

# Ville's Martingales (2 fragments)

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## Martingales as betting systems

Ville uses the term 'martingale' in two senses, first as a system for betting on Bernoulli trials, and second in the usual sense as a kind of random process. Since the random process describing the fortune of a gambler using a betting-system-martingale is a random-process-martingale, it appears that the first sense is a kind of special case of the second sense.

To complete the correspondence between these two senses, we imagine a game more general than Bernoulli trials, where on any turn the player is allowed to place any fair bet. We can arrange such a game as follows: On each turn a number  $X_n$  will be chosen uniformly at random between 0 and 1. (In fancier apparel, a point will be chosen from a standard Borel space according to a non-atomic measure of total mass 1.) Before each turn, the player will have been allowed to select any integrable function  $f_{X_1, \dots, X_{n-1}}$  on  $[0, 1]$  satisfying  $-1 \leq f_{X_1, \dots, X_{n-1}}$  a.e. and  $\int_0^1 f = 0$ . The payoff will be  $f_{X_1, \dots, X_{n-1}}(X_n)$  times his current fortune. Given a betting system, the random process describing the vicissitudes of his fortune is a martingale, and any martingale arises in this way.

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## What is a fair game?

It is usually asserted that a fair game is a martingale. Probably this is supposed to mean that no matter what system of play one adopts in a fair game, the random process describing one's fortune will be a martingale. So what *is* a fair game?

A *game* is a tree such that the children of every even-generation node form a probability space. (The even-generation nodes represent strategies, the odd-generation nodes represent outcomes. The root, being an even-generation node, represents a strategy, namely 'start'.) A *game for money* is gotten by defining a non-negative real-valued function of the odd-generation nodes that is a random variable when restricted to any of these probability spaces. It is a *fair game* if the expected value of each of these random variables equals the value at the grandparent node of the nodes in the domain. (Except, of course, for the random variable whose domain comprises the children of the root, since they have no grandparent.) (Pretty fuzzy, eh?)