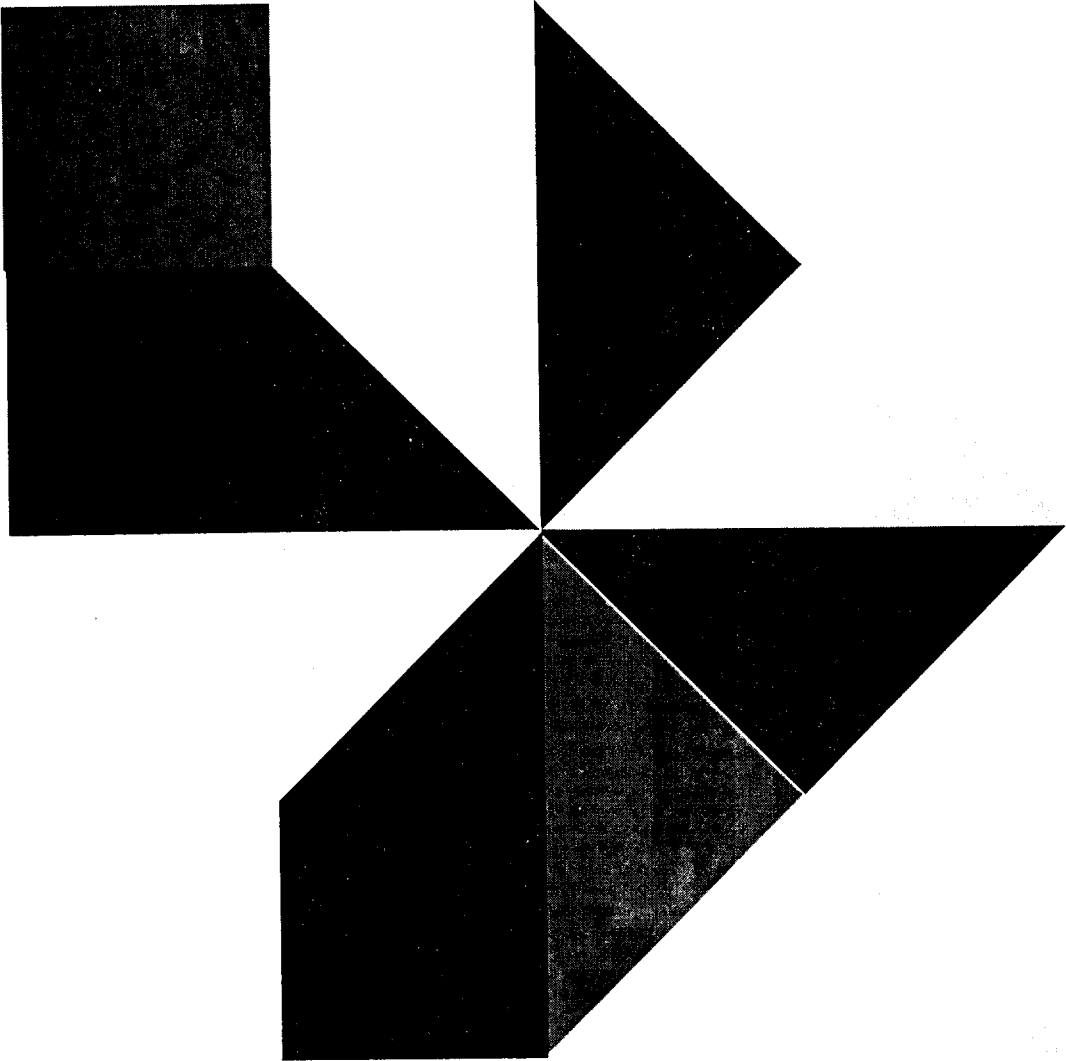


**Applications to the  
Behavioral and  
Managerial Sciences**



## 1 COMMUNICATION AND SOCIOMETRIC MATRICES

Matrices having only the entries 0 and 1 are useful in the analysis of graphs and networks. We shall not attempt to give a complete treatment of the subject here, but merely illustrate some of its more interesting applications.

A *communication network* consists of a set of people,  $A_1, A_2, \dots, A_n$ , such that between some pairs of persons there is a communication link. Such a link may be either one-way or two-way. A two-way communication link might be made by telephone or radio, and a one-way link by sending a messenger, lighting a signal light, setting off an explosion, etc. We shall use the symbol  $\gg$  to indicate the latter sort of connection; thus  $A_i \gg A_j$  shall mean that individual  $A_i$  can communicate with individual  $A_j$  (in that direction). The only requirement that we put on the symbol is:

- (i) It is false that  $A_i \gg A_i$  for any  $i$ ; that is, an individual cannot (or need not) communicate with himself.

It is convenient to use *directed graphs* to represent communication networks. Two such graphs are drawn in Figure 1. Individuals are represented on the graph as (lettered) points and a communication relation between two individuals as a directed line segment (line segment with an arrow) connecting the two individuals.

We can also represent communication networks by means of square matrices  $C$  having only 0 and 1 entries, which we call *communication matrices*. The entry in the  $i$ th row and  $j$ th column of  $C$  is equal to 1 if  $A_i$  can communicate with  $A_j$  (in that direction) and otherwise equal to 0. Thus

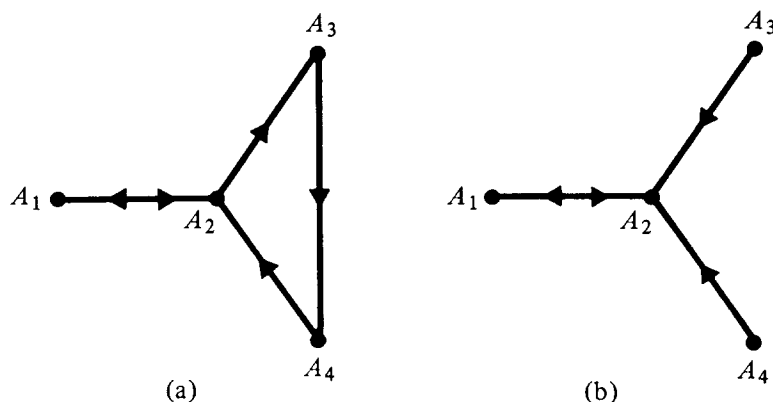


Figure 1

the communication matrices corresponding to the communication networks of Figure 1 are shown in Figure 2.

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

(a) (b)

Figure 2

Notice that the diagonal entries of the matrices in Figure 2 are all equal to 0. This is true in general for a communication matrix, since the matrix restatement of condition (i) is:

- (i) For all  $i$ ,  $c_{ii} = 0$ .

It is not hard to see that any matrix having only 0 and 1 entries, and with all zeros down the main diagonal, is the communication matrix of some network.

By a *dominance relation*\* we shall mean a special kind of communication relation in which, besides (i), the following condition holds:

- (ii) For each pair  $i, j$ , with  $i \neq j$ , either  $A_i \gg A_j$  or  $A_j \gg A_i$ , but not both; that is, in every pair of individuals, there is exactly one who is dominant.

It has been observed that in the pecking order of chickens a dominance relation holds. Also, in the play of one round of a round-robin contest among athletic teams, if ties are not allowed (as in baseball), then a dominance relation holds.

The reader may have been surprised that we did not assume that if  $A_i \gg A_j$  and  $A_j \gg A_k$  then  $A_i \gg A_k$ . This is the so-called transitive law for relations.

\*Recently dominance relations have been called *tournaments*. See the paper by H. A. David in the Suggested Readings at the end of the chapter.



Every dominance relation is also a communication relation; hence we shall concentrate on the latter, and what we say about them will also be true for the former.

Since a communication matrix  $C$  is square, we can compute its powers,  $C^2$ ,  $C^3$ , etc. Let  $E = C^2$ , and consider the entry in the  $i$ th row and  $j$ th column of  $E$ . It is

$$e_{ij} = c_{i1}c_{1j} + c_{i2}c_{2j} + \dots + c_{in}c_{nj}.$$

Now a term of the form  $c_{ik}c_{kj}$  can be nonzero only if both factors are nonzero, that is, only if both factors are equal to 1. But if  $c_{ik} = 1$ , then individual  $A_i$  communicates with  $A_k$ ; and if  $c_{kj} = 1$ , then individual  $A_k$  communicates with  $A_j$ . In other words,  $A_i \gg A_k \gg A_j$ . We shall call a communication of this kind a *two-stage communication*. (To keep ideas straight, let us call  $A_i \gg A_j$  a *one-stage communication*.) We can now see that the entry  $e_{ij}$  gives the total number of two-stage communication paths there are between  $A_i$  and  $A_j$  (in that direction). For example, let  $C$  be the matrix

$$C = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then  $C^2$  is the matrix

$$C^2 = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus we see that in this example  $A_1$  has one two-stage communication path with  $A_3$  and two two-stage communications with  $A_4$ ; similarly,  $A_2$  has one two-stage communication with  $A_4$ . These can be written down explicitly as

$$\begin{aligned} A_1 &\gg A_2 \gg A_3, \\ A_1 &\gg A_2 \gg A_4, \\ A_1 &\gg A_3 \gg A_4, \\ A_2 &\gg A_3 \gg A_4. \end{aligned}$$

The directed graph for this (dominance) situation is given in Figure 5. The reader should trace out on the graph of Figure 5 the two-stage communication paths given above.

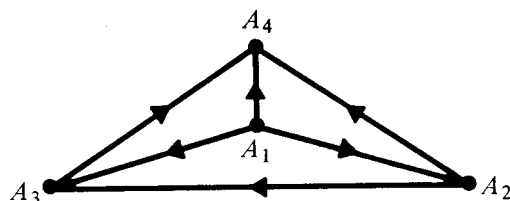


Figure 5

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**Theorem** Let a communication network of  $n$  individuals be such that, for every pair of individuals, at least one can communicate in one stage with the other. Then there is at least one person who can communicate with every other person in either one or two stages. Similarly, there is at least one person who can be communicated with in one or two stages by every other person.

Stated in matrix language, the above theorem is: Let  $C$  be the communication matrix for the network described above; then there is at least one row of  $S = C + C^2$  which has all its elements nonzero, except possibly the entry on the main diagonal. Similarly, there is at least one column having this property.

Notice that every dominance relation satisfies the hypotheses of the theorem, but there are communication networks, not dominance relations, that also satisfy these hypotheses.

---

*Proof* We shall prove only the first statement, since the proof of the second is analogous.

First we shall prove the following statement: If  $A_1$  cannot communicate in either one or two stages with  $A_i$ , where  $i \neq 1$ , then  $A_i$  can communicate in one stage with at least one more person than can  $A_1$ . We prove this in two steps. First by the hypothesis of the theorem, we see that:

- (a) If it is false that  $A_1 \gg A_i$ , then  $A_i \gg A_1$ .

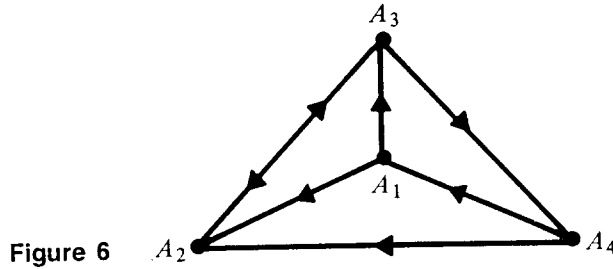
Second we can prove that:

- (b) Suppose that for all  $k$  it is false that  $A_1 \gg A_k \gg A_i$ ; it follows that, if  $A_1 \gg A_k$ , then also  $A_i \gg A_k$ .  
 For if  $A_1 \gg A_k$ , it is false that  $A_k \gg A_i$ , hence, by the hypothesis of the theorem, it is true that  $A_i \gg A_k$ .

Now (b) says that every one-stage communication possible for  $A_1$  is also possible for  $A_i$ . From this and (a), it then follows that  $A_i$  can make at least one more (one-stage) communication than can  $A_1$ .

We now return to the proof of the theorem. Let  $r_1, r_2, \dots, r_n$  be the row sums of the matrix  $C$ . By renaming the individuals, if necessary, we can assume that the largest row sum is  $r_1$ , that is,  $r_1 \geq r_k$  for  $k = 1, 2, \dots, n$ . We shall show that  $A_1$  can communicate with everyone else in one or two stages. (The proof is based on the indirect method.) Suppose, on the contrary, that there is an individual  $A_i$ , where  $i > 1$ , with whom  $A_1$  cannot so communicate. By the statement proved above,  $A_i$  can communicate in one stage with at least one more person than  $A_1$  can. But this implies that  $r_i > r_1$ , which contradicts the fact that we have named the individuals so that  $r_1 \geq r_i$ . This contradiction establishes the theorem.

An additional conclusion which can be made from the proof of the theorem is that the individual or individuals having the *largest* row sum in the matrix  $C$  can communicate with everyone else in one or two stages. Similarly, the individuals having the *largest* column sum can be communicated with by everyone in one or two stages.



The network shown in Figure 6 satisfies the hypothesis of the theorem, hence its conclusion. The communication matrix for this network is

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

Here the maximum row sum of 2 occurs in the first, third, and fourth rows, so that  $A_1$ ,  $A_3$ , and  $A_4$  can communicate with everyone else in one or two stages. (Find the necessary communication paths in Figure 6.) However, it requires three stages for  $A_2$  to communicate with  $A_1$ . The maximum sum of 3 occurs in the second column, so that  $A_2$  can be communicated with by everyone else in one or two stages (actually one stage is enough). It happens also that  $A_3$  and  $A_4$  can also be communicated with in one or two stages; however, as observed above,  $A_1$  cannot be.

Neither of the networks in Figure 1 satisfies the hypothesis of the theorem. It happens that the network in Figure 1(a) does satisfy the conclusion of the theorem, while the network in Figure 1(b) does not. (See Exercise 7.)

As a final application of dominance matrices, we shall define the power of an individual. By the *power* of an individual in a dominance situation we mean the total number of one-stage and two-stage dominances which he can exert. Since the total number of one-stage dominances exerted by  $A_i$  is the sum of the entries in the  $i$ th row of the matrix  $D$ , and the total number of two-stage dominances exerted by  $A_i$  is the sum of the entries in the  $i$ th row of the matrix  $D^2$ , we see that the power of  $A_i$  can be expressed as follows:

The power of  $A_i$  is the sum of the entries in the  $i$ th row of the matrix  $S = D + D^2$ .

In the example of Figure 7 it is easy to check that the powers of the various individuals are the following:

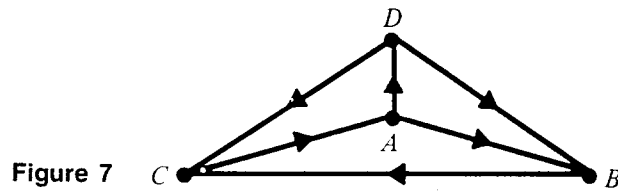


Figure 7

- The power of  $A$  is 5.
- The power of  $B$  is 2.
- The power of  $C$  is 3.
- The power of  $D$  is 4.

**EXAMPLE** (Athletic contest). The idea of the power of an individual can be used to judge athletic events. For example, the result of a single round of a round-robin athletic event results in the following data:

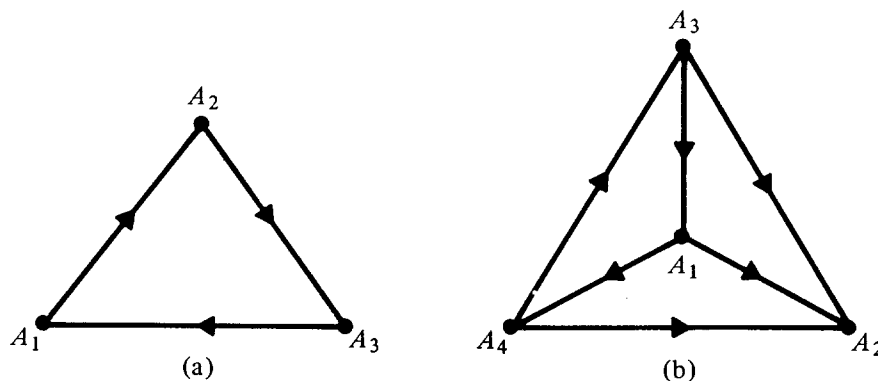
- Team  $A$  beats teams  $B$  and  $D$ .
- Team  $B$  beats team  $C$ .
- Team  $C$  beats team  $A$ .
- Team  $D$  beats teams  $C$  and  $B$ .

Then it is easy to check that this is precisely the dominance situation shown in Figure 7. By the analysis given above we can rate the teams in the following order according to their respective powers:  $A$ ,  $D$ ,  $C$ , and  $B$ .

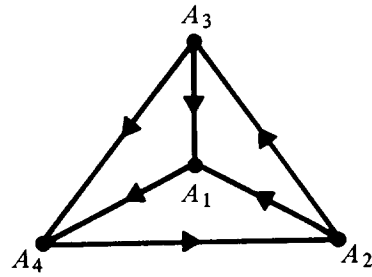
It should be remarked that the above definition of the power of an individual is not the only one possible. In Exercise 13 we suggest another definition of power which gives different results. Before using one or the other of these definitions, a sociologist should examine them carefully to see which (if either) fits his needs.

**EXERCISES**

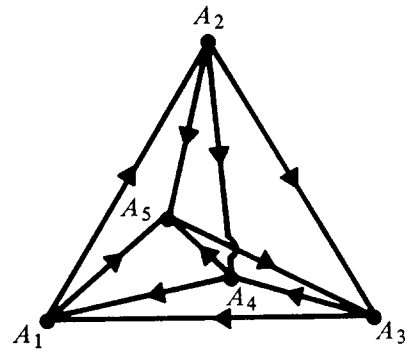
1. Show that there are only two essentially different pecking orders possible among three chickens, namely, those given in Figure 3. [Hint: Use directed graphs.]
2. Find the dominance matrices  $D$  corresponding to the following directed graphs:







(c)



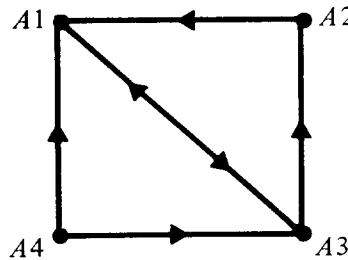
(d)

$$[\text{Ans. (b)} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}.]$$

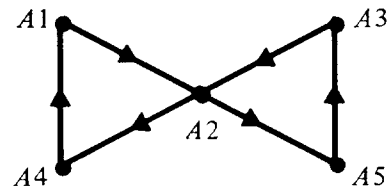
3. Compute the matrices  $D^2$  and  $S = D + D^2$  and determine the powers of each of the individuals in the examples of Exercise 2.

$$[\text{Ans. (b)} D^2 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}; S = \begin{pmatrix} 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix}; 4, 0, 4, 4.]$$

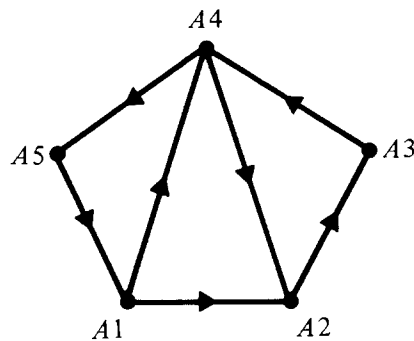
4. Find the communication matrices for the following communication networks:



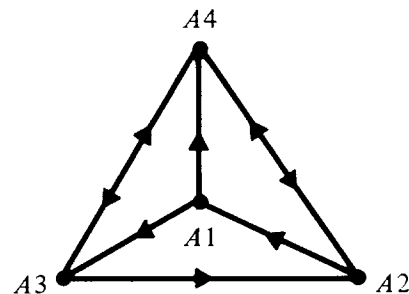
(a)



(b)



(c)



(d)

$$[\text{Ans. (a)} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}.]$$

5. Draw the directed graphs corresponding to the following communication matrices:

$$(a) \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

$$(b) \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

$$(c) \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

$$(d) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

6. Which of the communication networks whose matrices are given in Exercise 5 satisfy the hypothesis of the theorem of this section?

[Ans. (a) and (c).]

7. Show that the network in Figure 1(a) satisfies the conclusion of the theorem, while the network in Figure 1(b) does not.

8. By computing the matrix  $S$  in each case, find the persons who can communicate with everyone else in one or two stages and those who can be communicated with in one or two stages, for the communication matrices in Exercise 5. (In some cases such persons need not exist.)

[Ans. (a) Everyone; (b) everyone; (d) neither type of person exists.]

9. Find all the essentially different pecking orders that are possible among four chickens.

[Ans. There are four essentially different ones.]

10. If  $C$  is any communication matrix, give the interpretation of the entries in the columns of the matrix  $S = C + C^2$ . Also give the interpretation for the column sums of  $S$ .

11. Find all communication networks among three individuals which satisfy the hypothesis of the theorem of this section. How many of these are essentially different?

[Ans. There are seven.]

12. A round-robin tennis match among four people has produced the following results.

Smith has beaten Brown and Jones.

Jones has beaten Brown.

Taylor has beaten Smith, Brown, and Jones.

By finding the powers of each player, rank them into first, second, third, and fourth places. Does this ranking agree with your intuition?

[Ans. Taylor has power = 6, Smith has power = 3, Jones has power = 1, and Brown has power = 0.]

13. Let the power<sub>1</sub> of an individual be the power as defined in the text above. Define a new power, called power<sub>2</sub>, of an individual as follows: If  $D$  is the dominance matrix for a group of  $n$  individuals, then the power<sub>2</sub> of  $A_i$  is the sum of the  $i$ th row of the matrix

$$S' = D + \frac{1}{2}D^2.$$

Find the  $\text{power}_2$  of each of the teams in the athletic contest example in the text. Show that the  $\text{power}_2$  of an individual team need not equal its  $\text{power}_1$ . Comment on the result.

14. Find the  $\text{power}_2$  of the players in Exercise 12. Discuss its relation with the  $\text{power}_1$  of each of the players.

[Ans. Taylor has  $\text{power}_2 = \frac{9}{2}$ , Smith has  $\text{power}_2 = \frac{5}{2}$ , Jones has  $\text{power}_2 = 1$ , Brown has  $\text{power}_2 = 0$ .]

15. If  $C$  is a communication matrix, give an interpretation for the entries of the matrix  $C^3$ . Do the same for the matrix  $C^4$ .

[Ans. The entry in the  $i$ th row and the  $j$ th column of  $C^3$  gives the number of three-stage communications from  $i$  to  $j$ ; the same entry of  $C^4$  gives the number of four-stage communications from  $i$  to  $j$ .]

16. If  $C$  is a communication matrix, give an interpretation for the entries of the matrix  $S = C + C^2 + C^3 + \dots + C^m$ .

17. Prove the second statement of the theorem of the present section.

18. Prove that the following statement is true: In a communication network involving three individuals, it is possible for a message starting from any person to get to any other person if and only if the following condition is satisfied: each individual can send a message to at least one person and can receive a message from at least one person.

19. Show that the matrix form of the condition in Exercise 18 is: Every row and column of the communication matrix must have at least one nonzero entry.

20. Is the statement in Exercise 18 true for a communication network involving two individuals? For four or more individuals?

[Ans. Yes; no.]

## 2 EQUIVALENCE CLASSES IN COMMUNICATION NETWORKS

When considering communication networks, it becomes obvious that the various members of the network play different roles. Some members can only send messages, some can only receive them, and others can both send and receive. Subsets of members are also important. We shall consider subsets of members having the following two properties: (a) every member of the subset can both send and receive messages (not necessarily in one step) to and from every other member in the subset; and (b) the subset having property (a) is as large as possible. We shall show that it is possible to partition the set of all people in the network into subsets (called equivalence classes) having these two properties, and that between such equivalence classes there is at most a one-way communication link. We then apply our results to three different problems: (i) putting any nonnegative matrix into canonical form; (ii) the classification of states in a Markov chain; and (iii) the solution of an archeological problem.

As in the previous section, let  $A_1, \dots, A_n$ , be the members of the communication network. We define a relation  $R$  between some pairs of these

members as follows: Let  $A_i R A_j$  mean “ $A_i$  can send a message to  $A_j$  (in that direction and not necessarily in one step) or else  $i = j$ .” Then it is easy to show that the relation  $R$  has the following two properties:

- (1)  $A_i R A_i$  for every  $i$ . (Reflexive axiom)  
 (2)  $A_i R A_j$  and  $A_j R A_k$  imply  $A_i R A_k$ . (Transitive axiom)

To see this, note that property (1) follows from the definition of  $R$ , and (2) follows since if  $A_i$  can send a message to  $A_j$  and  $A_j$  can send a message to  $A_k$ , then  $A_i$  can send a message to  $A_k$  by routing it through  $A_j$ .

If  $S$  is any set and  $R$  is any relation defined for members of  $S$  that satisfies axioms (1) and (2), then  $R$  is called a *weak ordering* on  $S$ .

We next define another relation on the states of the network. Let  $A_i T A_j$  hold if and only if  $(A_i R A_j) \wedge (A_j R A_i)$ ; that is  $A_i T A_j$  holds if and only if “ $A_i$  has a two-way communication with  $A_j$  or else  $i = j$ .” It is easy to show that the relation  $T$  has the following three properties:

- (3)  $A_i T A_i$ . (Reflexive axiom)  
 (4)  $A_i T A_j$  if and only if  $A_j T A_i$ . (Symmetric axiom)  
 (5)  $A_i T A_j$  and  $A_j T A_k$  imply  $A_i T A_k$ . (Transitive axiom)

In Exercise 1 the reader is asked to establish these three axioms.

If  $S$  is any set and  $T$  is any relation defined for members of  $S$  that satisfies axioms (3), (4), and (5), then  $T$  is called an *equivalence relation* on  $S$ . The principal result about equivalence relations defined over a set  $S$  is that they partition  $S$  into equivalence classes.

---

**Definition** We say that  $A_i$  and  $A_j$  are *equivalent* if  $A_i T A_j$ . For any  $A_i$  the *equivalence class*  $E_i$  that it determines is the truth set of the statement  $A_i T A_k$ , i.e., it is the set of all  $A_k$  such that  $A_i T A_k$  is true.

---

**Theorem 1** The equivalence classes of  $T$  partition  $S$ , the set of members of the communication network.

---

**Proof** We must show that every member  $A_i$  of  $S$  belongs to one and only one equivalence class. Let  $S'$  be the equivalence class of  $A_i$ . Since  $A_i T A_i$  [from (3) above], we know that  $A_i$  belongs to  $S'$ , which shows that  $A_i$  belongs to some equivalence class, and also that  $S'$  is not empty.

Now let  $A_i$  and  $A_j$  be any two members of  $S$ , and let  $S'$  and  $S''$ , respectively, be their equivalence classes. We shall show that either  $S' \cap S'' = \mathcal{E}$  or else  $S' = S''$ . If  $S' \cap S'' = \mathcal{E}$ , then we are done. Hence suppose that there is an element  $X$  of  $S$  in  $S' \cap S''$ . Since  $X$  is in  $S'$ , we have  $A_i T X$ ; and since  $X$  is in  $S''$ , we have  $A_j T X$ . Using (4) we have  $X T A_j$ . But, by virtue of transitivity (5),  $A_i T X$  and  $X T A_j$  imply  $A_i T A_j$ ; hence  $A_j$  is in  $S'$ . Let  $Y$  be any element in  $S''$  so that  $A_j T Y$ . Using transitivity again, we have  $A_i T A_j$  and  $A_j T Y$ , so that  $Y$  is in  $S'$ . We have thus shown that every element of

$S''$  is in  $S'$ , i.e.,  $S'' \subset S'$ . In the same manner, one can show that  $S' \subset S''$ . Hence  $S' = S''$ .

Since we have shown that every member of  $S$  belongs to an equivalence class, and that every pair of equivalence classes are either identical or else disjoint, we have shown that they partition  $S$ , completing the proof of the theorem.

We now define a relation  $R$  on the equivalence classes of  $S$ . Namely, we let  $S'RS''$  mean, "either  $S' = S''$  or else some member of  $S'$  can send a message to some member of  $S''$ ." We leave it to the reader in Exercise 6 to show that  $R$  is a weak ordering of the set of equivalence classes of  $S$ .

---

**Theorem 2** Let  $S'$  and  $S''$  be two equivalence classes; then, if  $S'RS''$ , it is false that  $S''RS'$ . In other words, at most one-way communication is possible between equivalence classes.

---

**Proof** Suppose on the contrary, that  $S'$  and  $S''$  are two equivalence classes such that  $S'RS''$  and  $S''RS'$ . Then there is an element  $X$  in  $S'$  that can communicate with some element  $Y$  in  $S''$ ; and there is an element  $Z$  in  $S''$  that can communicate with some element  $U$  in  $S'$ . Since  $Y$  and  $Z$  are in  $S''$ , two-way communication is possible between them; and since  $X$  and  $U$  are in  $S'$ , they also have two-way communication. Hence  $Y$  can communicate with  $Z$ ,  $Z$  can communicate with  $U$ , and  $U$  can communicate with  $X$ . Therefore  $X$  and  $Y$  are in the same equivalence class, contradicting the assumption that they were in different (and hence disjoint) equivalence classes. This completes the proof.

For applications it is important to be able to find the equivalence classes for a given communication network. We develop an iterative method that constructs the following sets:

- (6)  $T_k$ , the set of states that  $A_k$  can *send* a message *to* (not necessarily in one step).
- (7)  $F_k$ , the set of states that  $A_k$  can *receive* a message *from* (not necessarily in one step).
- (8)  $E_k$ , the equivalence class of  $A_k$ .

It is easily seen (see Exercise 7) that  $E_k = T_k \cap F_k$ , so that we develop a step-by-step method for constructing the sets  $T_k$  and  $F_k$ . We illustrate the method with an example.

**EXAMPLE 1** We wish to get in contact with five alumni of a certain college, but do not know all their addresses. However, we have information of the form "Jones knows where Brown is," "Smith knows where Doe is," etc. We summarize this information in the communication matrix of Figure 8. In that figure for  $i \neq j$  we put 1 in the  $i,j$ th entry if the  $i$ th person knows where the  $j$ th

one is. What is the smallest number of people that we must contact in order to send a message to all of them?

Figure 8

	Brown	Jones	Smith	Adams	Doe
Brown	)	0	0	0	0
Jones		1	0	0	1
Smith		0	0	0	0
Adams		0	1	1	0
Doe		0	0	0	0

In order to solve this problem we first find the “send-to” lists for each person. We start by listing all the persons a person can contact in zero or one steps; these data come directly out of the communication matrix. These people form the “first-stage approximation” to the “send-to” lists. Next we go down the list of persons and add to each one’s “send-to” list all the people who can be contacted by people already on his first-stage approximate “send-to” list. The results are the “second-stage approximation to the send-to” lists. We continue this process, step by step, until for the first time we go through the list and do not add any member to any person’s “send-to” list. We then have the actual “send-to” sets for each person, since going through the process again would not change any list. The computations for the example in Figure 8 are shown in Figure 9.

Figure 9

Person	Zero- or One-Stage Communication	Send-to List
1 Brown	1	1
2 Jones	1, 2, 4	1, 2, 4, <b>3, 5</b>
3 Smith	3, 5	3, 5
4 Adams	2, 3, 4	2, 3, 4, <b>1, 5</b>
5 Doe	5	5

The first-stage approximation to the “send-to” list is shown in the second column of Figure 9. On the first pass through the list we add 3 to Jones’s list, which is indicated by boldface in the third column. We also add 1 and 5 to Adams’ list, also indicated by boldface numerals. On the next pass through the computation we add 5 to Jones’s list and make no other changes. The next pass through the computation produces no further changes so that the final lists shown in the third column of Figure 9 is the complete “send-to” list for each person.

We see that we have solved the problem posed above, for by contacting either Jones or Adams, we can relay a message to each of the five alumni members.

Let us go further and find the “receive-from” lists and the equivalence classes for each person in the network. The “receive-from” lists are easy, for we simply go down the “send-to” list and if we find member  $k$  on the  $i$ th person’s “send-to” list, we put  $i$  on the  $k$ th person’s “receive-from” list.

Person	Send-to List	Receive-from List	Equivalence Class
1 Brown	1	1, 2, 4	{1}
2 Jones	1, 2, 4, 3, 5	2, 4	{2, 4}
3 Smith	3, 5	2, 3, 4	{3}
4 Adams	2, 3, 4, 1, 5	2, 4	{2, 4}
5 Doe	5	2, 3, 4, 5	{5}

Figure 10

And we compute the equivalence classes from the relationship  $E_k = T_k \cap F_k$ . These computations are shown in Figure 10.

It is interesting to draw the graph of the weak-ordering relation  $R$  on the equivalence classes. To find the graph we simply check whether one-way communication is possible between each pair of equivalence classes. Then we connect two equivalence classes in the graph if such one-way communication is possible and if there is no intermediate class in the communication path. The graph of the weak-ordering relation for the matrix of Figure 8 is shown in Figure 11. Note that equivalence class  $\{2, 4\}$  can communicate

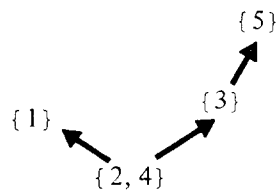


Figure 11

directly to  $\{1\}$  and  $\{3\}$  and to  $\{5\}$  through  $\{3\}$ . This graph shows very clearly the fact, noted above, that in order to contact all members of the group it is sufficient to contact either member of the equivalence class  $\{2, 4\}$ .

We can use the weak ordering of equivalence classes to put the matrix of Figure 8 in a canonical form, which is characterized by the following definition:

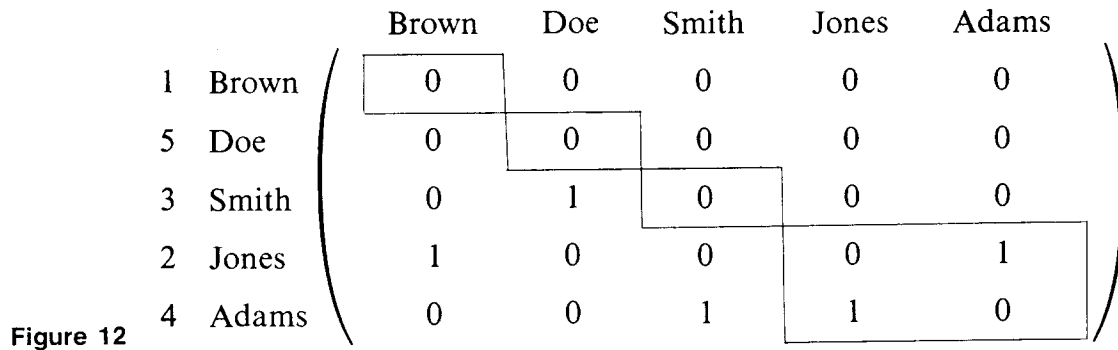
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**Definition** Let  $C$  be any communication matrix, and let  $S', S'', \dots$ , be the equivalence classes of its states. Then by a *canonical form* of  $C$  we shall mean a reordering of the rows and columns of  $C$  so that the following two properties are satisfied:

- (i) Members of a given equivalence class are listed next to each other.
  - (ii) No equivalence class  $S'$  is listed until all classes  $S''$  "above" it in the graph of the equivalence classes have already been listed; i.e.,  $S'$  is not listed until all classes such that  $S'RS''$  have already been listed.
- 

**EXAMPLE 1** We illustrate the above definition in terms of the matrix  $A$  of Figure 8.  
**(continued)** Using the weak ordering diagram of Figure 11, we see that the following

listing of the states (row indices) of  $A$  will satisfy the definition: 1, 5, 3, 2, 4. The resulting matrix is shown in Figure 12. In that figure dotted lines appear along the main diagonal, indicating the equivalence classes. Note that above the main diagonal blocks the only entries are zeros. Matrices having this property are called *block-triangular*.



The same kind of canonical form is possible for *any* nonnegative matrix  $A$ , if we let  $C(A)$  be the communication matrix derived from  $A$  by putting zeros on the main diagonal, and replacing positive off-diagonal entries by ones. We discuss this for Markov chain transition matrices. When the matrix under consideration is the transition matrix of a Markov chain, the classification of the states is extremely important in the study of the behavior of the chain, as the following definition indicates:

---

**Definition** Let  $P$  be the transition matrix of a Markov chain, and let  $C(P)$  be the matrix obtained from  $P$  by replacing each diagonal entry by 0 and replacing each positive off-diagonal entry by 1. Let  $S', S'', \dots$  be the equivalence classes of the states of  $C(P)$ ; then:

- (i) The maximal equivalence classes—that is, those classes that cannot send to other classes—are called *ergodic sets*. Members of ergodic sets are called *ergodic states*. If an ergodic set contains a single state, that state is an *absorbing state*.
- (ii) All equivalence classes that can send messages to other classes are called *transient sets*. Members of transient sets are called *transient states*.

---

**EXAMPLE 2** Consider the transition matrix

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \end{pmatrix}.$$

Changing the diagonal entries to zeros and the positive off-diagonal entries to ones gives



$$C(P) = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{matrix}.$$

In Exercise 8 the reader will be asked to show that the equivalence classes of  $C(P)$  are  $\{4\}$ ,  $\{1, 3\}$ ,  $\{5\}$ , and  $\{2\}$ . Moreover, the graph of the weak-ordering relation on these classes is as shown in Figure 13. As before, the

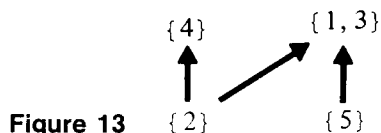


Figure 13

graph is obtained by checking whether one-way communication is possible between each pair of equivalence classes. From this diagram and the above definition we see that  $\{4\}$  and  $\{1, 3\}$  are ergodic sets and that  $\{4\}$  is an absorbing state; also,  $\{2\}$  and  $\{5\}$  are transient sets. A canonical form of the matrix found by listing the states in the order 4, 1, 3, 5, 2 is

$$P = \begin{pmatrix} \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & \boxed{0} & \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & \boxed{0} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \boxed{\frac{2}{3}} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & \boxed{\frac{1}{2}} \end{pmatrix}.$$

Note again that it is block-triangular, as indicated by the dotted lines. There are other orders in which to list the states, which lead to slightly different canonical forms for the matrix (see Exercise 9).

We conclude this section with an application of the above theory to an archeological problem.

**EXAMPLE 3** Recent archeological investigations in Asia Minor, between the Mediterranean and Black Seas, have disclosed the existence of an ancient Assyrian civilization dating back to at least the nineteenth century B.C. This civilization came to light when peasants working in fields turned up clay tablets having written inscriptions. Upon being translated, these tablets turned out to be letters written between merchants located at various cities and towns of the ancient civilization. The letters contained the name of the sender, the name of the receiver, and an order to buy, sell, or transport goods, to pay money, etc. But the *date* of the letter was not included. In addition, merchants

in different villages sometimes had the same name, and the location of the merchant was not always made clear in each of the letters. More than 2500 such tablets have been discovered; their contents give rise to two different problems. The first problem is to try to order the merchants according to their chronological dates. A second problem is to try to determine when the same name refers to more than one person. By studying the communication network that can be set up from the data of the tablets, we shall illustrate with small examples methods of trying to get partial answers to these questions.

	1	2	3	4	5	6	7	8	9	10
1	0	0	0	0	0	1	0	0	0	0
2	0	0	1	0	1	0	0	0	0	0
3	0	1	0	0	1	1	0	0	0	0
4	0	0	0	0	0	0	0	1	0	1
5	0	1	1	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	1	0	0
8	0	0	1	0	0	0	0	0	1	0
9	0	0	0	0	0	0	1	0	0	0
10	1	0	0	0	0	0	0	0	0	0

Figure 14

To illustrate an approach to the first problem, suppose that we set up a (hypothetical) communication matrix for a group of ten merchants, as indicated in the matrix of Figure 14. In that matrix an entry of 1 is made in the  $i,j$ th entry if merchant  $i$  sent a letter to merchant  $j$ . Carrying out the same analysis as in Example 1, we find the equivalence classes to be  $\{6\}$ ,  $\{1, 10\}$ ,  $\{2, 3, 5\}$ ,  $\{7, 8, 9\}$ , and  $\{4\}$ . The graph of the weak-ordering relation on these classes is shown in Figure 15. It was determined, as before,

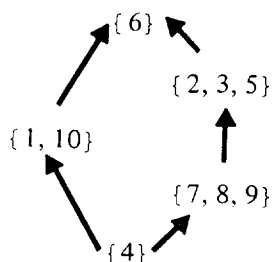


Figure 15

by seeing whether there is one-way communication between each pair of equivalence classes. It is clear that members of a given equivalence class are contemporaries. But it is not clear which of the equivalence classes is earlier, merely from the one-way communication between them. However, further analysis of the content of the messages might help to establish this. For instance, if one of the messages exchanged among merchants 7, 8, and 9 were related to one of the messages exchanged among merchants 2, 3, and 5, then it would be reasonable to assume that they are all contemporaries. We see that here is a case in which mathematics, although it cannot

furnish the complete answer to the problem, can indicate directions in which to search for more information.

To illustrate the second problem mentioned above, we use some actual data (see p. 865 of the Gardin-Garelli reference listed at the end of the chapter) summarized in the communication matrix of Figure 16. The matrix is symmetric, indicating that either there is a two-way (direct) communication between two individuals or else no (direct) communication at all. All the merchants belong to the same equivalence class, so that the previous analysis does not shed any light on their relative dates, except that they are all contemporaries. But is it possible that some names really stand for two different individuals? No definite answer can be provided to this question, but some indications can be provided by finding the *cliques* in the communication network.

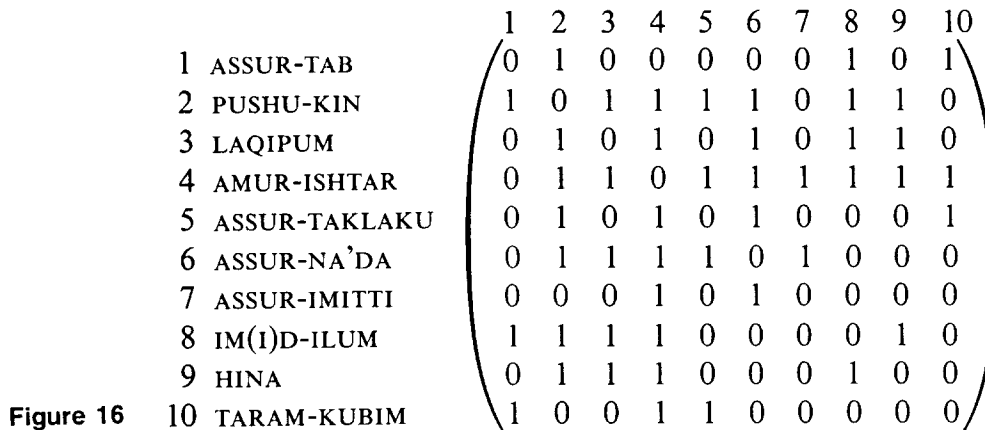


Figure 16

**Definition** A *clique* of a communication network is a subset *C* of individuals containing at least three members, with the following two properties:

- (i) Every pair of members of the clique has two-way communication.
- (ii) The subset *C* is as large as possible with every pair of members having property (i).

The problem of finding all cliques has been solved but is too lengthy to describe here. We content ourselves with listing all the maximal cliques for the data of Figure 16. They are

$$\{1, 2, 8\}, \{2, 3, 4, 6\}, \{2, 3, 4, 8, 9\}, \{2, 4, 5, 6\}, \{4, 5, 10\}, \{4, 6, 7\}.$$

From this list we can derive the frequency with which each merchant occurs in a clique, as shown in Figure 17. From that table it is evident that merchants 2(Pushu-Kin) and 4(Amur-Ishtar) occur most frequently in

Merchant	1	2	3	4	5	6	7	8	9	10
Number of times in a clique	1	4	2	4	1	3	1	2	1	0

Figure 17

cliques, and hence these names are most likely to be homonyms for two different people. Here again, mathematics does not completely solve the problem, but merely indicates the direction in which to look for further evidence.

The above calculations, though oversimplified, are illustrative of the kinds of calculations that must be done in order to study the complete communication network revealed by the 2500 tablets so far found at the archeological site.

## EXERCISES

1. Show that the relation  $T$  satisfies properties (3), (4), and (5).
2. Show that the relation " $\geq$ " is a weak-ordering relation on the set of integers. [Hint: Show that  $x \geq y$ , for  $x$  and  $y$  integers, satisfies (1) and (2).]
3. Show that the relation " $=$ " is an equivalence relation on the set of all rational numbers (fractions). What are the equivalence classes it determines?
4. Let  $x$  and  $y$  be any two words and let  $xRy$  mean "Word  $x$  occurs no later than word  $y$  in the dictionary." Show that  $R$  is a weak order on the set of words.
5. Let  $x$  and  $y$  be people and let  $xTy$  mean " $x$  is the same height as  $y$ ." Show that  $T$  is an equivalence relation. What are the equivalence classes it determines? Show that the relation "at least as tall as" is a weak-ordering relation on these equivalence classes.
6. Let  $R$  and  $T$  be the relations defined in the text; let  $S', S'', \dots$  be the equivalence classes determined by  $T$ ; and let  $S'RS''$  be as defined in the text. Show that  $R$  satisfies properties (1) and (2)—that is, it is a weak ordering on the set of equivalence classes.
7. Let  $E_k, T_k$ , and  $F_k$  be as defined in the text. Show that  $E_k = T_k \cap F_k$ .
8. Find the equivalence classes of the communication matrix given in Example 2.
9. Show that there are ten different canonical forms for the transition matrix of Example 2.
10. Show that if  $A$  can communicate with  $B$  in a communication network having  $n$  persons, then it must be possible to do this in not more than  $n - 1$  steps.
11. Suppose that there are six different individuals each of whom knows the location of certain others. This information is summarized in the following communication matrix:

$$\begin{array}{c}
 \\
 \\
 \\
 \\
 \\
 \\
 \end{array}
 \begin{array}{cccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 \\
 \begin{array}{c}
 1 \\
 2 \\
 3 \\
 4 \\
 5 \\
 6
 \end{array}
 & \left( \begin{array}{cccccc}
 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1 \\
 0 & 1 & 0 & 0 & 0 & 1 \\
 1 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 1 & 1 & 0
 \end{array} \right)
 \end{array}$$

- (a) Find the equivalence classes of  $T$ .
- (b) Draw the graph of the weak ordering relation on the equivalence classes.
- (c) Suppose you know where 3 is and you want to find out where 1 is. What is the shortest communication path from 3 to 1?  
[*Partial ans.* It has length 3.]
- (d) What is the longest such communication path?  
[*Partial ans.* It has length 5.]
12. Classify each of the states of the Markov chain whose transition matrix is given below, and put the matrix into a canonical form. [*Hint:* Use some of the results of Exercise 11.]

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{3}{4} & 0 & 0 & 0 & \frac{1}{4} \\ \frac{2}{5} & 0 & 0 & 0 & \frac{3}{5} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

[*Ans.* One canonical form is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{5} & \frac{3}{5} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix};$$

state 1 is absorbing; all other states are transient.]

13. If a matrix  $M$  can be put into the form

$$M = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix},$$

where  $0$  is the zero matrix, then  $M$  is said to be *reducible* or *decomposable*. If  $A$  and  $C$  are square and nonsingular show that

$$M^{-1} = \begin{pmatrix} A^{-1} & 0 \\ -C^{-1}BA^{-1} & C^{-1} \end{pmatrix}.$$

14. Use the results of Exercise 13 to show how a canonical form of a nonnegative matrix can be used to simplify the work of finding its inverse.
15. (a) Show that the Markov chain in Exercise 12 is an absorbing Markov chain.
- (b) Find the matrix  $Q$  in canonical form. Show that the matrix  $I - Q$  is block-triangular.
- (c) Use the results of Exercises 13 and 14 to find  $N = (I - Q)^{-1}$ .

[Ans. With the canonical form of the answer to Exercise 12, the inverse is

$$N = (I - Q)^{-1} = \begin{pmatrix} \frac{5}{2} & 0 & 0 & 0 & 0 \\ \frac{5}{4} & 1 & 0 & 0 & 0 \\ \frac{10}{7} & \frac{6}{7} & \frac{8}{7} & \frac{3}{7} & 0 \\ \frac{55}{28} & \frac{3}{7} & \frac{4}{7} & \frac{12}{7} & 0 \\ \frac{55}{28} & \frac{3}{7} & \frac{4}{7} & \frac{12}{7} & 1 \end{pmatrix}.$$

16. Draw the graph of a three-person clique. Also that of a four-person clique. Describe the graph of a clique containing  $n$  persons ( $n \geq 3$ ).
17. Verify that the cliques given in Example 3 satisfy the two properties given in the definition of a clique.
18. Let  $C_1$  and  $C_2$  be any two *distinct* cliques of the same communication network.
  - (a) Show by examples that  $C_1 \cap C_2$  may or may not be empty.
  - (b) Prove that the sets  $C_1 - C_2$  and  $C_2 - C_1$  are *never* empty.

### 3 STOCHASTIC PROCESSES IN GENETICS

The simplest type of inheritance of traits in animals occurs when a trait is governed by a pair of genes, each of which may be of two types, say  $G$  and  $g$ . An individual may have a  $GG$  combination or  $Gg$  (which is genetically the same as  $gG$ ) or  $gg$ . Very often the  $GG$  and  $Gg$  types are indistinguishable in appearance, and then we say that the  $G$  gene *dominates* the  $g$  gene. An individual is called *dominant* if he has  $GG$  genes, *recessive* if he has  $gg$ , and *hybrid* with a  $Gg$  mixture.

In the mating of two animals, the offspring inherits one gene of the pair from each parent, and the basic assumption of genetics is that these genes are selected at random, independently of each other. This assumption determines the probability of every type of offspring. Thus the offspring of two dominant parents must be dominant, of two recessive parents must be recessive, and of one dominant and one recessive parent must be hybrid. In the mating of a dominant and a hybrid animal, the offspring must get a  $G$  gene from the former and has probability  $\frac{1}{2}$  for getting  $G$  or  $g$  from the latter, hence the probabilities are even for getting a dominant or a hybrid offspring. Again in the mating of a recessive and a hybrid, there is an even chance of getting either a recessive or a hybrid. In the mating of two hybrids, the offspring has probability  $\frac{1}{2}$  for getting a  $G$  or a  $g$  from each parent. Hence the probabilities are  $\frac{1}{4}$  for  $GG$ ,  $\frac{1}{2}$  for  $Gg$ , and  $\frac{1}{4}$  for  $gg$ .

**EXAMPLE 1** Let us consider a process of continued crossings. We start with an individual of unknown genetic character, and cross it with a hybrid. The offspring is again crossed with a hybrid, etc. The resulting process is a Markov chain. The states are "dominant," "hybrid," and "recessive." The transition probabilities are

$$(1) \quad P = \begin{matrix} & \begin{matrix} D & H & R \end{matrix} \\ \begin{matrix} D \\ H \\ R \end{matrix} & \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{matrix}$$

as can be seen from the previous paragraph. The matrix  $P^2$  has all entries positive (see Exercise 1), hence we know from Chapter 4, Section 7, that there is a unique fixed-point probability vector, i.e., a vector  $p$  such that  $pP = p$ . By solving three equations, we find the fixed vector to be  $p = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$ . Hence, no matter what type the original animal was, after repeated crossing we have probability nearly  $\frac{1}{4}$  of having a dominant,  $\frac{1}{2}$  of having a hybrid, and  $\frac{1}{4}$  of having a recessive offspring.

In Example 1 we may ask a more difficult question. Suppose that we have a regular matrix  $P$  (as in Example 1), with states  $a_1, \dots, a_n$ . The process keeps going through all the states. If we are in  $a_i$ , how long, on the average, will it take for the process to return to  $a_i$ ? We can even ask the more general question of how long, on the average, it takes to go from  $a_i$  to  $a_j$ .

The average here is taken in the sense of an expected value. There is a probability  $p_1$  that we reach  $a_j$  for the first time in one step,  $p_2$  that we reach it first in two steps, etc. The expected value is  $p_1 \cdot 1 + p_2 \cdot 2 + \dots$  (see Chapter 3, Section 11). This, in general, requires a difficult computation. However, there is a much simpler way of finding the expected values. Let the expected number of steps required to go from state  $a_i$  to  $a_j$  be  $m_{ij}$ . How can we go from  $a_i$  to  $a_j$ ? We go from  $a_i$  to  $a_k$  with probability  $p_{ik}$  in one step. If  $k = j$ , we are there. If  $k \neq j$ , it takes an average of  $m_{kj}$  steps more to get to  $a_j$ . Hence  $m_{ij}$  is equal to 1 plus the sum of  $p_{ik}m_{kj}$  for all  $k \neq j$ . To state this as a matrix equation we define the matrix  $\bar{M}$  to be the matrix  $M$  but with all the diagonal entries  $m_{ii}$  being replaced by 0; also, let  $C$  be the square matrix having all entries equal to 1. Then the equations for  $m_{ij}$  can be written in matrix form as

$$(2) \quad M = P\bar{M} + C.$$

To see that this is so let us concentrate on the  $i, j$ th entry of equation (2). On the left-hand side it is  $m_{ij}$ . On the right-hand side it is the  $i, j$ th entry of  $P\bar{M}$  which is the sum of all products  $p_{ik}m_{kj}$  for  $k \neq j$  (since the main diagonal of  $\bar{M}$  is zero) plus the  $i, j$ th entry in  $C$ , which is 1. This is the same as before. Let us now multiply (2) by  $w$ , the fixed vector of  $P$ . Recalling that  $w$  is a probability vector we obtain

$$(3) \quad wM = w\bar{M} + (1, \dots, 1)$$

or

$$(4) \quad w(M - \bar{M}) = (1, \dots, 1).$$

But all components of  $M - \bar{M}$  except the diagonal ones are 0. Hence our

equation simply states that  $w_i m_{ii} = 1$  for each  $i$ . This tells us that  $m_{ii} = 1/w_i$ . The average time it takes to return from  $a_i$  to  $a_i$  is the reciprocal of limiting probability of being in  $a_i$ . In Example 1 this means that if we have a dominant offspring we shall have another dominant in an average of four steps, after a hybrid we have another hybrid in an average of two steps, and a recessive follows a recessive on the average in four steps.

**EXAMPLE 2** A more interesting, and also more complex, process is obtained by crossing a given population with itself, and then crossing the offspring with offspring, etc. Let us suppose that our population has a fraction  $d$  of dominants,  $h$  hybrids, and  $r$  recessives. Then  $d + h + r = 1$ . If the population is very large and they are mated at random, then (by the law of large numbers) we can expect  $d^2$  to be the fraction of matings in which both parents are dominant,  $2dh$  the fraction of mating a dominant with a hybrid, etc. The tree of logical possibilities with branch probabilities marked on it is shown in Figure 18. We use it to compute the fraction of each type. To do this we simply add together the path weights of the paths ending in  $D$ , in  $H$ ,

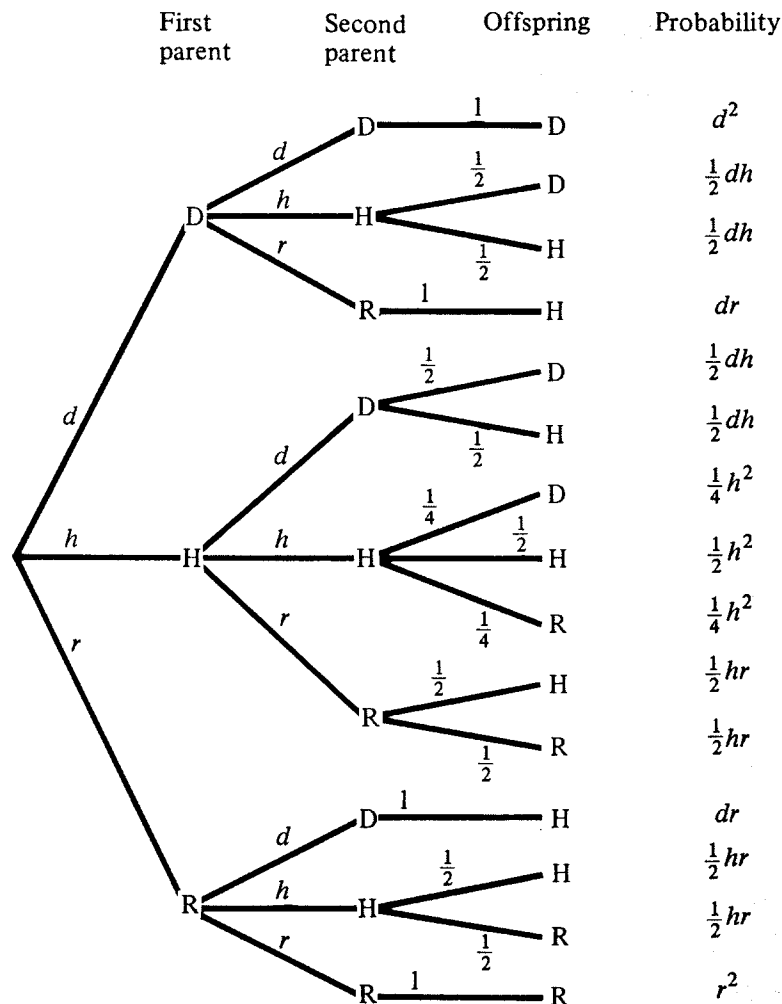


Figure 18



and in  $R$ . The results are:

$$D: d^2 + 2 \cdot \frac{1}{2}dh + \frac{1}{4}h^2 = d^2 + dh + \frac{1}{4}h^2$$

$$H: 2 \cdot \frac{1}{2}dh + 2dr + \frac{1}{2}h^2 + 2 \cdot \frac{1}{2}hr = dh + rh + 2dr + \frac{1}{2}h^2$$

$$R: \frac{1}{4}h^2 + 2 \cdot \frac{1}{2}hr + r^2 = r^2 + hr + \frac{1}{4}h^2.$$

If we represent the fractions in a given generation by a row vector, the process may be thought of as a transformation  $T$  which changes a row vector into another row vector.

$$(5) \quad (d, h, r) \cdot T = (d^2 + dh + \frac{1}{4}h^2, dh + rh + 2dr + \frac{1}{2}h^2, r^2 + rh + \frac{1}{4}h^2).$$

The trouble is that (see Exercise 2) the transformation  $T$  is not linear. Nevertheless, we know that after  $n$  crossings the distribution will be  $(d, h, r)T^n$ , so that, if we can get a simple formula for  $T^n$ , we can describe the results simply. And here luck is with us.

Let us compute  $T^2$ , i.e., find what happens if we apply twice the transformation specified above. The first generation of offspring is distributed according to the formula (5). We now take the first component on the right side as  $d$ , the second as  $h$ , and the third as  $r$ , and compute  $d^2 + dh + \frac{1}{4}h^2$ , etc. Here we find to our surprise that  $T^2 = T$ . Hence  $T^n = T$ .

This means that  $(d, h, r)T = (d, h, r)T^n$ , which in turn means that the distribution after many generations is the same as in the first generation of offspring. Hence we say that the process reaches an *equilibrium* in one step. It must, however, be remembered that our fractions are only approximate, and are a good approximation only for very large populations.

For the geneticist, this result is very interesting. It shows that, in a population in which no mutations occur and selection does not take place, "evolution" is all over in a single generation.

To the mathematician the process is interesting since it is an example of a quadratic transformation, a transformation more complex than the linear ones we have heretofore studied.

The next two examples give applications of absorbing Markov chains to genetics.

**EXAMPLE 3** If we keep crossing the offspring with a dominant animal, the result is quite different. The transition matrix is easily found to be

$$(6) \quad P' = \begin{matrix} & \begin{matrix} D & H & R \end{matrix} \\ \begin{matrix} D \\ H \\ R \end{matrix} & \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{matrix}.$$

This is an absorbing Markov chain with one absorbing state,  $D$ . Using the results of Chapter 4, Section 8, we have

$$Q' = \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & 0 \end{pmatrix}, \quad R' = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix},$$

so that

$$I - Q' = \begin{pmatrix} \frac{1}{2} & 0 \\ -1 & 1 \end{pmatrix}$$

and

$$N' = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}.$$

The absorption probabilities are

$$B = N'R' = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

as was to be expected, since there is only one absorbing state. This means that if we keep crossing the population with dominants, then after sufficiently many crossings we can expect only dominants. The mean number of steps to absorption are found by

$$t = N'c = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Hence we expect the process to be absorbed in two steps starting from state  $H$ , and three steps starting from state  $R$ .

**EXAMPLE 4** Let us construct a more complicated example of an absorbing Markov chain. We start with two animals of opposite sex, cross them, select two of their offspring of opposite sex and cross those, etc. To simplify the example we shall assume that the trait under consideration is independent of sex.

Here a state is determined by a pair of animals. Hence the states of our process will be:  $a_1 = (D, D)$ ,  $a_2 = (D, H)$ ,  $a_3 = (D, R)$ ,  $a_4 = (H, H)$ ,  $a_5 = (H, R)$ , and  $a_6 = (R, R)$ . Clearly, states  $a_1$  and  $a_6$  are absorbing, since if we cross two dominants or two recessives we must get one of the same type. The rest of the transition probabilities are easy to find. We illustrate their calculation in terms of state  $a_2$ . When the process is in this state, one parent has  $GG$  genes, the other  $Gg$ . Hence the probability of a dominant offspring or a hybrid offspring is  $\frac{1}{2}$  for each. Then the probability of transition to  $a_1$  (selection of two dominants) is  $\frac{1}{4}$ , the transition to  $a_2$  is  $\frac{1}{2}$ , and to  $a_4$  is  $\frac{1}{4}$ . The complete transition matrix is (listing the absorbing states first)

$$P'' = \begin{matrix} & \begin{matrix} a_1 & a_6 & a_2 & a_3 & a_4 & a_5 \end{matrix} \\ \begin{matrix} a_1 \\ a_6 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{16} & \frac{1}{16} & \frac{1}{4} & \frac{1}{8} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \end{matrix}.$$

Calculating the fundamental quantities for an absorbing chain, we obtain

$$Q'' = \begin{matrix} & a_2 & a_3 & a_4 & a_5 \\ a_2 & \left( \frac{1}{2} & 0 & \frac{1}{4} & 0 \right) \\ a_3 & \left( 0 & 0 & 1 & 0 \right) \\ a_4 & \left( \frac{1}{4} & \frac{1}{8} & \frac{1}{4} & \frac{1}{4} \right) \\ a_5 & \left( 0 & 0 & \frac{1}{4} & \frac{1}{2} \right) \end{matrix}, \quad R'' = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 0 \\ \frac{1}{16} & \frac{1}{16} \\ 0 & \frac{1}{4} \end{pmatrix},$$

and

$$I - Q'' = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{4} & 0 \\ 0 & 1 & -1 & 0 \\ -\frac{1}{4} & -\frac{1}{8} & \frac{3}{4} & -\frac{1}{4} \\ 0 & 0 & -\frac{1}{4} & \frac{1}{2} \end{pmatrix},$$

and

$$N'' = (I - Q'')^{-1} = \begin{pmatrix} \frac{3}{2} & \frac{1}{6} & \frac{4}{3} & \frac{2}{3} \\ \frac{4}{3} & \frac{4}{3} & \frac{4}{3} & \frac{4}{3} \\ \frac{5}{3} & \frac{1}{3} & \frac{5}{3} & \frac{4}{3} \\ \frac{4}{3} & \frac{1}{3} & \frac{5}{3} & \frac{4}{3} \\ \frac{2}{3} & \frac{1}{6} & \frac{4}{3} & \frac{5}{3} \end{pmatrix}.$$

The absorption probabilities are found to be

$$B'' = N''R'' = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}.$$

The genetic interpretation of absorption is that after a large number of inbreedings either the  $G$  gene or the  $g$  gene must disappear. It is also interesting to note that the probability of ending up entirely with  $G$  genes, if we start from a given state, is equal to the proportion of  $G$  genes in this state.

The mean number of steps to absorption is

$$t = N'' \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{5}{6} \\ \frac{6}{3} \\ \frac{5}{3} \\ \frac{4}{6} \end{pmatrix}.$$

Hence we see that, if we start in a state other than  $(D, D)$  or  $(R, R)$ , we can expect to reach one of these states in about five or six steps. The exact expected times are given by the entries of  $t$ . The matrix  $N''$  provides more detailed information, namely how many times we can expect to have offspring of the types  $(D, H)$ ,  $(D, R)$ ,  $(H, H)$ , and  $(H, R)$ , starting from a given nonabsorbing state. And the matrix  $B''$  gives the probabilities of ending up in  $a_1$  or  $a_6$ . These quantities jointly give us an excellent description of what we can expect of our process.

## EXERCISES

1. From matrix (1) compute  $P^2$ ,  $P^3$ ,  $P^4$ , and  $P^5$ . Verify that  $P^2 > 0$  and that the powers approach the expected form (see Chapter 4, Section 7).
2. Compute  $T^2$  by taking the first component of (5) as  $d$ , the second as  $h$ , the third as  $r$ , and substituting into the formula (5). Making use of the fact that  $d + h + r = 1$ , show that  $T^2 = T$ .
3. A fixed point of  $T$  is a vector such that  $(d, h, r)T = (d, h, r)$ . Write the conditions that such a vector must satisfy, and give three examples of such fixed vectors. What is the genetic meaning of such a distribution? [Ans. For example,  $(\frac{1}{3}, \frac{4}{9}, \frac{4}{9})$ .]
4. In the matrix  $P$  the second row is equal to the fixed-point vector. What significance does this have?
5. For Example 1 write the matrix  $M$  with unknown entries  $m_{ij}$ . Write  $M$  by replacing  $m_{11}$ ,  $m_{22}$ , and  $m_{33}$  by zeros. Then solve the nine simultaneous equations given by (3), to find the  $m_{ij}$ . Check that  $m_{ii} = 1/w_i$ . [Ans.  $m_{11} = 4$ ,  $m_{12} = 2$ ,  $m_{13} = 8$ .]
6. From the definition of a stochastic matrix (Chapter 4, Section 7), prove that  $PC = C$ .
7. Prove that, if  $P$  is a regular  $n \times n$  stochastic matrix having column sums equal to 1, then it takes an average of  $n$  steps to return from any state to itself. (See Chapter 4, Section 7, Exercise 8.)
8. It is raining in the Land of Oz. In how many days can the Wizard of Oz expect to go on a picnic? (See Chapter 4, Section 7, Exercise 12.) [Ans. 4.]

Exercises 9–14 develop a simpler method of treating the nonlinear transformation  $T$ , in the text above.

9. Let  $p$  be the ratio of  $G$  genes in the population, and  $q = 1 - p$  the ratio of  $g$  genes. Express  $p$  and  $q$  in terms of  $d$ ,  $h$ , and  $r$ . [Ans.  $p = d + \frac{1}{2}h$ ,  $q = r + \frac{1}{2}h$ .]
10. Suppose that we take all the genes in the population, mix them thoroughly, and select a pair at random for each offspring. Show, using the result of Exercise 9, that the resulting distribution of dominant, hybrid, and recessive individuals is precisely that given in formula (5). [Ans.  $(d, h, r) \cdot T = (p^2, 2pq, q^2)$ .]
11. If we write  $(d, h, r) \cdot T = (d', h', r')$ , show, using the result of Exercise 10, that  $h'^2 = 4d'r'$ .
12. Show that for equilibrium it is necessary that  $h^2 = 4dr$ .
13. Show that if  $h^2 = 4dr$ , then  $p^2 = d$ ,  $q^2 = r$ , and  $2pq = h$ . Hence show that this condition is also sufficient for equilibrium.
14. Use the results of Exercises 11–13 to show that the population reaches equilibrium in one generation.
15. Prove that in an absorbing Markov chain
  - (a) The probability of reaching a given absorbing state is independent of the starting state if and only if there is only one absorbing state.

- (b) The expected time for reaching an absorbing state is independent of the starting state if and only if every state is absorbing.
16. Suppose that hybrids have a high mortality rate; say that half of the hybrids die before maturity, while only a negligible number of dominants and recessives die before maturity.
- (a) In Example 4 above, modify the matrix  $P''$  to apply to this situation.
- (b) What are the absorbing states?
- (c) Verify that it is an absorbing chain.
- (d) Find the vectors  $d$  representing the probabilities of absorption in the various absorbing states.

$$[\text{Ans. For } a_1, d = \begin{pmatrix} 1 \\ 0 \\ \frac{9}{10} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{10} \end{pmatrix}.]$$

- (e) Find  $N$ , and interpret.

$$[\text{Ans. } t = \begin{pmatrix} 65 \\ 26 \\ 117 \\ 26 \\ 91 \\ 26 \\ 65 \\ 26 \end{pmatrix}.]$$

The remaining problems concern the inheritance of color blindness, which is a sex-linked characteristic. There is a pair of genes,  $C$  and  $S$ , of which the former tends to produce color blindness, the latter normal vision. The  $S$  gene is dominant. But a man has only one gene, and if this is  $C$ , he is color-blind. A man inherits one of his mother's two genes, while a woman inherits one gene from each parent. Thus a man may be of type  $C$  or  $S$ , while a woman may be of type  $CC$  or  $CS$  or  $SS$ . We shall study a process of inbreeding similar to that of Example 4.

17. List the states of the chain. [*Hint*: There are six.]
18. Compute the transition probabilities.
19. Show that the chain is absorbing, and interpret the absorbing states. [*Ans.* In one, the  $S$  gene disappears; in the other, the  $C$  gene is lost.]
20. Prove that the probability of absorption in the state having only  $C$  genes, if we start in a given state, is equal to the proportion of  $C$  genes in that state.
21. Find  $N$ , and interpret.
22. Find  $t$ , and interpret.

$$[\text{Ans. } \begin{pmatrix} 5 \\ 6 \\ 6 \\ 5 \end{pmatrix}; \text{ if we start with both } C \text{ and } S \text{ genes, we can expect one of these to disappear in five or six crossings.}]$$

#### 4 MARRIAGE RULES IN PRIMITIVE SOCIETIES

In some primitive societies there are rigid rules as to when marriages are permissible. These rules are designed to prevent very close relatives from marrying. The rules can be given precise mathematical formulation in terms of permutation matrices. Our discussion is based, in part, on the work of André Weil and Robert R. Bush.

The marriage rules found in these societies are characterized by the following axioms:

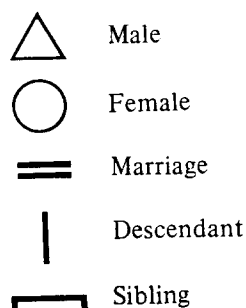
- Axiom 1.* Each member of the society is assigned a marriage type.
- Axiom 2.* Two individuals are permitted to marry only if they are of the same marriage type.
- Axiom 3.* The type of an individual is determined by the individual's sex and by the type of his parents.
- Axiom 4.* Two boys (or two girls) whose parents are of different types will themselves be of different types.
- Axiom 5.* The rule as to whether a man is allowed to marry a female relative of a given kind depends only on the kind of relationship.
- Axiom 6.* In particular, no man is allowed to marry his sister.
- Axiom 7.* For any two individuals it is permissible for some of their descendants to intermarry.

**EXAMPLE** Let us suppose that there are three marriage types,  $t_1, t_2, t_3$ . Two parents in a given family must be of the same type, since only then are they allowed to marry. Thus there are only three logical possibilities for marriages. For each case we have to state what the type of a son or a daughter will be:

<i>Type of both parents</i>	<i>Type of their son</i>	<i>Type of their daughter</i>
$t_1$	$t_2$	$t_3$
$t_2$	$t_3$	$t_1$
$t_3$	$t_1$	$t_2$

We must verify that all the axioms are satisfied. Some of the axioms are easy to check, others are harder to verify. We shall prove a general theorem which will show that this rule satisfies all the axioms.

In order to give a complete treatment to this problem, we must have a simple systematic method of representing relationships. For this we use family trees, as drawn by anthropologists. The following symbols are commonly used.



In Figure 19 we draw four family trees, representing the four kinds of first-cousin relationships between a man and a woman.

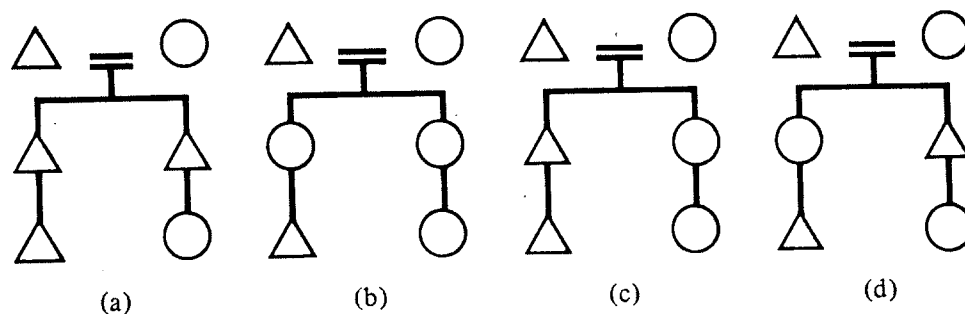


Figure 19

**EXAMPLE (continued)** Does our rule allow marriage between a man and his father's brother's daughter? This is the relationship in Figure 19(a). There are three possible types for the original couple (the grandparents), and in Figure 20, we work out the three cases. We find in each case that the man and woman are

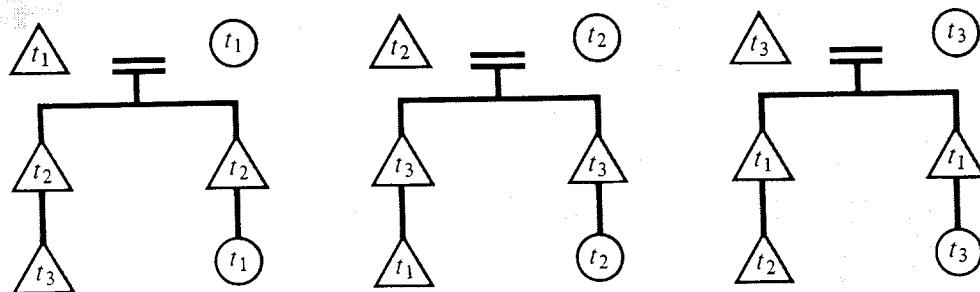


Figure 20

of different type; hence such marriages are *never* allowed. Can a man marry his mother's brother's daughter? This is the relationship in Figure 19(d). The three cases for this relationship are found in Figure 21. We find that such marriages are *always* allowed.

We are now ready to give the rules a mathematical formulation. The society chooses a number, say  $n$ , of marriage types (Axiom 1). We call these  $t_1, t_2, \dots, t_n$ . Our rule has two parts, one concerning sons, one concerning

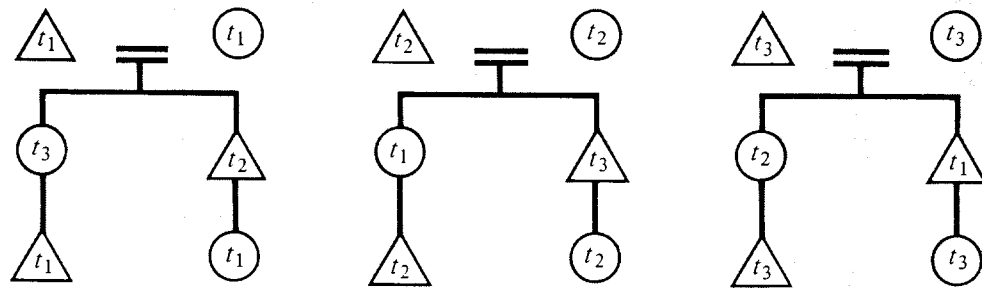


Figure 21

daughters. Let us consider the marriage type of sons. The parents must be of the same marriage type (Axiom 2). We must assign to a boy a type which depends only on the common type of his parents (Axiom 3). If his parents are of type  $t_i$ , he will be of type  $t_j$ . Furthermore, if some other boy has parents of a type different from  $t_i$ , then the boy will be of type different from  $t_j$  (Axiom 4). This defines a *permutation* of the marriage types (see Chapter 2, Section 4). The type of a son is obtained from the type of his parents by a permutation specified by the rule of the society. We shall find it convenient to represent these permutations by means of special permutation matrices.

---

**Definition** A *permutation matrix* is a square matrix having exactly one 1 in each row and each column, and having zeroes in all other entries.

---

As examples, consider the following permutation matrices:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$D = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Suppose that  $n = 3$  and  $t = (t_1, t_2, t_3)$  is the vector of marriage types and that under the son relation this type vector is sent into the vector  $(t_3, t_1, t_2)$ . We can represent this as

$$(t_1, t_2, t_3) \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = (t_3, t_1, t_2).$$

The following rule can be used for constructing the permutation matrix of a given permutation: If the permutation sends  $t_i$  into  $t_j$ , then make the entry in the  $j$ th row and  $i$ th column of the permutation matrix  $P$  equal to 1, and make all other entries in the  $i$ th column equal to 0.

It can easily be shown that every permutation matrix  $P$  is nonsingular



and also that  $P^{-1} = P'$ , where  $P'$  is the *transpose*—that is, the matrix obtained by interchanging the rows and columns of  $P$ . A proof of this fact is sketched in Exercise 3. In particular note that  $P^{-1}$  is itself a permutation matrix.

Let  $n$  be fixed and consider the set of all  $n \times n$  permutation matrices. Since there are  $n!$  permutations, it can be shown that there are  $n!$  permutation matrices. Note that the identity matrix  $I$  is in the set. We have just seen that if  $P$  is in the set, so is its inverse  $P^{-1}$ . Now consider two permutation matrices  $P$  and  $Q$ . In Exercise 4 you are asked to show that their product  $PQ$  is also a permutation matrix. Finally, if  $P$ ,  $Q$ , and  $R$  are permutation matrices, then they obey the associative law  $P(QR) = (PQ)R$ , since the operation of matrix multiplication obeys this law.

---

**Definition** A set of objects forms a *group* (with respect to multiplication) if

- (i) The product of two elements of the set is always an element of the set.
  - (ii) There is an identity element  $I$  in the set such that for every  $A$ ,  $IA = AI = A$ .
  - (iii) For every  $A$  in the set there is an element  $A^{-1}$  in the set such that  $AA^{-1} = A^{-1}A = I$ .
  - (iv) For every  $A, B, C$  in the set,  $A(BC) = (AB)C$ .
- 

By the remarks just above the definition it follows that the set of all  $n \times n$  permutation matrices forms a group.

A nonempty subset of a group that satisfies the definition of a group is called a *subgroup*. We shall be particularly interested in subgroups of the permutation group that consist of powers of a single matrix  $A$  or powers and products of powers of two permutation matrices  $A$  and  $B$ . These are called subgroups generated by  $A$ , or by the two elements  $A$  and  $B$ . (See Exercises 5 and 6.)

To return to the marriage rules, suppose that for each marriage type  $t_i$  of the parents we determine the marriage type  $t_j$  of the son. This determines a permutation of marriage types which we shall represent by an  $n \times n$  permutation matrix  $S$ . By a similar argument we arrive at the permutation matrix  $D$  giving the marriage types of daughters.

We have shown that the mathematical form of the first four axioms is to introduce the row vector  $t$  and the two permutation matrices  $S$  and  $D$ . The last three axioms restrict the choice of  $S$  and  $D$ . This will be considered in the next section.

We have repeatedly seen how the vector and matrix notation allows us to replace a series of equations by a single one. In the present problem this notation allows us to work out a given kind of relationship for *all* marriage types in a single diagram. As a matter of fact, this can be done

without knowing how many types there are in the given society, or knowing what the rules are. Let us illustrate this in terms of Figure 21. The couple at the top of the tree is of a given type, represented by our vector  $t$ . Their son is of type  $tS$ , their daughter of type  $tD$ . Then the son of a son is of type  $tSS$ , the son's daughter is of type  $tSD$ , etc. We arrive at the single vector diagram of Figure 22. If in this figure we take  $t$  to have three

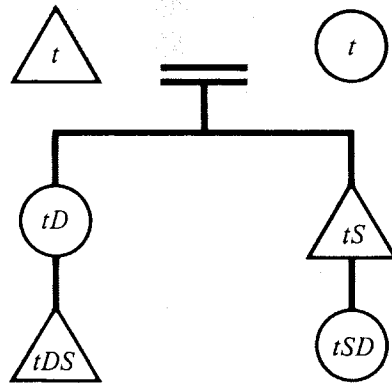


Figure 22

components, then the diagram is a shorthand for the three diagrams of Figure 21.

**EXAMPLE** Our  $t$  vector is  $(t_1, t_2, t_3)$  and  
(continued)

$$D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

We know from Figure 21 that a man is always allowed to marry his mother's brother's daughter. Can we see this in Figure 22? The marriage will always be permitted if  $tDS$  always equals  $tSD$ , which is equivalent to the matrix equation  $DS = SD$ . It so happens for our  $S$  and  $D$  that this equation is correct. But we can see more from Figure 22. No matter how many types there are, this kind of marriage will be permitted if and only if  $SD = DS$ , i.e., if the two matrices commute.

We have now seen one example of how the nature of  $S$  and  $D$  determines which kinds of relatives are allowed to marry. This question will be the subject of the next section.

## EXERCISES

1. Show that the rule for constructing a permutation matrix to represent a permutation is correct.

2. Use the rule to find the permutation matrices for the following permutations:
  - (a)  $(t_1, t_2, t_3)$  is sent into  $(t_2, t_3, t_1)$ .
  - (b)  $(t_1, t_2, t_3, t_4)$  is sent into  $(t_4, t_3, t_2, t_1)$ .
  - (c)  $(t_1, t_2)$  is sent into  $(t_1, t_2)$ .
  - (d)  $(t_1, t_2, t_3, t_4, t_5)$  is sent into  $(t_2, t_4, t_1, t_2, t_5)$ .
3. (a) Find the permutation matrix corresponding to the permutation that sends  $(t_2, t_3, t_1)$  into  $(t_1, t_2, t_3)$ .
  - (b) Show that the answer to part (a) is the inverse matrix to the answer you got in Exercise 2(a).
  - (c) Show that the answer you got in part (a) is the transposed matrix of the answer you got in 2(a).
  - (d) In general, show that if  $P$  is a permutation matrix then  $P^{-1} = P'$ .
4. Let  $P$  and  $Q$  be two permutation matrices. Use the fact that each matrix has exactly one 1 and all the rest zeroes in each row and column to prove that the product  $PQ$  is also a permutation matrix.
5. Let  $A$  be an  $n \times n$  permutation matrix and consider the powers  $A, A^2, A^3, \dots, A^k, \dots$ .
  - (a) Use the fact that there are  $n!$  permutation matrices to show that not all powers of  $A$  can be different.
  - (b) If  $A^k = A^{k+j}$ , multiply on the left by  $A^{-1}$  repeatedly to show that  $I = A^j$ .
  - (c) Show that the permutation matrices  $I, A, \dots, A^{i-1}$  form a subgroup, called the *subgroup generated by A*.
6. Let  $A$  and  $B$  be two permutation matrices. Consider the set of all products of powers of the matrices, such as  $A^3B^5AB^{-2}$ . Follow the following steps to show that this set is a group.
  - (a) Show that the product of two such matrices is again a product of powers.
  - (b) Show that the identity matrix can be written as a product of powers.
  - (c) Show that the inverse of such a matrix is again a product of powers. [Hint: E.g.,  $(A^3B^{-2})^{-1} = B^2A^{-3}$ .]
  - (d) Property (iv) is true for matrices in general.
7. In the marriage example in the text above, verify that the rule satisfies Axioms 1, 3, and 4.
8. In the example above, verify that the matrices  $S$  and  $D$  given represent the rule given.
9. Construct a diagram for the brother-sister relationship.
10. Using the diagram of Exercise 9, show that, in the above example, brother-sister marriages are never permitted.
11. Find the condition on  $S$  and  $D$  that would always allow brother-sister marriages. [Ans.  $S = D$ .]

In the *Kariera* society there are four marriage types, assigned according to the following rules:

<i>Parent type</i>	<i>Son type</i>	<i>Daughter type</i>
$t_1$	$t_3$	$t_4$
$t_2$	$t_4$	$t_3$
$t_3$	$t_1$	$t_2$
$t_4$	$t_2$	$t_1$

Exercises 12–17 refer to this society.

12. Find the  $t$ ,  $S$ , and  $D$  of the Kariera society.
13. Show that brother-sister marriages are never allowed in the Kariera society.
14. Show that  $S$  and  $D$  commute. What does this tell us about first-cousin marriages in the Kariera society?
15. Show that first cousins of the kinds in Figure 19(a) and (b) are never allowed to marry in the Kariera society.
16. Show that first cousins of the kind in Figure 19(c) are always allowed to marry in the Kariera society.
17. Find the group generated by  $S$  and  $D$  of the Kariera society.

In the *Tarau* society there are also four marriage types. A son is of the same type as his parents. A daughter's type is given by:

<i>Parent type</i>	<i>Daughter type</i>
$t_1$	$t_4$
$t_2$	$t_1$
$t_3$	$t_2$
$t_4$	$t_3$

Exercises 18–23 refer to this society.

18. Find the  $t$ ,  $S$ , and  $D$  of the Tarau society.
19. Show that brother-sister marriages are never allowed in the Tarau society.
20. Show that  $S$  and  $D$  commute. What does this tell us about first-cousin marriages in the Tarau society?
21. Show that first cousins of the kinds in Figures 19(a) and (b) are never allowed to marry in the Tarau society.
22. Show that first cousins of the kind in Figure 19(c) are never allowed to marry in the Tarau society.
23. Find the group generated by  $S$  and  $D$  of the Tarau society.

## 5 THE CHOICE OF MARRIAGE RULES

In the last section we saw that the marriage rules of a primitive society are determined by the vector  $t$  and the matrices  $S$  and  $D$ . The axioms make no mention of the number of types, and indeed, we shall find that we can have any number of types, as long as  $n > 1$ . But we shall find that the choices of  $S$  and  $D$  are severely limited. This shows that the rules of existing primitive societies required considerable ingenuity for their construction.

We must now consider the last three axioms. For Axiom 5 we need a simple way of describing a kind of relationship. The family tree is our basic tool, but we want to replace the family tree by a suitable matrix.

Let us consider Figure 22. Instead of starting with the grandparents and finding the types of the grandson and the granddaughter, we could start with the grandson, work up to the grandparents, and then down to the granddaughter. For this we must consider how we work "up." If a parent is of type  $t$ , the son is of type  $tS$ . Hence, if the son is of type  $t$ , then the parent is of type  $tS^{-1}$ . Similarly, if a daughter has type  $t$ , her parents have type  $tD^{-1}$ . In Figure 23 we find the new version of Figure 22.

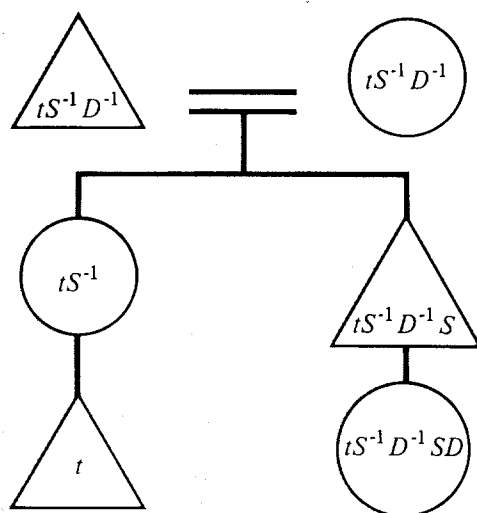


Figure 23

It is easily seen that we can follow this procedure for any relationship. Given a kind of relationship, it determines a matrix  $M$  such that if the male of the relationship is of type  $t$ , then the female is of type  $tM$ . From Figure 23 we see that for "mother's brother's daughter"  $M = S^{-1}D^{-1}SD$ . We shall speak of  $M$  as the *matrix of the relationship*. These matrices are all products of  $S$ ,  $D$ , and their inverses; hence each matrix is an element of the group generated by  $S$  and  $D$ .

Let us consider Axiom 5. Given any kind of relationship between a man and a woman, we form the matrix of the relationship  $M$ . The man will be permitted to marry this relation of his if and only if his type is the same as hers, i.e., if a certain component of  $t$  is the same as the corresponding

component of  $tM$ . This means that this component is left unchanged by the permutation  $M$ , which proves our first theorem.

---

**Theorem 1** A man is allowed to marry a female relative of a certain kind if and only if his marriage type is not changed by  $M$ .

---

We know that the permutation  $I$  leaves every element fixed. We shall also be interested in permutations that do not leave any element fixed, and we shall call them *complete permutations* (see Chapter 3, Section 4).

A second result now follows from Theorem 1.

---

**Theorem 2** Marriage between relatives of a given kind is always permitted if  $M = I$ , and is never permitted if  $M$  is a complete permutation.

---



---

**Theorem 3** Axiom 5 requires that in the group generated by  $S$  and  $D$  every element except  $I$  is a complete permutation.

---

*Proof* The axiom states that for a given relationship the marriage must always be allowed or must never be allowed. Hence the matrix of every relationship must either be  $I$  or a complete permutation matrix. The matrices are elements of the group generated by  $S$  and  $D$ . And given any element of this group, which can be written as a product of  $S$ 's and  $D$ 's, we can draw a family tree having this matrix. Hence the matrices of relationships are all the elements of the group. This means that all the elements of the group, other than the identity, must be complete permutations. This completes the proof.

---

**Theorem 4** Axiom 6 requires that  $S^{-1}D$  be a complete permutation.

This theorem is an immediate consequence of the fact that the matrix of the brother-sister relationship is  $S^{-1}D$ .

---



---

**Theorem 5** Axiom 7 requires that for every  $i$  and  $j$  there be a permutation in the group which carries  $t_i$  into  $t_j$ .

---

*Proof* Let us choose two individuals, one of type  $t_i$  and one of type  $t_j$ . There must be a descendant of the former who can marry a descendant of the latter. Hence the two descendants must have the same type. This means that we have permutations  $M_1$  and  $M_2$  such that  $t_i$  is carried by  $M_1$  into the same type as  $t_j$  by  $M_2$ . Then  $M_1M_2^{-1}$  carries  $t_i$  into  $t_j$ . Hence the theorem follows.

We have now translated Axioms 5-7 into the following three conditions on  $S$  and  $D$ : (1) The group generated by  $S$  and  $D$  consists of  $I$  and of complete permutations. (2)  $S^{-1}D$  is a complete permutation. (3) For every pair of types there is a permutation in the group that carries one type into the other.

---

**Definition** A permutation group is called *regular* if (a) it is complete, i.e., every element of the group other than  $I$  is a complete permutation, and if (b) for every pair from among the  $n$  objects there is a permutation in the group that carries one into the other.

---



---

**Basic Theorem** To satisfy the axioms we must choose two different  $n \times n$  permutation matrices  $S$  and  $D$  which generate a regular permutation group.

---

**Proof** Conditions (1) and (3) above state precisely that the group generated by  $S$  and  $D$  be regular. In a regular group every element other than  $I$  is a complete permutation; hence condition (2) requires only that  $S^{-1}D$  be different from  $I$ . Since  $S^{-1}D = I$  is equivalent to  $D = S$ , we need only require that  $D \neq S$ . This completes the proof.

It is important to be able to recognize regular permutation groups. Here we are helped by a very simple, well-known theorem: A subgroup of the group of permutations of degree  $n$  is regular if and only if it has  $n$  elements and is complete.

This leads to a relatively simple procedure. We choose  $n$ . Then we must pick a group of  $n \times n$  permutation matrices which has  $n$  elements and is complete, and select two different elements which generate the group. This is always possible if  $n > 1$  (see Exercise 11). One of these is chosen as  $S$  and one as  $D$ . Since there are not very many regular permutation groups for any  $n$ , the choice is very limited.

**EXAMPLE** Let us find all possibilities for a society having four marriage types. First of all we must find the regular subgroups of the symmetric group of degree 4, i.e., the groups of permutations on four objects that have four elements and are complete.

Among these we find cyclic groups. Any two of these groups have the same structure and hence lead to equivalent rules. Let us suppose that we choose the permutation group generated by

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The group consists of  $P$ ,  $P^2$ ,  $P^3$ , and  $I$ . Either  $P$  or  $P^3$  generates the group, and they play analogous roles. We may therefore assume that  $P$  is one of the two permutations chosen. This allows us  $(P, P^2)$ ,  $(P, P^3)$ , and  $(P, I)$  as possibilities. We must still ask which is  $S$  and which is  $D$ . In the second case it makes no difference, since  $P$  and  $P^3$  play analogous roles in the group, but there is a difference in the other two cases. This leads to five possibilities:

1.  $S = P, \quad D = P^2$
2.  $S = P^2, \quad D = P$
3.  $S = P, \quad D = P^3$
4.  $S = P, \quad D = I$
5.  $S = I, \quad D = P$ . This is the Tarau society; see Exercises 18–23 of Section 4.

There is only one noncyclic complete subgroup with four elements, consisting of  $I$  and the three permutations which interchange two pairs of elements. In this group we have essentially only one case, since all three permutations play the same role.

6. The Kariera society. (See Exercises 12–17 after the last section.)

Two of these six possibilities are actually exemplified in known primitive societies.

## EXERCISES

1. Figure 23 shows the matrix of one of the first-cousin relations. Find the matrices of the other three first-cousin relationships.
2. Prove that marriage between relations of a certain kind is permitted if and only if the matrix of the relation is  $I$ .
3. Use the result of Exercise 2 to prove that no society allows the marriage between cousins of the types in Figure 19(a) and (b).
4. Which of the six rules described above (in the example) allow marriage between a man and his father's sister's daughter? [Ans. 3, 6.]
5. Show that all six rules given in the example above allow marriages between a man and his mother's brother's daughter.
6. There are eight kinds of second-cousin relationships between a man and a woman. Draw their family trees.
7. Find the matrices of the eight second-cousin relationships.
8. Are there any second-cousin relationships for which marriage is forbidden by all possible rules? [Ans. Yes.]
9. Test the second-cousin relationships (other than those found in Exercise 8) for each of the six rules given in the example above.
10. For  $n$  objects, consider the permutation that carries object number  $i$  into position  $i + 1$ , except that the last object is put into first place. Show that the cyclic group generated by this permutation is regular.
11. Use the result of Exercise 10 to show that a society can have any number of marriage types, as long as the number is greater than one.
12. In the example of Section 4, prove that  $S$  and  $D$  generate a regular permutation group.
13. Prove that the following matrices lead to a rule satisfying all axioms.



$$S = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

14. Prove that the rule given in Exercise 13 allows no first-cousin marriages.

## 6 EXAMPLES FROM ECONOMICS AND FINANCE

Markov-chain models are probabilistic in nature. However, it turns out that sometimes the mathematics used to analyze them can also be used in analyzing deterministic models. In the present section we shall discuss two deterministic models that can be solved by imbedding them in absorbing Markov chains.

In what we are going to do the question of whether a given Markov chain is absorbing is crucial. Hence we need an algorithm for determining the answer to this question. Such an algorithm is given in Figure 24. The proof that the algorithm works is given in Exercises 1 and 2.

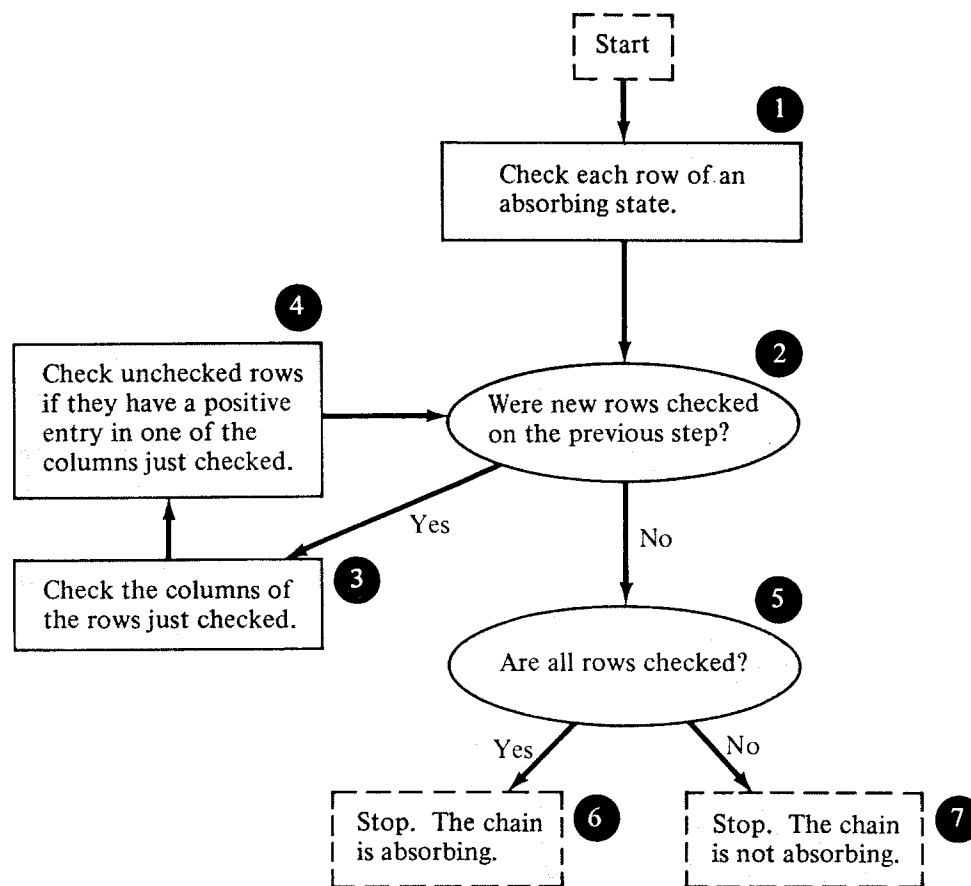


Figure 24

Flow diagram for testing to see whether a Markov chain is absorbing

**EXAMPLE 1** Apply the method in Figure 24 to the transition matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We carry out the checking process indicated by the algorithm, marking the checks in the order in which they were made. We obtain

$$\begin{array}{l} \phantom{1\checkmark} \phantom{3\checkmark} \phantom{2\checkmark} \phantom{3\checkmark} \\ 1\checkmark \phantom{3\checkmark} \phantom{2\checkmark} \phantom{3\checkmark} \\ 3\checkmark \phantom{2\checkmark} \phantom{3\checkmark} \\ 2\checkmark \phantom{3\checkmark} \\ 3\checkmark \end{array} \begin{pmatrix} \frac{1}{\checkmark} & \frac{3}{\checkmark} & \frac{2}{\checkmark} & \frac{3}{\checkmark} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Since all rows are checked, the Markov chain is absorbing. The reader should find the paths from each state to the absorbing state. The numbers of the checks will help in this regard (see Exercise 3).

**EXAMPLE 2** Let us apply the method of Figure 24 to the transition matrix

$$P' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{3}{4} & 0 & \frac{1}{4} \end{pmatrix}.$$

The checks produced by the algorithm are

$$\begin{array}{l} \phantom{1\checkmark} \phantom{2\checkmark} \\ 1\checkmark \\ 2\checkmark \end{array} \begin{pmatrix} \frac{1}{\checkmark} & \phantom{\frac{3}{\checkmark}} & \phantom{\frac{2}{\checkmark}} & \phantom{\frac{3}{\checkmark}} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{3}{4} & 0 & \frac{1}{4} \end{pmatrix}.$$

Here, since not all the rows are checked, the Markov chain whose transition matrix is  $P'$  is not absorbing. The reader should check that the states  $\{2, 4\}$  form a closed set, in the sense that it is impossible to leave them once entered. In particular it is impossible to go from either of these states to the absorbing state (1), so that the chain is not absorbing.

We now consider our first important applied example.

**EXAMPLE 3** *The Open Leontief Model.* Consider an economy with  $r$  industries and each industry produces just one kind of good. These industries are interconnected in the sense that each must buy a nonnegative amount of the other industries' products in order to operate. Let  $q_{ij}$  be the dollar amount of the output of industry  $j$  that must be purchased by industry  $i$  in order to produce \$1

of its own goods. Let  $Q$  be the  $r \times r$  matrix with entries  $q_{ij}$ . By definition we have

$$(1) \quad Q \geq 0.$$

It is not hard to see that for fixed  $i$  the sum  $q_{i1} + \dots + q_{ir}$  gives the total cost of the inputs needed by industry  $i$  in order to produce \$1 worth of output. Clearly it makes sense to require that  $q_{i1} + \dots + q_{ir} \leq 1$ ; that is, the total value of the inputs going into a dollar's worth of output must be less than or equal to a dollar. For obvious reasons we shall call the  $i$ th industry *profitable* if the strict inequality holds, and *profitless* if the equality holds. In order to rule out unprofitable industries we require

$$(2) \quad Qf \leq f,$$

where  $f$  is the  $r$ -component column vector of all 1's.

Suppose now that the total economy is to be run so that a vector  $d = (d_1, \dots, d_r)$  of goods can be supplied for consumption. Here  $d_j$  is the amount of good  $j$  to be consumed. At what levels shall we run each industry in order to supply total demand? Let  $x_i$  be the level at which industry  $i$  is to be run. To make economic sense,  $x_i \geq 0$ . Then  $x = (x_1, \dots, x_r)$  is the *activity vector* for the industries, and  $x \geq 0$ . The  $j$ th component of  $xQ$  is  $x_1q_{1j} + \dots + x_rq_{rj}$ , and this is the total output of industry  $j$  demanded by all the other industries when the economy uses the activity vector  $x$ . In the same manner one can see that  $xQ$  is the vector of internal demands when the economy uses the activity vector  $x$ .

Now the vector  $x$  must be chosen so as to provide the sum of the internal plus the external demand. That is,

$$(3) \quad x = xQ + d.$$

Equation (3) implies

$$(4) \quad x(I - Q) = d.$$

If the matrix  $(I - Q)$  has a nonnegative inverse, we can solve for (4) as

$$(5) \quad x = d(I - Q)^{-1},$$

and the result will be economically meaningful. If this inverse does not exist or if it has negative entries, then there will be some kinds of outside demand vectors  $d$  for which there is no nonnegative solution activity vector  $x$ .

Thus we need conditions on the matrix  $Q$  that  $(I - Q)$  have a nonnegative inverse. Here we apply the theory of absorbing Markov chains. We imbed the matrix  $Q$  into a Markov chain  $P$  having  $r + 1$  states as follows:

$$(6) \quad P = \left( \begin{array}{c|c} 1 & 0 \\ \hline R & Q \end{array} \right),$$

where the first state is absorbing, and  $R$  is an  $r \times 1$  matrix with components  $r_i = 1 - (q_{i1} + \dots + q_{ir})$  for  $i = 1, \dots, r$ . (We label the first state 0.) We now apply the flow diagram of Figure 24, but marking only state 0 when

in box 1 of the flow diagram. If we end up in box 6, we know that  $P$  is an absorbing Markov chain with fundamental matrix  $N = (I - Q)^{-1}$ . And the fundamental matrix is nonnegative. If we end up in box 7 of Figure 24, it can be shown (see Exercise 5) that  $I - Q$  is singular and has no inverse. Hence in this case there is no economic solution to the economy as it stands.

We summarize our results in a theorem:

---

**Theorem** Let  $Q$  be the input matrix of an open Leontief economy satisfying (1) and (2); then equation (3) can be solved for all demand vectors  $d$  if and only if the Markov chain  $P$  of (6) is an absorbing chain.

---

**EXAMPLE 3** As an example, suppose that there are three industries and  
(continued)

$$(7) \quad Q = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & 0 & \frac{1}{2} \end{pmatrix}.$$

Then the Markov chain  $P$  is given by

$$(8) \quad P = \left( \begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{6} & 0 & \frac{1}{2} \end{array} \right)$$

Applying the algorithm of Figure 24, we easily find that the Markov chain is absorbing. Hence  $(I - Q)^{-1}$  exists. It is

$$(9) \quad (I - Q)^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{6} & 0 & \frac{1}{2} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{3}{2} & \frac{9}{8} & \frac{3}{4} \\ 1 & \frac{9}{4} & \frac{3}{2} \\ \frac{1}{2} & \frac{3}{8} & \frac{9}{4} \end{pmatrix}$$

Thus for a demand vector  $d = (400, 200, 300)$  we have activity vector

$$(10) \quad x = d(I - Q)^{-1} = (400, 200, 300) \begin{pmatrix} \frac{3}{2} & \frac{9}{8} & \frac{3}{4} \\ 1 & \frac{9}{4} & \frac{3}{2} \\ \frac{1}{2} & \frac{3}{8} & \frac{9}{4} \end{pmatrix} \\ = (950, 1012.5, 1275),$$

so that \$950 worth of output must be produced by industry 1, \$1012.50 worth of output by industry 2, and \$1275 worth of output by industry 3 in order that consumptions of \$400, \$200, and \$300 of each (respective) industry's output may be realized.

**EXAMPLE 4** *A Cost-Accounting Model.* Consider a company that has  $r$  departments. It has adopted accounting conventions such that if department  $i$  performs services for department  $j$  then it charges a fraction  $q_{ij}$  of its costs to department  $j$ . It requires  $q_{ii} = 0$ , since it does not make sense for a department

to charge costs to itself. It also requires  $q_{ij} \geq 0$ , so that (1) holds. No department is permitted to charge more than 100 percent of its costs to other departments, so that  $q_{i1} + \dots + q_{ir} \leq 1$ , and (2) holds as well. Departments for which the equality holds are called *service departments*, since they charge away all of their costs, and departments for which the inequality holds are *profit centers*, since they actually pay some of their costs. Let  $d_i$  be the dollar amount of external costs charged to department  $i$  by outside firms, and let  $d = (d_1, \dots, d_r)$  be the corresponding vector. Finally, let  $x_i$  be the total costs assigned to department  $i$  and let  $x = (x_1, \dots, x_r)$  be the corresponding vector.

Since the costs assigned to a department must be the sum of the internally charged costs plus the external costs, it is easy to see by the same kind of analysis as for the Leontief model that equation (3) must be satisfied by the  $x$ 's. Therefore we have the same problem as before of determining whether the inverse of  $(I - Q)$  exists. The same solution technique of imbedding the matrix  $Q$  in a Markov chain and using Figure 24 to see if it is absorbing provides the solution technique.

It is remarkable that from two quite different interpretations in the Leontief input-output model and the cost-accounting model we have arrived at exactly the same mathematical model and corresponding solution technique.

**EXAMPLE 4** As a numerical example, consider the matrix  
(continued)

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}.$$

Here the first two departments are profit centers, since they do not charge any of their costs to other departments of the company. The last three departments are service centers, since they charge off all their costs to other departments. It is easy to show that the Markov chain  $P$  obtained by (6) is absorbing, hence  $(I - Q)^{-1}$  exists. A computer gave the following values for this inverse:

$$\begin{aligned} (I - Q)^{-1} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 1 & -\frac{1}{4} \\ 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ .7547 & .2453 & 1.2453 & .5283 & .3396 \\ .5472 & .4528 & .4528 & 1.2830 & .3962 \\ .4340 & .5660 & .5660 & .6038 & 1.2453 \end{pmatrix}. \end{aligned}$$

If we assume that  $d_i = 10,000$  for  $i = 1, \dots, 5$ —that is, each department incurs outside expenses of \$10,000—then from (5) we find that

$$x = (27,359 \quad 22,641 \quad 22,641 \quad 24,151 \quad 19,811).$$

Notice that the sum of the costs of the first two departments, which are profit centers, is  $\$27,359 + \$22,641 = \$50,000$ . In other words, the profit centers end up paying all the outside costs. This is always true (see Exercise 7).

## EXERCISES

- In Figure 24 show that each time we check a new row when we are in box 4 of the flow diagram, there is a way of getting from state  $i$  to an absorbing state. Hence show that if we end up in box 6, the chain is absorbing.
- In Figure 24 show that if we end up in box 7, there is at least one state that cannot reach an absorbing state.
- In Example 1 show that if we number each check as it is made, then the numbers on the rows are equal to 1 plus the minimum number of steps required to go from that state to an absorbing state.
- Apply the flow diagram of Figure 24 to the following examples of Markov chains to see if they are absorbing.

$$(a) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (b) \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

$$(c) \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad (d) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix}$$

- Assume that we started with an  $r \times r$  matrix  $Q$  satisfying (1) and (2), defined the Markov chain  $P$  as in (6), and applied the flow diagram of Figure 24, ending up in box 7.
  - Let  $h_i = 0$  when  $i$  is a checked row and  $h_i = 1$  when  $i$  is an unchecked row. Let  $h$  be the  $r$ -component column vector with components  $h_i$ . Show that  $h \neq 0$  and  $h \geq 0$ .
  - Show that  $Qh \geq h$ , and hence that  $(I - Q)h \leq 0$ .
  - If  $(I - Q)^{-1}$  exists, use (b) to show that  $h \leq 0$ .
  - Use (a) and (c) to prove that  $(I - Q)^{-1}$  does not exist.
- Let  $Q$  and  $d$  be as given below for Leontief input-output models. When possible, solve for  $x$  as in (5).

$$(a) \quad Q = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}, \quad d = (5000, 8000, 2000).$$

$$(b) \quad Q = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}, \quad d = (100, 500, 300).$$

$$(c) \quad Q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}, \quad d = (1000, 2000, 3000).$$

7. In the cost-accounting model of Example 4, show that no service center can pay any outside cost. Use this to show that the profit centers must ultimately pay all outside costs.
8. Let  $Q$  and  $d$  be as given below for cost-accounting models. When possible, solve for  $x$  as in (5).

$$(a) \quad Q = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{1}{3} \\ 1 & 0 & 0 \end{pmatrix}, \quad d = (200, 500, 700).$$

$$(b) \quad Q = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix}, \quad d = (3000, 7000, 5000).$$

$$(c) \quad Q = \begin{pmatrix} 0 & \frac{1}{5} & \frac{3}{5} \\ \frac{1}{6} & 0 & \frac{1}{3} \\ \frac{1}{6} & \frac{2}{3} & 0 \end{pmatrix}, \quad d = (1000, 1000, 1000).$$

## 7 OPTIMAL HARVESTING OF DEER

Linear programming models have frequently been used to solve production problems in industry such as determining optimal production schedules and making oil refinery operating decisions. More recently such models have been applied to the solution of local, state, and federal governmental decision problems. We illustrate here the latter type.

Consider a county located in an eastern state that has a large proportion of forested areas, and hence a large deer population. Every year the county forest service has to set the length of both the *regular deer season*, when only mature males may be killed, and the *unrestricted deer season*, during which any deer except a fawn may be killed. In making these decisions the service has conflicting objectives. If it makes the hunting seasons long it will attract many hunters, who will spend large amounts of money in the county and thus please local businessmen. But if the season is too long, then too many deer will be taken in a given year and the stock will be depleted for the next year's hunting. The forest service must also take into account the local property owners whose land is walked over by the hunters. On the one hand, the owners would like to keep the deer population low, since during the winter the deer eat the new growth on small trees, thus killing the trees. But the property owners also dislike the idea of hunters walking in the woods shooting at moving targets.

Still other considerations are involved in the presence of the deer and

the hunting season. For instance, every year in the county there are numerous accidents involving cars hitting deer on the roads. Also, during the hunting season there are numerous hunting accidents. Finally, if the deer population is permitted to become too large, then many will starve because of a limited food supply.

Every year the forest service makes a survey of the deer population and estimates the numbers of (a) *mature males* and (b) *other deer* (mature females and yearlings and fawns of both sexes). It then weighs the above factors and sets the number of days for the regular hunting season and for the unrestricted hunting season. Let us set up a linear programming model for the decision problem.

Suppose that for each day of the regular deer-hunting season 150 mature male deer are killed and hunters spend \$18,000 in the county for hunting licenses, motels, meals, gasoline, hunting equipment, etc. Also suppose that for each day of the unrestricted hunting season 85 mature males and 400 other kinds of deer are killed and hunters spend \$20,000. Let us also assume that during a year a mature male will cause \$5 worth of damage to young trees and that other kinds of deer will cause \$2 worth of damage.

The annual deer survey has revealed that there are 12,000 mature males and 33,600 other kinds of deer within the county limits. The service wants to leave at least 10,000 mature males and at least 27,000 of other kinds of deer as breeding stock for next year's season. Also, the service has set an upper limit of 5 days for the unrestricted hunting season because of pressure from a local ecological society. How shall it determine the number of days for each season in order to maximize the difference between the revenues to local businessmen and the damage done to the woodlands by the deer?

To set the problem up as a linear programming problem we first define four variables:

- $x_1$  is the number of days of the regular hunting season.
- $x_2$  is the number of days of the unrestricted hunting season.
- $x_3$  is the number of mature males remaining after both seasons.
- $x_4$  is the number of other deer remaining after both seasons.

We then set up the initial tableau of the linear program as in Figure 25. Let us check whether this captures the essence of the problem described above. The third constraint is just the upper bound on the unrestricted

$x_1$	$x_2$	$x_3$	$x_4$	
150	85	1	0	= 12,000
0	400	0	1	= 33,600
0	1	0	0	≤ 5
0	0	1	0	≥ 10,000
0	0	0	1	≥ 27,000
18,000	20,000	-5	-2	

Figure 25



hunting season mentioned above. The fourth and fifth  $\geq$  constraints are the lower bounds on the remaining mature males and other kinds of deer at the end of both hunting seasons. The first constraint is a kind of "accounting equation" that accounts for the total number of mature males (12,000) as those killed in each of the seasons plus the number remaining ( $x_3$ ). The second constraint similarly accounts for the number of other deer.

The linear programming problem of Figure 25 is so simple that we can solve it by inspection (see Exercises 2 and 3). The optimal primal and dual solutions are on the top and to the left of the tableau in Figure 26. The reader should check that the indicated primal and dual solutions are feasible and produce the same objective function values. By an extension of the duality theorem to the case of mixed constraints this will guarantee that the solutions are optimal (see Exercise 1).

	10.5	5	10,000	31,600	
120	150	85	1	0	= 12,000
-2	0	400	0	1	= 33,600
10,600	0	1	0	0	$\leq 5$
-125	0	0	1	0	$\geq 10,000$
0	0	0	0	1	$\geq 27,000$
	18,000	20,000	-5	-2	\$175,800

Figure 26

The answer indicates that there should be a regular hunting season of 10.5 days, which could be accomplished by starting it at noon on a given day and ending at sundown 10 days later. Also, there should be an unrestricted hunting season of 5 days. At the end of the seasons there will be 10,000 mature males and 31,600 other kinds of deer remaining to provide hunting for the next season. The total net income to the county is \$175,800.

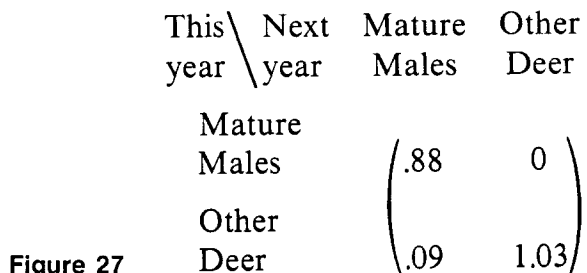
The dual solution variables provide very interesting information. It is easy to see that the first dual variable  $v_1$  indicates that a mature male deer has an imputed value of \$120. And the third dual variable  $v_3$  indicates that the imputed value of each day of the unrestricted hunting season is \$10,600. The interpretations of the other dual variables are indicated in Exercise 4.

Notice that in the solution indicated in Figure 26 there are just 10,000 mature males remaining at the end of the season (the lower bound), while there are 31,600 other kinds of deer remaining, which is a considerably greater number than the lower bound of 27,000. Perhaps this bound should be raised.

But there is a still more serious criticism of the solution and the problem formulation. Namely, it does not take into account the dynamic factors in the changes of the deer population from one year to the next. Thus we know that there will be 10,000 mature males and 31,600 other kinds of deer at the end of the hunting season, but between that date and the beginning of next year's hunting season there will be changes in these numbers due

to births, deaths, and aging of the deer population. We extend the model to include these factors next.

As part of its annual survey the forest service estimates that about 1 out of 8 mature males dies from natural causes such as old age, disease, predators, etc., during the entire year. About 9 percent of the other deer population become mature males during the year. And the other deer population increases by about 3 percent each year, even after removing from it the yearling males that mature during the year. These changes are summarized in the matrix of Figure 27.



We can use the data of Figure 27 to make predictions as to the numbers of mature males and other deer who will be alive next year, given the number alive at the end of the present hunting season.

We now set up a two-year optimal harvesting linear program as shown in Figure 28. Variables  $x_1, x_2, x_3,$  and  $x_4$  are defined as before for the first

$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	
150	85	1	0	0	0	0	0	= 12,000
0	400	0	1	0	0	0	0	= 33,600
0	1	0	0	0	0	0	0	≤ 5
0	0	1	0	0	0	0	0	≥ 10,000
0	0	0	1	0	0	0	0	≥ 27,000
0	0	-.88	-.09	150	85	1	0	= 0
0	0	0	-1.03	0	400	0	1	= 0
0	0	0	0	0	1	0	0	≤ 5
0	0	0	0	0	0	1	0	≥ 10,000
0	0	0	0	0	0	0	1	≥ 27,000
1	0	0	0	-1	0	0	0	= 0
18,000	20,000	-5	-2	18,000	20,000	-5	-2	

Figure 28

year, and variables  $x_5, x_6, x_7,$  and  $x_8$  are the corresponding variables for the second year. Similarly, the first five constraints indicate restrictions for the first year just like those in Figure 25, and the next five constraints indicate the corresponding constraints for the second year. Note that in columns 3 and 4 of constraints 6 and 7 the numbers of Figure 27 appear but with negative signs, and with the matrix transposed. Also, the right-hand sides

of constraints 6 and 7 are zeroes. In effect, what the minus signs on the coefficients do is to make predictions from the values of  $x_3$  and  $x_4$  as to the numbers of each of the kinds of deer that will be alive for the next season. In Exercise 5 you are asked to give a detailed explanation of these constraints.

The final constraint forces  $x_1 = x_5$ , i.e., the same number of regular hunting days both years. Unless the county follows this policy, it will receive many complaints.

The optimal primal solution to the problem in Figure 28 is

$$x = (9.24 \quad 5 \quad 10,189 \quad 31,600 \quad 9.24 \quad 5 \quad 10,000 \quad 30,548).$$

Note that the solution for the first year has changed. The number of regular hunting days is reduced from 10.5 to 9.24. As a result, more than 10,000 mature males are left (see  $x_3$ ). Also, we note from the values of  $x_4$  and  $x_8$  that the numbers of other kinds of deer are decreasing. Thus it is likely that the number of unrestricted hunting days should be reduced from 5 to 4; this would permit a somewhat longer regular hunting season each year.

The transpose of the optimal dual solution (rounded) is

$$v^{\text{TR}} = (110 \quad 7.7 \quad 7,610 \quad 0 \quad 0 \quad 130 \quad -2 \quad 9,723 \quad -135 \quad 0 \quad 1,548).$$

Here the last five dual variables are the same as in Figure 26, but the first five are different. Exercise 7 asks for interpretations of these variables. The value to the county over two years is \$307,310. This is less than twice the value we found for the first year (see Exercise 8).

It is obvious that this model could be extended to cover three years or even longer, which might be particularly useful when planning long-range changes in the deer population.

Although the above model is an extremely simplified version of a governmental decision-making problem, it brings out some of the essential features: in making its decision the public agency (in this case the forest service) has to consider the benefits (in this case added revenues to the county) versus the costs (the negative features of the deer hunting). We have illustrated one kind of cost-benefit calculation commonly carried out by the decision makers. It should be emphasized that there are many other kinds of cost-benefit analyses that are used.

## EXERCISES

1. (a) Check that the solution for  $x_1, x_2, x_3,$  and  $x_4$  shown in Figure 26 is feasible.
- (b) Check that the dual solution  $v_1, v_2, v_3, v_4,$  and  $v_5$  shown in Figure 26 is feasible.
- (c) Show that both solutions—in (a) and (b)—give the same objective value, and that nonnegative  $v$ -values correspond to  $\leq$  constraints, and nonpositive  $v$ -values to  $\geq$  constraints. (Either may occur

- for = constraints.) An extension of the duality theorem will then guarantee that the solutions are optimal.
2. Solve the problem in Figure 25 by carrying out the following steps:
    - (a) Show that the most attractive variable to increase from 0 is  $x_2$ . Show that it can be set equal to its upper bound of 5 without destroying feasibility.
    - (b) Show that if  $x_2 = 5$ , then  $x_4 = 31,600$ .
    - (c) Show that  $x_3$  may be set equal to 10,000, thus satisfying constraint 4.
    - (d) With  $x_2 = 5$  and  $x_3 = 10,000$ , show that  $x_1 = 10.5$ .
  3. By carrying out a series of steps similar to those outlined in Exercise 2, solve for the dual solution to the problem in Figure 25.
  4. For each of the dual variables in Figure 26 find their physical dimensions and give their economic interpretations.
  5. Consider the problem in Figure 28. Assume that  $x_3$  and  $x_4$  have been determined and write the equations corresponding to constraints 5 and 6. Move the terms involving  $x_3$  and  $x_4$  to the right-hand side and show that these terms give the predictions as to the numbers of mature males and other deer that will be alive at the beginning of the second year's hunting seasons.
  6. Check that the  $x$ - and  $v$ -vectors given are feasible for the problem in Figure 28. Show that they are optimal.
  7. Find the dimensions of the optimal dual variables for the problem in Figure 28 and give their economic interpretations.
  8. Explain why the value in the two-year model is less than twice the value for the single-year model.

## 8 KNAPSACK AND SEQUENCING PROBLEMS

In Section 7 of Chapter 2 we discussed combinatorial decision problems. We continue the discussion here and introduce the so-called *branch-and-bound method* for solving such problems.

Recall that a combinatorial decision problem is one of choosing the most desirable element from a finite set of possibilities. The branch-and-bound technique is a way of solving such problems that (usually) does not involve enumerating all the elements of the finite possibility set. We shall not try to give a formal description of branch-and-bound methods, but merely illustrate them in terms of examples.

**EXAMPLE 1** *The Knapsack Problem.* You are going on a camping trip and are considering taking five different items with you. To help decide which ones to take, you have attached a *value* to each item indicating how strongly you want to take that item. These values and the (physical) weights of the various items are listed in the table of Figure 29. The last column of the table lists the items' value-to-weight ratios. Notice that the items are listed in decreas-

Item	Value	Weight	Value/Weight
1	72	30 pounds	2.4
2	65	28 pounds	2.32
3	52	23 pounds	2.26
4	38	19 pounds	2.00
5	30	16 pounds	1.88

Figure 29

ing order of this ratio. Thus the first item has the greatest value per pound, item 2 has the next greatest value per pound, and so on.

Adding up all the items you find (to your horror) that the total weight is 116 pounds. Since you are willing to carry at most 65 pounds you must choose a subset of the items to take with you. You quickly calculate that there  $2^5 = 32$  subsets to consider. How can you find the best one?

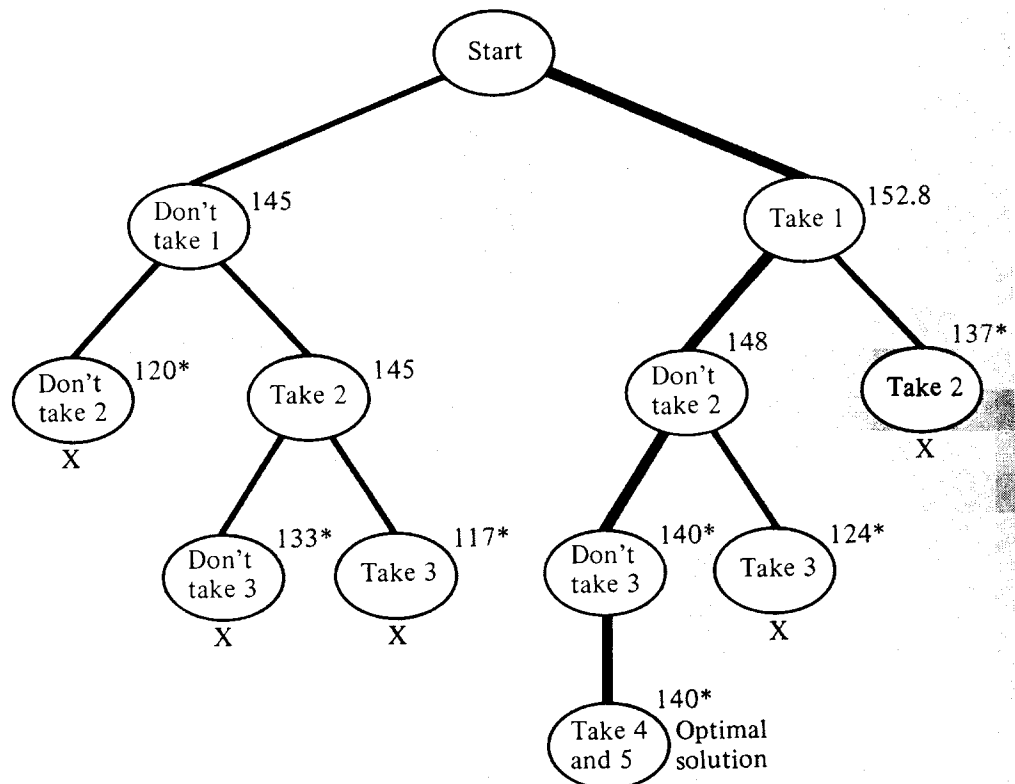
Here the combinatorial decision problem is clear. A proper subset is *feasible* if its total weight is less than or equal to 65 pounds. The *value of the subset* is the sum of the values of its elements. We want to select the feasible subset with largest value.

This problem, which finds numerous applications in many other forms, has been dubbed a *knapsack problem* from the above rather frivolous illustration.

One way of solving the problem is to list all proper subsets, eliminate those that are not feasible, calculate the values of those that are feasible, and select the one whose value is greatest. Clearly this is a lengthy procedure; it would be even longer if there were, say, 20 items under consideration, for then there would be over a million proper subsets to generate!

The branch-and-bound method is a way of finding the best subset without (usually) having to enumerate all possible subsets. In Figure 30 we have drawn a tree diagram of the computation of the branch-and-bound solution to our problem. Each node of the tree below "Start" represents a decision either to take or not to take a given item. Any path going from "Start" to a given decision node represents a series of decisions. The first node on the path indicates whether or not item 1 is to be included; the second node on the path indicates whether or not item 2 is to be included, etc. Above and to the right of each decision node is a number representing the *upper bound* to the maximum value of any subset which can be chosen so as to agree with the decisions on the path from "Start" to that decision node.

Let us describe how the upper bounds are calculated. For instance, consider the decision "Take 1" (just below and to the right of "Start"). Having made this decision, we know that we have achieved a value of 72 plus whatever else can be obtained from later decisions. We now go down the list of possible items in decreasing order of value-to-weight ratio and assume that we can take them, including (if possible) a fractional part of the last item. In the present case, in addition to 1 we can take all of item



\* Means a feasible solution was used to get bound.  
 X Means that the upper bound is less than a feasible value already achieved or the tree cannot be continued.

Figure 30

2; these two items add up to 58 pounds, leaving 7 more pounds, so we take  $\frac{7}{23}$  of item 3. Thus our upper bound is  $72 + 65 + 15.8 = 152.8$  for the decision. Of course, this value can't be achieved in practice because (we assume that) we can't physically take  $\frac{7}{23}$  of item 3. But no subset that includes item 1 can have a greater value (see Exercise 1).

Upper bounds marked with an asterisk indicate that they were calculated using a feasible solution. For instance, the "Take 2" decision branching out from the "Take 1" decision indicates that we have decided to take items 1 and 2, whose weight totals 58 pounds, and to stop there. (Since each of the remaining items weighs more than 7 pounds, none of them can be included in a feasible solution.) Hence the asterisk shows that the set of decisions "Take 1" and "Take 2"—and no other items—is feasible, and the figure indicates an upper bound of 137.

Finally, note that in Figure 30 an X appears below some of the decisions. What this means is that the tree does not need to be continued further because the upper bound is less than a bound already achieved. Thus the branch ending in the "Don't Take 2" node need not be continued further, since its upper bound of 120 is lower than the upper bound of 137 already calculated.

Thus the optimal decision is indicated by the decision path:

Start—Take 1—Don't Take 2—Don't Take 3—Take 4 and 5

which is darkened in Figure 30. This means that you should take just items 1, 4, and 5 on the trip, giving a total weight of 65 and the maximum value of 140.

Examination of Figure 30 indicates that the branch-and-bound process actually evaluated only 6 of the 30 possible proper subsets. The other possible subsets have been *implicitly* evaluated by means of the bounding rules. Thus the branch-and-bound procedure requires considerably less work to find the solution than does the complete enumeration method.

In the previous problem we were trying to find the most valuable subset of a five-element set. In the sequencing problem to be described next we have the problem of finding the least costly permutation of a set of four elements. Again the branch-and-bound procedure will allow us to find the solution with less work than by enumerating all the possible permutations.

**EXAMPLE 2** *A Sequencing Problem.* A manufacturer wants to make four different items on a single machine. If the machine has just completed item  $i$  and is to be used for item  $j$  next, a setup cost  $c_{ij}$  is incurred (the area around the machine must be cleaned up, a machinist must reset the machine to produce item  $j$ , etc.). The matrix of setup costs is as indicated in Figure 31. (The

		To			
		1	2	3	4
From	1	—	10	18	6
	2	10	—	5	15
	3	18	8	—	13
	4	6	12	9	—

Figure 31

costs  $c_{ii}$ , are indicated by dashes and not defined, since a given job will never be assigned to follow itself.) Notice that sometimes the setup cost of going from item  $i$  to item  $j$  is the same as the setup cost of going from  $j$  to  $i$ —for instance,  $c_{k1} = c_{1k}$  for  $k = 2, 3, 4$ . But in other cases this is not so—for instance,  $c_{23} = 5$  while  $c_{32} = 8$ .

The combinatorial decision problem is: In what order shall the manufacturer sequence the jobs to be produced on the machine in order to minimize the total setup time?

Since there are four items there are  $4! = 24$  possible sequences. We could, of course, enumerate each of these and choose one that minimizes total setup cost. This is practical with four items, but with, say, ten items there would be  $10! = 3,628,800$  permutations to enumerate, which would be possible only with a very fast computer. And for 20 items the enumeration of all  $20!$  permutations is even beyond the capability of contemporary computers.

The branch-and-bound method can be applied to solve the problem,

however—at least for a moderate number of items. For the example of Figure 31, the branch-and-bound tree is shown in Figure 32. Notice that on the first step the node of the tree has four branches (corresponding to starting with each one of the items). On the second step the nodes have three branches, on the third step at most two branches, etc.

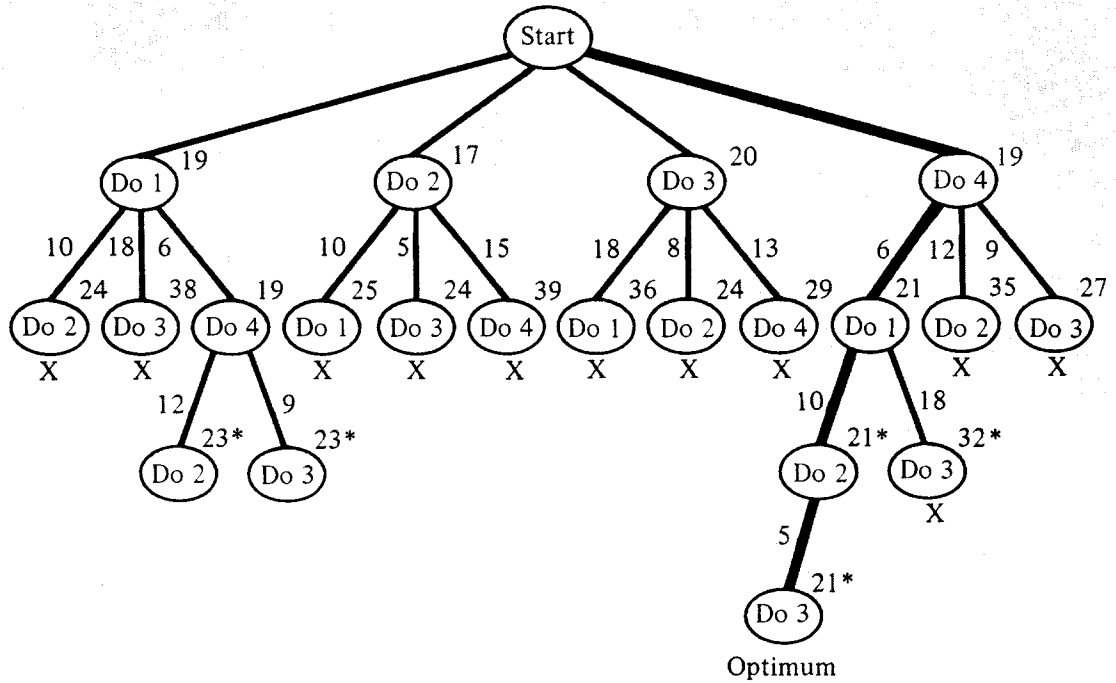


Figure 32

The numbers marked above and to the right of each decision are now *lower bounds* on the total setup cost, since our problem is to minimize this quantity. Those lower bounds marked with an asterisk were computed using feasible permutations, and the other lower bounds were calculated with permutations that are not necessarily feasible. Let us illustrate with the “Do 1” choice at the left of the second stage of choices. If we decide to “Do 1” first, then we shall *not* use any of the numbers in the first column of Figure 31, and we *must* use one number from the first row of Figure 31. Let’s assume that we can use the smallest number in the first row, which is 6. To find the rest of the sequence we are now reduced to the  $3 \times 3$  matrix in Figure 33(a), from which we must choose exactly two numbers. Without

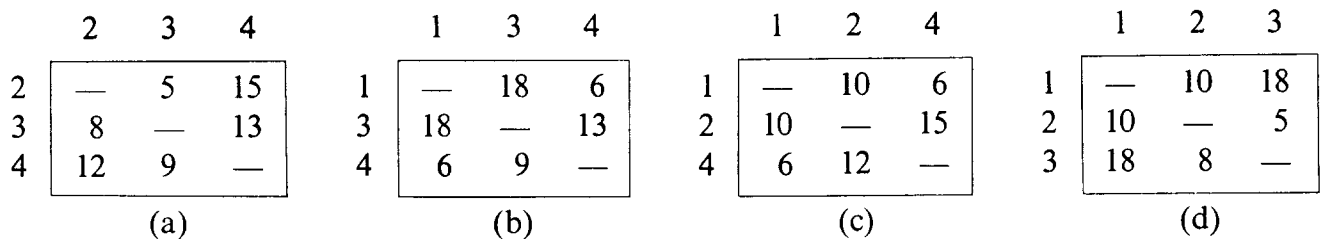


Figure 33



worrying about feasibility let's assume that we can get the two smallest numbers in Figure 33(a), which are 5 and 8. The sum of these three numbers is  $6 + 5 + 8 = 19$ , which is the lower bound marked on the second-level "Do 1" decision node. The other lower bounds for the second stage of the tree are calculated using Figures 33(b)–(d). Note that on the third stage of the decision tree we have marked on those branches of the tree the actual setup costs corresponding to the actual decision representing the sequence so far determined. These numbers are used in finding the third-stage lower bounds. As in the previous example, some nodes are marked with an X, indicating that the bound at that node is inferior to a bound already achieved, so that further search on that branch is unnecessary.

When the decision tree is completed it is found that the optimal sequence is to produce the items in the order 4, 1, 2, 3 and that the total setup cost for this sequence is 21. This optimal decision path is darkened in Figure 32. Comparing this value with the lower bounds on some of the terminated branches makes it clear that there is considerable cost saving by finding and using the optimal sequence rather than just choosing a production sequence at random.

In the examples above we have used the so-called "search in breadth" technique for selecting nodes to evaluate, since we found bounds for all decisions on the same level. There is another technique called "search in depth" in which we first go all the way down one path through the tree and actually construct a feasible solution, hopefully with a good bound. In many cases the latter technique has proved to be the better of the two. It is illustrated in the exercises.

## EXERCISES

1. In Example 1 use the fact that the items are listed in order of decreasing value/weight ratio to demonstrate that the upper bound calculation insures a true upper bound being found.
2. Solve the knapsack problem whose data is given by the following table:

Item	Value	Weight
1	70	30
2	61	27
3	54	25
4	40	20
5	31	18
6	25	15

The maximum weight to be carried is again 65 pounds.

3. Show that all the upper bounds in Figure 30 are correctly determined.
4. You have \$18 to spend on clothes, and have set up the following table of the kinds of clothes you want to buy with their values and costs:

Item(s)	Value	Cost
1 Shirt	15	\$ 6.50
2 Shirts	25	12.00
1 Tie	10	4.50
2 Ties	18	8.00
1 Pr. Socks	5	1.25
2 Pr. Socks	8	2.25

Use the branch-and-bound method to determine your optimal set of purchases. (This is an example of what in economics is called a *budget problem*.)

5. Show that the bounds indicated in Figure 32 are correct.
6. Work the four-item sequencing problem with the following data:

	1	2	3	4
1	—	7	11	4
2	8	—	12	7
3	6	13	—	12
4	9	5	4	—

7. In Figure 30 assume that you are doing a “search in depth” by following at each level only the alternative with the greatest upper bound. How does the tree of the branch-and-bound method change?
8. In Figure 32 assume that you are doing a “search in depth” by following at each level only the alternative with the smallest lower bound. How does the tree of the branch-and-bound method change?

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