The shape of distributions (2 fragments)

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Version 0.0, 5 October 1998

FRAGMENT 1. The distributions in question are the distributions of the number of heads obtained in flipping n coins that come up heads with probability $p_1, p_2, \ldots, p_n$. If $p_1 = p_2 = \ldots = p_n$ this is just a Bernoulli distribution, otherwise it is called a 'Poisson-binomial distribution.' I became interested in these distributions because of their connection with the van der Waerden permanent conjecture (cf. Andrew M. Gleason, 'Remarks on the van der Waerden permanent conjecture', J.C.T. 8 (1970), pp. 54-64). ... Here is a special case of one of my results:

Suppose that $p_1 + \ldots + p_n$, the expected number of heads flipped, is an integer:

$$p_i + \ldots + p_n = k.$$

Then it is known that k is the most probable number of heads, that is, k is not only the mean but also the mode of the distribution. It is also known that the probability P_k of flipping exactly k heads is smallest when $p_1 = \ldots = p_n = k/n$. So we might expect that when we 'average' two of the p_i's, that is, when we replace p_i and p_j by $p_i' = (1-t)p_i + tp_j$ and $p_j' = (1-t)p_j + tp_i$, $0 \leq t \leq 1$, that P_k should diminish. Gleser has given an example showing that this need not be the case (cf. L. J. Gleser, 'On the distribution of the number of successes in independent trials', Ann. Probab. 3, pp. 182–). However, I was able to show that when we 'head straight for the middle', that is, when we replace each p_i by $p_i' = (1-t)p_i + tk/n$, $0 \leq t \leq 1$, then P_k does in fact decrease. This is closely related to a generalization

*This Space Intentionally Left Blank. Contributors include: Peter Doyle. Last revised in the early 1980's.

of the van der Waerden conjecture having to do with monotonicity of the permanent (cf. S. Friedland and H. Minc, 'Monotonicity of permanents of doubly stochastic matrices', Linear and Multilinear Algebra, 6 (1978) pp. 227–).

PROOF? As I recall this had something to do with Hoeffding's work on the shape of Poisson-binomial distributions. Presumably this should now have a nice simple proof, maybe using Alexandroff-Fenchel.

FRAGMENT 2. Let n be a fixed positive integer, and let $p = (p_1, \ldots, p_n)$ describe a sequence of Poisson trials. The problem is to find

$$\min_{p} \max_{0 \leq j \leq n} P(\text{exactly } j \text{ successes} | p),$$

i.e. to find the Poisson-binomial distribution whose mode occurs least often.

Snell and I have shown that the minimum is attained when $p_1 = \ldots = p_n$. The common value of the p_i's is $1/2$ when n is odd and $1/2(1 \pm 1/(n+1))$ when n is even.

PROOF? 'Our method, while reminiscent of Hoeffding's Tchebychev method, is substantially different.' Is there some simple proof?