

Frustration solitaire

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Abstract

In this expository article, we discuss the rank-derangement problem, which asks for the number of permutations of a deck of cards such that each card is replaced by a card of a different rank. This combinatorial problem arises in computing the probability of winning the game of ‘frustration solitaire’, which was the subject of a recent column by Marilyn vos Savant. The solution by means of the method of inclusion and exclusion is a prime example of the use of this simple yet powerful method.

1 Frustration solitaire and rank-derangements

In the game of frustration solitaire, you shuffle a deck of cards thoroughly—at least 13 times—so that by the time you’re done all orderings of the deck are equally likely. Now you run through the deck, turning over the cards one at a time as you call out, ‘Ace, two, three, four, five, six, seven, eight, nine, ten, jack, queen, king, ace, two, three, . . .,’ and so on, so that you end up calling out the thirteen ranks four times each. If the card that comes up ever matches the rank you call out as you turn it over, then you lose.

The *rank-derangement problem* asks for the number of permutations of a deck of cards such that after the permutation, every card has been replaced

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by a card of a different rank. We denote this number by R_{13} , where the subscript 13 is there to remind us that there are 13 different ranks. More generally, the rank-derangement problem asks for the R_n , the number of rank-derangements of a $4n$ -card deck containing cards of n different ranks in each of four suits. Of course we could generalize the problem still further by varying the number of suits, but we won't get into that here.

To see the connection between the rank-derangement problem and frustration solitaire, imagine that the deck starts out in the order: ace through king of spades, ace through king of hearts, ace through king of diamonds, ace through king of clubs. Starting with this ordering, you win if the permutation you do when you shuffle is a rank-derangement, which happens with probability $R_{13}/52!$.

Now of course you probably didn't start with this particular ordering, unless you just won at one of the more conventional kinds of solitaire. But because we're assuming that after the shuffle all orderings are equally likely, your probability of winning doesn't depend on what order the deck started in, so your probability of winning is still going to be $R_{13}/52!$.

By the way, the special ordering we've just been discussing is not at all the ordering of a brand new deck of Bees or Bicycles, or practically any other brand of top-quality playing cards. New decks come in quite a different ordering, and the particular ordering that they come in is very important if you are going to be playing certain special games like *bore* or *new age solitaire*, and are only planning on shuffling the deck seven or eight times. We're not going to say anything more about that here. We just didn't want to give you the wrong impression about the order of the cards in a new deck.

2 Historical background

The roots of the rank-derangement go back to a gambling game closely resembling frustration solitaire that was studied long ago by the Chevalier de Montmort.

In 1708 Pierre de Montmort published the first edition of his book *Essay d'Analyse sur les Jeux de Hazard* [3] (Analytical Essay on Games of Chance). He was inspired by recent work of James Bernoulli in probability. Montmort hoped that, by applying techniques developed by Bernoulli to analyze common card and dice games, he could show people that certain of their methods of play based on superstitions should be replaced by more rational behavior.

One of the games that he analyzed was *Jeu de Treize* (Thirteen). This game was played as follows:

One person is chosen as dealer and the others are players. Each player puts up a stake. The dealer shuffles the cards and turns them up one at a time calling out, ‘Ace, two, three, . . . , king’, just as in frustration solitaire. If the dealer goes through the 13 cards without a match he (or she?) pays the players an amount equal to their stake, and the deal passes to someone else. If there is a match the dealer collects the players’ stakes; the players put up new stakes, and the dealer continues through the deck, calling out, ‘Ace, two, three, . . .’. If the dealer runs out of cards he reshuffles and continues the count where he left off. He continues until there is a run of 13 without a match and then a new dealer is chosen.

Montmort’s ‘Problème du Treize’ was to find the expected value of this game to the dealer. The answer, found by Peter Doyle in 1994,

26516072156010218582227607912734182784642120482136\
09144671537196208993152311343541724554334912870541\
44029923925160769411350008077591781851201382176876\
65356317385287455585936725463200947740372739557280\
74593843427478766496507606399053826118938814351354\
73663160170049455072017642788283066011710795363314\
27343824779227098352817532990359885814136883676558\
33113244761533107206274741697193018066491526987040\
84383914217907906954976036285282115901403162021206\
01549126920880824913325553882692055427830810368578\
18861208758248800680978640438118582834877542560955\
55066287892712304826997601700116233592793308297533\
64219350507454026892568319388782130144270519791882 /
33036929133582592220117220713156071114975101149831\
06336407213896987800799647204708825303387525892236\
58132301562800562114342729062565897443397165719454\
12290800708628984130608756130281899116735786362375\
60671849864913535355362219744889022326710115880101\
62859313519792943872232770333969677979706993347580\
24236769498736616051840314775615603933802570709707\
11959696412682424550133198797470546935178093837505\
93488858698672364846950539888686285826099055862710\
01318150621134407056983214740221851567706672080945\

86589378459432799868706334161812988630496327287254\
81845887935302449800322425586446741048147720934108\
06135061350385697304897121306393704051559533731591

which as you can see is something like .8. The solution, which we will not discuss here, involves having the computer solve something like 4 million variations on the frustration solitaire problem.

Montmort first considered the problem of finding the probability that there would be a match before getting through the first 13 cards. He started by assuming that the deck had only 13 cards all of the same suit and showed that the probability of getting no match is very close to $1/e = .3678\dots$, so the probability of getting a match is very close to .6321... He used the method of recursion to derive this result, thus giving the first solution to the problem we now call the ‘derangement problem’, or ‘hat-check problem’, which we will discuss below.

In later correspondence with Nicholas Bernoulli, Montmort found that, for a normal 52 card deck, the probability of getting through 13 cards without a match is .356... , so the dealer wins the first round of Treize with probability .643... (Note that the dealer does a little better on the first round with a 52 card deck than with a 13 card deck.) Thus Montmort showed that the dealer has a significant advantage even without considering the additional winnings from further rounds before giving up the deal.

The game of Treize is still mentioned in some books on card games, but in forms not so advantageous to the dealer. In one version, the dealer deals until there is a match or until 13 cards have come up without a match. If there is a match on card n , he or she wins $\$n$ from each player.

The current interest in the problem came from a column of Marilyn vos Savant [9]. Charles Price wrote to ask about his experience playing frustration solitaire. He found that he rarely won and wondered how often he should win. Marilyn answered by remarking that the expected number of matches is 4 so it should be difficult to get no matches.

Finding the chance of winning is a harder problem than Montmort solved because, when you go through the entire 52 cards, there are different patterns for the matches that might occur. For example matches may occur for two cards the same, say two aces, or for two different cards, say a two and a three.

We learned about the more recent history of the problem from Steven Landfelder. Steven was introduced to this form of solitaire by his gypsy

grandmother Ernestine Landfelder who had named it ‘frustration solitaire’. He was 15 in 1956 when he learned of the game and tried to find the chance of winning, but he found it too hard for him.

Steven became obsessed with finding this probability. As he grew older he became better able to read math books but this was certainly not his specialty. He found references that solved the problem, but the authors left out too many steps for him to follow their solutions. He persevered and, with hints from his reading, was finally able to carry out the computations to his satisfaction ending a lifelong obsession with this problem.

The most complete discussion that Steven found was in Riordan [6], who found the solution using the method of rook polynomials. Riordan also showed that R_n approaches $1/e^4$ as n tends to infinity.

3 The derangement problem

As noted above, the rank-derangement problem is a variation on the well-known *derangement problem*, which asks for the number D_n of permutations of an n -element set such that no object is left in its original position. The derangement problem is also known as the *hat-check problem*, for reasons that will suggest themselves.

In the current context, the derangement problem arises as follows: Again you shuffle the deck, and turn the cards over one at a time, only now you call out, ‘Ace of spades, two of spades, three of spades, . . . , king of spades, ace of hearts, two of hearts, three of hearts, . . . ,’ and so through the diamonds and clubs. If you ever name the card exactly, you lose. The problem of winning this game is $D_{52}/52!$.

4 The principle of inclusion and exclusion

The derangement problem can be solved using a standard method called the *principle of inclusion and exclusion*. The method is nothing more than a systematic application of the notion that if you want to know how many students belong to neither the French Club nor the German Club, you take the total number of students, subtract the number in the French Club, subtract the number in the German Club, and add back in the number who belong to both clubs.

The principle of inclusion and exclusion, together with its application to the derangement problem, is beautifully discussed in Ryser's Carus Monograph [8]. (This series also includes the critically acclaimed monograph of Doyle and Snell [5] on the fascinating connection between random walks and electric networks.)

5 Solution of the derangement problem

Applied to the derangement problem, the principle of inclusion and exclusion yields the following for the number of derangements:

$$\begin{aligned}
 D(n) &= \text{total number of permutations of } \{1, 2, \dots, n\} \\
 &\quad - \sum_{\{i\}} \text{number of permutations fixing } i \\
 &\quad + \sum_{\{i, j\}} \text{number of permutations fixing } i \text{ and } j \\
 &\quad - \dots \\
 &= n! - n(n-1)! + \binom{n}{2}(n-2)! - \dots \\
 &= n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \dots + \frac{(-1)^n}{n!} \right).
 \end{aligned}$$

We write the formula in this way to emphasize that the ratio $D(n)/n!$, which represents the probability that a randomly selected permutation of $\{1, 2, \dots, n\}$ turns out to have no fixed points, is approaching

$$1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots = \frac{1}{e}.$$

6 Solution of the rank-derangement problem

The derangement problem is tailor-made for applying the method of inclusion and exclusion. In the case of rank-derangements, a little care is needed in applying the principle, but we maintain that with sufficient experience you can pretty much just write down the answer to problems of this sort.

By analogy with the solution of the derangement problem, where we used inclusion and exclusion on the set of fixed points, here we will use inclusion

and exclusion on the set of rank-fixed points, that is, cards that get replaced by cards of the same rank. However in the present case, instead of classifying a set of cards only according to its size, we must keep track of how the set intersects the 13 different ranks.

Specifically, we associate to a set S of cards the parameters m_0, m_1, m_2, m_3, m_4 , where m_0 tells how many ranks are not represented at all in the set S , m_1 tells how many ranks are represented in the set S by a single card, and so on. For example, the set $S = \{AKQJ\spadesuit, AKQ\heartsuit, A\diamondsuit, A\clubsuit\}$ has parameters $m_0 = 9, m_1 = 1, m_2 = 2, m_3 = 0, m_4 = 1$. Evidently $m_0 + m_1 + m_2 + m_3 + m_4 = 13$, and $|S| = m_1 + 2m_2 + 3m_3 + 4m_4$.

For any set S of cards, denote by $n(S)$ the number of permutations having S as its set of rank-fixed points, and denote by $N(S) = \sum_{T \supset S} n(T)$ the number of permutations whose set of rank-fixed points includes S . We are trying to determine $n(\emptyset)$, the number of permutations whose rank-fixed set is empty. According to the principle of inclusion and exclusion,

$$n(\emptyset) = \sum_S (-1)^{|S|} N(S).$$

The parameters m_0, m_1, m_2, m_3, m_4 that we've chosen have two key properties. The first is that we can easily determine the number $s(m_0, m_1, m_2, m_3, m_4)$ of sets S having specified values of the parameters:

$$s(m_0, m_1, m_2, m_3, m_4) = \binom{13}{m_0, m_1, m_2, m_3, m_4} 1^{m_0} 4^{m_1} 6^{m_2} 4^{m_3} 1^{m_4}.$$

(For each of the 13 ranks, decide whether there will be 0, 1, 2, 3, or 4 rank-matches, and then decide which specific cards will be rank-matched.) The crucial second property of these parameters is that they are enough to determine $N(S)$:

$$N(S) = 1^{m_0} 4^{m_1} (4 \cdot 3)^{m_2} (4 \cdot 3 \cdot 2)^{m_3} (4 \cdot 3 \cdot 2 \cdot 1)^{m_4} (52 - |S|)!.$$

(Choose how the $|S|$ rank-matches come about, and then distribute the remaining $52 - |S|$ cards arbitrarily.)

Plugging into the inclusion-exclusion formula yields

$$\begin{aligned} R_{13} &= \sum_{m_0+m_1+m_2+m_3+m_4=13} (-1)^{|S|} s(m_0, m_1, m_2, m_3, m_4) N(S) \\ &= \sum_{m_0+m_1+m_2+m_3+m_4=13} (-1)^{|S|} \binom{13}{m_0, m_1, m_2, m_3, m_4} \\ &\quad \cdot 16^{m_1} 72^{m_2} 96^{m_3} 24^{m_4} (52 - |S|)!, \end{aligned}$$

where $|S| = m_1 + 2m_2 + 3m_3 + 4m_4$. Substituting in for $|S|$ gives

$$R_{13} = \sum_{m_0+m_1+m_2+m_3+m_4=13} (-1)^{m_1+m_3} \binom{13}{m_0, m_1, m_2, m_3, m_4} \cdot 16^{m_1} 72^{m_2} 96^{m_3} 24^{m_4} (52 - (m_1 + 2m_2 + 3m_3 + 4m_4))!.$$

In the more general case of n ranks we have

$$R_n = \sum_{m_0+m_1+m_2+m_3+m_4=n} (-1)^{m_1+m_3} \binom{n}{m_0, m_1, m_2, m_3, m_4} \cdot 16^{m_1} 72^{m_2} 96^{m_3} 24^{m_4} (4n - (m_1 + 2m_2 + 3m_3 + 4m_4))!.$$

7 Exact values and asymptotics

Evaluating the expression we have obtained for R_{13} gives

$$R_{13} = 1309302175551177162931045000259922525308763433362019257020678406144$$

and

$$\begin{aligned} R_{13}/52! &= \frac{4610507544750288132457667562311567997623087869}{284025438982318025793544200005777916187500000000} \\ &= 0.01623272746719463674 \dots \end{aligned}$$

Looking at what happens when n gets large, we find by a straight-forward analysis that

$$\begin{aligned} \lim_{n \rightarrow \infty} R_n/(4n)! &= e^{-4} \\ &= 0.01831563888873418029 \dots \end{aligned}$$

This makes good sense, since the expected number of rank-fixed cards is 4, and we would expect that when n is large the number of rank-matches would be roughly Poisson-distributed. Probably the current situation is covered by the theory of asymptotically independent events developed by Aspvall and Liang [1] in their analysis of the dinner table problem, but we haven't checked into this yet.

Evidently $p_n = R_n/(4n)!$ is still pretty far from its asymptotic value when $n = 13$. Checking larger values of n , we find that

$$\begin{aligned} p_{20} &= 0.01695430844136377527 \dots \\ p_{50} &= 0.01776805714328362582 \dots \end{aligned}$$

8 Other problems of the same ilk

The rank-derangement problem is a prime example of the use of the principle of inclusion and exclusion. We referred earlier to Ryser's book [8] as a good place to read about inclusion-exclusion. Further examples can be found in the beautifully-written and thought-provoking article 'Non-sexist solution of the ménage problem', by Bogart and Doyle [2], and the references cited there.

A key feature of the application of inclusion-exclusion to the rank-derangement problem is that the quantity $N(S)$ does not depend solely on $|S|$, and a little care is needed to identify the parameters to sum over. Other problems of this kind are the *dinner-table problem* (see Aspvall and Liang [1] and Robbins [7]), and the problem of enumerating Latin rectangles (see Doyle [4]).

References

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