6 Covering Properties and Compactness on $\mathbb{R}$

Definition 6.1. Let $S \subset \mathbb{R}$. A collection of open sets, $\mathcal{O} = \{G_\lambda : \lambda \in \Lambda\}$, is an *open cover* of S if $S \subset \bigcup_{G\in\mathcal{O}} G$. If $\mathcal{O}' \subset \mathcal{O}$ is also an open cover of S, then $\mathcal{O}'$ is an *open subcover of* S *from* $\mathcal{O}$.

Example 6.1. Let $S = (0,1)$ and $\mathcal{O} = \{(1/n,1) : n \in \mathbb{N}\}$. It is easy to see that $\mathcal{O}$ is an open cover of S. To prove this, let $x \in (0,1)$. Choose $n_0 \in \mathbb{N}$ such that $1/n_0 < x$. Then

$$x \in (1/n_0, 1) \subset \bigcup_{n\in\mathbb{N}} (1/n, 1) = \bigcup_{G\in\mathcal{O}} G.$$

Since x is an arbitrary element of $(0,1)$, it follows that $(0,1) = \bigcup_{G\in\mathcal{O}} G$.

Suppose $\mathcal{O}'$ is any infinite subset of $\mathcal{O}$ and $x \in (0,1)$. Since $\mathcal{O}'$ is infinite, there exists an $n \in \mathbb{N}$ such that $x \in (1/n,1) \in \mathcal{O}'$. The rest of the proof proceeds as above.

On the other hand, if $\mathcal{O}'$ is a finite subset of $\mathcal{O}$, then let $M = \max\{n : (1/n,1) \in \mathcal{O}'\}$. If $0 < x < 1/M$, it is clear that $x \notin \bigcup_{G\in\mathcal{O}'} G$, so $\mathcal{O}'$ is not an open cover of $(0,1)$.

Example 6.2. Let $T = [0,1)$ and $0 < \varepsilon < 1$. If $\mathcal{O} = \{(1/n,1) : n \in \mathbb{N}\} \cup (-\varepsilon, \varepsilon)$. It is easy to see that $\mathcal{O}$ is an open cover of T.

It is evident that any open subcover of T from $\mathcal{O}$ must contain $(-\varepsilon,\varepsilon)$, because that is the only element of $\mathcal{O}$ which contains 0. Choose $n \in \mathbb{N}$ such that $1/n < \varepsilon$. Then $\mathcal{O}' = \{(-\varepsilon,\varepsilon), (1/n,1)\}$ is an open subcover of T from $\mathcal{O}$ which contains only two elements.

Theorem 6.1 (Lindelöf Property). *If* $S \subset \mathbb{R}$ *and* $\mathcal{O}$ *is any open cover of* S, *then* $\mathcal{O}$ *contains a subcover with a countable number of elements.*

Proof. Let $\mathcal{O} = \{G_\lambda : \lambda \in \Lambda\}$ be an open cover of $S \subset \mathbb{R}$. Since $\mathcal{O}$ is an open cover of S, for each $x \in S$ there is a $\lambda_x \in \Lambda$ and numbers $p_x, q_x \in \mathbb{Q}$ satisfying $x \in (p_x, q_x) \subset G_{\lambda_x} \in \mathcal{O}$. The collection $\mathcal{T} = \{(p_x,q_x) : x \in S\}$ is an open cover of S.

Thinking of the collection $\{(p_x,q_x) : x \in S\}$ as a set of ordered pairs of rational numbers, it is seen that $\operatorname{card}(\mathcal{T}) \le \operatorname{card}(\mathbb{Q}\times\mathbb{Q}) = \aleph_0$, so $\mathcal{T}$ is countable.

For each interval $I \in \mathcal{T}$, choose a $\lambda_I \in \Lambda$ such that $I \subset G_{\lambda_I}$. Then

$$S \subset \bigcup_{I\in\mathcal{T}} I \subset \bigcup_{I\in\mathcal{T}} G_{\lambda_I}$$

shows $\mathcal{O}' = \{G_{\lambda_I} : I \in \mathcal{T}\} \subset \mathcal{O}$ is an open subcover of S from $\mathcal{O}$. Also, $\operatorname{card}(\mathcal{O}') \le \operatorname{card}(\mathcal{T}) \le \aleph_0$, so $\mathcal{O}'$ is a countable open subcover of S from $\mathcal{O}$. $\square$

Corollary 6.2. *Any open subset of* $\mathbb{R}$ *can be written as a countable union of pairwise disjoint open intervals.*

Proof. Let G be open in $\mathbb{R}$. For $x \in G$ let $\alpha_x = \text{glb}\,\{y : (y, x] \subset G\}$ and $\beta_x = \text{lub}\,\{y : ([x, y) \subset G\}$. The fact that G is open easily implies $\alpha_x < x < \beta_x$. Define $I_x = (\alpha_x, \beta_x)$.

Then $I_x \subset G$. To see this, suppose $x < w < \beta_x$. Choose $y \in (w, \beta_x)$. The definition of β_x guarantees $w \in (x, y) \subset G$. Similarly, if $\alpha_x < w < x$, it follows that $w \in G$.

This shows $\mathcal{O} = \{I_x : x \in G\}$ has the property that $G = \bigcup_{x \in G} I_x$.

Suppose $x, y \in G$ and $I_x \cap I_y \neq \emptyset$. There is no generality lost in assuming $x < y$. In this case, there must be a $w \in (x, y)$ such that $w \in I_x \cap I_y$. We know from above that both $[x, w] \subset G$ and $[w, y] \subset G$, so $[x, y] \subset G$. It follows easily from this that $\alpha_x = \alpha_y < x < y < \beta_x = \beta_y$ and $I_x = I_y$.

From this we conclude $\mathcal{O}$ consists of pairwise disjoint open intervals.

To finish, apply Theorem 6.1 to extract a countable subcover from $\mathcal{O}$. □

Corollary 6.2 can also be proved by a different strategy. Instead of using Theorem 6.1 to extract a countable subcover, we could just choose one rational number from each interval in the cover. The pairwise disjointness of the intervals in the cover guarantee that this will give a bijection between $\mathcal{O}$ and a subset of $\mathbb{Q}$. This method has the advantage of showing that $\mathcal{O}$ itself is countable from the start.

Definition 6.2. An open cover $\mathcal{O}$ of a set S is a *finite cover*, if $\mathcal{O}$ has only a finite number of elements. The definition of a *finite subcover* is analogous.

Definition 6.3. A set $K \subset \mathbb{R}$ is *compact*, if every open cover of K contains a finite subcover.

Theorem 6.3 (Heine-Borel). *A set $K \subset \mathbb{R}$ is compact iff it is closed and bounded.*

Proof. ($\Rightarrow$) Suppose K is unbounded. The collection $\mathcal{O} = \{(-n, n) : n \in \mathbb{N}\}$ is an open cover of K. If $\mathcal{O}'$ is any finite subset of $\mathcal{O}$, then $\bigcup_{G \in \mathcal{O}'} G$ is a bounded set and cannot cover the unbounded set K. This shows K cannot be compact, and every compact set must be bounded.

Suppose K is not closed. Then there is a limit point x of K such that $x \notin K$. Define $\mathcal{O} = \{[x - 1/n, x + 1/n]^c : n \in \mathbb{N}\}$. Then $\mathcal{O}$ is a collection of open sets and $K \subset \bigcup_{G \in \mathcal{O}} G = \mathbb{R} \setminus \{x\}$. Let $\mathcal{O}' = \{[x - 1/n_i, x + 1/n_i]^c : 1 \leq i \leq N\}$ be a finite subset of $\mathcal{O}$ and $M = \max\{n_i : 1 \leq i \leq N\}$. Since x is a limit point of K, there is a $y \in K \cap (x - 1/M, x + 1/M)$. Clearly, $y \notin \bigcup_{G \in \mathcal{O}'} G = [x - 1/M, x + 1/M]^c$, so $\mathcal{O}'$ cannot cover K. This shows every compact set must be closed.

($\Leftarrow$) Let K be closed and bounded and let $\mathcal{O}$ be an open cover of K. Applying Theorem 6.1, if necessary, we can assume $\mathcal{O}$ is countable. Thus, $\mathcal{O} = \{G_n : n \in \mathbb{N}\}$.

For each $n \in \mathbb{N}$, define

$$F_n = K \setminus \bigcup_{i=1}^{n} G_i = K \cap \bigcap_{i=1}^{n} G_i^c.$$

Then F_n is a sequence of nested, bounded and closed subsets of K. Since $\mathcal{O}$ covers K, it follows that

$$\bigcap_{n\in\mathbb{N}} F_n \subset K \setminus \bigcup_{n\in\mathbb{N}} G_n = \emptyset.$$

According to the Cauchy criterion, the only way this can happen is if $F_n = \emptyset$ for some $n \in \mathbb{N}$. Then $K \subset \bigcup_{i=1}^{n} G_i$, and $\mathcal{O}' = \{G_i : 1 \le i \le n\}$ is a finite subcover of K from $\mathcal{O}$. □

Compactness shows up in several different, but equivalent ways on $\mathbb{R}$. We've already seen most of them, but their equivalence is not obvious. The following theorem shows a few of the most common manifestations of compactness.

Theorem 6.4. *Let $K \subset \mathbb{R}$. The following statements are equivalent to each other.*

(a) K is compact.

(b) K is closed and bounded.

(c) Every infinite subset of K has a limit point.

(d) Every sequence $\{a_n : n \in \mathbb{N}\} \subset K$ has a convergent subsequence.

(e) If F_n is a nested sequence of nonempty relatively closed subsets of K, then $\bigcap_{n\in\mathbb{N}} F_n \neq \emptyset$.

Proof. (a) $\Longleftrightarrow$ (b) is the Heine-Borel Theorem.

That (b)$\Rightarrow$(c) is the Bolzano-Weierstrass Theorem.

(c)$\Rightarrow$(d) is contained in the sequence version of the Bolzano-Weierstrass theorem.

(d)$\Rightarrow$(e) is done the same as the proof of the Cauchy criterion.

To complete the proof, it suffices to show (e)$\Rightarrow$(b). So, suppose K is such that (e) is true.

Let $F_n = K \cap ((-\infty, -n] \cup [n, \infty))$. Then F_n is a sequence of sets which are relatively closed in K such that $\bigcap_{n\in\mathbb{N}} F_n = \emptyset$. If K is unbounded, then $F_n \neq \emptyset$, $\forall n \in \mathbb{N}$, and a contradiction of (e) is evident. Therefore, K must be bounded.

If K is not closed, then there must be a limit point x of K such that $x \notin K$. Define a sequence of relatively closed and nested subsets of K by $F_n = [x - 1/n, x + 1/n] \cap K$ for $n \in \mathbb{N}$. Then $\bigcap_{n\in\mathbb{N}} F_n = \emptyset$, because $x \notin K$. This contradiction of (e) shows that K must be closed. □

These various ways of looking at compactness have been given different names by topologists. Property (c) is called *limit point compactness* and (d) is called *sequential compactness*.

Problem 18. A closed subset of a compact set is compact.