15 Differentiable Functions

Definition 15.1. The function f is *differentiable on an open interval* I if $I \subset D(f)$. If f is differentiable on its domain, then it is said to be *differentiable*. In this case, the function f' is called the *derivative* of f.

Lemma 15.1 (Rolle's Theorem). *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, differentiable on (a, b) and $f(a) = 0 = f(b)$, then there is a $c \in (a, b)$ such that $f'(c) = 0$.*

Proof. Since $[a, b]$ is compact, Corollary 12.4 implies the existence of $x_m, x_M \in [a, b]$ such that $f(x_m) \leq f(x) \leq f(x_M)$ for all $x \in [a, b]$. If $f(x_m) = f(x_M)$, then f is constant on $[a, b]$ and any $c \in (a, b)$ satisfies the lemma. Otherwise, either $f(x_m) < 0$ or $f(x_M) > 0$. If $f(x_m) < 0$, then $x_m \in (a, b)$ and Theorem 14.7 implies $f'(x_m) = 0$. If $f(x_M) > 0$, then $x_M \in (a, b)$ and Theorem 14.7 implies $f'(x_M) = 0$. □

Theorem 15.2 (Cauchy Mean Value Theorem). [4] *If $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are such that f and g are continuous on $[a, b]$ and differentiable on (a, b), then there is a $c \in (a, b)$ such that*

$$g'(c)(f(b) - f(a)) = f'(c)(g(b) - g(a)).$$

Proof. Let

$$h(x) = (g(b) - g(a))(f(a) - f(x)) + (g(x) - g(a))(f(b) - f(a)).$$

Then h is continuous on $[a, b]$ and differentiable on (a, b) with $h(a) = h(b) = 0$. Theorem 15.1 yields a $c \in (a, b)$ such that $h'(c) = 0$. Then

$$\begin{aligned} 0 = h'(c) &= -(g(b) - g(a))f'(c) + g'(c)(f(b) - f(a)) \\ &\implies g'(c)(f(b) - f(a)) = f'(c)(g(b) - g(a)). \end{aligned}$$

□

Corollary 15.3 (Mean Value Theorem). *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there is a $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.*

Proof. Let $g(x) = x$ in Theorem 15.2. □

Theorem 15.4. *Suppose $f : (a, b) \to \mathbb{R}$ is a differentiable function. f is increasing on (a, b) iff $f'(x) \geq 0$ for all $x \in (a, b)$.*

Proof. Choose $\alpha, \beta \in (a, b)$ with $\alpha < \beta$. According to Corollary 15.3, there is a $c \in (\alpha, \beta)$ such that

$$f(\beta) - f(\alpha) = f'(c)(\beta - \alpha) \geq 0.$$

□

[4] Theorem 15.2 is also often called the "Generalized Mean Value Theorem."

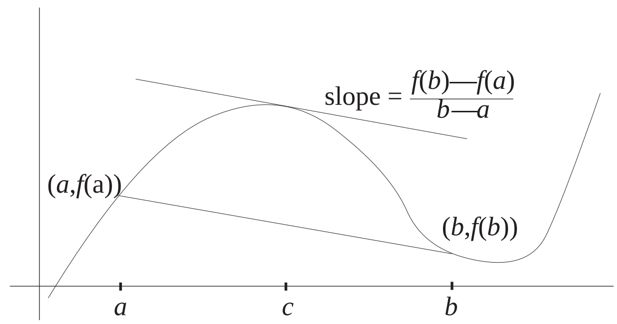

Figure 9: This is a “picture proof” of Corollary 15.3.

Corollary 15.5. *Let $f : (a, b) \to \mathbb{R}$ be a differentiable function. f is constant iff $f'(x) = 0$ for all $x \in (a, b)$.*

Theorem 15.6 (Darboux's Theorem). *If f is differentiable on an open set containing $[a, b]$ and γ is between $f'(a)$ and $f'(b)$, then there is a $c \in [a, b]$ such that $f'(c) = \gamma$.*

Proof. If $f'(a) = f'(b)$, then $c = a$ satisfies the theorem. So, we may as well assume $f'(a) \neq f'(b)$. There is no generality lost in assuming $f'(a) < f'(b)$, for, otherwise, we just replace f with $g = -f$.

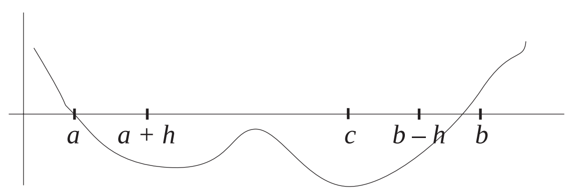

Figure 10: This could be the function h of Theorem 15.6.

Let $h(x) = f(x) - \gamma(x - \alpha)$ so that $D(f) = D(h)$ and $h'(x) = f'(x) - \gamma$. In particular, this implies $h'(a) < 0 < h'(b)$. Because of this, there must be an $h > 0$ small enough so that

$$\frac{f(a+h) - f(a)}{h} < 0 \implies f(a+h) < f(a)$$

and

$$\frac{f(b) - f(b-h)}{h} > 0 \implies f(b-h) < f(b).$$

(See Figure 10.) In light of these two inequalities and Theorem 12.4, there must be a $c \in (a, b)$ such that $f(c) = \text{glb}\,\{f(x) : x \in [a, b]\}$. Now Theorem 14.7 gives $0 = h'(c) = f'(c) - \gamma$, and the theorem follows. □