5 Cauchy Sequences

Often the biggest problem with showing that a sequence converges using the techniques we have seen so far is that we must know ahead of time to what it converges. This is often a chicken and egg type problem because to prove a sequence converges, we must seemingly already know it converges. An escape from this dilemma is provided by Cauchy sequences.

Definition 5.1. A sequence a_n is a *Cauchy sequence* if for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $n, m \geq N$ implies $|a_n - a_m| < \varepsilon$.

Theorem 5.1. *A sequence converges iff it is a Cauchy sequence.*

Proof. ($\Rightarrow$) Suppose $a_n \to L$ and $\varepsilon > 0$. There is an $N \in \mathbb{N}$ such that $g \geq N$ implies $|a_n - L| < \varepsilon/2$. If $m, n \geq N$, then

$$|a_m - a_n| = |a_m - L + L - a_n| \leq |a_m - L| + |L - a_m| < \varepsilon/e + \varepsilon/2 = \varepsilon.$$

This shows a_n is a Cauchy sequence.

($\Leftarrow$) Let a_n be a Cauchy sequence. First, we claim that a_n is bounded. To see this, let $\varepsilon = 1$ and choose $N \in \mathbb{N}$ such that $n, m \geq N$ implies $|a_n - a_m| < 1$. In this case, $a_N - 1 < a_n < a_N + 1$ for all $n \geq N$, so $\{a_n : n \geq N\}$ is a bounded set. The set $\{a_n : n < N\}$, being finite, is also bounded. Since $\{a_n : n \in \mathbb{N}\}$ is the union of these two bounded sets, it too must be bounded.

Because a_n is a bounded sequence, Corollary 4.6 implies it has a convergent subsequence $b_n \to L$. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ so that $n, m \geq N$ implies $|a_n - a_m| < \varepsilon/2$. There is a $b_k = a_{m_k}$ such that $m_k \geq N$ and $|b_{m_k} - L| < \varepsilon/2$. If $n \geq N$, then

$$\begin{aligned}|a_n - L| &= |a_n - b_k + b_k - L| \leq |a_n - b_k| + |b_k - L| \\ &< |a_n - a_{m_k}| + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

Therefore, $a_n \to L$. □

According to this theorem, we can prove that a sequence converges without ever knowing precisely to what it converges. An example of the usefulness of this idea is contained in the following definition and theorem.

Definition 5.2. A sequence a_n is *contractive* if there is a $c \in (0, 1)$ such that $|x_{k+1} - x_k| \leq c|x_k - x_{k-1}|$ for all $k > 1$.

Theorem 5.2. *If a sequence is contractive, then it converges.*

Proof. Let x_k be a contractive sequence with contraction constant $c \in (0, 1)$.

We first claim that if $n \in \mathbb{N}$, then

$$|x_n - x_{n+1}| \leq c^{n-1}|x_1 - x_2|. \tag{2}$$

This is proved by induction. When $n = 1$, the statement is $|x_1 - x_2| \leq c^0|x_1 - x_2| = |x_1 - x_2|$, which is trivially true. Suppose that $|x_n - x_{n+1}| \leq c^{n-1}|x_1 - x_2|$

for some $n \in \mathbb{N}$. Then, from the definition of a contractive sequence and the inductive hypothesis,

$$|x_{n+1} - x_{n+2}| \leq c|x_n - x_{n+1}| \leq c(c^{n-1}|x_1 - x_2|) = c^n|x_1 - x_2|.$$

This shows the claim is true in the case $n+1$. Therefore, by induction, the claim is true for all $n \in \mathbb{N}$.

To show x_n is a Cauchy sequence, let $\varepsilon > 0$. Since $c^n \to 0$, we can choose $N \in \mathbb{N}$ so that

$$\frac{c^{N+1}}{(1-c)} < \frac{\varepsilon}{|x_1 - x_2|}. \tag{3}$$

Let $n > m \geq N$. Then

$$\begin{aligned} |x_n - x_m| &= |x_n - x_{n-1} + x_{n-1} - x_{n-2} + x_{n-2} - \cdots - x_{m+1} + x_{m+1} - x_m| \\ &\leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m| \end{aligned}$$

Now, use (2) on each of these terms.

$$\begin{aligned} &\leq c^{n-2}|x_1 - x_2| + c^{n-3}|x_1 - x_2| + \cdots + c^{m-1}|x_1 - x_2| \\ &= |x_1 - x_2|(c^{n-2} + c^{n-3} + \cdots + c^{m-1}) \end{aligned}$$

Apply the formula for a geometric sum.

$$\begin{aligned} &= |x_1 - x_2| c^{m-1} \frac{1 - c^{n-m}}{1-c} \\ &< |x_1 - x_2| \frac{c^{m-1}}{1-c} \end{aligned}$$

Use (3) to estimate the following.

$$\begin{aligned} &\leq |x_1 - x_2| \frac{c^{N-1}}{1-c} \\ &< |x_1 - x_2| \frac{\varepsilon}{|x_1 - x_2|} \\ &= \varepsilon \end{aligned}$$

This shows x_n is a Cauchy sequence. □

Example 5.1. Let $-1 < r < 1$ and define the sequence $s_n = \sum_{k=0}^{n} r^k$. If $r = 0$, the convergent os s_n is trivial. So, suppose $r \neq 0$. In this case

$$\frac{|s_{n+1} - s_n|}{|s_n - s_{n-1}|} = \left| \frac{r^{n+1}}{r^n} \right| = |r| < 1.$$

This shows s_n is contractive, and Theorem 5.2 implies it converges.

Problem 17. If x_n is a sequence and there is a $c \geq 1$ such that $|x_{k+1} - x_k| > c|x_k - x_{k-1}|$ for all $k > 1$, then can x_n converge?