho, 0) &= f(\rho), \quad 0 \le \rho \le R \end{aligned}$$

Assume $u = y(\rho)g(t)$. Then the PDE and boundary conditions separate into the boundary value problem

$$y'' + (2/\rho)y' + \lambda y = 0, \quad y'(R) = -hy(R), \;\; y \text{ bounded}$$

and the differential equation

$$g- = -\lambda k g$$

The latter has solution $g(t) = exp(-\lambda kt)$. One can show that the eigenvalues are positive. So let $\lambda = p^2$ and make the substitution $Y = \rho y$, as in the text, to obtain

$$Y'' + p^2 Y = 0$$

This has solution

$$Y(\rho) = a \cos p\rho + b \sin p\rho$$

But $Y(0) = 0$ forces $a = 0$ (because $y$ is bounded). Then the other boundary condition forces $p$ to satisfy the nonlinear equation

$$\tan Rp = \frac{Rp}{1 - Rh}$$

If we graph both sides of this equation against $p$ we note that there are infinitely many intersections, giving infinitely many roots $p_n$, $n = 1, 2, \ldots$, and therefore infinitely many eigenvalues $\lambda_n = p_n^2$. The corresponding eigenfunctions are

$$y_n(\rho) = \rho^{-1} \sin(\sqrt{\lambda_n} \rho)$$

Thus we have

$$u(\rho, t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n k t} \rho^{-1} \sin(\sqrt{\lambda_n} \rho)$$

The $c_n$ are found from the initial condition. We have

$$u(\rho, 0) = f(\rho) = \sum_{n=1}^{\infty} c_n \rho^{-1} \sin(\sqrt{\lambda_n} \rho)$$

Thus

$$c_n = \frac{\int_0^R \rho f(\rho) \sin(\sqrt{\lambda_n} \rho) d\rho}{\int_0^R \sin^2(\sqrt{\lambda_n} \rho) \rho d\rho}$$

**Exercise 4.** Representing the Laplacian in spherical coordinates, the boundary value problem for $u = u(\rho, \phi)$, where $\rho \in (0, 1)$ and $\phi \in (0, \pi)$, is

$$\Delta u = u_{\rho\rho} + \frac{2}{\rho} u_\rho + \frac{1}{\rho^2 \sin \phi} (\sin \phi u_\phi) = 0$$
$$u(1, \phi) = f(\phi), \quad 0 \le \phi \le \pi$$

Observe, by symmetry of the boundary condition, $u$ cannot depend on the angle $\theta$. Now assume $u = R(\rho) Y(\phi)$. The PDE separates into two equations,

$$\rho^2 R'' + 2\rho R' - \lambda R = 0$$

and

$$\frac{1}{\sin \phi} (\sin \phi Y')' = \lambda Y$$

We transform the $Y$ equation by changing the independent variable to $x = \cos \phi$. Then we get, using the chain rule,

$$\frac{1}{\sin \phi} \frac{d}{d\phi} = -\frac{d}{dx}$$

So the $Y$–equation becomes

$$-\frac{d}{dx} \left( (1 - x^2) \frac{dy}{dx} \right) = \lambda y \quad -1 < x < 1$$

By the given facts, this equation has bounded, orthogonal, solutions $y_n(x) = P_n(x)$ on $[-1, 1]$ when $\lambda = \lambda_n = n(n+1)$, $n = 0, 1, 2, \ldots$. Here $P_n(x)$ are the Legendre polynomials.

Now, the $R$–equation then becomes

$$\rho^2 R'' + 2\rho R' - n(n+1) R = 0$$

This is a Cauchy-Euler equation (see the Appendix in the text) with characteristic equation

$$m(m-1)+m-n(n+1)=0$$

The roots are $m=n,-(n+1)$. Thus

$$R_n(\rho)=a_n\rho^n$$

are the bounded solutions (the other root gives the solution $\rho^{-n-1}$, which is unbounded at zero). Therefore we form

$$u(\rho,\phi)=\sum_{n=0}^{\infty}a_n\rho^nP_n(\cos\phi)$$

or, equivalently

$$u(\rho,x)=\sum_{n=0}^{\infty}a_n\rho^nP_n(x)$$

Applying the boundary condition gives the coefficients. We have

$$u(1,x)=f(\arccos x)=\sum_{n=0}^{\infty}a_nP_n(x)$$

By orthogonality we get

$$a_n=\frac{1}{||P_n||^2}\int_{-1}^{1}f(\arccos x)P_n(x)dx$$

or

$$a_n=\frac{1}{||P_n||^2}\int_{0}^{\pi}f(\phi)P_n(\cos\phi)\sin\phi\,d\phi$$

By direct differentiation we get

$$P_0(x)=1,\ P_1(x)=x,\ P_2(x)=\frac{1}{2}(3x^2-1),\ P_3(x)=\frac{5}{2}x^3-\frac{3}{2}x$$

Also the norms are given by

$$||P_0||^2=2,\ \ ||P_1||^2=\frac{2}{3},\ \ ||P_2||^2=\frac{2}{5}$$

When $f(\phi)=\sin\phi$ the first few Fourier coefficients are given by

$$a_0=\frac{\pi}{4},\ \ a_1=a_3=0,\ \ a_2=-0.49$$

Therefore a two-term approximation is given by

$$u(\rho,\phi)\approx\frac{\pi}{4}-\frac{0.49}{2}\rho^2(3\cos^2\phi-1)$$

**Exercise 5.** To determine the temperature of the earth we must derive the temperature formula for any radius $R$ (the calculation in the text uses $R=\pi$). The method is exactly the same, but now the eigenvalues are $\lambda_n=n^2\pi^2/R^2$ and the eigenfunctions are $y_n=\rho^{-1}\sin(n\pi\rho/R)$, for $n=1,2,\ldots$. Then the temperature is

$$u(\rho,t)=\frac{2RT_0}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-lam_nkt}\rho^{-1}\sin(n\pi\rho/R)$$

Now we compute the geothermal gradient at the surface, which is $u_\rho(\rho, t)$ at $\rho = R$. We obtain

$$u_\rho(R,t) = -\frac{2T_0}{R}\sum_{n=1}^{\infty} e^{-n^2\pi^2 kt/R^2}$$

If $G$ is the value of the geothermal gradient at the current time $t = t_c$, then

$$\frac{RG}{2T_0} = \sum_{n=1}^{\infty} e^{-n^2\pi^2 kt_c/R^2}$$

We must solve for $t_c$. Notice that the sum has the form

$$\sum_{n=1}^{\infty} e^{-an^2}$$

where $a = \pi^2 kt_c/R^2$. We can make an approximation by noting that the sum represents a Riemann sum approximation to the integral

$$\int_0^\infty e^{-ax^2}dx = \frac{1}{2}\sqrt{\pi/a}$$

So we use this value to approximate the sum, i.e.,

$$\sum_{n=1}^{\infty} e^{-n^2\pi^2 kt_c/R^2} \approx \frac{R}{2\sqrt{k\pi t_c}}$$

Solving for $t_c$ gives

$$t_c = \frac{T_0^2}{G^2 k\pi}$$

Substituting the numbers in from Exercise 5 in Section 2.4 gives $t_c = 5.15(10)^8$ years. This is the same approximation we found earlier.

## 5. Diffusion in a Disk

**Exercise 1.** The differential equation is $-(ry')' = \lambda ry$. Multiply both sides by $y$ and integrate over $[0, R]$ to get

$$\int_0^R -(ry')'ydr = \lambda\int_0^R ry^2dr$$

Integrating the left hand side by parts gives

$$-ryy' \mid_0^R \int_0^R r(y')^2dr = \lambda\int_0^R ry^2dr$$

But, since $y$ and $y'$ are assumed to be bounded, the boundary term vanishes. The remaining integrals are nonnegative and so $\lambda \geq 0$.

**Exercise 2.** Let $y, \lambda$ and $w, \mu$ be two eigenpairs. Then $-(ry')' = \lambda ry$ and $-(rw')' = \mu rw$. Multiply the first of these equations by $w$ and the second by $y$, and then subtract and integrate to get

$$\int_0^R [-(ry')'w + (rw')'y]dr = (\lambda - \mu)\int_0^R ruwdr$$

FIGURE 2. The Bessel functions $J_0(z_n r)$.

Now integrate both terms in the first integral on the left hand side by parts to get

$$(-ry'w + rw'y)\,|_0^R + \int_0^R (ry'w' - rw'y')dr = (\lambda - \mu)\int_0^R ruwdr$$

The left side of the equation is zero and so $y$ and $w$ are orthogonal with respect to the weight function $r$.

**Exercise 3.** We have

$$u(r,t) = \sum_{n=1}^{\infty} c_n e^{-0.25\lambda_n t} J_0(z_n r)$$

where

$$c_n = \frac{\int_0^1 5r^4(1-r)J_0(z_n r)dr}{\int_0^1 J_0(z_n r)^2 r dr}$$

We have $z_1 = 2.405,\ z_2 = 5.520,\ z_3 = 8.654$. Use a computer algebra program to calculate a three-term approximation.

**Exercise 4.** The eigenfunctions $J_0(z_n r)$, $n = 1, 2, 3, 4$ are sketched in the figure. For larger $n$ the number of oscillations increases.

**Exercise 5.** The Maple worksheet follows.

## 6. Sources on Bounded Domains

**Exercise 1.** Use Duhamel's principle to solve the problem

$$u_{tt} - c^2 u_{xx} = f(x,t), \quad 0 < x < \pi,\ t > 0$$
$$u(0,t) = u(\pi,t) = 0, \quad t > 0$$
$$u(x,0) = u_t(x,0) = 0, \quad 0 < x < 1$$

Consider the problem for $w = w(x,t,\tau)$, where $\tau$ is a parameter:

$$w_{tt} - c^2 w_{xx} = 0, \quad 0 < x < \pi,\ t > 0$$
$$w(0,t,\tau) = w(\pi,t,\tau) = 0, \quad t > 0$$
$$w(x,0,\tau) = 0,\ \ w_t(x,0,\tau) = 0, \quad 0 < x < 1$$

This problem was solved in Section 4.1 (see (4.14)–(4.14)). The solution is

$$w(x,t,\tau) = \sum_{n=1}^{\infty} c_n(\tau) \sin nct \sin nx$$

where

$$c_n(\tau) = \frac{2}{nc\pi} \int_0^{\pi} f(x,\tau) \sin nx\ dx$$

So the solution to the original problem is

$$u(x,t) = \int_0^t w(x, t-\tau, \tau) d\tau$$

**Exercise 2.** If $f = f(x)$, and does not depend on $t$, then the solution can be written

$$u(x,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \left( \int_0^{\pi} f(r) \sin nr\ dr \right) ( \left( \int_0^t e^{-n^2 k(t-\tau)} d\tau \right) \sin nx$$

But a straightforward integration gives

$$\int_0^t e^{-n^2 k(t-\tau)} d\tau = \frac{1}{kn^2}(1 - e^{-n^2 kt})$$

Therefore

$$u(x,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{kn^2}(1 - e^{-n^2 kt}) \left( \int_0^{\pi} f(r) \sin nr\ dr \right) \sin nx$$

Taking the limit as $t \to \infty$ gives

$$U(x) \equiv \lim u(x,t) = u(x,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{kn^2} \left( \int_0^{\pi} f(r) \sin nr\ dr \right) \sin nx$$

Now consider the steady state problem

$$-kv'' = x(\pi - x), \quad v(0) = v(\pi) = 0$$

This can be solved directly by integrating twice and using the boundary conditions to determine the constants of integration. One obtains

$$v(x) = -\frac{1}{12k}(2\pi x^2 - x^4 - \pi^3 x)$$

To observe that the solution $v(x)$ is the same as the limiting solution $U(x)$ we expand the right side of the $v$–equation in its Fourier sine series on $[0,\pi]$. Then

$$-kv'' = \sum_{n=1}^{\infty} c_n \sin nx$$

where

$$c_n = \frac{2}{\pi}\int_0^{\pi} r(\pi - r)\sin nr dr$$

Integrating the differential equation twice gives

$$-kv(x) + kv'(0)x = \sum_{n=1}^{\infty} c_n n^{-2}(\sin nx - x)$$

Evaluating at $x = \pi$ gives

$$kv'(0) = \sum_{n=1}^{\infty} c_n n^{-2}$$

Whence

$$kv(x) = \sum_{n=1}^{\infty} c_n n^{-2}\sin nx$$

or

$$v(x) = \frac{2}{k\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\int_0^{\pi} f(r)\sin nr\, dr\right)\sin nx$$

Hence $v(x) = U(x)$.

**Exercise 3.** Following the hint in the text we have

$$u(x,y) = \sum_{n=1}^{\infty} g_n(y)\sin nx$$

where we find

$$g_n'' - n^2 g_n = f_n(y), \quad g_n(0) = g_n(1) = 0$$

where $f_n(y)$ are the Fourier coefficients of $f(x,y)$. From the variation of parameters formula

$$g_n(y) = ae^{ny} + be^{-ny} - \frac{2}{n}\int_0^{y}\sinh(n(\xi - y))f_n(\xi)d\xi$$

Now $g_n(0) = 0$ implies $b = -a$. So we can write

$$g_n(y) = 2a\sinh ny - \frac{2}{n}\int_0^{y}\sinh(n(\xi - y))f_n(\xi)d\xi$$

which gives, using $g_n(1) = 0$,

$$a = \frac{1}{n\sinh n}\int_0^{1}\sinh(n(\xi - 1))f_n(\xi)d\xi$$

Thus the $g_n(y)$ are given by

$$g_n(y) = \frac{2\sinh ny}{n\sinh n}\int_0^{1}\sinh(n(\xi - 1))f_n(\xi)d\xi - \frac{2}{n}\int_0^{y}\sinh(n(\xi - y))f_n(\xi)d\xi$$

**Exercise 4.** The problem is

$$u_t = \Delta u + f(r,t) \quad 0 \le r < R,\ t > 0$$
$$u(R,t) = 0, \quad t > 0$$
$$u(r,0) = 0, \quad 0 < r < R$$

For $w = w(r,t,\tau)$ we consider the problem

$$w_t = \Delta w \quad 0 \le r < R,\ t > 0$$
$$w(R, t\tau) = 0, \quad t > 0$$
$$w(r,0,\tau) = f(r,\tau), \quad 0 < r < R$$

This is the model for heat flow in a disk of radius $R$; the solution is given by equation (4.53) in Section 4.5 of the text. It is

$$w(r,t,\tau) = \sum c_n(\tau) e^{-\lambda_n k t} J_0(z_n r/R)$$

where $z_n$ are the zeros of the Bessel function $J_0$, $\lambda_n = z_n^2/R^2$ and

$$c_n(\tau) = \frac{1}{||J_0(z_n r/R)||^2} \int_0^R f(r,\tau) J_0(z_n r/R) r dr$$

Then

$$u(r,t) = \int_0^R w(r, t-\tau, \tau) d\tau$$

## 7. Parameter Identification Problems

**Exercise 1.** We have

$$p_1 = p_0 e^{r_1 t^*}, \quad p_2 = p_0 e^{r_2 t^*}$$

Dividing the two equations and then taking natural logarithms gives

$$r_1 - r_2 = \frac{1}{t^*}(\ln p_1 - \ln p_2)$$

By the mean value theorem

$$\frac{|\ln a - \ln b|}{|a-b|} = \frac{d \ln x}{dx} |_{x=c} = \frac{1}{c}$$

for some $c$ between $a$ and $b$. Thus

$$|r_1 - r_2| = \frac{1}{t^*} |\ln p_1 - \ln p_2| \le \frac{1}{M t^*} |p_1 - p_2|$$

**Exercise 2.** We have $\rho(x) u_{tt} = u_{xx}$. Putting $u = Y(x) g(t)$ gives, upon separating variables,

$$-y'' = \rho(x) \lambda y, \quad y(0) = y(1) = 0$$

We have

$$-y_f'' = \rho(x) \lambda_f y_f, \quad y_f(0) = y_f(1) = 0$$

Integrating from $x = 0$ to $x = s$ gives

$$-y_f'(s) + y_f'(0) = \lambda_f \int_0^s \rho(x) y_f(x) dx$$

Now integrate from $s = 0$ to $s = 1$ to get

$$\begin{aligned} y_f'(0) &= \lambda_f \int_0^1 \int_0^s \rho(x) y_f(x) dx ds \\ &= \lambda_f \int_0^1 (1-x)\rho(x) y_f(x) dx \end{aligned}$$

The last step follows by interchanging the order of integration. If $\rho(x) =_r ho_0$ is a constant, then

$$\rho_0 = \frac{y_f'(0)}{\lambda_f \int_0^1 (1-x) y_f(x) dx}$$

**Exercise 3.** From Exercise 2 in Section 4.6 we have the solution

$$u(x,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{kn^2}(1 - e^{-n^2 kt}) \left( \int_0^{\pi} f(r) \sin nr \, dr \right) \sin nx$$

Therefore

$$U(t) = u(\pi/2, t) = \int_0^{\pi} \left( \frac{2}{\pi} \sum_{n=1}^{\infty} \sin nr \sin(n\pi/2) \frac{1}{kn^2}(1 - e^{-n^2 kt}) \right) f(r) dr$$

We want to recover $f(x)$ if we know $U(t)$. This problem is not stable, as the following example shows. Let

$$u(x,t) = m^{-3/2}(1 - e^{-m^2 t}) \sin mx, \quad f(x) = \sqrt{m} \sin mx$$

This pair satisfies the model. If $m$ is sufficiently large, then$u(x,t)$ is uniformly small; yet $f(0)$ is large. So a small error in measuring $U(t)$ will result in a large change in $f(x)$.

**Exercise 4.** The problem is

$$\begin{aligned} u_t = u_{xx}, &\quad x, t > 0 \\ u(x,0) = 0, &\quad x > 0 \\ u(0,t) = f(t), &\quad t > 0 \end{aligned}$$

Taking Laplace transforms and solving gives

$$U(x,s) = F(s) e^{-x\sqrt{s}}$$

Here we have discarded the unbounded part of the solution. So, by convolution,

$$u(x,t) = \int_0^t f(\tau) \frac{x}{\sqrt{4\pi(t-\tau)^3}} e^{-x^2/4(t-\tau)} d\tau$$

Hence, evaluating at $x = 1$,

$$U(t) = \int_0^t f(\tau) \frac{1}{\sqrt{4\pi(t-\tau)^3}} e^{-1/4(t-\tau)} d\tau$$

which is an integral equation for $f(t)$. Suppose $f(t) = f_0$ is constant and $U(5) = 10$. Then

$$\frac{20\sqrt{\pi}}{f_0} = \int_0^5 \frac{e^{-1/4(5-\tau)}}{\sqrt{4\pi(5-\tau)^3}} d\tau = 2 \; erfc \; (1/\sqrt{5})$$

where $erfc = 1 - erf$. Thus $f_0 = 59.9$ degrees.

**Exercise 5.** The problem

$$u_t = Du_{xx} - vu_x, \quad x \in R,\, t > 0; \quad u(x,0) = e^{-ax^2}, \;\; x \in R$$

can be solved by Fourier transforms to get

$$u(x,t) = \frac{\sqrt{a}}{\sqrt{a+4Dt}} e^{-(x-vt)^2/((a+4Dt)}$$

Thus, choosing $a = v = 1$ we get

$$U(t) = u(1,t) = \frac{1}{\sqrt{1+4Dt}} e^{-(1-t)^2/((1+4Dt)}$$

## 8. Finite Difference Methods

**Exercise 1.** The Cauchy-Euler algorithm for this problem is

$$Y_{n+1} = Y_n + h(-2nhY_n + Y_n^2), \quad n = 0, 1, 2, \ldots; Y_0 = 1$$

where $h$ is the step size. With $h = 0.1$ the values of $Y_n$ at the points $t_n = nh,\ n = 0, \ldots, 10$ are:

$$1, 1.1, 1.199, 1.295, 1.385, 1.466, 1.534, 1.585, 1.615, 1.617, 1.587$$

**Exercise 2.** A time snapshot of the solution surface at $t = 1$ is shown in the accompanying figure. The Maple program in Figure 4.8 of the text was run with $h = 0.1$ and $k = 0.1$, which gives $r = k/h^2 = 10$. This violates the stability condition, and one can observe the highly oscillatory behavior of the numerical scheme.

**Exercise 3.** The Maple worksheet and surface plot is given in the two figures.

**Exercise 4.** The Maple worksheet and surface plot is given in the two figures.

**Exercise 5.** The Maple worksheet and surface plot for the first problem in Exercise 5 is given in the accompanying figures.

FIGURE 3. Time $t = 1$ profile of the numerical solution in Exercise 2 when $h/k = 10$.

FIGURE 4. Maple program to solve Exercise 3.

Figure 5. Solution surface in Exercise 3.

FIGURE 6. Maple program to solve Exercise 4.

FIGURE 7. Solution surface in Exercise 4.

Figure 8. Maple program to solve Exercise 5.

Figure 9. Solution surface in Exercise 5.

CHAPTER 2

# Appendix

**1.** The equation $y' + 2y = e^{-x}$ is first order, linear. Multiply by the integrating factor $e^{2x}$ and the equation becomes $(ye^{2x})' = e^x$ Integrate both sides and multiply by $e^{-2x}$ to get

$$y(x) = Ce^{-2x} + e^{-x}$$

**2.** Here, $y' = -3y$. Separate variables and integrate to obtain

$$y(x) = Ce^{-3x}$$

**3.** The equation $y'' + 8y = 0$ is second-order, linear, with constant coefficients. The characteristic equation is $m^2 + 8 = 0$ which has roots $m = \pm\sqrt{8}$. Then

$$y(x) = A\cos\sqrt{8}x + B\sin\sqrt{8}x$$

**4.** The equation $y' - xy = x^2y^2$ is a Bernoulli equation. Make the substitution $w = 1/y$ and the equation turns into a linear equation$w' + xw = -x^2$. The integrating factor is $\exp(x^2/2)$. Multiplying by the integrating factor gives

$$(we^{x^2/2})' = -x^2e^{x^2/2}$$

Integrating both sides from 0 to $x$ gives

$$we^{x^2/2} - w(0) = -\int_0^x r^2e^{r^2/2}dr$$

Then

$$w(x) = 1/y(x) = e^{-x^2/2}w(0) - e^{-x^2/2}\int_0^x r^2e^{r^2/2}dr$$

**5.** The equation $x^2y'' - 3xy' + 4y = 0$ is a Cauchy-Euler equation. The characteristic equation is $m(m-1) - 3m + 4 = 0$ which has roots $m = 2, 2$. Thus

$$y(x) = ax^2 + bx^2\ln x$$

**6.** The equation $y'' + x(y')^2 = 0$ does not have $y$ appearing explicitly. So let $v = y'$ to obtain

$$v' + xv^2 = 0$$

We separate variables to get

$$\frac{1}{v} = \frac{1}{2}x^2 + A$$

Then

$$y(x) = \int \frac{dx}{x^2/2 + A} + B = \frac{2}{\sqrt{2A}} \arctan \frac{x}{\sqrt{2A}} + B$$

**7.** The equation $y'' + y' + y = 0$ is linear, second-order, with constant coefficients. The characteristic equation is $m^2 + m + 1 = 0$, which has roots $m = -1/2 \pm \sqrt{3}i/2$. Thus

$$y(x) = e^{-x/2} \left( a \cos \frac{\sqrt{3}x}{2} + b \sin \frac{\sqrt{3}x}{2} \right)$$

**8.** In the equation $yy'' - (y')^3 = 0$ the independent variable does not appear explicitly. So let $v = y'$ which gives $y'' = v\frac{dv}{dy}$. Then we get

$$yv\frac{dv}{dy} = v^3$$

Separating variables and solving gives $v = -1/(C + \ln y)$. Then

$$(C + \ln y)dy = -dx$$

Then

$$Cy + y \ln y - y = -x + B$$

**9.** The equation $2x^2y'' + 3xy' - y = 0$ is a Cauchy-Euler equation with characteristic equation $2m(m-1) + 3m - 1 = 0$. The roots are $m = -1, 1/2$. Then

$$y(x) = a\sqrt{x} + \frac{b}{x}$$

**10.** The equation $y'' - 3y' - 4y = 2 \sin x$ is a linear, nonhomogeneous equation. The homogeneous solution is $y_h(x) = ae^{4t} + be^{-t}$. Guess a particular solution of the form $y_p(x) = A \sin x + B \cos x$. Substitute into the equation to find $A = 5/8,\ B = -3/8$. Then

$$y(x) = ae^{4t} + be^{-t} + \frac{5}{8} \sin x - \frac{3}{8} \cos x$$

**11.** The homogeneous solution of $y'' + 4y = x \sin x$ is $y_h(x) = a \cos 2x + b \sin 2x$. A particular solution can be found by undetermined coefficients or using the variation of parameters formula from this appendix. We choose the latter. We have

$$y_p(x) = \int_0^x \frac{\cos 2s \sin 2x - \cos 2x \sin 2s}{2} s \sin 2s \, ds$$

The calculation is left as an exercise. The solution is the sum of $y_h$ and $y_p$.

**12.** The equation $y' - 2xy = 1$ is linear with integrating factor $exp(-x^2)$. Multiplying by the integrating factor leads to

$$(ye^{-x^2})' = e^{-x^2}$$

Integrating from 0 to $x$ then gives

$$y(x) = y(0)e^{x^2} + \int_0^x e^{x^2-r^2}\,dr$$

**13.** The second order linear equation $y'' + 5y" + 6y = 0$ has characteristic equation $m^2 + 5m + 6 = 0$ with roots $-3, -2$. Hence

$$y(x) = ae^{-3x} + be^{-2x}$$

**14.** Separate variables to obtain

$$(1 + 3y^3)dy = x^2 dx$$

Integrating gives the implicit solution

$$y + \frac{3}{4}y^4 = \frac{1}{3} + C$$