11 Unilateral Continuity

Definition 11.1. Let $f : D \to \mathbb{R}$ and $x_0 \in D$. f is *left-continuous* at x_0 if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $f((x_0 - \delta, x_0] \cap D) \subset (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$.

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Theorem 11.1. *Let $f : D \to \mathbb{R}$ and $x_0 \in D$. $x_0 \in C(f)$ iff f is both right and left-continuous at x_0.*

Proof. The proof of this theorem is left as an exercise. □

According to Theorem 9.1, when f is monotone on an interval (a, b), the unilateral limits of f exist at every point. In order for such a function to be continuous at $x_0 \in (a, b)$, it must be the case that

$$\lim_{x \uparrow x_0} f(x) = f(x_0) = \lim_{x \downarrow x_0} f(x).$$

If either of the two inequalities is violated, the function is not continuous at x_0.

In the case, when $\lim_{x \uparrow x_0} f(x) \neq \lim_{x \downarrow x_0} f(x)$, it is said that a *jump discontinuity* occurs at x_0.

Example 11.1. The function

$$f(x) = \begin{cases} |x|/x, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

has a jump discontinuity at $x = 0$.

In the case when $\lim_{x \uparrow x_0} f(x) = \lim_{x \downarrow x_0} f(x) \neq f(x_0)$, it is said that f has a *removable discontinuity* at x_0. The discontinuity is called "removable" because in this case, the function can be made continuous at x_0 merely by redefining its value at the single point, x_0, to be the value of the two one-sided limits.

Example 11.2. The function $f(x) = \frac{x^2-4}{x-2}$ is not continuous at $x = 2$ because 2 is not in the domain of f. Since $\lim_{x \to 2} f(x) = 4$, if the domain of f is extended by setting $f(2) = 4$, then this extended f is continuous everywhere. (See Figure 8.)

Theorem 11.2. *If $f : (a, b) \to \mathbb{R}$ is monotone, then $(a, b) \setminus C(f)$ is countable.*

Proof. In light of the discussion above and Theorem 9.1, it is apparent that the only types of discontinuities f can have are jump discontinuities.

To be specific, suppose f is increasing and $x_0, y_0 \in (a, b) \setminus C(f)$ with $x_0 < y_0$. In this case, the fact that f is increasing implies

$$\lim_{x \uparrow x_0} f(x) < \lim_{x \downarrow x_0} f(x) \leq \lim_{x \uparrow y_0} f(x) < \lim_{x \downarrow y_0} f(x).$$

This implies that for any two $x_0, y_0 \in (a, b) \setminus C(f)$, there are disjoint open intervals, $I_{x_0} = (\lim_{x \uparrow x_0} f(x), \lim_{x \downarrow x_0} f(x))$ and $I_{y_0} = (\lim_{x \uparrow y_0} f(x), \lim_{x \downarrow y_0} f(x))$.

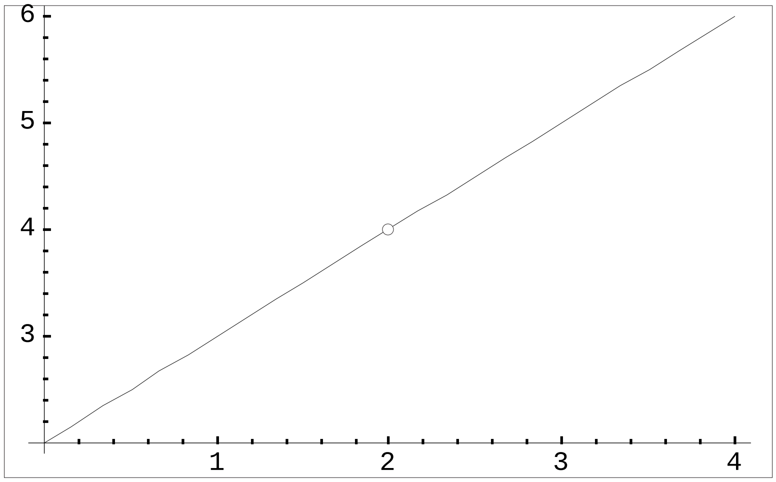

Figure 8: The function from Example 11.2. Note that the graph is a line with one "hole" in it. Plugging up the hole removes the discontinuity.

For each $x \in (a, b) \setminus C(f)$, choose $q_x \in I_x \cap \mathbb{Q}$. Because of the pairwise disjointness of the intervals $\{I_x : x \in (a, b) \setminus C(f)\}$, this defines an bijection between $(a, b) \setminus C(f)$ and a subset of $\mathbb{Q}$. Therefore, $(a, b) \setminus C(f)$ must be countable.

A similar argument holds for a decreasing function. □

Theorem 11.2 implies that a monotone function is continuous at "nearly every" point in its domain. Characterizing the points of discontinuity as countable is the best that can be hoped for. To see this, let $D = \{d_n : n \in \mathbb{N}\}$ be a countable set and define $J_x = \{n : d_n < x\}$. Using this, we define

$$f(x) = \sum_{n \in J_x} \frac{1}{2^n}. \tag{4}$$

Extra Credit 7. If f is defined as in (4), then $D = C(f)^c$.

Problem 21. If $f : \mathbb{R} \to \mathbb{R}$ is monotone, then there is a countable set D such that the values of f can be altered on D in such a way that the altered function is left-continuous at every point of $\mathbb{R}$.