CHAPTER TWO

Solutions for Section 2.1

Exercises

1. For t between 2 and 5, we have

$$\text{Average velocity} = \frac{\Delta s}{\Delta t} = \frac{400 - 135}{5 - 2} = \frac{265}{3} \text{ km/hr.}$$

The average velocity on this part of the trip was $265/3$ km/hr.

2. (a) Let $s = f(t)$.

(i) We wish to find the average velocity between $t = 1$ and $t = 1.1$. We have

$$\text{Average velocity} = \frac{f(1.1) - f(1)}{1.1 - 1} = \frac{3.63 - 3}{0.1} = 6.3 \text{ m/sec.}$$

(ii) We have

$$\text{Average velocity} = \frac{f(1.01) - f(1)}{1.01 - 1} = \frac{3.0603 - 3}{0.01} = 6.03 \text{ m/sec.}$$

(iii) We have

$$\text{Average velocity} = \frac{f(1.001) - f(1)}{1.001 - 1} = \frac{3.006003 - 3}{0.001} = 6.003 \text{ m/sec.}$$

(b) We see in part (a) that as we choose a smaller and smaller interval around $t = 1$ the average velocity appears to be getting closer and closer to 6, so we estimate the instantaneous velocity at $t = 1$ to be 6 m/sec.

3.

Slope	-3	-1	0	$1/2$	1	2
Point	F	C	E	A	B	D

4. The slope is positive at A and D; negative at C and F. The slope is most positive at A; most negative at F.

5. Using $h = 0.1, 0.01, 0.001$, we see

$$\frac{(3 + 0.1)^3 - 27}{0.1} = 27.91$$
$$\frac{(3 + 0.01)^3 - 27}{0.01} = 27.09$$
$$\frac{(3 + 0.001)^3 - 27}{0.001} = 27.009.$$

These calculations suggest that $\lim_{h \to 0} \frac{(3 + h)^3 - 27}{h} = 27$.

6. Using radians,

h	$(\cos h - 1)/h$
0.01	-0.005
0.001	-0.0005
0.0001	-0.00005

These values suggest that $\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$.

7. Using $h = 0.1, 0.01, 0.001$, we see

$$\frac{7^{0.1}-1}{0.1} = 2.148$$
$$\frac{7^{0.01}-1}{0.01} = 1.965$$
$$\frac{7^{0.001}-1}{0.001} = 1.948$$
$$\frac{7^{0.0001}-1}{0.0001} = 1.946.$$

This suggests that $\lim_{h\to 0} \frac{7^h - 1}{h} \approx 1.9\ldots$

8. Using $h = 0.1, 0.01, 0.001$, we see

h	$(e^{1+h} - e)/h$
0.01	2.7319
0.001	2.7196
0.0001	2.7184

These values suggest that $\lim_{h\to 0} \frac{e^{1+h} - e}{h} = 2.7\ldots$. In fact, this limit is e.

9. For $-0.5 \le \theta \le 0.5$, $0 \le y \le 3$, the graph of $y = \frac{\sin(2\theta)}{\theta}$ is shown in Figure 2.1. Therefore, $\lim_{\theta\to 0} \frac{\sin(2\theta)}{\theta} = 2$.

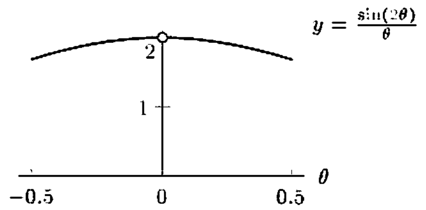


Figure 2.1

10. For $-1 \le \theta \le 1$, $-1 \le y \le 1$, the graph of $y = \frac{\cos\theta - 1}{\theta}$ is shown in Figure 2.2. Therefore, $\lim_{\theta\to 0} \frac{\cos\theta - 1}{\theta} = 0$.

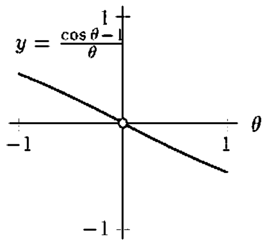


Figure 2.2

11. For $-90^\circ \le \theta \le 90^\circ$, $0 \le y \le 0.02$, the graph of $y = \frac{\sin\theta}{\theta}$ is shown in Figure 2.3. Therefore, by tracing along the curve, we see that in degrees, $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 0.01745\ldots$.

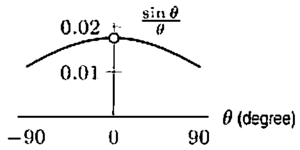


Figure 2.3

12. For $-0.5 \le \theta \le 0.5$, $0 \le y \le 0.5$, the graph of $y = \dfrac{\theta}{\tan(3\theta)}$ is shown in Figure 2.4. Therefore, by tracing along the curve, we see that $\lim\limits_{\theta \to 0} \dfrac{\theta}{\tan(3\theta)} = 0.3333\ldots$.

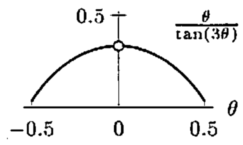
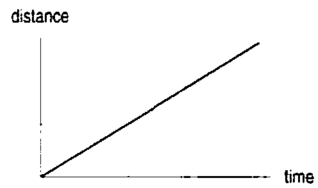


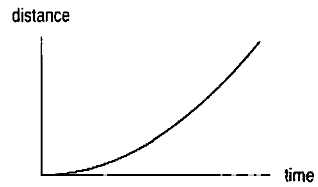
Figure 2.4

Problems

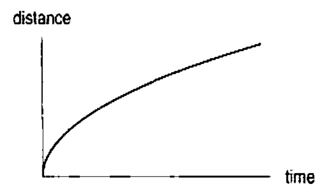
13.



14.



15.



16. $0 <$ slope at $C <$ slope at $B <$ slope of $AB < 1 <$ slope at A. (Note that the line $y = x$, has slope 1.)

17. Between 1804 and 1927, the world's population increased 1 billion people in 123 years, for an average rate of change of $1/123$ billion people per year. We convert this to people per minute:

$$\frac{1,000,000,000}{123} \text{ people/year} \cdot \frac{1}{60 \cdot 24 \cdot 365} \text{ years/minute} = 15.47 \text{ people/minute}.$$

Between 1804 and 1927, the population of the world increased at an average rate of 15.47 people per minute. Similarly, we find the following:
Between 1927 and 1960, the increase was 57.65 people per minute.
Between 1960 and 1974, the increase was 135.90 people per minute.
Between 1974 and 1987, the increase was 146.35 people per minute.
Between 1987 and 1999, the increase was 158.55 people per minute.

18. Since $f(t)$ is concave down between $t = 1$ and $t = 3$, the average velocity between the two times should be less than the instantaneous velocity at $t = 1$ but greater than the instantaneous velocity at time $t = 3$, so $D < A < C$. For analogous reasons, $F < B < E$. Finally, note that f is decreasing at $t = 5$ so $E < 0$, but increasing at $t = 0$, so $D > 0$. Therefore, the ordering from smallest to greatest of the given quantities is

$$F < B < E < 0 < D < A < C.$$

19. One possibility is shown in Figure 2.5.

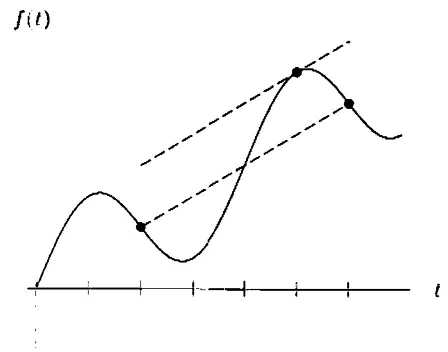
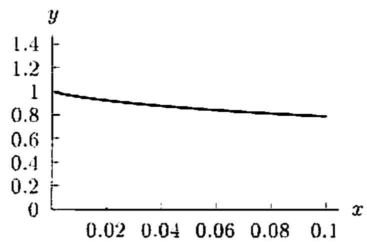


Figure 2.5

20. The limit appears to be 1; a graph and table of values is shown below.



x	x^x
0.1	0.7943
0.01	0.9550
0.001	0.9931
0.0001	0.9990
0.00001	0.9999

Solutions for Section 2.2

Exercises

1. **(a)** As x approaches -2 from either side, the values of $f(x)$ get closer and closer to 3, so the limit appears to be about 3.
(b) As x approaches 0 from either side, the values of $f(x)$ get closer and closer to 7. (Recall that to find a limit, we are interested in what happens to the function near x but not at x.) The limit appears to be about 7.
(c) As x approaches 2 from either side, the values of $f(x)$ get closer and closer to 3 on one side of $x = 2$ and get closer and closer to 2 on the other side of $x = 2$. Thus the limit does not exist.
(d) As x approaches 4 from either side, the values of $f(x)$ get closer and closer to 8. (Again, recall that we don't care what happens right at $x = 4$.) The limit appears to be about 8.

2. From the graphs of f and g, we estimate $\lim_{x\to 1^-} f(x) = 3$, $\lim_{x\to 1^-} g(x) = 5$, $\lim_{x\to 1^+} f(x) = 4$, $\lim_{x\to 1^+} g(x) = 1$.
(a) $\lim_{x\to 1^-} (f(x) + g(x)) = 3 + 5 = 8$
(b) $\lim_{x\to 1^+} (f(x) + 2g(x)) = \lim_{x\to 1^+} f(x) + 2 \lim_{x\to 1^+} g(x) = 4 + 2(1) = 6$
(c) $\lim_{x\to 1^-} (f(x)g(x)) = (\lim_{x\to 1^-} f(x))(\lim_{x\to 1^-} g(x)) = (3)(5) = 15$
(d) $\lim_{x\to 1^+} (f(x)/g(x)) = \left(\lim_{x\to 1^+} f(x)\right) / \left(\lim_{x\to 1^+} g(x)\right) = 4/1 = 4$

3. From Table 2.1, it appears the limit is 1. This is confirmed by Figure 2.6. An appropriate window is $-0.0033 < x < 0.0033$, $0.99 < y < 1.01$.

Table 2.1

x	$f(x)$
0.1	1.3
0.01	1.03
0.001	1.003
0.0001	1.0003

x	$f(x)$
−0.0001	0.9997
−0.001	0.997
−0.01	0.97
−0.1	0.7

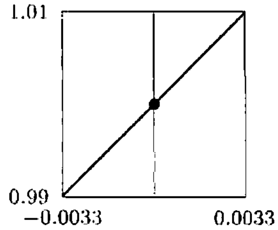


Figure 2.6

4. From Table 2.2, it appears the limit is -1. This is confirmed by Figure 2.7. An appropriate window is $-0.099 < x < 0.099$, $-1.01 < y < -0.99$.

Table 2.2

x	$f(x)$
0.1	−0.99
0.01	−0.9999
0.001	−0.999999
0.0001	−0.99999999

x	$f(x)$
−0.0001	−0.99999999
−0.001	−0.999999
−0.01	−0.9999
−0.1	−0.99

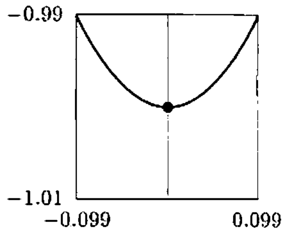


Figure 2.7

5. From Table 2.3, it appears the limit is 0. This is confirmed by Figure 2.8. An appropriate window is $-0.005 < x < 0.005$, $-0.01 < y < 0.01$.

Table 2.3

x	$f(x)$
0.1	0.1987
0.01	0.0200
0.001	0.0020
0.0001	0.0002

x	$f(x)$
−0.0001	−0.0002
−0.001	−0.0020
−0.01	−0.0200
−0.1	−0.1987

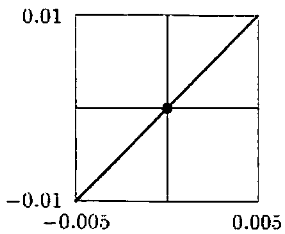


Figure 2.8

6. From Table 2.4, it appears the limit is 0. This is confirmed by Figure 2.9. An appropriate window is $-0.0033 < x < 0.0033$, $-0.01 < y < 0.01$.

Table 2.4

x	$f(x)$
0.1	0.2955
0.01	0.0300
0.001	0.0030
0.0001	0.0003

x	$f(x)$
−0.0001	−0.0003
−0.001	−0.0030
−0.01	−0.0300
−0.1	−0.2955

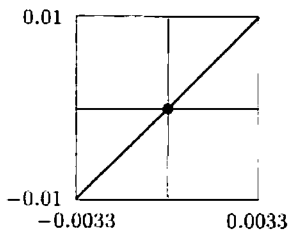


Figure 2.9

7. From Table 2.5, it appears the limit is 2. This is confirmed by Figure 2.10. An appropriate window is $-0.0865 < x < 0.0865$, $1.99 < y < 2.01$.

Table 2.5

x	$f(x)$
0.1	1.9867
0.01	1.9999
0.001	2.0000
0.0001	2.0000

x	$f(x)$
−0.0001	2.0000
−0.001	2.0000
−0.01	1.9999
−0.1	1.9867

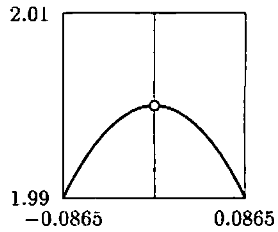


Figure 2.10

8. From Table 2.6, it appears the limit is 3. This is confirmed by Figure 2.11. An appropriate window is $-0.047 < x < 0.047$, $2.99 < y < 3.01$.

Table 2.6

x	$f(x)$
0.1	2.9552
0.01	2.9996
0.001	3.0000
0.0001	3.0000

x	$f(x)$
−0.0001	3.0000
−0.001	3.0000
−0.01	2.9996
−0.1	2.9552

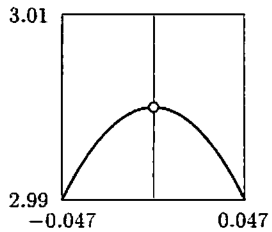


Figure 2.11

9. From Table 2.7, it appears the limit is 1. This is confirmed by Figure 2.12. An appropriate window is $-0.0198 < x < 0.0198$, $0.99 < y < 1.01$.

Table 2.7

x	$f(x)$
0.1	1.0517
0.01	1.0050
0.001	1.0005
0.0001	1.0001

x	$f(x)$
−0.0001	1.0000
−0.001	0.9995
−0.01	0.9950
−0.1	0.9516

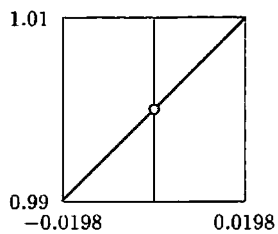


Figure 2.12

10. From Table 2.8, it appears the limit is 2. This is confirmed by Figure 2.13. An appropriate window is $-0.0049 < x < 0.0049$, $1.99 < y < 2.01$.

Table 2.8

x	$f(x)$
0.1	2.2140
0.01	2.0201
0.001	2.0020
0.0001	2.0002

x	$f(x)$
−0.0001	1.9998
−0.001	1.9980
−0.01	1.9801
−0.1	1.8127

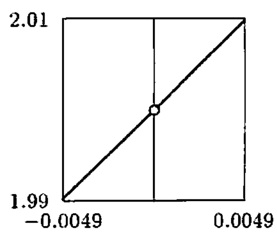


Figure 2.13

11. From Table 2.9, it appears the limit is 4. Figure 2.14 confirms this. An appropriate window is $1.99 < x < 2.01$, $3.99 < y < 4.01$.

Table 2.9

x	$f(x)$
2.1	4.1
2.01	4.01
2.001	4.001
2.0001	4.0001
1.9999	3.9999
1.999	3.999
1.99	3.99
1.9	3.9

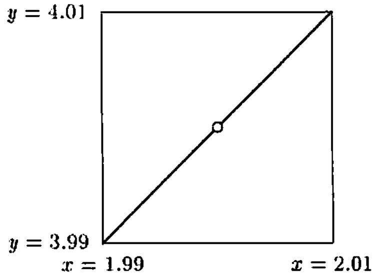


Figure 2.14

12. From Table 2.10, it appears the limit is 6. Figure 2.15 confirms this. An appropriate window is $2.99 < x < 3.01$, $5.99 < y < 6.01$.

Table 2.10

x	$f(x)$
3.1	6.1
3.01	6.01
3.001	6.001
3.0001	6.0001
2.9999	5.9999
2.999	5.999
2.99	5.99
2.9	5.9

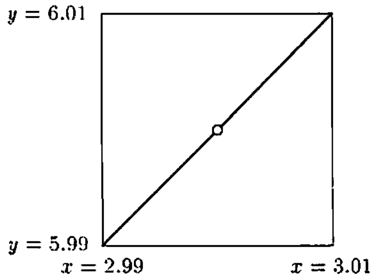


Figure 2.15

13. From Table 2.11, it appears the limit is 0. Figure 2.16 confirms this. An appropriate window is $1.55 < x < 1.59$, $-0.01 < y < 0.01$.

Table 2.11

x	$f(x)$
1.6708	−0.0500
1.5808	−0.0050
1.5718	−0.0005
1.5709	−0.0001
1.5707	0.0001
1.5698	0.0005
1.5608	0.0050
1.4708	0.0500

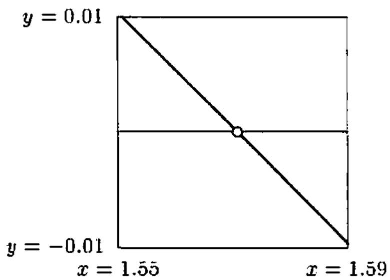


Figure 2.16

14. From Table 2.12, it appears the limit is 2. Figure 2.17 confirms this. An appropriate window is $0.995 < x < 1.004$, $1.99 < y < 2.01$.

Table 2.12

x	$f(x)$
1.1	2.2140
1.01	2.0201
1.001	2.0020
1.0001	2.0002
0.9999	1.9998
0.999	1.9980
0.99	1.9801
0.9	1.8127

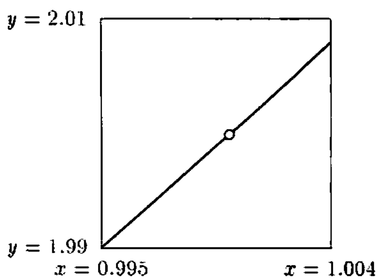


Figure 2.17

15. $\lim_{h\to 0}\frac{(2+h)^2-4}{h}=\lim_{h\to 0}\frac{4+4h+h^2-4}{h}=\lim_{h\to 0}(4+h)=4$

16. $\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{1+h}-1\right)=\lim_{h\to 0}\frac{1-(1+h)}{(1+h)h}=\lim_{h\to 0}\frac{-1}{1+h}=-1$

17. $\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{(1+h)^2}-1\right)=\lim_{h\to 0}\frac{1-(1+2h+h^2)}{h(1+h)^2}=\lim_{h\to 0}\frac{-2-h}{(1+h)^2}=-2$

18. $\sqrt{4+h}-2=\frac{(\sqrt{4+h}-2)(\sqrt{4+h}+2)}{\sqrt{4+h}+2}=\frac{4+h-4}{\sqrt{4+h}+2}=\frac{h}{\sqrt{4+h}+2}.$

Therefore $\lim_{h\to 0}\frac{\sqrt{4+h}-2}{h}=\lim_{h\to 0}\frac{1}{\sqrt{4+h}+2}=\frac{1}{4}$

19. $\frac{1}{\sqrt{4+h}}-\frac{1}{2}=\frac{2-\sqrt{4+h}}{2\sqrt{4+h}}=\frac{(2-\sqrt{4+h})(2+\sqrt{4+h})}{2\sqrt{4+h}(2+\sqrt{4+h})}=\frac{4-(4+h)}{2\sqrt{4+h}(2+\sqrt{4+h})}.$

Therefore $\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{\sqrt{4+h}}-\frac{1}{2}\right)=\lim_{h\to 0}\frac{-1}{2\sqrt{4+h}(2+\sqrt{4+h})}=-\frac{1}{16}$

20. $f(x)=\frac{|x-4|}{x-4}=\begin{cases}\frac{x-4}{x-4} & x>4\\ -\frac{x-4}{x-4} & x<4\end{cases}=\begin{cases}1 & x>4\\ -1 & x<4\end{cases}$

Figure 2.18 confirms that $\lim_{x\to 4^+}f(x)=1$, $\lim_{x\to 4^-}f(x)=-1$ so $\lim_{x\to 4}f(x)$ does not exist.

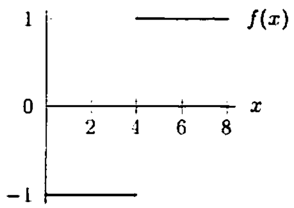


Figure 2.18

21. $f(x)=\frac{|x-2|}{x}=\begin{cases}\frac{x-2}{x}, & x>2\\ -\frac{x-2}{x}, & x<2\end{cases}$

Figure 2.19 confirms that $\lim_{x\to 2^+}f(x)=\lim_{x\to 2^-}f(x)=\lim_{x\to 2}f(x)=0.$

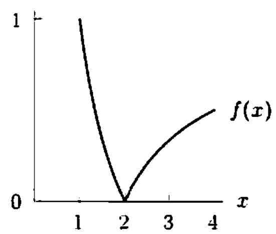


Figure 2.19

22. $f(x)=\begin{cases}x^2-2 & 0<x<3\\ 2 & x=3\\ 2x+1 & 3<x\end{cases}$

Figure 2.20 confirms that $\lim_{x\to 3^-}f(x)=\lim_{x\to 3^-}(x^2-2)=7$ and that $\lim_{x\to 3^+}f(x)=\lim_{x\to 3^+}(2x+1)=7$, so $\lim_{x\to 3}f(x)=7$. Note, however, that $f(x)$ is not continuous at $x=3$ since $f(3)=2$.

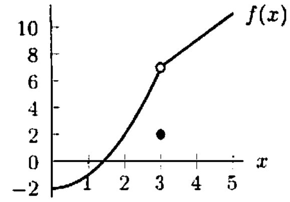


Figure 2.20

23. The graph in Figure 2.21 suggests that

$$\text{if } -0.05 < \theta < 0.05, \quad \text{then} \quad 0.999 < (\sin\theta)/\theta < 1.001.$$

Thus, if θ is within 0.05 of 0, we see that $(\sin\theta)/\theta$ is within 0.001 of 1.

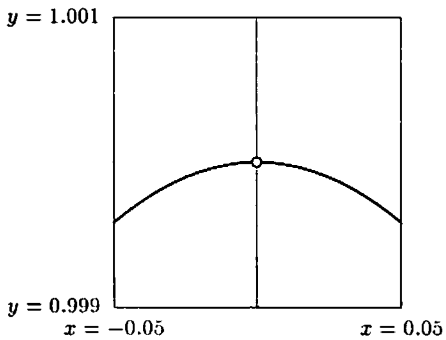


Figure 2.21: Graph of $(\sin\theta)/\theta$ with $-0.05 < \theta < 0.05$

24. The statement

$$\lim_{h\to a} g(h) = K$$

means that we can make the value of $g(h)$ as close to K as we want by choosing h sufficiently close to, but not equal to, a.

In symbols, for any $\epsilon > 0$, there is a $\delta > 0$ such that

$$|g(h) - K| < \epsilon \quad \text{for all } 0 < |h - a| < \delta.$$

Problems

25. The only change is that, instead of considering all x near c, we only consider x near to and greater than c. Thus the phrase "$|x - c| < \delta$" must be replaced by "$c < x < c + \delta$." Thus, we define

$$\lim_{x\to c^+} f(x) = L$$

to mean that for any $\epsilon > 0$ (as small as we want), there is a $\delta > 0$ (sufficiently small) such that if $c < x < c + \delta$, then $|f(x) - L| < \epsilon$.

26. The only change is that, instead of considering all x near c, we only consider x near to and less than c. Thus the phrase "$|x - c| < \delta$" must be replaced by "$c - \delta < x < c$." Thus, we define

$$\lim_{x\to c^-} f(x) = L$$

to mean that for any $\epsilon > 0$ (as small as we want), there is a $\delta > 0$ (sufficiently small) such that if $c - \delta < x < c$, then $|f(x) - L| < \epsilon$.

27. Instead of being "sufficiently close to c," we want x to be "sufficiently large." Using N to measure how large x must be, we define

$$\lim_{x\to\infty} f(x) = L$$

to mean that for any $\epsilon > 0$ (as small as we want), there is a $N > 0$ (sufficiently large) such that if $x > N$, then $|f(x) - L| < \epsilon$.

28. If $x > 1$ and x approaches 1, then $p(x) = 55$. If $x < 1$ and x approaches 1, then $p(x) = 34$. There is not a single number that $p(x)$ approaches as x approaches 1, so we say that $\lim_{x\to 1} p(x)$ does not exist.

29. We use values of h approaching, but not equal to, zero. If we let $h = 0.01, 0.001, 0.0001, 0.00001$, we calculate the values 2.7048, 2.7169, 2.7181, and 2.7183. If we let $h = -0.01\ -0.001, -0.0001, -0.00001$, we get values 2.7320, 2.7196, 2.7184, and 2.7183. These numbers suggest that the limit is the number $e = 2.71828\ldots$. However, these calculations cannot tell us that the limit is exactly e; for that a proof is needed.

30. Divide numerator and denominator by x:

$$f(x) = \frac{x+3}{2-x} = \frac{1+3/x}{2/x-1},$$

so

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1+3/x}{2/x-1} = \frac{\lim_{x\to\infty}(1+3/x)}{\lim_{x\to\infty}(2/x-1)} = \frac{1}{-1} = -1.$$

31. Divide numerator and denominator by x^2, giving

$$f(x) = \frac{x^2+2x-1}{3+3x^2} = \frac{1+2/x-1/x^2}{3/x^2+3},$$

so

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1+2/x-1/x^2}{3/x^2+3} = \frac{\lim_{x\to\infty}(1+2/x-1/x^2)}{\lim_{x\to\infty}(3/x^2+3)} = \frac{1}{3}.$$

32. Divide numerator and denominator by x, giving

$$f(x) = \frac{x^2+4}{x+3} = \frac{x+4/x}{1+3/x}.$$

so

$$\lim_{x\to\infty} f(x) = +\infty.$$

33. Divide numerator and denominator by x^3, giving

$$f(x) = \frac{2x^3-16x^2}{4x^2+3x^3} = \frac{2-16/x}{4/x+3},$$

so

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2-16/x}{4/x+3} = \frac{\lim_{x\to\infty}(2-16/x)}{\lim_{x\to\infty}(4/x+3)} = \frac{2}{3}.$$

34. Divide numerator and denominator by x^5, giving

$$f(x) = \frac{x^4+3x}{x^4+2x^5} = \frac{1/x+3/x^4}{1/x+2}.$$

so

$$\lim_{x\to\infty} f(x) = \frac{\lim_{x\to\infty}(1/x+3/x^4)}{\lim_{x\to\infty}(1/x+2)} = \frac{0}{2} = 0.$$

35. Divide numerator and denominator by e^x, giving

$$f(x) = \frac{3e^x+2}{2e^x+3} = \frac{3+2e^{-x}}{2+3e^{-x}}.$$

so

$$\lim_{x\to\infty} f(x) = \frac{\lim_{x\to\infty}(3+2e^{-x})}{\lim_{x\to\infty}(2+3e^{-x})} = \frac{3}{2}.$$

36. $f(x) = \dfrac{2e^{-x}+3}{3e^{-x}+2}$, so $\lim_{x\to\infty} f(x) = \dfrac{\lim_{x\to\infty}(2e^{-x}+3)}{\lim_{x\to\infty}(3e^{-x}+2)} = \dfrac{3}{2}$.

37. Because the denominator equals 0 when $x = 4$, so must the numerator. This means $k^2 = 16$ and the choices for k are 4 or -4.

38. Because the denominator equals 0 when $x = 1$, so must the numerator. So $1 - k + 4 = 0$. The only possible value of k is 5.

39. Because the denominator equals 0 when $x = -2$, so must the numerator. So $4 - 8 + k = 0$ and the only possible value of k is 4.

40. Division of numerator and denominator by x^2 yields

$$\frac{x^2+3x+5}{4x+1+x^k} = \frac{1+3/x+5/x^2}{4/x+1/x^2+x^{k-2}}.$$

As $x \to \infty$, the limit of the numerator is 1. The limit of the denominator depends upon k. If $k > 2$, the denominator approaches ∞ as $x \to \infty$, so the limit of the quotient is 0. If $k = 2$, the denominator approaches 1 as $x \to \infty$, so the limit of the quotient is 1. If $k < 2$ the denominator approaches 0^+ as $x \to \infty$, so the limit of the quotient is ∞. Therefore the values of k we are looking for are $k \geq 2$.

41. For the numerator, $\lim_{x\to-\infty}\left(e^{2x}-5\right) = -5$. If $k > 0$, $\lim_{x\to-\infty}\left(e^{kx}+3\right) = 3$, so the quotient has a limit of $-5/3$. If $k = 0$, $\lim_{x\to-\infty}\left(e^{kx}+3\right) = 4$, so the quotient has limit of $-5/4$. If $k < 0$, the limit of the quotient is given by $\lim_{x\to-\infty}(e^{2x}-5)/(e^{kx}+3) = 0$.

42. By tracing on a calculator or solving equations, we find the following values of δ:
For $\epsilon = 0.2$, $\delta \leq 0.1$.
For $\epsilon = 0.1$, $\delta \leq 0.05$.
For $\epsilon = 0.02$, $\delta \leq 0.01$.
For $\epsilon = 0.01$, $\delta \leq 0.005$.
For $\epsilon = 0.002$, $\delta \leq 0.001$.
For $\epsilon = 0.001$, $\delta \leq 0.0005$.

43. By tracing on a calculator or solving equations, we find the following values of δ:
For $\epsilon = 0.1$, $\delta \leq 0.46$.
For $\epsilon = 0.01$, $\delta \leq 0.21$.
For $\epsilon = 0.001$, $\delta < 0.1$. Thus, we can take $\delta \leq 0.09$.

44. The results of Problem 42 suggest that we can choose $\delta = \epsilon/2$. For any $\epsilon > 0$, we want to find the δ such that

$$|f(x) - 3| = |-2x + 3 - 3| = |2x| < \epsilon.$$

Then if $|x| < \delta = \epsilon/2$, it follows that $|f(x) - 3| = |2x| < \epsilon$. So $\lim_{x\to 0}(-2x+3) = 3$.

45. (a) Since $\sin(n\pi) = 0$ for $n = 1, 2, 3, \ldots$ the sequence of x-values

$$\frac{1}{\pi}, \frac{1}{2\pi}, \frac{1}{3\pi}, \ldots$$

works. These x-values $\to 0$ and are zeroes of $f(x)$.

(b) Since $\sin(n\pi/2) = 1$ for $n = 1, 5, 9\ldots$ the sequence of x-values

$$\frac{2}{\pi}, \frac{2}{5\pi}, \frac{2}{9\pi}, \ldots$$

works.

(c) Since $\sin(n\pi)/2 = -1$ for $n = 3, 7, 11, \ldots$ the sequence of x-values

$$\frac{2}{3\pi}, \frac{2}{7\pi}, \frac{2}{11\pi} \ldots$$

works.

(d) Any two of these sequences of x-values show that if the limit were to exist, then it would have to have two (different) values: 0 and 1, or 0 and -1, or 1 and -1. Hence, the limit can not exist.

46. (a) If $b = 0$, then the property says $\lim_{x\to c} 0 = 0$, which is easy to see is true.
(b) If $|f(x) - L| < \frac{\epsilon}{|b|}$, then multiplying by $|b|$ gives

$$|b||f(x) - L| < \epsilon.$$

Since

$$|b||f(x) - L| = |b(f(x) - L)| = |bf(x) - bL|,$$

we have

$$|bf(x) - bL| < \epsilon.$$

(c) Suppose that $\lim_{x\to c} f(x) = L$. We want to show that $\lim_{x\to c} bf(x) = bL$. If we are to have

$$|bf(x) - bL| < \epsilon,$$

then we will need

$$|f(x) - L| < \frac{\epsilon}{|b|}.$$

We choose δ small enough that

$$|x - c| < \delta \quad \text{implies} \quad |f(x) - L| < \frac{\epsilon}{|b|}.$$

By part (b), this ensures that

$$|bf(x) - bL| < \epsilon,$$

as we wanted.

47. Suppose $\lim_{x\to c} f(x) = L_1$ and $\lim_{x\to c} g(x) = L_2$. Then we need to show that

$$\lim_{x\to c} (f(x) + g(x)) = L_1 + L_2.$$

Let $\epsilon > 0$ be given. We need to show that we can choose $\delta > 0$ so that whenever $|x - c| < \delta$, we will have $|(f(x) + g(x)) - (L_1 + L_2)| < \epsilon$. First choose $\delta_1 > 0$ so that $|x - c| < \delta_1$ implies $|f(x) - L_1| < \frac{\epsilon}{2}$; we can do this since $\lim_{x\to c} f(x) = L_1$. Similarly, choose $\delta_2 > 0$ so that $|x - c| < \delta_2$ implies $|g(x) - L_2| < \frac{\epsilon}{2}$. Now, set δ equal to the smaller of δ_1 and δ_2. Thus $|x - c| < \delta$ will make both $|x - c| < \delta_1$ and $|x - c| < \delta_2$. Then, for $|x - c| < \delta$, we have

$$\begin{aligned} |f(x) + g(x) - (L_1 + L_2)| &= |(f(x) - L_1) + (g(x) - L_2)| \\ &\le |(f(x) - L_1)| + |(g(x) - L_2)| \\ &\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

This proves $\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$, which is the result we wanted to prove.

48. (a) We need to show that for any given $\epsilon > 0$, there is a $\delta > 0$ so that $|x - c| < \delta$ implies $|f(x)g(x)| < \epsilon$. If $\epsilon > 0$ is given, choose δ_1 so that when $|x - c| < \delta_1$, we have $|f(x)| < \sqrt{\epsilon}$. This can be done since $\lim_{x\to 0} f(x) = 0$. Similarly, choose δ_2 so that when $|x - c| < \delta_2$, we have $|g(x)| < \sqrt{\epsilon}$. Then, if we take δ to be the smaller of δ_1 and δ_2, we'll have that $|x - c| < \delta$ implies both $|f(x)| < \sqrt{\epsilon}$ and $|g(x)| < \sqrt{\epsilon}$. So when $|x - c| < \delta$, we have $|f(x)g(x)| = |f(x)|\,|g(x)| < \sqrt{\epsilon}\cdot\sqrt{\epsilon} = \epsilon$. Thus $\lim_{x\to c} f(x)g(x) = 0$.
(b) $(f(x) - L_1)(g(x) - L_2) + L_1 g(x) + L_2 f(x) - L_1 L_2$
$= f(x)g(x) - L_1 g(x) - L_2 f(x) + L_1 L_2 + L_1 g(x) + L_2 f(x) - L_1 L_2 = f(x)g(x)$.
(c) $\lim_{x\to c} (f(x) - L_1) = \lim_{x\to c} f(x) - \lim_{x\to c} L_1 = L_1 - L_1 = 0$, using the second limit property. Similarly, $\lim_{x\to c} (g(x) - L_2) = 0$.
(d) Since $\lim_{x\to c} (f(x) - L_1) = \lim_{x\to c} (g(x) - L_2) = 0$, we have that $\lim_{x\to c} (f(x) - L_1)(g(x) - L_2) = 0$ by part (a).
(e) From part (b), we have

$$\begin{aligned} \lim_{x\to c} f(x)g(x) &= \lim_{x\to c} ((f(x) - L_1)(g(x) - L_2) + L_1 g(x) + L_2 f(x) - L_1 L_2) \\ &= \lim_{x\to c} (f(x) - L_1)(g(x) - L_2) + \lim_{x\to c} L_1 g(x) + \lim_{x\to c} L_2 f(x) + \lim_{x\to c}(-L_1 L_2) \\ &\quad \text{(using limit property 2)} \\ &= 0 + L_1 \lim_{x\to c} g(x) + L_2 \lim_{x\to c} f(x) - L_1 L_2 \\ &\quad \text{(using limit property 1 and part (d))} \\ &= L_1 L_2 + L_2 L_1 - L_1 L_2 = L_1 L_2. \end{aligned}$$

Solutions for Section 2.3

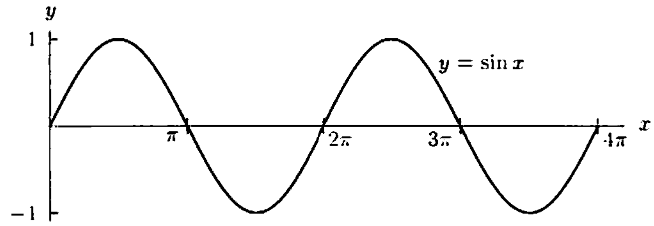
Exercises

1. The derivative, $f'(2)$, is the rate of change of x^3 at $x = 2$. Notice that each time x changes by 0.001 in the table, the value of x^3 changes by 0.012. Therefore, we estimate

$$f'(2) = \begin{matrix}\text{Rate of change}\\ \text{of } f \text{ at } x = 2\end{matrix} \approx \frac{0.012}{0.001} = 12.$$

The function values in the table look exactly linear because they have been rounded. For example, the exact value of x^3 when $x = 2.001$ is 8.012006001, not 8.012. Thus, the table can tell us only that the derivative is approximately 12. Example 5 on page 82 shows how to compute the derivative of $f(x)$ exactly.

2.



Since $\sin x$ is decreasing for values near $x = 3\pi$, its derivative at $x = 3\pi$ is negative.

3. (a) Using a calculator we obtain the values found in the table below:

x	1	1.5	2	2.5	3
e^x	2.72	4.48	7.39	12.18	20.09

(b) The average rate of change of $f(x) = e^x$ between $x = 1$ and $x = 3$ is

$$\text{Average rate of change} = \frac{f(3) - f(1)}{3 - 1} = \frac{e^3 - e}{3 - 1} \approx \frac{20.09 - 2.72}{2} = 8.69.$$

(c) First we find the average rates of change of $f(x) = e^x$ between $x = 1.5$ and $x = 2$, and between $x = 2$ and $x = 2.5$:

$$\text{Average rate of change} = \frac{f(2) - f(1.5)}{2 - 1.5} = \frac{e^2 - e^{1.5}}{2 - 1.5} \approx \frac{7.39 - 4.48}{0.5} = 5.82$$

$$\text{Average rate of change} = \frac{f(2.5) - f(2)}{2.5 - 2} = \frac{e^{2.5} - e^2}{2.5 - 2} \approx \frac{12.18 - 7.39}{0.5} = 9.58.$$

Now we approximate the instantaneous rate of change at $x = 2$ by averaging these two rates:

$$\text{Instantaneous rate of change} \approx \frac{5.82 + 9.58}{2} = 7.7.$$

4. (a)

Table 2.13

x	1	1.5	2	2.5	3
$\log x$	0	0.18	0.30	0.40	0.48

(b) The average rate of change of $f(x) = \log x$ between $x = 1$ and $x = 3$ is

$$\frac{f(3) - f(1)}{3 - 1} = \frac{\log 3 - \log 1}{3 - 1} \approx \frac{0.48 - 0}{2} = 0.24$$

(c) First we find the average rates of change of $f(x) = \log x$ between $x = 1.5$ and $x = 2$, and between $x = 2$ and $x = 2.5$.

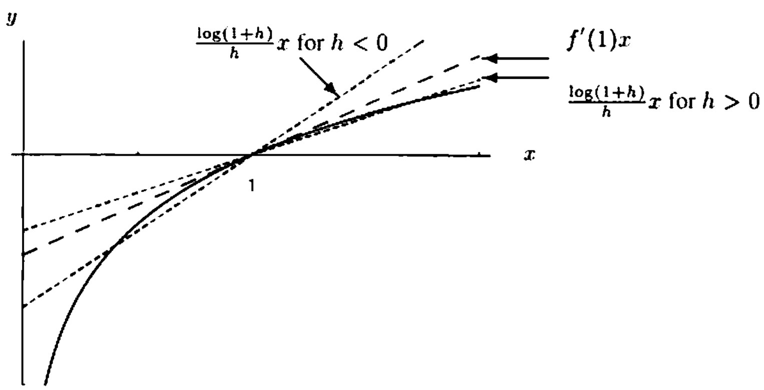
$$\frac{\log 2 - \log 1.5}{2 - 1.5} = \frac{0.30 - 0.18}{0.5} \approx 0.24$$
$$\frac{\log 2.5 - \log 2}{2.5 - 2} = \frac{0.40 - 0.30}{0.5} \approx 0.20$$

Now we approximate the instantaneous rate of change at $x = 2$ by finding the average of the above rates, i.e.

$$\left(\begin{array}{c}\text{the instantaneous rate of change} \\ \text{of } f(x) = \log x \text{ at } x = 2\end{array}\right) \approx \frac{0.24 + 0.20}{2} = 0.22.$$

5. $f'(1) = \lim_{h\to 0} \frac{\log(1 + h) - \log 1}{h} = \lim_{h\to 0} \frac{\log(1 + h)}{h}$

Evaluating $\frac{\log(1+h)}{h}$ for $h = 0.01, 0.001$, and 0.0001, we get $0.43214, 0.43408, 0.43427$, so $f'(1) \approx 0.43427$. The corresponding secant lines are getting steeper, because the graph of $\log x$ is concave down. We thus expect the limit to be more than 0.43427 . If we consider negative values of h, the estimates are too large. We can also see this from the graph below:



6. We estimate $f'(2)$ using the average rate of change formula on a small interval around 2. We use the interval $x = 2$ to $x = 2.001$. (Any small interval around 2 gives a reasonable answer.) We have

$$f'(2) \approx \frac{f(2.001) - f(2)}{2.001 - 2} = \frac{3^{2.001} - 3^2}{2.001 - 2} = \frac{9.00989 - 9}{0.001} = 9.89.$$

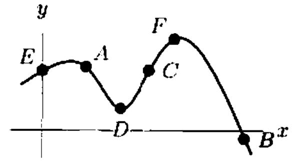
7. Since $f'(x) = 0$ where the graph is horizontal, $f'(x) = 0$ at $x = d$. The derivative is positive at points b and c, but the graph is steeper at $x = c$. Thus $f'(x) = 0.5$ at $x = b$ and $f'(x) = 2$ at $x = c$. Finally, the derivative is negative at points a and e but the graph is steeper at $x = e$. Thus, $f'(x) = -0.5$ at $x = a$ and $f'(x) = -2$ at $x = e$. See Table 2.14.

Thus, we have $f'(d) = 0, f'(b) = 0.5, f'(c) = 2, f'(a) = -0.5, f'(e) = -2$.

Table 2.14

x	$f'(x)$
d	0
b	0.5
c	2
a	−0.5
e	−2

8. One possible choice of points is shown below.



9. **(a)** The average rate of change from $x = a$ to $x = b$ is the slope of the line between the points on the curve with $x = a$ and $x = b$. Since the curve is concave down, the line from $x = 1$ to $x = 3$ has a greater slope than the line from $x = 3$ to $x = 5$, and so the average rate of change between $x = 1$ and $x = 3$ is greater than that between $x = 3$ and $x = 5$.

(b) Since f is increasing, $f(5)$ is the greater.

(c) As in part (a), f is concave down and f' is decreasing throughout so $f'(1)$ is the greater.

10. Using the definition of the derivative, we have

$$\begin{aligned} f'(10) &= \lim_{h\to 0} \frac{f(10+h)-f(10)}{h} \\ &= \lim_{h\to 0} \frac{5(10+h)^2 - 5(10)^2}{h} \\ &= \lim_{h\to 0} \frac{500+100h+5h^2-500}{h} \\ &= \lim_{h\to 0} \frac{100h+5h^2}{h} \\ &= \lim_{h\to 0} \frac{h(100+5h)}{h} \\ &= \lim_{h\to 0} 100+5h \\ &= 100. \end{aligned}$$

11. Using the definition of the derivative, we have

$$\begin{aligned} f'(-2) &= \lim_{h\to 0} \frac{f(-2+h)-f(-2)}{h} \\ &= \lim_{h\to 0} \frac{(-2+h)^3-(-2)^3}{h} \\ &= \lim_{h\to 0} \frac{(-8+12h-6h^2+h^3)-(-8)}{h} \\ &= \lim_{h\to 0} \frac{12h-6h^2+h^3}{h} \\ &= \lim_{h\to 0} \frac{h(12-6h+h^2)}{h} \\ &= \lim_{h\to 0} (12-6h+h^2), \end{aligned}$$

which goes to 12 as $h \to 0$. So $f'(-2) = 12$.

12. Using the definition of the derivative

$$\begin{aligned} g'(-1) &= \lim_{h\to 0} \frac{g(-1+h)-g(-1)}{h} \\ &= \lim_{h\to 0} \frac{(3(-1+h)^2+5(-1+h))-(3(-1)^2+5(-1))}{h} \\ &= \lim_{h\to 0} \frac{(3(1-2h+h^2)-5+5h)-(-2)}{h} \\ &= \lim_{h\to 0} \frac{3-6h+3h^2-3+5h}{h} \\ &= \lim_{h\to 0} \frac{(-h+3h^2)}{h} = \lim_{h\to 0}(-1+3h) = -1. \end{aligned}$$

13.

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{((1+h)^3+5)-(1^3+5)}{h} \\ &= \lim_{h\to 0} \frac{1+3h+3h^2+h^3+5-1-5}{h} = \lim_{h\to 0} \frac{3h+3h^2+h^3}{h} \\ &= \lim_{h\to 0}(3+3h+h^2) = 3. \end{aligned}$$

14.

$$\begin{aligned} g'(2) &= \lim_{h\to 0} \frac{g(2+h)-g(2)}{h} = \lim_{h\to 0} \frac{\frac{1}{2+h}-\frac{1}{2}}{h} \\ &= \lim_{h\to 0} \frac{2-(2+h)}{h(2+h)2} = \lim_{h\to 0} \frac{-h}{h(2+h)2} \\ &= \lim_{h\to 0} \frac{-1}{(2+h)2} = -\frac{1}{4} \end{aligned}$$

15.

$$\begin{aligned} g'(2) &= \lim_{h\to 0} \frac{g(2+h)-g(2)}{h} = \lim_{h\to 0} \frac{\frac{1}{(2+h)^2}-\frac{1}{2^2}}{h} \\ &= \lim_{h\to 0} \frac{2^2-(2+h)^2}{2^2(2+h)^2h} = \lim_{h\to 0} \frac{4-4-4h-h^2}{4h(2+h)^2} \\ &= \lim_{h\to 0} \frac{-4h-h^2}{4h(2+h)^2} = \lim_{h\to 0} \frac{-4-h}{4(2+h)^2} \\ &= \frac{-4}{4(2)^2} = -\frac{1}{4}. \end{aligned}$$

16. As we saw in the answer to Problem 10, the slope of the tangent line to $f(x) = 5x^2$ at $x = 10$ is 100. When $x = 10$, $f(x) = 500$ so $(10, 500)$ is a point on the tangent line. Thus $y = 100(x - 10) + 500 = 100x - 500$.

17. As we saw in the answer to Problem 11, the slope of the tangent line to $f(x) = x^3$ at $x = -2$ is 12. When $x = -2$, $f(x) = -8$ so we know the point $(-2, -8)$ is on the tangent line. Thus the equation of the tangent line is $y = 12(x + 2) - 8 = 12x + 16$.

18. We know that the slope of the tangent line to $f(x) = x$ when $x = 20$ is 1. When $x = 20$, $f(x) = 20$ so $(20, 20)$ is on the tangent line. Thus the equation of the tangent line is $y = 1(x - 20) + 20 = x$.

19. First find the derivative of $f(x) = 1/x^2$ at $x = 1$.

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{\frac{1}{(1+h)^2}-\frac{1}{1^2}}{h} \\ &= \lim_{h\to 0} \frac{1^2-(1+h)^2}{h(1+h)^2} = \lim_{h\to 0} \frac{1-(1+2h+h^2)}{h(1+h)^2} \\ &= \lim_{h\to 0} \frac{-2h-h^2}{h(1+h)^2} = \lim_{h\to 0} \frac{-2-h}{(1+h)^2} = -2 \end{aligned}$$

Thus the tangent line has a slope of -2 and goes through the point $(1, 1)$, and so its equation is

$$y - 1 = -2(x - 1) \quad \text{or} \quad y = -2x + 3.$$

Problems

20. The statements $f(100) = 35$ and $f'(100) = 3$ tell us that at $x = 100$, the value of the function is 35 and the function is increasing at a rate of 3 units for a unit increase in x. Since we increase x by 2 units in going from 100 to 102, the value of the function goes up by approximately $2 \cdot 3 = 6$ units, so

$$f(102) \approx 35 + 2 \cdot 3 = 35 + 6 = 41.$$

21. The coordinates of A are $(4, 25)$. See Figure 2.22. The coordinates of B and C are obtained using the slope of the tangent line. Since $f'(4) = 1.5$, the slope is 1.5

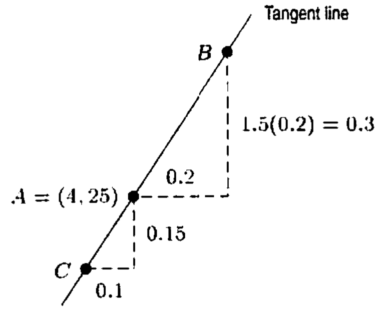


Figure 2.22

From A to B, $\Delta x = 0.2$, so $\Delta y = 1.5(0.2) = 0.3$. Thus, at C we have $y = 25 + 0.3 = 25.3$. The coordinates of B are $(4.2, 25.3)$.

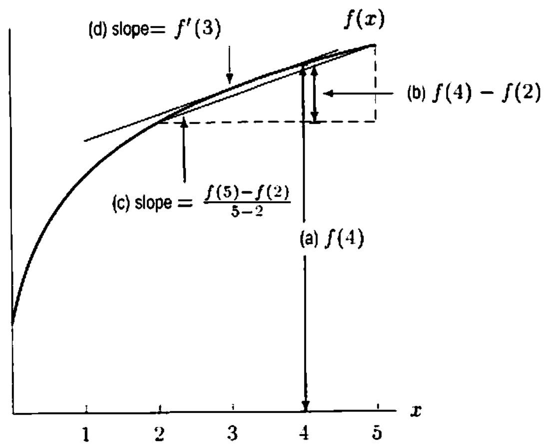
From A to C, $\Delta x = -0.1$, so $\Delta y = 1.5(-0.1) = -0.15$. Thus, at C we have $y = 25 - 0.15 = 24.85$. The coordinates of C are $(3.9, 24.85)$.

22. (a) Since the point $B = (2, 5)$ is on the graph of g, we have $g(2) = 5$.
(b) The slope of the tangent line touching the graph at $x = 2$ is given by

$$\text{Slope} = \frac{\text{Rise}}{\text{Run}} = \frac{5 - 5.02}{2 - 1.95} = \frac{-0.02}{0.05} = -0.4.$$

Thus, $g'(2) = -0.4$.

23.



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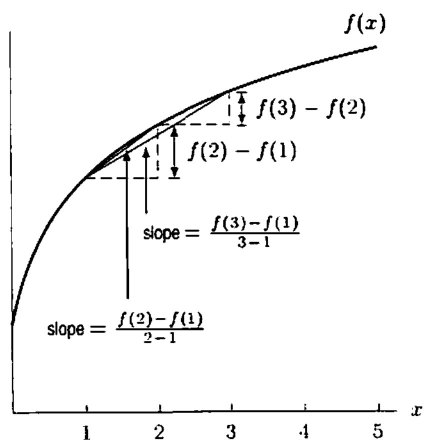


(a) $f(4) > f(3)$ since f is increasing.

(b) From the figure, it appears that $f(2) - f(1) > f(3) - f(2)$.

(c) $\dfrac{f(2) - f(1)}{2 - 1}$ represents the slope of the secant line connecting the graph at $x = 1$ and $x = 2$. This is greater than the slope of the secant line connecting the graph at $x = 1$ and $x = 3$ which is $\dfrac{f(3) - f(1)}{3 - 1}$.

(d) The function is steeper at $x = 1$ than at $x = 4$ so $f'(1) > f'(4)$.

25. Figure 2.23 shows the quantities in which we are interested.

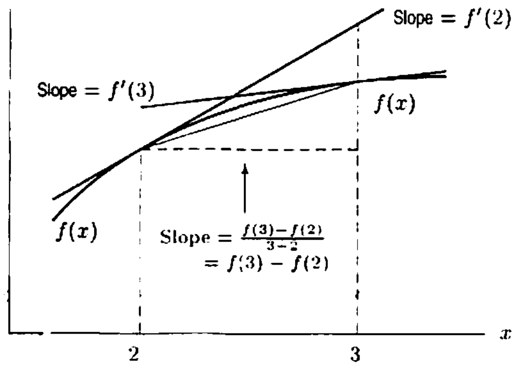


Figure 2.23

The quantities $f'(2)$, $f'(3)$ and $f(3) - f(2)$ have the following interpretations:

- $f'(2)$ = slope of the tangent line at $x = 2$
- $f'(3)$ = slope of the tangent line at $x = 3$
- $f(3) - f(2) = \frac{f(3)-f(2)}{3-2}$ = slope of the secant line from $f(2)$ to $f(3)$.

From Figure 2.23, it is clear that $0 < f(3) - f(2) < f'(2)$. By extending the secant line past the point $(3, f(3))$, we can see that it lies above the tangent line at $x = 3$.

Thus

$$0 < f'(3) < f(3) - f(2) < f'(2).$$

26. (a) $f(4)/4$ is the slope of the line connecting (0,0) to $(4, f(4))$. (See Figure 2.24.)

(b) It is clear from the picture for part (a) that $f(3)/3 > f(4)/4$.

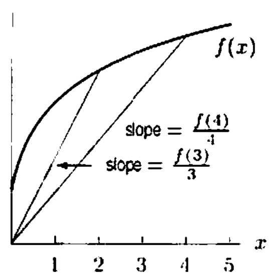


Figure 2.24

27.

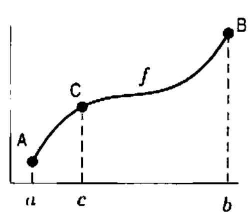


Figure 2.25

(a) For the line from A to B,
$$\text{Slope} = \frac{f(b) - f(a)}{b - a}.$$
(b) The tangent line at point C appears to be parallel to the line from A to B. Assuming this to be the case, the lines have the same slope.
(c) There is only one other point, labeled D in Figure 2.26, at which the tangent line is parallel to the line joining A and B.

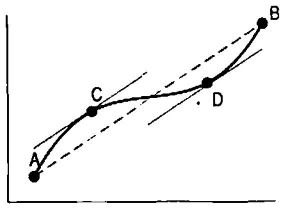


Figure 2.26

28. Using a difference quotient with $h = 0.001$, say, we find
$$f'(1) \approx \frac{1.001\ln(1.001) - 1\ln(1)}{1.001 - 1} = 1.0005$$
$$f'(2) \approx \frac{2.001\ln(2.001) - 2\ln(2)}{2.001 - 2} = 1.6934$$
The fact that f' is larger at $x = 2$ than at $x = 1$ suggests that f is concave up on the interval $[1, 2]$.

29. (a)
$$f'(0) = \lim_{h\to 0} \frac{\overbrace{\sin h}^{h \text{ in degrees}} - \overbrace{\sin 0}^{0}}{h} = \frac{\sin h}{h}.$$
To four decimal places,
$$\frac{\sin 0.2}{0.2} \approx \frac{\sin 0.1}{0.1} \approx \frac{\sin 0.01}{0.01} \approx \frac{\sin 0.001}{0.001} \approx 0.01745$$
so $f'(0) \approx 0.01745$.

(b) Consider the ratio $\frac{\sin h}{h}$. As we approach 0, the numerator, $\sin h$, will be much smaller in magnitude if h is in degrees than it would be if h were in radians. For example, if $h = 1^\circ$ radian, $\sin h = 0.8415$, but if $h = 1$ degree, $\sin h = 0.01745$. Thus, since the numerator is smaller for h measured in degrees while the denominator is the same, we expect the ratio $\frac{\sin h}{h}$ to be smaller.

30. We want $f'(2)$. The exact answer is
$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{(2+h)^{2+h} - 4}{h},$$
but we can approximate this. If $h = 0.001$, then
$$\frac{(2.001)^{2.001} - 4}{0.001} \approx 6.779$$
and if $h = 0.0001$ then
$$\frac{(2.0001)^{2.0001} - 4}{0.0001} \approx 6.773,$$
so $f'(2) \approx 6.77$.

31. Notice that we can't get all the information we want just from the graph of f for $0 \le x \le 2$, shown on the left in Figure 2.27. Looking at this graph, it looks as if the slope at $x = 0$ is 0. But if we zoom in on the graph near $x = 0$, we get the graph of f for $0 \le x \le 0.05$, shown on the right in Figure 2.27. We see that f does dip down quite a bit between $x = 0$ and $x \approx 0.11$. In fact, it now looks like $f'(0)$ is around -1. Note that since $f(x)$ is undefined for $x < 0$, this derivative only makes sense as we approach zero from the right.

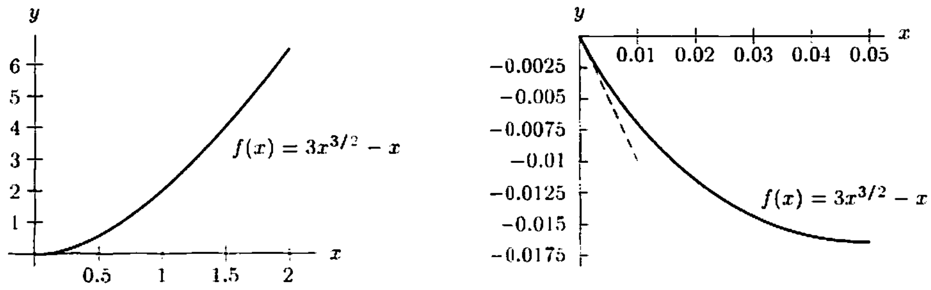


Figure 2.27

We zoom in on the graph of f near $x = 1$ to get a more accurate picture from which to estimate $f'(1)$. A graph of f for $0.7 \leq x \leq 1.3$ is shown in Figure 2.28. [Keep in mind that the axes shown in this graph don't cross at the origin!] Here we see that $f'(1) \approx 3.5$.

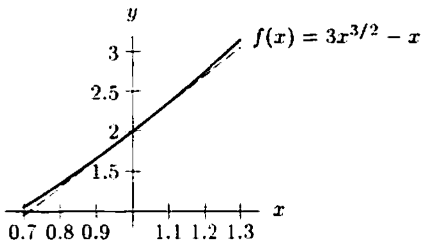


Figure 2.28

32.

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{\ln(\cos(1+h)) - \ln(\cos 1)}{h}$$

For $h = 0.001$, the difference quotient $= -1.55912$; for $h = 0.0001$, the difference quotient $= -1.55758$. The instantaneous rate of change of f therefore appears to be about -1.558 at $x = 1$.
At $x = \frac{\pi}{4}$, if we try $h = 0.0001$, then

$$\text{difference quotient} = \frac{\ln[\cos(\frac{\pi}{4} + 0.0001)] - \ln(\cos\frac{\pi}{4})}{0.0001} \approx -1.0001.$$

The instantaneous rate of change of f appears to be about -1 at $x = \frac{\pi}{4}$.

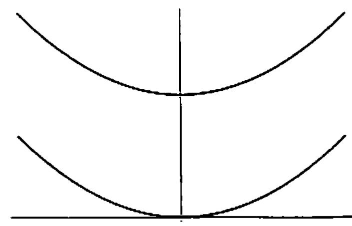
33. We want to approximate $P'(0)$ and $P'(2)$. Since for small h

$$P'(0) \approx \frac{P(h) - P(0)}{h},$$

if we take $h = 0.01$, we get

$$\begin{aligned} P'(0) &\approx \frac{1.15(1.014)^{0.01} - 1.15}{0.01} = 0.01599 \text{ billion/year} \\ &= 16.0 \text{ million people/year} \\ P'(2) &\approx \frac{1.15(1.014)^{2.01} - 1.15(1.014)^2}{0.01} = 0.0164 \text{ billion/year} \\ &= 16.4 \text{ million people/year} \end{aligned}$$

34.



(a) From the figure above, it appears that the slopes of the tangent lines to the two graphs are the same at each x. For $x = 0$, the slopes of the tangents to the graphs of $f(x)$ and $g(x)$ at 0 are

For $x =$

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h) - 0}{h} = \lim_{h\to 0} \frac{\frac{1}{2}h^2}{h} = \lim_{h\to 0} \frac{1}{2}h = 0,$$

$$g'(0) = \lim_{h\to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = \lim_{h\to 0} \frac{\frac{1}{2}h^2 + 3 - 3}{h} = \lim_{h\to 0} \frac{\frac{1}{2}h^2}{h} = \lim_{h\to 0} \frac{1}{2}h = 0.$$

2, the slopes of the tangents to the graphs of $f(x)$ and $g(x)$ are

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{\frac{1}{2}(2+h)^2 - \frac{1}{2}(2)^2}{h} = \lim_{h\to 0} \frac{\frac{1}{2}(4+4h+h^2) - 2}{h} = \lim_{h\to 0} \frac{2+2h+\frac{1}{2}h^2 - 2}{h} = \lim_{h\to 0} \frac{2h+\frac{1}{2}h^2}{h} = \lim_{h\to 0}\left(2 + \frac{1}{2}h\right) = 2.$$

$$g'(2) = \lim_{h\to 0} \frac{g(2+h) - g(2)}{h} = \lim_{h\to 0} \frac{\frac{1}{2}(2+h)^2 + 3 - (\frac{1}{2}(2)^2 + 3)}{h} = \lim_{h\to 0} \frac{\frac{1}{2}(2+h)^2 - \frac{1}{2}(2)^2}{h} = \lim_{h\to 0} \frac{\frac{1}{2}(4+4h+h^2) - 2}{h} = \lim_{h\to 0} \frac{2+2h+\frac{1}{2}(h^2) - 2}{h} = \lim_{h\to 0} \frac{2h+\frac{1}{2}(h^2)}{h} = \lim_{h\to 0}\left(2 + \frac{1}{2}h\right) = 2.$$

For $x = x_0$, the slopes of the tangents to the graphs of $f(x)$ and $g(x)$ are

$$\begin{aligned} f'(x_0) &= \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2}(x_0+h)^2 - \frac{1}{2}x_0^2}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2}(x_0^2+2x_0h+h^2) - \frac{1}{2}x_0^2}{h} \\ &= \lim_{h\to 0} \frac{x_0h+\frac{1}{2}h^2}{h} \\ &= \lim_{h\to 0} \left(x_0+\frac{1}{2}h\right) \\ &= x_0. \end{aligned}$$

$$\begin{aligned} g'(x_0) &= \lim_{h\to 0} \frac{g(x_0+h)-g(x_0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2}(x_0+h)^2+3-(\frac{1}{2}(x_0)^2+3)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2}(x_0+h)^2-\frac{1}{2}(x_0)^2}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2}(x_0^2+2x_0h+h^2) - \frac{1}{2}x_0^2}{h} \\ &= \lim_{h\to 0} \frac{x_0h+\frac{1}{2}h^2}{h} \\ &= \lim_{h\to 0} \left(x_0+\frac{1}{2}h\right) \\ &= x_0. \end{aligned}$$

(b)

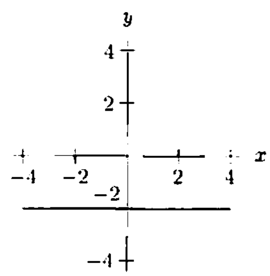
$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)+C-(f(x)+C)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= f'(x). \end{aligned}$$

35. As h gets smaller, round-off error becomes important. When $h = 10^{-12}$, the quantity $2^h - 1$ is so close to 0 that the calculator rounds off the difference to 0, making the difference quotient 0. The same thing will happen when $h = 10^{-20}$.

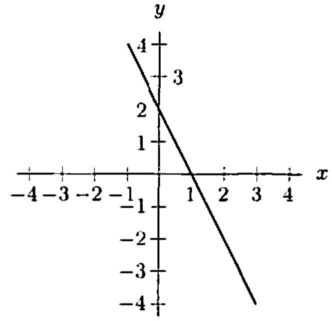
Solutions for Section 2.4

Exercises

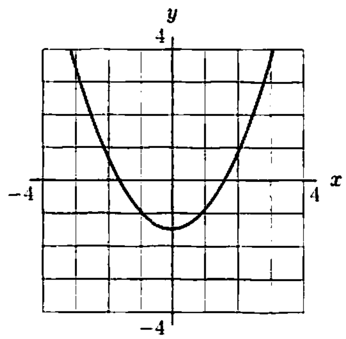
1. The graph is that of the line $y = -2x + 2$. The slope, and hence the derivative, is -2.



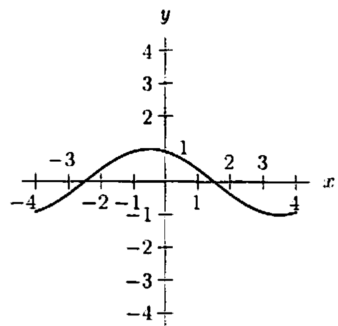
2.



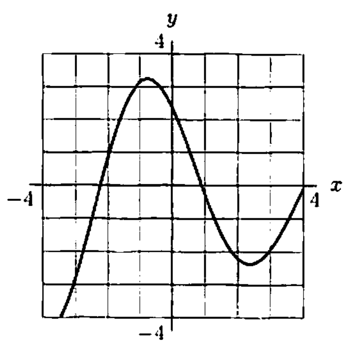
3.



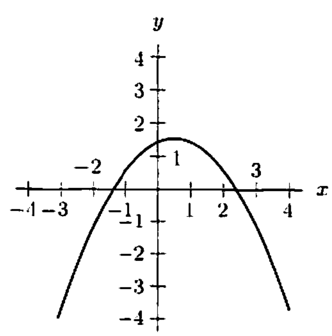
4. The slope of this curve is approximately -1 at $x = -4$ and at $x = 4$, approximately 0 at $x = -2.5$ and $x = 1.5$, and approximately 1 at $x = 0$.



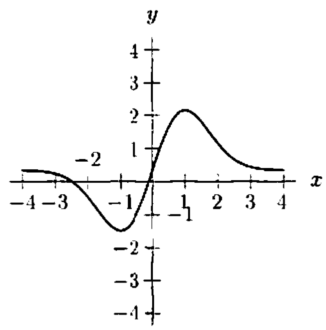
5.



6.



7.



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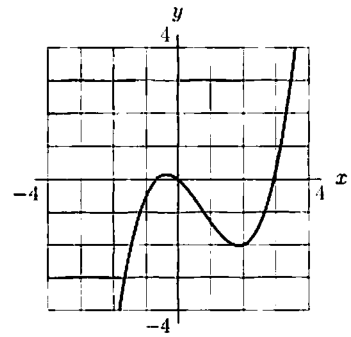
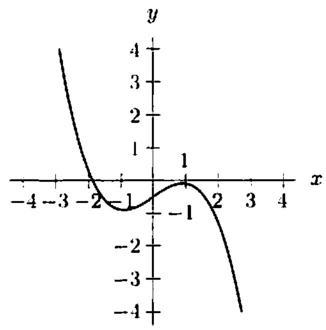


9.



10.

x	$\ln x$
0.998	−0.0020
0.999	−0.0010
1.000	0.0000
1.001	0.0010
1.002	0.0020

x	$\ln x$
1.998	0.6921
1.999	0.6926
2.000	0.6931
2.001	0.6936
2.002	0.6941

x	$\ln x$
4.998	1.6090
4.999	1.6092
5.000	1.6094
5.001	1.6096
5.002	1.6098

x	$\ln x$
9.998	2.3024
9.999	2.3025
10.000	2.3026
10.001	2.3027
10.002	2.3028

At $x = 1$, the values of $\ln x$ are increasing by 0.001 for each increase in x of 0.001, so the derivative appears to be 1. At $x = 2$, the increase is 0.0005 for each increase of 0.001, so the derivative appears to be 0.5. At $x = 5$, $\ln x$ increases by 0.0002 for each increase of 0.001 in x, so the derivative appears to be 0.2. And at $x = 10$, the increase is 0.0001 over intervals of 0.001, so the derivative appears to be 0.1. These values suggest an inverse relationship between x and $f'(x)$, namely $f'(x) = \frac{1}{x}$.

11. (a) We use the interval to the right of $x = 2$ to estimate the derivative. (Alternately, we could use the interval to the left of 2, or we could use both and average the results.) We have

$$f'(2) \approx \frac{f(4) - f(2)}{4 - 2} = \frac{24 - 18}{4 - 2} = \frac{6}{2} = 3.$$

We estimate $f'(2) \approx 3$.

(b) We know that $f'(x)$ is positive when $f(x)$ is increasing and negative when $f(x)$ is decreasing, so it appears that $f'(x)$ is positive for $0 < x < 4$ and is negative for $4 < x < 12$.

12. For $x = 0, 5, 10$, and 15, we use the interval to the right to estimate the derivative. For $x = 20$, we use the interval to the left. For $x = 0$, we have

$$f'(0) \approx \frac{f(5) - f(0)}{5 - 0} = \frac{70 - 100}{5 - 0} = \frac{-30}{5} = -6.$$

Similarly, we find the other estimates in Table 2.15.

Table 2.15

x	0	5	10	15	20
$f'(x)$	-6	-3	-1.8	-1.2	-1.2

13. Since $1/x = x^{-1}$, using the power rule gives

$$\frac{d}{dx}(x^{-1}) = (-1)x^{-2} = -\frac{1}{x^2}.$$

Using the definition of the derivative, we have

$$\begin{aligned} k'(x) &= \lim_{h \to 0} \frac{k(x+h) - k(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{x - (x+h)}{h(x+h)x} \\ &= \lim_{h \to 0} \frac{-h}{h(x+h)x} = \lim_{h \to 0} \frac{-1}{(x+h)x} = -\frac{1}{x^2}. \end{aligned}$$

14. Since $1/x^2 = x^{-2}$, using the power rule gives

$$\frac{d}{dx}(x^{-2}) = -2x^{-3} = -\frac{2}{x^3}.$$

Using the definition of the derivative, we have

$$\begin{aligned} l'(x) &= \lim_{h \to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} = \lim_{h \to 0} \frac{x^2 - (x+h)^2}{h(x+h)^2x^2} \\ &= \lim_{h \to 0} \frac{x^2 - (x^2 + 2xh + h^2)}{h(x+h)^2x^2} = \lim_{h \to 0} \frac{-2xh - h^2}{h(x+h)^2x^2} \\ &= \lim_{h \to 0} \frac{-2x - h}{(x+h)^2x^2} = \frac{-2x}{x^2x^2} = -\frac{2}{x^3}. \end{aligned}$$

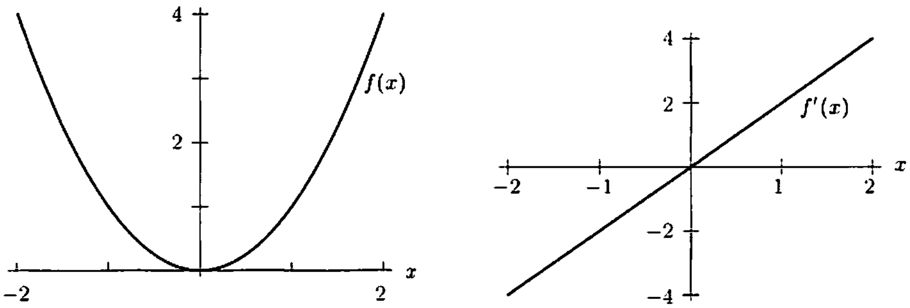
15. Using the definition of the derivative,

$$\begin{aligned} g'(x) &= \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{2(x+h)^2 - 3 - (2x^2 - 3)}{h} \\ &= \lim_{h \to 0} \frac{2(x^2 + 2xh + h^2) - 3 - 2x^2 + 3}{h} = \lim_{h \to 0} \frac{4xh + 2h^2}{h} \\ &= \lim_{h \to 0} (4x + 2h) = 4x. \end{aligned}$$

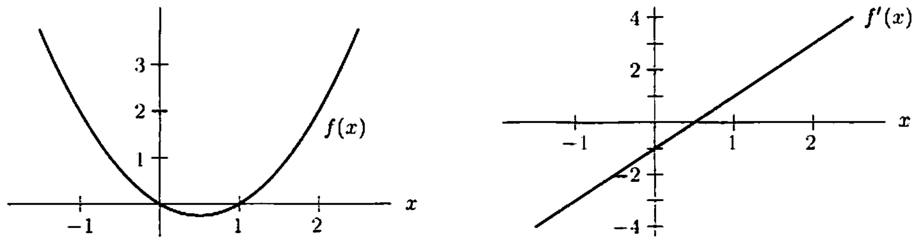
16. Using the definition of the derivative, we have

$$\begin{aligned} m'(x) &= \lim_{h \to 0} \frac{m(x+h) - m(x)}{h} = \lim_{h \to 0} \frac{1}{h}\left(\frac{1}{x+h+1} - \frac{1}{x+1}\right) \\ &= \lim_{h \to 0} \frac{1}{h}\left(\frac{x+1-x-h-1}{(x+1)(x+h+1)}\right) = \lim_{h \to 0} \frac{-h}{h(x+1)(x+h+1)} \\ &= \lim_{h \to 0} \frac{-1}{(x+1)(x+h+1)} \\ &= \frac{-1}{(x+1)^2}. \end{aligned}$$

17.



18.



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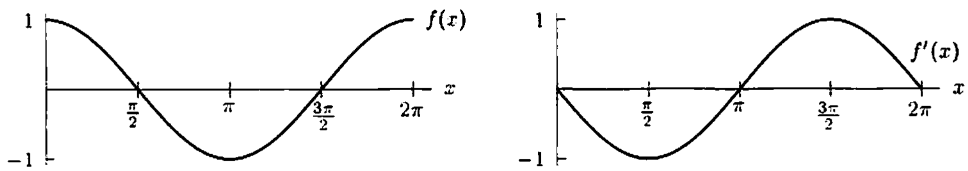

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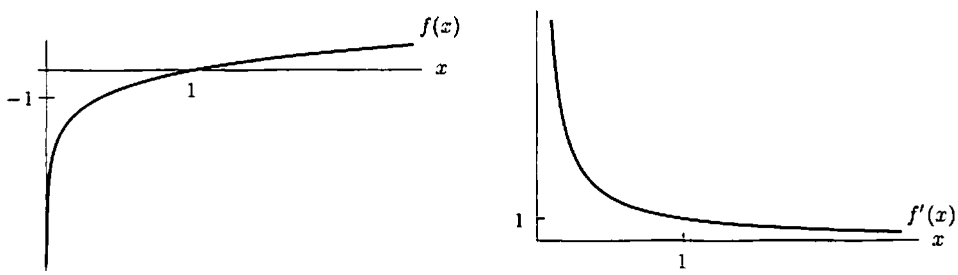


Problems

21. We know that $f'(x) \approx \dfrac{f(x+h)-f(x)}{h}$. For this problem, we'll take the average of the values obtained for $h = 1$ and $h = -1$; that's the average of $f(x+1) - f(x)$ and $f(x) - f(x-1)$ which equals $\dfrac{f(x+1)-f(x-1)}{2}$. Thus,

$f'(0) \approx f(1) - f(0) = 13 - 18 = -5$.
$f'(1) \approx [f(2) - f(0)]/2 = [10 - 18]/2 = -4$.
$f'(2) \approx [f(3) - f(1)]/2 = [9 - 13]/2 = -2$.
$f'(3) \approx [f(4) - f(2)]/2 = [9 - 10]/2 = -0.5$.
$f'(4) \approx [f(5) - f(3)]/2 = [11 - 9]/2 = 1$.
$f'(5) \approx [f(6) - f(4)]/2 = [15 - 9]/2 = 3$.
$f'(6) \approx [f(7) - f(5)]/2 = [21 - 11]/2 = 5$.
$f'(7) \approx [f(8) - f(6)]/2 = [30 - 15]/2 = 7.5$.
$f'(8) \approx f(8) - f(7) = 30 - 21 = 9$.

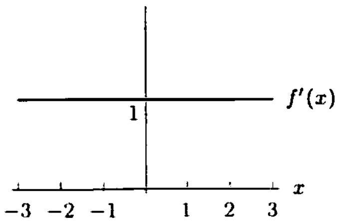
The rate of change of $f(x)$ is positive for $4 \le x \le 8$, negative for $0 \le x \le 3$. The rate of change is greatest at about $x = 8$.

22. The value of $g(x)$ is increasing at a decreasing rate for $2.7 < x < 4.2$ and increasing at an increasing rate for $x > 4.2$.

$$\frac{\Delta y}{\Delta x} = \frac{7.4 - 6.0}{5.2 - 4.7} = 2.8 \qquad \text{between } x = 4.7 \text{ and } x = 5.2$$
$$\frac{\Delta y}{\Delta x} = \frac{9.0 - 7.4}{5.7 - 5.2} = 3.2 \qquad \text{between } x = 5.2 \text{ and } x = 5.7$$

Thus $g'(x)$ should be close to 3 near $x = 5.2$.

23.



24. This is a line with slope -2, so the derivative is the constant function $f'(x) = -2$. The graph is a horizontal line at $y = -2$. See Figure 2.29.

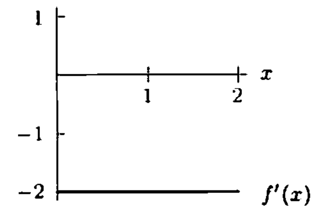


Figure 2.29

25. This function is decreasing for $x < 2$ and increasing for $x > 2$ and so the derivative is negative for $x < 2$ and positive for $x > 2$. One possible graph is shown in Figure 2.30.

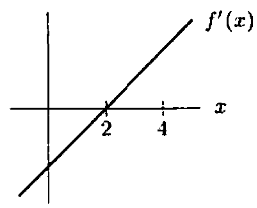
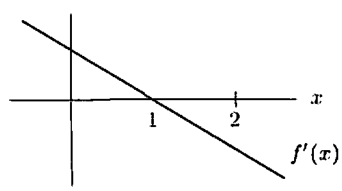


Figure 2.30

26.



27. This function is increasing for approximately $x < 1$ and $x > 4.5$ and is decreasing for approximately $1 < x < 4.5$. The derivative is positive for $x < 1$ and $x > 4.5$ and negative for $1 < x < 4.5$. One possible graph is shown in Figure 2.31.

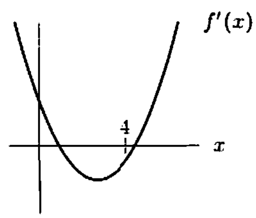
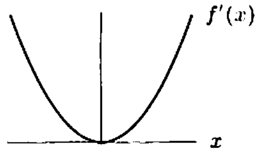
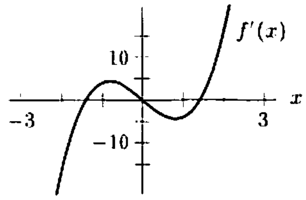


Figure 2.31

28.



29.



30. This function is increasing for $x < 1$ and is decreasing for $x > 1$ so the derivative is positive for $x < 1$ and negative for $x > 1$. In addition, as x gets large, the graph of $f(x)$ gets more and more horizontal. Thus, as x gets large, $f'(x)$ gets closer and closer to 0. One possible graph is shown in Figure 2.32.

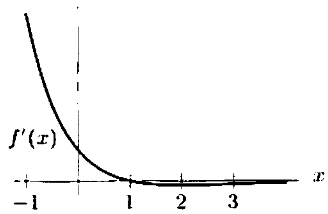
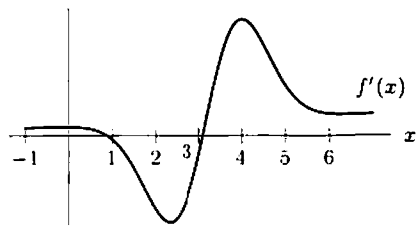
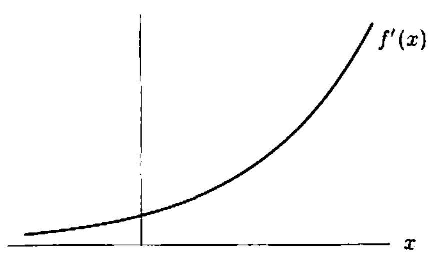


Figure 2.32

31.



32.



33. From the given information we know that f is increasing for values of x less than -2, is decreasing between $x = -2$ and $x = 2$, and is constant for $x > 2$. Figure 2.33 shows a possible graph—yours may be different.

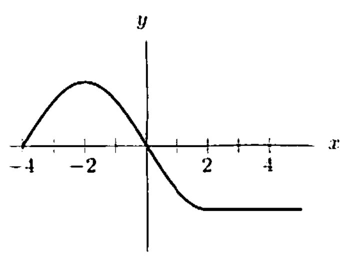


Figure 2.33

34. Figure 2.34 shows a possible graph – yours may be different.

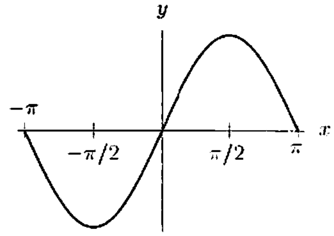


Figure 2.34

35. (a) x_3 (b) x_4 (c) x_5 (d) x_3

36. The derivative is zero whenever the graph of the original function is horizontal. Since the current is proportional to the derivative of the voltage, segments where the current is zero alternate with positive segments where the voltage is increasing and negative segments where the voltage is decreasing. See Figure 2.35. Note that the derivative does not exist where the graph has a corner.

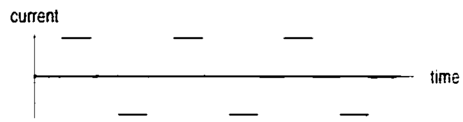
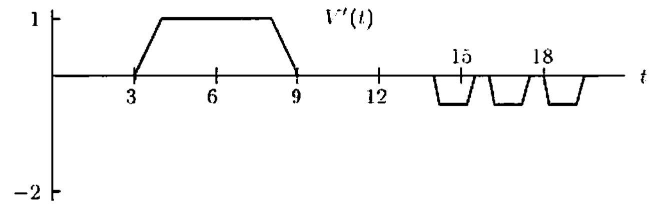


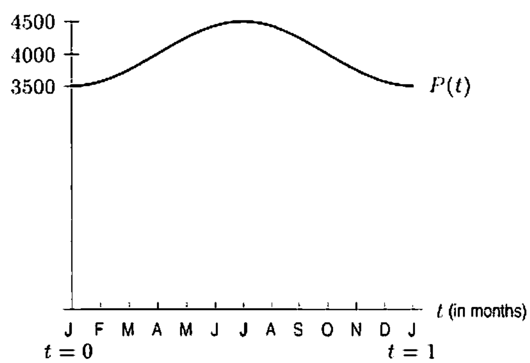
Figure 2.35

37. (a) Graph II
(b) Graph I
(c) Graph III

38. (a) $t = 3$
(b) $t = 9$
(c) $t = 14$
(d)



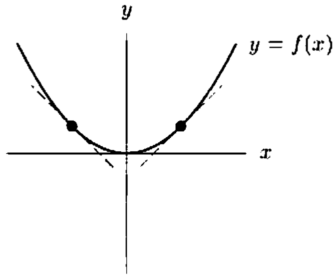
39. (a) The population varies periodically with a period of 1 year. See below.

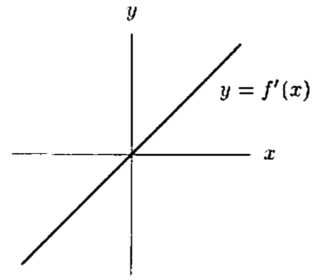


(b) The population is at a maximum on July 1^{st}. At this time $\sin(2\pi t - \frac{\pi}{2}) = 1$, so the actual maximum population is $4000 + 500(1) = 4500$. Similarly, the population is at a minimum on January 1^{st}. At this time, $\sin(2\pi t - \frac{\pi}{2}) = -1$, so the minimum population is $4000 + 500(-1) = 3500$.
(c) The rate of change is most positive about April 1^{st} and most negative around October 1^{st}.
(d) Since the population is at its maximum around July 1^{st}, its rate of change is about 0 then.

40. (a) The function f is increasing where f' is positive, so for $x_1 < x < x_3$.
(b) The function f is decreasing where f' is negative, so for $0 < x < x_1$ or $x_3 < x < x_5$.

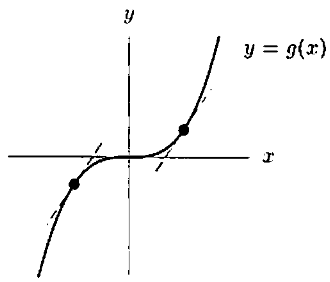
41. If $f(x)$ is even, its graph is symmetric about the y-axis. So the tangent line to f at $x = x_0$ is the same as that at $x = -x_0$ reflected about the y-axis.

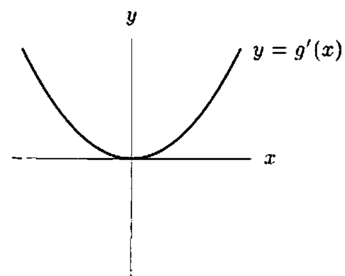




So the slopes of these two tangent lines are opposite in sign, so $f'(x_0) = -f'(-x_0)$, and f' is odd.

42. If $g(x)$ is odd, its graph remains the same if you rotate it 180° about the origin. So the tangent line to g at $x = x_0$ is the tangent line to g at $x = -x_0$, rotated 180°.





But the slope of a line stays constant if you rotate it 180°. So $g'(x_0) = g'(-x_0)$; g' is even.

Solutions for Section 2.5

Exercises

1. (a) As the cup of coffee cools, the temperature decreases, so $f'(t)$ is negative.
   (b) Since $f'(t) = dH/dt$, the units are degrees Celsius per minute. The quantity $f'(20)$ represents the rate at which the coffee is cooling, in degrees per minute, 20 minutes after the cup is put on the counter.
2. (Note that we are considering the average temperature of the yam, since its temperature is different at different points inside it.)
   (a) It is positive, because the temperature of the yam increases the longer it sits in the oven.
   (b) The units of $f'(20)$ are °F/min. $f'(20) = 2$ means that at time $t = 20$ minutes, the temperature T increases by approximately 2°F for each additional minute in the oven.
3. (a) The statement $f(200) = 350$ means that it costs \$350 to produce 200 gallons of ice cream.
   (b) The statement $f'(200) = 1.4$ means that when the number of gallons produced is 200, costs are increasing by about \$1.40 per gallon. In other words, it costs about \$1.40 to produce the next (the 201st) gallon of ice cream.
4. (a) The statement $f(5) = 18$ means that when 5 milliliters of catalyst are present, the reaction will take 18 minutes. Thus, the units for 5 are ml while the units for 18 are minutes.
   (b) As in part (a), 5 is measured in ml. Since f' tells how fast T changes per unit a, we have f' measured in minutes/ml. If the amount of catalyst increases by 1 ml (from 5 to 6 ml), the reaction time decreases by about 3 minutes.

5. Since B is measured in dollars and t is measured in years, dB/dt is measured in dollars per year. We can interpret dB as the extra money added to your balance in dt years. Therefore dB/dt represents how fast your balance is growing, in units of dollars/year.

6. (a) This means that investing the \$1000 at 5% would yield \$1649 after 10 years.
   (b) Writing $g'(r)$ as dB/dt, we see that the units of dB/dt are dollars per percent (interest). We can interpret dB as the extra money earned if interest rate is increased by dr percent. Therefore $g'(5) = \frac{dB}{dr}|_{r=5} \approx 165$ means that the balance, at 5% interest, would increase by about \$165 if the interest rate were increased by 1%. In other words, $g(6) \approx g(5) + 165 = 1649 + 165 = 1814$.

7. Units of $C'(r)$ are dollars/percent. Approximately, $C'(r)$ means the additional amount needed to pay off the loan when the interest rate is increased by 1%. The sign of $C'(r)$ is positive, because increasing the interest rate will increase the amount it costs to pay off a loan.

8. Units of $P'(t)$ are dollars/year. The practical meaning of $P'(t)$ is the rate at which the monthly payments change as the duration of the mortgage increases. Approximately, $P'(t)$ represents the change in the monthly payment if the duration is increased by one year. $P'(t)$ is negative because increasing the duration of a mortgage decreases the monthly payments.

9. The units of $f'(x)$ are feet/mile. The derivative, $f'(x)$, represents the rate of change of elevation with distance from the source, so if the river is flowing downhill everywhere, the elevation is always decreasing and $f'(x)$ is always negative. (In fact, there may be some stretches where the elevation is more or less constant, so $f'(x) = 0$.)

10. (a) If the price is \$150, then 2000 items will be sold.
    (b) If the price goes up from \$150 by \$1 per item, about 25 fewer items will be sold. Equivalently, if the price is decreased from \$150 by \$1 per item, about 25 more items will be sold.

Problems

11. (a) Since $W = f(c)$ where W is weight in pounds and c is the number of Calories consumed per day:

$f(1800) = 155$	means that	consuming 1800 Calories per day results in a weight of 155 pounds.
$f'(2000) = 0$	means that	consuming 2000 Calories per day causes neither weight gain nor loss.
$f^{-1}(162) = 2200$	means that	a weight of 162 pounds is caused by a consumption of 2200 Calories per day.

    (b) The units of dW/dc are pounds/(Calories/day).

12. The graph is increasing for $0 < t < 10$ and is decreasing for $10 < t < 20$. One possible graph is shown in Figure 2.36. The units on the horizontal axis are years and the units on the vertical axis are people.

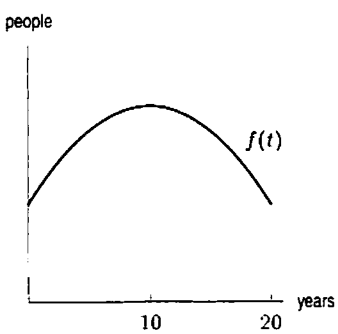
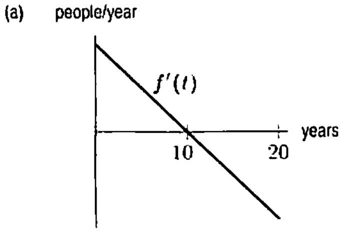


Figure 2.36

The derivative is positive for $0 < t < 10$ and negative for $10 < t < 20$. Two possible graphs are shown in Figure 2.37. The units on the horizontal axes are years and the units on the vertical axes are people per year.



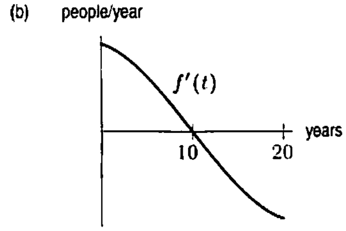


Figure 2.37

13. Since $f(t) = 1.15(1.014)^t$, we have
$$f(6) = 1.15(1.014)^6 = 1.25.$$
To estimate $f'(6)$, we use a small interval around 6:
$$f'(6) \approx \frac{f(6.001) - f(6)}{6.001 - 6} = \frac{1.15(1.014)^{6.001} - 1.15(1.014)^6}{0.001} = 0.0174.$$
We see that $f(6) = 1.25$ billion people and $f'(6) = 0.0174$ billion people per year. This model tells us that the population of China was about 1,250,000,000 people in 1999 and was growing at a rate of about 17,400,000 people per year at that time.

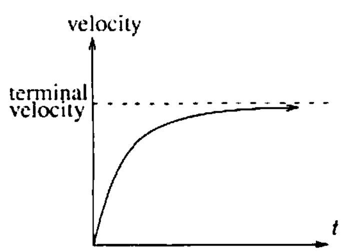
14. (a) The statement $f(140) = 120$ means that a patient weighing 140 pounds should receive a dose of 120 mg of the painkiller. The statement $f'(140) = 3$ tells us that if the weight of a patient increases by about one pound (from 140 pounds), the dose should be increased by about 3 mg.
(b) Since the dose for a weight of 140 lbs is 120 mg and at this weight the dose goes up by 3 mg for each pound, a 145 lb patient should get an additional $3(5) = 15$ mg. Thus, for a 145 lb patient, the correct dose is approximately
$$f(145) \approx 120 + 3(5) = 135 \text{ mg}.$$

15. (a) When $t = 10$, that is, at 10 am, 3.1 cm of rain has fallen.
(b) We are told that when 10 cm of rain has fallen, 16 hours have passed ($t = 16$); that is, 10 cm of rain has fallen by 4 pm.
(c) The rate at which rain is falling is 0.4 cm/hr at $t = 8$, that is, at 8 am.
(d) The units of $(f^{-1})'(5)$ are hours/cm. Thus, we are being told that when 5 cm of rain has fallen, rain is falling at a rate such that it will take 2 additional hours for another centimeter to fall.

16. (a) The pressure in dynes/cm^2 at a depth of 100 meters.
(b) The depth of water in meters giving a pressure of $1.2 \cdot 10^6$ dynes/cm^2.
(c) The pressure at a depth of h meters plus a pressure of 20 dynes/cm^2.
(d) The pressure at a depth of 20 meters below the diver.
(e) The rate of increase of pressure with respect to depth, at 100 meters, in units of dynes/cm^2 per meter. Approximately, $p'(100)$ represents the increase in pressure in going from 100 meters to 101 meters.
(f) The depth, in meters, at which the rate of change of pressure with respect to depth is 20 dynes/cm^2 per meter.

17. Units of $g'(55)$ are mpg/mph. The statement $g'(55) = -0.54$ means that at 55 miles per hour the fuel efficiency (in miles per gallon, or mpg) of the car decreases at a rate of approximately one half mpg as the velocity increases by one mph.

18. (a)



(b) The graph should be concave down because wind resistance decreases your acceleration as you speed up, and so the slope of the graph of velocity is decreasing.
(c) The slope represents the acceleration due to gravity.

19. (a) The company hopes that increased advertising always brings in more customers instead of turning them away. Therefore, it hopes $f'(a)$ is always positive.
(b) If $f'(100) = 2$, it means that if the advertising budget is \$100,000, each extra dollar spent on advertising will bring in \$2 worth of sales. If $f'(100) = 0.5$, each dollar above \$100 thousand spent on advertising will bring in \$0.50 worth of sales.
(c) If $f'(100) = 2$, then as we saw in part (b), spending slightly more than \$100,000 will increase revenue by an amount greater than the additional expense, and thus more should be spent on advertising. If $f'(100) = 0.5$, then the increase in revenue is less than the additional expense, hence too much is being spent on advertising. The optimum amount to spend is an amount that makes $f'(a) = 1$. At this point, the increases in advertising expenditures just pay for themselves. If $f'(a) < 1$, too much is being spent; if $f'(a) > 1$, more should be spent.

20. Since $\frac{P(67)-P(66)}{67-66}$ is an estimate of $P'(66)$, we may think of $P'(66)$ as an estimate of $P(67) - P(66)$, and the latter is the number of people between 66 and 67 inches tall. Alternatively, since $\frac{P(66.5)-P(65.5)}{66.5-65.5}$ is a better estimate of $P'(66)$, we may regard $P'(66)$ as an estimate of the number of people of height between 65.5 and 66.5 inches. The units for $P'(x)$ are people per inch. Since there were 250 million people at the 1990 census, we might guess that there are about 200 million full-grown persons in the US whose heights are distributed between 60″(5′) and 75″(6′3″). There are probably quite a few people of height 66″–perhaps $1\frac{1}{2}$ what you'd expect from an even, or uniform, distribution–because it's nearly average. An even distribution would yield $P'(66) = \frac{200 \text{ million}}{15''} \approx 13$ million per inch–so we can expect $P'(66)$ to be perhaps $13(1.5) \approx 20$.

$P'(x)$ is never negative because $P(x)$ is never decreasing. To see this, let's look at an example involving a particular value of x, say $x = 70$. The value $P(70)$ represents the number of people whose height is less than or equal to 70 inches, and $P(71)$ represents the number of people whose height is less than or equal to 71 inches. Since everyone shorter than 70 inches is also shorter than 71 inches, $P(70) \leq P(71)$. In general, $P(x)$ is 0 for small x, and increases as x increases, and is eventually constant (for large enough x).

21. (a) The units of compliance are units of volume per units of pressure, or liters per centimeter of water.
(b) The increase in volume for a 5 cm reduction in pressure is largest between 10 and 15 cm. Thus, the compliance appears maximum between 10 and 15 cm of pressure reduction. The derivative is given by the slope, so

$$\text{Compliance} \approx \frac{0.70 - 0.49}{15 - 10} = 0.042 \text{ liters per centimeter.}$$

(c) When the lung is nearly full, it cannot expand much more to accommodate more air.

Solutions for Section 2.6

Exercises

1. (a) Since the graph is below the x-axis at $x = 2$, $f(2)$ is negative.
(b) Since $f(x)$ is decreasing at $x = 2$, $f'(2)$ is negative.
(c) Since $f(x)$ is concave up at $x = 2$, $f''(2)$ is positive.

2. By noting whether $f(x)$ is positive or negative, increasing or decreasing, and concave up or down at each of the given points, we get the completed Table 2.16:

Table 2.16

Point	f	f'	f''
A	−	0	+
B	+	0	−
C	+	−	−
D	−	+	+

3. At B both dy/dx and d^2y/dx^2 are positive because at B the graph is increasing, so $dy/dx > 0$, and concave up, so $d^2y/dx^2 > 0$.

4. The velocity is the derivative of the distance, that is, $v(t) = s'(t)$. Therefore, we have

$$\begin{aligned} v(t) &= \lim_{h\to 0} \frac{s(t+h)-s(t)}{h} \\ &= \lim_{h\to 0} \frac{(5(t+h)^2+3)-(5t^2+3)}{h} \\ &= \lim_{h\to 0} \frac{10th+5h^2}{h} \\ &= \lim_{h\to 0} \frac{h(10t+5h)}{h} = \lim_{h\to 0}(10t+5h) = 10t \text{ km/minute.} \end{aligned}$$

The acceleration is the derivative of velocity, so $a(t) = v'(t)$:

$$\begin{aligned} a(t) &= \lim_{h\to 0} \frac{10(t+h)-10t}{h} \\ &= \lim_{h\to 0} \frac{10h}{h} = 10 \text{ km/(minute)}^2. \end{aligned}$$

5. The function is everywhere increasing and concave up. One possible graph is shown in Figure 2.38.

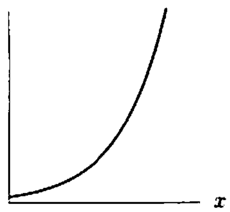


Figure 2.38

6. The graph must be everywhere decreasing and concave up on some intervals and concave down on other intervals. One possibility is shown in Figure 2.39.

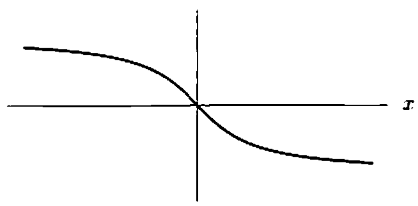


Figure 2.39

7. Since velocity is positive and acceleration is negative, we have $f' > 0$ and $f'' < 0$, and so the graph is increasing and concave down. See Figure 2.40.

Figure 2.40

8. $f'(x) > 0$
$f''(x) > 0$

9. $f'(x) = 0$
$f''(x) = 0$

10. $f'(x) < 0$
$f''(x) = 0$

11. $f'(x) < 0$
$f''(x) > 0$

12. $f'(x) > 0$
$f''(x) < 0$

13. $f'(x) < 0$
$f''(x) < 0$

Problems

14. (a) The derivative, $f'(t)$, appears to be positive since the number of cars is increasing. The second derivative, $f''(t)$, appears to be positive because the rate of change is increasing. For example, between 1940 and 1950, the rate of change is $(40.3 - 27.5)/10 = 1.28$ million cars per year, while between 1950 and 1960, the rate of change is 2.14 million cars per year.

(b) We use the average rate of change formula on the interval 1970 to 1980 to estimate $f'(1975)$:

$$f'(1975) \approx \frac{121.6 - 89.3}{1980 - 1970} = \frac{32.3}{10} = 3.23.$$

We see that $f'(1975) \approx 3.23$ million cars per year. The number of passenger cars in the US was increasing at a rate of about 3.23 million cars per year in 1975.

15. To measure the average acceleration over an interval, we calculate the average rate of change of velocity over the interval. The units of acceleration are ft/sec per second, or (ft/sec)/sec, written ft/sec^2.

$$\text{Average acceleration for } 0 \leq t \leq 1 = \frac{\text{Change in velocity}}{\text{Time}} = \frac{v(1) - v(0)}{1} = \frac{30 - 0}{1} = 30 \text{ ft/sec}^2$$

$$\text{Average acceleration for } 1 \leq t \leq 2 = \frac{52 - 30}{2 - 1} = 22 \text{ ft/sec}^2$$

16. To the right of $x = 5$, the function starts by increasing, since $f'(5) = 2 > 0$ (though f may subsequently decrease) and is concave down, so its graph looks like the graph shown in Figure 2.41. Also, the tangent line to the curve at $x = 5$ has slope 2 and lies above the curve for $x > 5$. If we follow the tangent line until $x = 7$, we reach a height of 24. Therefore, $f(7)$ must be smaller than 24, meaning 22 is the only possible value for $f(7)$ from among the choices given.

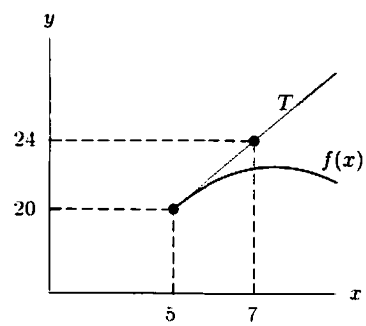
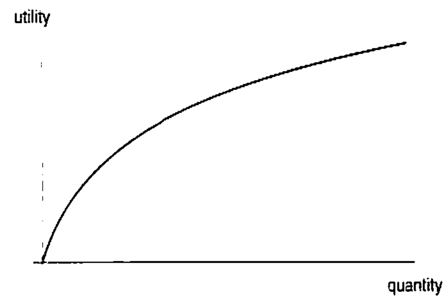


Figure 2.41

17. (a) $dP/dt > 0$ and $d^2P/dt^2 > 0$.

(b) $dP/dt < 0$ and $d^2P/dt^2 > 0$ (but dP/dt is close to zero).

18. (a)

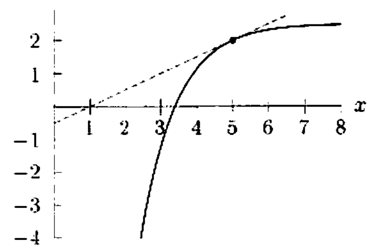


(b) As a function of quantity, utility is increasing but at a decreasing rate; the graph is increasing but concave down. So the derivative of utility is positive, but the second derivative of utility is negative.

19. Since all advertising campaigns are assumed to produce an increase in sales, a graph of sales against time would be expected to have a positive slope.

A positive second derivative means the rate at which sales are increasing is increasing. If a positive second derivative is observed during a new campaign, it is reasonable to conclude that this increase in the rate sales are increasing is caused by the new campaign–which is therefore judged a success. A negative second derivative means a decrease in the rate at which sales are increasing, and therefore suggests the new campaign is a failure.

20. (a)



(b) Exactly one. There can't be more than one zero because f is increasing everywhere. There does have to be one zero because f stays below its tangent line (dotted line in above graph), and therefore f must cross the x-axis.

(c) The equation of the (dotted) tangent line is $y = \frac{1}{2}x - \frac{1}{2}$, and so it crosses the x-axis at $x = 1$. Therefore the zero of f must be between $x = 1$ and $x = 5$.

(d) $\lim_{x\to-\infty} f(x) = -\infty$, because f is increasing and concave down. Thus, as $x \to -\infty$, $f(x)$ decreases, at a faster and faster rate.

(e) Yes.

(f) No. The slope is decreasing since f is concave down, so $f'(1) > f'(5)$, i.e. $f'(1) > \frac{1}{2}$.

21. (a) The EPA will say that the rate of discharge is still rising. The industry will say that the rate of discharge is increasing less quickly, and may soon level off or even start to fall.

(b) The EPA will say that the rate at which pollutants are being discharged is levelling off, but not to zero — so pollutants will continue to be dumped in the lake. The industry will say that the rate of discharge has decreased significantly.

22. Since f' is everywhere positive, f is everywhere increasing. Hence the greatest value of f is at x_6 and the least value of f is at x_1. Directly from the graph, we see that f' is greatest at x_3 and least at x_2. Since f'' gives the slope of the graph of f', f'' is greatest where f' is rising most rapidly, namely at x_6, and f'' is least where f' is falling most rapidly, namely at x_1.

23. (a) B (where $f', f'' > 0$) and E (where $f', f'' < 0$)

(b) A (where $f = f' = 0$) and D (where $f' = f'' = 0$)

Solutions for Section 2.7

Exercises

1. (a) Function f is not continuous at $x = 1$.

(b) Function f appears not differentiable at $x = 1, 2, 3$.

2. (a) Function g appears continuous at all x-values shown.
   (b) Function g appears not differentiable at $x = 2, 4$. At $x = 2$, the curve is vertical, so the derivative does not exist. At $x = 4$, the graph has a corner, so the derivative does not exist.
3. (a) The absolute value function is continuous everywhere. See Figure 2.42.
   (b) The absolute value function is not differentiable at $x = 0$. The graph has a corner at $x = 0$, which suggests f is not differentiable there. (See Figure 2.42.) This is confirmed by the fact that the limit of the difference quotient

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

does not exist for $x = 0$, since the following limit does not exist:

$$\lim_{h\to 0} \frac{|h|}{h}.$$

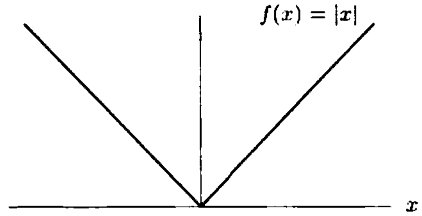


Figure 2.42

4. No, there are sharp turning points.
5. Yes.

Problems

6. We want to look at

$$\lim_{h\to 0} \frac{(h^2+0.0001)^{1/2}-(0.0001)^{1/2}}{h}.$$

As $h \to 0$ from positive or negative numbers, the difference quotient approaches 0. (Try evaluating it for $h = 0.001$, 0.0001, etc.) So it appears there is a derivative at $x = 0$ and that this derivative is zero. How can this be if f has a corner at $x = 0$?

The answer lies in the fact that what appears to be a corner is in fact smooth—when you zoom in, the graph of f looks like a straight line with slope 0! See Figure 2.43.

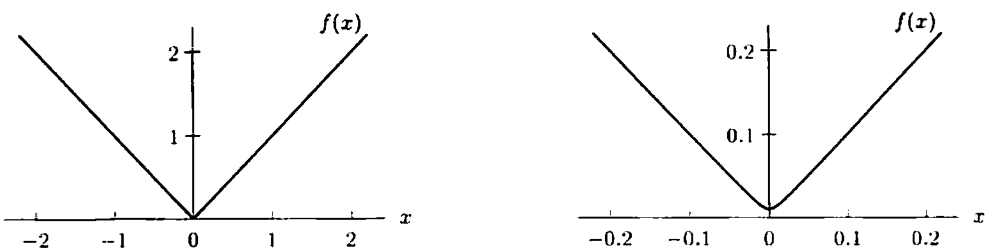
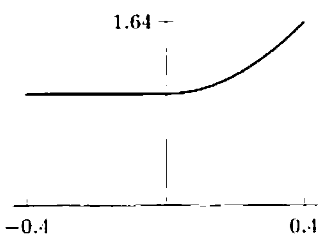


Figure 2.43: Close-ups of $f(x) = (x^2 + 0.0001)^{1/2}$ showing differentiability at $x = 0$

7. Yes, f is differentiable at $x = 0$, since its graph does not have a "corner" at $x = 0$. See below.



Another way to see this is by computing:

$$\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{(h+|h|)^2}{h}=\lim_{h\to 0}\frac{h^2+2h|h|+|h|^2}{h}.$$

Since $|h|^2 = h^2$, we have:

$$\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{2h^2+2h|h|}{h}=\lim_{h\to 0}2(h+|h|)=0.$$

So f is differentiable at 0 and $f'(0) = 0$.

8. As we can see in Figure 2.44, f oscillates infinitely often between the x-axis and the line $y = 2x$ near the origin. This means a line from $(0, 0)$ to a point $(h, f(h))$ on the graph of f alternates between slope 0 (when $f(h) = 0$) and slope 2 (when $f(h) = 2h$) infinitely often as h tends to zero. Therefore, there is no limit of the slope of this line as h tends to zero, and thus there is no derivative at the origin. Another way to see this is by noting that

$$\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{h\sin(\frac{1}{h})+h}{h}=\lim_{h\to 0}\left(\sin\left(\frac{1}{h}\right)+1\right)$$

does not exist, since $\sin(\frac{1}{h})$ does not have a limit as h tends to zero. Thus, f is not differentiable at $x = 0$.

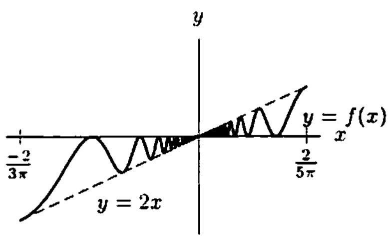


Figure 2.44

9. We can see from Figure 2.45 that the graph of f oscillates infinitely often between the curves $y = x^2$ and $y = -x^2$ near the origin. Thus the slope of the line from $(0, 0)$ to $(h, f(h))$ oscillates between h (when $f(h) = h^2$ and $\frac{f(h)-0}{h-0} = h$) and $-h$ (when $f(h) = -h^2$ and $\frac{f(h)-0}{h-0} = -h$) as h tends to zero. So, the limit of the slope as h tends to zero is 0, which is the derivative of f at the origin. Another way to see this is to observe that

$$\begin{aligned}\lim_{h\to 0}\frac{f(h)-f(0)}{h}&=\lim_{h\to 0}\left(\frac{h^2\sin(\frac{1}{h})}{h}\right)\\&=\lim_{h\to 0}h\sin(\frac{1}{h})\\&=0,\end{aligned}$$

since $\lim_{h\to 0} h = 0$ and $-1 \le \sin(\frac{1}{h}) \le 1$ for any h. Thus f is differentiable at $x = 0$, and $f'(0) = 0$.

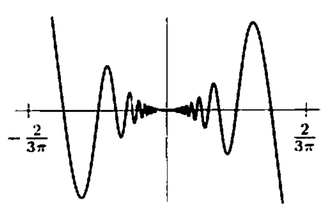


Figure 2.45

10. (a) The graph is concave up everywhere, except at $x = 2$ where the derivative is undefined. This is the case if the graph has a corner at $x = 2$. One possible graph is shown in Figure 2.46.

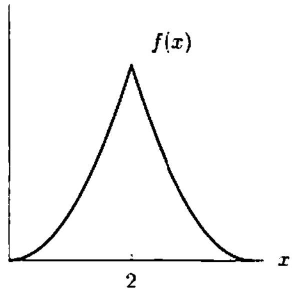


Figure 2.46

(b) The graph is concave up for $x < 2$ and concave down for $x > 2$, and the derivative is undefined at $x = 2$. This is the case if the graph is vertical at $x = 2$. One possible graph is shown in Figure 2.47.

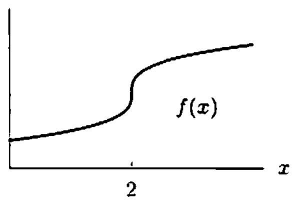
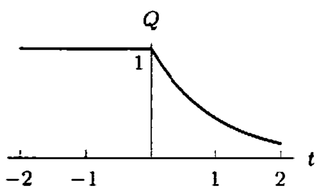


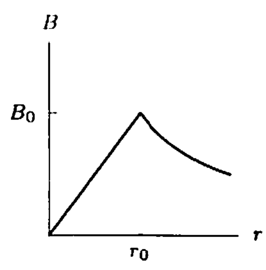
Figure 2.47

11. (a) The graph of Q against t does not have a break at $t = 0$, so Q appears to be continuous at $t = 0$. See below.



(b) The slope dQ/dt is zero for $t < 0$, and negative for all $t > 0$. At $t = 0$, there appears to be a corner, which does not disappear as you zoom in, suggesting that I is defined for all times t except $t = 0$.

12. (a) Notice that B is a linear function of r for $r \leq r_0$ and a reciprocal for $r > r_0$. The constant B_0 is the value of B at $r = r_0$ and the maximum value of B.



(b) B is continuous at $r = r_0$ because there is no break in the graph there. Using the formula for B, we have

$$\lim_{r \to r_0^-} B = \frac{r_0}{r_0} B_0 = B_0 \quad \text{and} \quad \lim_{r \to r_0^+} B = \frac{r_0}{r_0} B_0 = B_0.$$

(c) The function B is not differentiable at $r = r_0$ because the graph has a corner there. The slope is positive for $r < r_0$ and the slope is negative for $r > r_0$.

13. (a) Since

$$\lim_{r\to r_0^-} E = kr_0$$

and

$$\lim_{r\to r_0^+} E = \frac{kr_0^2}{r_0} = kr_0$$

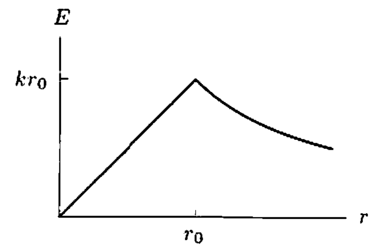
and

$$E(r_0) = kr_0.$$

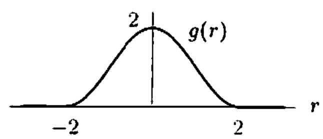
we see that E is continuous at r_0.

(b) The function E is not differentiable at $r = r_0$ because the graph has a corner there. The slope is positive for $r < r_0$ and the slope is negative for $r > r_0$.

(c)



14.



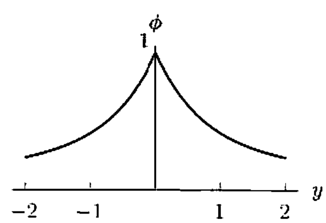
(a) The graph of $g(r)$ does not have a break or jump at $r = 2$, and so $g(r)$ is continuous there. This is confirmed by the fact that

$$g(2) = 1 + \cos(\pi 2/2) = 1 + (-1) = 0$$

so the value of $g(r)$ as you approach $r = 2$ from the left is the same as the value when you approach $r = 2$ from the right.

(b) The graph of $g(r)$ does not have a corner at $r = 2$, even after zooming in, so $g(r)$ appears to be differentiable at $r = 0$. This is confirmed by the fact that $\cos(\pi r/2)$ is at the bottom of a trough at $r = 2$, and so its slope is 0 there. Thus the slope to the left of $r = 2$ is the same as the slope to the right of $r = 2$.

15. (a) The graph of ϕ does not have a break at $y = 0$, and so ϕ appears to be continuous there. See figure below.



(b) The graph of ϕ has a corner at $y = 0$ which does not disappear as you zoom in. Therefore ϕ appears not be differentiable at $y = 0$.

16. We will show $f(x) = x$ is continuous at $x = c$. Since $f(c) = c$, we need to show that

$$\lim_{x\to c} f(x) = c$$

that is, since $f(x) = x$, we need to show

$$\lim_{x\to c} x = c.$$

Pick any $\epsilon > 0$, then take $\delta = \epsilon$. Thus,

$$|f(x) - c| = |x - c| < \epsilon \quad \text{for all} \quad |x - c| < \delta = \epsilon.$$

17. Since $f(x) = x$ is continuous, Theorem 2.2 on page 95 shows that products of the form $f(x) \cdot f(x) = x^2$ and $f(x) \cdot x^2 = x^3$, etc., are continuous. By a similar argument, x^n is continuous for any $n > 0$.

18. If c is in the interval, we know $\lim_{x\to c} f(x) = f(c)$ and $\lim_{x\to c} g(x) = g(c)$. Then,

$$\begin{aligned}\lim_{x\to c} (f(x) + g(x)) &= \lim_{x\to c} f(x) + \lim_{x\to c} g(x) \quad \text{by limit property 2} \\ &= f(c) + g(c), \quad \text{so } f + g \text{ is continuous at } x = c.\end{aligned}$$

Also,

$$\begin{aligned}\lim_{x\to c} (f(x)g(x)) &= \lim_{x\to c} f(x) \lim_{x\to c} g(x) \quad \text{by limit property 3} \\ &= f(c)g(c) \quad \text{so } fg \text{ is continuous at } x = c.\end{aligned}$$

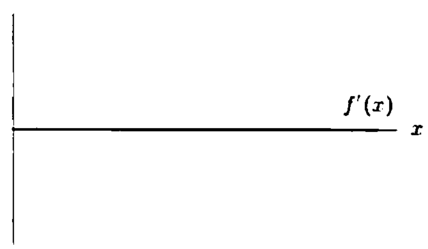
Finally,

$$\begin{aligned}\lim_{x\to c} \frac{f(x)}{g(x)} &= \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} \quad \text{by limit property 4} \\ &= \frac{f(c)}{g(c)}, \quad \text{so } \frac{f}{g} \text{ is continuous at } x = c.\end{aligned}$$

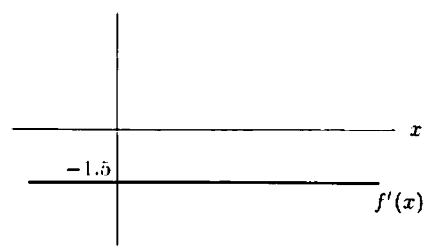
Solutions for Chapter 2 Review

Exercises

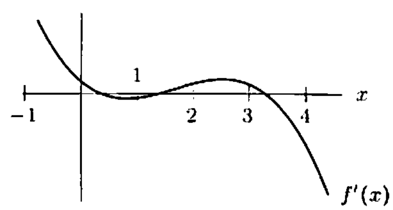
1.



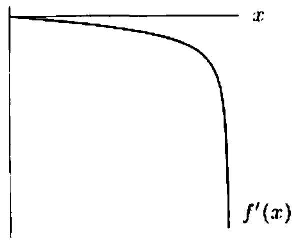
2.



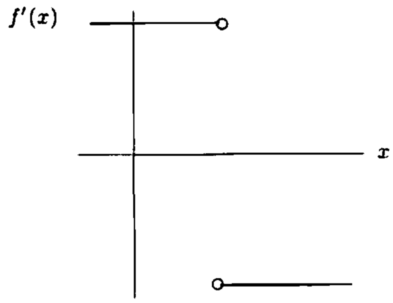
3.



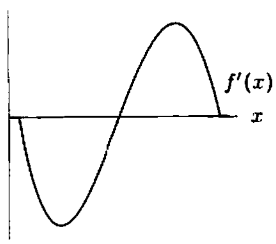
4.



5.



6.



7. Using the definition of the derivative

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{5(x+h)^2 + x + h - (5x^2 + x)}{h} \\
&= \lim_{h\to 0} \frac{5(x^2 + 2xh + h^2) + x + h - 5x^2 - x}{h} \\
&= \lim_{h\to 0} \frac{10xh + 5h^2 + h}{h} \\
&= \lim_{h\to 0} (10x + 5h + 1) = 10x + 1
\end{aligned}
$$

8. Using the definition of the derivative, we have

$$
\begin{aligned}
n'(x) &= \lim_{h\to 0} \frac{n(x+h)-n(x)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left[\left(\frac{1}{x+h}+1\right) - \left(\frac{1}{x}+1\right)\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right) \\
&= \lim_{h\to 0} \frac{x-(x+h)}{hx(x+h)} \\
&= \lim_{h\to 0} \frac{-h}{hx(x+h)} \\
&= \lim_{h\to 0} \frac{-1}{x(x+h)} = \frac{-1}{x^2}.
\end{aligned}
$$

9. From Table 2.17, it appears the limit is 0. This is confirmed by Figure 2.48. An appropriate window is $-0.015 < x < 0.015$, $-0.01 < y < 0.01$.

Table 2.17

x	$f(x)$
0.1	0.0666
0.01	0.0067
0.001	0.0007
0.0001	0

x	$f(x)$
−0.0001	−0.0001
−0.001	−0.0007
−0.01	−0.0067
−0.1	−0.0666

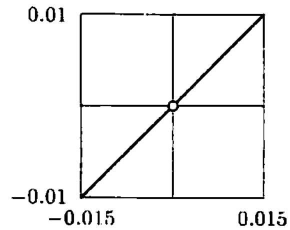


Figure 2.48

10. From Table 2.18, it appears the limit is 0. This is confirmed by Figure 2.49. An appropriate window is $-0.0029 < x < 0.0029$, $-0.01 < y < 0.01$.

Table 2.18

x	$f(x)$
0.1	0.3365
0.01	0.0337
0.001	0.0034
0.0001	0.0004

x	$f(x)$
−0.0001	−0.0004
−0.001	−0.0034
−0.01	−0.0337
−0.1	−0.3365

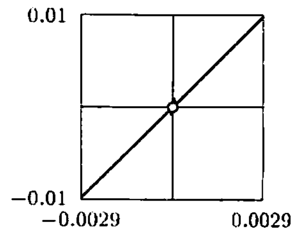


Figure 2.49

11. From Table 2.19, it appears the limit is 0. Figure 2.50 confirms this. An appropriate window is $1.570 < x < 1.5715$, $-0.01 < y < 0.01$.

Table 2.19

x	$f(x)$
1.6708	−1.2242
1.5808	−0.1250
1.5718	−0.0125
1.5709	−0.0013
1.5707	0.0012
1.5698	0.0125
1.5608	0.1249
1.4708	1.2241

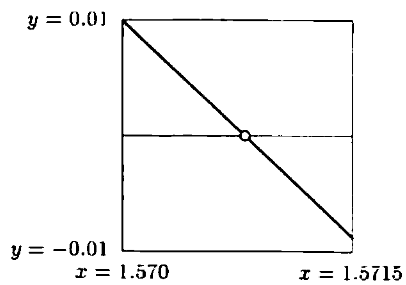


Figure 2.50

12. From Table 2.20, it appears the limit is 1/2. Figure 2.51 confirms this. An appropriate window is $1.92 < x < 2.07$, $0.49 < y < 0.51$.

Table 2.20

x	$f(x)$
2.1	0.5127
2.01	0.5013
2.001	0.5001
2.0001	0.5000
1.9999	0.5000
1.999	0.4999
1.99	0.4988
1.9	0.4877

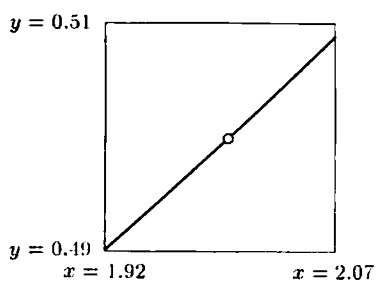


Figure 2.51

13. $\lim_{h\to 0}\frac{(a+h)^2-a^2}{h}=\lim_{h\to 0}\frac{a^2+2ah+h^2-a^2}{h}=\lim_{h\to 0}(2a+h)=2a$

14. $\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{a+h}-\frac{1}{a}\right)=\lim_{h\to 0}\frac{a-(a+h)}{(a+h)ah}=\lim_{h\to 0}\frac{-1}{(a+h)a}=\frac{-1}{a^2}$

15. $\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{(a+h)^2}-\frac{1}{a^2}\right)=\lim_{h\to 0}\frac{a^2-(a^2+2ah+h^2)}{(a+h)^2a^2h}=\lim_{h\to 0}\frac{(-2a-h)}{(a+h)^2a^2}=\frac{-2}{a^3}$

16. $\sqrt{a+h}-\sqrt{a}=\frac{(\sqrt{a+h}-\sqrt{a})(\sqrt{a+h}+\sqrt{a})}{\sqrt{a+h}+\sqrt{a}}=\frac{a+h-a}{\sqrt{a+h}+\sqrt{a}}=\frac{h}{\sqrt{a+h}+\sqrt{a}}.$

Therefore $\lim_{h\to 0}\frac{\sqrt{a+h}-\sqrt{a}}{h}=\lim_{h\to 0}\frac{1}{\sqrt{a+h}+\sqrt{a}}=\frac{1}{2\sqrt{a}}$

17. We combine terms in the numerator and multiply top and bottom by $\sqrt{a}+\sqrt{a+h}$.

$$\frac{1}{\sqrt{a+h}}-\frac{1}{\sqrt{a}}=\frac{\sqrt{a}-\sqrt{a+h}}{\sqrt{a+h}\sqrt{a}}=\frac{(\sqrt{a}-\sqrt{a+h})(\sqrt{a}+\sqrt{a+h})}{\sqrt{a+h}\sqrt{a}(\sqrt{a}+\sqrt{a+h})}$$
$$=\frac{a-(a+h)}{\sqrt{a+h}\sqrt{a}(\sqrt{a}+\sqrt{a+h})}$$

Therefore $\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{\sqrt{a+h}}-\frac{1}{\sqrt{a}}\right)=\lim_{h\to 0}\frac{-1}{\sqrt{a+h}\sqrt{a}(\sqrt{a}+\sqrt{a+h})}=\frac{-1}{2(\sqrt{a})^3}$

18. $f(x)=\frac{x^3|2x-6|}{x-3}=\begin{cases}\frac{x^3(2x-6)}{x-3}=2x^3, & x>3\\ \frac{x^3(-2x+6)}{x-3}=-2x^3, & x<3\end{cases}$

Figure 2.52 confirms that $\lim_{x\to 3^+}f(x)=54$ while $\lim_{x\to 3^-}f(x)=-54$; thus $\lim_{x\to 3}f(x)$ does not exist.

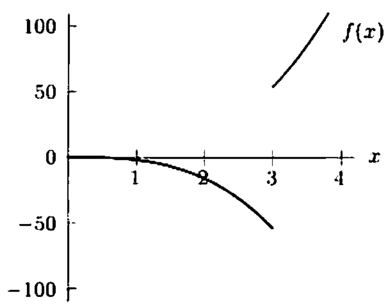


Figure 2.52

19. $f(x)=\begin{cases}e^x & -1<x<0\\ 1 & x=0\\ \cos x & 0<x<1\end{cases}$

Figure 2.53 confirms that $\lim_{x\to 0^-}f(x)=\lim_{x\to 0^-}e^x=e^0=1$, and that $\lim_{x\to 0+}f(x)=\lim_{x\to 0+}\cos x=\cos 0=1$, so $\lim_{x\to 0}f(x)=1$.

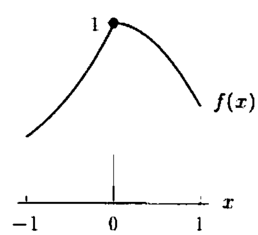


Figure 2.53

Problems

20. (a) A possible example is $f(x) = 1/|x-2|$ as $\lim_{x\to 2} 1/|x-2| = \infty$.
(b) A possible example is $f(x) = -1/(x-2)^2$ as $\lim_{x\to 2} -1/(x-2)^2 = -\infty$.

21. Since $f(2) = 3$ and $f'(2) = 1$, near $x = 2$ the graph looks like the segment shown in Figure 2.54.

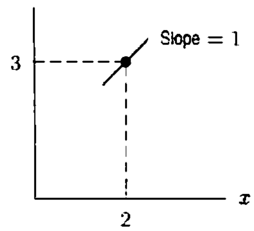


Figure 2.54

(a) If $f(x)$ is even, then the graph of $f(x)$ near $x = 2$ and $x = -2$ looks like Figure 2.55. Thus $f(-2) = 3$ and $f'(-2) = -1$.
(b) If $f(x)$ is odd, then the graph of $f(x)$ near $x = 2$ and $x = -2$ looks like Figure 2.56. Thus $f(-2) = -3$ and $f'(-2) = 1$.

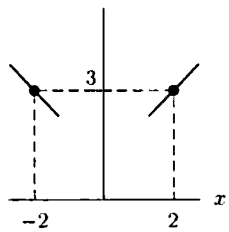


Figure 2.55: For f even

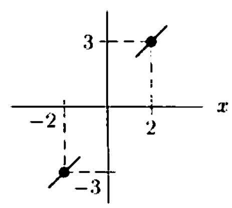


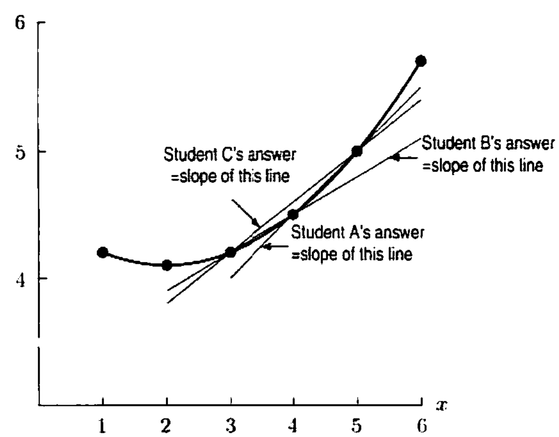
Figure 2.56: For f odd

22. The slopes of the lines drawn through successive pairs of points are negative but increasing, suggesting that $f''(x) > 0$ for $1 \le x \le 3.3$ and that the graph of $f(x)$ is concave up.

23. Using the approximation $\Delta y \approx f'(x)\Delta x$ with $\Delta x = 2$, we have $\Delta y \approx f'(20) \cdot 2 = 6 \cdot 2$, so

$$f(22) \approx f(20) + f'(20) \cdot 2 = 345 + 6 \cdot 2 = 357.$$

24. (a)

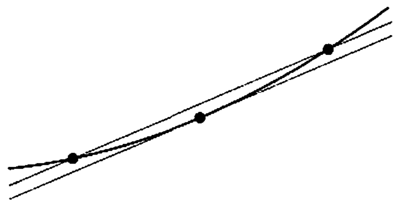


(b) The slope of f appears to be somewhere between student A's answer and student B's, so student C's answer, halfway in between, is probably the most accurate.

(c) Student A's estimate is $f'(x) \approx \frac{f(x+h)-f(x)}{h}$, while student B's estimate is $f'(x) \approx \frac{f(x)-f(x-h)}{h}$. Student C's estimate is the average of these two, or

$$f'(x) \approx \frac{1}{2}\left[\frac{f(x+h)-f(x)}{h} + \frac{f(x)-f(x-h)}{h}\right] = \frac{f(x+h)-f(x-h)}{2h}.$$

This estimate is the slope of the chord connecting $(x-h, f(x-h))$ to $(x+h, f(x+h))$. Thus, we estimate that the tangent to a curve is nearly parallel to a chord connecting points h units to the right and left, as shown below.



25. (a) Since the point $A = (7, 3)$ is on the graph of f, we have $f(7) = 3$.

(b) The slope of the tangent line touching the curve at $x = 7$ is given by

$$\text{Slope} = \frac{\text{Rise}}{\text{Run}} = \frac{3.8-3}{7.2-7} = \frac{0.8}{0.2} = 4.$$

Thus, $f'(7) = 4$.

26. At point A, we are told that $x = 1$ and $f(1) = 3$. Since $A = (x_2, y_2)$, we have $x_2 = 1$ and $y_2 = 3$. Since $h = 0.1$, we know $x_1 = 1 - 0.1 = 0.9$ and $x_3 = 1 + 0.1 = 1.1$.

Now consider Figure 2.57. Since $f'(1) = 2$, the slope of the tangent line AD is 2. Since $AB = 0.1$,

$$\frac{\text{Rise}}{\text{Run}} = \frac{BD}{0.1} = 2,$$

so $BD = 2(0.1) = 0.2$. Therefore $y_1 = 3 - 0.2 = 2.8$ and $y_3 = 3 + 0.2 = 3.2$.

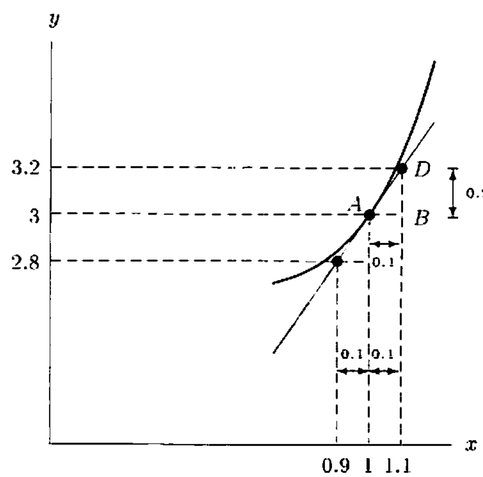


Figure 2.57

27. A possible graph of $y = f(x)$ is shown in Figure 2.58.

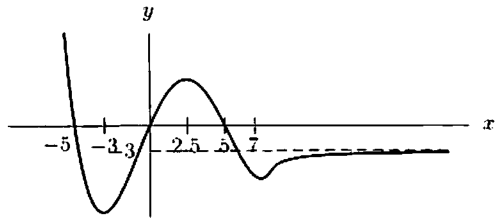


Figure 2.58

28. (a) The yam is cooling off so T is decreasing and $f'(t)$ is negative.
(b) Since $f(t)$ is measured in degrees Fahrenheit and t is measured in minutes, df/dt must be measured in units of °F/min.

29. $f(10) = 240.000$ means that if the commodity costs \$10, then 240,000 units of it will be sold. $f'(10) = -29.000$ means that if the commodity costs \$10 now, each \$1 increase in price will cause a decline in sales of 29.000 units.

30. The rate of change of the US population is $P'(t)$, so

$$P'(t) = 0.8\% \cdot \text{Current population } = 0.008P(t).$$

31. (a) $f'(0.6) \approx \dfrac{f(0.8) - f(0.6)}{0.8 - 0.6} = \dfrac{4.0 - 3.9}{0.2} = 0.5.$ $\quad f'(0.5) \approx \dfrac{f(0.6) - f(0.4)}{0.6 - 0.4} = \dfrac{0.4}{0.2} = 2.$
(b) Using the values of f' from part (a), we get $f''(0.6) \approx \dfrac{f'(0.6) - f'(0.5)}{0.6 - 0.5} = \dfrac{0.5 - 2}{0.1} = \dfrac{-1.5}{0.1} = -15.$
(c) The maximum value of f is probably near $x = 0.8$. The minimum value of f is probably near $x = 0.3$.

32. By tracing on a calculator or solving equations, we find the following values of δ:
For $\epsilon = 0.1$, $\delta \leq 0.1$
For $\epsilon = 0.05$, $\delta \leq 0.05$.
For $\epsilon = 0.0007$, $\delta \leq 0.00007$.

33. By tracing on a calculator or solving equations, we find the following values of δ:
For $\epsilon = 0.1$, $\delta \leq 0.45$.
For $\epsilon = 0.001$, $\delta \leq 0.0447$.
For $\epsilon = 0.00001$, $\delta \leq 0.00447$.

34. (a) Slope of tangent line $= \lim_{h\to 0} \frac{\sqrt{4+h}-\sqrt{4}}{h}$. Using $h = 0.001$, $\frac{\sqrt{4.001}-\sqrt{4}}{0.001} = 0.249984$. Hence the slope of the tangent line is about 0.25.
(b)

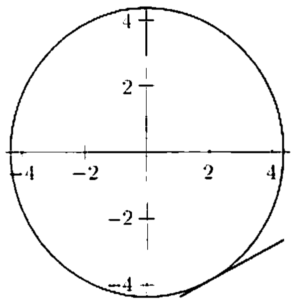
$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 2 &= 0.25(x - 4) \\ y - 2 &= 0.25x - 1 \\ y &= 0.25x + 1 \end{aligned}$$

(c) $f(x) = kx^2$
If $(4, 2)$ is on the graph of f, then $f(4) = 2$, so $k \cdot 4^2 = 2$. Thus $k = \frac{1}{8}$, and $f(x) = \frac{1}{8}x^2$.
(d) To find where the graph of f crosses then line $y = 0.25x + 1$, we solve:

$$\begin{aligned} \frac{1}{8}x^2 &= 0.25x + 1 \\ x^2 &= 2x + 8 \\ x^2 - 2x - 8 &= 0 \\ (x - 4)(x + 2) &= 0 \\ x = 4 &\text{ or } x = -2 \\ f(-2) &= \frac{1}{8}(4) = 0.5 \end{aligned}$$

Therefore, $(-2, 0.5)$ is the other point of intersection. (Of course, $(4, 2)$ is a point of intersection; we know that from the start.)

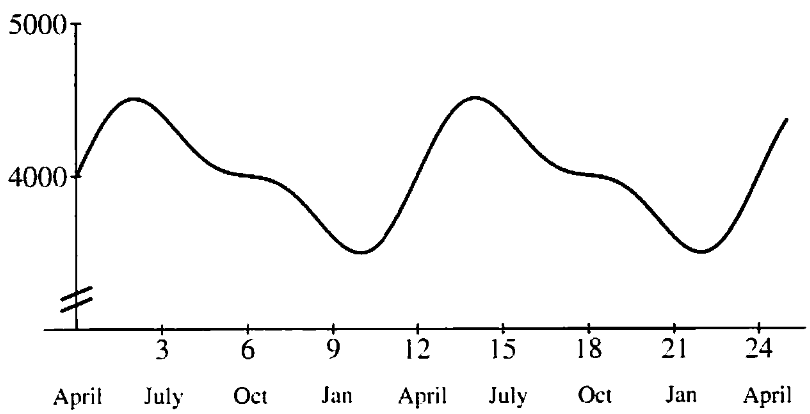
35. (a) The slope of the tangent line at $(0, \sqrt{19})$ is zero: it is horizontal.
The slope of the tangent line at $(\sqrt{19}, 0)$ is undefined: it is vertical.
(b) The slope appears to be about $\frac{1}{2}$. (Note that when x is 2, y is about -4, but when x is 4, y is approximately -3.)



(c) Using symmetry we can determine: Slope at $(-2, \sqrt{15})$: about $\frac{1}{2}$. Slope at $(-2, -\sqrt{15})$: about $-\frac{1}{2}$. Slope at $(2, \sqrt{15})$: about $-\frac{1}{2}$.

36. (a) IV, (b) III, (c) II, (d) I, (e) IV, (f) II

37. (a) The population varies periodically with a period of 12 months (i.e. one year).



(b) The herd is largest about June 1[st] when there are about 4500 deer.
(c) The herd is smallest about February 1[st] when there are about 3500 deer.
(d) The herd grows the fastest about April 1[st]. The herd shrinks the fastest about July 15 and again about December 15.
(e) It grows the fastest about April 1[st] when the rate of growth is about 400 deer/month, i.e about 13 new fawns per day.

38. (a) The graph looks straight because the graph shows only a small part of the curve magnified greatly.
(b) The month is March: We see that about the 21[st] of the month there are twelve hours of daylight and hence twelve hours of night. This phenomenon (the length of the day equaling the length of the night) occurs at the equinox, midway between winter and summer. Since the length of the days is increasing, and Madrid is in the northern hemisphere, we are looking at March, not September.
(c) The slope of the curve is found from the graph to be about 0.04 (the rise is about 0.8 hours in 20 days or 0.04 hours/day). This means that the amount of daylight is increasing by about 0.04 hours (about $2\frac{1}{2}$ minutes) per calendar day, or that each day is $2\frac{1}{2}$ minutes longer than its predecessor.

39. (a) A possible graph is shown in Figure 2.59. At first, the yam heats up very quickly, since the difference in temperature between it and its surroundings is so large. As time goes by, the yam gets hotter and hotter, its rate of temperature increase slows down, and its temperature approaches the temperature of the oven as an asymptote. The graph is thus concave down. (We are considering the average temperature of the yam, since the temperature in its center and on its surface will vary in different ways.)

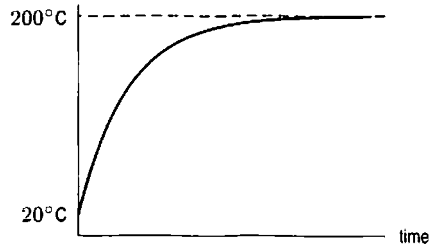


Figure 2.59

(b) If the rate of temperature increase were to remain 2°/min, in ten minutes the yam's temperature would increase 20°, from 120° to 140°. Since we know the graph is not linear, but concave down, the actual temperature is between 120° and 140°.
(c) In 30 minutes, we know the yam increases in temperature by 45° at an average rate of $45/30 = 1.5$°/min. Since the graph is concave down, the temperature at $t = 40$ is therefore between $120 + 1.5(10) = 135°$ and 140°.
(d) If the temperature increases at 2°/minute, it reaches 150° after 15 minutes, at $t = 45$. If the temperature increases at 1.5°/minute, it reaches 150° after 20 minutes, at $t = 50$. So t is between 45 and 50 mins.

40. (a) We construct the difference quotient using erf(0) and each of the other given values:

$$\operatorname{erf}'(0) \approx \frac{\operatorname{erf}(1) - \operatorname{erf}(0)}{1 - 0} = 0.84270079$$
$$\operatorname{erf}'(0) \approx \frac{\operatorname{erf}(0.1) - \operatorname{erf}(0)}{0.1 - 0} = 1.1246292$$
$$\operatorname{erf}'(0) \approx \frac{\operatorname{erf}(0.01) - \operatorname{erf}(0)}{0.01 - 0} = 1.128342.$$

Based on these estimates, the best estimate is $\operatorname{erf}'(0) \approx 1.12$; the subsequent digits have not yet stabilized.

(b) Using erf(0.001), we have

$$\operatorname{erf}'(0) \approx \frac{\operatorname{erf}(0.001) - \operatorname{erf}(0)}{0.001 - 0} = 1.12838$$

and so the best estimate is now 1.1283.

41. (a)

Table 2.21

x	$\frac{\sinh(x+0.001)-\sinh(x)}{0.001}$	$\frac{\sinh(x+0.0001)-\sinh(x)}{0.0001}$	so $f'(0) \approx$	$\cosh(x)$
0	1.00000	1.00000	1.00000	1.00000
0.3	1.04549	1.04535	1.04535	1.04534
0.7	1.25555	1.25521	1.25521	1.25517
1	1.54367	1.54314	1.54314	1.54308

(b) It seems that they are approximately the same, i.e. the derivative of $\sinh(x) = \cosh(x)$ for $x = 0, 0.3, 0.7$, and 1.

CAS Challenge Problems

42. The CAS says the derivative is zero. This can be explained by the fact that $f(x) = \sin^2 x + \cos^2 x = 1$, so $f'(x)$ is the derivative of the constant function 1. The derivative of a constant function is zero.

43. (a) The CAS gives $f'(x) = 2\cos^2 x - 2\sin^2 x$. Form of answers may vary.
(b) Using the double angle formulas for sine and cosine, we have

$$f(x) = 2\sin x \cos x = \sin(2x)$$
$$f'(x) = 2\cos^2 x - 2\sin^2 x = 2(\cos^2 x - \sin^2 x) = 2\cos(2x).$$

Thus we get

$$\frac{d}{dx}\sin(2x) = 2\cos(2x).$$

44. (a) The first derivative is $g'(x) = -2axe^{-ax^2}$, so the second derivative is

$$g''(x) = \frac{d^2}{dx^2}e^{-ax^2} = \frac{-2a}{e^{ax^2}} + \frac{4a^2x^2}{e^{ax^2}}.$$

Form of answers may vary.

(b) Both graphs get narrow as a gets larger; the graph of g'' is below the x-axis along the interval where g is concave down, and is above the x-axis where g is concave up. See Figure 2.60.

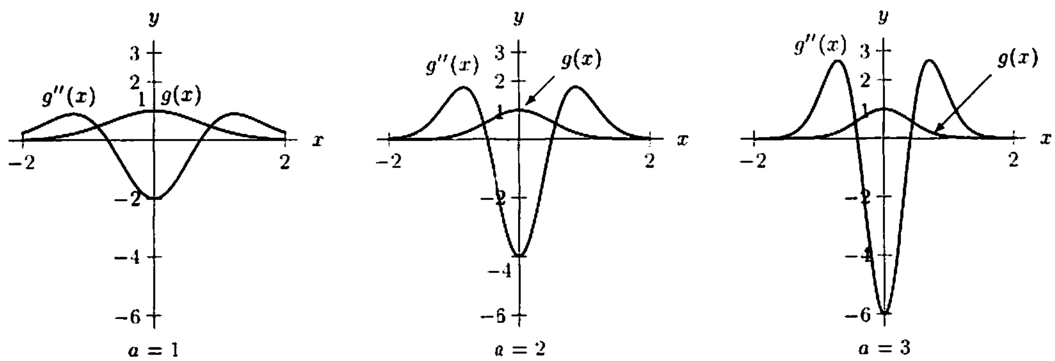


Figure 2.60

(c) The second derivative of a function is positive when the graph of the function is concave up and negative when it is concave down.

45. (a) The CAS gives the same derivative, $1/x$, in all three cases.

(b) From the properties of logarithms, $g(x) = \ln(2x) = \ln 2 + \ln x = f(x) + \ln 2$. So the graph of g is the same shape as the graph of f, only shifted up by $\ln 2$. So the graphs have the same slope everywhere, and therefore the two functions have the same derivative. By the same reasoning, $h(x) = f(x) + \ln 3$, so h and f have the same derivative as well.

46. (a) The computer algebra system gives

$$\frac{d}{dx}(x^2+1)^2 = 4x(x^2+1)$$
$$\frac{d}{dx}(x^2+1)^3 = 6x(x^2+1)^2$$
$$\frac{d}{dx}(x^2+1)^4 = 8x(x^2+1)^3$$

(b) The pattern suggests that

$$\frac{d}{dx}(x^2+1)^n = 2nx(x^2+1)^{n-1}.$$

Taking the derivative of $(x^2+1)^n$ with a CAS confirms this.

47. (a) Using a CAS, we find

$$\frac{d}{dx}\sin x = \cos x$$
$$\frac{d}{dx}\cos x = -\sin x$$
$$\frac{d}{dx}(\sin x \cos x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1.$$

(b) The product of the derivatives of $\sin x$ and $\cos x$ is $\cos x(-\sin x) = -\cos x \sin x$. On the other hand, the derivative of the product is $\cos^2 x - \sin^2 x$, which is not the same. So no, the derivative of a product is not always equal to the product of the derivatives.

CHECK YOUR UNDERSTANDING

1. False. For example, the car could slow down or even stop at one minute after 2 pm, and then speed back up to 60 mph at one minute before 3 pm. In this case the car would travel only a few miles during the hour, much less than 50 miles.
2. False. Its average velocity for the time between 2 pm and 4 pm is 40 mph, but the car could change its speed a lot during that time period. For example, the car might be motionless for an hour then go 80 mph for the second hour. In that case the velocity at 2 pm would be 0 mph.
3. True. During a short enough time interval the car can not change its velocity very much, and so it velocity will be nearly constant. It will be nearly equal to the average velocity over the interval.
4. True. The instantaneous velocity is a limit of the average velocities. The limit of a constant equals that constant.
5. True. By definition, Average velocity = Distance traveled/Time.
6. False. Instantaneous velocity equals a *limit* of difference quotients.
7. False. All we know is that if h is close enough to zero then $f(h)$ will be as close as we please to L. We do not know how close would be close enough to zero for $f(h)$ to be closer to L than is $f(0.01)$. It might be that we have to get a lot closer than 0.0001. It is even possible that $f(0.01) = L$ but $f(0.0001) \neq L$ so $f(h)$ could never get closer to L than $f(0.01)$.
8. True. This is seen graphically. The derivative $f'(a)$ is the slope of the line tangent to the graph of f at the point P where $x = a$. The difference quotient $(f(b) - f(a))/(b - a)$ is the slope of the secant line with endpoints on the graph of f at the points where $x = a$ and $x = b$. The tangent and secant lines cross at the point P. The secant line goes above the tangent line for $x > a$ because f is concave up, and so the secant line has higher slope.
9. True. The derivative of a function is the limit of difference quotients. A few difference quotients can be computed from the table, but the limit can not be computed from the table.
10. False. If $f'(x)$ is increasing then $f(x)$ is concave up. However, $f(x)$ may be either increasing or decreasing. For example, the exponential decay function $f(x) = e^{-x}$ is decreasing but $f'(x)$ is increasing because the graph of f is concave up.
11. False. A counterexample is given by $f(x) = 5$ and $g(x) = 10$, two different functions with the same derivatives: $f'(x) = g'(x) = 0$.
12. True. The graph of a linear function $f(x) = mx + b$ is a straight line with the same slope m at every point. Thus $f'(x) = m$ for all x.
13. True. Shifting a graph vertically does not change the shape of the graph and so it does not change the slopes of the tangent lines to the graph.
14. False. The function $f(x)$ may be discontinuous at $x = 0$, for instance $f(x) = \begin{cases} 0 \text{ if } x \le 0 \\ 1 \text{ if } x > 0 \end{cases}$. The graph of f may have a vertical tangent line at $x = 0$, for instance $f(x) = x^{1/3}$.
15. True. The two sides of the equation are different frequently used notations for the very same quantity, the derivative of f at the point a.
16. True. The derivative $f'(10)$ is the slope of the tangent line to the graph of $y = f(x)$ at the point where $x = 10$. When you zoom in on $y = f(x)$ close enough it is not possible to see the difference between the tangent line and the graph of f on the calculator screen. The line you see on the calculator is a little piece of the tangent line, so its slope is the derivative $f'(10)$.
17. True. The second derivative $f''(x)$ is the derivative of $f'(x)$. Thus the derivative of $f'(x)$ is positive, and so $f'(x)$ is increasing.
18. True. Instantaneous acceleration is a derivative, and all derivatives are limits of difference quotients. More precisely, instantaneous acceleration $a(t)$ is the derivative of the velocity $v(t)$, so
$$a(t) = \lim_{h \to 0} \frac{v(t+h) - v(t)}{h}.$$
19. True. The derivatives $f'(t)$ and $g'(t)$ measure the same thing, the rate of chemical production at the same time t, but they measure it in different units. The units of $f'(t)$ are grams per minute, and the units of $g'(t)$ are kilograms per minute. To convert from kg/min to g/min, multiply by 1000.
20. False. The derivatives $f'(t)$ and $g'(t)$ measure different things because they measure the rate of chemical production at different times. There is no conversion possible from one to the other.
21. True. Let $f(x) = |x - 3|$. Then $f(x)$ is continuous for all x but not differentiable at $x = 3$ because its graph has a corner there. Other answers are possible.

22. True. If a function is differentiable at a point, then it is continuous at that point. For example, $f(x) = x^2$ is both differentiable and continuous on any interval. However, *one* example does not establish the truth of this statement; it merely illustrates the statement.

23. False. Being continuous does not imply differentiability. For example, $f(x) = |x|$ is continuous but not differentiable at $x = 0$.

24. True. If a function were differentiable, then it would be continuous. For example,
$f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$ is neither differentiable nor continuous at $x = 0$. However, *one* example does not establish the truth of this statement; it merely illustrates the statement.

25. False. For example, $f(x) = |x|$ is not differentiable at $x = 0$, but it is continuous at $x = 0$.

26. False. For example, let $f(x) = 1/x$ and $g(x) = -1/x$, then $f(x) + g(x) = 0$. If $c = 0$, $\lim_{x\to 0^+} (f(x) + g(x))$ exists (it is 0), but $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^+} g(x)$ do not exist.

27. True, by Property 3 of limits in Theorem 2.1, since $\lim_{x\to 3} x = 3$.

28. False. If $\lim_{x\to 3} g(x)$ does not exist, then $\lim_{x\to 3} f(x)g(x)$ may not even exist. For example, let $f(x) = 2x + 1$ and define g by:

$$g(x) = \begin{cases} 1/(x-3) & \text{if } x \neq 3 \\ 4 & \text{if } x = 3 \end{cases}$$

Then $\lim_{x\to 3} f(x) = 7$ and $g(3) = 4$, but $\lim_{x\to 3} f(x)g(x) \neq 28$, since $\lim_{x\to 3}(2x + 1)/(x - 3)$ does not exist.

29. True, by Property 2 of limits in Theorem 2.1.

30. True, by Properties 2 and 3 of limits in Theorem 2.1.

$$\lim_{x\to 3} g(x) = \lim_{x\to 3}(f(x) + g(x) + (-1)f(x)) = \lim_{x\to 3}(f(x) + g(x)) + (-1)\lim_{x\to 3} f(x) = 12 + (-1)7 = 5.$$

31. False. For example, define f as follows:

$$f(x) = \begin{cases} 2x + 1 & \text{if } x \neq 2.99 \\ 1000 & \text{if } x = 2.99. \end{cases}$$

Then $f(2.9) = 2(2.9) + 1 = 6.8$, whereas $f(2.99) = 1000$.

32. False. For example, define f as follows:

$$f(x) = \begin{cases} 2x + 1 & \text{if } x \neq 3.01 \\ -1000 & \text{if } x = 3.01. \end{cases}$$

Then $f(3.1) = 2(3.1) + 1 = 7.2$, whereas $f(3.01) = -1000$.

33. True. Suppose instead that $\lim_{x\to 3} g(x)$ does not exist but $\lim_{x\to 3}(f(x)g(x))$ did exist. Since $\lim_{x\to 3} f(x)$ exists and is not zero, then $\lim_{x\to 3}((f(x)g(x))/f(x))$ exists, by Property 4 of limits in Theorem 2.1. Furthermore, $f(x) \neq 0$ for all x in some interval about 3, so $(f(x)g(x))/f(x) = g(x)$ for all x in that interval. Thus $\lim_{x\to 3} g(x)$ exists. This contradicts our assumption that $\lim_{x\to 3} g(x)$ does not exist.

34. False. For some functions we need to pick smaller values of δ. For example, if $f(x) = x^{1/3} + 2$ and $c = 0$ and $L = 2$, then $f(x)$ is within 10^{-3} of 2 if $|x^{1/3}| < 10^{-3}$. This only happens if x is within $(10^{-3})^3 = 10^{-9}$ of 0. If $x = 10^{-3}$ then $x^{1/3} = (10^{-3})^{1/3} = 10^{-1}$, which is too large.

35. False. The definition of a limit guarantees that, for any positive ϵ, there is a δ. This statement, which guarantees an ϵ for a specific $\delta = 10^{-3}$, is not equivalent to $\lim_{x\to c} f(x) = L$. For example, consider a function with a vertical asymptote within 10^{-3} of 0, such as $c = 0$, $L = 0$, $f(x) = x/(x - 10^{-4})$.

36. True. This is equivalent to the definition of a limit.

37. False. Although x may be far from c, the value of $f(x)$ could be close to L. For example, suppose $f(x) = L$, the constant function.

38. False. The definition of the limit says that if x is within δ of c, then $f(x)$ is within ϵ of L, not the other way round.

39. (a) This is not a counterexample, since it does not satisfy the conditions of the statement, and therefore does not have the potential to contradict the statement.
(b) This contradicts the statement, because it satisfies its conditions but not its conclusion. Hence it is a counterexample. Notice that this counterexample could not actually exist, since the statement is true.
(c) This is an example illustrating the statement; it is not a counterexample.
(d) This is not a counterexample, for the same reason as in part (a).

PROJECTS FOR CHAPTER TWO

1. (a) $S(0) = 12$ since the days are always 12 hours long at the equator.

(b) Since $S(0) = 12$ from part (a) and the formula gives $S(0) = a$, we have $a = 12$. Since $S(x)$ must be continuous at $x = x_0$, and the formula gives $S(x_0) = a + b\arcsin(1) = 12 + b\left(\frac{\pi}{2}\right)$ and also $S(x_0) = 24$, we must have $12 + b\left(\frac{\pi}{2}\right) = 24$ so $b\left(\frac{\pi}{2}\right) = 12$ and $b = \frac{24}{\pi} \approx 7.64$.

(c) $S(32°13') \approx 14.12$ and $S(46°4') \approx 15.58$.

(d)

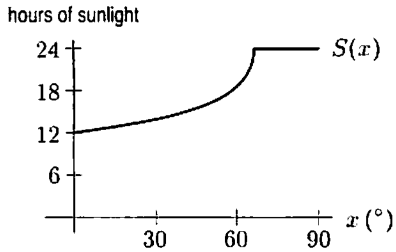


Figure 2.61

(e) The graph in Figure 2.61 appears to have a corner at $x_0 = 66°30'$. We compare the slope to the right of x_0 and to the left of x_0. To the right of S_0, the function is constant, so $S'(x) = 0$ for $x > 66°30'$.

We estimate the slope immediately to the left of x_0. We want to calculate the following:

$$\lim_{h\to 0^-} \frac{S(x_0 + h) - S(x_0)}{h}.$$

We approximate it by taking $x_0 = 66.5$ and $h = -0.1, \ -0.01, \ -0.001$:

$$\frac{S(66.49) - S(66.5)}{-0.1} \approx \frac{22.3633 - 24}{-0.1} = 16.38,$$

$$\frac{S(66.499) - S(66.5)}{-0.01} \approx \frac{23.4826 - 24}{-0.01} = 51.83,$$

$$\frac{S(66.4999) - S(66.5)}{-0.001} \approx \frac{23.8370 - 24}{-0.001} = 163.9.$$

These approximations suggest that, for $x_0 = 66.5$,

$$\lim_{h\to 0^-} \frac{S(x_0 + h) - S(x_0)}{h} \quad \text{does not exist.}$$

This evidence suggests that $S(x)$ is not differentiable at x_0. A proof requires the techniques found in Chapter 3.

2. (a) (i) Estimating derivatives using difference quotients (but other answers are possible):

$$P'(1900) \approx \frac{P(1910) - P(1900)}{10} = \frac{92.0 - 76.0}{10} = 1.6 \text{ million people per year}$$

$$P'(1945) \approx \frac{P(1950) - P(1940)}{10} = \frac{150.7 - 131.7}{10} = 1.9 \text{ million people per year}$$

$$P'(1990) \approx \frac{P(1990) - P(1980)}{10} = \frac{248.7 - 226.5}{10} = 2.22 \text{ million people per year}$$

(ii) The population growth was maximal somewhere between 1950 and 1960.

(iii) $P'(1950) \approx \frac{P(1960) - P(1950)}{10} = \frac{179.0 - 150.7}{10} = 2.83$ million people per year, so $P(1956) \approx P(1950) + P'(1950)(1956 - 1950) = 150.7 + 2.83(6) \approx 167.7$ million people.

(iv) If the growth rate between 1990 and 2000 was the same as the growth rate from 1980 to 1990, then the total population should be about 271 million people in 2000.

(b) (i) $f^{-1}(100)$ is the point in time when the population of the US was 100 million people (somewhere between 1910 and 1920).

(ii) The derivative of $f^{-1}(P)$ at $P = 100$ represents the ratio of change in time to change in population, and its units are years per million people. In other words, this derivative represents about how long it took for the population to increase by 1 million, when the population was 100 million.

(iii) Since the population increased by $105.7 - 92.0 = 13.7$ million people in 10 years, the average rate of increase is 1.37 million people per year. If the rate is fairly constant in that period, the amount of time it would take for an increase of 8 million people (100 million − 92.0 million) would be

$$\frac{8 \text{ million people}}{1.37 \text{ million people/year}} \approx 5.8 \text{ years} \approx 6 \text{ years}$$

Adding this to our starting point of 1910, we estimate that the population of the US reached 100 million around 1916, i.e. $f^{-1}(100) \approx 1916$.

(iv) Since it took 10 years between 1910 and 1920 for the population to increase by $105.7 - 92.0 = 13.7$ million people, the derivative of $f^{-1}(P)$ at $P = 100$ is approximately

$$\frac{10 \text{ years}}{13.7 \text{ million people}} = 0.73 \text{ years/million people}$$

(c) (i) Clearly the population of the US at any instant is an integer that varies up and down every few seconds as a child is born, a person dies, or a new immigrant arrives. So $f(t)$ has "jumps;" it is not a smooth function. But these jumps are small relative to the values of f, so f appears smooth unless we zoom in very closely on its graph (to within a few seconds).

Major land acquisitions such as the Louisiana Purchase caused larger jumps in the population, but since the census is taken only every ten years and the territories acquired were rather sparsely populated, we cannot see these jumps in the census data.

(ii) We can regard rate of change of the population for a particular time t as representing an estimate of how much the population will increase during the year after time t.

(iii) Many economic indicators are treated as smooth, such as the Gross National Product, the Dow Jones Industrial Average, volumes of trading, and the price of commodities like gold. But these figures only change in increments, not continuously.