CHAPTER 9. Vector Integral Calculus. Integral Theorems

Sec. 9.1 Line Integrals

Problem Set 9.1. Page 470

3. **Line integral in the plane.** This generalizes a definite integral of calculus. Instead of integrating along the x-axis you now integrate over a curve C, a quarter-circle, in the xy-plane.

Integrals of the form (3) have various applications, for instance, in connection with work done by a force in a displacement. The right side of (3) shows how such an integral is converted to a definite integral with t as the variable of integration. The conversion is done by using the representation of the path of integration C.

The problem parallels Example 1 in the text. The quarter-circle C has the radius 2 and can be represented by

$$\mathbf{r}(t) = [2\cos t, \quad 2\sin t], \qquad \text{in components,} \qquad x = 2\cos t, \qquad y = 2\sin t, \tag{I}$$

where t varies from $t = 0$ (the initial point of C on the x-axis) to $t = \pi/2$ (the terminal point of C on the y-axis). The given function is a vector function

$$\mathbf{F} = [xy, \quad x^2y^2]. \tag{II}$$

$\mathbf{F}$ defines a vector field in the xy-plane. At each point (x, y) it gives a certain vector, which you could draw as a little arrow. In particular, at each point of C the vector function $\mathbf{F}$ gives a vector. You can obtain these vectors simply by substituting x and y from (I) into (II). This gives

$$\mathbf{F}(\mathbf{r}(t)) = [4\cos t\sin t, \quad 16\cos^2 t\sin^2 t]. \tag{III}$$

This is now a vector function of t defined on the quarter-circle C.

Now comes an important point to observe. You do not integrate $\mathbf{F}$ itself, but you integrate the dot product of $\mathbf{F}$ in (III) and the tangent vector $\mathbf{r}'(t)$ of C. This dot product $\mathbf{F} \bullet \mathbf{r}'$ can be "visualized" because it is the component of $\mathbf{F}$ in the direction of the tangent of C (times the factor $|\mathbf{r}'(t)|$), as you can see from (11) in Sec. 8.2 with $\mathbf{F}$ playing the role of $\mathbf{a}$ and $\mathbf{r}'$ playing the role of $\mathbf{b}$. (Note that if t is the arc length s, then $\mathbf{r}'$ is a unit vector, so that that factor equals 1 and you get exactly that tangential projection.) Think this over before you go on calculating.

Differentiation with respect to t gives the tangent vector

$$\mathbf{r}'(t) = [-2\sin t, \quad 2\cos t].$$

Hence the dot product is

$$\begin{aligned} \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t) &= 4\cos t\sin t\,(-2\sin t) + 16\cos^2 t\sin^2 t\,(2\cos t) \\ &= -8\cos t\sin^2 t + 32\cos^3 t\sin^2 t \\ &= -8\cos t\sin^2 t + 32\cos t\,(1 - \sin^2 t)\sin^2 t. \end{aligned} \tag{IV}$$

Now by the chain rule,

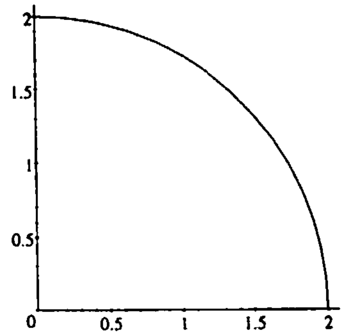
$$(\sin^3 t)' = 3\sin^2 t\cos t, \qquad (\sin^5 t)' = 5\sin^4 t\cos t.$$

This shows that the expression in the last line of (IV) can be readily integrated; you obtain

$$-\frac{8}{3}\sin^3 t + 32\left(\frac{1}{3}\sin^3 t - \frac{1}{5}\sin^5 t\right).$$

At 0 the sine is 0, and at the upper limit of integration $\pi/2$ it is 1. Hence the result is

$$-\frac{8}{3} + 32\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{8}{5}.$$



Section 9.1. Problem 3. Path of integration in the xy-plane

15. Line integral (7) in space. Line integrals in space are handled by the same method as line integrals in the plane. Quite generally, it is a great advantage of vector methods that methods in space and in the plane are very similar in most cases. Integrals (3), as just discussed, are suggested by work. Integrals (7) are conceptually a little simpler because the integrand is given directly (whereas in (3) it is obtained from a vector function by taking a dot product).

In the present problem, the path of integration C is a portion of a helix

$$C : \mathbf{r}(t) = [\cos t, \ \sin t, \ 2t]$$

with t varying from 0 to 4π. It lies on the circular cylinder of radius $a = 1$ with the z-axis as the axis, and has $c = 2$. Hence the arc length is (see Example 5 in Sec. 8.5)

$$s = t\sqrt{a^2 + c^2} = t\sqrt{5}.$$

It follows that with respect to s the limits of integration are 0 and $4\pi\sqrt{5}$. Furthermore, the integrand is

$$f = x^2 + y^2 + z^2 = \cos^2 t + \sin^2 t + 4t^2 = 1 + 4t^2 = 1 + \frac{4}{5}s^2.$$

You thus obtain the integral

$$\int_0^{4\pi\sqrt{5}} \left(1 + \frac{4s^2}{5}\right) ds = s + \frac{4s^3}{15}\bigg|_0^{4\pi\sqrt{5}} = \sqrt{5}\,(4\pi + \frac{4}{3}\cdot 64\pi^3),$$

as given on p. A23 in Appendix 2.

Sec. 9.2 Line Integrals Independent of Path

Problem Set 9.2. Page 477

3. Exactness and independence of path. In general, a line integral will depend on the path along which you integrate from a given point A to a given point B. The reason for the importance of independence of path is explained at the beginning of the section.

In the present problem the form under the integral sign is

$$3z^2\,dx + 6zx\,dz.$$

The limits of integration in the xz-plane are $A : (-1, 5)$ and $B : (4, 3)$. Theorem 1 relates path independence to the existence of a function f such that

$$[3z^2, \ 6\,zx] = \text{grad} f = [f_x, \ f_z],$$

thus

$$f_x = 3z^2, \quad \text{integrated with respect to } x: f = 3z^2x + g(z)$$
$$f_z = 6zx, \quad \text{integrated with respect to } z: f = 3z^2x + h(x).$$

Hence you can take $f(x, z) = 3z^2x$. Then at the lower limit of integration, $f(-1, 5) = 3 \cdot 25 \cdot (-1) = -75$ and at the upper, $f(4, 3) = 108$, so that (3) gives the answer $108 - (-75) = 183$.

More systematically, you can check for the existence of a function f by (6) in Theorem 3. Now, since y does not occur in the integrand or in the limits of integration, F_2 and the partial derivatives of **F** with respect to y are zero, (6′) reduces to its second relationship, that is,

$$(F_1)_z = (3z^2)_z = 6z = (F_3)_x.$$

This shows that the integral is independent of path in the xz-plane. Its value is now obtained as just explained.

11. Integral in space. The method is the same as for an integral in the plane. It involves somewhat more work. In (6′) you may have to check all three relationships; of course, you can stop if you arrive at a relationship that is not satisfied; then you know that you have path dependence and cannot use (3).

In the present problem the form under the integral sign is

$$2xy^2\,dx + 2x^2y\,dy + dz.$$

Hence

$$F_1 = 2xy^2, \quad F_2 = 2x^2y, \quad F_3 = 1.$$

In (6′) you thus obtain

$$(F_3)_y = 0 = (F_2)_z, \quad (F_1)_z = 0 = (F_3)_x, \quad (F_2)_x = 4xy = (F_1)_y.$$

Hence the differential form is exact in space, so that the integral is independent of path.

To evaluate the integral, you have to find f such that

$$\mathbf{F} = [F_1,\ F_2,\ F_3] = [2xy^2,\ 2x^2y,\ 1] = \text{grad} f = [f_x,\ f_y,\ f_z].$$

This gives the conditions

$$f_x = 2xy^2, \quad \text{integrated with respect to } x \quad f = x^2y^2 + g(y, z)$$
$$f_y = 2x^2y, \quad \text{integrated with respect to } y \quad f = x^2y^2 + h(x, z)$$
$$f_z = 1, \quad \text{integrated with respect to } z \quad f = z + k(x, y).$$

You see that the three functions on the right agree if you choose

$$g(y, z) = h(x, z) = z, \qquad k(x, y) = x^2y^2.$$

Hence your result is $f = x^2y^2 + z$. From (3) you now obtain the answer

$$f(a, b, c) - f(0, 0, 0) = a^2b^2 + c.$$

13. Path dependence. The integrand is $F_1\,dx + F_3\,dz$, where

$$F_1 = z \sinh xz, \qquad F_3 = -x \sinh xz.$$

Hence the second formula in (6′) gives on the left

$$(F_1)_z = \sinh xz + xz \cosh xz$$

but on the right

$$(F_3)_x = -\sinh xz - xz \cosh xz.$$

This shows path dependence. (The other two conditions in (6′) are satisfied because $F_2 = 0$, and F_1 and F_3 are independent of y.)

Sec. 9.3 From Calculus: Double Integrals. *Optional*

Problem Set 9.3. Page 484

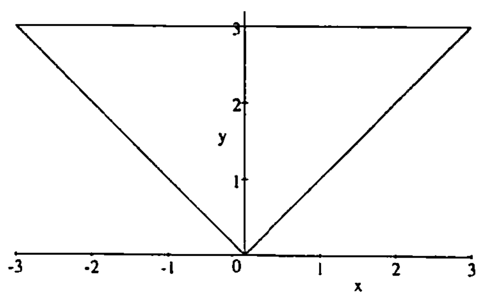
3. **Double integral.** Further work in this chapter will be concerned with basic integral theorems for transforming different kinds of integrals (line, double, surface, and triple integrals) into one another, and double integrals will occur in connection with line integrals (Sec. 9.3) and surface integrals (Secs. 9.6, 9.9).

The figure shows the triangular region of integration. Indeed, you first integrate with respect to x horizontally from $-y$ to y. The resulting integral is a function of y,

$$\int_{-y}^{y} (x^2 + y^2)\,dx = \frac{x^3}{3} + xy^2 \Big|_{-y}^{y} = \frac{y^3}{3} + y^3 - \left(\frac{(-y)^3}{3} + (-y)y^2\right) = \frac{8y^3}{3}.$$

You have to integrate this with respect to y from 0 to 3, obtaining

$$\int_0^3 \frac{8y^3}{3}\,dy = \frac{2y^4}{3}\Big|_0^3 = 54.$$



Section 9.3. Problem 3. Region of integration

15. **Center of gravity.** The circular disk of radius a has area πa^2. Hence $M = \pi a^2/4$ for the given region (a quarter of that disk). Since the mass density is 1, the formula in the text gives for the x-coordinate of the center of gravity

$$\bar{x} = \frac{1}{M} \iint x\,dx\,dy.$$

The form of the region of integration suggests the use of polar coordinates $x = r\cos\theta$, $y = r\sin\theta$. Then $dx\,dy = r\,dr\,d\theta$ and you have to integrate with respect to r from 0 to a and with respect to θ from 0 to $\pi/2$; this corresponds to the quarter of the circular disk. You obtain

$$\bar{x} = \frac{1}{M} \int_0^{\pi/2} \int_0^a r(\cos\theta)\,r\,dr\,d\theta.$$

The integral of r^2 is $r^3/3$, hence $a^3/3$ at the upper limit. The integral of $\cos\theta$ from 0 to $\pi/2$ gives 1. Hence, together, since $1/M = 4/(\pi a^2)$,

$$\bar{x} = (4/\pi a^2)(a^3/3) = 4a/(3\pi).$$

$\bar{y}$ has the same value, for reasons of symmetry.

Sec. 9.4 Green's Theorem in the Plane

Problem Set 9.4. Page 490

1. Transformation of a line integral into a double integral. Green's theorem in the plane transforms double integrals over a region R in the xy-plane into line integrals over the boundary curve C of R and conversely. These transformations are of practical and theoretical interest in both directions, depending on the purpose. In Probs. 1-10 the direct evaluation of the line integral would be much more involved than that of the double integral obtained by Green's theorem. Given

$$\mathbf{F} = [F_1,\ F_2] = [x^2 e^y,\ y^2 e^x].$$

In (1) you obtain on the left side

$$(F_2)_x - (F_1)_y = y^2 e^x - x^2 e^y. \tag{A}$$

Sketch the given rectangle (the region of integration R in the double integral). Then you see that you have to integrate over x from 0 to 2 and over y from 0 to 3. Integrating (A) over x gives

$$y^2 e^x - (x^3/3)e^y.$$

Substituting the upper limit $x = 2$ and then the lower limit $x = 0$ and taking the difference, you obtain

$$\left(y^2 e^2 - \frac{8}{3}e^y\right) - (y^2 \cdot 1 - 0) = y^2 e^2 - \frac{8}{3}e^y - y^2.$$

Integrating this over y gives

$$\frac{1}{3}y^3 e^2 - \frac{8}{3}e^y - \frac{1}{3}y^3.$$

Substituting the upper limit $y = 3$ and the lower limit $y = 0$, and taking the difference of the two expressions obtained you finally arrive at the answer

$$\left(\frac{27}{3}e^2 - \frac{8}{3}e^3 - \frac{27}{3}\right) - \left(0 - \frac{8}{3}\cdot 1 - 0\right) = 9e^2 - \frac{8}{3}e^3 - 9 + \frac{8}{3},$$

in agreement with the answer on p. A23 in Appendix 2 of the book.

13. Area. The given Pascal snail has the representation in polar coordinates

$$r = 1 + 2\cos\theta.$$

In the formula (5) for the area you need

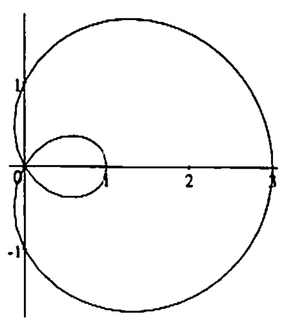
$$r^2 = (1 + 2\cos\theta)^2 = 1 + 4\cos\theta + 4\cos^2\theta.$$

Integration of the three terms on the right from 0 to $\pi/2$ gives

$$\pi/2 + 4 + 4(\pi/4) = 4 + 3\pi/2.$$

Now multiply by 1/2 as shown in (5) to obtain the answer given on p. A23 of the book.

The figure shows the entire curve which you get if you let θ range over a full interval of periodicity of $\cos\theta$, say, from 0 to 2π. For $\theta = 0$ you get $r = 3$ on the x-axis. For $\theta = \pi/2$ you have $r = 1$ because $\cos(\pi/2) = 0$; this is the point where the curve intersects the y-axis. Hence our answer gives the area under the curve in the first quadrant. For what values of θ does the curve pass through the origin? Through the point 1 on the x-axis?



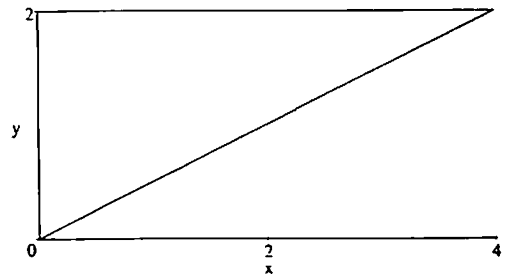
Section 9.4. Problem 13. Limaçon

15. **Consequences of Green's theorem in the plane** are given in formulas (9)-(12). Formula (9) shows that if you integrate the normal derivative of a function w over a closed curve and if w is harmonic, then you get zero.

For other functions you may use (9) to simplify the evaluation of integrals of the normal derivative. Such integrals occur, for instance, in connection with the flux of a fluid through a surface. For $w = \cosh x$ two partial differentiations with respect to x give $\nabla^2 w = w_{xx} = \cosh x$. The hypothenuse of the triangle has the representation

$$y = x/2. \qquad \text{Hence} \qquad x = 2y.$$

(Make a sketch.) The integral of $\cosh x$ is $\sinh x$, and you integrate from $x = 0$ horizontally to $x = 2y$. Since $\sinh 0 = 0$, this gives $\sinh 2y$. You now integrate over y, obtaining $(\cosh 2y)/2$. The limits are 0 and 2. This gives the answer $(\cosh 4 - 1)/2$.



Section 9.4. Problem 15. Region of integration in the double integral

Sec. 9.5 Surfaces for Surface Integrals

Problem Set 9.5. Page 495

3. **Parametric surface representations** have the advantage that the components x, y, z of the position vector $\mathbf{r}$ play the same role in the sense that none of them is an independent variable (as it is the case when we use $z = f(x, y)$), but all three are functions of two variables ("**parameters**") u and v (we need two of them because a surface is two-dimensional). Thus, in the present problem,

$$\mathbf{r}(u, v) = [x(u, v), \quad y(u, v), \quad z(u, v)] = [u \cos v, \quad u \sin v, \quad cu].$$

In components,

$$x = u \cos v, \quad y = u \sin v, \quad z = cu \quad (c \text{ constant}). \tag{A}$$

If cos and sin occur, you can often use $\cos^2 v + \sin^2 v = 1$. At present,

$$x^2 + y^2 = u^2(\cos^2 v + \sin^2 v) = u^2.$$

From this and $z = cu$ you see that

$$z = c\sqrt{x^2 + y^2}.$$

This is a representation of the cone of the form $z = f(x, y)$.

If you set $u = const$, you see that $z = const$, so these curves are the intersections of the cone with horizontal planes $u = const$. They are circles.

If you set $v = const$, then $y/x = \tan v = const$ (since u drops out in (A)). Hence $y = kx$, where $k = \tan v = const$. These are straight lines through the origin in the xy-plane, hence they are planes through the z-axis in space, which intersect the cone along straight lines.

To find a surface normal, you first have to calculate the partial derivatives of $\mathbf{r}$,

$$\mathbf{r}_u = [\cos v, \quad \sin v, \quad c],$$
$$\mathbf{r}_v = [-u \sin v, \quad u \cos v, \quad 0],$$

and then form their cross product **N** because this cross product is perpendicular to the two vectors, which span the tangent plane, so that **N** in fact is a normal vector. You obtain

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & c \\ -u \sin v & u \cos v & 0 \end{vmatrix}$$

$$= \mathbf{i} \begin{vmatrix} \sin v & c \\ u \cos v & 0 \end{vmatrix} - \mathbf{j} \begin{vmatrix} \cos v & c \\ -u \sin v & 0 \end{vmatrix} + \mathbf{k} \begin{vmatrix} \cos v & \sin v \\ -u \sin v & u \cos v \end{vmatrix}$$

$$= [-cu \cos v, \quad -cu \sin v, \quad u].$$

In this calculation, the third component resulted by simplification,

$$(\cos v)\, u \cos v - (\sin v)(-u \sin v) = u(\cos^2 v + \sin^2 v) = u.$$

23. Representation $z = f(x, y)$. The simplest way to convert this to a parametric representation $\mathbf{r}(u, v)$ is to set $x = u, y = v$. Then $z = f(u, v)$, and by substituting this into $\mathbf{r}(u, v)$ you obtain the first of the two formulas in (6),

$$\mathbf{r}(u, v) = [u, \quad v, \quad f(u, v)].$$

A normal vector **N** is now obtained by first calculating the partial derivatives

$$\mathbf{r}_u = [1, \quad 0, \quad f_u],$$
$$\mathbf{r}_v = [0, \quad 1, \quad f_v]$$

and then their cross product

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_u \\ 0 & 1 & f_v \end{vmatrix}$$

$$= \mathbf{i} \begin{vmatrix} 0 & f_u \\ 1 & f_v \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & f_u \\ 0 & f_v \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= [-f_u, \quad -f_v, \quad 1].$$

Sec. 9.6 Surface Integrals

Problem Set 9.6. Page 503

1. Surface integral over a plane in space. The surface S is given by

$$\mathbf{r}(u, v) = [u, \quad v, \quad 2u + 3v].$$

Hence $x = u, y = v, z = 2u + 3v = 2x + 3y$. This shows that this is a plane in space. The region of integration is a rectangle; u varies from 0 to 2 and v from -1 to 1. Since $x = u, y = v$, this is the same rectangle in the xy-plane.

In (3) on the right you need the normal vector $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$. Now

$$\mathbf{r}_u = [1, \quad 0, \quad 2],$$
$$\mathbf{r}_v = [0, \quad 1, \quad 3],$$

so that

$$\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{vmatrix} = -2\mathbf{i} - 3\mathbf{j} + \mathbf{k} = [-2, \quad -3, \quad 1].$$

Next calculate **F** on the surface by substituting the components of **r** into **F**. This gives

$$\mathbf{F} = [3x^2, \quad y^2, \quad 0] = [3u^2, \quad v^2, \quad 0].$$

Hence the dot product in (3) on the right is

$$\mathbf{F} \bullet \mathbf{N} = [3u^2, \quad v^2, \quad 0] \bullet [-2, \quad -3, \quad 1] = -6u^2 - 3v^2.$$

The following is quite interesting. Since **N** is a cross product, **F** • **N** is a scalar triple product (Sec. 8.3), given by the determinant

$$\begin{vmatrix} 3u^2 & v^2 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{vmatrix} = 3u^2(-2) - v^2 \cdot 3 = -6u^2 - 3v^2.$$

In this way you have done two steps in one.

Now integrate $-6u^2 - 3v^2$. Integration over u gives

$$-6u^3/3 - 3uv^2 = -2u^3 - 3uv^2.$$

At the upper limit of integration $u = 2$ this equals $-16 - 6v^2$, and at the lower limit 0 it is zero. Integration of this result $-16 - 6v^2$ over v gives $-16v - 2v^3$. At the upper limit $v = 1$ this equals $-16 - 2 = -18$. At the lower limit $v = -1$ you obtain the value $16 + 2 = 18$. The difference of these two values gives the answer $-18 - 18 = -36$.

17. **Surface integrals of the form (7).** Integrate $G = z$ over the upper half of the sphere of radius 3 and center at the origin

$$x^2 + y^2 + z^2 = 9.$$

It seems best to use a parametric representation, say, (3) in Sec. 9.5 with $a = 3$, that is,

$$\mathbf{r} = [3 \cos v \cos u, \quad 3 \cos v \sin u, \quad 3 \sin v].$$

Here, u varies from 0 to 2π and v from 0 to $\pi/2$. Also,

$$G = z = 3 \sin v.$$

You need $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$. Differentiation gives

$$\mathbf{r}_u = [-3 \cos v \sin u, \quad 3 \cos v \cos u, \quad 0],$$

$$\mathbf{r}_v = [-3 \sin v \cos u, \quad -3 \sin v \sin u, \quad 3 \cos v].$$

The cross product is

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = [9 \cos^2 v \cos u, \quad 9 \cos^2 v \sin u, \quad 9 \cos v \sin v],$$

where the last component has been obtained by simplification, namely,

$$\begin{aligned} &(-3 \cos v \sin u)(-3 \sin v \sin u) - (3 \cos v \cos u)(-3 \sin v \cos u) \\ &= 9(\cos v \sin v \sin^2 u + \cos v \sin v \cos^2 u) \\ &= 9 \cos v \sin v. \end{aligned}$$

N has the length

$$\begin{aligned} |\mathbf{N}| &= \sqrt{81 \cos^4 v \cos^2 u + 81 \cos^4 v \sin^2 u + 81 \cos^2 v \sin^2 v} \\ &= 9\sqrt{\cos^4 v + \cos^2 v \sin^2 v} \\ &= 9\sqrt{\cos^4 v + \cos^2 v (1 - \cos^2 v)} \\ &= 9 \cos v. \end{aligned}$$

Hence $G|\mathbf{N}| = 27 \cos v \sin v = 13.5 \sin 2v$. Integration of $\sin 2v$ gives $-(1/2)\cos 2v$. At the upper limit $v = \pi/2$ this is $-(1/2)(-1) = 1/2$. At the lower limit $v = 0$ it is $-1/2$. The difference of the two values is 1, so that your intermediate result is 13.5. Finally, integration over u from 0 to 2π produces a factor 2π, so that the answer is 27π.

21. **Moment of inertia.** The integrals for the moments of inertia in Prob. 19 involve the square of the distance of points (x, y, z) from the axis. In the present example, where the surface S is the cylinder $x^2 + y^2 = 1$ for z from 0 to h, that distance is simply $x^2 + y^2 = 1$. The density is $\sigma = 1$ by assumption. Integration over the angle from 0 to 2π gives 2π. Subsequent integration over z from 0 to h gives a factor h. You thus get the answer $2\pi\, h$.

Sec. 9.7 Triple Integrals. Divergence Theorem of Gauss

Problem Set 9.7. Page 509

5. **Triple integral.** The vertices of the region of integration are the unit points on the axes and the origin. The faces are triangles, three of them are portions of the coordinate planes and one of them is a portion of the plane $x + y + z = 1$, which intersects each of the coordinate axes at the point 1. Hence integration over z extends from 0 to $z = 1 - x - y$. The integrand is $12\,xy$ and is independent of z, so that the integration over z gives $12\,xyz$. At the upper limit of integration this equals

$$12xy(1 - x - y) = 12x(1 - x)y - 12xy^2. \tag{A}$$

At the lower limit $z = 0$ it is 0.

That plane intersects the xy-plane along the straight line $y = 1 - x$. (Make a sketch to be sure.) Hence the integration over y extends from 0 to $1 - x$. It gives $6x(1 - x)y^2 - 4xy^3$. At the upper limit $y = 1 - x$ this equals

$$6x(1 - x)^3 - 4x(1 - x)^3 = 2x(1 - x)^3. \tag{B}$$

At the lower limit $y = 0$ it is 0. The function in (B) must finally be integrated over x from 0 to 1. (Use $u = 1 - x$ as a new variable of integration.) After simplification this gives the answer 1/10.

19. **Divergence theorem.** In this problem you use the divergence theorem for evaluating the surface integral of the normal component $\mathbf{F} \bullet \mathbf{n}$ of

$$\mathbf{F} = [x^3, \quad y^3, \quad z^3]$$

over the sphere S of radius 3 and center at the origin. This integral is converted to a volume integral of the divergence

$$\operatorname{div}\mathbf{F} = 3x^3 + 3y^2 + 3z^2 = 3(x^2 + y^2 + z^2) = 3r^2.$$

Here $r^2 = x^2 + y^2 + z^2$ and you see that $\operatorname{div}\mathbf{F}$ is constant on any of these concentric spheres $r = const.$ Hence the integral of $\operatorname{div}\mathbf{F}$ over such a sphere is simply $3r^2$ times the area $4\pi r^2$ of the sphere. This gives $12\pi r^4$. All you still have to do is integrating over r from 0 to 3 (the radius of S). This gives

$$12\pi 3^5/5 = 2916\pi/5.$$

If you do not see this way, you can use a parametric representation, say, (3) in Sec. 9.5, that is,

$$\mathbf{r} = [r \cos v \cos u, \quad r \cos v \sin u, \quad r \sin v],$$

where you assume r to vary from 0 to 3. Then you have $\operatorname{div} \mathbf{F} = 3r^2$, as before. This must be multiplied by the volume element in spherical coordinates

$$dV = r^2 \cos v\, dr\, du\, dv, \qquad \text{thus} \qquad 3r^4 \cos v\, dr\, du\, dv,$$

and integrated. Integration over u from 0 to 2π gives a factor 2π. Integration of $\cos v$ over v from $-\pi/2$ to $\pi/2$ gives

$$\sin \pi/2 - (-\sin \pi/2) = 2.$$

Integration of $3r^4$ over r from 0 to 3 gives $3 \cdot 3^5/5$. Together, $12\pi 3^5/5$, as before.

Sec. 9.8 Divergence Theorem. Further Applications

Problem Set 9.8. Page 514

1. **Formula** (9) expresses a very remarkable property of harmonic functions stated in Theorem 1. Of course, the formula can also be used for other functions. The point of the problem is to gain confidence in the formula and to see how to organize more involved calculations so that errors are avoided as much as possible. The box has six faces $S_1, ..., S_6$. The normals to them (which you need in connection with normal derivatives) have the direction of the axes; this makes the calculation of the surface integrals simple, in addition to the fact that the normal derivatives will turn out to be constant on each face. Thus to the faces $x = 0$ and $x = 1$ there correspond the negative and positive x-directions, respectively, as outer normal direction, and so on. For the given function $f = 2z^2 - x^2 - y^2$ you thus obtain

$S_1 : x = 0$	$f_x = -2x = 0$	integrated	0
$S_2 : x = 1$	$f_x = -2$	"	$-2 \cdot 2 \cdot 4 = -16$
$S_3 : y = 0$	$f_y = -2y = 0$	"	0
$S_4 : y = 2$	$f_y = -4$	"	$-4 \cdot 1 \cdot 4 = -16$
$S_5 : z = 0$	$f_z = 4z = 0$	"	0
$S_6 : z = 4$	$f_z = 16$	"	$16 \cdot 1 \cdot 2 = 32.$

In each line the value of the normal derivative is multiplied by the area of the corresponding face of the box. For a more general f you would have to evaluate double integrals over these six faces. You see that the six integrals add up to zero, confirming formula (9) for our special case. Indeed, the Laplacian of f is $4 - 2 - 2 = 0$, as differentiation shows; hence f is harmonic.

5. **Divergence theorem.** This concerns the surface integral of the outer normal component of $\mathbf{F} = [9x, \quad y\cosh^2 x, \quad -z\sinh^2 x]$ over the ellipsoid

$$4x^2 + y^2 + 9z^2 = 36, \qquad \text{thus} \qquad x^2/9 + y^2/36 + z^2/4 = 1.$$

To evaluate the integral, use the divergence theorem for converting it to a volume integral of the divergence. The latter is found to be

$$\operatorname{div} \mathbf{F} = 9 + \cosh^2 x - \sinh^2 x.$$

The sum of the last two terms is 1. Hence the divergence is constant, $\operatorname{div} \mathbf{F} = 10$. It follows that the volume integral equals 10 times the volume of the region bounded by the ellipsoid, that is, $\frac{4}{3}\pi abc$, where $a = 3$, $b = 6$, and $c = 2$ are the lengths of the three semi-axes of the ellipsoid. This gives the answer $\frac{4}{3} \cdot 36 \cdot 10\pi = 480\pi$.

Sec. 9.9 Stokes's Theorem

Problem Set 9.9. Page 520

3. **Stokes's theorem** converts surface integrals into line integrals over the boundary of the (portion of the) surface and conversely. It will depend on the special problem which of the two integrals is simpler, the surface integral or the line integral. In the present problem, the application of Stokes's theorem proceeds as follows. Given are $\mathbf{F} = [e^z, \quad e^z \sin y, \quad e^z \cos y]$ and the surface $S : z = y^2$, with x varying from 0 to 4 and y from 0 to 2. (S is called a *cylinder.*) Using (1), calculate the curl of $\mathbf{F}$,

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^z & e^z \sin y & e^z \cos y \end{vmatrix}$$

$$= [-e^z \sin y + e^z \sin y, \quad e^z - 0, \quad 0 - 0]$$
$$= [0, \quad e^z, \quad 0]$$
$$= [0, \quad \exp y^2, \quad 0] \qquad \text{on } S$$

In Stokes's theorem you further need a normal vector of S. To get it, write S in the form

$$S : \mathbf{r} = [x, \quad y, \quad y^2].$$

You could also write $\mathbf{r} = [u, \quad v, \quad v^2]$, so that $x = u, y = v$; this would not make any difference in what follows. The partial derivatives are

$$\mathbf{r}_x = [1, \quad 0, \quad 0],$$
$$\mathbf{r}_y = [0, \quad 1, \quad 2y].$$

Their cross product is the normal vector

$$\mathbf{N} = \mathbf{r}_x \times \mathbf{r}_y = [0, \quad -2y, \quad 1].$$

From it you obtain

$$(\operatorname{curl}\mathbf{F}) \bullet \mathbf{n}\, dA = (\operatorname{curl}\mathbf{F}) \bullet \mathbf{N}\, dx\, dy = (0 - 2y \exp y^2 + 0)\, dx\, dy = -2y \exp y^2\, dx\, dy.$$

Integration over x from 0 to 4 gives a factor 4. Integration over y gives $-\exp y^2$. At the upper limit $y = 2$ this equals $-e^4$, and at the lower limit $y = 0$ it equals -1. Together with the factor 4 you thus obtain the answer

$$4(1 - e^4) \tag{A}$$

(or $-4\,(1 - e^4)$ if you reverse the direction of the normal vector).

Confirm this result by Stokes's theorem, as follows. Sketch the surface S, so that you see what is going on. The boundary curve of S has four portions. The first, C_1, is the segment of the x-axis from the origin to $x = 4$. On it, $y = 0$, $z = 0$, x varies from 0 to 4, $\mathbf{F} = [1, \quad 0, \quad 1]$, $\mathbf{r} = [x, \quad 0, \quad 0]$ (because $s = x$ on C_1), $\mathbf{r}' = [1, \quad 0, \quad 0]$, $\mathbf{F} \bullet \mathbf{r}' = 1$, integrated from 0 to 4 gives 4. The third, C_3, is the upper straight-line edge from $(0, 2, 4)$ to $(4, 2, 4)$. On it, $y = 2$, $z = 4$, x varies from 4 to 0,

$$\mathbf{F} = [e^4, \quad e^4 \sin 2, \quad e^4 \cos 2], \qquad \mathbf{r} = [x, \quad 2, \quad 4], \qquad \mathbf{r}' = [1, 0, 0].$$

$\mathbf{F} \bullet \mathbf{r}' = e^4$, integrated from 4 to 0 gives $-4e^4$, the minus sign appearing because you integrate in the negative x-direction. The sum of the two integrals equals $4 - 4e^4$; this is your result in (A). Now show that the sum of the other two integrals over the portions of parabolas is zero. The second portion, C_2, is the parabola $z = y^2$ in the plane $x = 4$, which you can represent by $\mathbf{r} = [4, \quad y, \quad y^2]$. The derivative with respect to y is $\mathbf{r}' = [0, \quad 1, \quad 2y]$. Furthermore, $\mathbf{F}$ on C_2 is

$$\mathbf{F} = [\exp y^2, \quad (\exp y^2) \sin y, \quad (\exp y^2) \cos y].$$

This gives the dot product

$$\mathbf{F} \bullet \mathbf{r}' = 0 + (\exp y^2) \sin y + 2y (\exp y^2) \cos y.$$

This must be integrated over y from 0 to 2. But for the fourth portion, C_4 , you obtain exactly the same expression because C_4 can be given by $\mathbf{r} = [0, \quad y, \quad y^2]$, so that $\mathbf{r}' = [0, \quad 1, \quad 2y]$ is exactly as for C_2 , and so is $\mathbf{F} \bullet \mathbf{r}'$ because $\mathbf{F}$ does not involve x. Now on C_4 you have to integrate over y in the opposite sense, from 2 to 0, so that the two integrals do indeed cancel each other and their sum is zero. This remains true if the arc length s is introduced as a variable of integration. This completes the integration over the boundary of S, the result being the same as in (A).

9. Evaluation of a line integral. If you want to do this by Stokes's theorem, you have to find a surface S whose boundary is the path of integration C of the line integral. In Prob. 9 you can use the plane of the ellipse, given by

$$\mathbf{r} = [u, \quad v, \quad v + 1].$$

For $\mathbf{F} = [4z, \ -2x, \ 2x]$ you get curl $\mathbf{F} = [0, \ -2+4, \ -2] = [0, \ 2, \ -2]$. A normal vector to the plane is obtained from

$$\mathbf{r}_u = [1, \quad 0, \quad 0],$$

$$\mathbf{r}_v = [0, \quad 1, \quad 1]$$

in the form

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = [0, \ -1, \ 1].$$

(You get this more quickly by writing the plane as $0 \cdot x - y + z = -1$ and remembering Example 6 in Sec. 8.2.) From this you see that

$$(\text{curl}\,\mathbf{F}) \bullet \mathbf{N} = [0, \ 2, \ -2] \bullet [0, \ -1, \ 1] = -4 = const.$$

Hence the answer is -4 times the area of the region of integration in the uv-plane, which is the interior of the circle $u^2 + v^2 = 1$ and has the area π. This gives the answer -4π.

Alternatively, instead of $\mathbf{N}\,du\,dv$ used you can use $\mathbf{n}\,dA$, where $\mathbf{n}$ is the unit normal vector

$$\mathbf{n} = [0, \ -1/\sqrt{2}, \ 1/\sqrt{2}].$$

Then (curl $\mathbf{F}$) $\bullet$ $\mathbf{n} = -4/\sqrt{2} = const.$ This must now be multiplied by the area of the ellipse in that plane, with semi-axes $\sqrt{2}$ and 1, which equals $\pi\sqrt{2}$, so that $\sqrt{2}$ cancels and the result is the same as before.