Real Hypersurfaces and Complex Analysis

Howard Jacobowitz

The theory of functions (what we now call the theory of functions of a complex variable) was one of the great achievements of nineteenth century mathematics. Its beauty and range of applications were immense and immediate. The desire to generalize to higher dimensions must have been correspondingly irresistible. In this desire to generalize, there were two ways to proceed. One was to focus on functions of several complex variables as the generalization of functions of one complex variable. The other was to consider a function of one complex variable as a map of a domain in $\mathbb{C}$ to another domain in $\mathbb{C}$ and to study, as a generalization, maps of domains in $\mathbb{C}^n$. Both approaches immediately led to surprises and both are still active and important. The study of real hypersurfaces arose within these generalizations. This article surveys some contemporary results about these hypersurfaces and also briefly places the subject in its historical context. We organize our survey by considering separately these two roads to generalization.

We start with a hypersurface M^{2n-1} of $\mathbb{R}^{2n}$ and consider it as a hypersurface of $\mathbb{C}^n$, using an identification of $\mathbb{R}^{2n}$ with $\mathbb{C}^n$. We call M a real hypersurface of the complex space $\mathbb{C}^n$ to distinguish it from a complex hypersurface, that is, a complex $n-1$ dimensional submanifold of $\mathbb{C}^n$. This said, the dimensions in statements like

$$M^{2n-1} \subset \mathbb{C}^n$$

should not cause any concern. The best example to keep in mind is the boundary of an open subset of $\mathbb{C}^n$ (whenever this boundary is smooth). Indeed, much of the excitement in the study of real hypersurfaces comes from the interplay between the domain and the boundary and between the geometry and the analysis.

Howard Jacobowitz is professor of mathematics at Rutgers University-Camden. His e-mail address is `jacobowi@crab.rutgers.edu`.

Work supported in part by NSF Grant #DMS 94-04494.

Functions

It is natural to begin by considering a function on $\mathbb{C}^n$ as holomorphic if it is holomorphic in each variable separately (that is, it is holomorphic when restricted to each of the special complex lines $\{z = (z_1, \ldots, z_n) \in \mathbb{C}^n | z_k$ fixed for all k except for $k = j$ and z_j arbitrary$\}$). For continuous functions this coincides with any other reasonable generalization (say by convergent power series or by the solution of the Cauchy-Riemann equations). Almost at once, we encounter a striking difference between functions of one and more complex variables. (Contrast this to the theory of functions of real variables, where one must delve deeply before the dimension is relevant.) For instance, consider the domain obtained by poking a balloon gently with your finger, but in $\mathbb{C}^2$, of course. More concretely, consider a domain in $\mathbb{C}^2$ that contains the set

$$H = \{|z| < 2, |w| < 1\} \bigcup \left\{\frac{1}{2} < |z| < 2, |w| < 2\right\}. \tag{1}$$

We show that every function holomorphic on this set is also holomorphic on the larger set (see Figure 1)

$$P = \{|z| < 2, |w| < 2\}.$$

It follows, by using an appropriate modification of H, that every function holomorphic on the interior of the poked balloon is also holomorphic on a somewhat larger set (but perhaps not on all of the interior of the original balloon). There is no similar extension phenomenon for functions of one complex variable.

It is very easy to prove that any function holomorphic on H is also holomorphic on P. In doing so, we see how the extra dimension is used. Let $f(z)$ be holomorphic on H and for $|z| < 1$ set

$$h(z,w) = \frac{1}{2\pi i}\oint_{|\zeta|=1} \frac{f(\zeta,w)}{z-\zeta}d\zeta.$$

Then h is holomorphic on $\{|z| < 1, |w| < 2\}$. Further, h agrees with f on $\{|z| < 1, |w| < 1\}$ and thus h agrees with f also on $\{\frac{1}{2} < |z| < 1, |w| < 2\}$. Hence h is the sought-after extension of f to P.

In this way, we have "extended" the original domain H and it becomes of interest to characterize those domains that cannot be further extended. This leads to the main topics of several complex variables: domains of holomorphy (those domains which cannot be extended), pseudoconvex domains, holomorphic convexity, etc. Most of this theory developed without consideration of the boundaries of the domains, so it is not strictly about real hypersurfaces—we skip over it in this survey.

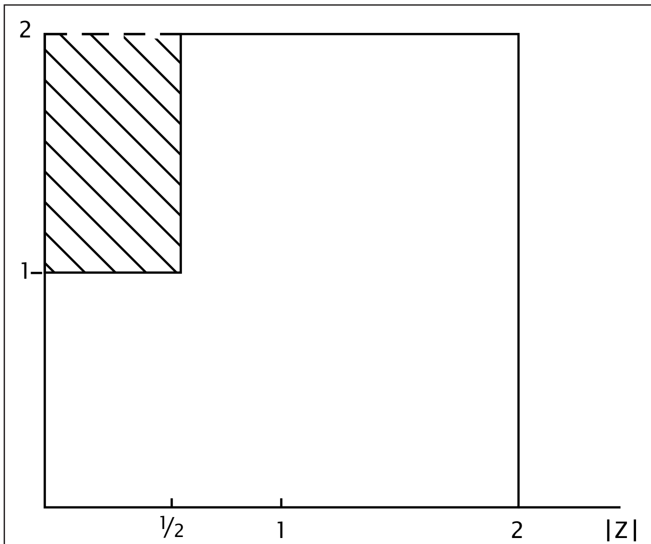
E. E. Levi was apparently the first (1909) to try to characterize those domains of holomorphy that have smooth boundaries. It is easy to see that a convex domain must be a domain of holomorphy. But convexity is not preserved under biholomorphisms while the property of being a domain of holomorphy is so preserved. Levi discovered the analog of convexity appropriate for complex analysis. Let $\Omega \subset \mathbb{C}^n$ have smooth boundary M. Let r be any defining function for Ω; so, $r \in C^\infty$ in a neighborhood of Ω, $r < 0$ in Ω, $r = 0$ on M, and $dr(p) \neq 0$ for each $p \in M$.

Let $V^0 \subset \mathbb{C} \otimes T\mathbb{C}^n$ consist of all tangent vectors of the form

$$L = \sum_{j=1}^{n} \alpha_j \frac{\partial}{\partial \bar{z}_j}$$

and let

$$V = (\mathbb{C} \otimes TM) \bigcap V^0.$$



The point (a, b) represents the torus {|z| = a, |w| = b}

Figure 1

Definition. The Levi form is the hermitian form $\mathcal{L} : V \times \bar{V} \to \mathbb{C}$ given by

$$\mathcal{L}(L,\bar{L}) = \frac{\partial^2 r}{\partial \bar{z}_j \partial z_k}\alpha_j \bar{\alpha}_k$$

for $L = \sum \alpha_j \frac{\partial}{\partial \bar{z}_j} \in V$.

The derivatives are computed according to the rules

$$\frac{\partial}{\partial z}f = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial}{\partial \bar{z}}f = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

(note that in this notation the Cauchy-Riemann equations are just $\frac{\partial f}{\partial \bar{z}} = 0$). $\mathcal{L}$ depends upon the choice of the defining function r in that it is multiplied by a positive function when r is replaced by another defining function for Ω. Since $\mathcal{L}$ is hermitian, its eigenvalues are real and the numbers of positive, negative, and zero eigenvalues do not depend on the choice of r. These numbers are also unchanged under a holomorphic change of coordinates $z \to \zeta(z)$.

Levi's Theorem. If Ω is a domain of holomorphy, then $\mathcal{L}$ is positive semi-definite ($\mathcal{L}(L,\bar{L}) \geq 0$ for all $L \in V_p$ and all $p \in M$).

We abbreviate the conclusion as $\mathcal{L} \geq 0$ and say that Ω is pseudoconvex if this condition holds at all boundary points. If instead we have that $\mathcal{L}$ is positive definite, $\mathcal{L} > 0$, at all boundary points, we say that Ω is strictly pseudoconvex.

To see that this condition generalizes convexity, recall that $X = \{r = 0\}$ is a convex hypersurface in R^n if

$$\sum_{j,k} \frac{\partial^2 r}{\partial x_j \partial x_k} a_j a_k > 0$$

for all vectors

$$\sum a_j \frac{\partial}{\partial x_j}$$

tangent to X.

We have already seen an example of Levi's theorem. The sphere is strictly pseudoconvex. The "poked" sphere has points where $\mathcal{L} < 0$. Given F holomorphic on the poked sphere, we can place a domain like (1) right near the poke and extend F to a somewhat larger open set. This is how Levi's Theorem is proved; the geometry for any open set at points where $\mathcal{L} < 0$ is similar to that of the poked sphere.

The Levi problem is to prove the converse of this theorem. It is easy to show where the difficulty arises. Early work on the problem, by mathematicians such as Behnke, H. Cartan, Stein, and Thullen, show it is enough to prove that if Ω is strictly pseudoconvex, then for each $p \in$ boundary Ω there exists a function F holomorphic on Ω with $|F(z)| \to \infty$ as $z \to p$. Given p, with $\mathcal{L} > 0$ at p, there is an open neighborhood U of p and a function holomorphic on $U \cap \Omega$ that blows up at p. This function is given explicitly in terms of the defining function of the domain. For the unit sphere and $p = (0, 1)$,

$$F = \frac{1}{1 - w}$$

works, where a point in $\mathbb{C}^2$ is designated (z, w). The entire difficulty in general is to go from F holomorphic on $U \cap \Omega$ to some other function G holomorphic on all of Ω in such a way that $|G|$ still blows up at p. (Of course, for the sphere, $\frac{1}{1-w}$ does work globally.) What is needed is a way to patch local analytic information to end up with a global analytic object. This can be done in two general ways; the mantras are "sheaf theory" and "partial differential equations". Note that if Ω is convex, then an explicit F works globally, just as in the case of the sphere. But strictly pseudoconvex domains definitely do not have to be convex. For instance, see [11, page 110] for a strictly pseudoconvex solid torus.

The Levi problem was solved in 1953 by Oka. Thus, pseudoconvexity characterizes domains of holomorphy. An immediate corollary is that pseudoconvexity is of basic importance. We shall see this again below, when we investigate its relation to partial differential equations.

Levi's theorem gives an extension theorem. If $\mathcal{L}$ is not positive semi-definite at some point $p \in$ boundary Ω, then Ω is not a domain of holomorphy and, as for our poked balloon, any function holomorphic on Ω is also holomorphic on $\Omega \cup U$ where U is a neighborhood of p. This is a local result. That is, if f is holomorphic on some $\Omega \cap U$ where U is a neighborhood of p, and $\mathcal{L}_p < 0$, then f is also holomorphic on $\Omega \cup V$ where V is a (perhaps smaller) neighborhood of p. There is also a global extension result of Hartogs (also around 1909). This does not depend on pseudoconvexity.

Hartogs's Extension Theorem. Let Ω be any open set in $\mathbb{C}^n$ and let K be a compact subset of Ω such that $\Omega - K$ is connected. Then any function holomorphic on $\Omega - K$ is the restriction of a function holomorphic on Ω.

This theorem is the most compelling evidence that function theory in $\mathbb{C}^n$ is not just a straightforward generalization of that in $\mathbb{C}^1$. In particular, it implies that only in $\mathbb{C}^1$ can holomorphic functions have isolated singularities.

There is a version of Hartogs's theorem that focuses on real hypersurfaces. Let us return to

$$V = (\mathbb{C} \otimes TM) \bigcap V^0.$$

Geometrically, V at a point $p \in M$ is the set of those vectors of the form

$$L = \sum_j \alpha_j \frac{\partial}{\partial \bar{z}_j}$$

that are tangent to the boundary of Ω at p (in the sense that ReL and ImL are tangent to the boundary M of Ω at p). From the viewpoint of analysis, it is more natural to consider L as a first-order partial differential operator acting on functions.

Recall that F is holomorphic if $\frac{\partial F}{\partial \bar{z}_j} = 0$ for all j, since these are just the Cauchy-Riemann equations in each variable. Since $L \in V_p$ is a combination of the operators $\frac{\partial}{\partial \bar{z}_j}$, $LF = 0$. On the other hand, L is tangential and so operates on functions defined on M. Thus, L annihilates the restriction of F to M. This is true even if F is only holomorphic on one side of M, and smooth up to M.

So $Lf = 0$ is a necessary condition for a function f on M to extend to a function holomorphic in a possibly one-sided neighborhood of M.

Definition. A C^1 function f on M is called a CR function if $Lf = 0$ for all $L \in V$.

CR stands for Cauchy-Riemann and signifies that f satisfies the induced Cauchy-Riemann equations (those equations induced on M by the Cauchy-Riemann equations on $\mathbb{C}^n$).

Theorem (Bochner). Let Ω be a bounded open set in $\mathbb{C}^n$ with smooth boundary M^{2n-1} and connected complement. For each CR function f on M there is some function F, necessarily unique, holomorphic on Ω, and differentiable up to the boundary M, such that $f = F|_M$.

What about a local version of this extension theorem? We have seen that if F is holomorphic in a neighborhood of $p \in M$, then $f = F|_M$ is annihilated by each $L \in V$. The converse is true when M and f are real analytic (but not in general) and can be proved by complexifying $M$and f.

Theorem. Let M be a real analytic hypersurface in $\mathbb{C}^n$ and let f be a real analytic CR function on M. Then there exists an open neighborhood U of M and a function F, holomorphic on U, such that $F = f$ on M.

However, a C^∞ CR function need not be the restriction of a holomorphic function, even if M is real analytic. For example, consider

$$\begin{aligned} M &= \{(z, w) \in \mathbb{C}^2 : \operatorname{Im} w = 0\} \\ &= \{(x, y, u, 0) \in \mathbb{R}^4\}. \end{aligned}$$

Here V is spanned by

$$L = \frac{\partial}{\partial \bar{z}}.$$

So any function $f = f(u)$ is a CR function on M. But such an f can be extended as a holomorphic function only if $f(u)$ is real analytic, and can be extended as a holomorphic function to one side of M only if $f(u)$ is the boundary value of a holomorphic function of one variable.

Now we come to two extremely important and influential results of Hans Lewy. The first brings to completion the study of extensions for definite Levi forms. The second, only four pages long, revolutionized the study of partial differential equations.

Lewy Extension Theorem. [13] Let M be a strictly pseudoconvex real hypersurface in $\mathbb{C}^n$ and let f be a CR function on M. For each $p \in M$ there exists a ball U, centered at p and open in $\mathbb{C}^n$, such that f extends to a holomorphic function on the pseudoconvex component of $U - M$.

The ideas in the proof can be seen by letting M be a piece of the unit sphere S^3 in $\mathbb{C}^2$. Let p be any point of M. Consider a complex line, close to the complex tangent line at p, intersecting M nontangentially. This intersection is a circle and the values of f on this circle determine a holomorphic function on the disc bounded by this circle. We have to show that this holomorphic function takes on the boundary values f and that the collection of holomorphic functions agree and give a well-defined holomorphic function on some open subset of the ball containing M in its boundary. The CR equations are used to establish both of these facts. (Lewy actually only considered $n = 2$.)

Next we consider the simplest real hypersurface in $\mathbb{C}^2$ with definite Levi form. It is, as could be guessed, the sphere S^3. However, in order to write it in an especially useful way, we need to let one point go to infinity. We obtain the hyperquadric:

$$Q = \{(z, w) | \operatorname{Im} w = |z|^2\}.$$

(There exists a biholomorphism defined in a neighborhood of S^3 - {one point} taking S to Q.)

For Q, V has complex dimension one and is generated by

$$L = \frac{\partial}{\partial \bar{z}} - iz\frac{\partial}{\partial u}$$

where $u = \operatorname{Re} w$. We can think of L as a partial differential operator on $\mathbb{R}^3$ and try to solve the equation $Lu = f$. Here f is a C^∞ function in a neighborhood of the origin and we seek a function u, say $u \in C^1$, satisfying this equation in a perhaps smaller neighborhood of the origin. This is one equation with one unknown. The simplest partial differential equations, those with constant coefficients, are always solvable. Since the coefficients of L, while not constant, are merely linear, this is an example of the next simplest type of equation. Further, when f is real analytic, there is a real analytic solution u.

Lewy Nonsolvability Theorem. [14] There exists a C^∞ function f defined on all of $\mathbb{R}^3$ such that there do not exist (p, U, u) where p is a point of $\mathbb{R}^3$, U is an open neighborhood of p, and u is a C^1 function with $Lu = f$ on U.

The idea that a differential equation might not even have local solutions was extremely surprising, and Lewy's example had an enormous effect. Consider this convincing testimonial [22]:

> Allow me to insert a personal anecdote: in 1955 I was given the following thesis problem: prove that every linear partial differential equation with smooth coefficients, not vanishing identically at some point, is locally solvable at that point. My thesis director was, and still is, a leading analyst; his suggestion simply shows that, at the time, nobody had any inkling of the structure underlying the local solvability problem, as it is now gradually revealed.

We conclude our discussion of extension theorems with Trepreau's condition of extendability. This necessary and sufficient condition leaves unanswered a curious question. So again, let M be a real hypersurface in $\mathbb{C}^n$ and p a point on M. Assume there is one side of M, call it Ω^+, such that every CR function on M in a neighborhood of p extends to some $B_\epsilon \cap \Omega^+$, where B_ϵ is the ball of radius ϵ centered at p. The Baire Category Theorem then can be used to show that there is one such ball B with the property that each CR function extends to $B \cap \Omega^+$. But no such B can exist if M contains a complex hypersurface $\{f(z) = 0\}$, for then $f(z) - \lambda$ is nonzero on M for various values of λ converging to zero, and the reciprocal functions are not holomorphic on a common one-sided neighborhood of p. Thus if M contains a complex hypersurface, then there exist CR functions that do not extend to either side. In [21] it is shown that if there is no such complex hypersurface, then there is one side of M to which all such CR functions extend as holomorphic functions. The question left unanswered is to use the defining equation for M to determine to which side the extensions are possible.

Mappings

A function $f(z)$ holomorphic on a domain $\Omega \subset \mathbb{C}$ can be thought of as a mapping of Ω to some other domain in $\mathbb{C}$. Indeed, as every graduate student knows, f preserves angles at all points where $f' \neq 0$, and so the theory of holomorphic functions coincides, more or less, with the theory of conformal maps. How should this be generalized to higher dimensions? We could look at maps of domains in $\mathbb{C}^n$ that preserve angles. But then the connection to complex variables is destroyed and we end up by generalizing complex analysis to $\mathbb{R}^3$ and its finite dimensional group of conformal transformations.

It is more fruitful to look at maps $\Phi : \mathcal{O}_1 \to \mathcal{O}_2$ of domains in $\mathbb{C}^n$ with $\Phi = (f_1, \ldots, f_n)$ and each f_j is holomorphic. Thus we are again using holomorphic functions of several variables but now we are focusing on the mapping Φ rather than on the individual functions. Note that Φ preserves some angles but not others. Classically such maps were called "pseudo-conformal" following Severi and Segre.

From the viewpoint of maps, the Riemann Mapping Theorem is the fundamental result in the study of one complex variable. The unit ball in $\mathbb{C}^1$, which acts as the source domain for the mappings, can reasonably be generalized to either the unit ball in $\mathbb{C}^2$

$$\{(z, w) : |z|^2 + |w|^2 < 1\}$$

or to the polydisc

$$\{(z, w) : |z| < 1, |w| < 1\}.$$

In a profound paper in 1907, Poincaré computed, among many other results, the group of biholomorphic self-mappings of the ball [17]. By comparing this group to the more easily computed corresponding group of the polydisc, it follows that these two domains are not biholomorphically equivalent. Thus the Riemann Mapping Theorem does not hold for several complex variables and, moreover, fails for the two "simplest" domains. (Actually, we have already seen earlier in this article a failure of the Riemann Mapping Theorem. If one domain can be "extended" and the other cannot, then the two domains are not biholomorphically equivalent. This can be seen using relatively simple properties of holomorphic convexity.) Further, Poincaré provided a wonderful counting argument to indicate the extent to which the Riemann Mapping Theorem fails to hold. He did this by asking this question: Given two real hypersurfaces M_1 and M_2 in $\mathbb{C}^2$ and points $p \in M_1$ and $q \in M_2$, when do there exist open sets U and V in $\mathbb{C}^2$, with $p \in U$ and $q \in V$ and a biholomorphism $\Phi : U \to V$ such that $\Phi(p) = q$ and $\Phi(M_1 \cap U) = M_2 \cap V$?

More particularly, Poincaré asked: What are the invariants of a real hypersurface M? That is, what are the quantities preserved when M is mapped by a biholomorphism? We already know one invariant. The Levi form for a real hypersurface in $\mathbb{C}^2$ is a number and it is necessary, in order that Φ exists, that the Levi forms at p and q both are zero or both are nonzero.

There are infinitely many other invariants. A consequence is that there is a zero probability that two randomly given real hypersurfaces are equivalent. Here is the counting argument used by Poincaré to show this. How many real hypersurfaces are there and how many local biholomorphisms? There are

$$\binom{N+k}{k}$$

coefficients in the Taylor series expansion, to order N, of a function of k variables. So, we see that there are

$$\binom{N+3}{3}$$

N-jets of hypersurfaces of the form $v = f(x, y, u)$.

Similarly, there are

$$\binom{N+2}{2}$$

N-jets of a holomorphic function $F(z,w)$ but these coefficients are complex, so there are

$$2\binom{N+2}{2}$$

real N-jets. Finally, for a map $\Phi = (F(z,w,),G(z,w))$, there are

$$4\binom{N+2}{2}$$

real N-jets. Thus, since

$$\binom{N+3}{3}$$

is eventually greater than

$$4\binom{N+2}{2},$$

there are more real hypersurfaces than local biholomorphisms. From this, we see that there should be an infinite number of invariants.

Poincaré outlined a method of producing these invariants. Given two hypersurfaces s and S written as graphs over the (x,y,u) plane, the coefficients of the Taylor series must be related in certain ways in order for there to exist a biholomorphism under which S becomes tangent to s to some order n at a particular point. Having made this observation, Poincaré implied that there would be no difficulty in actually finding the invariants:

> These relations express the fact that the two surfaces S and s can be transformed so as to have nth order contact. If s is given, then the coefficients of S satisfy N conditions, that is to say, N functions of the coefficients, which we call the invariants of nth order of our surface S, have the appropriate values; I do not dwell on the details of the proof, which ought to be done as in all analogous problems.

Here Poincaré somewhat underestimated the difficulties involved and perhaps would have been surprised by the geometric structure, described below, underlying these invariants.

In 1932, Cartan found these invariants by a new and completely different method, namely as an application of his method of equivalences. Starting with the real hypersurface M in $\mathbb{C}^2$, Cartan constructed a bundle B of dimension eight along with independent differential 1-forms $\omega_1,\dots,\omega_8$ defined on the bundle. He did this using only information derivable from the complex structure of $\mathbb{C}^2$. Thus there is a biholomorphism of open sets in $\mathbb{C}^2$ taking M_1 to M_2 only if there is a map $\Phi : B_1 \to B_2$ such that $\Phi^*(\omega_j^2) = \omega_j^1$. Conversely, any real analytic map $\Phi : B_1 \to B_2$ such that $\Phi^*(\omega_j^2) = \omega_j^1$ arises from such a biholomorphism. (This is stated loosely; to be more precise, one would have to specify points and neighborhoods.) So, one can find properties of a hypersurface that are invariant under the infinite pseudogroup of local biholomorphisms by studying a finite dimensional structure bundle.

The structure (M,B,ω) is an example of a Cartan connection. When this connection has zero curvature, M locally maps by a biholomorphism to the hyperquadric Q (and so also to the sphere S^3, but, in this context, it is much easier to work with Q). So we obtain a geometry based on Q in the same way that Riemannian geometry is based on the Euclidean structure of $\mathbb{R}^n$. In particular, there is a distinguished family of curves, called chains by Cartan, that play the role of geodesics, and projective parametrizations of these chains, that play the role of arc length. The two papers [6] developing this theory are still relatively difficult going, even after Cartan's approach to geometry has become part of the mathematical language. They were quite demanding at the time he wrote them. The theorems of Hans Lewy are one surprising consequence of this difficulty; Professor Lewy remarked to the present author that he became interested in the CR vector fields as partial differential operators as he struggled to understand Cartan's papers.

In about 1974, Moser determined the invariants explicitly in the manner indicated by Poincaré. Moser first considered this problem following a question in a seminar talk. He was not discouraged by Poincaré's opinion that the determination would be routine (and the inference that it would be uninteresting) because he was, fortunately, unaware of Poincaré's paper. (However, once Moser became interested in this question, it is not a coincidence that he rediscovered Poincaré's approach, since Moser had learned similar techniques from Poincaré's work in celestial mechanics.)

As we indicated above, the determination of the invariants proceeds from a study of order of contact of biholomorphic images of the given hypersurface with a standard hypersurface. Here is the basic result.

Theorem (Moser Normal Form). Let p be a point on $M^3 \subset \mathbb{C}^2$ at which the Levi form is nonzero. There exists a local biholomorphism Φ taking p to 0 such that $\Phi(M)$ is given by

$$v = |z|^2 + 2\mathrm{Re}(F_{42}(u)z^4\bar{z}^2) + \sum_{\substack{j+k\geq 7\\ j\geq 2, k\geq 2}} F_{jk}(u)z^j\bar{z}^k$$

where (z, w) are the coordinates for $\mathbb{C}^2$, with $w{=}u{+}iv$.

There is an eight parameter family of local biholomorphisms taking M^3 to Moser Normal Form. Thus F_{42} and the higher order coefficients are not true invariants. To decide if a hypersurface M_1 can be mapped onto another hypersurface M_2 by a local biholomorphism, we choose one mapping of M_1 to normal form and ask if this normal form belongs to the eight parameter set of normal forms associated to M_2. This should remind us of Cartan's reduction to a finite dimensional structure bundle, also of dimension eight.

This is actually only part of the story, and not even the most interesting part. To obtain this normal form, Moser discovered and exploited a rich geometric structure. Let $L = \alpha_1 \frac{\partial}{\partial \bar{z}_1} + \alpha_2 \frac{\partial}{\partial \bar{z}_2}$ belong to $\mathbb{C} \otimes TM$, i.e., let L generate the one-dimensional bundle V, and set $H =$ linear span $\{\mathrm{Re}L, \mathrm{Im}L\}$. So, H is a 2-plane distribution on M. For each direction Γ transverse to H at some point q, there exist a curve γ in the direction Γ and a projective parametrization of γ that are invariant under biholomorphisms. Further, any vector in H_q has an invariantly defined parallel transport along γ. These are precisely the geometric structures found by Cartan!

Moser's work was a second solution to the problem of invariants and quite different in method and spirit from Cartan's. Chern and Moser [7] then generalized the results of Cartan and of Moser to higher dimensions. In [7], the problem of invariants is solved twice (once using Cartan's approach and once using Moser's) for hypersurfaces with nondegenerate Levi form. All the geometric properties discovered by Cartan and by Moser carry over to higher dimensions. (In [9], it is shown how to directly use the Moser normal form for M^{2n-1} and the trivial Cartan connection on Q to obtain the Cartan connection (M, B, ω). In [19] and [20] two other methods of generalizing Cartan's work to higher dimensions are given; however, these apply to a somewhat restricted class of hypersurfaces.)

Now we return to the theme of how the boundary affects analysis on a domain. In the first half of our survey, we have seen how the theory of functions on a domain is influenced by the boundary of the domain. Now, in turn, we discuss how the boundary affects the mappings of a domain. The starting point is a result of Fefferman establishing that the boundary is indeed potentially useful in studying biholomorphisms. Let Ω and Ω' be bounded strictly pseudoconvex domains in $\mathbb{C}^n$ with C^∞ boundaries.

Theorem. [8] If $\Phi : \Omega \to \Omega'$ is a biholomorphism, then Φ extends to a C^∞ diffeomorphism $\Phi : \bar{\Omega} \to \bar{\Omega}'$ of manifolds with boundary.

This theorem generalizes the fact that in $\mathbb{C}^1$ the Riemann mapping of the disk to a smoothly bounded domain extends smoothly to a diffeomorphism of the closures.

It follows from Fefferman's theorem that for two strictly pseudoconvex domains to be biholomorphically equivalent, it is necessary that all of the infinite number of Cartan-Moser invariants match up. Burns, Schnider, and Wells [4] used this to show that any strictly pseudoconvex domain can be deformed by an arbitrarily small perturbation into a nonbiholomorphically equivalent domain. So here is another failure of the Riemann Mapping Theorem.

Now consider strictly pseudoconvex domains Ω and Ω' with real analytic boundaries. Once a biholomorphism Φ is known to give a diffeomorphism of the boundaries (as in Fefferman's theorem), the extendability of Φ to a biholomorphism of larger domains is immediate. For then

$$\Phi : \text{ boundary } \Omega \to \text{ boundary } \Omega'$$

preserves the Cartan connections and these connections are real analytic. It follows that Φ is real analytic. This in turn implies that Φ is holomorphic in a neighborhood of boundary Ω.

What can be said about real analytic hypersurfaces that need not be strictly pseudoconvex? Let M be a (piece of a) real analytic surface in $\mathbb{C}^n$. (It does not even need to be of codimension one.) Let $\Phi = (f_1, \ldots, f_n)$ be holomorphic on some open set Ω with

$$M \subset \text{ boundary } \Omega$$

and let Φ extend differentiably to M and be a diffeomorphism of M to $\Phi(M)$. Then, as long as M (or $\Phi(M)$) satisfies a very general condition called "essentially finite", Φ is holomorphic on an open set containing M [2].

Thus, although a holomorphic function need not extend across M, those holomorphic functions that fit together to give a mapping Φ do extend. Why should this be so? Clearly, it must be because the components satisfy an equation.

The simplest example of a surface that is not "essentially finite", and for which a one-sided biholomorphism need not extend to the other side, is

$$M = \{(z, w) : \mathrm{Im} w = 0\}.$$

Here the defining function is not strong enough to relate the components and their conjugates by an appropriate equation.

The geometric concept of holomorphic nondegeneracy, introduced in [18], is related to essential finiteness and has been used to generalize results from [2] (see [3]).

Abstract CR Structures and the Realization Problem

Just as Riemannian manifolds abstract the induced metric structure on a submanifold of Euclidean space, we abstract the structure relevant to a hypersurface in $\mathbb{C}^{n+1}$. So recall the bundle

$$V = (\mathbb{C} \otimes TM) \bigcap V^0,$$

defined in section 1, where

$$V^0 = \text{lin span} \left\{ \frac{\partial}{\partial \bar{z}_1} \cdots \frac{\partial}{\partial \bar{z}_{n+1}} \right\}.$$

This is to be our model. Thus, for the abstract definition, we start with a manifold M and a subbundle of the complexified tangent bundle of M. Now what properties of V do we want to abstract? Our first observation is that M should have odd dimension, say $2n+1$, and that the complex dimension of the fibers of V should be n. So, this is our first assumption:

(1) M is a manifold of dimension $2n+1$ and V is a subbundle of $\mathbb{C} \otimes TM$ with fibers of complex dimension n.

The next key fact for hypersurfaces in $\mathbb{C}^n$ is that none of the induced CR operators is a real vector field. This gives us our second assumption:

(2) $V \cap \bar{V} = \{0\}$.

Our final assumption is a restriction on how V varies from point to point. This restriction is easily justified if we first discuss the realization problem. We start with a pair (M, V) satisfying (1).

Definition. An embedding $\Phi : M \to \mathbb{C}^N$ is a realization of (M, V) if its differential $\Phi_* : \mathbb{C} \otimes TM \to C \otimes T\mathbb{C}^N$ maps V into V^0

Let $\Phi : M \to \mathbb{C}^N$ be a realization of (M, V). Note that condition (2) must hold for V since it does for V^0. Let p be a point of M. By using an appropriate linear projection of $\mathbb{C}^N$ into some $\mathbb{C}^{n+1}$, we obtain an embedding of a neighborhood of p into $\mathbb{C}^{n+1}$ that realizes (M, V) in that neighborhood. The image of this neighborhood is now a real hypersurface. Thus, for local realizability, there is no loss of generality in taking $N = n+1$ in the definition.

The definition of a CR function given on page 1482 applies also to the present case. Just as the restriction of a holomorphic function to a hypersurface $M \subset \mathbb{C}^{n+1}$ gives a CR function, the pull-back via a realization of any holomorphic function to a function on the abstract manifold M is also a CR function.

Applying this to the coordinate functions on $\mathbb{C}^{n+1}$, we see that each component Φ_i of Φ is a CR function. Since these functions are independent and vanish on V, their differentials $d\Phi_i$ span the annihilator of V. Thus, a necessary condition for there to be a local embedding is that the annihilator of V has a basis of exact differentials. This is the *integrability condition*, and can be restated in the formally equivalent form:

(3) The space $\mathcal{V}$ of vector fields with values in V is closed under brackets: $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$.

Note that in the case when V is a subbundle of the tangent space of M (rather than the complexified tangent space), condition (3) is just the Frobenius condition and then M is foliated by submanifolds that at each point have V for their tangent space. There is no similar foliation when (2) holds.

Definition. (M, V) is called a CR structure if it satisfies conditions (1), (2), and (3).

We emphasize that each real hypersurface in a complex manifold satisfies these conditions and so is a CR structure. The following result tells us that we should be satisfied with these three conditions and not seek to abstract other properties of real hypersurfaces.

Lemma. A real analytic CR structure is locally realizable.

Proof: Complexify M and V. Then V becomes a bundle of holomorphic tangent vectors and condition (3) becomes the Frobenius condition for a holomorphic foliation of the complexification of M. Holomorphic functions parametrizing the leaves of this foliation restrict to CR functions on M.

We know now what to take as the abstract CR structure and we ask if every abstract CR structure can be realized locally as a real hypersurface. Because of our experience with the boundaries of open sets in $\mathbb{C}^n$, it is natural to at first limit ourselves to strictly pseudoconvex abstract CR hypersurfaces. Lewy seems to be the first to have posed this question [13]. Nirenberg was certainly the first to answer [16]: There exists a C^∞ strictly pseudoconvex CR structure defined in a neighborhood of $0 \in R^3$ such that the only CR functions are the constants. Of course, this rules out realizability.

Said another way, there is a complex vector field L such that the only functions satisfying $Lf = 0$ in a neighborhood of the origin are the constant functions, and this vector field can be constructed as a perturbation of the standard Lewy operator.

There are several reasons (having to do with the technical structure of the partial differential system) to conjecture that when we restrict attention to strictly pseudoconvex structures counterexamples such as the one of Nirenberg would be possible only in dimension 3.

After attempts by many mathematicians, Kuranishi showed in 1982 that a strictly pseudoconvex CR structure of dimension at least nine is locally realizable. This was improved in 1987 by Akahori to include the case of dimension seven. See [12] and [1]. The five dimensional problem remains open. The technical reasons alluded to above suggest realizations are always possible in this dimension; other reasons such as the argument in [15] hint that it is not always possible. A simpler proof of the known dimensions was given in [23]. Recently, Catlin has found a new proof that also includes many other signatures of the Levi form [5]. However, there is one special signature where realizability is not always possible: Nirenberg's counterexample was generalized in [10] to the so-called aberrant signature of one eigenvalue of a given sign and the other eigenvalues all of the other sign. Catlin's results, together with these counterexamples, leave open the case of precisely two eigenvalues of one sign. This includes, of course, the strictly pseudoconvex CR manifolds of dimension five.

References

[1] T. Akahori, *A new approach to the local embedding theorem of CR-structures for $n \geq 4$*, Memoirs Amer. Math. Soc., Number 366, Amer. Math. Soc., Providence, 1987.

[2] M. S. Baouendi, H. Jacobowitz, and F. Treves, *On the analyticity of CR mappings*, Ann. of Math. **122** (1985), 365–400.

[3] M. S. Baouendi, and L. P. Rothschild, *Mappings of real algebraic hypersurfaces*, to appear in Journal of Amer. Math. Soc.

[4] D. Burns, S. Shnider, and R. Wells, *On deformations of strictly pseudo-convex domains*, Inventiones Math. **46** (1978), 237–253.

[5] D. Catlin, *Sufficient conditions for the extension of CR structures*, to appear.

[6] E. Cartan, *Sur l'équivalence pseudo-conforme des hypersurfaces de l'espace de deux variables complexes*, I and II, Oeuvres Complétes, Part II, 1232–1305 and Part III 1218–1238.

[7] S. S. Chern, and J. Moser, *Real hypersurfaces in complex manifolds*, Acta Math. **133** (1974), 219–271.

[8] C. Fefferman, *The Bergman kernel and biholomorphic mappings of pseudoconvex domains*, Inventiones Math. **26** (1974), 1–65.

[9] H. Jacobowitz, *Induced connections on hypersurfaces in $\mathbb{C}^{n+1}$*, Inventiones Math., **43** (1977), 109–123.

[10] H. Jacobowitz, and F. Treves, *Aberrant CR structures*, Hokkaido Math. J. **12** (1983), 276–292.

[11] S. G. Krantz, *Function theory of several complex variables*, John Wiley and Sons, New York, 1982.

[12] M. Kuranishi, *Strongly pseudo-convex CR structures over small balls*, Part III, Ann. Math. **116** (1982), 249–330.

[13] H. Lewy, *On the local character of the solutions of an atypical linear differential equation in three variables and a related theorem for regular functions of two complex variables*, Ann. Math. **64** (1956), 514–522.

[14] H. Lewy, *An example of a smooth linear partial differential equation without solution*, Ann. Math. **66** (1957), 155–158.

[15] A. Nagel, and J. -P. Rosay, *Nonexistence of homotopy formula for (0,1) forms on hypersurfaces in $\mathbb{C}^3$*, Duke Math J. **58** (1989), 823-827.

[16] L. Nirenberg, *Lectures on Linear Partial Differential Equations*, Amer. Math. Soc., Providence, 1973.

[17] H. Poincaré, *Les fonctions analytiques de deux variables et la représentation conforme*, Rend. Circ. Mat. Palermo (1907), 185–220.

[18] N. Stanton, *Infinitesimal CR automorphisms of rigid hypersurfaces*, Am. J. Math. **117** (1995), 141–167.

[19] N. Tanaka, *On the pseudoconformal geometry of hypersurfaces of the space of n complex variables*, J. Math. Soc. Japan **14** (1962), 397–429.

[20] N. Tanaka, *On nondegenerate real hypersurfaces, grades Lie algebras and Cartan connections*, Japanese J. Math. **2** (1976), 131–190.

[21] J. Trepreau, *Sur le prolongement holomorphe de fonctiones CR défines sur une hypersurface reélle de classe C^2 dans $\mathbb{C}^n$*, Inventiones Math. **83** (1986), 583–592.

[22] F. Treves, *On the local solvability of linear partial differential equations*, Bull. Amer. Math. Soc. **76** (1970), 552–571.

[23] S. Webster, S., *On the proof of Kuranishi's embedding theorem*, Ann. Inst. H. Poincaré **6** (1989), 183–207.