8 Limits of Functions

Definition 8.1. Let $D \subset \mathbb{R}$, x_0 be a limit point of D and $f : D \to \mathbb{R}$. The limit of $f(x)$ at x_0 is L if for each $\varepsilon > 0$ there is a $\delta > 0$ such that when $x \in D$ with $0 < |x - x_0| < \delta$, then $|f(x) - L| < \varepsilon$. When this is the case, we write $\lim_{x\to x_0} f(x) = L$.

A useful way of rewording this is to say that $\lim_{x\to x_0} f(x) = L$ iff for every $\varepsilon > 0$ there is a $\delta > 0$ such that $x \in (x_0 - \delta, x_0 + \delta) \cap D \setminus \{x_0\}$ implies $f(x) \in (L - \varepsilon, L + \varepsilon)$.

Example 8.1. If $f(x) = c$ is a constant function and $x_0 \in \mathbb{R}$, then for any positive numbers ε and δ,

$$x \in (x_0 - \delta, x_0 + \delta) \cap D \setminus \{x_0\} \Rightarrow |f(x) - c| = |c - c| = 0 < \varepsilon.$$

This shows the limit of every constant function exists at every point, and the limit is just the value of the function.

Example 8.2. Let $f(x) = x$, $x_0 \in \mathbb{R}$, and $\varepsilon = \delta > 0$. Then

$$x \in (x_0 - \delta, x_0 + \delta) \cap D \setminus \{x_0\} \Rightarrow |f(x) - f(x_0)| = |x - x_0| < \delta = \varepsilon.$$

This shows that the identity function has a limit at every point and its limit is just the value of the function at that point.

Example 8.3. Let $f(x) = \frac{2x^2-8}{x-2}$. In this case, the implied domain of f is $D = \mathbb{R} \setminus \{2\}$. We claim that $\lim_{x\to 2} f(x) = 8$.

To see this, let $\varepsilon > 0$ and choose $\delta \in (0, \varepsilon/2)$. If $0 < |x - 2| < \delta$, then

$$|f(x) - 8| = \left|\frac{2x^2 - 8}{x - 2} - 8\right| = |2(x + 2) - 8| = 2|x - 2| < \varepsilon.$$

Example 8.4. Let $f(x) = \sqrt{x + 1}$. Then the implied domain of f is $D = [-1, \infty)$. We claim that $\lim_{x\to -1} f(x) = 0$.

To see this, let $\varepsilon > 0$ and choose $\delta \in (0, \varepsilon^2)$. If $0 < x - (-1) = x + 1 < \delta$, then

$$|f(x) - 0| = \sqrt{x + 1} < \sqrt{\delta} < \sqrt{\varepsilon^2} = \varepsilon.$$

There is an obvous similarity between the definition of limit of a sequence and limit of a function. The following theorem makes this similarity explicit, and gives another way to prove facts about limits of functions.

Theorem 8.1. *Let $f : D \to \mathbb{R}$ and x_0 be a limit point of D. $\lim_{x\to x_0} f(x) = L$ iff whenever x_n is a sequence from $D \setminus \{x_0\}$ such that $x_n \to x_0$, then $f(x_n) \to L$.*

Proof. ($\Rightarrow$) Suppose $\lim_{x\to x_0} f(x) = L$ and x_n is a sequence from $D \setminus \{x_0\}$ such that $x_n \to x_0$. Let $\varepsilon > 0$. There exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$

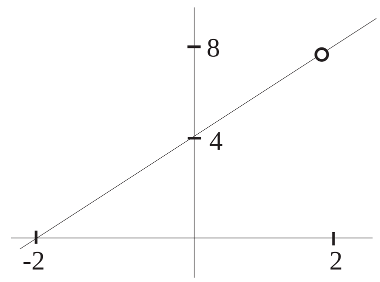

Figure 4: The function from Example 8.4. Note that the graph is a line with one "hole" in it.

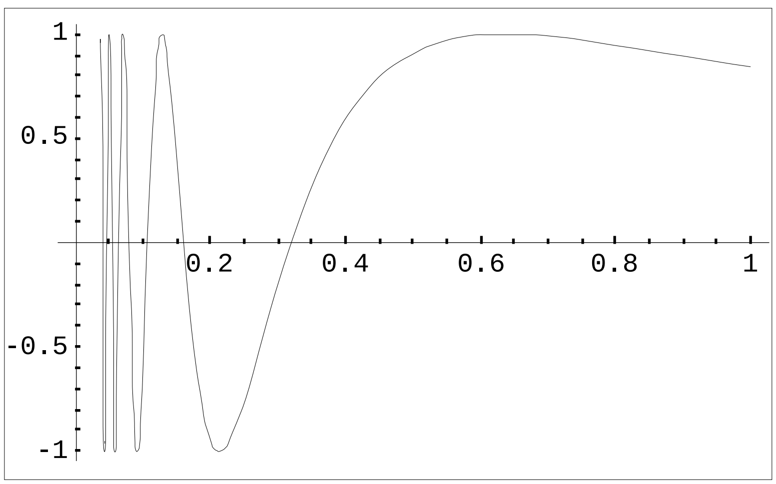

Figure 5: This is the function from Example 8.5. The graph shown here is on the interval $[0.05, 1]$. There are an infinite number of oscillations from -1 to 1 on any open interval containing the origin.

whenever $x \in (x - \delta, x + \delta) \cap D \setminus \{x_0\}$. Since $x_n \to x_0$, there is an $N \in \mathbb{N}$ such that $n \geq N$ implies $0 < |x_n - x_0| < \delta$. In this case, $|f(x_n) - L| < \varepsilon$. This shows $f(x_n) \to x_0$.

($\Leftarrow$) Suppose that whenever x_n is a sequence from $D \setminus \{x_0\}$ such that $x_n \to x_0$, then $f(x_n) \to L$, but $\lim_{x \to x_0} f(x) \neq L$. Then there exists an $\varepsilon > 0$ such that for all $\delta > 0$ there is an $x \in (x_0 - \delta, x_0 + \delta) \cap D \setminus \{x_0\}$ such that $|f(x) - L| \geq \varepsilon$. In particular, for each $n \in \mathbb{N}$, there must exist $x_n \in (x_0 - 1/n, x_0 + 1/n) \cap D \setminus \{x_0\}$ such that $|f(x_n) - L| \geq \varepsilon$. Since $x_n \to x_0$, this is a contradiction. Therefore, $\lim_{x \to x_0} f(x) = L$. □

Example 8.5. Let $f(x) = \sin(1/x)$, $a_n = \frac{1}{n\pi}$ and $b_n = \frac{1}{(2n-1)\pi}$. Then $a_n \downarrow 0$, $b_n \downarrow 0$, $f(a_n) = 0$ and $f(b_n) = 1$ for all $n \in \mathbb{N}$. An application of Theorem 8.1 shows $\lim_{x \to 0} f(x)$ does not exist.

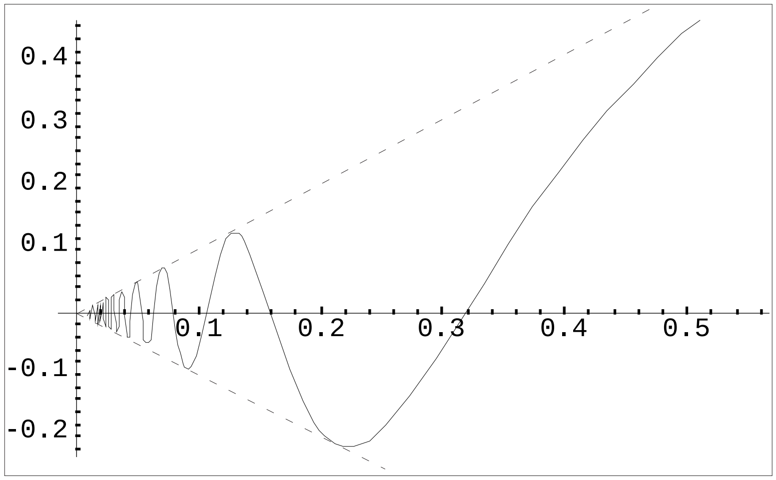

Figure 6: This is the function from Example 8.6. The graph shown here is on the interval $[0.01, 0.5]$. There are an infinite number of oscillations from $-x$ to x on any open interval containing the origin.

Theorem 8.2 (Squeeze Theorem). *Suppose f, g and h are all functions defined on $D \subset \mathbb{R}$ with $f(x) \leq g(x) \leq h(x)$ for all $x \in D$. If x_0 is a limit point of D and $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = L$, then $\lim_{x \to x_0} g(x) = L$.*

Proof. Let x_n be any sequence from $D \setminus \{x_0\}$ such that $x_n \to x_0$. According to Theorem 8.1, both $f(x_n) \to L$ and $h(x_n) \to L$. Since $f(x_n) \leq g(x_n) \leq h(x_n)$, an application of the sandwich theorem for sequences shows $g(x_n) \to L$. Now, another use of Theorem 8.1 shows $\lim_{x \to x_0} g(x) = L$. □

Example 8.6. Let $f(x) = x \sin(1/x)$. Since $-1 \leq \sin(1/x) \leq 1$ when $x \neq 0$, we see that $-x \leq \sin(1/x) \leq x$ for $x \neq 0$. Since $\lim_{x \to 0} x = \lim_{x \to 0} -x = 0$, Theorem 8.2 implies $\lim_{x \to 0} x \sin(1/x) = 0$. See Figure 6

Theorem 8.3. *Suppose $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ and x_0 is a limit point of D. If $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} g(x) = M$, then*

(a) $\lim_{x \to x_0} (f + g)(x) = L + M$,

(b) $\lim_{x \to x_0} (af)(x) = aL, \ \forall x \in \mathbb{R}$,

(c) $\lim_{x \to x_0} (fg)(x) = LM$, *and*

(d) $\lim_{x \to x_0} (1/f)(x) = 1/L$, *as long as* $L \neq 0$.

Proof. Suppose a_n is a sequence from $D \setminus \{x_0\}$ converging to x_0. Then Theorem 8.1 implies $f(a_n) \to L$ and $g(a_n) \to M$. (a)-(d) follow at once from the corresponding properties for sequences. □

Example 8.7. Let $f(x) = 3x + 2$. If $g_1(x) = 3$, $g_2(x) = x$ and $g_3(x) = 2$, then $f(x) = g_1(x)g_2(x) + g_3(x)$. Examples 8.1 and 8.2 along with parts (a) and (c) of Theorem 8.3 immediately show that for every $x \in \mathbb{R}$, $\lim_{x\to x_0} f(x) = f(x_0)$.

In the same manner as Example 8.7, it can be shown for every rational function $f(x)$, that $\lim_{x\to x_0} f(x) = f(x_0)$ whenever $f(x_0)$ exists.

Extra Credit 7. If $\mathbb{Q} = \{q_n : n \in \mathbb{N}\}$ is an enumeration of the rational numbers and

$$f(x) = \begin{cases} 1/n, & x = q_n \\ 0, & x \in \mathbb{Q}^c \end{cases}$$

then $\lim_{x\to a} f(x) = 0$, for all $a \in \mathbb{Q}^c$.