Student Solutions Manual to accompany

Advanced Engineering Mathematics

Eighth Edition

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John Wiley & Sons, Inc.
New York • Chichester • Weinheim • Brisbane • Singapore • Toronto

CONTENTS

PART A. ORDINARY DIFFERENTIAL EQUATIONS

CHAPTER 1. First-Order Differential Equations

Sec. 1.1 Basic Concepts and Ideas

Comment on (5). $y = cx - c^2$, hence $y' = c$, and $y'^2 - xy' + y = c^2 - xc + (cx - c^2) = 0$.

Problem Set 1.1. Page 8

1. **Calculus.** This is a problem of calculus, namely, to integrate x^2, giving $\frac{1}{3}x^3 + c$, where the constant of integration c is arbitrary. This is essential. It means that the differential equation $y' = x^2$ has infinitely many solutions, each of these cubical parabolas corresponding to a certain value of c. Sketch some of them.

13. **Initial value problem.** $y' = -2ce^{-2x}$ by differentiation. Hence the left side becomes
$$y' + 2y = -2ce^{-2x} + 2(ce^{-2x} + 1.4) = 2.8.$$
This verifies the given solution $y = ce^{-2x} + 1.4$. For $x = 0$ you have $e^0 = 1$ and thus $y(0) = c + 1.4$, which is required to be equal to 1.0. Hence $1.0 = c + 1.4, c = -0.4$, and the answer is $y = -0.4e^{-2x} + 1.4$.

23. **Falling body.** $s = gt^2/2 = 100$ [m]. Here $g = 9.80$ m/sec^2 since s is measured in meters. Using $s = 100$ and solving for t gives
$$t = \sqrt{\frac{100}{g/2}} = 10\sqrt{\frac{1}{4.9}} = 4.52 \text{ [sec]}.$$
The second result, 6.389 sec, is less than twice the first because the motion is accelerated, the velocity increases.

Sec. 1.2 Geometrical Meaning of $y' = f(x,y)$. Direction Fields

Problem Set 1.2. Page 12

1. **Calculus.** Note that the solution curves are *not* congruent because c is a factor, not an additive constant (as, for instance, in Prob. 5).

5. **Verification of solution.** Geometrically, the solution curves are obtained from each other by translations in the y-direction; they are congruent because c is an additive constant.

7. **Verification of solution.** At each point (x, y) the tangent direction of the solution is $-x/y$, hence perpendicular to the slope y/x of the ray from (0, 0) to (x, y), suggesting that the solutions are concentric circles about the origin. You can prove this by calculus, as follows. Multiply the equation by y, obtaining $yy' = -x$. Then integrate on both sides with respect to x. This gives
$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + c \quad \text{or} \quad y^2 + x^2 = 2c.$$

15. **Initial value problem.** The idea from calculus just applied in Prob. 7 here gives $(9/2)y^2 + 2x^2 = c$ or $4x^2 + 9y^2 = 2c$; these are the ellipses $x^2/9 + y^2/4 = c/18$.

17. Initial value problem. In this section the usual notation is (2), that is, $y' = f(x,y)$, and the direction field lies in the xy-plane. In Prob. 17 the equation is $v' = f(t,v) = g - bv^2/m$. Hence the direction field lies in the tv-plane. With $m = 1$ and $b = 1$ the equation becomes $v' = g - v^2$. Then $v = 3.13$ gives $g - v^2 = 9.80 - 3.13^2 = 0$, approximately. The differential equation now shows that v' must be identically zero. Conclude that $v = 3.13$ must be a solution. For $v < 3.13$ you have $v' > 0$ (increasing curves) and for $v > 3.13$ you have $v' < 0$ (decreasing curves). Note that the isoclines are the horizontal parallel straight lines $g - v^2 = const$, thus $v = const$.

Sec. 1.3 Separable Differential Equations

Problem Set 1.3. Page 18

3. General solution by separation. Dividing by the right side gives

$$\frac{y'}{1+0.01\,y^2} = 1 \quad \text{or} \quad \frac{dy}{1+0.01\,y^2} = dx. \tag{A}$$

Now integrate. This is one of the more important integrals; set $v = 0.1y$ to get $y = 10\,v$, $dy = 10\,dv$, and from (A),

$$10\,dv/(1+v^2) = dx, \qquad \text{integrated} \qquad 10 \arctan v = x + C.$$

Recalling that $v = 0.1\,y$ gives $10 \arctan 0.1y = x + C$. This implies

$$y = 10 \tan(0.1(x+C)) = 10 \tan(0.1x + c), \qquad c = 0.1C.$$

15. Initial value problem. Separate variables and integrate on both sides (by parts on the right) to get

$$dy/y^2 = 2(x+1)e^{-x}\,dx, \qquad -1/y = (-2x-4)e^{-x} + c.$$

Multiply by -1 and take the reciprocal,

$$y = 1/[(2x+4)e^{-x} - c].$$

From the initial condition $y(0) = 1/6$ obtain by setting $x = 0$

$$1/6 = y(0) = 1/(4-c), \qquad \text{hence} \qquad 6 = 4 - c, \quad c = -2.$$

Inserting this into y gives the answer.

23. Initial value problem. Dividing the given equation by x^2 and setting $y/x = u$, hence $y = xu$ and $y' = u + xu'$, gives

$$(y/x)y' = u(u + xu') = 2u^2 + 4.$$

Subtracting u^2 on both sides gives $xuu' = u^2 + 4$. Separate variables, then multiply both sides by 2, and integrate with respect to x on both sides,

$$2u\,du/(u^2+4) = 2\,dx/x, \qquad \ln(u^2+4) = \ln(x^2) + C, \qquad u^2 + 4 = cx^2.$$

Solving for u^2 and taking roots gives $y/x = u = \sqrt{cx^2 - 4}$, so that

$$y = ux = \sqrt{cx^4 - 4x^2}\,.$$

From this and the initial condition,

$$y(2) = 4 = \sqrt{16c - 16} = 4\sqrt{c-1}, \qquad c - 1 = 1, \qquad c = 2.$$

This gives the answer in Appendix 2.

26. Team Project. (b) In finding a differential equation you always have to get rid of the arbitrary constant c. For $xy = c$ this is very simple because this equation is solved for c (differentiate this equation implicitly with respect to x); in other cases it is usually best to first solve algebraically for c.
(d) This orthogonality condition is usually considered in calculus. You will need it again in Sec. 1.8.

27. CAS Project. This integral (the error function, except for a constant factor; see (35) in Appendix A3.1) is important in heat conduction (see Sec. 11.6). A similar integral is basic in statistics (see Sec. 22.8).

Sec. 1.4 Modeling: Separable Equations

Problem Set 1.4. Page 23

1. Exponential growth. Let $y(0) = y_0$ be the initial amount at $t = 0$. The model equation $y' = ky$ has the solution $y = ce^{kt}$. For the given initial amount y_0 this becomes $y = y_0 e^{kt}$. For $t = 1$ (1 day) this gives $y(1) = y_0 e^k$. By assumption this is twice the initial amount (doubling in 1 day). Hence $y_0 e^k = 2y_0$. Divide this by y_0 to get $e^k = 2$. After 3 days you have $y(3) = y_0 e^{3k} = y_0 \cdot 2^3$, where we used $e^{ab} = (e^a)^b$. Similarly for 1 week ($t = 7$).

11. Sugar inversion. $y' = ky$, $y(t) = 0.01e^{kt}$ from the first condition and $y(4) = 0.01e^{4k} = 1/300 = 0.01/3$ from the second. Hence $e^{4k} = 1/3$, $k = 1/4 \ln(1/3) = -0.275$.

15. Curves (ellipses) From calculus you know that the slope of the tangent of a curve $y = y(x)$ is the derivative $y'(x)$. From the given data you thus obtain immediately the differential equation $y' = -4x/y$. Solve it by separation of variables (multiply by y),

$$y\,dy = -4x\,dx, \qquad y^2/2 = -2x^2 + c, \qquad y^2/4 + x^2 = c/2.$$

For instance, $c = 2$ gives the ellipse with semi-axes 1 (in the x-direction) and 2 (in the y-direction). Sketch this ellipse and some of the others.

Sec. 1.5 Exact Differential Equations. Integrating Factors

Example 3. A nonexact equation. You can write the given equation as $y' = y/x$. Separate variables, obtaining $dy/y = dx/x$, $\ln y = \ln x + \tilde{c}$, $y = cx$.

Problem Set 1.5. Page 31

17. Test for exactness. Initial value problem. Exactness is seen from

$$\frac{\partial}{\partial y} M = \frac{\partial}{\partial y}((x+1)e^x - e^y) = -e^y,$$

$$\frac{\partial}{\partial x} N = \frac{\partial}{\partial x}(-xe^y) \qquad = -e^{-y},$$

where the minus sign in the second line results from taking the dy-term to the left in order to have the standard form of the equation. You see that the equation is exact. Integrating M with respect to x gives $u = xe^x - xe^y + k(y)$ with arbitrary $k(y)$. Differentiating this with respect to y and equating the result to N gives $-xe^y + k'(y) = -xe^y$, hence $k'(y) = 0$ and $k = const$. This shows that a general solution is $u = xe^x - xe^y = c$. Because of the initial condition set $x = 1$ and $y = 0$, obtaining $u = e - 1$. This gives the answer $u = xe^x - xe^y = e - 1$.

23. Several integrating factors. From this problem you can learn that if an equation has an integrating factor, it has many such factors, giving essentially the same (implicit) general solution. Taking $F = y$, you obtain the equation $y^2\,dx + 2xy\,dy = 0$. To check exactness, calculate $\frac{\partial}{\partial y}(y^2) = 2y$ and $\frac{\partial}{\partial x}(2xy) = 2y$, which proves exactness. Integrating y^2 with respect to x gives $xy^2 + k(y)$. Differentiating this with respect to y and equating the result to $2xy$, you obtain for $k(y)$ the condition $2xy + k'(y) = 2xy$, $k'(y) = 0$, $k(y) = const$. The solution is $xy^2 = const$.

Choosing $F = xy^3$ as an integrating factor gives the exact equation $xy^4\,dx + 2x^2y^3\,dy = 0$. Proceeding as before, you obtain

$$u = (1/2)x^2y^4 + k(y), \qquad 2x^2y^3 + k'(y) = 2x^2y^3, \qquad u = (1/2)x^2y^4 = C,$$

which implies $xy^2 = c$, as before.

25. Integrating factor. $P\,dx + Q\,dy = 0$ in (12) is the nonexact equation. $FP\,dx + FQ\,dy = 0$ is the exact equation obtained by multiplying with an integrating factor F. Hence $FP = M$ and $FQ = N$ play the role of M and N in an exact equation. Accordingly, the exactness condition is $\partial(FP)/\partial y = \partial(FQ)/\partial x$. In the present problem,

$$\frac{\partial}{\partial y}(FP) = \frac{\partial}{\partial y}(e^x \sin y) = e^x \cos y,$$

$$\frac{\partial}{\partial x}(FQ) = \frac{\partial}{\partial x}(e^x \cos y) = e^x \cos y,$$

which shows exactness. Integrating FP with respect to x gives $u = e^x \sin y + k(y)$. To determine $k(y)$, differentiate u with respect to y and equate the result to FQ (which now plays the role of N). This gives

$$e^x \cos y + k'(y) = e^x \cos y, \qquad k'(y) = 0, \qquad k(y) = const.$$

Hence the answer is

$$u = e^x \sin y = c = const.$$

Note that in the present case you can solve this for y; this gives

$$y = \arcsin(ce^{-x}).$$

Sec. 1.6 Linear Differential Equations. Bernoulli Equation

Example 2. The integral can be solved by integration by parts or more simply by "undetermined coefficients", that is, by setting

$$\int e^{0.05t} \cos t\,dt = e^{0.05t}(A \cos t + B \sin t)$$

and differentiating on both sides. This gives

$$e^{0.05t} \cos t = e^{0.05t}[0.05\,(A \cos t + B \sin t) - A \sin t + B \cos t].$$

Now equate the coefficients of $\sin t$ and $\cos t$ on both sides. The sine terms give $0 = 0.05B - A$, hence $A = 0.05B$. The cosine terms give

$$1 = 0.05A + B = 0.05^2B + B,$$

hence $B = 1/1.0025 = 0.997506$ and $A = 0.05B = 0.049875$. Multiplying A and B by 50 (the factor that we did not carry along) gives a and b in Example 2. The integrals in Example 3 can be handled similarly.

Problem Set 1.6. Page 38.

7. General solution. Multiplying the given equation by e^{kx}, you obtain

$$(y' + ky)\,e^{kx} = (ye^{kx})' = e^{kx}e^{-kx} = 1$$

and by integration, $ye^{kx} = x + c$. Division by e^{kx} gives the solution $y = (x + c)e^{-kx}$. Note that in (4) you have the integral of $e^{kx}e^{-kx} = 1$, which has the value $x + c$, so that the use of (4) is very simple, too.

17. Initial value problem. In any case the first task is to write the equation in the form (1). In the present problem,

$$y' - 2\,y \tanh 2x = -2 \tanh 2x.$$

In (4) you thus have $p = -2 \tanh 2x = -(\ln \cosh 2x)'$. Hence the integral h of p is $h = -\ln(\cosh 2x)$. In (4) you need $e^{-h} = \cosh 2x$ and under the integral sign $e^h = 1/(\cosh 2x)$. Since $r = -2 \tanh 2x$, the integrand is

$$-2\tanh 2x/\cosh 2x = -2\sinh 2x/(\cosh 2x)^2 = (1/\cosh 2x)'.$$

Hence the integral equals $1/(\cosh 2x) + c$. Multiplying this by $e^{-h} = \cosh 2x$ gives the general solution $y = 1 + c\cosh 2x$. From this and the initial condition, $y(0) = 1 + c = 4$, $c = 3$. *Answer*: $y = 1 + 3\cosh 2x$.

33. Bernoulli equation. This is a Bernoulli equation with $a = 4$. Hence you have to set $u = 1/y^3$. By differentiation (chain rule!) $u' = -3y^{-4}y'$. This suggests multiplying the given equation by $-3y^{-4}$, obtaining

$$-3y^{-4}y' - y^{-3} = -1 + 2x.$$

The first term is u' and the second is $-u$; thus $u' - u = 2x - 1$. Formula (4) with u instead of y gives the general solution $u = ce^x - 2x - 1$. Hence the answer is

$$y = u^{-1/3} = (ce^x - 2x - 1)^{-1/3}.$$

Sec. 1.7 Modeling: Electric Circuits

Example 1 Step 5. For the idea of evaluating the integral by undetermined coefficients, see this Manual, Sec. 1.6.

Problem Set 1.7. Page 47

7. Choice of L. This is a problem on the exponential approach to the limit, as it also occurs in various other applications. For constant $E = E_0$ the model of the circuit is $I' + (R/L)I = E_0/L$. The initial condition is $I(0) = 0$ since the current is supposed to start from zero. The general solution and the particular solution are

$$I = ce^{-Rt/L} + \frac{E_0}{R}, \qquad I = \frac{E_0}{R}(1 - e^{-Rt/L}).$$

25% of the final value of I is reached if the exponential term has the value 0.75, that is, $\exp(-Rt/L) = 0.75$. With $R = 1000$, $t = 1/10000$ by taking logarithms you obtain $0.1/L = \ln(1/0.75) = 0.2877$, so that $L = 0.1/0.2877 = 0.3476$.

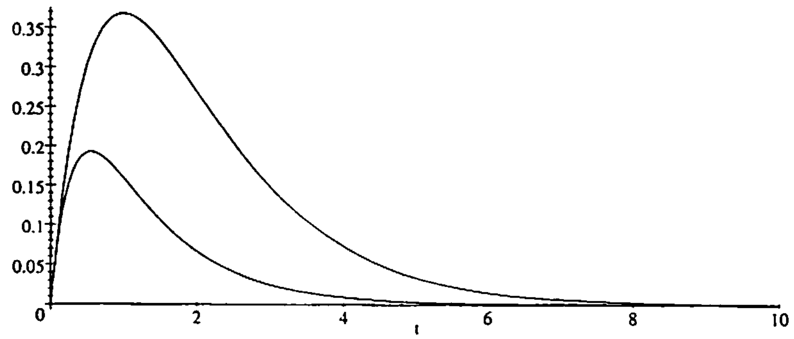
9. *RL*-circuit. The two cases can first be handled jointly; the difference will appear in evaluating the integral. The model is $I' + RI/L = e^{-t}/L$. You can solve it by (4) in Sec. 1.6. Since $p = R/L$, integration gives $h = Rt/L$. Hence $e^{-h} = e^{-Rt/L}$ and $e^h = e^{Rt/L}$. This yields the integrand $(1/L)\exp(Rt/L)\exp(-t) = (1/L)\exp[(R/L - 1)t]$. If $R/L - 1 = 0$, the integrand is $1/L$, and the integral is $t/L + c$. This is Case (b), the solution being

$$I = (t/L + c)e^{-t}.$$

If $R/L - 1$ is not zero, you have to integrate an exponential function, obtaining $\exp[(R/L - 1)t]/(R - L)$. This is Case (a), the solution being

$$I = \frac{e^{-t}}{R - L} + ce^{-Rt/L},$$

where the first term became simple because $\exp(-h)\exp h = 1$. The figure shows the two solutions for $I_0 = 0$, $L = 1$ and (a) $R = 3$, (b) $R = 1$. Find out which curve corresponds to (a) and which to (b). Sketch the solutions when $L = 1$, $R = 3$, and $I_0 = 1$, and compare.



Section 1.7. Problem 9. Solutions in both cases

19. Periodic electromotive forces are particularly important in practice. The simplest way of obtaining steady-state solutions is by substituting an expression of the form of the electromotive force with undetermined coefficients and determining the latter by equating corresponding coefficients on both sides of the equation. In the problem, the model equation, divided by a common factor 25, is

$$2Q' + Q = 4 \cos 2t + \sin 2t + 8 \cos 4t + \sin 4t.$$

The right side suggests setting

$$Q = a \cos 2t + b \sin 2t + c \cos 4t + k \sin 4t$$

By differentiation and multiplication by 2,

$$2Q' = -4a \sin 2t + 4b \cos 2t - 8c \sin 4t + 8k \cos 4t.$$

Hence you must have $a + 4b = 4$ (from $\cos 2t$), $-4a + b = 1$ (from $\sin 2t$). The solution is $a = 0$, $b = 1$. Similarly, $c + 8k = 8$ (from $\cos 4t$), $-8c + k = 1$ (from $\sin 4t$). The solution is $c = 0$, $k = 1$. Hence there are no cosine terms. The answer is $Q = \sin 2t + \sin 4t$. This "method of undetermined coefficients" will be very important in connection with vibrations in the next chapter.

Sec. 1.8 Orthogonal Trajectories of Curves. *Optional*

Problem Set 1.8. Page 51

3. Family of curves. $\cosh (x - c)$ is a translate of $\cosh x$ through the distance c to the right ($x - c = 0$ or $x = c$ corresponds to the lowest point of the curve, which is now at $x = c$, $y = 1$). Adding $-c$ moves the translated curve down. Thus, $y = \cosh (x - c) - c$. If $x = c$, then $y = -c + 1$; this is the lowest point of the corresponding curve. Make a sketch.

9. Differential equation of a family of curves. The differential equation to be derived must not contain c. This is quite essential. You accomplish this as follows. Solve the given equation algebraically for c^2,

$$c^2 (x^2 - 1) + y^2 = 0, \qquad -c^2 = y^2/(x^2 - 1).$$

Differentiation with respect to x gives (chain rule!)

$$0 = \frac{2yy'}{x^2 - 1} - \frac{y^2}{(x^2 - 1)^2} \cdot 2x.$$

Dividing by $2y$ and solving algebraically for y' yields the answer shown in Appendix 2 of the text.

21. Orthogonal trajectories derive their importance from applications in electrostatics, fluid flow, heat flow, and so on. The given curves $xy = c$ are the familiar hyperbolas with the coordinate axes as asymptotes (the solid curves in Fig. 30 of the text). Differentiation with respect to x gives their differential equation

$y + xy' = 0$ or $y' = -y/x$. Formula (2) in Sec. 1.8 gives the differential equation of the trajectories $y' = +x/y$ or $yy' = x$. By integration on both sides you obtain $y^2/2 = x^2/2 + C$ or $x^2 - y^2 = c^*$, the dashed hyperbolas in Fig. 30, whose asymptotes are $y = x$ and $y = -x$ (the latter in the quadrants not shown in the figure).

Sec. 1.9 Existence and Uniqueness of Solutions. Picard Iteration

Problem Set 1.9. Page 58

1. No solution. Obtain the general solution by separating variables.

3. Vertical strip. α is the smaller of the numbers a and b/K. Since K is constant and you can now choose b as large as you please (there is no restriction in the y-direction), the smaller number is a, as claimed.

7. Linear differential equation. $y' = f(x,y) = r - p(x)\,y$ shows that the continuity of r and p makes both f and $\partial f/\partial y = -p(x)$ continuous.

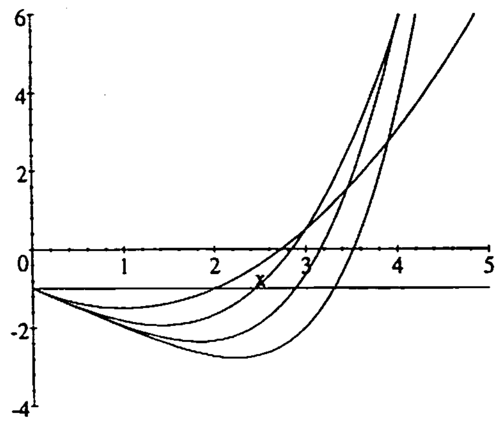
11. Picard iteration. Proof by induction. You have to show that $y_n = 1 + x + \dots + x^n/n!$. This is true for $n = 0$ because $y_0 = y(0) = 1$; see (6) in Sec. 1.9. Since $y' = f(x,y) = y$, the integrand in (6) is $y_{n-1}(t)$. Make the induction hypothesis that this equals $1 + t + \dots + t^{n-1}/(n-1)!$ According to (6) you have to integrate this expression from 0 to x, obtaining $x + x^2/2 + \dots + x^n/n!$ (because $(n-1)!n = n!$), and to add $y_0 = 1$. This gives y_n, the next partial sum of the Maclaurin series of e^x, and completes the proof.

13. Picard iteration. $y' = x + y,\ y_0 = -1.$

$$y_n = -1 + \int_0^x (t + y_{n-1}(t))\,dt = -1 + \int_0^x y_{n-1}(t)\,dt + \frac{x^2}{2},$$

thus

$$y_1 = -1 - x + \frac{x^2}{2},$$
$$y_2 = -1 - x - \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^2}{2} = -1 - x + \frac{x^3}{6}, \quad \text{etc.}$$



Section 1.9. Problem 13. Picard approximations of the solution $y = -1 - x$