10 Continuity

Definition 10.1. Let $f : D \to \mathbb{R}$ and $x_0 \in D$. f is *continuous at* x_0 if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that when $x \in D$ with $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$. The set of all points at which f is continuous is denoted $C(f)$.

Several useful ways of rephrasing this are contained in the following theorem, the proof of which is left to the reader.

Theorem 10.1. *Let $f : D \to \mathbb{R}$ and $x_0 \in D$. The following statements are equivalent.*

(a) $x_0 \in C(f)$,

(b) For all $\varepsilon > 0$ there is a $\delta > 0$ such that $x \in (x_0 - \delta, x_0 + \delta) \cap D \Rightarrow f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$, and

(c) For all $\varepsilon > 0$ there is a $\delta > 0$ such that $f((x_0 - \delta, x_0 + \delta)) \subset (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$.

Example 10.1. Define

$$f(x) = \begin{cases} \frac{2x^2-8}{x-2}, & x \neq 2 \\ 8, & x = 2 \end{cases}.$$

It follows easily from Example 8.3 that $2 \in C(f)$.

There is a subtle difference between the treatment of the domain of the function between the definitions of limit and continuity. In the definition of limit, the "target point," x_0 is required to be a limit point of the domain. There is no such stipulation in the definition of continuity. To see a consequence of this difference, consider the following example.

Example 10.2. If $f : \mathbb{Z} \to \mathbb{R}$ is an arbitrary function, then $C(f) = \mathbb{Z}$. To see this, let $n_0 \in \mathbb{Z}$, $\varepsilon > 0$ and $\delta = 1$. If $x \in \mathbb{Z}$ with $|x - n_0| < \delta$, then $x = n_0$. It's now obvious that $|f(x) - f(n_0)| = 0 < \varepsilon$, so f is continuous at n_0.

This leads to the following theorem.

Theorem 10.2. *Let $f : D \to \mathbb{R}$ and $x_0 \in D$. If x_0 is an isolated point of D, then $x_0 \in C(f)$. If x_0 is a limit point of D, then $x_0 \in C(f)$ iff $\lim_{x \to x_0} f(x) = f(x_0)$.*

Proof. If x_0 is isolated in D, then there is an $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \cap D = \{x_0\}$. For any $\varepsilon > 0$, the definition of continuity is satisfied with this δ.

Next, suppose x_0 is a limit point of D.

The definition of continuity says that f is continuous at x_0 iff for all $\varepsilon > 0$ there is a $\delta > 0$ such that when $x \in (x_0 - \delta, x_0 + \delta) \cap D$, then $f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$.

The definition of limit says that $\lim_{x\to x_0} f(x) = f(x_0)$ iff for all $\varepsilon > 0$ there is a $\delta > 0$ such that when $x \in (x_0 - \delta, x_0 + \delta) \cap D \setminus \{x_0\}$, then $f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$.

Comparing these two definitions, it is clear that $x_0 \in C(f)$ implies

$$\lim_{x\to x_0} f(x) = f(x_0).$$

On the other hand, suppose $\lim_{x\to x_0} f(x) = f(x_0)$ and $\varepsilon > 0$. Choose δ according to the definition of limit. When $x \in (x_0 - \delta, x_0 + \delta) \cap D \setminus \{x_0\}$, then $f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$. It is easy to see from this that when $x = x_0$, then $f(x) - f(x_0) = f(x_0) - f(x_0) = 0 < \varepsilon$. Therefore, when $x \in (x_0 - \delta, x_0 + \delta) \cap D$, then $f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$, and $x_0 \in C(f)$, as desired. □

Example 10.3. If $f(x) = c$, for some $c \in \mathbb{R}$, then Example 8.1 and Theorem 10.2 show that f is continuous at every point.

Example 10.4. If $f(x) = x$, then Example 8.2 and Theorem 10.2 show that f is continuous at every point.

Corollary 10.3. *Let $f : D \to \mathbb{R}$ and $x_0 \in D$. $x_0 \in C(f)$ iff whenever x_n is a sequence from D with $x_n \to x_0$, then $f(x_n) \to f(x_0)$.*

Proof. Combining Theorem 10.2 with Theorem 8.1 shows this to be true. □

Example 10.5. Suppose

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}.$$

For each $x \in \mathbb{Q}$, there is a sequence of irrational numbers converging to x, and for each $y \in \mathbb{Q}^c$ there is a sequence of rational numbers converging to y. Corollary 10.3 shows $C(f) = \emptyset$.

Example 10.6 (Salt and Pepper Function). Since $\mathbb{Q}$ is a countable set, it can be written as a sequence, $\mathbb{Q} = \{q_n : n \in \mathbb{N}\}$. Define

$$f(x) = \begin{cases} 1/n, & x = q_n, \\ 0, & x \in \mathbb{Q}^c. \end{cases}$$

If $x \in \mathbb{Q}$, then $x = q_n$, for some n and $f(x) = 1/n > 0$. There is a sequence x_n from $\mathbb{Q}^c$ such that $x_n \to x$ and $f(x_n) = 0 \not\to f(x) = 1/n$. Therefore $C(f) \cap \mathbb{Q} = \emptyset$.

On the other hand, let $x \in \mathbb{Q}^c$ and $\varepsilon > 0$. Choose $N \in \mathbb{N}$ large enough so that $1/N < \varepsilon$ and let $\delta = \min\{|x - q_n| : 1 \le n \le N\}$. If $|x - y| < \delta$, there are two cases to consider. If $y \in \mathbb{Q}^c$, then $|f(y) - f(x)| = |0 - 0| = 0 < \varepsilon$. If $y \in \mathbb{Q}$, then the choice of δ guarantees $y = q_n$ for some $n > N$. In this case, $|f(y) - f(x)| = f(y) = f(q_n) = 1/n < 1/N < \varepsilon$. Therefore, $x \in C(f)$.

This shows that $C(f) = \mathbb{Q}^c$.

It is a consequence of an advanced result known as the Baire category theorem that there is no function f such that $C(f) = \mathbb{Q}$.

The following theorem is an almost immediate consequence of Theorem 8.3.

Theorem 10.4. *Let $f : D_f \to \mathbb{R}$ and $g : D_g \to \mathbb{R}$. If $x_0 \in C(f) \cap C(G)$, then*

(a) $x_0 \in C(f + g)$,

(b) $x_0 \in C(\alpha f)$, $\forall \alpha \in \mathbb{R}$,

(c) $x_0 \in C(fg)$, *and*

(d) $x_0 \in C(f/g)$ *when* $g(x_0) \neq 0$.

Corollary 10.5. *If f is a rational function, then f is continuous at each point of its domain.*

Proof. This is a consequence of Examples 10.3 and 10.4 used with Theorem 10.4. □

Theorem 10.6. *Suppose $f : D_f \to \mathbb{R}$ and $g : D_g \to \mathbb{R}$ such that $f(D_f) \subset D_g$. If there is an $x_0 \in C(f)$ such that $f(x_0) \in C(g)$, then $x_0 \in C(g \circ f)$.*

Proof. Let $\varepsilon > 0$ and choose $\delta_1 > 0$ such that $g((f(x_0) - \delta_1, f(x_0) + \delta_1) \cap D_g) \subset (g \circ f(x_0) - \varepsilon, g \circ f(x_0) + \varepsilon)$. Choose $\delta_2 > 0$ such that $f((x_0 - \delta_2, x_0 + \delta_2) \cap D_f) \subset (f(x_0) - \delta_1, f(x_0) + \delta_1)$. Then

$$\begin{aligned} g \circ f((x_0 - \delta_2, x_0 + \delta_2) \cap D_f) &\subset g((f(x_0) - \delta_1, f(x_0) + \delta_1) \cap D_g) \\ &\subset (g \circ f(x_0) - \delta_2, g \circ f(x_0) + \delta_2) \cap D_f). \end{aligned}$$

Since this shows Theorem 10.1(c) is satisfied at x_0 with the function $g \circ f$, it follows that $x_0 \in C(g \circ f)$. □

Problem 20. Prove that $f(x) = \sqrt{x}$ is continuous on $[0, \infty)$.

Example 10.7. If f is as in Problem 20, then Theorem 10.6 shows $f \circ f = \sqrt[4]{x}$ is continuous on $[0, \infty)$.

In the same way, it can be shown by induction that $f(x) = x^{m/2^n}$ is continuous on $[0, \infty)$ for all $m, n \in \mathbb{Z}$.