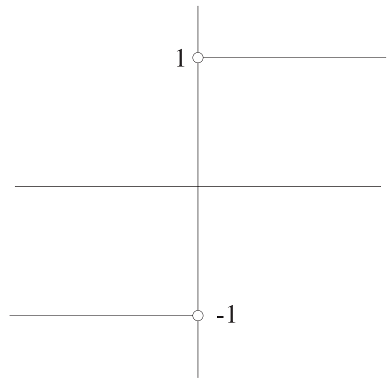

Figure 7: The function $f(x) = |x|/x$ from Example 9.1.

9 Unilateral Limits

Definition 9.1. Let $f : D \to \mathbb{R}$ and x_0 be a limit point of $D \cap (-\infty, x_0)$. f has L as its *left-hand limit* at x_0 if for all $\varepsilon > 0$ there is a $\delta > 0$ such that $f((x_0 - \delta, x_0) \cap D) \subset (L - \varepsilon, L + \varepsilon)$. In this case, we write $\lim_{x \uparrow x_0} f(x) = L$.

Let $f : D \to \mathbb{R}$ and x_0 be a limit point of $D \cap (x_0, \infty)$. f has L as its *right-hand limit* at x_0 if for all $\varepsilon > 0$ there is a $\delta > 0$ such that $f((x_0, x_0 + \delta) \cap D) \subset (L - \varepsilon, L + \varepsilon)$. In this case, we write $\lim_{x \downarrow x_0} f(x) = L$.

Another standard notation for the unilateral limits is

$$\lim_{x \uparrow x_0} f(x) = \lim_{x \to x_0-} f(x) \text{ and } \lim_{x \downarrow x_0} f(x) = \lim_{x \to x_0+} f(x).$$

Example 9.1. Let $f(x) = |x|/x$. Then $\lim_{x \downarrow 0} f(x) = 1$ and $\lim_{x \uparrow 0} f(x) = -1$. (See Figure 7.)

Theorem 9.1. *Let $f : D \to \mathbb{R}$ and x_0 be a limit point of D.*

$$\lim_{x \to x_0} f(x) = L \quad \iff \quad \lim_{x \uparrow x_0} f(x) = L = \lim_{x \downarrow x_0} f(x)$$

Proof. This proof is left as an exercise. □

Theorem 9.2. *If $f : (a, b) \to \mathbb{R}$ is monotone, then both unilateral limits of f exist at every point of (a, b).*

Proof. To be specific, suppose f is increasing and $x_0 \in (a, b)$. Let $\varepsilon > 0$ and $L = \operatorname{lub}\{f(x) : a < x < x_0\}$. According to Corollary 11, there must exist an $x \in (a, x_0)$ such that $L - \varepsilon < f(x) \le L$. Define $\delta = x_0 - x$. If $y \in (x_0 - \delta, x_0)$, then $L - \varepsilon = f(x) < f(y) \le L$. This shows $\lim_{x \uparrow x_0} f(x) = L$.

The proof that $\lim_{x \downarrow x_0} f(x)$ exists is similar.

To handle the case when f is decreasing, consider $-f$ instead of f. □