CHAPTER 5

PDEs in The Life Sciences

1. Age-Structured Models

Exercise 1: Write

$$1=\int_3^8 4e^{-(r+0.03)a}da=-\frac{4}{r+0.03}\left(e^{-8(r+0.03)}-e^{-3(r+0.03)}\right),$$

and use a software package or calculator to solve for r.

Exercise 2: First note that $u(a,t)=0$ for $a>t+\delta$, since $f(a)=0$ for $a>\delta$. For (a) observe that the renewal equation (5.9) is

$$B(t)=\int_0^t \beta B(t-a)e^{-\gamma a}da+\int_0^\infty \beta f(a-t)e^{-\gamma t}da.$$

The first integral becomes, upon changing variables to $s=t-a$,

$$\int_0^t \beta B(t-a)e^{-\gamma a}da=\int_0^t \beta B(s)e^{-\gamma(t-s)}ds.$$

The second integral is

$$\int_0^\infty \beta f(a-t)e^{-\gamma t}da=\int_t^{t+\delta}\beta u_0 e^{-\gamma t}da=\beta u_0\delta e^{-\gamma t}.$$

For (b), differentiate (using Leibniz rule)

$$B(t)=\int_0^t \beta B(s)e^{-\gamma(t-s)}ds+\beta u_0\delta e^{-\gamma t}$$

to get

$$B'(t)=\int_0^t \beta B(s)e^{-\gamma(t-s)}ds(-\gamma)+\beta B(t)-\gamma\beta u_0\delta e^{-\gamma t}=(\beta-\gamma)B(t).$$

For (c) note that the last equation is the differential equation for growth-decay and has solution

$$B(t)=B(0)e^{(\beta-\gamma)t}.$$

Therefore the solution from (5.7)–(5.8) is given by

$$u(a,t)=\begin{cases} 0, & a>t+\delta \\ u_0e^{-\gamma t}, & t<a<t+\delta \\ B(0)e^{(\beta-\gamma)t}e^{-\beta a}, & 0<a<t. \end{cases}$$

Finally, for part (d) we have, using part (c),

$$\begin{aligned} N(t) &= \int_0^t u(a,t)da + \int_t^{t+\delta} u(a,t)da \\ &= \int_0^t B(0)e^{(\beta-\gamma)t}e^{-\beta a}da + \int_t^{t+\delta} u_0 e^{-\gamma t}da \\ &= \frac{B(0)}{\beta}e^{(\beta-\gamma)t}(1-e^{-\beta t}) + \delta u_0 e^{-\gamma t}. \end{aligned}$$

Exercise 3: Integrate the PDE from $a = 0$ to $a = \infty$ to get

$$N(t) = -\int_0^\infty u_a da - m(N)N = B(t) - m(N)N.$$

To get an equation for B we differentiate the $B(t)$ equation to get

$$\begin{aligned} B'(t) &= \int_0^\infty b_0 e^{-\gamma a}u_t da = \int_0^\infty b_0 e^{-\gamma a}(-u_a - m(N)u)da \\ &= -m(N)B(t) - \int_0^\infty b_0 e^{-\gamma a}(u_a)da \\ &= -m(N)B(t) - \left[b_0 e^{-\gamma a}u|_0^\infty + \int_0^\infty b_0 \gamma e^{-\gamma a}u da \right] \\ &= -m(N)B(t) - b_0 B(t) - \gamma B(t). \end{aligned}$$

To obtain the next-to-last line we used integration by parts. In summary we have the dynamical system

$$\begin{aligned} N' &= B - m(N)N, \\ B' &= (b_0 - \gamma - m(N))B. \end{aligned}$$

In the phase plane the paths or integral curves are defined by

$$\frac{dB}{dN} = \frac{(b_0 - \gamma - m(N))B}{B - m(N)N}.$$

Observe that $B = (b_0 - \gamma)N$ is easily shown to be a solution to this equation. It represents a straight line in the NB plane. The line B=0 is a horizontal nullcline where the vector field points to the left. Another horizontal nullcline is the vertical line $N = N^*$, where N^* is the root of $m(N) = (b_0 - \gamma)$. The point $P = (N^*, (b_0 - \gamma)N^*)$ is an equilibrium that lies on the straight line solution curve $B = (b_0 - \gamma)N$. The solution cannot oscillate since that it would require it cross the straight line, violating uniqueness. On the straight line solution, the direction is toward the point P.

Exercise 4: In preparation.

Exercise 5: Let $\xi = a - t, \tau = t$. In these characteristic coordinates the PDE becomes

$$U_\tau = -\frac{c}{d - \xi - \tau}U.$$

Separating variable and integrating gives

$$U = (d - \xi - \tau)^c \varphi(\xi)$$

or

$$u(a, t) = (d - a)^c \varphi(a - t),$$

which is the general solution. Now, for the region $a > t$ we use the initial condition to determine φ. We have

$$u(a, 0) = (d - a)^c \varphi(a) = f(a),$$

which gives $\varphi(a) = f(a)(d - a)^c$. Hence

$$u(a, t) = (d - a)^c f(a - t)(d - a - t)^{-c}, \quad a > t.$$

For the region $a < t$ we use the boundary conditon. to determine φ. Thus,

$$u(0, t) = d^c \varphi(-t) = B(t),$$

or

$$\varphi(t) = B(-t) d^{-c}.$$

Whence

$$u(a, t) = (d - a)^c B(t - a) d^{-c}, \quad 0 < a < t.$$

Exercise 6: Using Taylor's expansion to write

$$u(a + da, t + dt) = u(a, t) + u_t(a, t)da + u_t(a, t)dt + \text{ higher order terms.}$$

2. Traveling Wave Fronts

Exercise 1: The traveling wave equation can be written

$$-cU' = DU'' - \frac{1}{2}(U^2)'.$$

Integrating, we get

$$-cU = DU' - \frac{1}{2}U^2 + A.$$

Using the boundary condition at $z = +\infty$ forces $A = 0$. Using the boundary condition at $z = -\infty$ gives the wave speed $c = 1/2$. Therefore

$$DU' = \frac{1}{2}U(U - 1).$$

This DE has equilibria at $U = 0, 1$; the solution can be found by separating variables or noting it is a Bernoulli equation (see the Appendix on Differential Equations). The graph falls from left to right (decreasing), approaching 0 at plus infinity and 1 at minus infinity.

Exercise 2: The traveling wave equation may be written

$$-cU' = U'' - \frac{1}{3}(U^3)'.$$

Integrating, we find that the constant of integration is zero from the $z = +\infty$ boundary condition. Then

$$-cU = U' - \frac{1}{3}U^3.$$

Applying the condition at $z = -\infty$ we get

$$-cU_l = -\frac{1}{3}U_l^3,$$

or

$$-cU_l + \frac{1}{3}U_l^3 = -\frac{1}{3}U_l(3c - U_l^2) = 0.$$

Therefore $U_l = \sqrt{3c}$.

Exercise 3: To have constant states at infinity we must have $F(0, v_r) = 0 = F(u_l, 0) = 0$. The traveling wave equations are

$$\begin{aligned} -cU' &= DU'' - \gamma U' - aF(U, V), \\ -cV' &= -bF(U, V). \end{aligned}$$

Clearly we may write a single equation

$$-cU' = DU'' - \gamma U' - \frac{ac}{b}V'.$$

Now we may integrate to get

$$-cU = DU' - \gamma U - \frac{ac}{b}V + A.$$

The right boundary condition forces $A = acv_r/b$. The left boundary condition then gives

$$-cu_l = -\gamma u_l + \frac{ac}{b}v_r$$

or

$$(\gamma - c)u_l = \frac{ac}{b}v_r > 0.$$

Therefore $c < \gamma$

Exercise 4: In preparation.

Exercise 5: The traveling wave equation is

$$-c\left[(1 + b)U - mU^2\right]' = U'' - U'.$$

Integrating gives

$$-c\left[(1 + b)U - mU^2\right] = U' - U + A.$$

¿From the boundary condition $U(+\infty) = 0$ we get $A = 0$. Since $U(-\infty) = 1$, we get

$$c = \frac{1}{1 + b - m} > 0.$$

The differential equation then simplifies to

$$U' = (1 - c - cb)U + cmU^2,$$

which is a Bernoulli equation. It is also separable.

3. Equilibria and Stability

Exercise 1: The equilibria are roots of

$$f(u) = ru(1 - u/K) - hu = u(r - \frac{r}{K}u - h) = 0.$$

So the equilibria are

$$u_1 = 0, \quad u_2 = \frac{r-h}{r}K.$$

To check stability we calculate $f'(u_1) = r - \frac{2r}{K}u - h$. Then $f'(0) = r > 0$, so $u_1 = 0$ is unstable. Next

$$f'(\frac{r-h}{r}K) = r - \frac{2r}{K}\frac{r-h}{r}K - h = r - 2r + 2h - h = -r + h.$$

Therefore u_2 is stable if $r > h$ and unstable if $r < h$.

Exercise 2: In preparation.

Exercise 3: In preparation.

Exercise 4: To obtain (a) just substitute $u_e(x)$ into the PDE and check the boundary conditions. To get (b) substitute $u = u_e(x) + U(x,t)$ into the PDE to obtain

$$U_t = u_e'' + U_{xx} + (u_e(x) + U(x,t))(1 - u_e(x) - U(x,t)),$$

or

$$U_t = u_e'' + U_{xx} + u_e(x)(1 - u_e(x)) - u_e(x)U + U(1 - u_e(x)) - U^2(x,t).$$

But $u_e'' + u_e(x)(1 - u_e(x)) = 0$, and neglecting the nonlinear term gives

$$U_t = U_{xx} + (1 - 2u_e(x))U,$$

which is the linearized perturbation equation. The boundary conditions are $U(\pm\pi/2) = 0$. For part (c) assume that $U = e^{\sigma t}g(x)$ and substitute to get

$$\sigma g = g'' + (1 - 2u_e(x))g,$$

or

$$g'' + \frac{\cos x - 5}{1 + \cos x}g = \sigma g,$$

with $g = 0$ at $x = \pm\pi/2$. Finally, to prove (d), we proceed as in the hint. If this BVP has a nontrivial solution, then it must be, say, positive somewhere in the interval. (The negative case can be treated similarly). So it must have a positive maximum in the interval. At this maximum, $g > 0$, $g'' < 0$. Therefore

$$\frac{\cos x - 5}{1 + \cos x}g < 0.$$

So the left side of the DE is negative, so $\sigma < 0$.