13 Uniform Continuity

Most of the ideas contained in this section will not be needed until we begin developing the theory of integration.

Definition 13.1. A function $f : D \to \mathbb{R}$ is *uniformly continuous* if for all $\varepsilon > 0$ there is a $\delta > 0$ such that when $x, y \in D$ with $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

The idea here is that in the ordinary definition of continuity, the δ in the definition depends on both the ε and the x at which continuity is being tested. With uniform continuity, δ only depends on ε; i. e., the same δ works uniformly across the whole domain.

Theorem 13.1. *If $f : D \to \mathbb{R}$ is uniformly continuous, then it is continuous.*

Proof. This proof is left as an exercise. □

Example 13.1. Let $f(x) = 1/x$ on $D = (0, 1)$ and $\varepsilon > 0$. It's clear that f is continuous on D. Let $\delta > 0$ and choose $m, n \in \mathbb{N}$ such that $m > 1/\delta$ and $n - m > \varepsilon$. If $x = 1/m$ and $y = 1/n$, then $0 < y < x < \delta$ and $f(y) - f(x) = n - m > \varepsilon$. Therefore, f is not uniformly continuous.

Theorem 13.2. *If $f : D \to \mathbb{R}$ is continuous and D is compact, then f is uniformly continuous.*

Proof. Suppose f is not uniformly continuous. Then there is an $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ there are $x_n, y_n \in D$ with $|x_n - y_n| < 1/n$ and $|f(x_n) - f(y_n)| \geq \varepsilon$. An application of the Bolzano-Weierstrass theorem yields a subsequence x_{n_k} of x_n such that $x_{n_k} \to x_0 \in D$.

Since f is continuous at x_0, there is a $\delta > 0$ such that whenever $x \in (x_0 - \delta, x_0 + \delta) \cap D$, then $|f(x) - f(x_0)| < \varepsilon/2$. Choose $n_k \in \mathbb{N}$ such that $1/n_k < \delta/2$ and $x_{n_k} \in (x_0 - \delta/2, x_0 + \delta/2)$. Then both $x_{n_k}, y_{n_k} \in (x_0 - \delta, x_0 + \delta)$ and

$$\begin{aligned}\varepsilon \leq |f(x_{n_k}) - f(y_{n_k})| &= |f(x_{n_k}) - f(x_0) + f(x_0) - f(y_{n_k})| \\ &\leq |f(x_{n_k}) - f(x_0)| + |f(x_0) - f(y_{n_k})| < \varepsilon/2 + \varepsilon/2 = \varepsilon,\end{aligned}$$

which is a contradiction.

Therefore, f must be uniformly continuous. □

The following corollary is an immediate consequence of Theorem 13.2.

Corollary 13.3. *If $f : [a, b] \to \mathbb{R}$ is continuous, then f is uniformly continuous.*

Problem 24. Prove that an unbounded function on a bounded open interval cannot be uniformly continuous.

Problem 25. Prove Theorem 13.1.