CHAPTER 13
Geometry in the Plane, Vectors

13.1 Concepts Review

1. simple; closed; simple

2. parametric; parameter

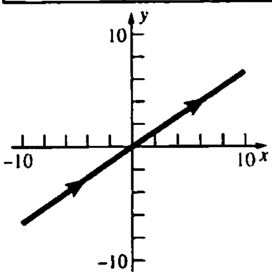
3. cycloid

4. $\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Problem Set 13.1

1. a.

t	x	y
–2	–6	–4
–1	–3	–2
0	0	0
1	3	2
2	6	4



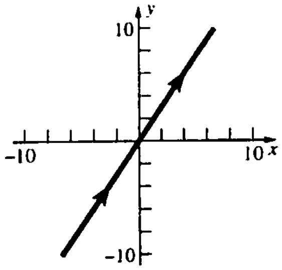
b. Simple; not closed

c. $t = \dfrac{x}{3}$

$y = \dfrac{2}{3}x$

2. a.

t	x	y
–2	–4	–6
–1	–2	–3
0	0	0
1	2	3
2	4	6



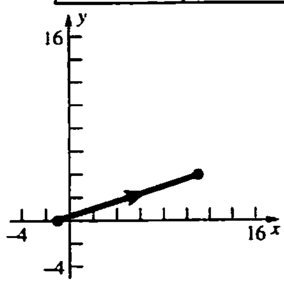
b. Simple; not closed

c. $t = \dfrac{x}{2}$

$y = \dfrac{3}{2}x$

3. a.

t	x	y
0	–1	0
1	2	1
2	5	2
3	8	3
4	11	4



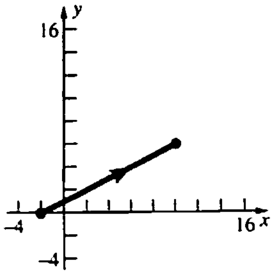
b. Simple; not closed

c. $t = \dfrac{1}{3}(x+1)$

$y = \dfrac{1}{3}(x+1)$

4. a.

t	x	y
0	–2	0
1	2	2
2	6	4
3	10	6



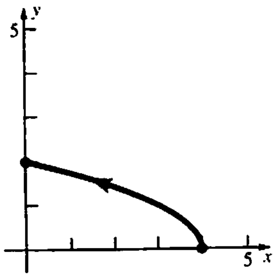
b. Simple; not closed

c. $t = \frac{1}{4}(x+2)$

$y = \frac{1}{2}(x+2)$

5. a.

t	x	y
0	4	0
1	3	1
2	2	$\sqrt{2}$
3	1	$\sqrt{3}$
4	0	2



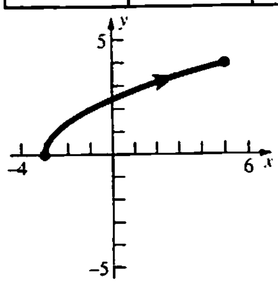
b. Simple; not closed

c. $t = 4 - x$

$y = \sqrt{4-x}$

6. a.

t	x	y
0	–3	0
2	–1	2
4	1	$2\sqrt{2}$
6	3	$2\sqrt{3}$
8	5	4



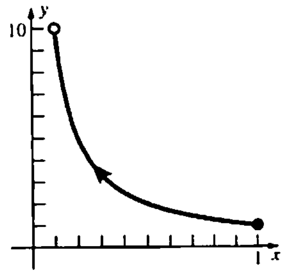
b. Simple; not closed

c. $t = x + 3$

$y = \sqrt{2x+6}$

7. a.

s	x	y
1	1	1
3	$\frac{1}{3}$	3
5	$\frac{1}{5}$	5
7	$\frac{1}{7}$	7
9	$\frac{1}{9}$	9



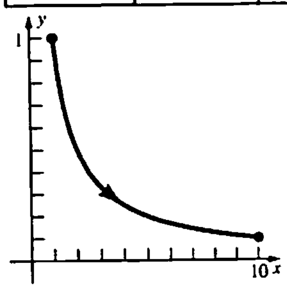
b. Simple; not closed

c. $s = \frac{1}{x}$

$y = \frac{1}{x}$

8. a.

s	x	y
1	1	1
3	3	$\frac{1}{3}$
5	5	$\frac{1}{5}$
7	7	$\frac{1}{7}$
9	9	$\frac{1}{9}$



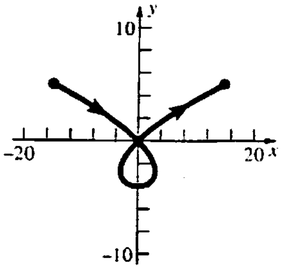
b. Simple; not closed

c. $s = x$

$y = \frac{1}{x}$

9. a.

t	x	y
–3	–15	5
–2	0	0
–1	3	–3
0	0	–4
1	–3	–3
2	0	0
3	15	5



b. Not simple; not closed

c. $x^2 = t^6 - 8t^4 + 16t^2$

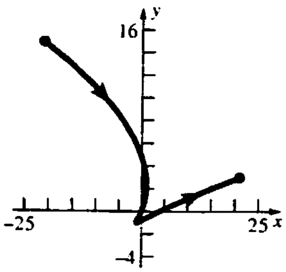
$t^2 = y + 4$

$x^2 = (y+4)^3 - 8(y+4)^2 + 16(y+4)$

$x^2 = y^3 + 4y^2$

10. a.

t	x	y
–3	–21	15
–2	–4	8
–1	1	3
0	0	0
1	–1	–1
2	4	0
3	21	3



b. Simple; not closed

c. $t^2 - 2t - y = 0$

$t = 1 \pm \sqrt{1+y}$

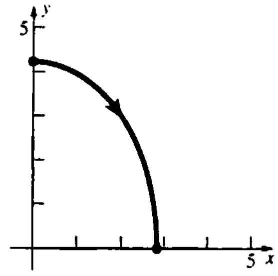
$x = \left(1 \pm \sqrt{1+y}\right)^3 - 2\left(1 \pm \sqrt{1+y}\right)$

$x = 2 + 3y \pm (y+2)\sqrt{1+y}$

$(x - 3y - 2)^2 = (y+1)(y+2)^2$

11. a.

t	x	y
2	0	$3\sqrt{2}$
3	2	3
4	$2\sqrt{2}$	0



b. Simple; not closed

c. $t = \frac{1}{4}x^2 + 2$

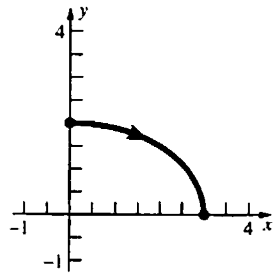
$t = 4 - \frac{1}{9}y^2$

$\frac{1}{4}x^2 + 2 = 4 - \frac{1}{9}y^2$

$\frac{x^2}{8} + \frac{y^2}{18} = 1$

12. a.

t	x	y
3	0	2
$\frac{7}{2}$	$\frac{3}{\sqrt{2}}$	$\sqrt{2}$
4	3	0



b. Simple; not closed

c. $t = \frac{1}{9}x^2 + 3$

$t = 4 - \frac{1}{4}y^2$

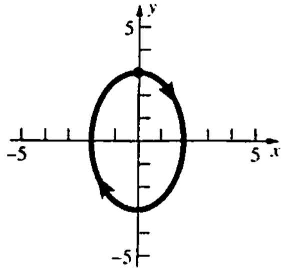
$\frac{1}{9}x^2 + 3 = 4 - \frac{1}{4}y^2$

$\frac{x^2}{9} + \frac{y^2}{4} = 1$

13. a.

t	x	y
0	0	3
$\frac{\pi}{2}$	2	0

π	0	−3
$\frac{3\pi}{2}$	−2	0
2π	0	3



b. Simple; closed

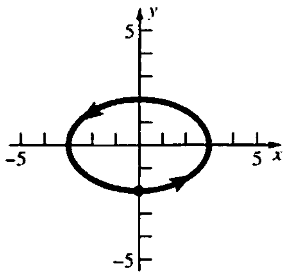
c. $\sin^2 t = \frac{x^2}{4}$

$\cos^2 t = \frac{y^2}{9}$

$\frac{x^2}{4} + \frac{y^2}{9} = 1$

14. a.

r	x	y
0	0	−2
$\frac{\pi}{2}$	3	0
π	0	2
$\frac{3\pi}{2}$	−3	0
2π	0	−2



b. Simple; closed

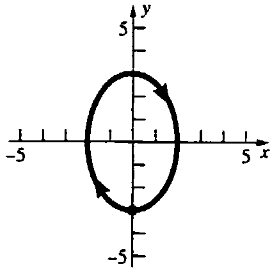
c. $\sin^2 r = \frac{x^2}{9}$

$\cos^2 r = \frac{y^2}{4}$

$\frac{x^2}{9} + \frac{y^2}{4} = 1$

15. a.

r	x	y
0	0	−3
$\frac{\pi}{2}$	−2	0
π	0	3
$\frac{3\pi}{2}$	2	0
2π	0	−3
$\frac{5\pi}{2}$	−2	0
3π	0	3
$\frac{7\pi}{2}$	2	0
4π	0	−3



b. Not simple; closed

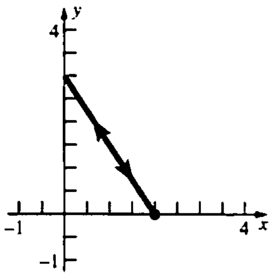
c. $\sin^2 r = \frac{x^2}{4}$

$\cos^2 r = \frac{y^2}{9}$

$\frac{x^2}{4} + \frac{y^2}{9} = 1$

16. a.

r	x	y
0	2	0
$\frac{\pi}{2}$	0	3
π	2	0
$\frac{3\pi}{2}$	0	3
2π	2	0



b. Not simple; closed

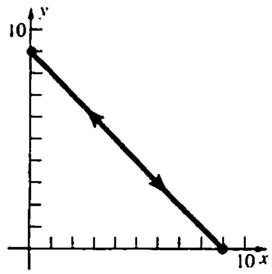
c. $\cos^2 r = \frac{x}{2}$

$\sin^2 r = \frac{y}{3}$

$\frac{x}{2} + \frac{y}{3} = 1$

17. a.

θ	x	y
0	0	9
$\frac{\pi}{4}$	$\frac{9}{2}$	$\frac{9}{2}$
$\frac{\pi}{2}$	0	9
$\frac{3\pi}{4}$	$\frac{9}{2}$	$\frac{9}{2}$
π	0	9



b. Not simple; closed

c. $\sin^2\theta = \frac{x}{9}$

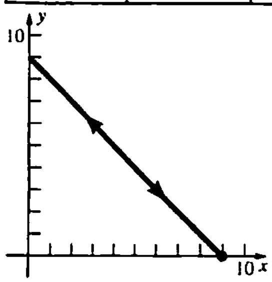
$\cos^2\theta = \frac{y}{9}$

$\frac{x}{9} + \frac{y}{9} = 1$

$x + y = 9$

18. a.

θ	x	y
0	9	0
$\frac{\pi}{4}$	$\frac{9}{2}$	$\frac{9}{2}$
$\frac{\pi}{2}$	0	9
$\frac{3\pi}{4}$	$\frac{9}{2}$	$\frac{9}{2}$
π	9	0



b. Not simple; closed

c. $\cos^2\theta = \frac{x}{9}$

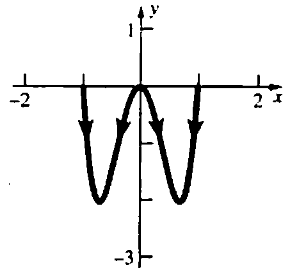
$\sin^2\theta = \frac{y}{9}$

$\frac{x}{9} + \frac{y}{9} = 1$

$x + y = 9$

19. a.

θ	x	y
$0 + 2\pi n$	1	0
$\frac{\pi}{3} + 2\pi n$	$\frac{1}{2}$	$-\frac{3}{2}$
$\frac{\pi}{2} + 2\pi n$	0	0
$\frac{2\pi}{3} + 2\pi n$	$-\frac{1}{2}$	$-\frac{3}{2}$
$\pi + 2\pi n$	-1	0
$\frac{4\pi}{3} + 2\pi n$	$-\frac{1}{2}$	$-\frac{3}{2}$
$\frac{3\pi}{2} + 2\pi n$	0	0
$\frac{5\pi}{3} + 2\pi n$	$\frac{1}{2}$	$-\frac{3}{2}$



b. Not simple; not closed

c. $\cos\theta = x$

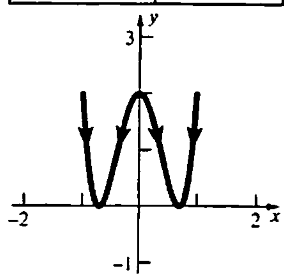
$\sin\theta = \sqrt{1 - x^2}$

$y = -8\sin^2\theta\cos^2\theta$

$y = -8x^2(1 - x^2)$

20. a.

θ	x	y
$0 + 2\pi n$	0	2
$\frac{\pi}{6} + 2\pi n$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{\pi}{2} + 2\pi n$	1	2
$\frac{5\pi}{6} + 2\pi n$	$\frac{1}{2}$	$\frac{1}{2}$
$\pi + 2\pi n$	0	2
$\frac{7\pi}{6} + 2\pi n$	$-\frac{1}{2}$	$\frac{1}{2}$
$\frac{3\pi}{2} + 2\pi n$	-1	2
$\frac{11\pi}{6} + 2\pi n$	$-\frac{1}{2}$	$\frac{1}{2}$



b. Not simple; not closed

c. $\sin\theta = x$

$\cos\theta = \sqrt{1-x^2}$

$y = 2(\cos^2\theta - \sin^2\theta)^2$

$y = 2(2x^2-1)^2$

21. $\frac{dx}{d\tau} = 6\tau$

$\frac{dy}{d\tau} = 12\tau^2$

$\frac{dy}{dx} = 2\tau$

$\frac{dy'}{d\tau} = 2$

$\frac{d^2y}{dx^2} = \frac{1}{3\tau}$

22. $\frac{dx}{ds} = 12s$

$\frac{dy}{ds} = -6s^2$

$\frac{dy}{dx} = -\frac{1}{2}s$

$\frac{dy'}{ds} = -\frac{1}{2}$

$\frac{d^2y}{dx^2} = -\frac{1}{24s}$

23. $\frac{dx}{d\theta} = 4\theta$

$\frac{dy}{d\theta} = 3\sqrt{5}\theta^2$

$\frac{dy}{dx} = \frac{3\sqrt{5}}{4}\theta$

$\frac{dy'}{d\theta} = \frac{3\sqrt{5}}{4}$

$\frac{d^2y}{dx^2} = \frac{3\sqrt{5}}{16\theta}$

24. $\frac{dx}{d\theta} = 2\sqrt{3}\theta$

$\frac{dy}{d\theta} = -3\sqrt{3}\theta^2$

$\frac{dy}{dx} = -\frac{3}{2}\theta$

$\frac{dy'}{d\theta} = -\frac{3}{2}$

$\frac{d^2y}{dx^2} = -\frac{\sqrt{3}}{4\theta}$

25. $\frac{dx}{dt} = \sin t$

$\frac{dy}{dt} = \cos t$

$\frac{dy}{dx} = \cot t$

$\frac{dy'}{dt} = -\csc^2 t$

$\frac{d^2y}{dx^2} = -\csc^3 t$

26. $\frac{dx}{dt} = 2\sin t$

$\frac{dy}{dt} = 5\cos t$

$\frac{dy}{dx} = \frac{5}{2}\cot t$

$\frac{dy'}{dt} = -\frac{5}{2}\csc^2 t$

$\frac{d^2y}{dx^2} = -\frac{5}{4}\csc^3 t$

27. $\frac{dx}{dt} = 3\sec^2 t$

$\frac{dy}{dt} = 5\sec t\tan t$

$\frac{dy}{dx} = \frac{5}{3}\sin t$

$\frac{dy'}{dt} = \frac{5}{3}\cos t$

$\frac{d^2y}{dx^2} = \frac{5}{9}\cos^3 t$

28. $\frac{dx}{dt} = -\csc^2 t$

$\frac{dy}{dt} = 2\csc t\cot t$

$\frac{dy}{dx} = -2\cos t$

$\frac{dy'}{dt} = 2\sin t$

$\frac{d^2y}{dx^2} = -2\sin^3 t$

29. $\frac{dx}{dt} = -\frac{2t}{(1+t^2)^2}$

$\frac{dy}{dt} = \frac{2t-1}{t^2(1-t)^2}$

$\frac{dy}{dx} = \frac{(1-2t)(1+t^2)^2}{2t^3(1-t)^2}$

$\frac{dy'}{dt} = -\frac{3t^5+7t^4-6t^3+10t^2-9t+3}{2t^4(1-t)^3}$

$\frac{d^2y}{dx^2} = \frac{(3t^5+7t^4-6t^3+10t^2-9t+3)(1+t^2)^2}{4t^5(1-t)^3}$

30. $\frac{dx}{dt} = -\frac{4t}{(1+t^2)^2}$

$\frac{dy}{dt} = -\frac{2(3t^2+1)}{t^2(1+t^2)^2}$

$\frac{dy}{dx} = \frac{3t^2+1}{2t^3}$

$\frac{dy'}{dt} = -\frac{3(t^2+1)}{2t^4}$

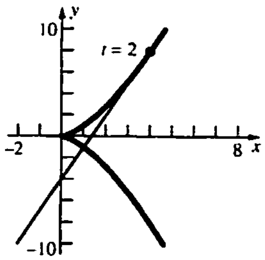
$\frac{d^2y}{dx^2} = \frac{3(t^2+1)^3}{8t^5}$

31. $\frac{dx}{dt} = 2t, \frac{dy}{dt} = 3t^2$

$\frac{dy}{dx} = \frac{3}{2}t$

At $t = 2, x = 4, y = 8$, and $\frac{dy}{dx} = 3.$

Tangent line: $y - 8 = 3(x - 4)$ or $3x - y - 4 = 0$

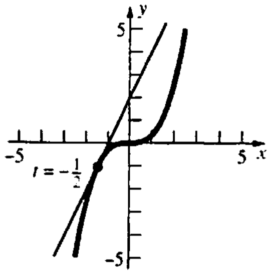


32. $\frac{dx}{dt} = 3, \frac{dy}{dt} = 24t^2$

$\frac{dy}{dx} = 8t^2$

At $t = -\frac{1}{2}, x = -\frac{3}{2}, y = -1$, and $\frac{dy}{dx} = 2$.

Tangent line: $y+1 = 2\left(x+\frac{3}{2}\right)$ or $2x - y + 2 = 0$



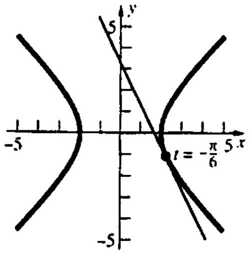
33. $\frac{dx}{dt} = 2\sec t\tan t, \frac{dy}{dt} = 2\sec^2 t$

$\frac{dy}{dx} = \csc t$

At $t = -\frac{\pi}{6}, x = \frac{4}{\sqrt{3}}, y = -\frac{2}{\sqrt{3}}$, and $\frac{dy}{dx} = -2.$

Tangent line: $y+\frac{2}{\sqrt{3}} = -2\left(x-\frac{4}{\sqrt{3}}\right)$ or

$2\sqrt{3}x + \sqrt{3}y - 6 = 0$



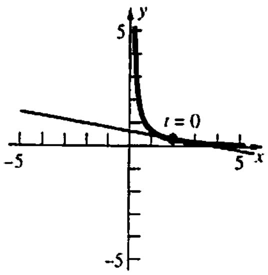
34. $\frac{dx}{dt} = 2e^t, \frac{dy}{dt} = -\frac{1}{3}e^{-t}$

$\frac{dy}{dx} = -\frac{1}{6}e^{-2t}$

At $t = 0, x = 2, y = \frac{1}{3}$, and $\frac{dy}{dx} = -\frac{1}{6}$.

Tangent line: $y-\frac{1}{3} = -\frac{1}{6}(x-2)$ or

$x + 6y - 4 = 0$



35. $\frac{dx}{dt} = 2, \frac{dy}{dt} = 3$

$L = \int_0^3 \sqrt{4+9}\,dt = \sqrt{13}\int_0^3 dt = \sqrt{13}[t]_0^3 = 3\sqrt{13}$

36. $\frac{dx}{dt} = -1, \frac{dy}{dt} = 2$

$$L = \int_{-3}^{3} \sqrt{1+4}\,dt = \sqrt{5}\int_{-3}^{3} dt = \sqrt{5}[t]_{-3}^{3} = 6\sqrt{5}$$

37. $\frac{dx}{dt} = 1, \frac{dy}{dt} = \frac{3}{2}t^{1/2}$

$$L = \int_0^3 \sqrt{1+\frac{9}{4}t}\,dt$$

$$= \frac{1}{2}\int_0^3 \sqrt{4+9t}\,dt$$

$$= \frac{1}{18}\left[\frac{2}{3}(4+9t)^{3/2}\right]_0^3$$

$$= \frac{1}{27}(31^{3/2} - 8) = \frac{1}{27}(31\sqrt{31} - 8)$$

38. $\frac{dx}{dt} = 2\cos t, \frac{dy}{dt} = -2\sin t$

$$L = \int_0^{\pi} \sqrt{4\cos^2 t + 4\sin^2 t}\,dt = 2\int_0^{\pi} dt = 2[t]_0^{\pi} = 2\pi$$

39. $\frac{dx}{dt} = 6t, \frac{dy}{dt} = 3t^2$

$$L = \int_0^2 \sqrt{36t^2 + 9t^4}\,dt = 3\int_0^2 t\sqrt{4+t^2}\,dt$$

$$3\left[\frac{1}{3}(4+t^2)^{3/2}\right]_0^2 = 16\sqrt{2} - 8$$

40. $\frac{dx}{dt} = 1 - \frac{1}{t^2}, \frac{dy}{dt} = \frac{2}{t}$

$$L = \int_1^4 \sqrt{\left(1 - \frac{2}{t^2} + \frac{1}{t^4}\right) + \frac{4}{t^2}}\,dt$$

$$= \int_1^4 \sqrt{1 + \frac{2}{t^2} + \frac{1}{t^4}}\,dt = \int_1^4 \sqrt{\left(1+\frac{1}{t^2}\right)^2}\,dt$$

$$= \int_1^4 \left(1 + \frac{1}{t^2}\right)dt = \left[t - \frac{1}{t}\right]_1^4 = \frac{15}{4}$$

41. $\frac{dx}{dt} = 2e^t, \frac{dy}{dt} = \frac{9}{2}e^{3t/2}$

$$L = \int_{\ln 3}^{2\ln 3} \sqrt{4e^{2t} - \frac{81}{4}e^{3t}}\,dt = \int_{\ln 3}^{2\ln 3} e^t\sqrt{4 - \frac{81}{4}e^t}\,dt$$

$$= \left[-\frac{8}{243}\left(4 - \frac{81}{4}e^t\right)^{3/2}\right]_{\ln 3}^{2\ln 3}$$

$$= \frac{713\sqrt{713} - 227\sqrt{227}}{243}$$

42. $\frac{dx}{dt} = -\frac{t}{\sqrt{1-t^2}}, \frac{dy}{dt} = -1$

$$L = \int_0^{1/4} \sqrt{\frac{t^2}{1-t^2} + 1}\,dt = \int_0^{1/4} \frac{1}{\sqrt{1-t^2}}\,dt$$

$$= [\sin^{-1} t]_0^{1/4} = \sin^{-1}\frac{1}{4}$$

43. $\frac{dx}{dt} = \frac{2}{\sqrt{t}}, \frac{dy}{dt} = 2t - \frac{1}{2t^2}$

$$L = \int_{1/4}^{1} \sqrt{\frac{4}{t} + \left(4t^2 - \frac{2}{t} + \frac{1}{4t^4}\right)}\,dt$$

$$= \int_{1/4}^{1} \sqrt{4t^2 + \frac{2}{t} + \frac{1}{4t^4}}\,dt$$

$$= \int_{1/4}^{1} \sqrt{\left(2t + \frac{1}{2t^2}\right)^2}\,dt$$

$$= \int_{1/4}^{1} \left(2t + \frac{1}{2t^2}\right)dt = \left[t^2 - \frac{1}{2t}\right]_{1/4}^{1} = \frac{39}{16}$$

44. $\frac{dx}{dt} = \operatorname{sech}^2 t, \frac{dy}{dt} = 2\tanh t$

$$L = \int_{-3}^{3} \sqrt{\operatorname{sech}^4 t + 4\tanh^2 t}\,dt$$

$$= \int_{-3}^{3} \sqrt{4 - 4\operatorname{sech}^2 t + \operatorname{sech}^4 t}\,dt$$

$$= \int_{-3}^{3} \sqrt{(2 - \operatorname{sech}^2 t)^2}\,dt = \int_{-3}^{3} (2 - \operatorname{sech}^2 t)\,dt$$

$$= [2t - \tanh t]_{-3}^{3} = 12 - 2\tanh 3$$

45. $\frac{dx}{dt} = -\sin t,$

$$\frac{dy}{dt} = \frac{\sec t \tan t + \sec^2 t}{\sec t + \tan t} - \cos t = \sec t - \cos t$$

$$L = \int_0^{\pi/4} \sqrt{\sin^2 t + (\sec^2 t - 2 + \cos^2 t)}\,dt$$

$$= \int_0^{\pi/4} \tan t\,dt$$

$$= \left[-\ln|\cos t|\right]_0^{\pi/4} = -\ln\frac{1}{\sqrt{2}} = \frac{1}{2}\ln 2$$

46. $\frac{dx}{dt} = t\sin t, \frac{dy}{dt} = t\cos t$

$$L = \int_{\pi/4}^{\pi/2} \sqrt{t^2\sin^2 t + t^2\cos^2 t}\,dt$$

$$= \int_{\pi/4}^{\pi/2} t\,dt = \left[\frac{1}{2}t^2\right]_{\pi/4}^{\pi/2} = \frac{3\pi^2}{32}$$

47. a. $\frac{dx}{d\theta} = \cos\theta, \frac{dy}{d\theta} = -\sin\theta$

$L = \int_0^{2\pi} \sqrt{\cos^2\theta + \sin^2\theta}\, d\theta = \int_0^{2\pi} d\theta$

$= [\theta]_0^{2\pi} = 2\pi$

b. $\frac{dx}{d\theta} = 3\cos 3\theta, \frac{dy}{d\theta} = -3\sin 3\theta$

$L = \int_0^{2\pi} \sqrt{9\cos^2 3\theta + 9\sin^2 3\theta}\, d\theta$

$= 3\int_0^{2\pi} d\theta = 3[\theta]_0^{2\pi} = 6\pi$

c. The curve in part a goes around the unit circle once, while the curve in part b goes around the unit circle three times.

48. $\Delta S = 2\pi x \Delta s$

$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$S = \int_a^b 2\pi x\, ds = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

See Section 6.4 of the text

49. $\frac{dx}{dt} = -\sin t, \frac{dy}{dt} = \cos t$

$S = \int_0^{2\pi} 2\pi(1+\cos t)\sqrt{\sin^2 t + \cos^2 t}\, dt$

$= 2\pi \int_0^{2\pi} (1+\cos t)\, dt = 2\pi[t + \sin t]_0^{2\pi} = 4\pi^2$

50. $\frac{dx}{dt} = -\sin t, \frac{dy}{dt} = \cos t$

$S = \int_0^{2\pi} 2\pi(3+\sin t)\sqrt{\sin^2 t + \cos^2 t}\, dt$

$= 2\pi \int_0^{2\pi} (3+\sin t)\, dt = 2\pi[3t - \cos t]_0^{2\pi} = 12\pi^2$

51. $\frac{dx}{dt} = -\sin t, \frac{dy}{dt} = \cos t$

$S = \int_0^{2\pi} 2\pi(1+\sin t)\sqrt{\sin^2 t + \cos^2 t}\, dt$

$= 2\pi \int_0^{2\pi} (1+\sin t)\, dt = 2\pi[t - \cos t]_0^{2\pi} = 4\pi^2$

52. $\frac{dx}{dt} = \sqrt{t}, \frac{dy}{dt} = \frac{1}{\sqrt{t}}$

$S = \int_0^{2\sqrt{3}} 2\pi\left(\frac{2}{3}t^{3/2}\right)\sqrt{t + \frac{1}{t}}\, dt$

$= \frac{4\pi}{3}\int_0^{2\sqrt{3}} t\sqrt{t^2+1}\, dt$

$= \frac{4\pi}{3}\left[\frac{1}{3}(t^2+1)^{3/2}\right]_0^{2\sqrt{3}} = \frac{4\pi}{9}(13\sqrt{13} - 1)$

53. $\frac{dx}{dt} = 1, \frac{dy}{dt} = t + \sqrt{7}$

$S = \int_{-\sqrt{7}}^{\sqrt{7}} 2\pi\left(t+\sqrt{7}\right)\sqrt{1 + \left(t+\sqrt{7}\right)^2}\, dt$

$= 2\pi\left[\frac{1}{3}\left(1 + \left(t+\sqrt{7}\right)^2\right)^{3/2}\right]_{-\sqrt{7}}^{\sqrt{7}}$

$= \frac{2\pi}{3}\left(29\sqrt{29} - 1\right)$

54. $\frac{dx}{dt} = t + a, \frac{dy}{dt} = 1$

$S = \int_{-\sqrt{a}}^{\sqrt{a}} 2\pi(t+a)\sqrt{(t+a)^2 + 1}\, dt$

$= 2\pi\left[\frac{1}{3}((t+a)^2+1)^{3/2}\right]_{-\sqrt{a}}^{\sqrt{a}}$

$= \frac{2\pi}{3}\left[\left(a^2 + 2a\sqrt{a} + a + 1\right)^{3/2} - \left(a^2 - 2a\sqrt{a} + a + 1\right)^{3/2}\right]$

55. $dx = dt$; when $x = 0, t = -1$; when $x = 1, t = 0$.

$\int_0^1 (x^2 - 4y)\, dx = \int_{-1}^0 [(t+1)^2 - 4(t^3+4)]\, dt$

$= \int_{-1}^0 (-4t^3 + t^2 + 2t - 15)\, dt$

$= \left[-t^4 + \frac{1}{3}t^3 + t^2 - 15t\right]_{-1}^0 = -\frac{44}{3}$

56. $dy = \sec^2 t\, dt$; when $y = 1, t = \frac{\pi}{4}$;

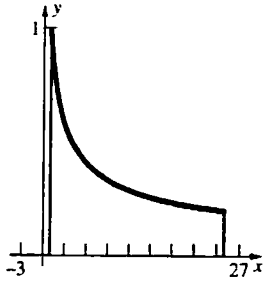
when $y = \sqrt{3}, t = \frac{\pi}{3}$.

$\int_1^{\sqrt{3}} xy\, dy = \int_{\pi/4}^{\pi/3} (\sec t)(\tan t)\sec^2 t\, dt$

$= \left[\frac{1}{3}\sec^3 t\right]_{\pi/4}^{\pi/3} = \frac{8}{3} - \frac{2\sqrt{2}}{3}$

57. $dx = 2e^{2t}dt$

$A = \int_1^{25} y\,dx = \int_0^{\ln 5} 2e^t dt = [2e^t]_0^{\ln 5} = 8$



58. a. $t = \dfrac{x}{v_0 \cos\alpha}$

$y = -16\left(\dfrac{x}{v_0 \cos\alpha}\right)^2 + (v_0 \sin\alpha)\left(\dfrac{x}{v_0 \cos x}\right)$

$y = -\left(\dfrac{16}{v_0^2 \cos^2 \alpha}\right)x^2 + (\tan\alpha)x$

This is an equation for a parabola.

b. Solve for t when $y = 0$.

$-16t^2 + (v_0 \sin\alpha)t = 0$

$t(-16t + v_0 \sin\alpha) = 0$

$t = 0, \dfrac{v_0 \sin\alpha}{16}$

The time of flight is $\dfrac{v_0 \sin\alpha}{16}$ seconds.

c. At $t = \dfrac{v_0 \sin\alpha}{16}, x = (v_0 \cos\alpha)\left(\dfrac{v_0 \sin\alpha}{16}\right)$

$= \dfrac{v_0^2 \sin\alpha\cos\alpha}{16} = \dfrac{v_0^2 \sin 2\alpha}{32}.$

d. Let R be the range as a function of α.

$R = \dfrac{v_0^2 \sin 2\alpha}{32}$

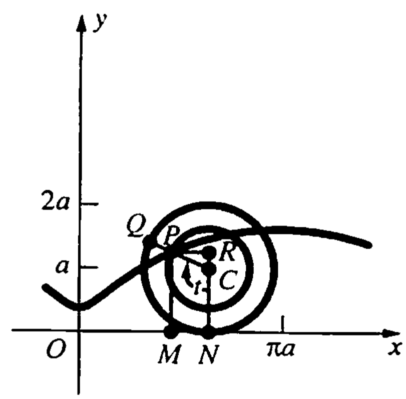
$\dfrac{dR}{d\alpha} = \dfrac{v_0^2 \cos 2\alpha}{16}$

$\dfrac{v_0^2 \cos 2\alpha}{16} = 0, \cos 2\alpha = 0, \alpha = \dfrac{\pi}{4}$

$\dfrac{d^2R}{d\alpha^2} = -\dfrac{v_0^2 \sin 2\alpha}{8}; \dfrac{d^2R}{d\alpha^2} < 0$ at $\alpha = \dfrac{\pi}{4}$.

The range is the largest possible when $\alpha = \dfrac{\pi}{4}$.

59.

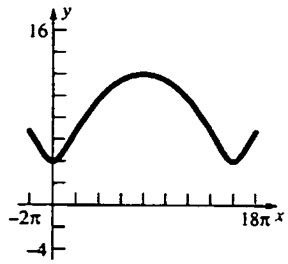


Let the wheel roll along the x-axis with P initially at $(0, a - b)$.

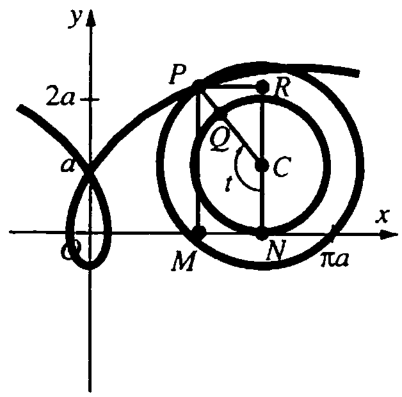
$|ON| = \text{arc } NQ = at$

$x = |OM| = |ON| - |MN| = at - b\sin t$

$y = |MP| = |RN| = |NC| + |CR| = a - b\cos t$



60.

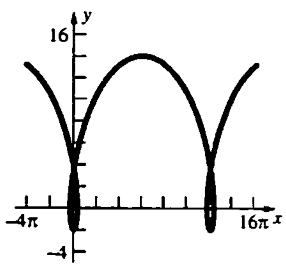


Let the wheel roll along the x-axis with P initially at $(0, a - b)$.

$|ON| = \text{arc } NQ = at$

$x = |OM| = |ON| - |MN| = at - b\sin t$

$y = |MP| = |RN| = |NC| + |CR| = a - b\cos t$



61. The x- and y-coordinates of the center of the circle of radius b are $(a-b)\cos t$ and $(a-b)\sin t$, respectively. The angle measure (in a clockwise direction) of arc BP is $\frac{a}{b}t$. The horizontal change from the center of the circle of radius b to P is $b\cos\left(-\left(\frac{a}{b}t-t\right)\right)=b\cos\left(\frac{a-b}{b}t\right)$ and the vertical change is $b\sin\left(-\left(\frac{a}{b}t-t\right)\right)=-b\sin\left(\frac{a-b}{b}t\right)$. Therefore,

$x=(a-b)\cos t+b\cos\left(\frac{a-b}{b}t\right)$ and

$y=(a-b)\sin t-b\sin\left(\frac{a-b}{b}t\right)$.

62. From Problem 61,

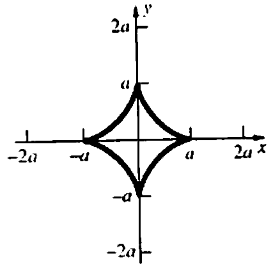
$x=(a-b)\cos t+b\cos\left(\frac{a-b}{b}t\right)$ and

$y=(a-b)\sin t-b\sin\left(\frac{a-b}{b}t\right)$.

Substitute $b=\frac{a}{4}$.

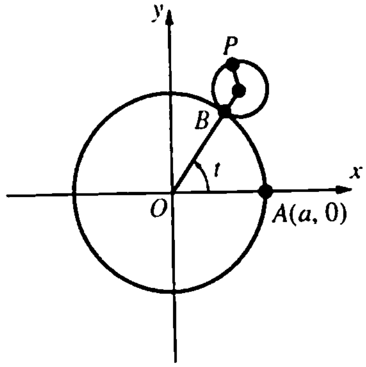
$$x=\left(\frac{3a}{4}\right)\cos t+\left(\frac{a}{4}\right)\cos(3t)$$
$$=\left(\frac{3a}{4}\right)\cos t+\left(\frac{a}{4}\right)\cos(2t+t)$$
$$=\left(\frac{3a}{4}\right)\cos t+\left(\frac{a}{4}\right)(\cos 2t\cos t-\sin 2t\sin t)$$
$$=\left(\frac{3a}{4}\right)\cos t+\left(\frac{a}{4}\right)(\cos^3 t-\sin^2 t\cos t-2\sin^2 t\cos t)$$
$$=\left(\frac{3a}{4}\right)\cos t+\left(\frac{a}{4}\right)\cos^3 t-\left(\frac{3a}{4}\right)\cos t\sin^2 t$$
$$=\left(\frac{3a}{4}\right)(\cos t)(1-\sin^2 t)+\left(\frac{a}{4}\right)\cos^3 t=a\cos^3 t$$

$$y=\left(\frac{3a}{4}\right)\sin t-\left(\frac{a}{4}\right)\sin(3t)$$
$$=\left(\frac{3a}{4}\right)\sin t-\left(\frac{a}{4}\right)\sin(2t+t)$$
$$=\left(\frac{3a}{4}\right)\sin t-\left(\frac{a}{4}\right)(\sin 2t\cos t+\cos 2t\sin t)$$
$$=\left(\frac{3a}{4}\right)\sin t-\left(\frac{a}{4}\right)(2\sin t\cos^2 t+\cos^2 t\sin t-\sin^3 t)$$
$$=\left(\frac{3a}{4}\right)\sin t-\left(\frac{3a}{4}\right)\sin t\cos^2 t+\left(\frac{a}{4}\right)\sin^3 t$$
$$=\left(\frac{3a}{4}\right)(\sin t)(1-\cos^2 t)+\left(\frac{a}{4}\right)\sin^3 t=a\sin^3 t$$



$$x^{2/3}+y^{2/3}=a^{2/3}\cos^2 t+a^{2/3}\sin^2 t=a^{2/3}$$

63. Consider the following figure similar to the one in the text for Problem 61.



The x- and y-coordinates of the center of the circle of radius b are $(a+b)\cos t$ and $(a+b)\sin t$ respectively. The angle measure (in a counter-clockwise direction) of arc PB is $\frac{a}{b}t$. The horizontal change from the center of the circle of radius b to P is

$b\cos\left(\frac{a}{b}t+t+\pi\right)=-b\cos\left(\frac{a+b}{b}t\right)$ and the vertical change is

$b\sin\left(\frac{a}{b}t+t+\pi\right)=-b\sin\left(\frac{a+b}{b}t\right)$. Therefore,

$x=(a+b)\cos t-b\cos\left(\frac{a+b}{b}t\right)$ and

$y=(a+b)\sin t-b\sin\left(\frac{a+b}{b}t\right)$.

64. $x=2a\cos t-a\cos 2t=2a\cos t-2a\cos^2 t+a$

$=2a\cos t(1-\cos t)+a$

$y=2a\sin t-a\sin 2t=2a\sin t-2a\sin t\cos t$

$=2a\sin t(1-\cos t)$

$x-a=2a\cos t(1-\cos t)$

$(x-a)^2+y^2=4a^2(1-\cos t)^2$

$(x-a)^2+y^2+2a(x-a)=$

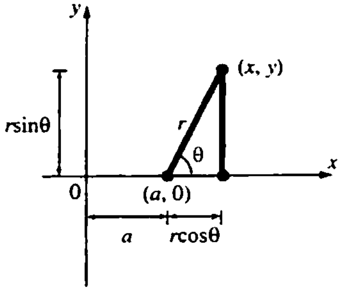
$4a^2(1-\cos t)^2+4a^2(1-\cos t)\cos t$

$(x-a)^2+y^2+2a(x-a)=4a^2(1-\cos t)$

$[(x-a)^2+y^2+2a(x-a)]^2=4a^2\cdot 4a^2(1-\cos t)^2$

$[(x-a)^2+y^2+2a(x-a)]^2=4a^2[(x-a)^2+y^2]$

If $(a, 0)$ is the pole with the polar axis in the direction of the positive x-axis, $x=r\cos\theta+a$ and $y=r\sin\theta$. (See figure below.)



Substitute $x-a=r\cos\theta$ and $y=r\sin\theta$ into the Cartesian equation.

$[(r\cos\theta)^2+(r\sin\theta)^2+2a(r\cos\theta)]^2$
$\quad=4a^2[(r\cos\theta)^2+(r\sin\theta)^2]$

$(r^2+2ar\cos\theta)^2=4a^2r^2$

$r^2(r+2a\cos\theta)^2=4a^2r^2$

$r+2a\cos\theta=2a$

$r=2a(1-\cos\theta)$

65. $\frac{dx}{dt}=\left(\frac{a}{3}\right)(-2\sin t-2\sin 2t)$,

$\frac{dy}{dt}=\left(\frac{a}{3}\right)(2\cos t-2\cos 2t)$

$\left(\frac{dx}{dt}\right)^2=\left(\frac{a}{3}\right)^2(4\sin^2 t+8\sin t\sin 2t+4\sin^2 2t)$

$\left(\frac{dy}{dt}\right)^2=\left(\frac{a}{3}\right)^2(4\cos^2 t-8\cos t\cos 2t+4\cos^2 2t)$

$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=\left(\frac{a}{3}\right)^2(8+8\sin t\sin 2t-8\cos t\cos 2t)$

$=\left(\frac{a}{3}\right)^2(8+16\sin^2 t\cos t-8\cos^3 t+8\sin^2 t\cos t)$

$=\left(\frac{a}{3}\right)^2(8+24\cos t\sin^2 t-8\cos^3 t)$

$=\left(\frac{a}{3}\right)^2(8+24\cos t-32\cos^3 t)$

$L=3\int_0^{2\pi/3}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

$=a\int_0^{2\pi/3}\sqrt{8+24\cos t-32\cos^3 t}\,dt$

Using a CAS to evaluate the length, $L=\frac{16a}{3}$.

66. **a.** Let $x=a\cos t$ and $y=b\sin t$.

$\frac{dx}{dt}=-a\sin t, \frac{dy}{dt}=b\cos t$

$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=a^2\sin^2 t+b^2\cos^2 t$

$=a^2+(b^2-a^2)\cos^2 t$

$=a^2-c^2\cos^2 t=a^2\left(1-\frac{c^2}{a^2}\cos^2 t\right)$

$=a^2\left(1-\left(\frac{c}{a}\right)^2\cos^2 t\right)=a^2(1-e^2\cos^2 t)$

$P=4\int_0^{\pi/2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

$=4a\int_0^{\pi/2}\sqrt{1-e^2\cos^2 t}\,dt$

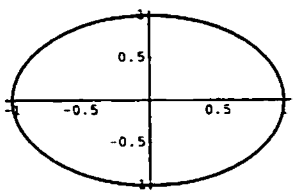
b. $P=4\int_0^{\pi/2}\sqrt{1-\frac{\cos^2 t}{16}}\,dt$

$=\int_0^{\pi/2}\sqrt{16-\cos^2 t}\,dt\approx 6.1838$

(The answer is near 2π because it is slightly smaller than a circle of radius 1 whose perimeter is 2π).

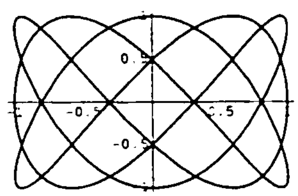
c. $P=\int_0^{\pi/2}\sqrt{16-\cos^2 t}dt\approx 6.1838$

67. **a.**



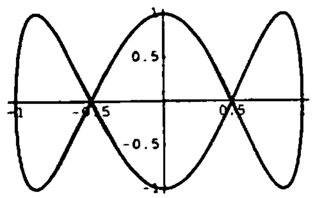
The curve touches a horizontal border once and touches a vertical border once.

b.



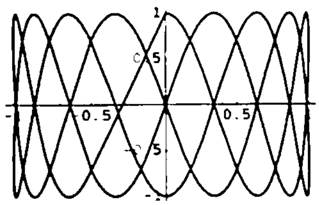
The curve touches a horizontal border five times and touches a vertical border three times.

c.



The curve touches a horizontal border three times and touches a vertical border once. Note that the curve is traced out three times.

d.



The curve touches a horizontal border nine times and touches a vertical border twice.

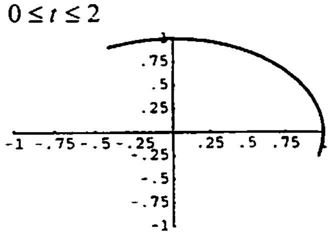
68. a. Figure 15

b. Figure 16

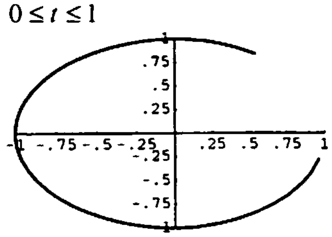
c. Figure 17

d. Figure 14

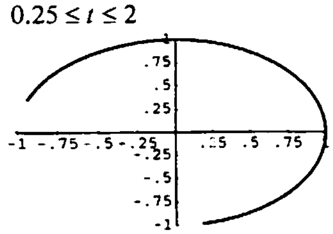
69. a. $0 \le t \le 2$



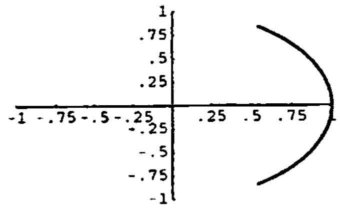
b. $0 \le t \le 1$



c. $0.25 \le t \le 2$

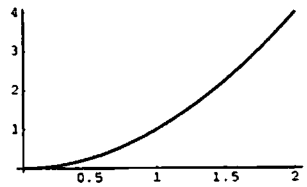


d. $0 \le t \le 2\pi$



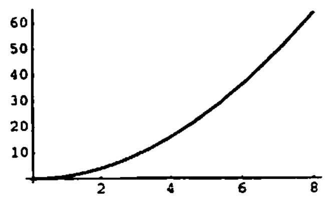
Given a parameterization of the form $x = \cos f(t)$ and $y = \sin f(t)$, the point moves around the curve (which is a circle of radius 1) at a speed of $|f'(t)|$. The point travels clockwise around the circle when $f(t)$ is decreasing and counterclockwise when $f(t)$ is increasing. Note that in part d, only part of the circle will be traced out since the range of $f(t) = \sin t$ is $[-1, 1]$.

70. a.



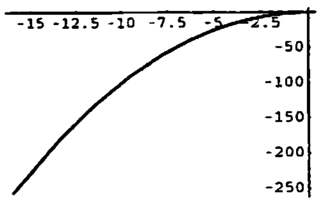
The curve traced out is the graph of $y = x^2$ for $0 \le x \le 2$

b.



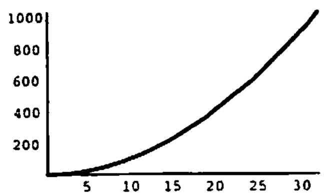
The curve traced out is the graph of $y = x^2$ for $0 \le x \le 8$.

c.



The curve traced out is the graph of $y = -x^2$ for $-16 \le x \le 0$.

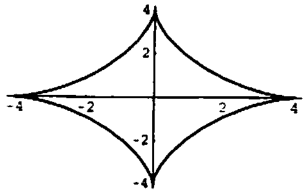
d.



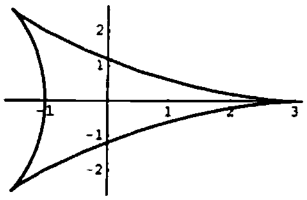
The curve traced out is the graph of $y = x^2$ for $0 \le x \le 32$

All of the curves lie on the graph of $y = \pm x^2$, but trace out different parts because of the parameterization

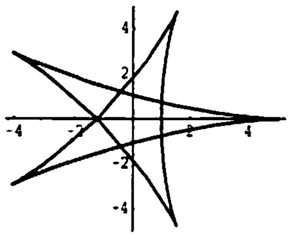
71. **a.** $0 \le t \le 2\pi$



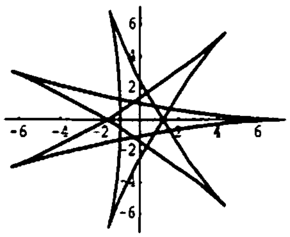
b. $0 \le t \le 2\pi$



c. $0 \le t \le 4\pi$



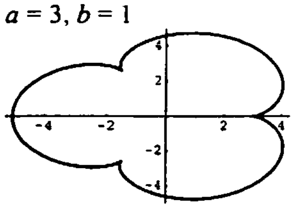
d. $0 \le t \le 8\pi$



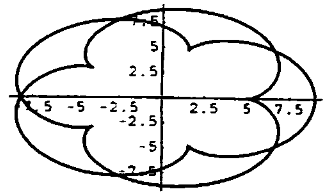
Let $\frac{p}{q} = \frac{a}{b}$ where $\frac{p}{q}$ is the reduced fraction of $\frac{a}{b}$. The length of the t-interval is $2q\pi$. The number of times the graph would touch the circle of radius a during the t-interval is p.

72. Some possible graphs for different a and b are shown below.

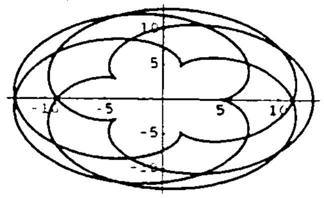
$a = 3, b = 1$



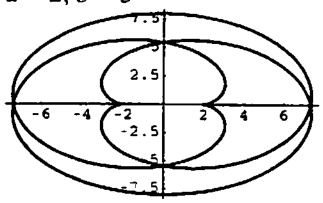
$a = 5, b = 2$



$a = 5, b = 4$

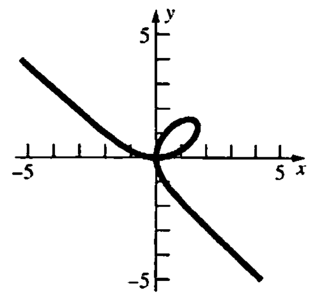


$a = 2, b = 3$



Let $\frac{p}{q} = \frac{a}{b}$ where $\frac{p}{q}$ is the reduced fraction of $\frac{a}{b}$. The length of the t-interval is $2q\pi$. The number of times the graph would touch the circle of radius a during the t-interval is p.

73. $x = \frac{3t}{t^3 + 1}, y = \frac{3t^2}{t^3 + 1}$



When $x > 0$, $t > 0$ or $t < -1$.
When $x < 0$, $-1 < t < 0$.
When $y > 0$, $t > -1$. When $y < 0$, $t < -1$.
Therefore the graph is in quadrant I for $t > 0$, quadrant II for $-1 < t < 0$, quadrant III for no t, and quadrant IV for $t < -1$.

13.2 Concepts Review

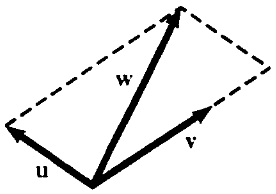
1. magnitude; direction

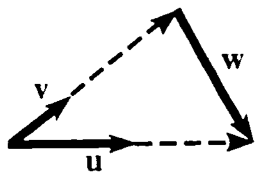
2. they have the same magnitude and direction.

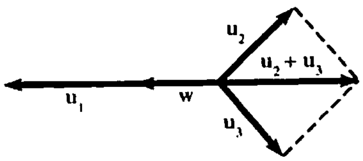
3. the tail of **u**; the head of **v**

4. force; velocity

Problem Set 13.2

1. 

2. 

3. 

4. 0

5. $\mathbf{w} = \frac{1}{2}(\mathbf{u}+\mathbf{v}) = \frac{1}{2}\mathbf{u} + \frac{1}{2}\mathbf{v}$

6. $\mathbf{n} = \frac{1}{2}(\mathbf{v}-\mathbf{u}) = \frac{1}{2}\mathbf{v} - \frac{1}{2}\mathbf{u}$

   $\mathbf{m} = \mathbf{v} - \mathbf{n} = \mathbf{v} - \frac{1}{2}(\mathbf{v}-\mathbf{u}) = \frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{u}$

7. $|\mathbf{w}| = |\mathbf{u}|\cos 60° + |\mathbf{v}|\cos 60° = \frac{1}{2} + \frac{1}{2} = 1$

8. $|\mathbf{w}| = |\mathbf{u}|\cos 45° + |\mathbf{v}|\cos 45° = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$

9. Let θ be the angle of **w** measured clockwise from south.

   $|\mathbf{w}|\cos\theta = |\mathbf{u}|\cos 30° + |\mathbf{v}|\cos 45° = 25\sqrt{3} + 25\sqrt{2}$

   $= 25(\sqrt{3}+\sqrt{2})$

   $|\mathbf{w}|\sin\theta = |\mathbf{v}|\sin 45° - |\mathbf{u}|\sin 30° = 25\sqrt{2} - 25$

   $= 25(\sqrt{2}-1)$

   $|\mathbf{w}|^2 = |\mathbf{w}|^2\cos^2\theta + |\mathbf{w}|^2\sin^2\theta$

   $= 625(\sqrt{3}+\sqrt{2})^2 + 625(\sqrt{2}-1)^2$

   $= 625(8 - 2\sqrt{2} + 2\sqrt{6})$

   $|\mathbf{w}| = \sqrt{625(8-2\sqrt{2}+2\sqrt{6})} = 25\sqrt{8-2\sqrt{2}+2\sqrt{6}}$

   ≈ 79.34

   $\tan\theta = \frac{|\mathbf{w}|\sin\theta}{|\mathbf{w}|\cos\theta} = \frac{\sqrt{2}-1}{\sqrt{3}+\sqrt{2}}$

   $\theta = \tan^{-1}\left(\frac{\sqrt{2}-1}{\sqrt{3}+\sqrt{2}}\right) = 7.5°$

   w has magnitude 79.34 lb in the direction S 7.5° W.

10. Let **v** be the resulting force. Let θ be the angle of **v** measured clockwise from south.

    $|\mathbf{v}|\cos\theta = 60\cos 30° + 80\cos 60° = 30\sqrt{3} + 40$

    $= 10(3\sqrt{3}+4)$

    $|\mathbf{v}|\sin\theta = 80\sin 60° - 60\sin 30° = 40\sqrt{3} - 30$

    $= 10(4\sqrt{3}-3)$

    $|\mathbf{v}|^2 = |\mathbf{v}|^2\cos^2\theta + |\mathbf{v}|^2\sin^2\theta$

    $= 100(3\sqrt{3}+4)^2 + 100(4\sqrt{3}-3)^2$

    $= 100(100) = 10{,}000$

    $|\mathbf{v}| = \sqrt{10{,}000} = 100$

    $\tan\theta = \frac{|\mathbf{v}|\sin\theta}{|\mathbf{v}|\cos\theta} = \frac{4\sqrt{3}-3}{3\sqrt{3}+4}$

    $\theta = \tan^{-1}\left(\frac{4\sqrt{3}-3}{3\sqrt{3}+4}\right) \approx 23.13°$

    The resultant force has magnitude 100 lb in the direction S 23.13° W.

11. The force of 300 N parallel to the plane has magnitude 300 sin 30° = 150 N. Thus, a force of 150 N parallel to the plane will just keep the weight from sliding.

12. Let a be the magnitude of the rope that makes an angle of 27.34°. Let b be the magnitude of the rope that makes an angle of 39.22°.
    1. $a \sin 27.34° = b \sin 39.22°$
    2. $a \cos 27.34° + b \cos 39.22° = 258.5$

    Solve 1 for b and substitute in 2.

$$a\cos 27.34° + a\frac{\sin 27.34°}{\sin 39.22°}\cos 39.22° = 258.5$$

$$a = \frac{258.5}{\cos 27.34° + \sin 27.34° \cot 39.22°} \approx 178.15$$

$$b = \frac{a\sin 27.34°}{\sin 39.22°} \approx 129.40$$

The magnitudes of the forces exerted by the ropes making angles of 27.34° and 39.22° are 178.15 lb and 129.40 lb, respectively.

13. Let θ be the angle the plane makes from north, measured clockwise.

$425 \sin \theta = 45 \sin 20°$

$$\sin\theta = \frac{9}{85}\sin 20°$$

$$\theta = \sin^{-1}\left(\frac{9}{85}\sin 20°\right) \approx 2.08°$$

Let x be the speed of airplane with respect to the ground.

$x = 45 \cos 20° + 425 \cos \theta \approx 467$

The plane flies in the direction N 2.08° E, flying 467 mi/h with respect to the ground.

14. Let v be his velocity relative to the surface. Let θ be the angle that his velocity relative to the surface makes with south, measured clockwise.

$|v|\cos\theta = 20, |v|\sin\theta = 3$

$$|v|^2 = |v|^2\cos^2\theta + |v|^2\sin^2\theta = 400 + 9 = 409$$

$$|\mathbf{v}| = \sqrt{409} \approx 20.22$$

$$\tan\theta = \frac{|\mathbf{v}|\sin\theta}{|\mathbf{v}|\cos\theta} = \frac{3}{20}$$

$$\theta = \tan^{-1}\frac{3}{20} \approx 8.53°$$

His velocity has magnitude 20.22 mi/h in the direction S 8.53° W.

15. Let x be the air speed.

$x \cos 60° = 40$

$$x = \frac{40}{\cos 60°} = 80$$

The air speed of the plane is 80 mi/hr

16. Let x be the air speed. Let θ be the angle that the plane makes with north measured counter-clockwise.

$x \cos \theta = 837 + 63 \cos 11.5°$

$x \sin \theta = 63 \sin 11.5°$

$$x^2 = x^2\cos^2\theta + x^2\sin^2\theta$$

$$= (837 + 63\cos 11.5°)^2 + (63\sin 11.5°)^2$$

$$= 704{,}538 + 105{,}462 \cos 11.5°$$

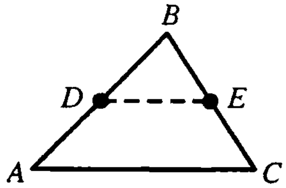
$$x = \sqrt{704{,}538 + 105{,}462\cos 11.5°} \approx 898.82$$

$$\tan\theta = \frac{x\sin\theta}{x\cos\theta} = \frac{63\sin 11.5°}{837 + 63\cos 11.5°}$$

$$\theta = \tan^{-1}\left(\frac{63\sin 11.5°}{837 + 63\cos 11.5°}\right) \approx 0.80°$$

The plane should fly in the direction N 0.80° W at an air speed of 898.82 mi/h.

17. Given triangle ABC, let D be the midpoint of AB and E be the midpoint of BC. $\mathbf{u} = \overline{AB}$, $\mathbf{v} = \overline{BC}$, $\mathbf{w} = \overline{AC}$, $\mathbf{z} = \overline{DE}$



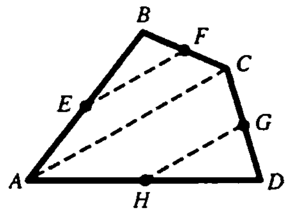
$\mathbf{u} + \mathbf{v} = \mathbf{w}$

$$\mathbf{z} = \frac{1}{2}\mathbf{u} + \frac{1}{2}\mathbf{v} = \frac{1}{2}(\mathbf{u} + \mathbf{v}) = \frac{1}{2}\mathbf{w}$$

Thus, DE is parallel to AC.

18. Given quadrilateral $ABCD$, let E be the midpoint of AB, F the midpoint of BC, G the midpoint of CD, and H the midpoint of AD.

ABC and ACD are triangles. From Problem 17, EF and HG are parallel to AC. Thus, EF is parallel to HG. By similar reasoning using triangles ABD and BCD, EH is parallel to FG. Therefore, $EFGH$ is parallelogram.

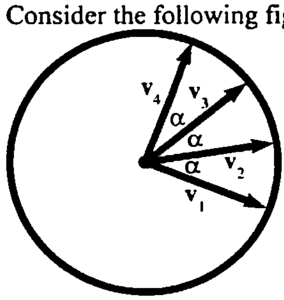


19. Let P_i be the tail of $\mathbf{v}_i$. Then

$$\mathbf{v}_1 + \mathbf{v}_2 + \ldots + \mathbf{v}_n = \overrightarrow{P_1P_2} + \overrightarrow{P_2P_3} + \cdots + \overrightarrow{P_nP_1}$$

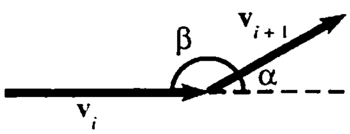
$$= \overrightarrow{P_1P_1} = \mathbf{0}.$$

20. Consider the following figure of the circle.



$$\alpha = \frac{2\pi}{n}$$

The vectors have the same length. Consider the following figure for adding vectors $\mathbf{v}_i$ and $\mathbf{v}_{i+1}$.



Then $\beta = \pi - \frac{2\pi}{n}$. Note that the interior angle of a regular n-gon is $\pi - \frac{2\pi}{n}$. Thus the vectors (placed head to tail from $\mathbf{v}_1$ to $\mathbf{v}_n$) form a regular n-gon. From Problem 19, the sum of the vectors is 0.

21. The components of the forces along the lines containing AP, BP, and CP are in equilibrium; that is,
$W = W\cos\alpha + W\cos\beta$
$W = W\cos\beta + W\cos\gamma$
$W = W\cos\alpha + W\cos\gamma$
Thus, $\cos\alpha + \cos\beta = 1$, $\cos\beta + \cos\gamma = 1$, and $\cos\alpha + \cos\gamma = 1$. Solving this system of equations results in $\cos\alpha = \cos\beta = \cos\gamma = \frac{1}{2}$.
Hence $\alpha = \beta = \gamma = 60°$.
Therefore, $\alpha + \beta = \alpha + \gamma = \beta + \gamma = 120°$.

22. Let A', B', C' be the points where the weights are attached. The center of gravity is located $\frac{|AA'| + |BB'| + |CC'|}{3}$ units below the plane of the triangle. Then, using the hint, the system is in equilibrium when $|AA'| + |BB'| + |CC'|$ is maximum. Hence, it is in equilibrium when $|AP| + |BP| + |CP|$ is minimum, because the total length of the string is $|AP| + |AA'| + |BP| + |BB'| + |CP| + |CC'|$.

23. The components of the forces along the lines containing AP, BP, and CP are in equilibrium; that is,
$5w\cos\alpha + 4w\cos\beta = 3w$
$3w\cos\beta + 5w\cos\gamma = 4w$
$3w\cos\alpha + 4w\cos\gamma = 5w$
Thus, $5\cos\alpha + 4\cos\beta = 3$, $3\cos\beta + 5\cos\gamma = 4$, and $3\cos\alpha + 4\cos\gamma = 5$. Solving this system of equations results in
$\cos\alpha = \frac{3}{5}, \cos\beta = 0, \cos\gamma = \frac{4}{5}$, from which it follows that $\sin\alpha = \frac{4}{5}, \sin\beta = 1, \sin\gamma = \frac{3}{5}$.
Therefore, $\cos(\alpha+\beta) = -\frac{4}{5}, \cos(\alpha+\gamma) = 0$,
$\cos(\beta+\gamma) = -\frac{3}{5}$, so
$\alpha + \beta = \cos^{-1}\left(-\frac{4}{5}\right) \approx 143.13°$, $\alpha + \gamma = 90°$,
$\beta + \gamma = \cos^{-1}\left(-\frac{3}{5}\right) \approx 126.87°$.
This problem can be modeled with three strings going through A, four strings through B, and five strings through C, with equal weights attached to the twelve strings. Then the quantity to be minimized is $3|AP| + 4|BP| + 5|CP|$.

24. Written response.

13.3 Concepts Review

1. $\langle u_1 + v_1, u_2 + v_2\rangle; \langle cu_1, cu_2\rangle; \sqrt{u_1^2 + u_2^2}$

2. $u_1v_1 + u_2v_2; |\mathbf{u}||\mathbf{v}|\cos\theta$

3. basis vectors

4. $\mathbf{F}\cdot\mathbf{D}$

Problem Set 13.3

1. a. $2\mathbf{a} - 4\mathbf{b} = (-4\mathbf{i} + 6\mathbf{j}) + (-8\mathbf{i} + 12\mathbf{j})$
$= -12\mathbf{i} + 18\mathbf{j}$

b. $\mathbf{a}\cdot\mathbf{b} = (-2)(2) + (3)(-3) = -13$

c. $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = (-2\mathbf{i} + 3\mathbf{j})\cdot(2\mathbf{i} - 8\mathbf{j})$
$= (-2)(2) + (3)(-8) = -28$

d. $(-2\mathbf{a} + 3\mathbf{b})\cdot 5\mathbf{c} = 5[(10\mathbf{i} - 15\mathbf{j})\cdot(-5\mathbf{j})]$
$= 5[(10)(0) + (-15)(-5)] = 375$

e. $|\mathbf{a}|\mathbf{c}\cdot\mathbf{a} = \sqrt{4+9}[(0)(-2) + (-5)(3)] = -15\sqrt{13}$

f. $\mathbf{b}\cdot\mathbf{b}-|\mathbf{b}|=(2)(2)+(-3)(-3)-\sqrt{4+9}$
$=13-\sqrt{13}$

2. a. $-4\mathbf{a}+3\mathbf{b}=\langle -12,4\rangle+\langle 3,-3\rangle=\langle -9,\ 1\rangle$

b. $\mathbf{b}\cdot\mathbf{c}=(1)(0)+(-1)(5)=-5$

c. $(\mathbf{a}+\mathbf{b})\cdot\mathbf{c}=\langle 4,-2\rangle\cdot\langle 0,5\rangle$
$=(4)(0)+(-2)(5)=-10$

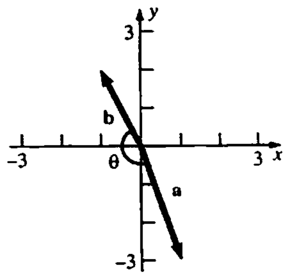
d. $2\mathbf{c}\cdot(3\mathbf{a}+4\mathbf{b})=2\langle 0,\ 5\rangle\cdot(\langle 9,\ -3\rangle+\langle 4,\ -4\rangle)$
$=2\langle 0,\ 5\rangle\cdot\langle 13,\ -7\rangle=2[(0)(13)+(5)(-7)]$
$=-70$

e. $|\mathbf{b}|\mathbf{b}\cdot\mathbf{a}=\sqrt{1+1}[(1)(3)+(-1)(-1)]=4\sqrt{2}$

f. $|\mathbf{c}|^2-\mathbf{c}\cdot\mathbf{c}=\left(\sqrt{0+25}\right)^2-[(0)(0)+(5)(5)]$
$=0$

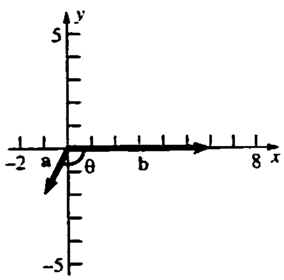
3. a. $\cos\theta=\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}=\dfrac{(1)(-1)+(-3)(2)}{\left(\sqrt{10}\right)\left(\sqrt{5}\right)}=-\dfrac{7}{\sqrt{50}}$

$=-\dfrac{7}{5\sqrt{2}}\approx -0.9899$

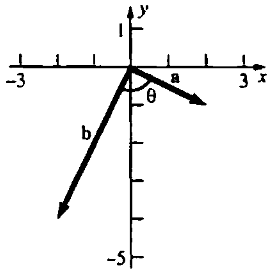


b. $\cos\theta=\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}=\dfrac{(-1)(6)+(-2)(0)}{\left(\sqrt{5}\right)(6)}=-\dfrac{6}{6\sqrt{5}}$

$=-\dfrac{1}{\sqrt{5}}\approx -0.4472$

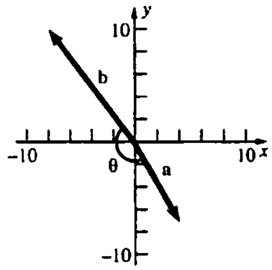


c. $\cos\theta=\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}=\dfrac{(2)(-2)+(-1)(-4)}{\left(\sqrt{5}\right)\left(2\sqrt{5}\right)}=\dfrac{0}{10}=0$



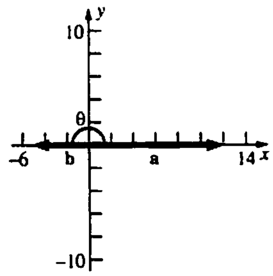
d. $\cos\theta=\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}=\dfrac{(4)(-8)+(-7)(10)}{\left(\sqrt{65}\right)\left(2\sqrt{41}\right)}$

$=\dfrac{-102}{2\sqrt{2665}}=-\dfrac{51}{\sqrt{2665}}\approx -0.9879$



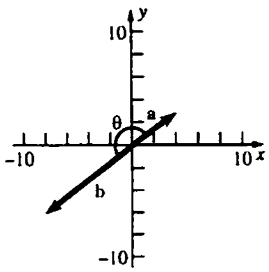
4. a. $\cos\theta=\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}=\dfrac{(12)(-5)+(0)(0)}{(12)(5)}=\dfrac{-60}{60}=-1$

$\theta=\cos^{-1}-1=180°$



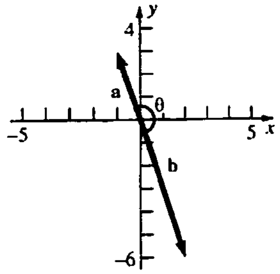
b. $\cos\theta=\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}=\dfrac{(4)(-8)+(3)(-6)}{(5)(10)}=\dfrac{-50}{50}=-1$

$\theta=\cos^{-1}-1=180°$



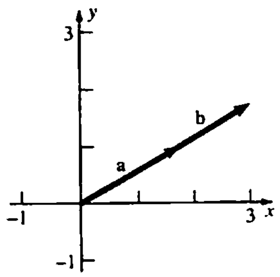
c. $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \dfrac{(-1)(2)+(3)(-6)}{\left(\sqrt{10}\right)\left(2\sqrt{10}\right)} = \dfrac{-20}{20} = -1$

$\theta = \cos^{-1} -1 = 180°$



d. $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \dfrac{\left(\sqrt{3}\right)(3)+(1)\left(\sqrt{3}\right)}{(2)\left(2\sqrt{3}\right)} = \dfrac{4\sqrt{3}}{4\sqrt{3}} = 1$

$\theta = \cos^{-1} 1 = 0°$



5. a. $\mathbf{a} = (-3-2)\mathbf{i} + (4-2)\mathbf{j} = -5\mathbf{i} + 2\mathbf{j}$

b. $\mathbf{a} = (-6-0)\mathbf{i} + (0-4)\mathbf{j} = -6\mathbf{i} - 4\mathbf{j}$

c. $\mathbf{a} = \left(0-\sqrt{2}\right)\mathbf{i} + (0+\pi)\mathbf{j} = -\sqrt{2}\mathbf{i} + \pi\mathbf{j}$

d. $\mathbf{a} = (-4+7)\mathbf{i} + \left(\dfrac{1}{3}-e\right)\mathbf{j} = 3\mathbf{i} + \left(\dfrac{1}{3}-e\right)\mathbf{j}$

6. a. $\mathbf{u} = \dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{-3\mathbf{i}-4\mathbf{j}}{5} = -\dfrac{3}{5}\mathbf{i} - \dfrac{4}{5}\mathbf{j}$

b. $\mathbf{u} = \dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{\mathbf{i}-7\mathbf{j}}{5\sqrt{2}} = \dfrac{1}{5\sqrt{2}}\mathbf{i} - \dfrac{7}{5\sqrt{2}}\mathbf{j}$

c. $\mathbf{u} = \dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{-4\mathbf{j}}{4} = -\mathbf{j}$

d. $\mathbf{u} = \dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{-5\mathbf{i}-12\mathbf{j}}{13} = -\dfrac{5}{13}\mathbf{i} - \dfrac{12}{13}\mathbf{j}$

7. $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} - \mathbf{v}\cdot\mathbf{v}$

$= |\mathbf{u}|^2 - |\mathbf{v}|^2 = 0$

Thus, $|\mathbf{u}|^2 = |\mathbf{v}|^2$ or $|\mathbf{u}| = |\mathbf{v}|$.

8. $(\mathbf{u}+\mathbf{v})\cdot(3\mathbf{u}-\mathbf{v}) = \mathbf{u}\cdot(3\mathbf{u}) - \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot(3\mathbf{u}) - \mathbf{v}\cdot\mathbf{v}$

$= 3(\mathbf{u}\cdot\mathbf{u}) - \mathbf{u}\cdot\mathbf{v} + 3(\mathbf{u}\cdot\mathbf{v}) - \mathbf{v}\cdot\mathbf{v}$

$= 3|\mathbf{u}|^2 - |\mathbf{v}|^2 + 2\mathbf{u}\cdot\mathbf{v}$

9. $a\mathbf{u} + b\mathbf{u} = a\langle u_1, u_2\rangle + b\langle u_1, u_2\rangle$

$= \langle au_1, au_2\rangle + \langle bu_1, bu_2\rangle = \langle au_1 + bu_1, au_2 + bu_2\rangle$

$\langle (a+b)u_1, (a+b)u_2\rangle = (a+b)\langle u_1, u_2\rangle$

$= (a+b)\mathbf{u}$

10. $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \langle u_1, u_2\rangle \cdot (\langle v_1, v_2\rangle + \langle w_1, w_2\rangle)$

$= \langle u_1, u_2\rangle \cdot \langle v_1 + w_1, v_2 + w_2\rangle$

$= u_1(v_1 + w_1) + u_2(v_2 + w_2)$

$= (u_1v_1 + u_2v_2) + (u_1w_1 + u_2w_2) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$

11. $c(\mathbf{u}\cdot\mathbf{v}) = c(\langle u_1, u_2\rangle \cdot \langle v_1, v_2\rangle)$

$= c(u_1v_1 + u_2v_2) = c(u_1v_1) + c(u_2v_2)$

$= (cu_1)v_1 + (cu_2)v_2 = \langle cu_1, cu_2\rangle \cdot \langle v_1, v_2\rangle$

$= (c\langle u_1, u_2\rangle)\cdot\langle v_1, v_2\rangle = (c\mathbf{u})\cdot\mathbf{v}$

12. $\mathbf{u}+\mathbf{v} = \mathbf{u} \Rightarrow (-\mathbf{u}) + (\mathbf{u}+\mathbf{v}) = (-\mathbf{u}) + \mathbf{u}$

$\Rightarrow [(-\mathbf{u}) + \mathbf{u}] + \mathbf{v} = \mathbf{u} + (-\mathbf{u})$

Theorem A, (1) and (2)

$\Rightarrow [\mathbf{u} + (-\mathbf{u})] + \mathbf{v} = \mathbf{0}$ Theorem A, (1) and (4)

$\Rightarrow \mathbf{0} + \mathbf{v} = \mathbf{0}$ Theorem A, (4)

$\Rightarrow \mathbf{v} = \mathbf{0}$ Theorem A, (3)

13. $3(6\mathbf{i} - 8\mathbf{j}) = 18\mathbf{i} - 24\mathbf{j}$

14. $\dfrac{-(-5\mathbf{i}+12\mathbf{j})}{|-5\mathbf{i}+12\mathbf{j}|} = \dfrac{5\mathbf{i}-12\mathbf{j}}{13} = \dfrac{5}{13}\mathbf{i} - \dfrac{12}{13}\mathbf{j}$

15. $\langle 6, 3\rangle \cdot \langle -1, 2\rangle = (6)(-1) + (3)(2) = -6 + 6 = 0$

By Theorem C, they are perpendicular.

16. $\langle -5, \sqrt{3}\rangle \cdot \langle \sqrt{27}, 15\rangle = (-5)\left(\sqrt{27}\right) + \left(\sqrt{3}\right)(15)$

$= -15\sqrt{3} + 15\sqrt{3} = 0$

By Theorem C, they are perpendicular.

17. $\langle c, 6\rangle \cdot \langle c, -4\rangle = (c)(c) + (6)(-4) = c^2 - 24 = 0$

$c^2 = 24$

$c = \pm 2\sqrt{6}$

18. $(2c\mathbf{i} - 8\mathbf{j})\cdot(3\mathbf{i} + c\mathbf{j}) = (2c)(3) + (-8)(c)$

$= 6c - 8c = -2c = 0$

$c = 0$

19. $\mathbf{r} = k\mathbf{a} + m\mathbf{b} \Rightarrow 7 = k(3) + m(-3)$ and

$-8 = k(-2) + m(4)$

$3k - 3m = 7$

$-2k + 4m = -8$

Solve the system of equations to get
$k = \frac{2}{3}, m = -\frac{5}{3}.$

20. $\mathbf{r} = k\mathbf{a} + m\mathbf{b} \Rightarrow 6 = k(-4) + m(2)$ and
$-7 = k(3) + m(-1)$
$-4k + 2m = 6$
$3k - m = -7$
Solve the system of equations to get
$k = -4, m = -5.$

21. $r_1\mathbf{i} + r_2\mathbf{j} = k(a_1\mathbf{i} + a_2\mathbf{j}) + m(b_1\mathbf{i} + b_2\mathbf{j})$
$= (ka_1 + mb_1)\mathbf{i} + (ka_2 + mb_2)\mathbf{j}$
$\Leftrightarrow r_1 = ka_1 + mb_1$ and $r_2 = ka_2 + mb_2$.
Solve these two equations simultaneously (noting that $a_1b_2 - a_2b_1 \neq 0$ since **a** and **b** are noncollinear) and obtain
$k = \frac{b_2r_1 - b_1r_2}{a_1b_2 - a_2b_1}$ and $m = \frac{a_1r_2 - a_2r_1}{a_1b_2 - a_2b_1}$.

22. a and b cannot both be zero. If $a = 0$, then the line $ax + by = c$ is horizontal and $\mathbf{n} = b\mathbf{j}$ is vertical, so $\mathbf{n}$ is perpendicular to the line. Use a similar argument if $b = 0$. If $d \neq 0$ and $b \neq 0$, then
$P_1\left(\frac{c}{a}, 0\right)$ and $P_2\left(0, \frac{c}{b}\right)$ are points on the line.
$\mathbf{n} \cdot \overrightarrow{P_1P_2} = (a\mathbf{i} + b\mathbf{j}) \cdot \left(-\frac{c}{a}\mathbf{i} + \frac{c}{b}\mathbf{j}\right) = -c + c = 0$

23. Work $= \mathbf{F} \cdot \mathbf{D} = (3\mathbf{i} + 10\mathbf{j}) \cdot (10\mathbf{j})$
$= 0 + 100 = 100$ joules

24. $\mathbf{F} = 100 \sin 70°\mathbf{i} - 100 \cos 70°\mathbf{j}$
$\mathbf{D} = 30\mathbf{i}$
Work $= \mathbf{F} \cdot \mathbf{D}$
$= (100 \sin 70°)(30) + (-100 \cos 70°)(0)$
$= 3000 \sin 70° \approx 2819$ joules

25. $\mathbf{D} = 5\mathbf{i} + 8\mathbf{j}$
Work $= \mathbf{F} \cdot \mathbf{D} = (6)(5) + (8)(8) = 94$ ft-lb

26. $\mathbf{D} = 12\mathbf{j}$
Work $= \mathbf{F} \cdot \mathbf{D} = (-5)(0) + (8)(12) = 96$ joules

27. $|\mathbf{u} \cdot \mathbf{v}| = |\cos\theta||\mathbf{u}||\mathbf{v}| \leq |\mathbf{u}||\mathbf{v}|$ since $|\cos\theta| \leq 1$.

28. $|\mathbf{u} + \mathbf{v}|^2 = |\mathbf{u}|^2 + 2\mathbf{u} \cdot \mathbf{v} + |\mathbf{v}|^2 \leq |\mathbf{u}|^2 + 2|\mathbf{u} \cdot \mathbf{v}| + |\mathbf{v}|^2$
$\leq |\mathbf{u}|^2 + 2|\mathbf{u}||\mathbf{v}| + |\mathbf{v}|^2 = (|\mathbf{u}| + |\mathbf{v}|)^2$
Therefore, $|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$.

29. a. $|\mathbf{u}|^2 + |\mathbf{v}|^2 = 2\mathbf{u} \cdot \mathbf{v} \Rightarrow \mathbf{u} \cdot \mathbf{u} - 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} = 0$
$\Rightarrow (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = 0 \Rightarrow |\mathbf{u} - \mathbf{v}|^2 = 0$
$\Rightarrow \mathbf{u} - \mathbf{v} = 0 \Rightarrow \mathbf{u} = \mathbf{v}$

b. $|\mathbf{u}|^2 = |\mathbf{v}|^2 = 2\mathbf{u} \cdot \mathbf{v} \Rightarrow |\mathbf{u}| = |\mathbf{v}|$ and
$|\mathbf{u}|^2 = 2|\mathbf{u}||\mathbf{v}|\cos\theta \Rightarrow |\mathbf{u}|^2 = 2|\mathbf{u}|^2\cos\theta$
$\Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$
Thus, **u** and **v** have the same length and the angle between them is $\frac{\pi}{3}$.

30. If **u** and **v** are adjacent edges, the diagonals are **u** + **v** and **u** – **v**.
$(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v}$
$= |\mathbf{u}|^2 - |\mathbf{v}|^2 = 0$
Therefore, **u** + **v** and **u** – **v** are perpendicular.

31. Let $x^2 + y^2 = r^2$ be the equation of the circle and let $A(-r, 0)$, $B(r, 0)$, and $C(x, y)$. Then
$\overrightarrow{AC} \cdot \overrightarrow{BC} = \langle x + r, y\rangle \cdot \langle x - r, y\rangle$
$= (x + r)(x - r) + y^2 = (x^2 + y^2) - r^2$
$= r^2 - r^2 = 0.$

32. **w** is between **u** and **v** since it is a linear combination of **u** and **v** with positive coefficients. Let α and β be the respective angles between **u** and **w** and between **v** and **w**.
$$\cos\alpha = \frac{\mathbf{u} \cdot \mathbf{w}}{|\mathbf{u}||\mathbf{w}|} = \frac{\mathbf{u} \cdot (|\mathbf{v}|\mathbf{u} + |\mathbf{u}|\mathbf{v})}{|\mathbf{u}||\mathbf{w}|}$$
$$= \frac{|\mathbf{v}|\mathbf{u} \cdot \mathbf{u} + |\mathbf{u}|\mathbf{v} \cdot \mathbf{u}}{|\mathbf{u}||\mathbf{w}|} = \frac{|\mathbf{v}||\mathbf{u}|^2 + |\mathbf{u}|\mathbf{v} \cdot \mathbf{u}}{|\mathbf{u}||\mathbf{w}|}$$
$$= \frac{|\mathbf{v}||\mathbf{u}| + \mathbf{v} \cdot \mathbf{u}}{|\mathbf{w}|}$$
$$\cos\beta = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}||\mathbf{w}|} = \frac{\mathbf{v} \cdot (|\mathbf{v}|\mathbf{u} + |\mathbf{u}|\mathbf{v})}{|\mathbf{v}||\mathbf{w}|} = \frac{|\mathbf{v}|\mathbf{v} \cdot \mathbf{u} + |\mathbf{u}|\mathbf{v} \cdot \mathbf{v}}{|\mathbf{v}||\mathbf{w}|}$$
$$= \frac{|\mathbf{v}|\mathbf{v} \cdot \mathbf{u} + |\mathbf{u}||\mathbf{v}|^2}{|\mathbf{v}||\mathbf{w}|} = \frac{\mathbf{v} \cdot \mathbf{u} + |\mathbf{u}||\mathbf{v}|}{|\mathbf{w}|}$$
Therefore, $\cos\alpha = \cos\beta$, so $\alpha = \beta$ since α and β are between 0° and 180°.

33. a. $\text{pr}_\mathbf{v}\mathbf{u} = \left(\frac{\langle 0, 5\rangle \cdot \langle 3, 4\rangle}{|\langle 3, 4\rangle|^2}\right)\langle 3, 4\rangle$
$= \frac{20}{9 + 16}\langle 3, 4\rangle = \left\langle \frac{12}{5}, \frac{16}{5}\right\rangle$

b. $\text{pr}_\mathbf{v}\mathbf{u} = \left(\frac{\langle -3, 2\rangle \cdot \langle 3, 4\rangle}{|\langle 3, 4\rangle|}\right)\langle 3, 4\rangle$

$= \frac{-9+8}{9+16}\langle 3,4\rangle = \left\langle -\frac{3}{25}, -\frac{4}{25}\right\rangle$

34. a. $\text{pr}_{\mathbf{u}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{u}}{|\mathbf{u}|^2}\right)\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} = \mathbf{u}$

b. $\text{pr}_{-\mathbf{u}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot(-\mathbf{u})}{|-\mathbf{u}|^2}\right)\mathbf{u} = \left(\frac{-\mathbf{u}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} = -\mathbf{u}$

c. $\text{pr}_{\mathbf{u}}(-\mathbf{u}) = \left(\frac{(-\mathbf{u})\cdot\mathbf{u}}{|\mathbf{u}|^2}\right)(-\mathbf{u})$

$= \left(\frac{-\mathbf{u}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)(-\mathbf{u}) = \mathbf{u}$

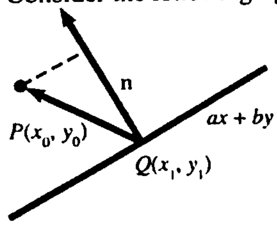
d. $\text{pr}_{-\mathbf{u}}(-\mathbf{u}) = \left(\frac{(-\mathbf{u})\cdot(-\mathbf{u})}{|-\mathbf{u}|^2}\right)(-\mathbf{u})$

$= \left(\frac{\mathbf{u}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)(-\mathbf{u}) = -\mathbf{u}$

35. a. a and b cannot both be zero. If $a = 0$, then the line $ax + by + c = 0$, is horizontal and $\mathbf{n} = b\mathbf{j}$ is vertical, so $\mathbf{n}$ is perpendicular to the line. Use a similar argument if $b = 0$.

If $a \neq 0$ and $b \neq 0$, then $\left(0, -\frac{c}{b}\right)$ and $\left(-\frac{c}{a}, 0\right)$ are on the line. Therefore, $\left\langle -\frac{c}{a}, \frac{c}{b}\right\rangle$ is a vector in the direction of the line. $\left\langle -\frac{c}{a}, \frac{c}{b}\right\rangle\cdot\langle a,b\rangle = -c + c = 0$, so $\left\langle -\frac{c}{a}, \frac{c}{b}\right\rangle$ is perpendicular to $\langle a, b\rangle$. Hence, $\langle a, b\rangle$ is perpendicular to the line.

b. Consider the following figure.



Observe that $ax_1 + by_1 = -c$.

$$\text{distance} = \left|\text{pr}_{\mathbf{n}}\overrightarrow{QP}\right| = \left|\frac{\overrightarrow{QP}\cdot\mathbf{n}}{|\mathbf{n}|^2}\mathbf{n}\right|$$

$$= \frac{\left|\overrightarrow{QP}\cdot\mathbf{n}\right|}{|\mathbf{n}|^2}|\mathbf{n}| = \frac{\left|\overrightarrow{QP}\cdot\mathbf{n}\right|}{|\mathbf{n}|}$$

$$= \frac{|\langle x_0 - x_1, y_0 - y_1\rangle\cdot\langle a,b\rangle|}{\sqrt{a^2+b^2}}$$

$$= \frac{|(ax_0 + by_0) - (ax_1 + by_1)|}{\sqrt{a^2+b^2}}$$

$$= \frac{|ax_0 + by_0 + c|}{\sqrt{a^2+b^2}}$$

13.4 Concepts Review

1. a vector-valued function of a real variable

2. f and g are continuous at c; $f'(t)\mathbf{i} + g'(t)\mathbf{j}$

3. position

4. $\mathbf{r}'(t)$; $\mathbf{r}''(t)$; tangent; concave

Problem Set 13.4

1. $\lim_{t\to1}[2t\mathbf{i} - t^2\mathbf{j}] = \lim_{t\to1}(2t)\mathbf{i} - \lim_{t\to1}(t^2)\mathbf{j} = 2\mathbf{i} - \mathbf{j}$

2. $\lim_{t\to3}[2(t-3)^2\mathbf{i} - 7t^3\mathbf{j}]$

$= \lim_{t\to3}[2(t-3)^2]\mathbf{i} - \lim_{t\to3}(7t^3)\mathbf{j} = -189\mathbf{j}$

3. $\lim_{t\to1}\left[\frac{t-1}{t^2-1}\mathbf{i} - \frac{t^2+2t-3}{t-1}\mathbf{j}\right]$

$= \lim_{t\to1}\left(\frac{t-1}{t^2-1}\right)\mathbf{i} - \lim_{t\to1}\left(\frac{t^2+2t-3}{t-1}\right)\mathbf{j}$

$= \lim_{t\to1}\left(\frac{1}{t+1}\right)\mathbf{i} - \lim_{t\to1}(t+3)\mathbf{j} = \frac{1}{2}\mathbf{i} - 4\mathbf{j}$

4. $\lim_{t\to-2}\left[\frac{2t^2-10t-28}{t+2}\mathbf{i} - \frac{7t^3}{t-3}\mathbf{j}\right]$

$= \lim_{t\to-2}\left(\frac{2t^2-10t-28}{t+2}\right)\mathbf{i} - \lim_{t\to-2}\left(\frac{7t^3}{t-3}\right)\mathbf{j}$

$= \lim_{t\to-2}(2t-14)\mathbf{i} - \frac{56}{5}\mathbf{j} = -18\mathbf{i} - \frac{56}{5}\mathbf{j}$

5. $\lim_{t\to 0}\left[\frac{\sin t\cos t}{t}\mathbf{i}-\frac{7t^3}{e^t}\mathbf{j}\right]$

$= \lim_{t\to 0}\left(\frac{\sin t\cos t}{t}\right)\mathbf{i}-\lim_{t\to 0}\left(\frac{7t^3}{e^t}\right)\mathbf{j}=\mathbf{i}$

6. $\lim_{t\to\infty}\left[\frac{t\sin t}{t^2}\mathbf{i}-\frac{7t^3}{t^3-3t}\mathbf{j}\right]$

$= \lim_{t\to\infty}\left(\frac{t\sin t}{t^2}\right)\mathbf{i}-\lim_{t\to\infty}\left(\frac{7t^3}{t^3-3t}\right)\mathbf{j}$

$= \lim_{t\to\infty}\left(\frac{\sin t}{t}\right)\mathbf{i}-7\mathbf{j}=-7\mathbf{j}$

7. $\lim_{t\to 0^+}\left\langle \ln(t^3), t^2\ln t\right\rangle$ does not exist because $\lim_{t\to 0^+}\ln(t^3)=-\infty$.

8. $\lim_{t\to 0^-}\left\langle e^{-1/t^2}, \frac{t}{|t|}\right\rangle=\left\langle \lim_{t\to 0^-} e^{-1/t^2}, \lim_{t\to 0^-}\frac{t}{|t|}\right\rangle$
$=\langle 0, -1\rangle$

9. **a.** The domain of $f(t)=\frac{2}{t-4}$ is $(-\infty,4)\cup(4,\infty)$. The domain of $g(t)=\sqrt{3-t}$ is $(-\infty, 3]$. Thus, the domain of **r** is $(-\infty, 3]$ or $\{t\in\mathbb{R}: t\le 3\}$.

b. The domain of $f(t)=[\![t^2]\!]$ is $(-\infty,\infty)$. The domain of $g(t)=\sqrt{20-t}$ is $(-\infty, 20]$. Thus, the domain of **r** is $(-\infty, 20]$ or $\{t\in\mathbb{R}: t\le 20\}$.

10. **a.** The domain of $f(t)=\ln(t-1)$ is $(1,\infty)$. The domain of $g(t)=\sqrt{20-t}$ is $(-\infty, 20]$. Thus, the domain of **r** is $(1, 20]$ or $\{t\in\mathbb{R}: 1<t\le 20\}$.

b. The domain of $f(t)=\ln(t^{-1})$ is $(0,\infty)$. The domain of $g(t)=\tan^{-1}t$ is $(-\infty,\infty)$. Thus, the domain of **r** is $(0,\infty)$ or $\{t\in\mathbb{R}: t>0\}$.

11. **a.** $f(t)=\frac{2}{t-4}$ is continuous on $(-\infty,4)\cup(4,\infty)$. $g(t)=\sqrt{3-t}$ is continuous on $(-\infty, 3)$. Thus, **r** is continuous on $(-\infty, 3)$ or $\{t\in\mathbb{R}: t<3\}$.

b. $f(t)=[\![t^2]\!]$ is continuous on $\left(-\sqrt{n+1},-\sqrt{n}\right)\cup\left(\sqrt{n},\sqrt{n+1}\right)$ where n is a non-negative integer. $g(t)=\sqrt{20-t}$ is continuous on $(-\infty, 20)$ or $\{t\in\mathbb{R}: t<20\}$. Thus, **r** is continuous on $\left(-\sqrt{n+1},-\sqrt{n}\right)\cup\left(\sqrt{k},\sqrt{k+1}\right)$ where n and k are non-negative integers and $k<400$ or $\{t\in\mathbb{R}: t<20, t^2 \text{ not an integer}\}$.

12. **a.** $f(t)=\ln(t-1)$ is continuous on $(1,\infty)$. $g(t)=\sqrt{20-t}$ is continuous on $(-\infty, 20)$. Thus, **r** is continuous on $(1, 20)$ or $\{t\in\mathbb{R}: 1<t<20\}$.

b. $f(t)=\ln(t^{-1})$ is continuous on $(0,\infty)$. $g(t)=\tan^{-1}t$ is continuous on $(-\infty,\infty)$. Thus, **r** is continuous on $(0,\infty)$ or $\{t\in\mathbb{R}: t>0\}$.

13. **a.** $D_t\mathbf{r}(t)=9(3t+4)^2\mathbf{i}+2te^{t^2}\mathbf{j}$

$D_t^2\mathbf{r}(t)=54(3t+4)\mathbf{i}+2(2t^2+1)e^{t^2}\mathbf{j}$

b. $D_t\mathbf{r}(t)=\sin 2t\mathbf{i}-3\sin 3t\mathbf{j}$

$D_t^2\mathbf{r}(t)=2\cos 2t\mathbf{i}-9\cos 3t\mathbf{j}$

14. **a.** $\mathbf{r}'(t)=\left(e^t-2te^{-t^2}\right)\mathbf{i}+(\ln 2)2^t\mathbf{j}$

$\mathbf{r}''(t)=\left(e^t+4t^2e^{-t^2}-2e^{-t^2}\right)\mathbf{i}+(\ln 2)^2 2^t\mathbf{j}$

b. $\mathbf{r}'(t)=2\sec^2 2t\mathbf{i}+\frac{1}{1+t^2}\mathbf{j}$

$\mathbf{r}''(t)=8\tan 2t\sec^2 2t\mathbf{i}-\frac{2t}{(1+t^2)^2}\mathbf{j}$

15. $\mathbf{r}'(t)=-e^{-t}\mathbf{i}-\frac{2}{t}\mathbf{j}$

$\mathbf{r}(t)\cdot\mathbf{r}'(t)=-e^{-2t}+\frac{2}{t}\ln(t^2)$

$D_t[\mathbf{r}(t)\cdot\mathbf{r}'(t)]=2e^{-2t}+\frac{4}{t^2}-\frac{2}{t^2}\ln(t^2)$

16. $\mathbf{r}'(t)=3\cos 3t\mathbf{i}+3\sin 3t\mathbf{j}$

$\mathbf{r}(t)\cdot\mathbf{r}'(t)=0$

$D_t[\mathbf{r}(t)\cdot\mathbf{r}'(t)]=0$

17. $h(t)\mathbf{r}(t) = e^{-3t}\sqrt{t-1}\mathbf{i} + e^{-3t}\ln(2t^2)\mathbf{j}$

$$D_t[h(t)\mathbf{r}(t)] = -\frac{e^{-3t}}{2}\left(\frac{6t-7}{\sqrt{t-1}}\right)\mathbf{i} + e^{-3t}\left(\frac{2}{t} - 3\ln(2t^2)\right)\mathbf{j}$$

18. $h(t)\mathbf{r}(t) = \ln(3t-2)\sin 2t\mathbf{i} + \ln(3t-2)\cosh t\mathbf{j}$

$$D_t[h(t)\mathbf{r}(t)] = \left[2\ln(3t-2)\cos 2t + \frac{3\sin 2t}{3t-2}\right]\mathbf{i} + \left[\ln(3t-2)\sinh t + \frac{3\cosh t}{3t-2}\right]\mathbf{j}$$

19. $\mathbf{f}'(u) = -\sin u\mathbf{i} + 3e^{3u}\mathbf{j};\ g'(t) = 6t$

$\mathbf{F}'(t) = \mathbf{f}'(g(t))g'(t)$

$= -6t\sin(3t^2 - 4)\mathbf{i} + 18te^{9t^2-12}\mathbf{j}$

20. $\mathbf{f}'(u) = 2u\mathbf{i} + \sin 2u\mathbf{j};\ g'(t) = \sec^2 t$

$\mathbf{F}'(t) = \mathbf{f}'(g(t))g'(t)$

$= 2\tan t\sec^2 t\mathbf{i} + \sec^2 t\sin(2\tan t)\mathbf{j}$

21. $\int_0^1 (e^t\mathbf{i} + e^{-t}\mathbf{j})dt = [e^t\mathbf{i} - e^{-t}\mathbf{j}]_0^1$

$= (e-1)\mathbf{i} + (1 - e^{-1})\mathbf{j}$

22. $\int_{-1}^1 [(1+t)^{3/2}\mathbf{i} + (1-t)^{3/2}\mathbf{j}]dt$

$= \left[\frac{2}{5}(1+t)^{5/2}\mathbf{i} - \frac{2}{5}(1-t)^{5/2}\mathbf{j}\right]_{-1}^1$

$= \frac{8\sqrt{2}}{5}\mathbf{i} + \frac{8\sqrt{2}}{5}\mathbf{j}$

23. $\mathbf{r}'(t) = \mathbf{v}(t) = -e^{-t}\mathbf{i} + e^t\mathbf{j}$

$\mathbf{r}''(t) = \mathbf{a}(t) = e^{-t}\mathbf{i} + e^t\mathbf{j}$

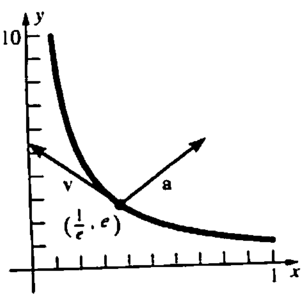
$|\mathbf{v}(t)| = \sqrt{e^{-2t} + e^{2t}}$

$\mathbf{r}(1) = e^{-1}\mathbf{i} + e\mathbf{j}$

$\mathbf{v}(1) = -e^{-1}\mathbf{i} + e\mathbf{j}$

$\mathbf{a}(1) = e^{-1}\mathbf{i} + e\mathbf{j}$

$|\mathbf{v}(1)| = \sqrt{e^{-2} + e^2}$



24. $\mathbf{r}'(t) = \mathbf{v}(t) = 6t\mathbf{i} + \mathbf{j}$

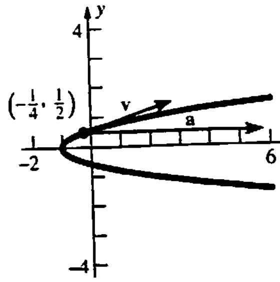
$\mathbf{r}''(t) = \mathbf{a}(t) = 6\mathbf{i}$

$\mathbf{r}\left(\frac{1}{2}\right) = -\frac{1}{4}\mathbf{i} + \frac{1}{2}\mathbf{j}$

$\mathbf{v}\left(\frac{1}{2}\right) = 3\mathbf{i} + \mathbf{j}$

$\mathbf{a}\left(\frac{1}{2}\right) = 6\mathbf{i}$

$\left|\mathbf{v}\left(\frac{1}{2}\right)\right| = \sqrt{9+1} = \sqrt{10}$



25. $\mathbf{r}'(t) = \mathbf{v}(t) = -2\sin t\mathbf{i} - 3\sin 2t\mathbf{j}$

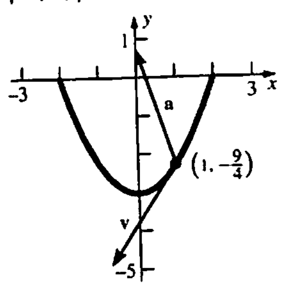
$\mathbf{r}''(t) = \mathbf{a}(t) = -2\cos t\mathbf{i} - 6\cos 2t\mathbf{j}$

$\mathbf{r}\left(\frac{\pi}{3}\right) = \mathbf{i} - \frac{9}{4}\mathbf{j}$

$\mathbf{v}\left(\frac{\pi}{3}\right) = -\sqrt{3}\mathbf{i} - \frac{3\sqrt{3}}{2}\mathbf{j}$

$\mathbf{a}\left(\frac{\pi}{3}\right) = -\mathbf{i} + 3\mathbf{j}$

$\left|\mathbf{v}\left(\frac{\pi}{3}\right)\right| = \sqrt{3 + \frac{27}{4}} = \frac{\sqrt{39}}{2}$



26. $\mathbf{r}'(t) = \mathbf{v}(t) = \sec^2 t\mathbf{i} + \cos t\mathbf{j}$

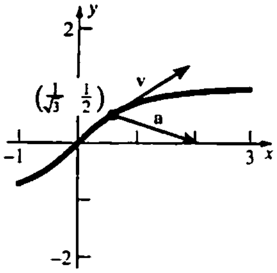
$\mathbf{r}''(t) = \mathbf{a}(t) = 2\sec^2 t\tan t\mathbf{i} - \sin t\mathbf{j}$

$\mathbf{r}\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{2}\mathbf{j}$

$\mathbf{v}\left(\frac{\pi}{6}\right) = \frac{4}{3}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$

$\mathbf{a}\left(\frac{\pi}{6}\right) = \frac{8}{3\sqrt{3}}\mathbf{i} - \frac{1}{2}\mathbf{j}$

$$\left|\mathbf{v}\left(\frac{\pi}{6}\right)\right| = \sqrt{\frac{16}{9}+\frac{3}{4}} = \frac{\sqrt{91}}{6}$$



27. $\mathbf{r}'(t) = \mathbf{v}(t) = 6t\mathbf{i} + 3t^2\mathbf{j}$

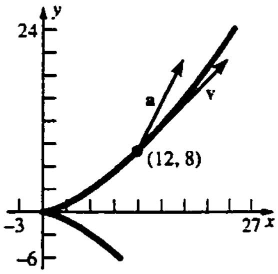
$\mathbf{r}''(t) = \mathbf{a}(t) = 6\mathbf{i} + 6t\mathbf{j}$

$\mathbf{r}(2) = 12\mathbf{i} + 8\mathbf{j}$

$\mathbf{v}(2) = 12\mathbf{i} + 12\mathbf{j}$

$\mathbf{a}(2) = 6\mathbf{i} + 12\mathbf{j}$

$|\mathbf{v}(2)| = \sqrt{144+144} = 12\sqrt{2}$



28. $\mathbf{r}'(t) = \mathbf{v}(t) = a\cos t\mathbf{i} - 2a\sin t\mathbf{j}$

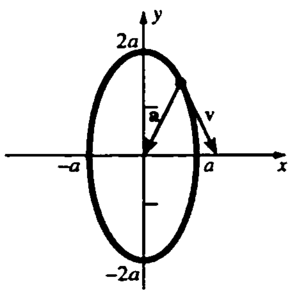
$\mathbf{r}''(t) = \mathbf{a}(t) = -a\sin t\mathbf{i} - 2a\cos t\mathbf{j}$

$$\mathbf{r}\left(\frac{\pi}{4}\right) = \frac{a}{\sqrt{2}}\mathbf{i} + \sqrt{2}a\mathbf{j}$$

$$\mathbf{v}\left(\frac{\pi}{4}\right) = \frac{a}{\sqrt{2}}\mathbf{i} - \sqrt{2}a\mathbf{j}$$

$$\mathbf{a}\left(\frac{\pi}{4}\right) = -\frac{a}{\sqrt{2}}\mathbf{i} - \sqrt{2}a\mathbf{j}$$

$$|\mathbf{v}(2)| = \sqrt{\frac{a^2}{2}+2a^2} = \sqrt{\frac{5}{2}}a$$



29. $\mathbf{r}'(t) = \mathbf{v}(t) = -\sin t\mathbf{i} - 2\sec^2 t\mathbf{j}$

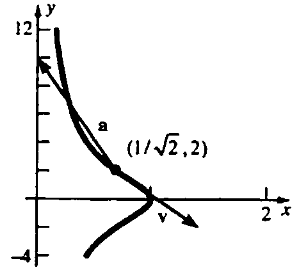
$\mathbf{r}''(t) = \mathbf{a}(t) = -\cos t\mathbf{i} - 4\sec^2 t\tan t\mathbf{j}$

$$\mathbf{r}\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\mathbf{i} + 2\mathbf{j}$$

$$\mathbf{v}\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\mathbf{i} - 4\mathbf{j}$$

$$\mathbf{a}\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}\mathbf{i} + 8\mathbf{j}$$

$$\left|\mathbf{v}\left(-\frac{\pi}{4}\right)\right| = \sqrt{\frac{1}{2}+16} = \sqrt{\frac{33}{2}}$$



30. $\mathbf{r}'(t) = \mathbf{v}(t) = \frac{1}{2}e^{t/2}\mathbf{i} - e^{-t}\mathbf{j}$

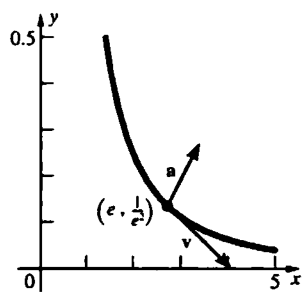
$\mathbf{r}''(t) = \mathbf{a}(t) = \frac{1}{4}e^{t/2}\mathbf{i} + e^{-t}\mathbf{j}$

$\mathbf{r}(2) = e\mathbf{i} + e^{-2}\mathbf{j}$

$\mathbf{v}(2) = \frac{1}{2}e\mathbf{i} - e^{-2}\mathbf{j}$

$\mathbf{a}(2) = \frac{1}{4}e\mathbf{i} + e^{-2}\mathbf{j}$

$$|\mathbf{v}(2)| = \sqrt{\frac{1}{4}e^2 + e^{-4}} = \frac{\sqrt{e^6+4}}{2e^2}$$



31. $\mathbf{v}(t) = \int \mathbf{a}(t)dt = C_1\mathbf{i} + (-32t + C_2)\mathbf{j}$

$\mathbf{v}(0) = \mathbf{0} = C_1\mathbf{i} + C_2\mathbf{j}$

$C_1 = C_2 = 0$

$\mathbf{v}(t) = -32t\mathbf{j}$

$\mathbf{r}(t) = \int \mathbf{v}(t)dt = C_1\mathbf{i} + (-16t^2 + C_2)\mathbf{j}$

$\mathbf{r}(0) = \mathbf{0} = C_1\mathbf{i} + C_2\mathbf{j}$

$C_1 = C_2 = 0$

$\mathbf{r}(t) = -16t^2\mathbf{j}$

32. $\mathbf{v}(t) = \int \mathbf{a}(t)dt = C_1\mathbf{i} + \left(\frac{1}{2}t^2 + C_2\right)\mathbf{j}$

$\mathbf{v}(0) = \mathbf{i} + 2\mathbf{j} = C_1\mathbf{i} + C_2\mathbf{j}$

$C_1 = 1, C_2 = 2$

$$\mathbf{v}(t) = \mathbf{i} + \left(\frac{1}{2}t^2 + 2\right)\mathbf{j}$$

$$\mathbf{r}(t) = \int \mathbf{v}(t)dt = (t + C_1)\mathbf{i} + \left(\frac{1}{6}t^3 + 2t + C_2\right)\mathbf{j}$$

$$\mathbf{r}(0) = \mathbf{0} = C_1\mathbf{i} + C_2\mathbf{j}$$

$$C_1 = C_2 = 0$$

$$\mathbf{v}(t) = t\mathbf{i} + \left(\frac{1}{6}t^3 + 2t\right)\mathbf{j}$$

33. $\mathbf{v}(t) = \int \mathbf{a}(t)dt = (t + C_1)\mathbf{i} + (-e^{-t} + C_2)\mathbf{j}$

$$\mathbf{v}(0) = 2\mathbf{i} + \mathbf{j} = C_1\mathbf{i} + (-1 + C_2)\mathbf{j}$$

$$C_1 = C_2 = 2$$

$$\mathbf{v}(t) = (t + 2)\mathbf{i} + (-e^{-t} + 2)\mathbf{j}$$

$$\mathbf{r}(t) = \int \mathbf{v}(t)dt = \left(\frac{1}{2}t^2 + 2t + C_1\right)\mathbf{i} + (e^{-t} + 2t + C_2)\mathbf{j}$$

$$\mathbf{r}(0) = \mathbf{i} + \mathbf{j} = C_1\mathbf{i} + (1 + C_2)\mathbf{j}$$

$$C_1 = 1, C_2 = 0$$

$$\mathbf{r}(t) = \left(\frac{1}{2}t^2 + 2t + 1\right)\mathbf{i} + (e^{-t} + 2t)\mathbf{j}$$

34. $\mathbf{v}(t) = \int \mathbf{a}(t)dt = (-\sin t + C_1)\mathbf{i} + (-\cos t + C_2)\mathbf{j}$

$$\mathbf{v}(0) = \mathbf{i} = C_1\mathbf{i} + (-1 + C_2)\mathbf{j}$$

$$C_1 = C_2 = 1$$

$$\mathbf{v}(t) = (-\sin t + 1)\mathbf{i} + (-\cos t + 1)\mathbf{j}$$

$$\mathbf{r}(t) = \int \mathbf{v}(t)dt = (\cos t + t + C_1)\mathbf{i} + (-\sin t + t + C_2)\mathbf{j}$$

$$\mathbf{r}(0) = \mathbf{i} + 3\mathbf{j} = (1 + C_1)\mathbf{i} + C_2\mathbf{j}$$

$$C_1 = 0, C_2 = 3$$

$$\mathbf{v}(t) = (\cos t + t)\mathbf{i} + (-\sin t + t + 3)\mathbf{j}$$

35. $\mathbf{r}(t) = 5\cos 6t\mathbf{i} + 5\sin 6t\mathbf{j}$

$$\mathbf{v}(t) = -30\sin 6t\mathbf{i} + 30\cos 6t\mathbf{j}$$

$$|\mathbf{v}(t)| = \sqrt{900\sin^2 6t + 900\cos^2 6t} = 30$$

$$\mathbf{a}(t) = -180\cos 6t\mathbf{i} - 180\sin 6t\mathbf{j}$$

36. $D_t[\mathbf{v}(t)\cdot\mathbf{v}(t)] = \mathbf{v}(t)\cdot\mathbf{a}(t) + \mathbf{a}(t)\cdot\mathbf{v}(t)$

$= 2\mathbf{v}(t)\cdot\mathbf{a}(t) = D_t[c] = 0$, so $\mathbf{v}(t)\cdot\mathbf{a}(t) = 0$.

Thus, $\mathbf{v}(t)$ and $\mathbf{a}(t)$ are perpendicular.

37. Substitute $\theta = 30°$ and $v_0 = 96$ ft/s into the equations for $\mathbf{r}(t)$ and $\mathbf{v}(t)$ in Example 5.

$$\mathbf{r}(t) = 48\sqrt{3}t\mathbf{i} + (48t - 16t^2)\mathbf{j},$$

$$\mathbf{v}(t) = 48\sqrt{3}\mathbf{i} + (48 - 32t)\mathbf{j}$$

The projectile is at ground level when $48t - 16t^2 = 0$, so $t = 0$ or $t = 3$. Thus, the projectile hits the ground after 3 seconds. Its speed is $|\mathbf{v}(3)| = \left|48\sqrt{3}\mathbf{i} - 48\mathbf{j}\right| = 96$ ft/s. It will land $48\sqrt{3}(3) = 144\sqrt{3}$ ft from the origin.

38. $\mathbf{v}(t) = \int \mathbf{a}(t)dt = b\mathbf{i} + (at + c)\mathbf{j}$

$$\mathbf{r}(t) = \int \mathbf{v}(t)dt = (bt + d)\mathbf{i} + \left(\frac{1}{2}at^2 + ct + e\right)\mathbf{j}$$

Thus, $x = bt + d$ and $y = \frac{1}{2}at^2 + ct + e$. If $b = 0$, then $x = d$, so the graph will be a vertical line. If $b \neq 0$, $t = \frac{1}{b}(x - d)$, and

$y = \frac{1}{2}a\left[\frac{1}{b}(x - d)\right]^2 + c\left[\frac{1}{b}(x - d)\right] + e$ is a quadratic equation, so the graph is a parabola. Note that if $a = 0$, the graph is a line.

39. $\mathbf{r}'(t) = \mathbf{v}(t) = \omega\sinh\omega t\mathbf{i} + \omega\cosh\omega t\mathbf{j}$

$$\mathbf{r}''(t) = \mathbf{a}(t) = \omega^2\cosh\omega t\mathbf{i} + \omega^2\sinh\omega t\mathbf{j}$$

$$= \omega^2(\cosh\omega t\mathbf{i} + \sinh\omega t\mathbf{j}) = \omega^2\mathbf{r}(t)$$

Thus, $\mathbf{a}(t) = c\mathbf{r}(t)$ where $c = \omega^2$.

40. $\mathbf{r}'(t) = \mathbf{v}(t) = -a\omega\sin\omega t\mathbf{i} + b\omega\cos\omega t\mathbf{j}$

$$\mathbf{r}''(t) = \mathbf{a}(t) = -a\omega^2\cos\omega t\mathbf{i} - b\omega^2\sin\omega t\mathbf{j}$$

$$= -\omega^2(a\cos\omega t\mathbf{i} + b\sin\omega t\mathbf{j}) = -\omega^2\mathbf{r}(t)$$

Thus, $\mathbf{a}(t) = c\mathbf{r}(t)$ where $c = -\omega^2$.

41. Substitute $\theta = 45°$ into the equation for $\mathbf{r}(t)$ in Example 5.

$$\mathbf{r}(t) = \frac{v_0}{\sqrt{2}}t\mathbf{i} + \left(\frac{v_0}{\sqrt{2}}t - 16t^2\right)\mathbf{j},$$

If t_1 is the time that the ball is caught,

$\frac{v_0}{\sqrt{2}}t_1 = 300$ and $\frac{v_0}{\sqrt{2}}t_1 - 16t_1^2 = 0$. Thus, from the first equation $t_1 = \frac{300\sqrt{2}}{v_0}$. Substitute into the second equation and solve for v_0.

$$v_0 = \sqrt{9600} = 40\sqrt{6} \approx 97.98 \text{ ft/s}$$

42. Consider the motion from when the ball rolls off the table at time $t = 0$. The only acceleration is due to gravity, so $\mathbf{a}(t) = -32\mathbf{j}$, $\mathbf{v}(0) = 20\mathbf{i}$, $\mathbf{r}(0) = 4\mathbf{j}$. $\mathbf{v}(t) = \int \mathbf{a}(t)dt = C_1\mathbf{i} + (-32t + C_2)\mathbf{j}$

$\mathbf{v}(0) = 20\mathbf{i} = C_1\mathbf{i} + C_2\mathbf{j}$, so $C_1 = 20, C_2 = 0$, and $\mathbf{v}(t) = 20\mathbf{i} - 32t\mathbf{j}$.

$$\mathbf{r}(t) = \int \mathbf{v}(t)dt = (20t + C_1)\mathbf{i} + (-16t^2 + C_2)\mathbf{j}$$

$\mathbf{r}(0) = 4\mathbf{j} = C_1\mathbf{i} + C_2\mathbf{j}$, so $C_1 = 0, C_2 = 4$, and $\mathbf{r}(t) = 20t\mathbf{i} + (-16t^2 + 4)\mathbf{j}$.

The ball will hit the floor when $-16t^2 + 4 = 0$ or $t = \frac{1}{2}$. $\mathbf{v}\left(\frac{1}{2}\right) = 20\mathbf{i} - 16\mathbf{j}$, which is in the direction of the tangent when the ball hits the ground. If θ is the angle at which the ball hits the ground, θ is also the angle between the vectors **i** and $20\mathbf{i} - 16\mathbf{j}$ or $5\mathbf{i} - 4\mathbf{j}$.

$$\cos\theta = \frac{\mathbf{i}\cdot(5\mathbf{i}-4\mathbf{j})}{|\mathbf{i}||5\mathbf{i}-4\mathbf{j}|} = \frac{5}{\sqrt{41}}, \text{ so}$$

$$\theta = \cos^{-1}\frac{5}{\sqrt{41}} \approx 38.66°.$$

The speed at which the ball hits the ground is

$$\left|\mathbf{v}\left(\frac{1}{2}\right)\right| = 4\sqrt{41} \approx 25.61 \text{ ft/s.}$$

43. From Example 3, the acceleration due to rotation is $\mathbf{a}(t) = -60\omega^2\cos\omega t\mathbf{i} - 60\omega^2\sin\omega t\mathbf{j}$. Since it must be at least greater than gravity, $980 \le |\mathbf{a}(t)| = 60\omega^2$. Thus, $\omega^2 \ge \frac{49}{3}$ or

$\omega \ge \frac{7}{\sqrt{3}} \approx 4.04$ rad/s. The angular speed must be at least 4.04 rad/s.

44. a. From Example 3, $a = \frac{v^2}{r}$, and from Newton's Inverse Square Law, $a = \frac{k}{r^2}$, so $\frac{v^2}{r} = \frac{k}{r^2}$. Therefore, $v^2 = \frac{k}{r}$.

b. $v = \frac{2\pi r}{T}$; substitute into $v^2 = \frac{k}{r}$, so $\frac{4\pi^2 r^2}{T^2} = \frac{k}{r}$. Therefore, $T^2 = \left(\frac{4\pi^2}{k}\right)r^3$.

c. At the surface of the earth, $r = R$ and $a = g$. Substitute into $a = \frac{k}{r^2}$, so $g = \frac{k}{R^2}$. Therefore, $k = gR^2$.

45. a. From Problem 44a and c, $v^2 = \frac{k}{r} = \frac{gR^2}{r}$.

Convert gravity into mi/h² so

$$g \approx (32.17)\left(\frac{1}{5280}\right)\left(\frac{3600}{1}\right)^2 \approx 78,963 \text{ mi/h}^2$$

$$v^2 \approx \frac{(78,963)(3960)^2}{(4160)}$$

$\approx 297,660,140$ mi²/h²

$v \approx 17,253$ mi/h

b. From Problem 44b and c, $r^3 = \left(\frac{k}{4\pi^2}\right)T^2 = \left(\frac{gR^2}{4\pi^2}\right)T^2$. Since the satellite is in synchronous orbit, $T = 24$ h.

$$r^3 \approx \left[\frac{(78,963)(3960)^2}{4\pi^2}\right](24)^2$$

$\approx 1.80666 \times 10^{13}$ mi³

$r \approx 26,240$ mi

46. From Problem 44b and c,

$$r^3 = \left(\frac{k}{4\pi^2}\right)T^2 = \left(\frac{gR^2}{4\pi^2}\right)T^2.$$

$T = (27.32)(24) = 655.68$ h

$$r^3 \approx \left[\frac{(78,963)(3960)^2}{4\pi^2}\right](655.68)^2$$

$\approx 1.348360 \times 10^{16}$ mi³

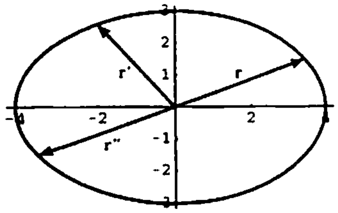
$r \approx 238,000$ mi

47. $\mathbf{r}'(t) = -4\sin t\mathbf{i} + 3\cos t\mathbf{j}$, $\mathbf{r}''(t) = -4\cos t\mathbf{i} - 3\sin t\mathbf{j}$

Observe that the graphs of $\mathbf{r}(t)$, $\mathbf{r}'(t)$, and $\mathbf{r}''(t)$ are the same.

a. $\mathbf{r}\left(\frac{\pi}{6}\right) = 2\sqrt{3}\mathbf{i} + \frac{3}{2}\mathbf{j}$, $\mathbf{r}'\left(\frac{\pi}{6}\right) = -2\mathbf{i} + \frac{3\sqrt{3}}{2}\mathbf{j}$,

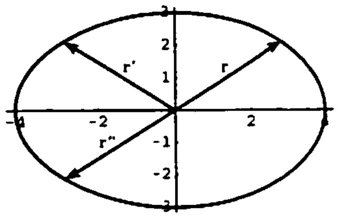
$\mathbf{r}''\left(\frac{\pi}{6}\right) = -2\sqrt{3}\mathbf{i} - \frac{3}{2}\mathbf{j}$



b. $\mathbf{r}\left(\frac{\pi}{4}\right) = 2\sqrt{2}\mathbf{i} + \frac{3\sqrt{2}}{2}\mathbf{j}$,

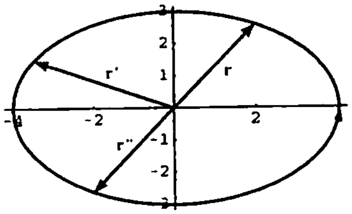
$\mathbf{r}'\left(\frac{\pi}{4}\right) = -2\sqrt{2}\mathbf{i} + \frac{3\sqrt{2}}{2}\mathbf{j}$,

$\mathbf{r}''\left(\frac{\pi}{4}\right) = -2\sqrt{2}\mathbf{i} - \frac{3\sqrt{2}}{2}\mathbf{j}$

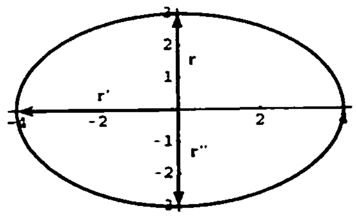


c. $\mathbf{r}\left(\frac{\pi}{3}\right) = 2\mathbf{i} + \frac{3\sqrt{3}}{2}\mathbf{j}$, $\mathbf{r}'\left(\frac{\pi}{3}\right) = -2\sqrt{3}\mathbf{i} + \frac{3}{2}\mathbf{j}$,

$\mathbf{r}''\left(\frac{\pi}{3}\right) = -2\mathbf{i} - \frac{3\sqrt{3}}{2}\mathbf{j}$

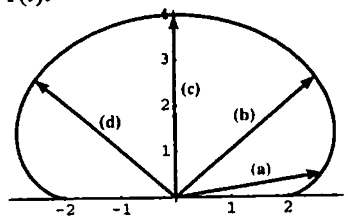


d. $\mathbf{r}\left(\frac{\pi}{2}\right) = 3\mathbf{j}$, $\mathbf{r}'\left(\frac{\pi}{2}\right) = -4\mathbf{i}$, $\mathbf{r}''\left(\frac{\pi}{2}\right) = -3\mathbf{j}$

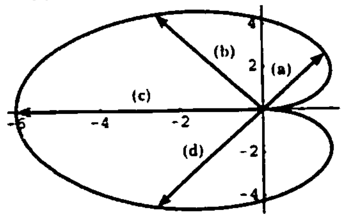


48. The paths of $\mathbf{r}(t)$, $\mathbf{r}'(t)$, and $\mathbf{r}''(t)$ are along the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

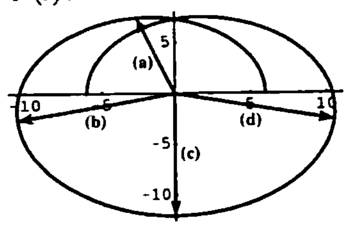
49. $\mathbf{r}(t)$:



$\mathbf{r}'(t)$:



$\mathbf{r}''(t)$:



13.5 Concepts Review

1. $\left|\frac{d\mathbf{T}}{ds}\right|$

2. $\frac{1}{a}$; 0

3. $\kappa = \frac{1}{R}$

4. $\frac{d^2s}{dt^2}\mathbf{T} + \left(\frac{ds}{dt}\right)^2 \kappa\mathbf{N}$

Problem Set 13.5

1. $\mathbf{r}'(t) = 3\mathbf{i} + 6t\mathbf{j}$

$|\mathbf{r}'(t)| = 3\sqrt{1+4t^2}$

$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{1+4t^2}}\mathbf{i} + \frac{2t}{\sqrt{1+4t^2}}\mathbf{j}$

$\mathbf{T}\left(\frac{1}{3}\right) = \frac{3}{\sqrt{13}}\mathbf{i} + \frac{2}{\sqrt{13}}\mathbf{j}$

$x'(t) = 3$ $\quad$ $y'(t) = 6t$

$x''(t) = 0$ $\quad$ $y''(t) = 6$

$\kappa(t) = \frac{|x'y'' - y'x''|}{\left[x'^2 + y'^2\right]^{3/2}} = \frac{18}{27(1+4t^2)^{3/2}}$

$= \frac{2}{3(1+4t^2)^{3/2}}$

$\kappa\left(\frac{1}{3}\right) = \frac{2}{3\left(\frac{13}{9}\right)^{3/2}} = \frac{18}{13\sqrt{13}}$

2. $\mathbf{s}'(t) = 6t\mathbf{i} + 3\mathbf{j}$

$|\mathbf{s}'(t)| = 3\sqrt{4t^2+1}$

$\mathbf{T}(t) = \frac{\mathbf{s}'(t)}{|\mathbf{s}'(t)|} = \frac{2t}{\sqrt{4t^2+1}}\mathbf{i} + \frac{1}{\sqrt{4t^2+1}}\mathbf{j}$

$\mathbf{T}\left(\frac{1}{3}\right) = \frac{2}{\sqrt{13}}\mathbf{i} + \frac{3}{\sqrt{13}}\mathbf{j}$

$x'(t) = 6t$ $\quad$ $y'(t) = 3$

$x''(t) = 6$ $\quad$ $y''(t) = 0$

$\kappa(t) = \frac{|x'y'' - y'x''|}{\left|x'^2 + y'^2\right|^{3/2}} = \frac{18}{27(4t^2+1)^{3/2}}$

$$=\frac{2}{3(4t^2+1)^{3/2}}$$

$$\kappa\left(\frac{1}{3}\right)=\frac{2}{3\left(\frac{13}{9}\right)^{3/2}}=\frac{18}{13\sqrt{13}}$$

3. $\mathbf{u}'(t)=8t\mathbf{i}+4\mathbf{j}$

$$|\mathbf{u}'(t)|=4\sqrt{4t^2+1}$$

$$\mathbf{T}(t)=\frac{\mathbf{u}'(t)}{|\mathbf{u}'(t)|}=\frac{2t}{\sqrt{4t^2+1}}\mathbf{i}+\frac{1}{\sqrt{4t^2+1}}\mathbf{j}$$

$$\mathbf{T}\left(\frac{1}{2}\right)=\frac{1}{\sqrt{2}}\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{j}$$

$x'(t)=8t \qquad y'(t)=4$

$x''(t)=8 \qquad y''(t)=0$

$$\kappa(t)=\frac{|x'y''-y'x''|}{\left[x'^2+y'^2\right]^{3/2}}=\frac{32}{64(4t^2+1)^{3/2}}$$

$$=\frac{1}{2(4t^2+1)^{3/2}}$$

$$\kappa\left(\frac{1}{2}\right)=\frac{1}{2(2)^{3/2}}=\frac{1}{4\sqrt{2}}$$

4. $\mathbf{r}'(t)=t^2\mathbf{i}+t\mathbf{j}$

$$|\mathbf{r}'(t)|=t\sqrt{t^2+1}$$

$$\mathbf{T}(t)=\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}=\frac{t}{\sqrt{t^2+1}}\mathbf{i}+\frac{1}{\sqrt{t^2+1}}\mathbf{j}$$

$$\mathbf{T}(1)=\frac{1}{\sqrt{2}}\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{j}$$

$x'(t)=t^2 \qquad y'(t)=t$

$x''(t)=2t \qquad y''(t)=1$

$$\kappa(t)=\frac{|x'y''-y'x''|}{\left[x'^2+y'^2\right]^{3/2}}=\frac{t^2}{t^3(t^2+1)^{3/2}}$$

$$=\frac{1}{t(t^2+1)^{3/2}}$$

$$\kappa(1)=\frac{1}{1(2)^{3/2}}=\frac{1}{2\sqrt{2}}$$

5. $\mathbf{z}'(t)=-3\sin t\mathbf{i}+4\cos t\mathbf{j}$

$$|\mathbf{z}'(t)|=\sqrt{9+7\cos^2 t}$$

$$\mathbf{T}(t)=\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}=-\frac{3\sin t}{\sqrt{9+7\cos^2 t}}\mathbf{i}+\frac{4\cos t}{\sqrt{9+7\cos^2 t}}\mathbf{j}$$

$$\mathbf{T}\left(\frac{\pi}{4}\right)=-\frac{3}{5}\mathbf{i}+\frac{4}{5}\mathbf{j}$$

$x'(t)=-3\sin t \qquad y'(t)=4\cos t$

$x''(t)=-3\cos t \qquad y''(t)=-4\sin t$

$$\kappa(t)=\frac{|x'y''-y'x''|}{\left[x'^2+y'^2\right]^{3/2}}=\frac{12}{(9+7\cos^2 t)^{3/2}}$$

$$\kappa\left(\frac{\pi}{4}\right)=\frac{12}{\left(\frac{25}{2}\right)^{3/2}}=\frac{24\sqrt{2}}{125}$$

6. $\mathbf{r}'(t)=e^t\mathbf{i}+e^t\mathbf{j}$

$$|\mathbf{r}'(t)|=e^t\sqrt{2}$$

$$\mathbf{T}(t)=\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}=\frac{1}{\sqrt{2}}\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{j}$$

$$\mathbf{T}(\ln 2)=\frac{1}{\sqrt{2}}\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{j}$$

$x'(t)=e^t \qquad y'(t)=e^t$

$x''(t)=e^t \qquad y''(t)=e^t$

$$\kappa(t)=\frac{|x'y''-y'x''|}{\left[x'^2+y'^2\right]^{3/2}}=0$$

$\kappa(\ln 2)=0$

7. $\mathbf{r}'(t)=(\cos t-\sin t)e^t\mathbf{i}+(\sin t+\cos t)e^t\mathbf{j}$

$$|\mathbf{r}'(t)|=\sqrt{2}e^t$$

$$\mathbf{T}(t)=\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}=\frac{\cos t-\sin t}{\sqrt{2}}\mathbf{i}+\frac{\sin t+\cos t}{\sqrt{2}}\mathbf{j}$$

$$\mathbf{T}\left(\frac{\pi}{2}\right)=-\frac{1}{\sqrt{2}}\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{j}$$

$x'(t)=(\cos t-\sin t)e^t \qquad y'(t)=(\sin t+\cos t)e^t$

$x''(t)=(-2\sin t)e^t \qquad y''(t)=(2\cos t)e^t$

$$\kappa(t)=\frac{|x'y''-y'x''|}{\left[x'^2+y'^2\right]^{3/2}}=\frac{2e^{2t}}{2\sqrt{2}e^{3t}}=\frac{1}{\sqrt{2}e^t}$$

$$\kappa\left(\frac{\pi}{2}\right)=\frac{1}{\sqrt{2}e^{\pi/2}}$$

8. $\mathbf{r}'(t)=-3e^{-3t}\mathbf{i}+e^t\mathbf{j}$

$$|\mathbf{r}'(t)|=e^{-3t}\sqrt{9+e^{8t}}$$

$$\mathbf{T}(t)=\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}=-\frac{3}{\sqrt{9+e^{8t}}}\mathbf{i}+\frac{e^{4t}}{\sqrt{9+e^{8t}}}\mathbf{j}$$

$$\mathbf{T}(0)=-\frac{3}{\sqrt{10}}\mathbf{i}+\frac{1}{\sqrt{10}}\mathbf{j}$$

$x'(t)=-3e^{-3t} \qquad y'(t)=e^t$

$x''(t)=9e^{-3t} \qquad y''(t)=e^t$

$$\kappa(t)=\frac{|x'y''-y'x''|}{\left[x'^2+y'^2\right]^{3/2}}=\frac{12e^{-2t}}{e^{-9t}(9+e^{8t})^{3/2}}$$

$$= \frac{12e^{7t}}{(9+e^{8t})^{3/2}}$$

$$\kappa(0) = \frac{12}{10\sqrt{10}} = \frac{6}{5\sqrt{10}}$$

9. $\mathbf{r}'(t) = -2t\mathbf{i} - 3t^2\mathbf{j}$

$$|\mathbf{r}'(t)| = t\sqrt{4+9t^2}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = -\frac{2}{\sqrt{4+9t^2}}\mathbf{i} - \frac{3t}{\sqrt{4+9t^2}}\mathbf{j}$$

$$\mathbf{T}(1) = -\frac{2}{\sqrt{13}}\mathbf{i} - \frac{3}{\sqrt{13}}\mathbf{j}$$

$x'(t) = -2t$ $\quad$ $y'(t) = -3t^2$

$x''(t) = -2$ $\quad$ $y''(t) = -6t$

$$\kappa(t) = \frac{|x'y'' - y'x''|}{\left[x'^2 + y'^2\right]^{3/2}} = \frac{6t^2}{t^3(4+9t^2)^{3/2}}$$

$$= \frac{6}{t(4+9t^2)^{3/2}}$$

$$\kappa(1) = \frac{6}{13\sqrt{13}}$$

10. $\mathbf{r}'(t) = \cosh t\mathbf{i} + \sinh t\mathbf{j}$

$$|\mathbf{r}'(t)| = \sqrt{\sinh^2 t + \cosh^2 t}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\cosh t}{\sqrt{\sinh^2 t + \cosh^2 t}}\mathbf{i} + \frac{\sinh t}{\sqrt{\sinh^2 t + \cosh^2 t}}\mathbf{j}$$

$$\mathbf{T}(\ln 3) = \frac{5}{\sqrt{41}}\mathbf{i} + \frac{4}{\sqrt{41}}\mathbf{j}$$

$x'(t) = \cosh t$ $\quad$ $y'(t) = \sinh t$

$x''(t) = \sinh t$ $\quad$ $y''(t) = \cosh t$

$$\kappa(t) = \frac{|x'y'' - y'x''|}{\left[x'^2 + y'^2\right]^{3/2}} = \frac{1}{(\sinh^2 t + \cosh^2 t)^{3/2}}; \ \kappa(\ln 3) = \frac{1}{\left(\frac{41}{9}\right)^{3/2}} = \frac{27}{41\sqrt{41}}$$

11. $\mathbf{r}'(t) = -(\cos t + \sin t)e^{-t}\mathbf{i} + (\cos t - \sin t)e^{t}\mathbf{j}$

$$|\mathbf{r}'(t)| = \sqrt{2}e^{-t}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = -\frac{\cos t + \sin t}{\sqrt{2}}\mathbf{i} + \frac{\cos t - \sin t}{\sqrt{2}}\mathbf{j}$$

$$\mathbf{T}(0) = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

$x'(t) = -(\cos t - \sin t)e^{-t}$ $\quad$ $y'(t) = (\cos t - \sin t)e^{-t}$

$x''(t) = (2\sin t)e^{-t}$ $\quad$ $y''(t) = (-2\cos t)e^{-t}$

$$\kappa(t) = \frac{|x'y'' - y'x''|}{\left[x'^2 + y'^2\right]^{3/2}} = \frac{2e^{-2t}}{2\sqrt{2}e^{-3t}} = \frac{e^t}{\sqrt{2}}$$

$$\kappa(0) = \frac{1}{\sqrt{2}}$$

12. $\mathbf{r}'(t) = (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j}$

$|\mathbf{r}'(t)| = \sqrt{t^2+1}$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\cos t - t\sin t}{\sqrt{t^2+1}}\mathbf{i} + \frac{\sin t + t\cos t}{\sqrt{t^2+1}}\mathbf{j}$$

$$\mathbf{T}(1) = \frac{\cos 1 - \sin 1}{\sqrt{2}}\mathbf{i} + \frac{\sin 1 + \cos 1}{\sqrt{2}}\mathbf{j}$$

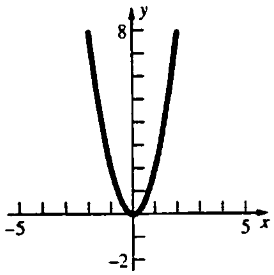
$x'(t) = \cos t - t\sin t \qquad y'(t) = \sin t + t\cos t$

$x''(t) = -2\sin t - t\cos t \qquad y''(t) = 2\cos t - t\sin t$

$$\kappa(t) = \frac{|x'y'' - y'x''|}{\left[x'^2 + y'^2\right]^{3/2}} = \frac{t^2+2}{(t^2+1)^{3/2}}$$

$$\kappa(1) = \frac{3}{2\sqrt{2}}$$

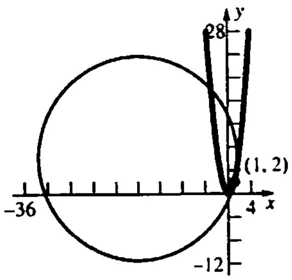
13.



$y' = 4x,\ y'' = 4$

$$\kappa = \frac{4}{(1+16x^2)^{3/2}}$$

At $(1, 2)$, $\kappa = \dfrac{4}{17\sqrt{17}}$ and $R = \dfrac{17\sqrt{17}}{4}$.



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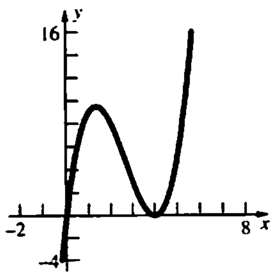
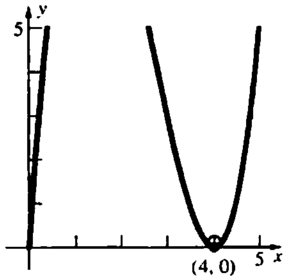


$y' = 3x^2 - 16x + 16,\ y'' = 6x - 16$

$$\kappa = \frac{|6x-16|}{\left[1+(3x^2-16x+16)^2\right]^{3/2}}$$

At $(4, 0)$, $\kappa = \dfrac{8}{1} = 8$ and $R = \dfrac{1}{8}$.



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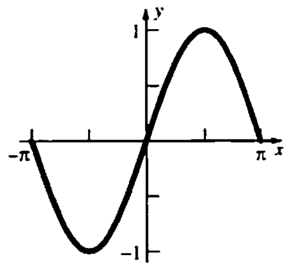
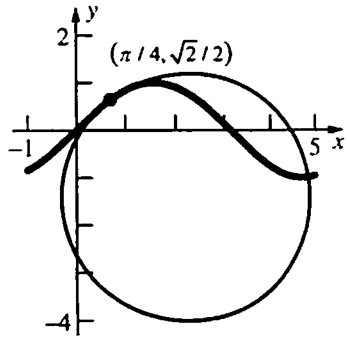


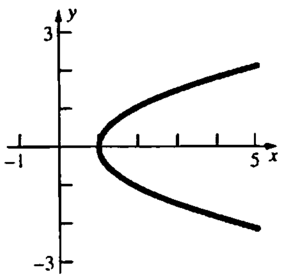
$y' = \cos x,\ y'' = -\sin x$

$$\kappa = \frac{|\sin x|}{(1+\cos^2 x)^{3/2}}$$

At $\left(\dfrac{\pi}{4}, \dfrac{\sqrt{2}}{2}\right)$, $\kappa = \dfrac{\frac{1}{\sqrt{2}}}{\left(\frac{3}{2}\right)^{3/2}} = \dfrac{2}{3\sqrt{3}}$ and $R = \dfrac{3\sqrt{3}}{2}$.



16.

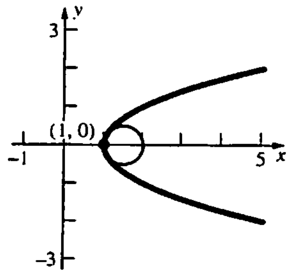


$2yy' = 1$

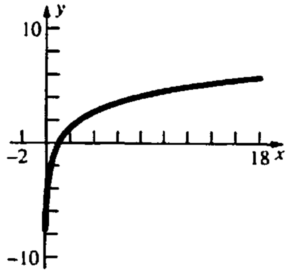
$$y' = \frac{1}{2y},\ y'' = -\frac{y'}{2y^2} = -\frac{1}{4y^3}$$

$$\kappa = \frac{\left|\frac{1}{4y^3}\right|}{\left(1+\frac{1}{4y^2}\right)^{3/2}} = \frac{2}{(4y^2+1)^{3/2}}$$

At (1, 0), $\kappa = \frac{2}{1} = 2$ and $R = \frac{1}{2}$.



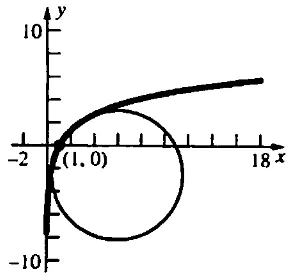
17.



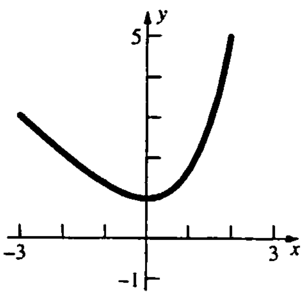
$$y' = \frac{2}{x},\ y'' = -\frac{2}{x^2}$$

$$\kappa = \frac{\frac{2}{x^2}}{\left(1+\frac{4}{x^2}\right)^{3/2}} = \frac{2|x|}{(x^2+4)^{3/2}}$$

At (1, 0), $\kappa = \frac{2}{5\sqrt{5}}$ and $R = \frac{5\sqrt{5}}{2}$.



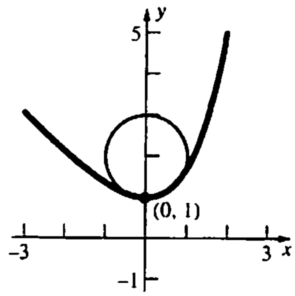
18.



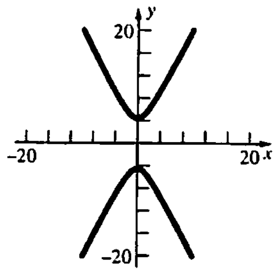
$$y' = e^x - 1,\ y'' = e^x$$

$$\kappa = \frac{e^x}{(e^{2x} - 2e^x + 2)^{3/2}}$$

At (0, 1), $\kappa = \frac{1}{1} = 1$ and $R = 1$.



19.

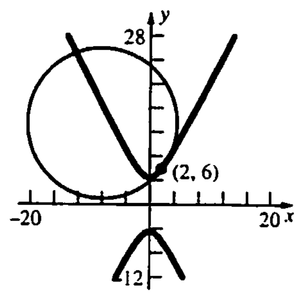


$$2yy' - 8x = 0$$

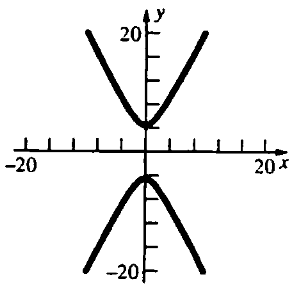
$$y' = \frac{4x}{y},\ y'' = \frac{4(y - xy')}{y^2} = \frac{4(y^2 - 4x^2)}{y^3}$$

$$\kappa = \frac{\frac{4y^2 - 16x^2}{y^3}}{\left(1+\frac{16x^2}{y^2}\right)^{3/2}} = \frac{4(y^2 - 4x^2)}{(y^2 + 16x^2)^{3/2}}$$

At (2, 6), $\kappa = \frac{80}{1000} = \frac{2}{25}$ and $R = \frac{25}{2}$.



20.

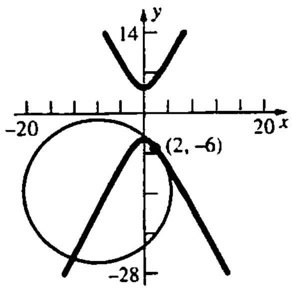


$2yy' - 8x = 0$

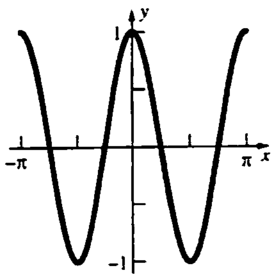
$$y' = \frac{4x}{y},\ y'' = \frac{4(y - xy')}{y^2} = \frac{4(y^2 - 4x^2)}{y^3}$$

$$\kappa = \frac{\frac{4(y^2-4x^2)}{y^3}}{\left(1+\frac{16x^2}{y^2}\right)^{3/2}} = \frac{4(y^2 - 4x^2)}{(y^2 + 16x^2)^{3/2}}$$

At $(2, -6)$, $\kappa = \frac{80}{1000} = \frac{2}{25}$ and $R = \frac{25}{2}$.



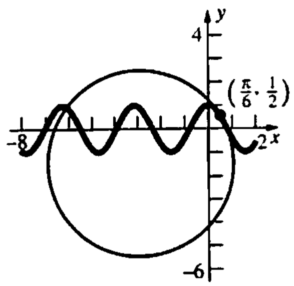
21.



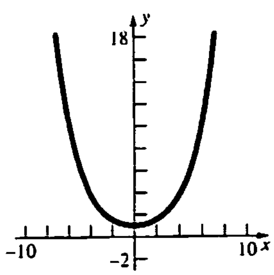
$y' = -2\sin 2x,\ y'' = -4\cos 2x$

$$\kappa = \frac{|4\cos 2x|}{(1 + 4\sin^2 2x)^{3/2}}$$

At $\left(\frac{\pi}{6}, \frac{1}{2}\right)$, $\kappa = \frac{2}{8} = \frac{1}{4}$ and $R = 4$.



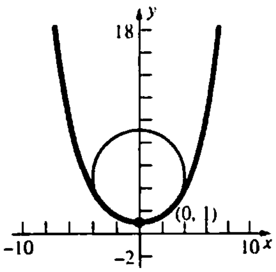
22.



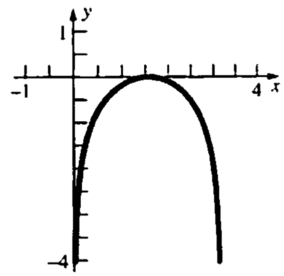
$$y' = \frac{1}{2}\sinh\frac{x}{2},\ y'' = \frac{1}{4}\cosh\frac{x}{2}$$

$$\kappa = \frac{\left|\frac{1}{4}\cosh\frac{x}{2}\right|}{\left(1 + \frac{1}{4}\sinh^2\frac{x}{2}\right)^{3/2}} = \frac{2\left|\cosh\frac{x}{2}\right|}{\left(4 + \sinh^2\frac{x}{2}\right)^{3/2}}$$

At $(0, 1)$, $\kappa = \frac{2}{8} = \frac{1}{4}$ and $R = 4$.



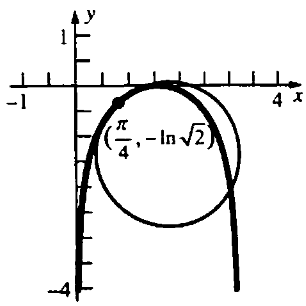
23.



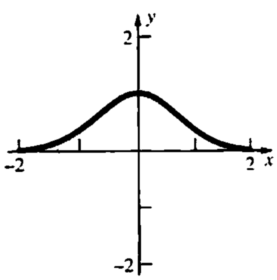
$y' = \cot x,\ y'' = -\csc^2 x$

$$\kappa = \frac{|\csc^2 x|}{(1 + \cot^2 x)^{3/2}} = \frac{1}{|\csc x|}$$

At $\left(\frac{\pi}{4}, -\ln\sqrt{2}\right)$, $\kappa = \frac{1}{\sqrt{2}}$ and $R = \sqrt{2}$.



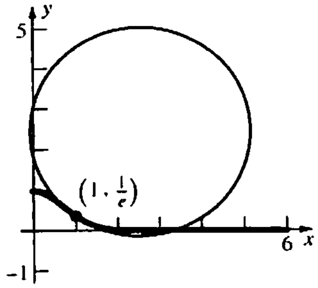
24.



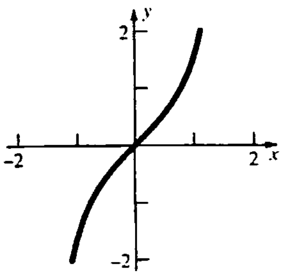
$y' = -2xe^{-x^2}$, $y'' = (4x^2 - 2)e^{-x^2}$

$$\kappa = \frac{\left|(4x^2-2)e^{-x^2}\right|}{(1+4x^2e^{-2x^2})^{3/2}} = \frac{e^{2x^2}\left|4x^2-2\right|}{(e^{2x^2}+4x^2)^{3/2}}$$

At $\left(1, \frac{1}{e}\right)$, $\kappa = \dfrac{2e^2}{(e^2+4)^{3/2}}$ and $R = \dfrac{(e^2+4)^{3/2}}{2e^2}$.



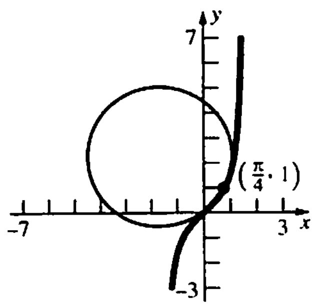
25.



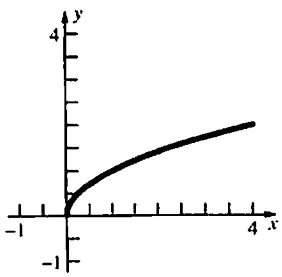
$y' = \sec^2 x$, $y'' = 2\sec^2 x \tan x$

$$\kappa = \frac{\left|2\sec^2 x \tan x\right|}{(1+\sec^4 x)^{3/2}}$$

At $\left(\frac{\pi}{4}, 1\right)$, $\kappa = \dfrac{4}{5\sqrt{5}}$ and $R = \dfrac{5\sqrt{5}}{4}$.



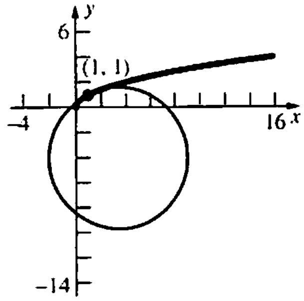
26.



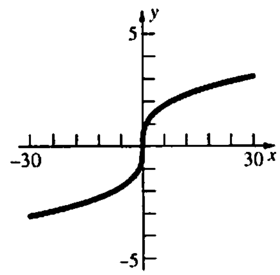
$y' = \dfrac{1}{2\sqrt{x}}$, $y'' = -\dfrac{1}{4x^{3/2}}$

$$\kappa = \frac{\left|\frac{1}{4x^{3/2}}\right|}{\left(1+\frac{1}{4x}\right)^{3/2}} = \frac{2}{(4x+1)^{3/2}}$$

At (1, 1), $\kappa = \dfrac{2}{5\sqrt{5}}$ and $R = \dfrac{5\sqrt{5}}{2}$.



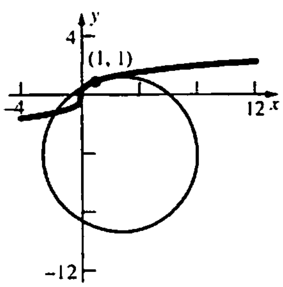
27.



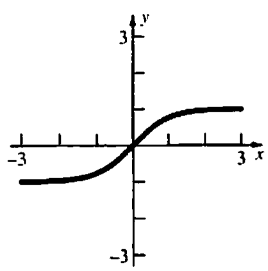
$y' = \dfrac{1}{3x^{2/3}}$, $y'' = -\dfrac{2}{9x^{5/3}}$

$$\kappa = \frac{\left|\frac{2}{9x^{5/3}}\right|}{\left(1+\frac{1}{9x^{4/3}}\right)^{3/2}} = \frac{6x^{1/3}}{(9x^{4/3}+1)^{3/2}}$$

At (1, 1), $\kappa = \dfrac{6}{10\sqrt{10}} = \dfrac{3}{5\sqrt{10}}$ and $R = \dfrac{5\sqrt{10}}{3}$.



28.

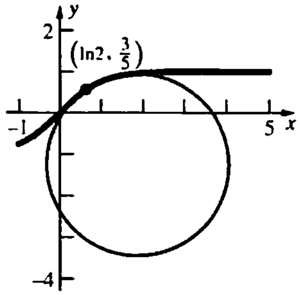


$y' = \text{sech}^2 x,\ y'' = -2\text{sech}^2 x \tanh x$

$\kappa = \dfrac{\left|2\text{sech}^2 x \tanh x\right|}{(1+\text{sech}^4 x)^{3/2}}$

At $\left(\ln 2, \dfrac{3}{5}\right)$, $\kappa = \dfrac{\frac{96}{125}}{\left(\frac{881}{625}\right)^{3/2}} = \dfrac{12{,}000}{881\sqrt{881}}$ and

$R = \dfrac{881\sqrt{881}}{12{,}000}$.



29. $y' = \dfrac{1}{x},\ y'' = -\dfrac{1}{x^2}$

$\kappa = \dfrac{\left|\frac{1}{x^2}\right|}{\left(1+\frac{1}{x^2}\right)^{3/2}} = \dfrac{|x|}{(x^2+1)^{3/2}}$

Since $0 < x < \infty$, $\kappa = \dfrac{x}{(x^2+1)^{3/2}}$.

$\kappa' = \dfrac{(x^2+1)^{3/2} - 3x^2(x^2+1)^{1/2}}{(x^2+1)^3} = \dfrac{-2x^2+1}{(x^2+1)^{5/2}}$

$\kappa' = 0$ when $x = \dfrac{1}{\sqrt{2}}$. Since $\kappa' > 0$ on $\left(0, \dfrac{1}{\sqrt{2}}\right)$ and $\kappa' < 0$ on $\left(\dfrac{1}{\sqrt{2}}, \infty\right)$, so κ is maximum when

$x = \dfrac{1}{\sqrt{2}},\ y = \ln\dfrac{1}{\sqrt{2}} = -\dfrac{\ln 2}{2}$. The point of maximum curvature is $\left(\dfrac{1}{\sqrt{2}}, -\dfrac{\ln 2}{2}\right)$.

30. $y' = \cos x,\ y'' = -\sin x$

$\kappa = \dfrac{|\sin x|}{(1+\cos^2 x)^{3/2}}$

$\kappa' = \dfrac{\frac{|\sin x|}{\sin x}\cos x(1+\cos^2 x)^{3/2} + 3|\sin x|\cos x \sin x(1+\cos^2 x)^{1/2}}{(1+\cos^2 x)^3}$

$= \dfrac{2|\sin x|\cot x(2+\cos^2 x)}{(1+\cos^2 x)^{5/2}}$

$k' = 0$ when $x = -\dfrac{\pi}{2}, \dfrac{\pi}{2}$. κ' is not defined when $x = -\pi, 0$. Since $\kappa' > 0$ on $\left(-\pi, -\dfrac{\pi}{2}\right) \cup \left(0, \dfrac{\pi}{2}\right)$ and $\kappa' < 0$ on $\left(-\dfrac{\pi}{2}, 0\right) \cup \left(\dfrac{\pi}{2}, \pi\right)$, so κ has local maxima when $x = -\dfrac{\pi}{2}, y = -1$ and $x = \dfrac{\pi}{2}, y = 1$.

$\kappa\left(-\dfrac{\pi}{2}\right) = \kappa\left(\dfrac{\pi}{2}\right) = 1$

The points of maximum curvature are $(0, 1)$ and $\left(\dfrac{\pi}{2}, 1\right)$.

31. $y' = \sinh x,\ y'' = \cosh x$

$\kappa = \dfrac{\cosh x}{(1+\sinh^2 x)^{3/2}} = \text{sech}^2 x$

$\kappa' = -2\text{sech}^2 x \tanh x$

$\kappa' = 0$ when $x = 0$. Since $\kappa' > 0$ on $(-\infty, 0)$ and $\kappa' < 0$ on $(0, \infty)$, so κ is maximum when $x = 0, y = 1$. The point of maximum curvature is $(0, 1)$.

32. $y' = \cosh x, y'' = \sinh x$

$$\kappa = \frac{|\sinh x|}{(1+\cosh^2 x)^{3/2}}$$

$$\kappa' = \frac{\frac{|\sinh x|}{\sinh x}\cosh x(1+\cosh^2 x)^{3/2} - 3|\sinh x|\cosh x \sinh x(1+\cosh^2 x)^{1/2}}{(1+\cosh^2 x)^3} = \frac{2|\sinh x|\coth x(2-\cosh^2 x)}{(1+\cosh^2 x)^{5/2}}$$

κ' is not defined when $x = 0$ and $\kappa' = 0$ when $\cosh x = \sqrt{2}$ or $x = \pm\ln\left(\sqrt{2}+1\right)$. Since $\kappa' > 0$ on $\left(-\infty, -\ln\left(\sqrt{2}+1\right)\right) \cup \left(0, \ln\left(\sqrt{2}+1\right)\right)$ and $\kappa' < 0$ on $\left(-\ln\left(\sqrt{2}+1\right), 0\right) \cup \left(\ln\left(\sqrt{2}+1\right), \infty\right)$, κ has local maxima when $x = -\ln\left(\sqrt{2}+1\right), y = -1$ and $x = \ln\left(\sqrt{2}+1\right), y = 1$.

$$\kappa\left(-\ln\left(\sqrt{2}+1\right)\right) = \kappa\left(\ln\left(\sqrt{2}+1\right)\right) = \frac{1}{3\sqrt{3}}$$

The points of maximum curvature are $\left(-\ln\left(\sqrt{2}+1\right), -1\right)$ and $\left(\ln\left(\sqrt{2}+1\right), 1\right)$.

33. $y' = e^x, y'' = e^x$

$$\kappa = \frac{e^x}{(1+e^{2x})^{3/2}}$$

$$\kappa' = \frac{e^x(1+e^{2x})^{3/2} - 3e^{3x}(1+e^{2x})^{1/2}}{(1+e^{2x})^3}$$

$$= \frac{e^x(1-2e^{2x})}{(1+e^{2x})^{5/2}}$$

$\kappa' = 0$ when $x = -\frac{1}{2}\ln 2$. Since $\kappa' > 0$ on $\left(-\infty, -\frac{1}{2}\ln 2\right)$ and $\kappa' < 0$ on $\left(-\frac{1}{2}\ln 2, \infty\right)$, so κ is maximum when $x = -\frac{1}{2}\ln 2$, $y = \frac{1}{\sqrt{2}}$. The point of maximum curvature is $\left(-\frac{1}{2}\ln 2, \frac{1}{\sqrt{2}}\right)$.

34. $y' = -\tan x, y'' = -\sec^2 x$

$$\kappa = \frac{|\sec^2 x|}{(1+\tan^2 x)^{3/2}} = |\cos x|$$

Since $-\frac{\pi}{2} < x < \frac{\pi}{2}$, $\kappa = \cos x$.

$\kappa' = -\sin x$

$\kappa' = 0$ when $x = 0$. Since $\kappa' > 0$ on $\left(-\frac{\pi}{2}, 0\right)$ and $\kappa' < 0$ on $\left(0, \frac{\pi}{2}\right)$, κ is maximum when $x = 0, y = 0$. The point of maximum curvature is $(0, 0)$.

35. $\mathbf{r}'(t) = 3\mathbf{i} + 6t\mathbf{j}$

$\mathbf{r}''(t) = 6\mathbf{j}$

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = 3\sqrt{1+4t^2}$$

$$a_T = \frac{d^2s}{dt^2} = \frac{12t}{\sqrt{1+4t^2}}$$

$$a_N^2 = |\mathbf{r}''(t)|^2 - a_T^2 = 36 - \frac{144t^2}{1+4t^2} = \frac{36}{1+4t^2}$$

$$a_N = \frac{6}{\sqrt{1+4t^2}}$$

At $t_1 = \frac{1}{3}$, $a_T = \frac{12}{\sqrt{13}}$ and $a_N = \frac{18}{\sqrt{13}}$.

36. $\mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$

$\mathbf{r}''(t) = 2\mathbf{i}$

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{4t^2+1}$$

$$a_T = \frac{d^2s}{dt^2} = \frac{4t}{\sqrt{4t^2+1}}$$

$$a_N^2 = |\mathbf{r}''(t)|^2 - a_T^2 = 4 - \frac{16t^2}{4t^2+1} = \frac{4}{4t^2+1}$$

$$a_N = \frac{2}{\sqrt{4t^2+1}}$$

At $t_1 = 1$, $a_T = \frac{4}{\sqrt{5}}$ and $a_N = \frac{2}{\sqrt{5}}$.

37. $\mathbf{r}'(t) = 2\mathbf{i} + 2t\mathbf{j}$

$\mathbf{r}''(t) = 2\mathbf{j}$

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = 2\sqrt{1+t^2}$$

$$a_T = \frac{d^2s}{dt^2} = \frac{2t}{\sqrt{1+t^2}}$$

$$a_N^2 = |\mathbf{r}''(t)|^2 - a_T^2 = 4 - \frac{4t^2}{1+t^2} = \frac{4}{1+t^2}$$

$$a_N = \frac{2}{\sqrt{1+t^2}}$$

At $t_1 = -1$, $a_T = -\sqrt{2}$ and $a_N = \sqrt{2}$.

38. $\mathbf{r}'(t) = -a\sin t\mathbf{i} + a\cos t\mathbf{j}$

$\mathbf{r}''(t) = -a\cos t\mathbf{i} - a\sin t\mathbf{j}$

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = a$$

$$a_T = \frac{d^2s}{dt^2} = 0$$

$$a_N^2 = |\mathbf{r}''(t)|^2 - a_T^2 = a^2$$

$a_N = a$

At $t_1 = \frac{\pi}{6}$, $a_T = 0$ and $a_N = a$.

39. $\mathbf{r}'(t) = a\sinh t\mathbf{i} + a\cosh t\mathbf{j}$

$\mathbf{r}''(t) = a\cosh t\mathbf{i} + a\sinh t\mathbf{j}$

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = a\sqrt{\sinh^2 t + \cosh^2 t}$$

$$a_T = \frac{d^2s}{dt^2} = \frac{2a\cosh t\sinh t}{\sqrt{\sinh^2 t + \cosh^2 t}}$$

$$a_N^2 = |\mathbf{r}''(t)|^2 - a_T^2 = a^2(\cosh^2 t + \sinh^2 t) - \frac{4a^2\cosh^2 t\sinh^2 t}{\sinh^2 t + \cosh^2 t} = \frac{a^2(\cosh^2 t - \sinh t^2)^2}{\sinh^2 t + \cosh^2 t}$$

$$a_N = \frac{a(\cosh^2 t - \sinh^2 t)}{\sqrt{\sinh^2 t + \cosh^2 t}}$$

At $t_1 = \ln 3$, $\cosh t = \frac{5}{3}$, $\sinh t = \frac{4}{3}$, and $\sqrt{\sinh^2 t + \cosh^2 t} = \frac{\sqrt{41}}{3}$, so $a_T = \frac{40a}{3\sqrt{41}}$ and $a_N = \frac{3a}{\sqrt{41}}$.

40. $\mathbf{r}'(t) = -3\cos^2 t\sin t\mathbf{i} + 3\sin^2 t\cos t\mathbf{j}$

$\mathbf{r}''(t) = 3\cos t(2\sin^2 t - \cos^2 t)\mathbf{i} + 3\sin t(2\cos^2 t - \sin^2 t)\mathbf{j}$

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = 3\cos t\sin t = \frac{3}{2}\sin 2t \text{ (Note that } t > 0.)$$

$$a_T = \frac{d^2s}{dt^2} = 3\cos 2t$$

$$a_N^2 = |\mathbf{r}''(t)|^2 - a_T^2 = 9(\cos^6 t + \sin^6 t) - 9\cos^2 2t = 9(\cos^6 t + \sin^6 t - \cos^2 2t)$$

$$a_N = 3\sqrt{\cos^6 t + \sin^6 t - \cos^2 2t}$$

At $t_1 = \frac{\pi}{3}$, $a_T = -\frac{3}{2}$ and $a_N = \frac{3\sqrt{3}}{4}$.

41. $\mathbf{r}'(t) = e^t\mathbf{i} - e^{-t}\mathbf{j}$; $\mathbf{r}''(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}$

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{e^{2t} + e^{-2t}}$$

$$a_T = \frac{d^2s}{dt^2} = \frac{e^{2t} - e^{-2t}}{\sqrt{e^{2t} + e^{-2t}}}$$

$a_N^2 = |\mathbf{r}''(t)|^2 - a_T^2$

$= e^{2t} + e^{-2t} - \dfrac{e^{4t} - 2 + e^{-4t}}{e^{2t} + e^{-2t}} = \dfrac{4}{e^{2t} + e^{-2t}}$

$a_N = \dfrac{2}{\sqrt{e^{2t} + e^{-2t}}}$

At $t_1 = 0, a_T = 0$ and $a_N = \sqrt{2}$.

42. $x'(t) = 3 \qquad y'(t) = -6$

$x''(t) = 0 \qquad y''(t) = 0$

$\dfrac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2} = 3\sqrt{5}$

$a_T = \dfrac{d^2s}{dt^2} = 0$

$a_N^2 = [x''(t)^2 + y''(t)^2] - a_T^2 = 0;\ a_N = 0$

At $t_1 = 0, a_T = 0$ and $a_N = 0$.

43. $x'(t) = (\sin t + \cos t)e^t \qquad y'(t) = (\cos t - \sin t)e^t$

$x''(t) = (2\cos t)e^t \qquad y''(t) = (-2\sin t)e^t$

$\dfrac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2} = \sqrt{2}e^t$

$a_T = \dfrac{d^2s}{dt^2} = \sqrt{2}e^t$

$a_N^2 = [x''(t)^2 + y''(t)^2] - a_T^2 = 4e^{2t} - 2e^{2t} = 2e^{2t}$

$a_N = \sqrt{2}e^t$

At $t_1 = \dfrac{\pi}{3}, a_T = \sqrt{2}e^{\pi/3}$ and $a_N = \sqrt{2}e^{\pi/3}$.

44. $\tan\phi = y'$

$(1 + y'^2)^{3/2} = (1 + \tan^2\phi)^{3/2} = \sec^3\phi$

$\kappa = \dfrac{|y''|}{\left[1 + y'^2\right]^{3/2}} = \dfrac{|y''|}{|\sec^3\phi|} = \left|y''\cos^3\phi\right|$

45. See the discussion dealing with s and a parameter t in the text. The important ingredients are that s increases as t increases and t can be expressed as a function of s. Use the Chain Rule.

$$\mathbf{N} = \frac{\frac{d\mathbf{T}}{ds}}{\left|\frac{d\mathbf{T}}{ds}\right|} = \frac{\frac{\mathbf{T}'(t)}{|\mathbf{v}(t)|}}{\left|\frac{\mathbf{T}'(t)}{|\mathbf{v}(t)|}\right|} = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{\frac{d\mathbf{T}}{dt}}{\left|\frac{d\mathbf{T}}{dt}\right|}$$

46. $\mathbf{r}(t) = a\cos\omega t\,\mathbf{i} + b\sin\omega t\,\mathbf{j}$;

$\mathbf{v}(t) = -a\omega\sin\omega t\,\mathbf{i} + b\omega\cos\omega t\,\mathbf{j}$

$\mathbf{a}(t) = \left\langle -a\omega^2\cos\omega t, -b\omega^2\sin\omega t\right\rangle = -\omega^2\mathbf{r}(t)$

$\mathbf{T} = \dfrac{\mathbf{v}}{(\mathbf{v}\cdot\mathbf{v})^{1/2}}$;

$\dfrac{d\mathbf{T}}{dt} = \dfrac{(\mathbf{v}\cdot\mathbf{v})\mathbf{a} - (\mathbf{v}\cdot\mathbf{a})\mathbf{v}}{(\mathbf{v}\cdot\mathbf{v})^{3/2}}$

$= \dfrac{-ab\omega}{(a^2\sin^2\omega t + b^2\cos\omega t)^{3/2}}(b\cos\omega t\mathbf{i} + a\sin\omega t\mathbf{j})$

$\left|\dfrac{d\mathbf{T}}{dt}\right| = \dfrac{ab\omega(b^2\cos^2\omega t + a^2\sin^2\omega t)^{1/2}}{(a^2\sin^2\omega t + b^2\cos\omega t)^{3/2}}$

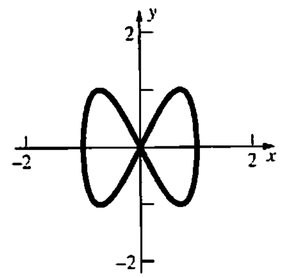
$= \dfrac{ab\omega}{a^2\sin^2\omega t + b^2\cos^2\omega t}$

Then

$\dfrac{\frac{d\mathbf{T}}{dt}}{\left|\frac{d\mathbf{T}}{dt}\right|} = \dfrac{-1}{(a^2\sin^2\omega t + b^2\cos^2\omega t)^{1/2}}(b\cos\omega t\mathbf{i} + a\sin\omega t\mathbf{j})$

Note that this was done assuming $ab > 0$; if $ab < 0$, drop the negative sign in the numerator.

47.



$\mathbf{v}(t) = \langle\cos t, 2\cos 2t\rangle$, $\mathbf{a}(t) = \langle -\sin t, -4\sin 2t\rangle$

$\mathbf{a}(t) = 0$ if and only if $-\sin t = 0$ and $-4\sin 2t = 0$, which occurs if and only if $t = 0, \pi, 2\pi$, so it occurs only at the origin.

$\mathbf{a}(t)$ points to the origin if and only if $\mathbf{a}(t) = -k\mathbf{r}(t)$ for some k and $\mathbf{r}(t)$ is not $\mathbf{0}$. This occurs if and only if $t = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$, so it occurs only at (1, 0) and (-1, 0).

48. $\mathbf{v}(t) = \langle -\sin t + t\cos t + \sin t, \cos t + t\sin t - \cos t\rangle$

$= t\cos t\mathbf{i} + t\sin t\mathbf{j}$

$\mathbf{a}(t) = \langle -t\sin t + \cos t, t\cos t + \sin t\rangle$

a. $\dfrac{ds}{dt} = |\mathbf{v}(t)| = \left|t(\cos^2 t + \sin^2 t)^{1/2}\right| = t$

(since $t \ge 0$)

b. $a_T = \dfrac{d^2s}{dt^2} = \left(\dfrac{d}{dt}\right)(t) = 1$

$a_N^2 = |a|^2 - a_T^2$

$= [t^2(\sin^2 t + \cos^2 t) + (\cos^2 t + \sin^2 t)] - 1 = t^2$

Therefore, $a_N = t$.

49. $s''(t) = a_T = 0 \Rightarrow$ speed $= s'(t) = c$ (a constant)

$\kappa\left(\frac{ds}{dt}\right)^2 = a_N = 0 \Rightarrow \kappa = 0$ or $\frac{ds}{dt} = 0$

$\Rightarrow \kappa = 0$

50. $\mathbf{N} = \frac{\frac{d\mathbf{T}}{ds}}{\left|\frac{d\mathbf{T}}{ds}\right|} = \frac{\langle -\sin\phi, \cos\phi\rangle\left(\frac{d\phi}{ds}\right)}{(\sin^2\phi + \cos^2\phi)^{1/2}\left|\frac{d\phi}{ds}\right|}$

$= \frac{\frac{d\phi}{ds}}{\left|\frac{d\phi}{ds}\right|}\langle -\sin\phi, \cos\phi\rangle$

If $\frac{d\phi}{ds} > 0, \mathbf{N} = \langle -\sin\phi, \cos\phi\rangle$ and if $\frac{d\phi}{ds} < 0, \mathbf{N} = \langle \sin\phi, -\cos\phi\rangle$, so **N** points to the concave side of the curve in either case.

51. $\mathbf{T}\cdot\mathbf{T} = 1 \Rightarrow \mathbf{T}\cdot\left(\frac{d\mathbf{T}}{ds}\right) + \mathbf{T}\cdot\left(\frac{d\mathbf{T}}{ds}\right) = 0$

$\Rightarrow \mathbf{T}\cdot\left(\frac{d\mathbf{T}}{ds}\right) = 0 = \mathbf{T}\cdot\kappa\mathbf{N} = 0$

Therefore, **T** is perpendicular to κ**N**, and hence to **N**.

52. $a_N = 0$ wherever $\kappa = 0$ or $\frac{ds}{dt} = 0$. κ, the curvature, is 0 at the inflection points, which occur at multiples of $\frac{\pi}{2}$. However, $\frac{ds}{dt} \neq 0$ on this curve. Therefore, $a_N = 0$ at multiples of $\frac{\pi}{2}$.

53. It is given that at (–12, 16), $s'(t) = 10$ ft/s and $s''(t) = 5$ ft/s². From Example 2, $\kappa = \frac{1}{20}$.

Therefore, $a_T = 5$ and $a_N = \left(\frac{1}{20}\right)(10)^2 = 5$, so $\mathbf{a} = 5\mathbf{T} + 5\mathbf{N}$. Let $\mathbf{r}(t) = \langle 20\cos t, 20\sin t\rangle$ describe the circle.

$\mathbf{r}(t) = \langle -12, 16\rangle \Rightarrow \cos t = -\frac{3}{5}$ and $\sin t = \frac{4}{5}$.

$\mathbf{v}(t) = \langle -20\sin t, 20\cos t\rangle$, so $|\mathbf{v}(t)| = 20$.

Then $\mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \langle -\sin t, \cos t\rangle$.

Thus, at (–12, 16), $\mathbf{T} = \left\langle -\frac{4}{5}, -\frac{3}{5}\right\rangle$ and

$\mathbf{N} = \left\langle \frac{3}{5}, -\frac{4}{5}\right\rangle$ since **N** is a unit vector perpendicular to **T** and pointing to the concave side of the curve.

Therefore,

$\mathbf{a} = 5\left\langle -\frac{4}{5}, -\frac{3}{5}\right\rangle + 5\left\langle \frac{3}{5}, -\frac{4}{5}\right\rangle = -\mathbf{i} - 7\mathbf{j}.$

54. $s'(t) = 4$ and $s''(t) = 0$.

$\kappa = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{2}{(1 + 4x^2)^{3/2}}$

Therefore,

$\mathbf{a} = (0)\mathbf{T} + (4)^2 \frac{2}{(1 + 4x^2)^{3/2}}\mathbf{N} = \frac{32}{(1 + 4x^2)^{3/2}}\mathbf{N}.$

55. Let $\mu mg = \frac{mv_R^2}{R}$. Then $v_R = \sqrt{\mu gR}$. At the values given,

$v_R = \sqrt{(0.4)(32)(400)} = \sqrt{5120} \approx 71.55$ ft/s (about 49.79 mi/h).

56. a. $\frac{R|\mathbf{F}|\sin\theta}{v_R^2} = \frac{|\mathbf{F}|\cos\theta}{g}$ (from the given equations, equating m in each.)

Therefore, $v_R = \sqrt{Rg\tan\theta}$.

b. For the values given,

$v_R = \sqrt{(400)(32)(\tan 10°)} \approx 47.51$ ft/s.

57. Let the polar coordinate equation of the curve be $r = f(\theta)$. Then the curve is parametrized by $x = r\cos\theta$ and $y = r\sin\theta$.

$x' = -r\sin\theta + r'\cos\theta$ $\qquad y' = r\cos\theta + r'\sin\theta$

$x'' = -r\cos\theta - 2r'\sin\theta + r''\cos\theta$ $\qquad y'' = -r\sin\theta + 2r'\cos\theta + r''\sin\theta$

By Theorem A, the curvature is

$\kappa = \frac{|x'y'' - y'x''|}{\left|x'^2 + y'^2\right|^{3/2}}$

$$= \frac{|(-r\sin\theta + r'\cos\theta)(-r\sin\theta + 2r'\cos\theta + r''\sin\theta) - (r\cos\theta + r'\sin\theta)(-r\cos\theta - 2r'\sin\theta + r''\cos\theta)|}{[(-r\sin\theta + r'\cos\theta)^2 + (r\cos\theta + r'\sin\theta)^2]^{3/2}}$$

$$= \frac{\left|r^2 + 2r'^2 - rr''\right|}{(r^2 + r'^2)^{3/2}}.$$

58. $r' = -4\sin\theta, r'' = -4\cos\theta$

$$\kappa = \frac{\left|16\cos^2\theta + 32\sin^2\theta + 16\cos^2\theta\right|}{(16\cos^2\theta + 16\sin^2\theta)^{3/2}} = \frac{32}{64} = \frac{1}{2}$$

59. $r' = -\sin\theta, r'' = -\cos\theta$

At $\theta = 0, r = 2, r' = 0$, and $r'' = -1$.

$$\kappa = \frac{|4+0+2|}{(4+0)^{3/2}} = \frac{6}{8} = \frac{3}{4}$$

60. $r' = 1, r'' = 0$

At $\theta = 1, r = 1, r' = 1$, and $r'' = 0$.

$$\kappa = \frac{|1+2-0|}{(1+1)^{3/2}} = \frac{3}{2\sqrt{2}}$$

61. $r' = -4\sin\theta, r'' = -4\cos\theta$

At $\theta = \frac{\pi}{2}, r = 4, r' = -4$, and $r'' = 0$.

$$\kappa = \frac{|16+32-0|}{(16+16)^{3/2}} = \frac{48}{128\sqrt{2}} = \frac{3}{8\sqrt{2}}$$

62. $r' = 3e^{3\theta}, r'' = 9e^{3\theta}$

At $\theta = 1,\ r = e^3, r' = 3e^3$, and $r'' = 9e^3$.

$$\kappa = \frac{\left|e^6 + 18e^6 - 9e^6\right|}{(e^6 + 9e^6)^{3/2}} = \frac{10e^6}{10\sqrt{10e^9}} = \frac{1}{\sqrt{10e^9}}$$

63. $r' = 4\cos\theta, r'' = -4\sin\theta$

At $\theta = \frac{\pi}{2}, r = 8, r' = 0, r'' = -4$.

$$\kappa = \frac{|64+0+32|}{(64+0)^{3/2}} = \frac{96}{512} = \frac{3}{16}$$

64. $r' = 6e^{6\theta}, r'' = 36e^{6\theta}$

$$\kappa = \frac{\left|e^{12\theta} + 72e^{12\theta} - 36e^{12\theta}\right|}{(e^{12\theta} + 36e^{12\theta})^{3/2}} = \frac{37e^{12\theta}}{37\sqrt{37}e^{18\theta}}$$

$$= \frac{1}{\sqrt{37}}\left(\frac{1}{e^{6\theta}}\right) = \frac{1}{\sqrt{37}}\left(\frac{1}{r}\right)$$

65. $r = \sqrt{\cos 2\theta}$

$$r' = -\frac{\sin 2\theta}{\sqrt{\cos 2\theta}}, r'' = -\frac{\cos^2 2\theta + 1}{(\cos 2\theta)^{3/2}}$$

$$\kappa = \frac{\left|\cos 2\theta + \frac{2\sin^2 2\theta}{\cos 2\theta} + \frac{\cos^2 2\theta + 1}{\cos 2\theta}\right|}{\left(\cos 2\theta + \frac{\sin^2 2\theta}{\cos 2\theta}\right)^{3/2}} = \frac{\left|\frac{3}{\cos 2\theta}\right|}{\left(\frac{1}{\cos 2\theta}\right)^{3/2}}$$

$$= 3\left(\sqrt{\cos 2\theta}\right) = 3r$$

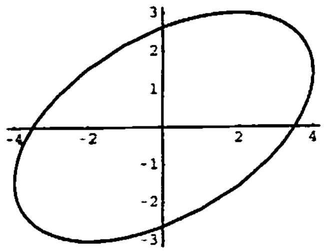
66. $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, where $x = f(t)$ and $y = g(t)$; $\mathbf{v}(t) = \mathbf{r}'(t)$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{x'}{\sqrt{x'^2 + y'^2}}\mathbf{i} + \frac{y'}{\sqrt{x'^2 + y'^2}}\mathbf{j} = \frac{x''(x'^2 + y'^2) - x'(x'x'' + y'y'')}{(x'^2 + y'^2)^{3/2}}\mathbf{i} + \frac{y''(x'^2 + y'^2) - y'(x'x'' + y'y'')}{(x'^2 + y'^2)^{3/2}}\mathbf{j}$$

$$\mathbf{T}'(t) = \frac{(x''y' - x'y'')y'}{(x'^2 + y'^2)^{3/2}}\mathbf{i} + \frac{(y''x' - y'x'')x'}{(x'^2 + y'^2)^{3/2}}\mathbf{j} = \frac{(x'y'' - y'x'')}{(x'^2 + y'^2)^{3/2}}(-y'\mathbf{i} + x'\mathbf{j})$$

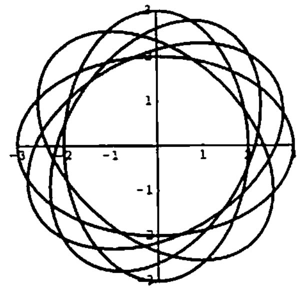
$$|\mathbf{T}'(t)| = \frac{|x'y'' - y'x''|}{(x'^2 + y'^2)^{3/2}}\sqrt{x'^2 + y'^2} = \frac{|x'y'' - y'x''|}{x'^2 + y'^2}$$

$$\kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|x'y'' - y'x''|}{(x'^2 + y'^2)^{3/2}}$$

67. 

Maximum curvature ≈ 0.7606, minimum curvature ≈ 0.1248

68. The following graph was plotted using Mathematica. The "AspectRatio" option was set to "Automatic."



The curves are ellipses with major axis length = 6 and minor axis length = 4. θ determines the angle of the major axis with the positive x-axis. From Example 4, the maximum curvature of such curves is $\frac{3}{4}$ and the minimum curvature is $\frac{2}{9}$.

69. If $a > b$, then the graph is an ellipse centered at the origin with the length of the major axis $2a$ and the length of the minor axis $2b$. The angle of the major axis with the positive x-axis is θ.

13.6 Chapter Review

Concepts Test

1. False: For example, $x = 0, y = t$, and $x = 0$, $y = -t$ both represent the line $x = 0$.

2. True: Eliminating the parameter gives $x = 2y$.

3. False: For example, the graph of $x = t^2, y = t$ does not represent y as a function of x. $y = \pm\sqrt{x}$, but $h(x) = \pm\sqrt{x}$ is not a function.

4. True: When $t = 1, x = 0$, and $y = 0$.

5. False: For example, if $x = t^3, y = t^3$ then $y = x$ so $\frac{d^2y}{dx^2} = 0$, but $\frac{g''(t)}{f''(t)} = 1$.

6. True: For example, the graph of the four-leaved rose has two tangent lines at the origin.

7. True: $(2\mathbf{i} - 3\mathbf{j}) \cdot (6\mathbf{i} + 4\mathbf{j}) = 0$ if and only if the vectors are perpendicular.

8. True: Since $\mathbf{u}$ and $\mathbf{v}$ are unit vectors, $\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \mathbf{u} \cdot \mathbf{v}$.

9. False: The dot product for three vectors $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$ is not defined.

10. True: $|\mathbf{u} \cdot \mathbf{v}| = |\mathbf{u}||\mathbf{v}||\cos\theta| \le |\mathbf{u}||\mathbf{v}|$ since $|\cos\theta| \le 1$.

11. True: If $|\mathbf{u} \cdot \mathbf{v}| = |\mathbf{u}||\mathbf{v}|$, then $|\cos\theta| = 1$ since $|\mathbf{u} \cdot \mathbf{v}| = |\mathbf{u}||\mathbf{v}||\cos\theta|$. Thus, $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$. If $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$, $|\mathbf{u} \cdot \mathbf{v}| = |\mathbf{u} \cdot k\mathbf{u}| = k|\mathbf{u}|^2 = |\mathbf{u}||k\mathbf{u}| = |\mathbf{u}||\mathbf{v}|$.

12. False: If $\mathbf{u} = -\mathbf{i}$ and $\mathbf{v} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$, then $\mathbf{u} + \mathbf{v} = -\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$ and $|\mathbf{u}| = |\mathbf{v}| = |\mathbf{u} + \mathbf{v}| = 1$.

13. True: $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = |\mathbf{u}|^2 - |\mathbf{v}|^2 = 0$, so $|\mathbf{u}|^2 = |\mathbf{v}|^2$ or $|\mathbf{u}| = |\mathbf{v}|$.

14. True: $|\mathbf{u} + \mathbf{v}|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v})$ $= |\mathbf{u}|^2 + |\mathbf{v}|^2 + 2\mathbf{u} \cdot \mathbf{v}$

15. True: Theorem 13.4A

16. True: $D_t[\mathbf{F}(t)\cdot\mathbf{F}(t)]$
$= \mathbf{F}(t)\cdot\mathbf{F}'(t)+\mathbf{F}(t)\cdot\mathbf{F}'(t)$
$= 2\mathbf{F}(t)\cdot\mathbf{F}'(t)$

17. True: $x' = 3 \quad y' = 2$
$x'' = 0 \quad y'' = 0$
Thus, $\kappa = \dfrac{|x'y''-y'x''|}{\left[x'^2+y'^2\right]^{3/2}} = 0.$

18. False: $x' = -2\sin t \quad y' = 2\cos t$
$x'' = -2\cos t \quad y'' = -2\sin t$
Thus, $\kappa = \dfrac{|x'y''-y'x''|}{[x'^2+y'^2]^{3/2}} = \dfrac{4}{8} = \dfrac{1}{2}.$

19. True: $\mathbf{T}(t) = \dfrac{1}{|\mathbf{r}'(t)|}\mathbf{r}'(t);$
$\mathbf{T}'(t) = -\dfrac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|^3}\mathbf{r}'(t)+\dfrac{1}{|\mathbf{r}'(t)|}\mathbf{r}''(t);$
$\mathbf{T}(t)\cdot\mathbf{T}'(t)$
$= \dfrac{1}{|\mathbf{r}'(t)|}\mathbf{r}'(t)\cdot \left[-\dfrac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|^3}\mathbf{r}'(t)+\dfrac{1}{|\mathbf{r}'(t)|}\mathbf{r}''(t)\right]$
$= -\dfrac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|^2}+\dfrac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|^2} = 0$

20. False: See Example 3 of Section 13.4.
$|\mathbf{v}| = 0$ but $|\mathbf{a}| = r\omega^2$.

21. True: $\kappa = \dfrac{|y''|}{[1+y'^2]^{3/2}} = 0$

22. False: If $y'' = k$ then $y' = kx + C$ and
$\kappa = \dfrac{|y''|}{[1+y'^2]^{3/2}} = \dfrac{k}{[1+(kx+C)^2]^{3/2}}$
is not constant.

23. False: For example, if $\mathbf{u} = \mathbf{i}$ and $\mathbf{v} = \mathbf{j}$, then $\mathbf{u}\cdot\mathbf{v} = 0$.

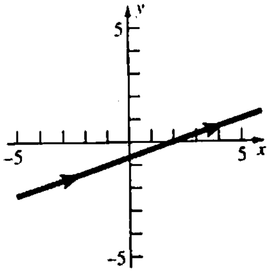
24. False: For example, if $\mathbf{r}(t) = \cos t^2\mathbf{i}+\sin t^2\mathbf{j}$, then $\mathbf{r}'(t) = -2t\sin t^2\mathbf{i}+2t\cos t^2\mathbf{j}$, so $|\mathbf{r}(t)| = 1$ but $|\mathbf{r}'(t)| = 2t$.

25. True: If $\mathbf{v}\cdot\mathbf{v} =$ constant, differentiate both sides to get $\mathbf{v}\cdot\mathbf{v}'+\mathbf{v}\cdot\mathbf{v}' = 2\mathbf{v}\cdot\mathbf{v}' = 0$, so $\mathbf{v}\cdot\mathbf{v}' = 0$.

Sample Test Problems

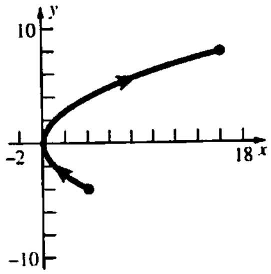
1. $t = \dfrac{1}{6}(x-2)$

$y = \dfrac{1}{3}(x-2)$



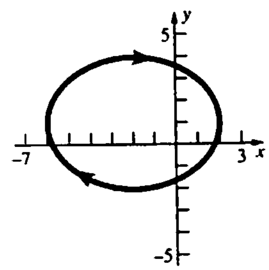
2. $t = \dfrac{y}{4}$

$x = \dfrac{y^2}{4}$ or $y^2 = 4x$



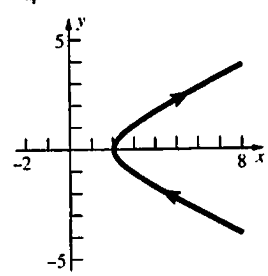
3. $\sin t = \dfrac{x+2}{4}, \cos t = \dfrac{y-1}{3}$

$\dfrac{(x+2)^2}{16}+\dfrac{(y-1)^2}{9} = 1$



4. $\dfrac{x^2}{4} = \sec^2 t, y^2 = \tan^2 t$

$\dfrac{x^2}{4}-y^2 = 1$



5. $\frac{dx}{dt} = 6t^2 - 4, \frac{dy}{dt} = 1 + \frac{1}{t+1} = \frac{t+2}{t+1}$

$\frac{dy}{dx} = \frac{\frac{t+2}{t+1}}{6t^2 - 4} = \frac{t+2}{(t+1)(6t^2 - 4)}$

At $t = 0$, $x = 7$, $y = 0$, and $\frac{dy}{dx} = -\frac{1}{2}$.

Tangent line: $y = -\frac{1}{2}(x-7)$ or $x + 2y - 7 = 0$

Normal line: $y = 2(x - 7)$ or $2x - y - 14 = 0$.

6. $\frac{dx}{dt} = -3e^{-t}, \frac{dy}{dt} = \frac{1}{2}e^t$

$\frac{dy}{dx} = \frac{\frac{1}{2}e^t}{-3e^{-t}} = -\frac{1}{6}e^{2t}$

At $t = 0$, $x = 3$, $y = \frac{1}{2}$, and $\frac{dy}{dx} = -\frac{1}{6}$.

Tangent line: $y - \frac{1}{2} = -\frac{1}{6}(x-3)$ or $x + 6y - 6 = 0$

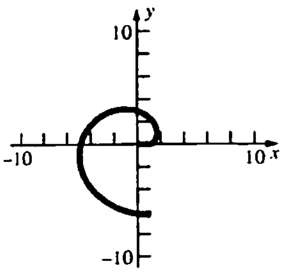
Normal line: $y - \frac{1}{2} = 6(x-3)$ or $12x - 2y - 35 = 0$.

7. $\frac{dx}{dt} = -\sin t + \sin t + t\cos t = t\cos t$

$\frac{dy}{dt} = \cos t - \cos t + t\sin t = t\sin t$

$L = \int_0^{2\pi} \sqrt{(t\cos t)^2 + (t\sin t)^2}\, dt = \int_0^{2\pi} t\, dt$

$= \left[\frac{1}{2}t^2\right]_0^{2\pi} = 2\pi^2$



8. a. $\mathbf{u} = \langle -3, 1 \rangle$, $|\mathbf{u}| = \sqrt{10} \approx 3.1623$

b. $\mathbf{u} = \langle -1, 5 \rangle$, $|\mathbf{u}| = \sqrt{26} \approx 5.0990$

9. a. $3\langle 2,-5\rangle - 2\langle 1,1\rangle = \langle 6,-15\rangle - \langle 2,2\rangle = \langle 4,-17\rangle$

b. $\langle 2,-5\rangle \cdot \langle 1,1\rangle = 2 + (-5) = -3$

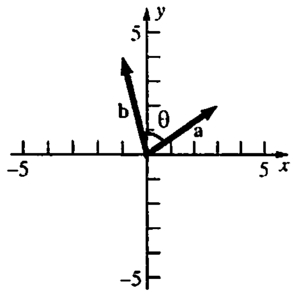
c. $\langle 2,-5\rangle \cdot (\langle 1,1\rangle + \langle -6,0\rangle) = \langle 2,-5\rangle \cdot \langle -5,1\rangle$
$= -10 + (-5) = -15$

d. $(4\langle 2,-5\rangle + 5\langle 1,1\rangle) \cdot 3\langle -6,0\rangle$
$= \langle 13,-15\rangle \cdot \langle -18,0\rangle$
$= -234 + 0 = -234$

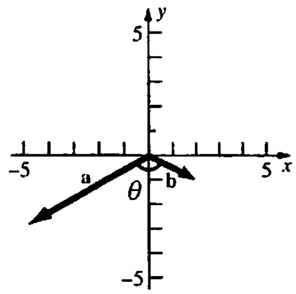
e. $\sqrt{36+0}\langle -6,0\rangle \cdot \langle 1,-1\rangle = 6(-6+0) = -36$

f. $\langle -6,0\rangle \cdot \langle -6,0\rangle - \sqrt{36+0}$
$= (36+0) - 6 = 30$

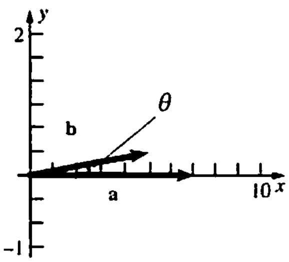
10. a. $\cos\theta = \frac{(3)(-1)+(2)(4)}{\sqrt{9+4}\sqrt{1+16}} = \frac{5}{\sqrt{221}} \approx 0.3363$



b. $\cos\theta = \frac{(-5)(2)+(-3)(-1)}{\sqrt{25+9}\sqrt{4+1}} = -\frac{7}{\sqrt{170}}$
≈ -0.5369



c. $\cos\theta = \frac{(7)(5)+(0)(1)}{\sqrt{49+0}\sqrt{25+1}} = \frac{5}{\sqrt{26}} \approx 0.9806$



11. $5\mathbf{i} - 4\mathbf{j} = k(-2\mathbf{i}) + m(3\mathbf{i} - 2\mathbf{j})$
$= (-2k + 3m)\mathbf{i} - (2m)\mathbf{j}$, so $5 = -2k + 3m$ and $4 = 2m$. Therefore, $m = 2$ and $k = \frac{1}{2}$.

12. $y = x^2$; $y' = 2x$, so the slope at $(-1, 1)$ is -2 (y changes by -2 as x changes by 1). Therefore, a

vector parallel to the tangent line is $\langle 1,-2\rangle$. To obtain a vector of length 3, first divide $\langle 1,-2\rangle$ by its length and then multiply that result by 3.

$$\frac{\langle 1,-2\rangle}{|\langle 1,-2\rangle|}(3)=\frac{3\langle 1,-2\rangle}{\sqrt{1+4}}=\left\langle \frac{3}{\sqrt{5}},-\frac{6}{\sqrt{5}}\right\rangle$$

13. One vector that makes such an angle is $\langle \cos 150°, \sin 150°\rangle$. The required vector is then
$$10\frac{\langle \cos 150°, \sin 150°\rangle}{|\langle \cos 150°, \sin 150°\rangle|}=5\langle -\sqrt{3},1\rangle.$$

14. $-(\mathbf{F}_1+\mathbf{F}_2)=-5\mathbf{i}-9\mathbf{j}$

15. Let the wind vector be
$\mathbf{w}=\langle 100\cos 30°, 100\sin 30°\rangle=\langle 50\sqrt{3}, 50\rangle$.
Let $\mathbf{p}=\langle p_1, p_2\rangle$ be the plane's air velocity vector.
We want $\mathbf{w}+\mathbf{p}=450\,\mathbf{j}=\langle 0,450\rangle$.
$\langle 50\sqrt{3}, 50\rangle+\langle p_1, p_2\rangle=\langle 0, 450\rangle$
$\Rightarrow 50\sqrt{3}+p_1=0, 50+p_2=450$
$\Rightarrow p_1=-50\sqrt{3}, p_2=400$
Therefore, $\mathbf{p}=\langle -50\sqrt{3},400\rangle$. The angle β formed with the vertical satisfies
$\cos\beta=\frac{\mathbf{p}\cdot\mathbf{j}}{|\mathbf{p}||\mathbf{j}|}=\frac{400}{\sqrt{167{,}500}}$; $\beta\approx 12.22°$. Thus, the heading is N12.22°W. The air speed is
$|\mathbf{p}|=\sqrt{167{,}500}\approx 409.27$ mi/h.

16. $\mathbf{r}(t)=\langle e^{2t},e^{-t}\rangle;\mathbf{r}'(t)=\langle 2e^{2t},-e^{-t}\rangle$

a. $\lim_{t\to 0}\langle e^{2t},e^{-t}\rangle=\left\langle \lim_{t\to 0}e^{2t},\lim_{t\to 0}e^{-t}\right\rangle=\langle 1,1\rangle$

b. $\lim_{h\to 0}\frac{\mathbf{r}(0+h)-\mathbf{r}(0)}{h}=\mathbf{r}'(0)=\langle 2,-1\rangle$

c. $\int_0^{\ln 2}\langle e^{2t},e^{-t}\rangle dt=\left[\left\langle\left(\frac{1}{2}\right)e^{2t},-e^{-t}\right\rangle\right]_0^{\ln 2}$
$=\left\langle 2,-\frac{1}{2}\right\rangle-\left\langle\frac{1}{2},-1\right\rangle=\left\langle\frac{3}{2},\frac{1}{2}\right\rangle$

d. $D_t[t\mathbf{r}(t)]=t\mathbf{r}'(t)+\mathbf{r}(t)$
$=t\langle 2e^{2t},-e^{-t}\rangle+\langle e^{2t},e^{-t}\rangle$
$=\langle e^{2t}(2t+1),e^{-t}(1-t)\rangle$

e. $D_t[\mathbf{r}(3t+10)]=[\mathbf{r}'(3t+10)](3)$
$=3\langle 2e^{6t+20},-e^{-3t-10}\rangle$
$=\langle 6e^{6t+20},-3e^{-3t-10}\rangle$

f. $D_t[\mathbf{r}(t)\cdot\mathbf{r}'(t)]=D_t[2e^{4t}-e^{-2t}]$
$=8e^{4t}+2e^{-2t}$

17. a. $\mathbf{r}'(t)=\left\langle\frac{1}{t},-6t\right\rangle;\mathbf{r}''(t)=\langle -t^{-2},-6\rangle$

b. $\mathbf{r}'(t)=\langle\cos t,-2\sin 2t\rangle$;
$\mathbf{r}''(t)=\langle -\sin t,-4\cos 2t\rangle$

c. $\mathbf{r}'(t)=\langle\sec^2 t,-4t^3\rangle$;
$\mathbf{r}''(t)=\langle 2\sec^2 t\tan t,-12t^2\rangle$

18. $L=\int_0^2|\mathbf{r}'(t)|dt=\int_0^2 3(4t+1)^{1/2}dt$
$=\left[\frac{(4t+1)^{3/2}}{2}\right]_0^2=\frac{27}{2}-\frac{1}{2}=13$

19. $\mathbf{v}(t)=\langle 4t,4\rangle$; $\mathbf{a}(t)=\langle 4,0\rangle$; $\mathbf{v}(-1)=\langle -4,4\rangle$;
$|\mathbf{v}(-1)|=4\sqrt{2}$; $\mathbf{a}(-1)=\langle 4,0\rangle$

20. $\mathbf{v}(t)=\langle -4\cos t,4(1-\sin t)\rangle$;
$\mathbf{a}(t)=\langle 4\sin t,-4\cos t\rangle$;
$\mathbf{v}\left(\frac{2\pi}{3}\right)=2\mathbf{i}+\left(4-2\sqrt{3}\right)\mathbf{j}$;
$\left|\mathbf{v}\left(\frac{2\pi}{3}\right)\right|=\left(32-16\sqrt{3}\right)^{1/2}\approx 2.0706$;
$\mathbf{a}\left(\frac{2\pi}{3}\right)=2\sqrt{3}\mathbf{i}+2\mathbf{j}$

21. a. $y=x^2-x; y'=2x-1; y''=2$;
$y'(1)=1; y''(1)=2$;
$\kappa(1)=\frac{|2|}{(1+1)^{3/2}}=\frac{1}{\sqrt{2}}\approx 0.7071$

b. Let $x=t+t^3; x'=1+3t^2; x''=6t$;
$y=t+t^2; y'=1+2t; y''=2$.
At (2, 2), $t=1$, so $x'=4, x''=6$,
$y'=3, y''=2$.
Therefore, $\kappa=\frac{|(4)(2)-(3)(6)|}{(16+9)^{3/2}}=0.08$.

c. $y = a\cosh\left(\frac{x}{a}\right); y' = \sinh\left(\frac{x}{a}\right);$

$y'' = \left(\frac{1}{a}\right)\cosh\left(\frac{x}{a}\right).$

At $(a, a\cosh 1)$,

$y' = \sinh 1, y'' = \left(\frac{1}{a}\right)(\cosh 1).$

Therefore,

$$\kappa = \frac{\left(\frac{1}{a}\right)(\cosh 1)}{(1+\sinh^2 1)^{3/2}} = \frac{\cosh 1}{a(\cosh^2 1)^{3/2}}$$

$$= \frac{1}{a\cosh^2 1} \approx \frac{0.4120}{a}.$$

22. $x(t) = t,\ x'(t) = 1, x''(t) = 0; y(t) = \frac{1}{3}t^3,$

$y'(t) = t^2, y''(t) = 2t$

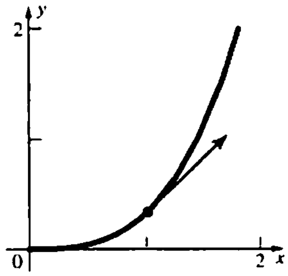
$x'(1) = 1, x''(1) = 0$

$y'(1) = 1, y''(1) = 2$

$$\mathbf{T}(1) = \frac{\mathbf{i}+\mathbf{j}}{[(1)^2+(1)^2]} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

$$\kappa(1) = \frac{2-0}{[1+1]^{3/2}} = \frac{1}{\sqrt{2}}$$

The Cartesian equation is $y = \left(\frac{1}{3}\right)x^3.$



23. $\mathbf{v}(t) = -2t\mathbf{i} + 2\mathbf{j};\ |\mathbf{v}(t)| = 2(t^2+1)^{1/2};\ \mathbf{a}(t) = -2\mathbf{i};$

$|\mathbf{a}(t)| = 2$

$$a_T = \frac{d^2s}{dt^2} = \left(\frac{d}{dt}\right)[2(t^2+1)^{1/2}] = 2t(t^2+1)^{-1/2}$$

At (0, 2):

$t = 1$

$a_T = \sqrt{2}$

$a_N = |a|^2 - a_T = 4 - 2 = 2,$ so $a_N = \sqrt{2}.$