CHAPTER 3

The Derivative

3.1 Concepts Review

1. tangent line

2. secant line

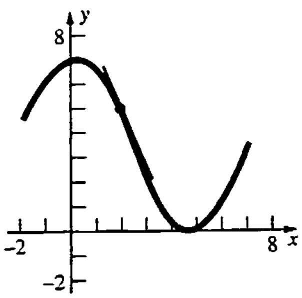
3. $\dfrac{f(c+h)-f(c)}{h}$

4. average velocity

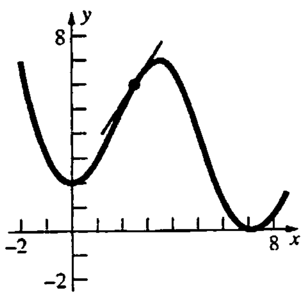
Problem Set 3.1

1. Slope $= \dfrac{5-3}{2-\frac{3}{2}} = 4$

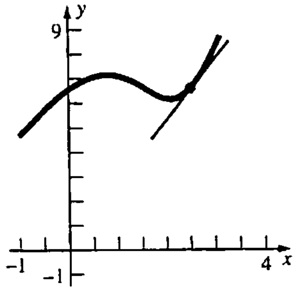
2. Slope $= \dfrac{6-4}{4-6} = -2$

3. 

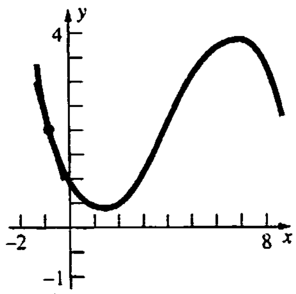
Slope ≈ -2

4. 

Slope ≈ 1.5

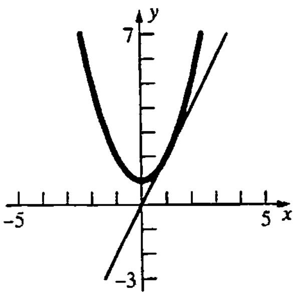
5. 

Slope $\approx \dfrac{5}{2}$

6. 

Slope $\approx -\dfrac{3}{2}$

7. $y = x^2 + 1$

a., b. 

c. $m_{\tan} = 2$

d. $m_{\sec} = \dfrac{(1.01)^2 + 1.0 - 2}{1.01 - 1} = \dfrac{0.0201}{.01} = 2.01$

e. $m_{\tan} = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$

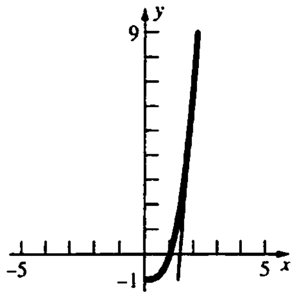
$= \lim_{h\to 0} \frac{[(1+h)^2+1]-(1^2+1)}{h}$

$= \lim_{h\to 0} \frac{2+2h+h^2-2}{h} = \lim_{h\to 0} \frac{h(2+h)}{h}$

$= \lim_{h\to 0}(2+h) = 2$

8. $y = x^3 - 1$

a., b.



c. $m_{\tan} = 12$

d. $m_{\sec} = \frac{[(2.01)^3 - 1.0] - 7}{2.01 - 2} = \frac{0.120601}{0.01}$

$= 12.0601$

e. $m_{\tan} = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$

$= \lim_{h\to 0} \frac{[(2+h)^3-1]-(2^3-1)}{h}$

$= \lim_{h\to 0} \frac{12h+6h^2+h^3}{h} = \lim_{h\to 0} \frac{h(12+6h+h^2)}{h}$

$= 12$

9. $f(x) = x^2 - 1$

$m_{\tan} = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$

$= \lim_{h\to 0} \frac{[(c+h)^2-1]-(c^2-1)}{h}$

$= \lim_{h\to 0} \frac{c^2+2ch+h^2-1-c^2+1}{h}$

$= \lim_{h\to 0} \frac{h(2c+h)}{h} = 2c$

At $x = -2$, $m_{\tan} = -4$
$x = -1$, $m_{\tan} = -2$
$x = 1$, $m_{\tan} = 2$
$x = 2$, $m_{\tan} = 4$

10. $f(x) = x^3 - 3x$

$m_{\tan} = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$

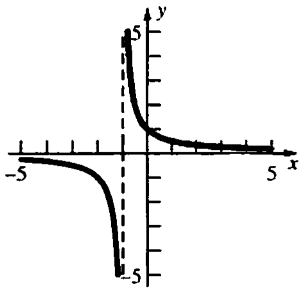
$= \lim_{h\to 0} \frac{[(c+h)^3 - 3(c+h)] - (c^3 - 3c)}{h}$

$= \lim_{h\to 0} \frac{c^3+3c^2h+3ch^2+h^3-3c-3h-c^3+3c}{h}$

$= \lim_{h\to 0} \frac{h(3c^2+3ch+h^2-3)}{h} = 3c^2 - 3$

At $x = -2$, $m_{\tan} = 9$
$x = -1$, $m_{\tan} = 0$
$x = 0$, $m_{\tan} = -3$
$x = 1$, $m_{\tan} = 0$
$x = 2$, $m_{\tan} = 9$

11.



$f(x) = \frac{1}{x+1}$

$m_{\tan} = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$

$= \lim_{h\to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h} = \lim_{h\to 0} \frac{-\frac{h}{2(2+h)}}{h} = \lim_{h\to 0} -\frac{1}{2(2+h)}$

$= -\frac{1}{4}$

$y - \frac{1}{2} = -\frac{1}{4}(x-1)$

12. $f(x) = \frac{1}{x-1}$

$m_{\tan} = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{\frac{1}{h-1}+1}{h}$

$= \lim_{h\to 0} \frac{\frac{h}{h-1}}{h} = \lim_{h\to 0} \frac{1}{h-1} = -1$

$y + 1 = -1(x - 0);\ y = -x - 1$

13. a. $16(1^2) - 16(0^2) = 16$ ft

b. $16(2^2) - 16(1^2) = 48$ ft

c. $V_{\text{ave}} = \frac{144-64}{3-2} = 80$ ft/sec

d. $V_{ave} = \frac{16(3.01)^2 - 16(3)^2}{3.01-3} = \frac{0.9616}{0.01}$
$= 96.16$ ft/s

e. $f(t) = 16t^2; v = 32c$
$v = 32(3) = 96$ ft/s

14. a. $V_{ave} = \frac{(3^2+1)-(2^2+1)}{3-2} = 5$ m/sec

b. $V_{ave} = \frac{[(2.003)^2+1]-(2^2+1)}{2.003-2} = \frac{0.012009}{0.003}$
$= 4.003$ m/sec

c. $V_{ave} = \frac{[(2+h)^2+1]-(2^2+1)}{2+h-2} = \frac{4h+h^2}{h}$
$= 4+h$

d. $f(t) = t^2 + 1$

$$v = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$
$$= \lim_{h\to 0}\frac{[(2+h)^2+1]-(2^2+1)}{h}$$
$$= \lim_{h\to 0}\frac{4h+h^2}{h} = \lim_{h\to 0}(4+h) = 4$$

15. a. $$v = \lim_{h\to 0}\frac{f(\alpha+h)-f(\alpha)}{h}$$
$$= \lim_{h\to 0}\frac{\sqrt{2(\alpha+h)+1}-\sqrt{2\alpha+1}}{h}$$
$$= \lim_{h\to 0}\frac{\sqrt{2\alpha+2h+1}-\sqrt{2\alpha+1}}{h}$$
$$= \lim_{h\to 0}\frac{(\sqrt{2\alpha+2h+1}-\sqrt{2\alpha+1})(\sqrt{2\alpha+2h+1}+\sqrt{2\alpha+1})}{h(\sqrt{2\alpha+2h+1}+\sqrt{2\alpha+1})}$$
$$= \lim_{h\to 0}\frac{2h}{h(\sqrt{2\alpha+2h+1}+\sqrt{2\alpha+1})}$$
$$= \frac{2}{\sqrt{2\alpha+1}+\sqrt{2\alpha+1}} = \frac{1}{\sqrt{2\alpha+1}} \text{ ft/s}$$

b. $\frac{1}{\sqrt{2\alpha+1}} = \frac{1}{2}$
$\sqrt{2\alpha+1} = 2$
$2\alpha+1 = 4;\ \alpha = \frac{3}{2}$
The object reaches a velocity of $\frac{1}{2}$ ft/s when $t = \frac{3}{2}$.

16. $f(t) = -t^2 + 4t$

$$v = \lim_{h\to 0}\frac{[-(c+h)^2+4(c+h)]-(-c^2+4c)}{h}$$
$$= \lim_{h\to 0}\frac{-c^2-2ch-h^2+4c+4h+c^2-4c}{h}$$
$$= \lim_{h\to 0}\frac{h(-2c-h+4)}{h} = -2c+4$$

$-2c + 4 = 0$ when $c = 2$
The particle comes to a momentary stop at $t = 2$.

17. a. $\left[\frac{1}{2}(2.01)^2+1\right]-\left[\frac{1}{2}(2)^2+1\right] = 0.02005$ g

b. $r_{ave} = \frac{0.02005}{2.01-2} = 2.005$ g/hr

c. $f(t) = \frac{1}{2}t^2 + 1$

$$r = \lim_{h\to 0}\frac{\left[\frac{1}{2}(2+h)^2+1\right]-\left[\frac{1}{2}2^2+1\right]}{h}$$
$$= \lim_{h\to 0}\frac{2+2h+\frac{1}{2}h^2+1-2-1}{h}$$
$$= \lim_{h\to 0}\frac{h\left(2+\frac{1}{2}h\right)}{h} = 2$$

At $t = 2$, $r = 2$

18\. a. $1000(3)^2 - 1000(2)^2 = 5000$

b. $\frac{1000(2.5)^2 - 1000(2)^2}{2.5-2} = \frac{2250}{0.5} = 4500$

c. $f(t) = 1000t^2$

$$r = \lim_{h \to 0} \frac{1000(2+h)^2 - 1000(2)^2}{h}$$
$$= \lim_{h \to 0} \frac{4000 + 4000h + 1000h^2 - 4000}{h}$$
$$= \lim_{h \to 0} \frac{h(4000 + 1000h)}{h} = 4000$$

19\. a. $d_{\text{ave}} = \frac{5^3 - 3^3}{5-3} = \frac{98}{2} = 49$ g/cm

b. $f(x) = x^3$

$$d = \lim_{h \to 0} \frac{(3+h)^3 - 3^3}{h}$$
$$= \lim_{h \to 0} \frac{27 + 27h + 9h^2 + h^3 - 27}{h}$$
$$= \lim_{h \to 0} \frac{h(27 + 9h + h^2)}{h} = 27 \text{ g/cm}$$

20\. $$MR = \lim_{h \to 0} \frac{R(c+h) - R(c)}{h}$$
$$= \lim_{h \to 0} \frac{[0.4(c+h) - 0.001(c+h)^2] - (0.4c - 0.001c^2)}{h}$$
$$= \lim_{h \to 0} \frac{0.4c + 0.4h - 0.001c^2 - 0.002ch - 0.001h^2 - 0.4c + 0.001c^2}{h}$$
$$= \lim_{h \to 0} \frac{h(0.4 - 0.002c - 0.001h)}{h} = 0.4 - 0.002c$$

When $n = 10$, $MR = 0.38$; when $n = 100$, $MR = 0.2$

21\. $$a = \lim_{h \to 0} \frac{2(1+h)^2 - 2(1)^2}{h}$$
$$= \lim_{h \to 0} \frac{2 + 4h + 2h^2 - 2}{h}$$
$$= \lim_{h \to 0} \frac{h(4 + 2h)}{h} = 4$$

22\. $$r = \lim_{h \to 0} \frac{p(c+h) - p(c)}{h}$$
$$= \lim_{h \to 0} \frac{[120(c+h)^2 - 2(c+h)^3] - (120c^2 - 2c^3)}{h}$$
$$= \lim_{h \to 0} \frac{h(240c - 6c^2 + 120h - 6ch - 2h^2)}{h}$$
$$= 240c - 6c^2$$

When $t = 10,\ r = 240(10) - 6(10)^2 = 1800$

$t = 20,\ r = 240(20) - 6(20)^2 = 2400$

$t = 40,\ r = 240(40) - 6(40)^2 = 0$

23\. $r_{\text{ave}} = \frac{100 - 800}{24 - 0} = -\frac{175}{6} \approx -29.167$

29,167 gal/hr

At 8 o'clock, $r \approx \frac{700 - 400}{6 - 10} \approx -75$

75,000 gal/hr

24\. a. The elevator reached the seventh floor at time $t = 80$. The average velocity is $v_{avg} = (84 - 0)/80 = 1.05$ feet per second

b. The slope of the line is approximately $\frac{60 - 12}{55 - 15} = 1.2$. The velocity is approximately 1.2 feet per second.

c. The building averages 84/7=12 feet from floor to floor. Since the velocity is zero for two intervals between time 0 and time 85, the elevator stopped twice. The heights are approximately 12 and 60. Thus, the elevator stopped at floors 1 and 5.

25\. a. A tangent line at $t = 91$ has slope approximately $(63 - 48)/(91 - 61) = 0.5$. The normal high temperature increases at the rate of 0.5 degree F per day.

b. A tangent line at $t = 191$ has approximate slope $(90-88)/30 \approx 0.067$. The normal high temperature increases at the rate of 0.067 degree per day.

c. There is a time in January, about January 15, when the rate of change is zero. There is also a time in July, about July 15, when the rate of change is zero.

d. The greatest rate of increase occurs around day 61, that is, some time in March. The greatest rate of decrease occurs between day 301 and 331, that is, sometime in November.

26. The slope of the tangent line at $t = 1930$ is approximately $(10-3.5)/(1980-1919) \approx 0.107$. The rate of growth in 1930 is approximately 0.107 million, or 107,000, persons per year. In 1990, the tangent line has approximate slope $(24-16)/(20000-1980) \approx 0.4$. Thus, the rate of growth in 1990 is 0.4 million, or 400,000, persons per year. The approximate percentage growth in 1930 is $0.107/6 \approx 0.018$ and in 1990 it is approximately $0.4/20 \approx 0.02$.

27. In both (a) and (b), the tangent line is always positive. In (a) the tangent line becomes steeper and steeper as t increases; thus, the velocity is increasing. In (b) the tangent line becomes flatter and flatter as t increases; thus, the velocity is decreasing.

28. $f(t) = \frac{1}{3}t^3 + t$

$$\text{current} = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$$

$$= \lim_{h\to 0} \frac{\left[\frac{1}{3}(c+h)^3 + (c+h)\right] - \left(\frac{1}{3}c^3 + c\right)}{h}$$

$$= \lim_{h\to 0} \frac{h\left(c^2 + ch + \frac{1}{3}h^2 + 1\right)}{h} = c^2 + 1$$

When $t = 3$, the current $=10$

$c^2 + 1 = 20$

$c^2 = 19$

$c = \sqrt{19} \approx 4.4$

A 20-amp fuse will blow at $t = 4.4$ s.

29. $A = \pi r^2$, $r = 2t$

$A = 4\pi t^2$

$$\text{rate} = \lim_{h\to 0} \frac{4\pi(3+h)^2 - 4\pi(3)^2}{h}$$

$$= \lim_{h\to 0} \frac{h(24\pi + 4\pi h)}{h} = 24\pi \text{ km}^2\text{/day}$$

30. $V = \frac{4}{3}\pi r^3, r = \frac{1}{4}t$

$V = \frac{1}{48}\pi t^3$

$$\text{rate} = \frac{1}{48}\pi \lim_{h\to 0} \frac{(3+h)^3 - 3^3}{h} = \frac{27}{48}\pi$$

$$= \frac{9}{16}\pi \text{ inch}^3/\text{sec}$$

31. $y = f(x) = x^3 - 2x^2 + 1$

a. $m_{\tan} = 7$ b. $m_{\tan} = 0$

c. $m_{\tan} = -1$ d. $m_{\tan} = 17.92$

32. $y = f(x) = \sin x \sin^2 2x$

a. $m_{\tan} = -1.125$ b. $m_{\tan} \approx -1.0315$

c. $m_{\tan} = 0$ d. $m_{\tan} \approx 1.1891$

33. $s = f(t) = t + t\cos^2 t$

At $t = 3$, $v \approx 2.818$

34. $s = f(t) = \frac{(t+1)^3}{t+2}$

At $t = 1.6$, $v \approx 4.277$

3.2 Concepts Review

1. $\frac{f(c+h)-f(c)}{h}; \frac{f(t)-f(c)}{t-c}$

2. $\frac{f(x+h)-f(x)}{h}$

3. continuous; $f(x) = |x|$

4. $2x^2$; c

Problem Set 3.2

1. $f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$

$= \lim_{h\to 0} \frac{(1+h)^2 - 1^2}{h} = \lim_{h\to 0} \frac{2h + h^2}{h}$

$= \lim_{h\to 0} (2+h) = 2$

2. $f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$

$= \lim_{h\to 0} \frac{[2(2+h)]^2 - [2(2)]^2}{h}$

$= \lim_{h\to 0} \frac{16h + 4h^2}{h} = \lim_{h\to 0} (16 + 4h) = 16$

3. $f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h}$

$= \lim_{h\to 0} \frac{[(3+h)^2 - (3+h)] - (3^2 - 3)}{h}$

$= \lim_{h\to 0} \frac{5h + h^2}{h} = \lim_{h\to 0} (5+h) = 5$

4. $f'(4) = \lim_{h\to 0} \frac{f(4+h) - f(4)}{h}$

$= \lim_{h\to 0} \frac{\frac{1}{3+h} - \frac{1}{4-1}}{h} = \lim_{h\to 0} \frac{\frac{3-(3+h)}{3(3+h)}}{h} = \lim_{h\to 0} \frac{-1}{3(3+h)}$

$= -\frac{1}{9}$

5. $s'(x) = \lim_{h\to 0} \frac{s(x+h) - s(x)}{h}$

$= \lim_{h\to 0} \frac{[2(x+h)+1] - (2x+1)}{h}$

$= \lim_{h\to 0} \frac{2h}{h} = 2$

6. $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{[\alpha(x+h) + \beta] - (\alpha x + \beta)}{h}$

$= \lim_{h\to 0} \frac{\alpha h}{h} = \alpha$

7. $r'(x) = \lim_{h\to 0} \frac{r(x+h) - r(x)}{h}$

$= \lim_{h\to 0} \frac{[3(x+h)^2 + 4] - (3x^2 + 4)}{h}$

$= \lim_{h\to 0} \frac{6xh + 3h^2}{h} = \lim_{h\to 0} (6x + 3h) = 6x$

8. $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{[(x+h)^2 + (x+h) + 1] - (x^2 + x + 1)}{h}$

$= \lim_{h\to 0} \frac{2xh + h^2 + h}{h} = \lim_{h\to 0} (2x + h + 1) = 2x + 1$

9. $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{[a(x+h)^2 + b(x+h) + c] - (ax^2 + bx + c)}{h}$

$= \lim_{h\to 0} \frac{2axh + ah^2 + bh}{h} = \lim_{h\to 0} (2ax + ah + b)$

$= 2ax + b$

10. $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{(x+h)^4 - x^4}{h}$

$= \lim_{h\to 0} \frac{4hx^3 + 6h^2x^2 + 4h^3x + h^4}{h}$

$= \lim_{h\to 0} (4x^3 + 6hx^2 + 4h^2x + h^3) = 4x^3$

11. $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{[(x+h)^3 + 2(x+h)^2 + 1] - (x^3 + 2x^2 + 1)}{h}$

$= \lim_{h\to 0} \frac{3hx^2 + 3h^2x + h^3 + 4hx + 2h^2}{h}$

$= \lim_{h\to 0} (3x^2 + 3hx + h^2 + 4x + 2h) = 3x^2 + 4x$

12. $g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$

$= \lim_{h\to 0} \frac{[(x+h)^4 + (x+h)^2] - (x^4 + x^2)}{h}$

$= \lim_{h\to 0} \frac{4hx^3 + 6h^2x^2 + 4h^3x + h^4 + 2hx + h^2}{h}$

$= \lim_{h\to 0} (4x^3 + 6hx^2 + 4h^2x + h^3 + 2x + h)$

$= 4x^3 + 2x$

13. $h'(x)=\lim_{h\to 0}\frac{h(x+h)-h(x)}{h}$

$=\lim_{h\to 0}\left[\left(\frac{2}{x+h}-\frac{2}{x}\right)\cdot\frac{1}{h}\right]$

$=\lim_{h\to 0}\left[\frac{-2h}{x(x+h)}\cdot\frac{1}{h}\right]=\lim_{h\to 0}\frac{-2}{x(x+h)}$

$=-\frac{2}{x^2}$

14. $S'(x)=\lim_{h\to 0}\frac{S(x+h)-S(x)}{h}$

$=\lim_{h\to 0}\left[\left(\frac{1}{x+h+1}-\frac{1}{x+1}\right)\cdot\frac{1}{h}\right]$

$=\lim_{h\to 0}\left[\frac{-h}{(x+1)(x+h+1)}\cdot\frac{1}{h}\right]$

$=\lim_{h\to 0}\frac{-1}{(x+1)(x+h+1)}=-\frac{1}{(x+1)^2}$

15. $F'(x)=\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}$

$=\lim_{h\to 0}\left[\left(\frac{6}{(x+h)^2+1}-\frac{6}{x^2+1}\right)\cdot\frac{1}{h}\right]$

$=\lim_{h\to 0}\left[\frac{6(x^2+1)-6(x^2+2hx+h^2+1)}{(x^2+1)(x^2+2hx+h^2+1)}\cdot\frac{1}{h}\right]$

$=\lim_{h\to 0}\left[\frac{-12hx-6h^2}{(x^2+1)(x^2+2hx+h^2+1)}\cdot\frac{1}{h}\right]$

$=\lim_{h\to 0}\frac{-12x-6h}{(x^2+1)(x^2+2hx+h^2+1)}=-\frac{12x}{(x^2+1)^2}$

16. $F'(x)=\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}$

$=\lim_{h\to 0}\left[\left(\frac{x+h-1}{x+h+1}-\frac{x-1}{x+1}\right)\cdot\frac{1}{h}\right]$

$=\lim_{h\to 0}\left[\frac{x^2+hx+h-1-(x^2+hx-h-1)}{(x+h+1)(x+1)}\cdot\frac{1}{h}\right]$

$=\lim_{h\to 0}\left[\frac{2h}{(x+h+1)(x+1)}\cdot\frac{1}{h}\right]=\frac{2}{(x+1)^2}$

17. $G'(x)=\lim_{h\to 0}\frac{G(x+h)-G(x)}{h}$

$=\lim_{h\to 0}\left[\left(\frac{2(x+h)-1}{x+h-4}-\frac{2x-1}{x-4}\right)\cdot\frac{1}{h}\right]$

$=\lim_{h\to 0}\left[\frac{2x^2+2hx-9x-8h+4-(2x^2+2hx-9x-h+4)}{(x+h-4)(x-4)}\cdot\frac{1}{h}\right]=\lim_{h\to 0}\left[\frac{-7h}{(x+h-4)(x-4)}\cdot\frac{1}{h}\right]$

$=\lim_{h\to 0}\frac{-7}{(x+h-4)(x-4)}=-\frac{7}{(x-4)^2}$

18. $G'(x)=\lim_{h\to 0}\frac{G(x+h)-G(x)}{h}$

$=\lim_{h\to 0}\left[\left(\frac{2(x+h)}{(x+h)^2-(x+h)}-\frac{2x}{x^2-x}\right)\cdot\frac{1}{h}\right]=\lim_{h\to 0}\left[\frac{(2x+2h)(x^2-x)-2x(x^2+2xh+h^2-x-h)}{(x^2+2hx+h^2-x-h)(x^2-x)}\cdot\frac{1}{h}\right]$

$=\lim_{h\to 0}\left[\frac{-2h^2x-2hx^2}{(x^2+2hx+h^2-x-h)(x^2-x)}\cdot\frac{1}{h}\right]$

$=\lim_{h\to 0}\frac{-2hx-2x^2}{(x^2+2hx+h^2-x-h)(x^2-x)}$

$=\frac{-2x^2}{(x^2-x)^2}=-\frac{2}{(x-1)^2}$

19. $g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$

$= \lim_{h \to 0} \frac{\sqrt{3(x+h)} - \sqrt{3x}}{h}$

$= \lim_{h \to 0} \frac{(\sqrt{3x+3h} - \sqrt{3x})(\sqrt{3x+3h} + \sqrt{3x})}{h(\sqrt{3x+3h} + \sqrt{3x})}$

$= \lim_{h \to 0} \frac{3h}{h(\sqrt{3x+3h} + \sqrt{3x})} = \lim_{h \to 0} \frac{3}{\sqrt{3x+3h} + \sqrt{3x}} = \frac{3}{2\sqrt{3x}}$

20. $g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$

$= \lim_{h \to 0} \left[\left(\frac{1}{\sqrt{3(x+h)}} - \frac{1}{\sqrt{3x}} \right) \cdot \frac{1}{h} \right]$

$= \lim_{h \to 0} \left[\frac{\sqrt{3x} - \sqrt{3x+3h}}{\sqrt{9x(x+h)}} \cdot \frac{1}{h} \right]$

$= \lim_{h \to 0} \left[\frac{(\sqrt{3x} - \sqrt{3x+3h})(\sqrt{3x} + \sqrt{3x+3h})}{\sqrt{9x(x+h)}(\sqrt{3x} + \sqrt{3x+3h})} \cdot \frac{1}{h} \right]$

$= \lim_{h \to 0} \frac{-3h}{h\sqrt{9x(x+h)}(\sqrt{3x} + \sqrt{3x+3h})} = \frac{-3}{3x \cdot 2\sqrt{3x}} = -\frac{1}{2x\sqrt{3x}}$

21. $H'(x) = \lim_{h \to 0} \frac{H(x+h) - H(x)}{h}$

$= \lim_{h \to 0} \left[\left(\frac{3}{\sqrt{x+h-2}} - \frac{3}{\sqrt{x-2}} \right) \cdot \frac{1}{h} \right]$

$= \lim_{h \to 0} \left[\frac{3\sqrt{x-2} - 3\sqrt{x+h-2}}{\sqrt{(x+h-2)(x-2)}} \cdot \frac{1}{h} \right]$

$= \lim_{h \to 0} \frac{3(\sqrt{x-2} - \sqrt{x+h-2})(\sqrt{x-2} + \sqrt{x+h-2})}{h\sqrt{(x+h-2)(x-2)}(\sqrt{x-2} + \sqrt{x+h-2})}$

$= \lim_{h \to 0} \frac{-3h}{h[(x-2)\sqrt{x+h-2} + (x+h-2)\sqrt{x-2}]}$

$= \lim_{h \to 0} \frac{-3}{(x-2)\sqrt{x+h-2} + (x+h-2)\sqrt{x-2}}$

$= -\frac{3}{2(x-2)\sqrt{x-2}} = -\frac{3}{2(x-2)^{3/2}}$

22. $H'(x) = \lim_{h \to 0} \frac{H(x+h) - H(x)}{h}$

$= \lim_{h \to 0} \frac{\sqrt{(x+h)^2 + 4} - \sqrt{x^2 + 4}}{h}$

$= \lim_{h \to 0} \frac{\left(\sqrt{x^2 + 2hx + h^2 + 4} - \sqrt{x^2 + 4}\right)\left(\sqrt{x^2 + 2hx + h^2 + 4} + \sqrt{x^2 + 4}\right)}{h\left(\sqrt{x^2 + 2hx + h^2 + 4} + \sqrt{x^2 + 4}\right)}$

$$= \lim_{h\to 0} \frac{2hx+h^2}{h\left(\sqrt{x^2+2hx+h^2+4}+\sqrt{x^2+4}\right)}$$

$$= \lim_{h\to 0} \frac{2x+h}{\sqrt{x^2+2hx+h^2+4}+\sqrt{x^2+4}}$$

$$= \frac{2x}{2\sqrt{x^2+4}} = \frac{x}{\sqrt{x^2+4}}$$

23. $f'(x) = \lim_{t\to x} \frac{f(t)-f(x)}{t-x}$

$$= \lim_{t\to x} \frac{(t^2-3t)-(x^2-3x)}{t-x}$$

$$= \lim_{t\to x} \frac{t^2-x^2-(3t-3x)}{t-x}$$

$$= \lim_{t\to x} \frac{(t-x)(t+x)-3(t-x)}{t-x}$$

$$= \lim_{t\to x} \frac{(t-x)(t+x-3)}{t-x} = \lim_{t\to x}(t+x-3)$$

$$= 2x-3$$

24. $f'(x) = \lim_{t\to x} \frac{f(t)-f(x)}{t-x}$

$$= \lim_{t\to x} \frac{(t^3+5t)-(x^3+5x)}{t-x}$$

$$= \lim_{t\to x} \frac{t^3-x^3+5t-5x}{t-x}$$

$$= \lim_{t\to x} \frac{(t-x)(t^2+tx+x^2)+5(t-x)}{t-x}$$

$$= \lim_{t\to x} \frac{(t-x)(t^2+tx+x^2+5)}{t-x}$$

$$= \lim_{t\to x}(t^2+tx+x^2+5) = 3x^2+5$$

25. $f'(x) = \lim_{t\to x} \frac{f(t)-f(x)}{t-x}$

$$= \lim_{t\to x}\left[\left(\frac{t}{t-5}-\frac{x}{x-5}\right)\left(\frac{1}{t-x}\right)\right]$$

$$= \lim_{t\to x} \frac{tx-5t-tx+5x}{(t-5)(x-5)(t-x)}$$

$$= \lim_{t\to x} \frac{-5(t-x)}{(t-5)(x-5)(t-x)} = \lim_{t\to x} \frac{-5}{(t-5)(x-5)}$$

$$= -\frac{5}{(x-5)^2}$$

26. $f'(x) = \lim_{t\to x} \frac{f(t)-f(x)}{t-x}$

$$= \lim_{t\to x}\left[\left(\frac{t+3}{t}-\frac{x+3}{x}\right)\left(\frac{1}{t-x}\right)\right]$$

$$= \lim_{t\to x} \frac{3x-3t}{xt(t-x)} = \lim_{t\to x} \frac{-3}{xt} = -\frac{3}{x^2}$$

27. $f(x) = 2x^3$ at $x = 5$

28. $f(x) = x^2 + 2x$ at $x = 3$

29. $f(x) = x^2$ at $x = 2$

30. $f(x) = x^3 + x$ at $x = 3$

31. $f(x) = x^2$ at x

32. $f(x) = x^3$ at x

33. $f(t) = \frac{2}{t}$ at t

34. $f(y) = \sin y$ at y

35. $f(x) = \cos x$ at x

36. $f(t) = \tan t$ at t

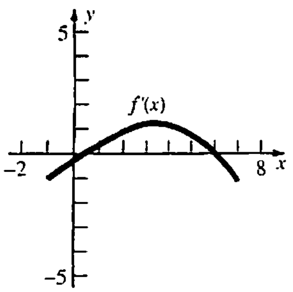
37. $f'(0) \approx -\frac{1}{2}$; $f'(2) \approx 1$

$f'(5) \approx \frac{2}{3}$; $f'(7) \approx -3$

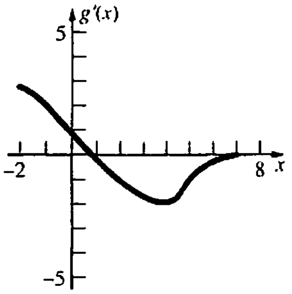
38. $g'(-1) \approx 2$; $g'(1) \approx 0$

$g'(4) \approx -2$; $g'(6) \approx -\frac{1}{3}$

39.



40.



41. a. $f(2) \approx \frac{5}{2}; f'(2) \approx \frac{3}{2}$

$f(0.5) \approx 1.8; f'(0.5) \approx -0.6$

b. $\frac{2.9 - 1.9}{2.5 - 0.5} = 0.5$

c. $x = 5$

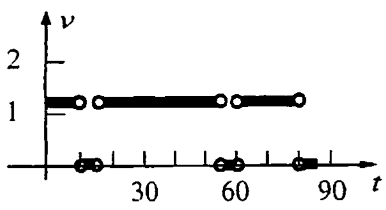
d. $x = 3, 5$

e. $x = 1, 3, 5$

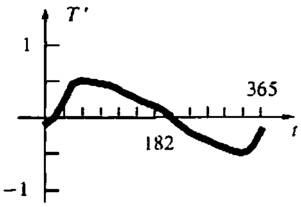
f. $x = 0$

g. $x \approx -0.7, \frac{3}{2}$ and $5 < x < 7$

42. The derivative fails to exist at the corners of the graph; that is, at $t = 10, 15, 55, 60, 80$. The derivative exists at all other points on the interval $(0,85)$.



43. The derivative is 0 at approximately $t = 15$ and $t = 201$. The greatest rate of increase occurs at about $t = 61$ and it is about 0.5 degree F per day. The greatest rate of decrease occurs at about $t = 320$ and it is about 0.5 degree F per day. The derivative is positive on (15,201) and negative on (0,15) and (201,365).



44. The slope of a tangent line for the dashed function is zero when x is approximately 0.3 or 1.9. The solid function is zero at both of these points. The graph indicates that the solid function is negative when the dashed function has a tangent line with a negative slope and positive when the dashed function has a tangent line with a positive slope. Thus, the solid function is the derivative of the dashed function.

45. The short-dash function has a tangent line with zero slope at about $x = 2.1$, where the solid function is zero. The solid function has a tangent line with zero slope at about $x = 0.4, 1.2$ and 3.5. The long-dash function is zero at these points. The graph shows that the solid function is positive (negative) when the slope of the tangent line of the short-dash function is positive (negative). Also, the long-dash function is positive (negative) when the slope of the tangent line of the solid function is positive (negative). Thus, the short-dash function is f, the solid function is $f' = g$, and the dash function is g'.

46. Note that since $x = 0 + x, f(x) = f(0 + x) = f(0)f(x)$, hence $f(0) = 1$.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{f(a)f(h) - f(a)}{h}$$

$$= f(a) \lim_{h \to 0} \frac{f(h) - 1}{h} = f(a) \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= f(a) f'(0)$$

$f'(a)$ exists since $f'(0)$ exists.

47. If f is differentiable everywhere, then it is continuous everywhere, so

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (mx + b) = 2m + b = f(2) = 4$

and $b = 4 - 2m$.

For f to be differentiable everywhere,

$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$ must exist.

$$\lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^+} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2^+} (x + 2) = 4$$

$$\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^-} \frac{mx + b - 4}{x - 2}$$

$= \lim_{x \to 2^-} \frac{mx + 4 - 2m - 4}{x - 2} = \lim_{x \to 2^-} \frac{m(x-2)}{x-2} = m$

Thus $m = 4$ and $b = 4 - 2(4) = -4$

48. $f_s(x) = \lim_{h \to 0} \frac{f(x+h) - f(x) + f(x) - f(x-h)}{2h}$

$= \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{2h} + \frac{f(x-h) - f(x)}{-2h} \right]$

$= \frac{1}{2} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \frac{1}{2} \lim_{-h \to 0} \frac{f[x+(-h)] - f(x)}{-h}$

$= \frac{1}{2} f'(x) + \frac{1}{2} f'(x) = f'(x).$

For the converse, let $f(x) = |x|$. Then

$f_s(0) = \lim_{h \to 0} \frac{|h| - |-h|}{2h} = \lim_{h \to 0} \frac{|h| - |h|}{2h} = 0$

but $f'(0)$ does not exist.

49. $f'(x_0) = \lim_{t \to x_0} \frac{f(t) - f(x_0)}{t - x_0}$, so

$f'(-x_0) = \lim_{t \to -x_0} \frac{f(t) - f(-x_0)}{t - (-x_0)}$

$= \lim_{t \to -x_0} \frac{f(t) - f(-x_0)}{t + x_0}$

a. If f is an odd function,

$f'(-x_0) = \lim_{t \to -x_0} \frac{f(t) - [-f(-x_0)]}{t + x_0}$

$= \lim_{t \to -x_0} \frac{f(t) + f(-x_0)}{t + x_0}.$

Let $u = -t$. As $t \to -x_0$, $u \to x_0$ and so

$f'(-x_0) = \lim_{u \to x_0} \frac{f(-u) + f(x_0)}{-u + x_0}$

$= \lim_{u \to x_0} \frac{-f(u) + f(x_0)}{-(u - x_0)} = \lim_{u \to x_0} \frac{-[f(u) - f(x_0)]}{-(u - x_0)}$

$= \lim_{u \to x_0} \frac{f(u) - f(x_0)}{u - x_0} = f'(x_0) = m.$

b. If f is an even function,

$f'(-x_0) = \lim_{t \to -x_0} \frac{f(t) - f(x_0)}{t + x_0}$. Let $u = -t$, as

above, then $f'(-x_0) = \lim_{u \to x_0} \frac{f(-u) - f(x_0)}{-u + x_0}$

$= \lim_{u \to x_0} \frac{f(u) - f(x_0)}{-(u - x_0)} = -\lim_{u \to x_0} \frac{f(u) - f(x_0)}{u - x_0}$

$= -f'(x_0) = -m.$

50. Say $f(-x) = -f(x)$. Then

$f'(-x) = \lim_{h \to 0} \frac{f(-x+h) - f(-x)}{h}$

$= \lim_{h \to 0} \frac{-f(x-h) + f(x)}{h} = -\lim_{h \to 0} \frac{f(x-h) - f(x)}{h}$

$= \lim_{-h \to 0} \frac{f[x+(-h)] - f(x)}{-h} = f'(x)$ so $f'(x)$ is an even function if $f(x)$ is an odd function.

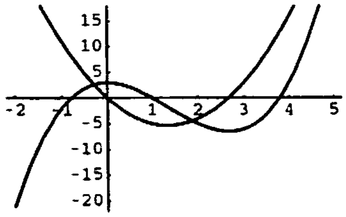
Say $f(-x) = f(x)$. Then

$f'(-x) = \lim_{h \to 0} \frac{f(-x+h) - f(-x)}{h}$

$= \lim_{h \to 0} \frac{f(x-h) - f(x)}{h}$

$= -\lim_{-h \to 0} \frac{f[x+(-h)] - f(x)}{-h} = -f'(x)$ so $f'(x)$ is an odd function if $f(x)$ is an even function.

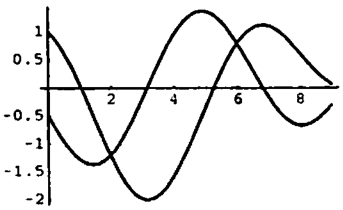
51.



a. $0 < x < \frac{8}{3}$ **b.** $0 < x < \frac{8}{3}$

c. A function $f(x)$ decreases as x increases when $f'(x) < 0$.

52.



a. $\pi < x < 6.8$ **b.** $\pi < x < 6.8$

c. A function $f(x)$ increases as x increases when $f'(x) > 0$.

3.3 Concepts Review

1. the derivative of the second; second; $f(x)g'(x)+g(x)f'(x)$

2. denominator; denominator; square of the denominator; $\frac{g(x)f'(x)-f(x)g'(x)}{g^2(x)}$

3. $nx^{n-1}h$; nx^{n-1}

4. $kL(f)$; $L(f)+L(g)$; D_x

Problem Set 3.3

1. $D_x(2x^2)=2D_x(x^2)=2\cdot 2x=4x$

2. $D_x(3x^3)=3D_x(x^3)=3\cdot 3x^2=9x^2$

3. $D_x(\pi x)=\pi D_x(x)=\pi\cdot 1=\pi$

4. $D_x(\pi x^3)=\pi D_x(x^3)=\pi\cdot 3x^2=3\pi x^2$

5. $D_x(2x^{-2})=2D_x(x^{-2})=2(-2x^{-3})=-4x^{-3}$

6. $D_x(-3x^{-4})=-3D_x(x^{-4})=-3(-4x^{-5})=12x^{-5}$

7. $D_x\left(\frac{\pi}{x}\right)=\pi D_x(x^{-1})=\pi(-1x^{-2})=-\pi x^{-2}$
$=-\frac{\pi}{x^2}$

8. $D_x\left(\frac{\alpha}{x^3}\right)=\alpha D_x(x^{-3})=\alpha(-3x^{-4})=-3\alpha x^{-4}$
$=-\frac{3\alpha}{x^4}$

9. $D_x\left(\frac{100}{x^5}\right)=100D_x(x^{-5})=100(-5x^{-6})$
$=-500x^{-6}=-\frac{500}{x^6}$

10. $D_x\left(\frac{3\alpha}{4x^5}\right)=\frac{3\alpha}{4}D_x(x^{-5})=\frac{3\alpha}{4}(-5x^{-6})$
$=-\frac{15\alpha}{4}x^{-6}=-\frac{15\alpha}{4x^6}$

11. $D_x(x^2+2x)=D_x(x^2)+2D_x(x)=2x+2$

12. $D_x(3x^4+x^3)=3D_x(x^4)+D_x(x^3)$
$=3(4x^3)+3x^2=12x^3+3x^2$

13. $D_x(x^4+x^3+x^2+x+1)$
$=D_x(x^4)+D_x(x^3)+D_x(x^2)+D_x(x)+D_x(1)$
$=4x^3+3x^2+2x+1$

14. $D_x(3x^4-2x^3-5x^2+\pi x+\pi^2)$
$=3D_x(x^4)-2D_x(x^3)-5D_x(x^2)$
$+\pi D_x(x)+D_x(\pi^2)$
$=3(4x^3)-2(3x^2)-5(2x)+\pi(1)+0$
$=12x^3-6x^2-10x+\pi$

15. $D_x(\pi x^7-2x^5-5x^{-2})$
$=\pi D_x(x^7)-2D_x(x^5)-5D_x(x^{-2})$
$=\pi(7x^6)-2(5x^4)-5(-2x^{-3})$
$=7\pi x^6-10x^4+10x^{-3}$

16. $D_x(x^{12}+5x^{-2}-\pi x^{-10})$
$=D_x(x^{12})+5D_x(x^{-2})-\pi D_x(x^{-10})$
$=12x^{11}+5(-2x^{-3})-\pi(-10x^{-11})$
$=12x^{11}-10x^{-3}+10\pi x^{-11}$

17. $D_x\left(\frac{3}{x^3}+x^{-4}\right)=3D_x(x^{-3})+D_x(x^{-4})$
$=3(-3x^{-4})+(-4x^{-5})=-\frac{9}{x^4}-4x^{-5}$

18. $D_x(2x^{-6}+x^{-1})=2D_x(x^{-6})+D_x(x^{-1})$
$=2(-6x^{-7})+(-1x^{-2})=-12x^{-7}-x^{-2}$

19. $D_x\left(\frac{2}{x}-\frac{1}{x^2}\right)=2D_x(x^{-1})-D_x(x^{-2})$
$=2(-1x^{-2})-(-2x^{-3})=-\frac{2}{x^2}+\frac{2}{x^3}$

20. $D_x\left(\frac{3}{x^3}-\frac{1}{x^4}\right)=3D_x(x^{-3})-D_x(x^{-4})$
$=3(-3x^{-4})-(-4x^{-5})=-\frac{9}{x^4}+\frac{4}{x^5}$

21. $D_x\left(\frac{1}{2x}+2x\right)=\frac{1}{2}D_x(x^{-1})+2D_x(x)$
$=\frac{1}{2}(-1x^{-2})+2(1)=-\frac{1}{2x^2}+2$

22. $D_x\left(\frac{2}{3x}-\frac{2}{3}\right)=\frac{2}{3}D_x(x^{-1})-D_x\left(\frac{2}{3}\right)$

$=\frac{2}{3}(-1x^{-2})-0=-\frac{2}{3x^2}$

23. $D_x[x(x^2+1)]=x\,D_x(x^2+1)+(x^2+1)D_x(x)$

$=x(2x)+(x^2+1)(1)=3x^2+1$

24. $D_x[3x(x^3-1)]=3x\,D_x(x^3-1)+(x^3-1)D_x(3x)$

$=3x(3x^2)+(x^3-1)(3)=12x^3-3$

25. $D_x[(2x+1)^2]$

$=(2x+1)D_x(2x+1)+(2x+1)D_x(2x+1)$

$=(2x+1)(2)+(2x+1)(2)=8x+4$

26. $D_x[(-3x+2)^2]$

$=(-3x+2)D_x(-3x+2)+(-3x+2)D_x(-3x+2)$

$=(-3x+2)(-3)+(-3x+2)(-3)=18x-12$

27. $D_x[(x^2+2)(x^3+1)]$

$=(x^2+2)D_x(x^3+1)+(x^3+1)D_x(x^2+2)$

$=(x^2+2)(3x^2)+(x^3+1)(2x)$

$=3x^4+6x^2+2x^4+2x$

$=5x^4+6x^2+2x$

28. $D_x[(x^4-1)(x^2+1)]$

$=(x^4-1)D_x(x^2+1)+(x^2+1)D_x(x^4-1)$

$=(x^4-1)(2x)+(x^2+1)(4x^3)$

$=2x^5-2x+4x^5+4x^3=6x^5+4x^3-2x$

29. $D_x[(x^2+17)(x^3-3x+1)]$

$=(x^2+17)D_x(x^3-3x+1)+(x^3-3x+1)D_x(x^2+17)$

$=(x^2+17)(3x^2-3)+(x^3-3x+1)(2x)$

$=3x^4+48x^2-51+2x^4-6x^2+2x$

$=5x^4+42x^2+2x-51$

30. $D_x[(x^4+2x)(x^3+2x^2+1)]=(x^4+2x)D_x(x^3+2x^2+1)+(x^3+2x^2+1)D_x(x^4+2x)$

$=(x^4+2x)(3x^2+4x)+(x^3+2x^2+1)(4x^3+2)$

$=7x^6+12x^5+12x^3+12x^2+2$

31. $D_x[(5x^2-7)(3x^2-2x+1)]=(5x^2-7)D_x(3x^2-2x+1)+(3x^2-2x+1)D_x(5x^2-7)$

$=(5x^2-7)(6x-2)+(3x^2-2x+1)(10x)$

$=60x^3-30x^2-32x+14$

32. $D_x[(3x^2+2x)(x^4-3x+1)]=(3x^2+2x)D_x(x^4-3x+1)+(x^4-3x+1)D_x(3x^2+2x)$

$=(3x^2+2x)(4x^3-3)+(x^4-3x+1)(6x+2)$

$=18x^5+10x^4-27x^2-6x+2$

33. $D_x\left(\frac{1}{3x^2+1}\right)=\frac{(3x^2+1)D_x(1)-(1)D_x(3x^2+1)}{(3x^2+1)^2}$

$=\frac{(3x^2+1)(0)-(6x)}{(3x^2+1)^2}=-\frac{6x}{(3x^2+1)^2}$

34. $D_x\left(\frac{2}{5x^2-1}\right)=\frac{(5x^2-1)D_x(2)-(2)D_x(5x^2-1)}{(5x^2-1)^2}$

$=\frac{(5x^2-1)(0)-2(10x)}{(5x^2-1)^2}=-\frac{20x}{(5x^2-1)^2}$

35. $D_x\left(\frac{1}{4x^2-3x+9}\right)=\frac{(4x^2-3x+9)D_x(1)-(1)D_x(4x^2-3x+9)}{(4x^2-3x+9)^2}$

$=\frac{(4x^2-3x+9)(0)-(8x-3)}{(4x^2-3x+9)^2}=-\frac{8x-3}{(4x^2-3x+9)^2}$

$=\frac{-8x+3}{(4x^2-3x+9)^2}$

36. $D_x\left(\frac{4}{2x^3-3x}\right)=\frac{(2x^3-3x)D_x(4)-(4)D_x(2x^3-3x)}{(2x^3-3x)^2}$

$=\frac{(2x^3-3x)(0)-4(6x^2-3)}{(2x^3-3x)^2}=\frac{-24x^2+12}{(2x^3-3x)^2}$

37. $D_x\left(\frac{x-1}{x+1}\right)=\frac{(x+1)D_x(x-1)-(x-1)D_x(x+1)}{(x+1)^2}$

$=\frac{(x+1)(1)-(x-1)(1)}{(x+1)^2}=\frac{2}{(x+1)^2}$

38. $D_x\left(\frac{2x-1}{x-1}\right)=\frac{(x-1)D_x(2x-1)-(2x-1)D_x(x-1)}{(x-1)^2}$

$=\frac{(x-1)(2)-(2x-1)(1)}{(x-1)^2}=-\frac{1}{(x-1)^2}$

39. $D_x\left(\frac{2x^2-1}{3x+5}\right)=\frac{(3x+5)D_x(2x^2-1)-(2x^2-1)D_x(3x+5)}{(3x+5)^2}$

$=\frac{(3x+5)(4x)-(2x^2-1)(3)}{(3x+5)^2}$

$=\frac{6x^2+20x+3}{(3x+5)^2}$

40. $D_x\left(\frac{5x-4}{3x^2+1}\right)=\frac{(3x^2+1)D_x(5x-4)-(5x-4)D_x(3x^2+1)}{(3x^2+1)^2}$

$=\frac{(3x^2+1)(5)-(5x-4)(6x)}{(3x^2+1)^2}$

$=\frac{-15x^2+24x+5}{(3x^2+1)^2}$

41. $D_x\left(\frac{2x^2-3x+1}{2x+1}\right)=\frac{(2x+1)D_x(2x^2-3x+1)-(2x^2-3x+1)D_x(2x+1)}{(2x+1)^2}$

$=\frac{(2x+1)(4x-3)-(2x^2-3x+1)(2)}{(2x+1)^2}$

$=\frac{4x^2+4x-5}{(2x+1)^2}$

42. $D_x\left(\frac{5x^2+2x-6}{3x-1}\right)=\frac{(3x-1)D_x(5x^2+2x-6)-(5x^2+2x-6)D_x(3x-1)}{(3x-1)^2}$

$=\frac{(3x-1)(10x+2)-(5x^2+2x-6)(3)}{(3x-1)^2}$

$=\frac{15x^2-10x+16}{(3x-1)^2}$

43. $D_x\left(\frac{x^2-x+1}{x^2+1}\right)=\frac{(x^2+1)D_x(x^2-x+1)-(x^2-x+1)D_x(x^2+1)}{(x^2+1)^2}$

$=\frac{(x^2+1)(2x-1)-(x^2-x+1)(2x)}{(x^2+1)^2}$

$=\frac{x^2-1}{(x^2+1)^2}$

44. $D_x\left(\frac{x^2-2x+5}{x^2+2x-3}\right)=\frac{(x^2+2x-3)D_x(x^2-2x+5)-(x^2-2x+5)D_x(x^2+2x-3)}{(x^2+2x-3)^2}$

$=\frac{(x^2+2x-3)(2x-2)-(x^2-2x+5)(2x+2)}{(x^2+2x-3)^2}$

$=\frac{4x^2-16x-4}{(x^2+2x-3)^2}$

45. a. $(f\cdot g)'(0)=f(0)g'(0)+g(0)f'(0)$
$=4(5)+(-3)(-1)=23$

b. $(f+g)'(0)=f'(0)+g'(0)=-1+5=4$

c. $(f/g)'(0)=\frac{g(0)f'(0)-f(0)g'(0)}{g^2(0)}$
$=\frac{-3(-1)-4(5)}{(-3)^2}=-\frac{17}{9}$

46. a. $(f-g)'(3)=f'(3)-g'(3)=2-(-10)=12$

b. $(f\cdot g)'(3)=f(3)g'(3)+g(3)f'(3)$
$=7(-10)+6(2)=-58$

c. $(g/f)'(3)=\frac{f(3)g'(3)-g(3)f'(3)}{f^2(3)}$
$=\frac{7(-10)-6(2)}{(7)^2}=-\frac{82}{49}$

47. $D_x[f(x)]^2 = D_x[f(x)f(x)]$
$= f(x)D_x[f(x)] + f(x)D_x[f(x)]$
$= 2 \cdot f(x) \cdot D_x f(x)$

48. $D_x[f(x)g(x)h(x)] = f(x)D_x[g(x)h(x)] + g(x)h(x)D_x f(x)$
$= f(x)[g(x)D_x h(x) + h(x)D_x g(x)] + g(x)h(x)D_x f(x)$
$= f(x)g(x)D_x h(x) + f(x)h(x)D_x g(x) + g(x)h(x)D_x f(x)$

49. $D_x(x^2 - 2x + 2) = 2x - 2$
At $x = 1$: $m_{\tan} = 2(1) - 2 = 0$
Tangent line: $y = 1$

50. $D_x\left(\frac{1}{x^2+4}\right) = \frac{(x^2+4)D_x(1) - (1)D_x(x^2+4)}{(x^2+4)^2}$
$= \frac{(x^2+4)(0) - (2x)}{(x^2+4)^2} = -\frac{2x}{(x^2+4)^2}$
At $x = 1$: $m_{\tan} = -\frac{2(1)}{(1^2+4)^2} = -\frac{2}{25}$
Tangent line: $y - \frac{1}{5} = -\frac{2}{25}(x-1)$
$y = -\frac{2}{25}x + \frac{7}{25}$

51. $D_x(x^3 - x^2) = 3x^2 - 2x$
The tangent line is horizontal when $m_{\tan} = 0$:
$m_{\tan} = 3x^2 - 2x = 0$
$x(3x - 2) = 0$
$x = 0$ and $x = \frac{2}{3}$
(0, 0) and $\left(\frac{2}{3}, -\frac{4}{27}\right)$

52. $D_x\left(\frac{1}{3}x^3 + x^2 - x\right) = x^2 + 2x - 1$
$m_{\tan} = x^2 + 2x - 1 = 1$
$x^2 + 2x - 2 = 0$
$x = \frac{-2 \pm \sqrt{4 - 4(1)(-2)}}{2} = \frac{-2 \pm \sqrt{12}}{2}$
$= -1 - \sqrt{3}, -1 + \sqrt{3}$
$x = -1 \pm \sqrt{3}$
$\left(-1 + \sqrt{3}, \frac{5}{3} - \sqrt{3}\right), \left(-1 - \sqrt{3}, \frac{5}{3} + \sqrt{3}\right)$

53. $y = 100/x^5 = 100x^{-5}$
$y' = -500x^{-6}$

Set y' equal to -1, the negative reciprocal of the slope of the line $y = x$. Solving for x gives
$x = \pm 500^{1/6} \approx \pm 2.817$
$y = \pm 100(500)^{-5/6} \approx \pm 0.563$

The points are (2.817, 0.563) and (–2.817, –0.563).

54. Proof #1:

$D_x\left[f(x) - g(x)\right] = D_x\left[f(x) + (-1)g(x)\right]$
$= D_x\left[f(x)\right] + D_x\left[(-1)g(x)\right]$
$= D_x f(x) - D_x g(x)$

Proof #2:

Let $F(x) = f(x) - g(x)$. Then

$F'(x) = \lim_{h \to 0} \frac{\left[f(x+h) - g(x+h)\right] - \left[f(x) - g(x)\right]}{h}$
$= \lim_{h \to 0}\left[\frac{f(x+h) - f(x)}{h} - \frac{g(x+h) - g(x)}{h}\right]$
$= f'(x) - g'(x)$

55. a. $D_t(-16t^2 + 40t + 100) = -32t + 40$
$v = -32(2) + 40 = -24$ ft/s

b. $v = -32t + 40 = 0$
$t = \frac{5}{4}$ s

56. $D_t(4.5t^2 + 2t) = 9t + 2$
$9t + 2 = 30$
$t = \frac{28}{9}$ s

57. $m_{\tan} = D_x(4x - x^2) = 4 - 2x$

The line through (2,5) and (x_0, y_0) has slope $\frac{y_0 - 5}{x_0 - 2}$.

$4 - 2x_0 = \frac{4x_0 - x_0^2 - 5}{x_0 - 2}$

$-2x_0^2 + 8x_0 - 8 = -x_0^2 + 4x_0 - 5$

$x_0^2 - 4x_0 + 3 = 0$

$(x_0 - 3)(x_0 - 1) = 0$

$x_0 = 1,\ x_0 = 3$

At $x_0 = 1$: $y_0 = 4(1) - (1)^2 = 3$

$m_{\tan} = 4 - 2(1) = 2$

Tangent line: $y - 3 = 2(x - 1)$; $y = 2x + 1$

At $x_0 = 3 : y_0 = 4(3) - (3)^2 = 3$

$m_{\tan} = 4 - 2(3) = -2$

Tangent line: $y - 3 = -2(x - 3)$; $y = -2x + 9$

58. $D_x(x^2) = 2x$

The line through (4, 15) and (x_0, y_0) has slope $\frac{y_0 - 15}{x_0 - 4}$. If (x_0, y_0) is on the curve $y = x^2$, then

$m_{\tan} = 2x_0 = \frac{x_0^2 - 15}{x_0 - 4}$.

$2x_0^2 - 8x_0 = x_0^2 - 15$

$x_0^2 - 8x_0 + 15 = 0$

$(x_0 - 3)(x_0 - 5) = 0$

At $x_0 = 3 : y_0 = (3)^2 = 9$

She should shut off the engines at (3, 9). (At $x_0 = 5$ she would not go to (4, 15) since she is moving left to right.)

59. $D_x(7 - x^2) = -2x$

The line through (4, 0) and (x_0, y_0) has slope $\frac{y_0 - 0}{x_0 - 4}$. If the fly is at (x_0, y_0) when the spider sees it, then $m_{\tan} = -2x_0 = \frac{7 - x_0^2 - 0}{x_0 - 4}$.

$-2x_0^2 + 8x_0 = 7 - x_0^2$

$x_0^2 - 8x_0 + 7 = 0$

$(x_0 - 7)(x_0 - 1) = 0$

At $x_0 = 1 : y_0 = 6$

$d = \sqrt{(4-1)^2 + (0-6)^2} = \sqrt{9+36} = \sqrt{45} = 3\sqrt{5}$

≈ 6.7

They are 6.7 units apart when they see each other.

60. $P(a, b)$ is $\left(a, \frac{1}{a}\right)$. $D_x y = -\frac{1}{x^2}$ so the slope of the tangent line at P is $-\frac{1}{a^2}$. The tangent line is $y - \frac{1}{a} = -\frac{1}{a^2}(x - a)$ or $y = -\frac{1}{a^2}(x - 2a)$ which has x-intercept $(2a, 0)$.

$d(O,P) = \sqrt{a^2 + \frac{1}{a^2}}, d(P, A) = \sqrt{(a - 2a)^2 + \frac{1}{a^2}}$

$= \sqrt{a^2 + \frac{1}{a^2}} = d(O, P)$ so AOP is an isosceles triangle. The height of AOP is a while the base, $\overline{OA}$ has length $2a$, so the area is $\frac{1}{2}(2a)(a) = a^2$.

61. The watermelon has volume $\frac{4}{3}\pi r^3$; the volume of the rind is

$V = \frac{4}{3}\pi r^3 - \frac{4}{3}\pi\left(r - \frac{r}{10}\right)^3 = \frac{271}{750}\pi r^3$.

At the end of the fifth week $r = 10$, so

$D_r V = \frac{271}{250}\pi r^2 = \frac{271}{250}\pi(10)^2 = \frac{542\pi}{5} \approx 340 \text{ cm}^3$

per cm of radius growth. Since the radius is growing 2 cm per week, the volume of the rind is growing at the rate of $\frac{542\pi}{5}(2) \approx 681 \text{ cm}^3$ per week.

3.4 Concepts Review

1. $\frac{\sin(x+h) - \sin(x)}{h}$

2. 0; 1

3. $\cos x$; $-\sin x$

4. $\cos\frac{\pi}{3} = \frac{1}{2}; y - \frac{\sqrt{3}}{2} = \frac{1}{2}\left(x - \frac{\pi}{3}\right)$

Problem Set 3.4

1. $D_x(2\sin x + 3\cos x) = 2\,D_x(\sin x) + 3\,D_x(\cos x)$
$= 2\cos x - 3\sin x$

2. $D_x(\sin^2 x) = \sin x\,D_x(\sin x) + \sin x\,D_x(\sin x)$
$= \sin x\cos x + \sin x\cos x = 2\sin x\cos x = \sin 2x$

3. $D_x(\sin^2 x + \cos^2 x) = D_x(1) = 0$

4. $D_x(1-\cos^2 x) = D_x(\sin^2 x)$
$= \sin x\,D_x(\sin x) + \sin x\,D_x(\sin x)$
$= \sin x\cos x + \sin x\cos x$
$= 2\sin x\cos x = \sin 2x$

5. $D_x(\sec x) = D_x\left(\dfrac{1}{\cos x}\right)$
$= \dfrac{\cos x\,D_x(1) - (1)D_x(\cos x)}{\cos^2 x}$
$= \dfrac{\sin x}{\cos^2 x} = \dfrac{1}{\cos x}\cdot\dfrac{\sin x}{\cos x} = \sec x\tan x$

6. $D_x(\csc x) = D_x\left(\dfrac{1}{\sin x}\right)$
$= \dfrac{\sin x\,D_x(1) - (1)D_x(\sin x)}{\sin^2 x}$
$= \dfrac{-\cos x}{\sin^2 x} = \dfrac{-1}{\sin x}\cdot\dfrac{\cos x}{\sin x} = -\csc x\cot x$

7. $D_x(\tan x) = D_x\left(\dfrac{\sin x}{\cos x}\right)$
$= \dfrac{\cos x\,D_x(\sin x) - \sin x\,D_x(\cos x)}{\cos^2 x}$
$= \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x} = \sec^2 x$

8. $D_x(\cot x) = D_x\left(\dfrac{\cos x}{\sin x}\right)$
$= \dfrac{\sin x\,D_x(\cos x) - \cos x\,D_x(\sin x)}{\sin^2 x}$
$= \dfrac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \dfrac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$
$= -\dfrac{1}{\sin^2 x} = -\csc^2 x$

9. $D_x\left(\dfrac{\sin x + \cos x}{\cos x}\right)$
$= \dfrac{\cos x\,D_x(\sin x + \cos x) - (\sin x + \cos x)D_x(\cos x)}{\cos^2 x}$
$= \dfrac{\cos x(\cos x - \sin x) - (-\sin^2 x - \sin x\cos x)}{\cos^2 x}$
$= \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x} = \sec^2 x$

10. $D_x\left(\dfrac{\sin x + \cos x}{\tan x}\right)$
$= \dfrac{\tan x\,D_x(\sin x + \cos x) - (\sin x + \cos x)D_x(\tan x)}{\tan^2 x}$
$= \dfrac{\tan x(\cos x - \sin x) - \sec^2 x(\sin x + \cos x)}{\tan^2 x}$
$= \left(\sin x - \dfrac{\sin^2 x}{\cos x} - \dfrac{\sin x}{\cos^2 x} - \dfrac{1}{\cos x}\right) \div \left(\dfrac{\sin^2 x}{\cos^2 x}\right)$
$= \left(\sin x - \dfrac{\sin^2 x}{\cos x} - \dfrac{\sin x}{\cos^2 x} - \dfrac{1}{\cos x}\right)\left(\dfrac{\cos^2 x}{\sin^2 x}\right)$
$= \dfrac{\cos^2 x}{\sin x} - \cos x - \dfrac{1}{\sin x} - \dfrac{\cos x}{\sin^2 x}$

11. $D_x(x^2\cos x) = x^2 D_x(\cos x) + \cos x\,D_x(x^2)$
$= -x^2\sin x + 2x\cos x$

12. $D_x\left(\dfrac{x\cos x + \sin x}{x^2+1}\right)$
$= \dfrac{(x^2+1)D_x(x\cos x + \sin x) - (x\cos x + \sin x)D_x(x^2+1)}{(x^2+1)^2}$
$= \dfrac{(x^2+1)(-x\sin x + \cos x + \cos x) - 2x(x\cos x + \sin x)}{(x^2+1)^2} = \dfrac{-x^3\sin x - 3x\sin x + 2\cos x}{(x^2+1)^2}$

13. $y = \tan^2 x = (\tan x)(\tan x)$
$y' = (\tan x)(\sec^2 x) + (\tan x)(\sec^2 x)$
$= 2\tan x\sec^2 x$

14. $y=\sec^3 x=(\sec^2 x)(\sec x)$

$y' = (\sec^2 x)\sec x\tan x + (\sec x)(\sec^2 x)'$

$= \sec^3 x\tan x + \sec x(\sec x\sec x\tan x$

$+\sec x\sec x\tan x)$

$= \sec^3 x\tan x + 2\sec^3 x\tan x$

$= 3\sec^2 x\tan x$

15. $D_x(\cos x) = -\sin x$

At $x = 1$: $m_{\tan} = -\sin 1 \approx -0.8415$

$y = \cos 1 \approx 0.5403$

Tangent line: $y - 0.5403 = -0.8415(x - 1)$

16. $D_x(\cot x) = -\csc^2 x$

At $x = \frac{\pi}{4}$: $m_{\tan} = -2$;

$y = 1$

Tangent line: $y-1=-2\left(x-\frac{\pi}{4}\right)$

17. $D_t(30\cos 2t) = 30D_t(\cos^2 t - \sin^2 t)$

$= 30[\cos t\, D_t(\cos t) + \cos t\, D_t(\cos t) - D_t(\sin^2 t)]$

$= 30[-2\sin t\cos t - \sin t D_t(\sin t) - \sin t D_t(\sin t)]$

$= 30[-2\sin t\cos t - 2\sin t\cos t]$

$= -60\sin 2t$

At $t=\frac{\pi}{4}$; $-60\sin\left(2\cdot\frac{\pi}{4}\right) = -60$ ft/s

The seat is moving to the left at the rate of 60 ft/s.

18. The coordinates of the seat at time t are $(20\cos t, 20\sin t)$.

a. $\left(20\cos\frac{\pi}{6}, 20\sin\frac{\pi}{6}\right) = (10\sqrt{3}, 10)$

$\approx (17.32,\ 10)$

b. $D_t(20\sin t) = 20\cos t$

At $t=\frac{\pi}{6}$: rate $= 20\cos\frac{\pi}{6} = 10\sqrt{3}$

≈ 17.32 ft/s

c. The fastest rate $20\cos t$ can obtain is 20 ft/s.

19. $y = \tan x$

$y' = \sec^2 x$

When $y=0$, $y=\tan 0 = 0$ and $y' = \sec^2 0 = 1$.

The tangent line at $x = 0$ is $y = x$.

20. $y = \tan^2 x = (\tan x)(\tan x)$

$y' = (\tan x)(\sec^2 x) + (\tan x)(\sec^2 x)$

$= 2\tan x\sec^2 x$

Now, $\sec^2 x$ is never 0, but $\tan x = 0$ at $x = k\pi$ where k is an integer.

21. $y = 9\sin x\cos x$

$y' = 9[\sin x(-\sin x) + \cos x(\cos x)]$

$= 9\left[\sin^2 x - \cos^2 x\right]$

$= 9[-\cos 2x]$

The tangent line is horizontal when $y' = 0$ or, in this case, where $\cos 2x = 0$. This occurs when $x = \frac{\pi}{4} + k\frac{\pi}{2}$ where k is an integer.

22. $f(x) = x - \sin x$

$f'(x) = 1 - \cos x$

$f'(x) = 0$ when $\cos x = 1$; i.e. when $x = 2k\pi$ where k is an integer.

$f'(x) = 2$ when $x = (2k+1)\pi$ where k is an integer.

23. The curves intersect when $\sqrt{2}\sin x = \sqrt{2}\cos x$, $\sin x = \cos x$ at $x = \frac{\pi}{4}$ for $0 < x < \frac{\pi}{2}$.

$D_x(\sqrt{2}\sin x) = \sqrt{2}\cos x$; $\sqrt{2}\cos\frac{\pi}{4} = 1$

$D_x(\sqrt{2}\cos x) = -\sqrt{2}\sin x$; $-\sqrt{2}\sin\frac{\pi}{4} = -1$

$1(-1) = -1$ so the curves intersect at right angles.

24. $v = D_t(2\sin t) = 2\cos t$

At $t = 0$: $v = 2$ cm/s

$t = \frac{\pi}{2}$: $v = 0$ cm/s

$t=\pi$: $v = -2$ cm/s

25. $D_x(\sin x^2) = \lim_{h\to 0} \frac{\sin(x+h)^2 - \sin x^2}{h}$

$= \lim_{h\to 0} \frac{\sin(x^2 + 2xh + h^2) - \sin x^2}{h}$

$= \lim_{h\to 0} \frac{\sin x^2 \cos(2xh+h^2) + \cos x^2 \sin(2xh+h^2) - \sin x^2}{h} = \lim_{h\to 0} \frac{\sin x^2[\cos(2xh+h^2)-1] + \cos x^2 \sin(2xh+h^2)}{h}$

$= \lim_{h\to 0}(2x+h)\left[\sin x^2 \frac{\cos(2xh+h^2)-1}{2xh+h^2} + \cos x^2 \frac{\sin(2xh+h^2)}{2xh+h^2}\right] = 2x(\sin x^2 \cdot 0 + \cos x^2 \cdot 1) = 2x\cos x^2$

26. $D_x(\sin 5x) = \lim_{h\to 0} \frac{\sin(5(x+h)) - \sin 5x}{h}$

$= \lim_{h\to 0} \frac{\sin(5x+5h) - \sin 5x}{h}$

$= \lim_{h\to 0} \frac{\sin 5x \cos 5h + \cos 5x \sin 5h - \sin 5x}{h}$

$= \lim_{h\to 0}\left[\sin 5x \frac{\cos 5h - 1}{h} + \cos 5x \frac{\sin 5h}{h}\right]$

$= \lim_{h\to 0}\left[5\sin 5x \frac{\cos 5h - 1}{5h} + 5\cos 5x \frac{\sin 5h}{5h}\right]$

$= 0 + 5\cos 5x \cdot 1 = 5\cos 5x$

27. $\sin x_0 = \sin 2x_0$

$\sin x_0 = 2\sin x_0 \cos x_0$

$\cos x_0 = \frac{1}{2}$ [if $\sin x_0 \neq 0$]

$x_0 = \frac{\pi}{3}$

$D_x(\sin x) = \cos x$, $D_x(\sin 2x) = 2\cos 2x$, so at x_0, the tangent lines to $y = \sin x$ and $y = \sin 2x$ have slopes of $m_1 = \frac{1}{2}$ and $m_2 = 2\left(-\frac{1}{2}\right) = -1$, respectively. From Problem 40 of Section 2.3, $\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$ where θ is the angle between the tangent lines. $\tan\theta = \frac{-1-\frac{1}{2}}{1+\left(\frac{1}{2}\right)(-1)} = \frac{-\frac{3}{2}}{\frac{1}{2}} = -3$, so $\theta \approx -1.25$. The curves intersect at an angle of 1.25 radians.

28. $\frac{1}{2}\overline{AB} = \overline{OA}\sin\frac{t}{2}$

$D = \frac{1}{2}\overline{OA}\cos\frac{t}{2}\cdot\overline{AB} = \overline{OA}^2 \cos\frac{t}{2}\sin\frac{t}{2}$

$E = D$ + area (semi-circle)

$= \overline{OA}^2 \cos\frac{t}{2}\sin\frac{t}{2} + \frac{1}{2}\pi\left(\frac{1}{2}\overline{AB}\right)^2$

$= \overline{OA}^2 \cos\frac{t}{2}\sin\frac{t}{2} + \frac{1}{2}\pi\overline{OA}^2 \sin^2\frac{t}{2}$

$= \overline{OA}^2 \sin\frac{t}{2}\left(\cos\frac{t}{2} + \frac{1}{2}\pi\sin\frac{t}{2}\right)$

$\frac{D}{E} = \frac{\cos\frac{t}{2}}{\cos\frac{t}{2} + \frac{1}{2}\pi\sin\frac{t}{2}}$

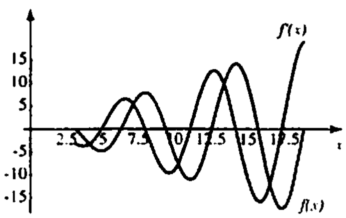
$\lim_{t\to 0^+} \frac{D}{E} = \frac{1}{1+0} = 1$

$\lim_{t\to \pi^-} \frac{D}{E} = \lim_{t\to\pi^-} \frac{\cos(t/2)}{\cos(t/2) + \frac{\pi}{2}\sin(t/2)}$

$= \frac{0}{0+\frac{\pi}{2}} = 0$

29. $f(x) = x\sin x$

a.



b. $f(x) = 0$ has 6 solutions on $[\pi, 6\pi]$

$f'(x) = 0$ has 5 solutions on $[\pi, 6\pi]$

c. $f(x) = x\sin x$ is a counterexample.

d. The maximum value of $|f(x) - f'(x)|$ on $[\pi, 6\pi]$ is about 24.93.

30. $x_0 \approx 1.95$

$f'(x_0) \approx -1.24$

3.5 Concepts Review

1. $D_t u$; $f'(g(t))g'(t)$

2. $D_v w$; $G'(H(s))H'(s)$

3. $(f(x))^2$; $(f(x))^2$

4. $2x\cos(x^2)$; $6(2x+1)^2$

Problem Set 3.5

1. $y = u^{15}$ and $u = 1 + x$
$$D_x y = D_u y \cdot D_x u$$
$$= (15u^{14})(1)$$
$$= 15(1+x)^{14}$$

2. $y = u^5$ and $u = 7 + x$
$$D_x y = D_u y \cdot D_x u$$
$$= (5u^4)(1)$$
$$= 5(7+x)^4$$

3. $y = u^5$ and $u = 3 - 2x$
$$D_x y = D_u y \cdot D_x u$$
$$= (5u^4)(-2) = -10(3-2x)^4$$

4. $y = u^7$ and $u = 4 + 2x^2$
$$D_x y = D_u y \cdot D_x u$$
$$= (7u^6)(4x) = 28x(4+2x^2)^6$$

5. $y = u^{11}$ and $u = x^3 - 2x^2 + 3x + 1$
$$D_x y = D_u y \cdot D_x u$$
$$= (11u^{10})(3x^2 - 4x + 3)$$
$$= 11(3x^2 - 4x + 3)(x^3 - 2x^2 + 3x + 1)^{10}$$

6. $y = u^{101}$ and $u = x^5 - 5x^3 + \pi x + 1$
$$D_x y = D_u y \cdot D_x u$$
$$= (101u^{100})(5x^4 - 15x^2 + \pi)$$
$$= 101(5x^4 - 15x^2 + \pi)(x^5 - 5x^3 + \pi x + 1)^{100}$$

7. $y = u^{111}$ and $u = x^3 - 2x^2 + 3x + 1$
$$D_x y = D_u y \cdot D_x u$$
$$= (111u^{110})(3x^2 - 4x + 3)$$
$$= 111(3x^2 - 4x + 3)(x^3 - 2x^2 + 3x + 1)^{110}$$

8. $y = u^{-7}$ and $u = x^2 - x + 1$
$$D_x y = D_u y \cdot D_x u$$
$$= (-7u^{-8})(2x - 1)$$
$$= -7(2x-1)(x^2 - x + 1)^{-8}$$

9. $y = u^{-5}$ and $u = x + 3$
$$D_x y = D_u y \cdot D_x u$$
$$= (-5u^{-6})(1) = -5(x+3)^{-6} = -\frac{5}{(x+3)^6}$$

10. $y = u^{-9}$ and $u = 3x^2 + x - 3$
$$D_x y = D_u y \cdot D_x u$$
$$= (-9u^{-10})(6x + 1)$$
$$= -9(6x+1)(3x^2 + x - 3)^{-10}$$
$$= -\frac{9(6x+1)}{(3x^2 + x - 3)^{10}}$$

11. $y = \sin u$ and $u = x^2 + x$
$$D_x y = D_u y \cdot D_x u$$
$$= (\cos u)(2x + 1)$$
$$= (2x+1)\cos(x^2 + x)$$

12. $y = \cos u$ and $u = 3x^2 - 2x$
$$D_x y = D_u y \cdot D_x u$$
$$= (-\sin u)(6x - 2)$$
$$= -(6x-2)\sin(3x^2 - 2x)$$

13. $y = u^3$ and $u = \cos x$
$$D_x y = D_u y \cdot D_x u$$
$$= (3u^2)(-\sin x)$$
$$= -3\sin x\cos^2 x$$

14. $y = u^4$, $u = \sin v$, and $v = 3x^2$
$$D_x y = D_u y \cdot D_v u \cdot D_x v$$
$$= (4u^3)(\cos v)(6x)$$
$$= 24x\sin^3(3x^2)\cos(3x^2)$$

15. $y = u^3$ and $u = \frac{x+1}{x-1}$
$$D_x y = D_u y \cdot D_x u$$
$$= (3u^2)\frac{(x-1)D_x(x+1) - (x+1)D_x(x-1)}{(x-1)^2}$$
$$= 3\left(\frac{x+1}{x-1}\right)^2\left(\frac{-2}{(x-1)^2}\right) = -\frac{6(x+1)^2}{(x-1)^4}$$

16. $y = u^{-3}$ and $u = \dfrac{x-2}{x-\pi}$

$D_x y = D_u y \cdot D_x u$

$= (-3u^{-4}) \cdot \dfrac{(x-\pi)D_x(x-2)-(x-2)D_x(x-\pi)}{(x-\pi)^2}$

$= -3\left(\dfrac{x-2}{x-\pi}\right)^{-4} \dfrac{(2-\pi)}{(x-\pi)^2} = -3\dfrac{(x-\pi)^2}{(x-2)^4}(2-\pi)$

17. $y = \cos u$ and $u = \dfrac{3x^2}{x+2}$

$D_x y = D_u y \cdot D_x u \; = (-\sin u)\dfrac{(x+2)D_x(3x^2)-(3x^2)D_x(x+2)}{(x+2)^2}$

$= -\sin\left(\dfrac{3x^2}{x+2}\right)\dfrac{(x+2)(6x)-(3x^2)(1)}{(x+2)^2} \; = -\dfrac{3x^2+12x}{(x+2)^2}\sin\left(\dfrac{3x^2}{x+2}\right)$

18. $y = u^3$, $u = \cos v$, and $v = \dfrac{x^2}{1-x}$

$D_x y = D_u y \cdot D_v u \cdot D_x v \; = (3u^2)(-\sin v)\dfrac{(1-x)D_x(x^2)-(x^2)D_x(1-x)}{(1-x)^2}$

$= -3\cos^2\left(\dfrac{x^2}{1-x}\right)\sin\left(\dfrac{x^2}{1-x}\right)\dfrac{(1-x)(2x)-(x^2)(-1)}{(1-x)^2} \; = \dfrac{-3(2x-x^2)}{(1-x)^2}\cos^2\left(\dfrac{x^2}{1-x}\right)\sin\left(\dfrac{x^2}{1-x}\right)$

19. $D_x[(3x-2)^2(3-x^2)^2] = (3x-2)^2 D_x(3-x^2)^2 + (3-x^2)^2 D_x(3x-2)^2$

$= (3x-2)^2(2)(3-x^2)(-2x) + (3-x^2)^2(2)(3x-2)(3)$

$= 2(3x-2)(3-x^2)[(3x-2)(-2x)+(3-x^2)(3)] = 2(3x-2)(3-x^2)(9+4x-9x^2)$

20. $D_x[(2-3x^2)^4(x^7+3)^3] = (2-3x^2)^4 D_x(x^7+3)^3 + (x^7+3)^3 D_x(2-3x^2)^4$

$= (2-3x^2)^4(3)(x^7+3)^2(7x^6) + (x^7+3)^3(4)(2-3x^2)^3(-6x) \; = 3x(3x^2-2)^3(x^7+3)^2(29x^7-14x^5+24)$

21. $D_x\left[\dfrac{(x+1)^2}{3x-4}\right] = \dfrac{(3x-4)D_x(x+1)^2-(x+1)^2D_x(3x-4)}{(3x-4)^2} \; = \dfrac{(3x-4)(2)(x+1)(1)-(x+1)^2(3)}{(3x-4)^2} = \dfrac{3x^2-8x-11}{(3x-4)^2}$

$= \dfrac{(x+1)(3x-11)}{(3x-4)^2}$

22. $D_x\left[\dfrac{2x-3}{(x^2+4)^2}\right] = \dfrac{(x^2+4)^2D_x(2x-3)-(2x-3)D_x(x^2+4)^2}{(x^2+4)^4}$

$= \dfrac{(x^2+4)^2(2)-(2x-3)(2)(x^2+4)(2x)}{(x^2+4)^4} \; = \dfrac{-6x^2+12x+8}{(x^2+4)^3}$

23. $D_t\left(\dfrac{3t-2}{t+5}\right)^3 = 3\left(\dfrac{3t-2}{t+5}\right)^2 \dfrac{(t+5)D_t(3t-2)-(3t-2)D_t(t+5)}{(t+5)^2}$

$= 3\left(\dfrac{3t-2}{t+5}\right)^2 \dfrac{(t+5)(3)-(3t-2)(1)}{(t+5)^2} \; = \dfrac{51(3t-2)^2}{(t+5)^4}$

24. $D_s\left(\dfrac{s^2-9}{s+4}\right) = \dfrac{(s+4)D_s(s^2-9)-(s^2-9)D_s(s+4)}{(s+4)^2} \; = \dfrac{(s+4)(2s)-(s^2-9)(1)}{(s+4)^2} = \dfrac{s^2+8s+9}{(s+4)^2}$

25. $D_t\left(\frac{(3t-2)^3}{t+5}\right)=\frac{(t+5)D_t(3t-2)^3-(3t-2)^3D_t(t+5)}{(t+5)^2}=\frac{(t+5)(3)(3t-2)^2(3)-(3t-2)^3(1)}{(t+5)^2}$

$=\frac{(6t+47)(3t-2)^2}{(t+5)^2}$

26. $D_\theta(\sin^3\theta)=3\sin^2\theta\cos\theta$

27. $D_x\left(\frac{\sin x}{\cos 2x}\right)^3=3\left(\frac{\sin x}{\cos 2x}\right)^2\frac{(\cos 2x)D_x(\sin x)-(\sin x)D_x(\cos 2x)}{\cos^2 2x}$

$=3\left(\frac{\sin x}{\cos 2x}\right)^2\frac{\cos x\cos 2x+2\sin x\sin 2x}{\cos^2 2x}=\frac{3\sin^2 x\cos x\cos 2x+6\sin^3 x\sin 2x}{\cos^4 2x}$

$=\frac{3(\sin^2 x)(\cos x\cos 2x+2\sin x\sin 2x)}{\cos^4 2x}$

28. $D_t[\sin t\tan(t^2+1)]=\sin t\,D_t[\tan(t^2+1)]+\tan(t^2+1)D_t(\sin t)$

$=(\sin t)[\sec^2(t^2+1)](2t)+\tan(t^2+1)\cos t\ =2t\sin t\sec^2(t^2+1)+\cos t\tan(t^2+1)$

29. $f'(x)=3\left(\frac{x^2+1}{x+2}\right)^2\frac{(x+2)D_x(x^2+1)-(x^2+1)D_x(x+2)}{(x+2)^2}$

$=3\left(\frac{x^2+1}{x+2}\right)^2\frac{2x^2+4x-x^2-1}{(x+2)^2}=\frac{3(x^2+1)^2(x^2+4x-1)}{(x+2)^4}$

$f'(3)=9.6$

30. $G'(t)=(t^2+9)^3D_t(t^2-2)^4+(t^2-2)^4D_t(t^2+9)^3\ =(t^2+9)^3(4)(t^2-2)^3(2t)+(t^2-2)^4(3)(t^2+9)^2(2t)$

$=2t(7t^2+30)(t^2+9)^2(t^2-2)^3$

$G'(1)=-7400$

31. $F'(t)=[\cos(t^2+3t+1)](2t+3)\ =(2t+3)\cos(t^2+3t+1)$

$F'(1)=5\cos 5\approx 1.4183$

32. $g'(s)=(\cos\pi s)D_s(\sin^2\pi s)+(\sin^2\pi s)D_s(\cos\pi s)\ =(\cos\pi s)(2\sin\pi s)(\cos\pi s)(\pi)+(\sin^2\pi s)(-\sin\pi s)(\pi)$

$=\pi\sin\pi s[2\cos^2\pi s-\sin^2\pi s]$

$g'\left(\frac{1}{2}\right)=-\pi$

33. $D_x[\sin^4(x^2+3x)]=4\sin^3(x^2+3x)D_x\sin(x^2+3x)\ =4\sin^3(x^2+3x)\cos(x^2+3x)D_x(x^2+3x)$

$=4\sin^3(x^2+3x)\cos(x^2+3x)(2x+3)\ =4(2x+3)\sin^3(x^2+3x)\cos(x^2+3x)$

34. $D_t[\cos^5(4t-19)]=5\cos^4(4t-19)D_t\cos(4t-19)\ =5\cos^4(4t-19)[-\sin(4t-19)]D_t(4t-19)$

$=-5\cos^4(4t-19)\sin(4t-19)(4)\ =-20\cos^4(4t-19)\sin(4t-19)$

35. $D_t[\sin^3(\cos t)]=3\sin^2(\cos t)D_t\sin(\cos t)\ =3\sin^2(\cos t)\cos(\cos t)D_t(\cos t)$

$=3\sin^2(\cos t)\cos(\cos t)(-\sin t)\ =-3\sin t\sin^2(\cos t)\cos(\cos t)$

36. $D_u\left[\cos^4\left(\frac{u+1}{u-1}\right)\right]=4\cos^3\left(\frac{u+1}{u-1}\right)D_u\cos\left(\frac{u+1}{u-1}\right)=4\cos^3\left(\frac{u+1}{u-1}\right)\left[-\sin\left(\frac{u+1}{u-1}\right)\right]D_u\left(\frac{u+1}{u-1}\right)$

$=-4\cos^3\left(\frac{u+1}{u-1}\right)\sin\left(\frac{u+1}{u-1}\right)\frac{(u-1)D_u(u+1)-(u+1)D_u(u-1)}{(u-1)^2}=\frac{8}{(u-1)^2}\cos^3\left(\frac{u+1}{u-1}\right)\sin\left(\frac{u+1}{u-1}\right)$

37. $D_\theta[\cos^4(\sin\theta^2)]=4\cos^3(\sin\theta^2)D_\theta\cos(\sin\theta^2)=4\cos^3(\sin\theta^2)[-\sin(\sin\theta^2)]D_\theta(\sin\theta^2)$

$=-4\cos^3(\sin\theta^2)\sin(\sin\theta^2)(\cos\theta^2)D_\theta(\theta^2)=-8\theta\cos^3(\sin\theta^2)\sin(\sin\theta^2)(\cos\theta^2)$

38. $D_x[x\sin^2(2x)]=x\,D_x\sin^2(2x)+\sin^2(2x)D_x x=x[2\sin(2x)D_x\sin(2x)]+\sin^2(2x)(1)$

$=x[2\sin(2x)\cos(2x)D_x(2x)]+\sin^2(2x)=x[4\sin(2x)\cos(2x)]+\sin^2(2x)=2x\sin(4x)+\sin^2(2x)$

39. $D_x\{\sin[\cos(\sin 2x)]\}=\cos[\cos(\sin 2x)]D_x\cos(\sin 2x)=\cos[\cos(\sin 2x)][-\sin(\sin 2x)]D_x(\sin 2x)$

$=-\cos[\cos(\sin 2x)]\sin(\sin 2x)(\cos 2x)D_x(2x)=-2\cos[\cos(\sin 2x)]\sin(\sin 2x)(\cos 2x)$

40. $D_t\{\cos^2[\cos(\cos t)]\}=2\cos[\cos(\cos t)]D_t\cos[\cos(\cos t)]=2\cos[\cos(\cos t)]\{-\sin[\cos(\cos t)]\}D_t\cos(\cos t)$

$=-2\cos[\cos(\cos t)]\sin[\cos(\cos t)][-\sin(\cos t)]D_t(\cos t)=2\cos[\cos(\cos t)]\sin[\cos(\cos t)]\sin(\cos t)(-\sin t)$

$=-2\sin t\cos[\cos(\cos t)]\sin[\cos(\cos t)]\sin(\cos t)$

41. $D_x y=(x^2+1)^3D_x(x^4+1)^2+(x^4+1)^2D_x(x^2+1)^3=(x^2+1)^3(2)(x^4+1)(4x^3)+(x^4+1)^2(3)(x^2+1)(2x)$

$=8x^3(x^2+1)^3(x^4+1)+6x(x^2+1)(x^4+1)^2$

At $x=1$: $m_{\tan}=224$

Tangent line: $y-32=224(x-1)$

42. a. $\left(\frac{x}{4}\right)^2+\left(\frac{y}{7}\right)^2=\left(\frac{4\cos 2t}{4}\right)^2+\left(\frac{7\sin 2t}{7}\right)^2=\cos^2 2t+\sin^2 2t=1$

b. $L=\sqrt{(x-0)^2+(y-0)^2}=\sqrt{x^2+y^2}=\sqrt{(4\cos 2t)^2+(7\sin 2t)^2}=\sqrt{16\cos^2 2t+49\sin^2 2t}$

c. $D_tL=\frac{1}{2\sqrt{16\cos^2 2t+49\sin^2 2t}}D_t(16\cos^2 2t+49\sin^2 2t)=\frac{32\cos 2t\,D_t(\cos 2t)+98\sin 2t\,D_t(\sin 2t)}{2\sqrt{16\cos^2 2t+49\sin^2 2t}}$

$=\frac{-64\cos 2t\sin 2t+196\sin 2t\cos 2t}{2\sqrt{16\cos^2 2t+49\sin^2 2t}}=\frac{-16\sin 4t+49\sin 4t}{\sqrt{16\cos^2 2t+49\sin^2 2t}}=\frac{33\sin 4t}{\sqrt{16\cos^2 2t+49\sin^2 2t}}$

At $t=\frac{\pi}{8}$: rate $=\frac{33}{\sqrt{16\cdot\frac{1}{2}+49\cdot\frac{1}{2}}}\approx 5.8$ ft/sec

43. a. $(10\cos 8\pi t, 10\sin 8\pi t)$

b. $D_t(10\sin 8\pi t)=10\cos(8\pi t)D_t(8\pi t)$

$=80\pi\cos(8\pi t)$

At $t=1$: rate $=80\pi\approx 251$ cm/s

P is rising at the rate of 251 cm/s.

44. a. $(\cos 2t, \sin 2t)$

b. $(0-\cos 2t)^2+(y-\sin 2t)^2=5^2$, so

$y=\sin 2t+\sqrt{25-\cos^2 2t}$

c. $D_t\left(\sin 2t+\sqrt{25-\cos^2 2t}\right)$

$=2\cos 2t+\frac{1}{2\sqrt{25-\cos^2 2t}}\cdot 4\cos 2t\sin 2t$

$=2\cos 2t\left(1+\frac{\sin 2t}{\sqrt{25-\cos^2 2t}}\right)$

45. 60 revolutions per minute is 120π radians per minute or 2π radians per second.

a. $(\cos 2\pi t, \sin 2\pi t)$

b. $(0-\cos 2\pi t)^2+(y-\sin 2\pi t)^2=5^2$, so
$y=\sin 2\pi t+\sqrt{25-\cos^2 2\pi t}$

c. $D_t\left(\sin 2\pi t+\sqrt{25-\cos^2 2\pi t}\right)$
$=2\pi\cos 2\pi t$
$+\frac{1}{2\sqrt{25-\cos^2 2\pi t}}\cdot 4\pi\cos 2\pi t\sin 2\pi t$
$=2\pi\cos 2\pi t\left(1+\frac{\sin 2\pi t}{\sqrt{25-\cos^2 2\pi t}}\right)$

46. $V(t)=\frac{4}{3}\pi[r(t)]^3$
$V'(t)=4\pi[r(t)]^2r'(t)$
Let t_0 be the time when $r=6$.
$V'(t_0)=4\pi\cdot 6^2\cdot 0.75$
$=108\pi\ \text{cm}^3/\text{sec}$

47. $V(t)=\frac{4}{3}\pi[r(t)]^3$
$[r(t)]^3=\frac{3V(t)}{4\pi}$
$r(t)=\left(\frac{3V(t)}{4\pi}\right)^{1/3}$
$r'(t)=\frac{1}{3}\left(\frac{3V(t)}{4\pi}\right)^{-2/3}\frac{3}{4\pi}V'(t)$
Let t_0 be the time when $V=60$.
$r'(t_0)=\left(\frac{3\cdot 60}{4\pi}\right)^{-2/3}\frac{1}{4\pi}\cdot 3$
≈ 0.0405 cm/sec

48. $y=\sqrt{u}$ and $u=x^2$
$D_xy=D_uy\cdot D_xu$
$=\frac{1}{2\sqrt{u}}\cdot 2x=\frac{2x}{2\sqrt{x^2}}=\frac{x}{|x|}=\frac{|x|}{x}$

49. a. $D_x\left|x^2-1\right|=\frac{\left|x^2-1\right|}{x^2-1}D_x(x^2-1)$
$=\frac{\left|x^2-1\right|}{x^2-1}(2x)=\frac{2x\left|x^2-1\right|}{x^2-1}$

b. $D_x|\sin x|=\frac{|\sin x|}{\sin x}D_x(\sin x)$
$=\frac{|\sin x|}{\sin x}\cos x=\cot x|\sin x|$

50. a. $D_xL\left(\frac{x}{x+1}\right)=\left(\frac{x+1}{x}\right)D_x\left(\frac{x}{x+1}\right)$
$=\left(\frac{x+1}{x}\right)\frac{(x+1)D_xx-xD_x(x+1)}{(x+1)^2}$
$=\frac{x+1}{x(x+1)^2}=\frac{1}{x(x+1)}=\frac{1}{x^2+x}$

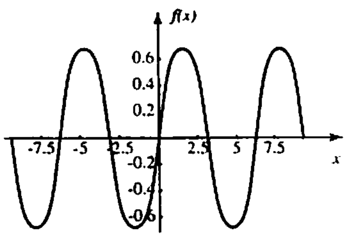
b. $D_xL(\cos^4 x)=\sec^4 x\,D_x(\cos^4 x)$
$=\sec^4 x(4\cos^3 x)D_x(\cos x)$
$=4\sec^4 x\cos^3 x(-\sin x)$
$=-4\sec x\sin x=-4\tan x$

51. $[f(f(f(f(0))))]'=f'(f(f(f(0))))\cdot$
$f'(f(f(0)))\cdot f'(f(0))\cdot f'(0)$
$=2\cdot 2\cdot 2\cdot 2=16$

52. Let $g(x)=-x$, so $g'(x)=-1$
Suppose $f(x)$ is odd. Then $-f(x)=f(f=g(x))$. Take the derivative of both sides.
$-f'(x)=f'(g(x))g'(x)$, so
$-f'(x)=-f'(-x)$ or $f'(x)=f'(-x)$. Thus, the derivative is an even function.
Suppose $f(x)$ is even. Then $f(x)=f(g(x))$. Take the derivative of both sides. $f'(x)=f'(g(x))g'(x)$, so $f'(x)=-f'(-x)$. Thus, the derivative is an odd function.
$f'(-x)=f'(-x)\cdot(-1)=-f'(-x)$
If $f(x)$ is odd, then $-f'(-x)=-f'(x)$ is even.
If $f(x)$ is even, then $-f'(-x)=f'(x)$ is odd.

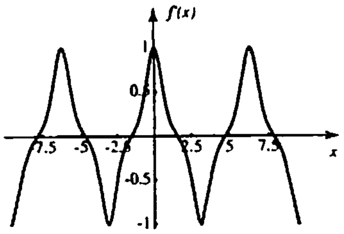
53. $f(x)=\sin(\sin(\sin(\sin x)))$

a.



Odd function
$f(-x)=\sin(\sin(\sin(\sin(-x))))$
$=\sin(\sin(\sin(-\sin x)))=\sin(\sin(-\sin(\sin x)))$
$=\sin(-\sin(\sin(\sin x)))=-\sin(\sin(\sin(\sin x)))$

b.



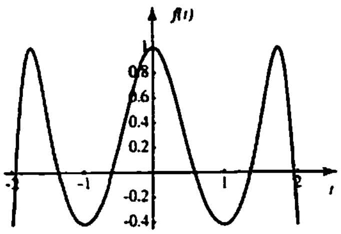
Even; the derivative of an odd function is even.

c. Largest value of $f(x) \approx 0.678$

d. Largest value of $|f'(x)| = 1$

54. $f(t) = \cos(t^3 - 3t)$

a.



Even function

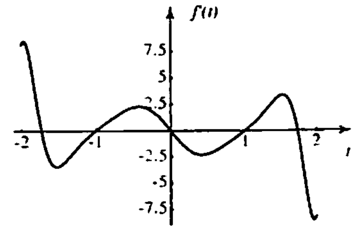
$f(-t) = \cos[(-t)^3 - 3(-t)]$

$= \cos[-t^3 + 3t]$

$= \cos[-(t^3 - 3t)]$

$= \cos(t^3 - 3t)$

b.



Odd

The derivative of an even function is odd.

c. Largest value of $f(t) = 1$

d. Largest value of $|f'(t)| \approx 8.53$

3.6 Concepts Review

1. Increment; $\dfrac{\Delta y}{\Delta x}$; $\dfrac{dy}{dx}$

2. $f'(x)$; $D_x y$; $\dfrac{dy}{dx}$

3. $\dfrac{dy}{du}\dfrac{du}{dx}$

4. $\dfrac{dw}{dt}\dfrac{dt}{ds}\dfrac{ds}{dr}$

Problem Set 3.6

1. $\Delta y = [3(1.5) + 2] - [3(1) + 2] = 1.5$

2. $\Delta y = [3(0.1)^2 + 2(0.1) + 1] - [3(0.0)^2 + 2(0.0) + 1]$
$= 0.23$

3. $\Delta y = \dfrac{3}{2.31+1} - \dfrac{3}{2.34+1} \approx 0.0081$

4. $\Delta y = \cos[2(0.573)] - \cos[2(0.571)] \approx -0.0036$

5. $\dfrac{\Delta y}{\Delta x} = \dfrac{(x+\Delta x)^2 - x^2}{\Delta x} = \dfrac{2x\Delta x + (\Delta x)^2}{\Delta x} = 2x + \Delta x$

$\dfrac{dy}{dx} = \lim_{\Delta x \to 0}(2x + \Delta x) = 2x$

6. $\dfrac{\Delta y}{\Delta x} = \dfrac{[(x+\Delta x)^3 - 3(x+\Delta x)^2] - (x^3 - 3x^2)}{\Delta x}$

$= \dfrac{3x^2\Delta x + 3x(\Delta x)^2 - 6x\Delta x - 3(\Delta x)^2 + \Delta x^3}{\Delta x}$

$= 3x^2 + 3x\Delta x - 6x - 3\Delta x + (\Delta x)^2$

$\dfrac{dy}{dx} = \lim_{\Delta x \to 0}(3x^2 + 3x\Delta x - 6x - 3\Delta x + (\Delta x)^2)$

$= 3x^2 - 6x$

7. $\dfrac{\Delta y}{\Delta x} = \dfrac{\frac{1}{x+\Delta x+1} - \frac{1}{x+1}}{\Delta x}$

$= \left(\dfrac{x+1-(x+\Delta x+1)}{(x+\Delta x+1)(x+1)}\right)\left(\dfrac{1}{\Delta x}\right)$

$= \dfrac{-\Delta x}{(x+\Delta x+1)(x+1)\Delta x}$

$= -\dfrac{1}{(x+\Delta x+1)(x+1)}$

$\dfrac{dy}{dx} = \lim_{\Delta x \to 0}\left[-\dfrac{1}{(x+\Delta x+1)(x+1)}\right] = -\dfrac{1}{(x+1)^2}$

8. $\frac{\Delta y}{\Delta x}=\left[\frac{(x+\Delta x)^3+(x+\Delta x)^2}{(x+\Delta x)^3}-\frac{x^3+x^2}{x^3}\right]\frac{1}{\Delta x}$

$=\left(1+\frac{1}{x+\Delta x}-1-\frac{1}{x}\right)\left(\frac{1}{\Delta x}\right)$

$=\frac{-\Delta x}{x(x+\Delta x)\Delta x}=-\frac{1}{x(x+\Delta x)}$

$\frac{dy}{dx}=\lim_{\Delta x\to 0}\left(-\frac{1}{x(x+\Delta x)}\right)=-\frac{1}{x^2}$

9. $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=(2u)(\cos x)=2\sin x\cos x=\sin 2x$

10. $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=(-\sin u)\frac{(x+1)(0)-(1)(1)}{(x+1)^2}$

$=-\sin\left(\frac{1}{x+1}\right)\left(-\frac{1}{(x+1)^2}\right)=\frac{1}{(x+1)^2}\sin\left(\frac{1}{x+1}\right)$

11. $y=\tan u$ and $u=x^2$

$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=(\sec^2 u)(2x)=2x\sec^2(x^2)$

12. $y=u^2$ and $u=\tan x$

$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=(2u)(\sec^2 x)=2\tan x\sec^2 x$

13. $y=u^4$ and $u=\frac{x^2+1}{\cos x}$

$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=(4u^3)\frac{(\cos x)(2x)-(x^2+1)(-\sin x)}{\cos^2 x}$

$=4\left(\frac{x^2+1}{\cos x}\right)^3\left(\frac{2x\cos x+\sin x+x^2\sin x}{\cos^2 x}\right)$

$=\frac{4(x^2+1)^3(2x\cos x+\sin x+x^2\sin x)}{\cos^5 x}$

14. $y=u^3$ and $u=(x^2+1)\sin x$

$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=(3u^2)[(x^2+1)(\cos x)+(\sin x)(2x)]$

$=[(x^2+1)\sin x]^2(3x^2\cos x+3\cos x+6x\sin x)$

15. $\frac{dy}{dx}=\cos(x^2)\frac{d}{dx}(\sin^2 x)+\sin^2 x\frac{d}{dx}[\cos(x^2)]$

$=\cos(x^2)2\sin x\cos x+\sin^2 x[-\sin(x^2)](2x)$

$=\sin 2x\cos(x^2)-2x\sin^2 x\sin(x^2)$

16. $\frac{dy}{dx}=\frac{(x^4+1)\frac{d}{dx}(x^3+2x)^4-(x^3+2x)^4\frac{d}{dx}(x^4+1)}{(x^4+1)^2}$

$=\frac{(x^4+1)4(x^3+2x)^3(3x^2+2)-(x^3+2x)^4(4x^3)}{(x^4+1)^2}$

$=\frac{4x^3(x^2+2)^3(2x^6+3x^2+2)}{(x^4+1)^2}$

17. $y=u^4$, $u=\sin v$, and $v=x^2+3$

$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}=(4u^3)(\cos v)(2x)$

$=4\sin^3(x^2+3)\cos(x^2+3)(2x)$

$=8x\sin^3(x^2+3)\cos(x^2+3)$

18. $y=\sin u$, $u=v^4$, and $v=x^2+3$

$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}=(\cos u)(4v^3)(2x)$

$=8x\cos[(x^2+3)^4](x^2+3)^3$

19. $y=u^2$, $u=\cos v$, and $v=\frac{x^2+2}{x^2-2}$

$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}$

$=(2u)(-\sin v)\frac{(x^2-2)(2x)-(x^2+2)(2x)}{(x^2-2)^2}$

$=-2\cos\left(\frac{x^2+2}{x^2-2}\right)\sin\left(\frac{x^2+2}{x^2-2}\right)\left[\frac{-8x}{(x^2-2)^2}\right]$

$=\frac{16x}{(x^2-2)^2}\cos\left(\frac{x^2+2}{x^2-2}\right)\sin\left(\frac{x^2+2}{x^2-2}\right)$

20. $y=t^2$, $t=\sin u$, $u=v^2$, $v=\cos w$, and $w=x^2$

$\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{du}\frac{du}{dv}\frac{dv}{dw}\frac{dw}{dx}$

$=(2t)(\cos u)(2v)(-\sin w)(2x)$

$=-8x\sin(\cos^2(x^2))\cos(\cos^2(x^2))(\cos(x^2))(\sin(x^2))$

$=-8x\cos(x^2)\sin(x^2)\sin(\cos^2(x^2))\cos(\cos^2(x^2))$

21. $\frac{d}{dt}(\sin^3 t+\cos^3 t)=3\sin^2 t\cos t+3\cos^2 t(-\sin t)$

$=3(\sin^2 t\cos t-\sin t\cos^2 t)$

22. $\frac{d}{ds}[(s^2+3)^3-(s^2+3)^{-3}]$

$=3(s^2+3)^2(2s)-(-3)(s^2+3)^{-4}(2s)$

$=6s[(s^2+3)^2+(s^2+3)^{-4}]$

23. $D_r[\pi(r+3)^2 - 3\pi r(r+2)^2]$
$= 2\pi(r+3)(1) - [3\pi r(2)(r+2) + (r+2)^2(3\pi)]$
$= 2\pi r + 6\pi - 6\pi r^2 - 12\pi r - 3\pi r^2 - 12\pi r - 12\pi$
$= -9\pi r^2 - 22\pi r - 6\pi$

24. $D_t[u^3 + 3u] = 3u^2 D_t u + 3D_t u$
$= 3(t^2)^2(2t) + 3(2t) = 6t^5 + 6t$

25. $f'(x) = 4\left(x + \frac{1}{x}\right)^3 [1 + (-1)x^{-2}]$
$= 4\left(x + \frac{1}{x}\right)^3 \left(1 - \frac{1}{x^2}\right)$
$f'(2) = 4\left(\frac{5}{2}\right)^3 \left(\frac{3}{4}\right) = \frac{1500}{32} = 46.875$

26. $F'(t) = \cos(t^2)(\cos 3t)(3) + (\sin 3t)[-\sin(t^2)(2t)]$
$= 3\cos(t^2)(\cos 3t) - 2t\sin(t^2)(\sin 3t)$
$F'(0) = 3$

27. a. $(f+g)'(3) = f'(3) + g'(3) = -1 + (-4) = -5$

b. $(f \cdot g)'(3) = f(3)g'(3) + g(3)f'(3)$
$= (2)(-4) + (3)(-1) = -11$

c. $(f/g)'(3) = \frac{g(3)f'(3) - f(3)g'(3)}{g^2(3)}$
$= \frac{(3)(-1) - (2)(-4)}{(3)^2} = \frac{5}{9}$

d. $(f \circ g)'(3) = f'(g(3)) \cdot g'(3)$
$= f'(3) \cdot g'(3) = (-1)(-4) = 4$

28. a. $\frac{d}{dx}[f(x)]^3 = 3[f(x)]^2 f'(x)$
At $x = 2$, $3[f(2)]^2 f'(2) = 3(4)^2(-2) = -96$

b. $\frac{d}{dx}\left[\frac{3}{f(x)}\right] = \frac{f(x)(0) - 3f'(x)}{f^2(x)} = -\frac{3f'(x)}{f^2(x)}$
At $x = 2$, $-\frac{3f'(2)}{f^2(2)} = -\frac{3(-2)}{(4)^2} = \frac{3}{8}$

c. $(f \circ f)'(2) = f'(f(2))f'(2) = f'(4)f'(2)$
$= (6)(-2) = -12$

29. a. $(f+g)'(4) = f'(4) + g'(4)$
$\approx \frac{1}{2} + \frac{3}{2} \approx 2$

b. $(f \circ g)'(6) = f'(g(6))g'(6)$
$= f'(2)g'(6) \approx (1)(-1) = -1$

30. a. $(f/g)'(2) = \frac{g(2)f'(2) - f(2)g'(2)}{g^2(2)}$
$\approx \frac{(1)(1) - (3)(0)}{(1)^2} = 1$

b. $(g \circ f)'(3) = g'(f(3))f'(3)$
$= g'(4)f'(3) \approx \left(\frac{3}{2}\right)(1) = \frac{3}{2}$

31. a. $V = s^3$ where s is the length of an edge of the cube. $\frac{ds}{dt} = 16$ cm/min.
$\frac{dV}{dt} = \frac{dV}{ds}\frac{ds}{dt} = (3s^2)(16) = 48s^2 \text{ cm}^3/\text{min}.$
When $s = 20$,
$\frac{dV}{dt} = 48(20)^2 = 19,200 \text{ cm}^3/\text{min}.$

b. $A = 6s^2 \text{ cm}^2.$
$\frac{dA}{dt} = \frac{dA}{ds}\frac{ds}{dt} = (12s)(16) = 192s \text{ cm}^2/\text{min}.$
When $s = 15$,
$\frac{dA}{dt} = 192(15) = 2880 \text{ cm}^2/\text{min}.$

32. $D = \sqrt{(20t)^2 + (12t)^2} = \sqrt{400t^2 + 144t^2}$
$= \sqrt{544t^2} = t\sqrt{544}$
$D_t(t\sqrt{544}) = \sqrt{544} \approx 23.3$ mph at $t = 3$ and $t = 6$ hours

33. $\frac{dy}{dx} = (x^2)(2)\sin(x^2)\cos(x^2)(2x) + 2x\sin^2(x^2)$
$= 4x^3\sin(x^2)\cos(x^2) + 2x\sin^2(x^2)$
At $x = \sqrt{\frac{\pi}{2}}$: $m_{\tan} = \sqrt{2\pi} \approx 2.507$
$y = \frac{\pi}{2}$
Tangent line: $y - \frac{\pi}{2} = \sqrt{2\pi}\left(x - \sqrt{\frac{\pi}{2}}\right)$
The line intersects the x-axis when
$-\frac{\pi}{2} = \sqrt{2\pi}x - \pi$
$\frac{\pi}{2} = \sqrt{2\pi}x$
$x = \frac{\sqrt{2\pi}}{4}$

34. The minute hand makes 1 revolution every hour, so at t minutes after the hour, it makes an angle of $\frac{\pi t}{30}$ radians with the vertical. By the Law of Cosines, the length of the elastic string is

$$s=\sqrt{10^2+10^2-2(10)(10)\cos\frac{\pi t}{30}}$$
$$=10\sqrt{2-2\cos\frac{\pi t}{30}}$$

$$\frac{ds}{dt}=10\cdot\frac{1}{2\sqrt{2-2\cos\frac{\pi t}{30}}}\cdot\frac{\pi}{15}\sin\frac{\pi t}{30}$$
$$=\frac{\pi\sin\frac{\pi t}{30}}{3\sqrt{2-2\cos\frac{\pi t}{30}}}$$

At 12:15, the string is stretching at the rate of

$$\frac{\pi\sin\frac{\pi}{2}}{3\sqrt{2-2\cos\frac{\pi}{2}}}=\frac{\pi}{3\sqrt{2}}\approx 0.74 \text{ cm/min}$$

35. $g'(x)=f'(f(f(f(x))))\cdot f'(f(f(x)))\cdot f'(f(x))\cdot f'(x)$

$g'(x_1)=f'(f(f(f(x_1))))\cdot f'(f(f(x_1)))\cdot f'(f(x_1))\cdot f'(x_1)$
$=f'(f(f(x_2)))\cdot f'(f(x_2))\cdot f'(x_2)\cdot f'(x_1)$
$=f'(f(x_1))\cdot f'(x_1)\cdot f'(x_2)\cdot f'(x_1)$
$=f'(x_2)f'(x_1)f'(x_2)f'(x_1)=[f'(x_1)f'(x_2)]^2$

$g'(x_2)=f'(f(f(f(x_2))))\cdot f'(f(f(x_2)))\cdot f'(f(x_2))\cdot f'(x_2)$
$=f'(f(f(x_1)))\cdot f'(f(x_1))\cdot f'(x_1)\cdot f'(x_2)$
$=f'(f(x_2))\cdot f'(x_2)\cdot f'(x_1)\cdot f'(x_2)$
$=f'(x_1)f'(x_2)f'(x_1)f'(x_2)=[f'(x_1)f'(x_2)]^2$

36. a. $f'(x)=x^2\left(\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)+\left(\sin\frac{1}{x}\right)(2x)$

$=-\cos\frac{1}{x}+2x\sin\frac{1}{x}$

b. $f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$

$=\lim_{h\to 0}\frac{h^2\sin\frac{1}{h}-0}{h}=\lim_{h\to 0}h\sin\frac{1}{h}=0$

c. $f'(x)$ is discontinuous at $x=0$ because $\lim_{x\to 0}f'(x)$ does not exist. $\lim_{x\to 0}2x\sin\frac{1}{x}=0$ but $\cos\frac{1}{x}$ oscillates between -1 and 1 increasingly rapidly as $x\to 0$.

37. The minute hand makes 1 revolution every hour, so at t minutes after noon it makes an angle of $\frac{\pi t}{30}$ radians with the vertical. Similarly, at t minutes after noon the hour hand makes an angle of $\frac{\pi t}{360}$ with the vertical. Thus, by the Law of Cosines, the distance between the tips of the hands is

$$s=\sqrt{6^2+8^2-2\cdot 6\cdot 8\cos\left(\frac{\pi t}{30}-\frac{\pi t}{360}\right)}$$
$$=\sqrt{100-96\cos\frac{11\pi t}{360}}$$
$$\frac{ds}{dt}=\frac{1}{2\sqrt{100-96\cos\frac{11\pi t}{360}}}\cdot\frac{44\pi}{15}\sin\frac{11\pi t}{360}$$
$$=\frac{22\pi\sin\frac{11\pi t}{360}}{15\sqrt{100-96\cos\frac{11\pi t}{360}}}$$

At 12:20,

$$\frac{ds}{dt}=\frac{22\pi\sin\frac{11\pi}{18}}{15\sqrt{100-96\cos\frac{11\pi}{18}}}\approx 0.38 \text{ in./min}$$

38. From Problem 37, $\frac{ds}{dt}=\frac{22\pi\sin\frac{11\pi t}{360}}{15\sqrt{100-96\cos\frac{11\pi t}{360}}}$.

Using a computer algebra system or graphing utility to view $\frac{ds}{dt}$ for $0\le t\le 60$, $\frac{ds}{dt}$ is largest when $t\approx 7.5$. Thus, the distance between the tips of the hands is increasing most rapidly at about 12:08.

39. $D_x\left(\frac{f(x)}{g(x)}\right) = D_x\left(f(x)\cdot\frac{1}{g(x)}\right)$

$= D_x\left(f(x)\cdot(g(x))^{-1}\right)$

$= f(x)D_x\left((g(x))^{-1}\right)$

$\quad + (g(x))^{-1}D_xf(x)$

$= f(x)\cdot(-1)(g(x))^{-2}D_xg(x)$

$\quad + (g(x))^{-1}D_xf(x)$

$= -f(x)(g(x))^{-2}D_xg(x)$

$\quad + (g(x))^{-1}D_xf(x)$

$= \frac{-f(x)D_xg(x)}{g^2(x)} + \frac{D_xf(x)}{g(x)}$

$= \frac{-f(x)D_xg(x)}{g^2(x)} + \frac{g(x)}{g(x)}\cdot\frac{D_xf(x)}{g(x)}$

$= \frac{-f(x)D_xg(x)}{g^2(x)} + \frac{g(x)D_xf(x)}{g^2(x)}$

$= \frac{g(x)D_xf(x) - f(x)D_xg(x)}{g^2(x)}$

40. a. $\frac{d}{dx}f^{[2]} = f'(f(x))\cdot f'(x)$

$= f'(f^{[1]})\cdot\frac{d}{dx}f^{[1]}(x)$

b. $\frac{d}{dx}f^{[3]} = f'(f(f(x)))\cdot f'(f(x))\cdot f'(x)$

$= f'(f^{[2]}(x))\cdot f'(f^{[1]}(x))\cdot\frac{d}{dx}f^{[1]}(x)$

$= f'(f^{[2]}(x))\cdot\frac{d}{dx}f^{[2]}(x)$

c. $\frac{d}{dx}f^{[4]} = f'(f(f(f(x))))\cdot f'(f(f(x)))$

$\cdot f'(f(x))\cdot f'(x)$

$= f'(f^{[3]}(x))\cdot f'(f^{[2]}(x))$

$\cdot f'(f^{[1]}(x))\cdot\frac{d}{dx}f^{[1]}(x)$

$= f'(f^{[3]}(x))\cdot\frac{d}{dx}f^{[3]}(x)$

d. Conjecture:

$\frac{d}{dx}f^{[n]}(x) = f'(f^{[n-1]}(x))\cdot\frac{d}{dx}f^{[n-1]}(x)$

Proof Applying the chain rule gives

$\frac{d}{dx}f^{[k+1]}(x) = \frac{d}{dx}f(f^{[k]}(x))$

$= f'(f^{[k]}(x))\frac{d}{dx}f^{[k]}(x)$

3.7 Concepts Review

1. $f'''(x), D_x^3y, \frac{d^3y}{dx^3}$

2. $\frac{ds}{dt}; \left|\frac{ds}{dt}\right|; \frac{d^2s}{dt^2}$

3. 0; < 0

4. positive; negative

Problem Set 3.7

1. $\frac{dy}{dx} = 3x^2 + 6x + 6$

$\frac{d^2y}{dx^2} = 6x + 6$

$\frac{d^3y}{dx^3} = 6$

2. $\frac{dy}{dx} = 5x^4 + 4x^3$

$\frac{d^2y}{dx^2} = 20x^3 + 12x^2$

$\frac{d^3y}{dx^3} = 60x^2 + 24x$

3. $\frac{dy}{dx} = 3(3x+5)^2(3) = 9(3x+5)^2$

$\frac{d^2y}{dx^2} = 18(3x+5)(3) = 162x + 270$

$\frac{d^3y}{dx^3} = 162$

4. $\frac{dy}{dx} = 5(3-5x)^4(-5) = -25(3-5x)^4$

$\frac{d^2y}{dx^2} = -100(3-5x)^3(-5) = 500(3-5x)^3$

$\frac{d^3y}{dx^3} = 1500(3-5x)^2(-5) = -7500(3-5x)^2$

5. $\frac{dy}{dx} = 7\cos(7x)$

$\frac{d^2y}{dx^2} = -7^2 \sin(7x)$

$\frac{d^3y}{dx^3} = -7^3 \cos(7x) = -343\cos(7x)$

6. $\frac{dy}{dx} = 3x^2 \cos(x^3)$

$\frac{d^2y}{dx^2} = 3x^2[-3x^2 \sin(x^3)] + 6x\cos(x^3) \;\; = -9x^4 \sin(x^3) + 6x\cos(x^3)$

$\frac{d^3y}{dx^3} = -9x^4 \cos(x^3)(3x^2) + \sin(x^3)(-36x^3) + 6x[-\sin(x^3)(3x^2)] + 6\cos(x^3)$

$= -27x^6 \cos(x^3) - 36x^3 \sin(x^3) - 18x^3 \sin(x^3) + 6\cos(x^3) \;\; = (6 - 27x^6)\cos(x^3) - 54x^3 \sin(x^3)$

7. $\frac{dy}{dx} = \frac{(x-1)(0)-(1)(1)}{(x-1)^2} = -\frac{1}{(x-1)^2}$

$\frac{d^2y}{dx^2} = -\frac{(x-1)^2(0)-2(x-1)}{(x-1)^4} = \frac{2}{(x-1)^3}$

$\frac{d^3y}{dx^3} = \frac{(x-1)^3(0)-2[3(x-1)^2]}{(x-1)^6}$

$= -\frac{6}{(x-1)^4}$

8. $\frac{dy}{dx} = \frac{(1-x)(3)-(3x)(-1)}{(1-x)^2} = \frac{3}{(x-1)^2}$

$\frac{d^2y}{dx^2} = \frac{(x-1)^2(0)-3[2(x-1)]}{(x-1)^4} = -\frac{6}{(x-1)^3}$

$\frac{d^3y}{dx^3} = -\frac{(x-1)^3(0)-6(3)(x-1)^2}{(x-1)^6}$

$= \frac{18}{(x-1)^4}$

9. $f'(x) = 2x; f''(x) = 2; f''(2) = 2$

10. $f'(x) = 15x^2 + 4x + 1$

$f''(x) = 30x + 4$

$f''(2) = 64$

11. $f'(t) = -\frac{2}{t^2}$

$f''(t) = \frac{4}{t^3}$

$f''(2) = \frac{4}{8} = \frac{1}{2}$

12. $f'(u) = \frac{(5-u)(4u)-(2u^2)(-1)}{(5-u)^2} = \frac{20u - 2u^2}{(5-u)^2}$

$f''(u) = \frac{(5-u)^2(20-4u)-(20u-2u^2)2(5-u)(-1)}{(5-u)^4}$

$= \frac{100}{(5-u)^3}$

$f''(2) = \frac{100}{3^3} = \frac{100}{27}$

13. $f'(\theta) = -2(\cos\theta\pi)^{-3}(-\sin\theta\pi)\pi \;\; = 2\pi(\cos\theta\pi)^{-3}(\sin\theta\pi)$

$f''(\theta) = 2\pi[(\cos\theta\pi)^{-3}(\pi)(\cos\theta\pi) + (\sin\theta\pi)(-3)(\cos\theta\pi)^{-4}(-\sin\theta\pi)(\pi)] \;\; = 2\pi^2[(\cos\theta\pi)^{-2} + 3\sin^2\theta\pi(\cos\theta\pi)^{-4}]$

$f''(2) = 2\pi^2[1 + 3(0)(1)] = 2\pi^2$

14. $f'(t) = t\cos\left(\frac{\pi}{t}\right)\left(-\frac{\pi}{t^2}\right) + \sin\left(\frac{\pi}{t}\right) = \left(-\frac{\pi}{t}\right)\cos\left(\frac{\pi}{t}\right) + \sin\left(\frac{\pi}{t}\right)$

$f''(t) = \left(-\frac{\pi}{t}\right)\left[-\sin\left(\frac{\pi}{t}\right)\left(-\frac{\pi}{t^2}\right)\right] + \left(\frac{\pi}{t^2}\right)\cos\left(\frac{\pi}{t}\right) + \left(-\frac{\pi}{t^2}\right)\cos\left(\frac{\pi}{t}\right) = -\frac{\pi^2}{t^3}\sin\left(\frac{\pi}{t}\right)$

$f''(2) = -\frac{\pi^2}{8}\sin\left(\frac{\pi}{2}\right) = -\frac{\pi^2}{8} \approx -1.23$

15. $f'(s) = s(3)(1-s^2)^2(-2s) + (1-s^2)^3 = -6s^2(1-s^2)^2 + (1-s^2)^3 = -7s^6 + 15s^4 - 9s^2 + 1$

$f''(s) = -42s^5 + 60s^3 - 18s$

$f''(2) = -900$

16. $f'(x) = \frac{(x-1)2(x+1) - (x+1)^2}{(x-1)^2} = \frac{x^2 - 2x - 3}{(x-1)^2}$

$f''(x) = \frac{(x-1)^2(2x-2) - (x^2 - 2x - 3)2(x-1)}{(x-1)^4} = \frac{(x-1)(2x-2) - (x^2 - 2x - 3)(2)}{(x-1)^3} = \frac{8}{(x-1)^3}$

$f''(2) = \frac{8}{1^3} = 8$

17. $D_x(x^n) = nx^{n-1}$

$D_x^2(x^n) = n(n-1)x^{n-2}$

$D_x^3(x^n) = n(n-1)(n-2)x^{n-3}$

$D_x^4(x^n) = n(n-1)(n-2)(n-3)x^{n-4}$

$\vdots$

$D_x^{n-1}(x^n) = n(n-1)(n-2)(n-3)...(2)x$

$D_x^n(x^n) = n(n-1)(n-2)(n-3)...2(1)x^0$

$= n!$

18. Let $k < n$.

$D_x^n(x^k) = D_x^{n-k}[D_x^k(x^k)] = D_x(k!) = 0$

so $D_x^n[a_n x^{n-1} + ... + a_1 x + a_0] = 0$

19. a. $D_x^4(3x^3 + 2x - 19) = 0$

b. $D_x^{12}(100x^{11} - 79x^{10}) = 0$

c. $D_x^{11}(x^2 - 3)^5 = 0$

20. $D_x\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

$D_x^2\left(\frac{1}{x}\right) = D_x(-x^{-2}) = 2x^{-3} = \frac{2}{x^3}$

$D_x^3\left(\frac{1}{x}\right) = D_x(2x^{-3}) = -\frac{3(2)}{x^4}$

$D_x^4\left(\frac{1}{x}\right) = \frac{4(3)(2)}{x^5}$

$D_x^n\left(\frac{1}{x}\right) = \frac{(-1)^n n!}{x^{n+1}}$

21. $f'(x) = 3x^2 + 6x - 45 = 3(x+5)(x-3)$

$3(x + 5)(x - 3) = 0$

$x = -5, x = 3$

$f''(x) = 6x + 6$

$f''(-5) = -24$

$f''(3) = 24$

22. $g(1) = a + b + c = 5$

$g'(t) = 2at + b$

$g'(1) = 2a + b = 3$

$g''(t) = 2a$

$g''(1) = 2a = -4$

$a = -2$

$2(-2) + b = 3$

$b = 7$

$(-2) + (7) + c = 5$

$c = 0$

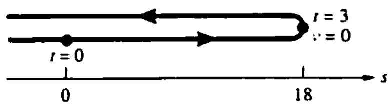
23. a. $v(t) = \frac{ds}{dt} = 12 - 4t$

$a(t) = \frac{d^2s}{dt^2} = -4$

b. $12 - 4t > 0$
$4t < 12$
$t < 3;\ (-\infty, 3)$

c. $12 - 4t < 0$
$t > 3;\ (3, \infty)$

d. $a(t) = -4 < 0$ for all t

e. 

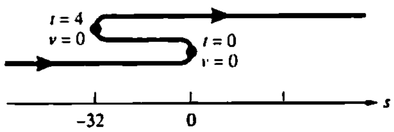
24. a. $v(t) = \frac{ds}{dt} = 3t^2 - 12t$

$a(t) = \frac{d^2s}{dt^2} = 6t - 12$

b. $3t^2 - 12t > 0$
$3t(t-4) > 0;\ (-\infty, 0) \cup (4, \infty)$

c. $3t^2 - 12t < 0$
$(0, 4)$

d. $6t - 12 < 0$
$6t < 12$
$t < 2;\ (-\infty, 2)$

e. 

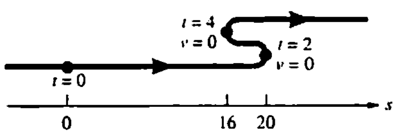
25. a. $v(t) = \frac{ds}{dt} = 3t^2 - 18t + 24$

$a(t) = \frac{d^2s}{dt^2} = 6t - 18$

b. $3t^2 - 18t + 24 > 0$
$3(t-2)(t-4) > 0$
$(-\infty, 2) \cup (4, \infty)$

c. $3t^2 - 18t + 24 < 0$
$(2, 4)$

d. $6t - 18 < 0$
$6t < 18$
$t < 3;\ (-\infty, 3)$

e. 

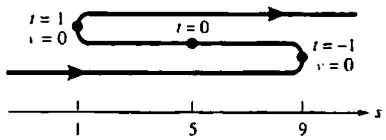
26. a. $v(t) = \frac{ds}{dt} = 6t^2 - 6$

$a(t) = \frac{d^2s}{dt^2} = 12t$

b. $6t^2 - 6 > 0$
$6(t+1)(t-1) > 0$
$(-\infty, -1) \cup (1, \infty)$

c. $6t^2 - 6 < 0$
$(-1, 1)$

d. $12t < 0$
$t < 0$
The acceleration is negative for negative t.

e. 

27. a. $v(t) = \frac{ds}{dt} = 2t - \frac{16}{t^2}$

$a(t) = \frac{d^2s}{dt^2} = 2 + \frac{32}{t^3}$

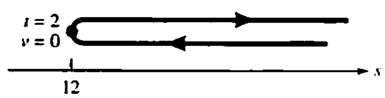
b. $2t - \frac{16}{t^2} > 0$

$\frac{2t^3 - 16}{t^2} > 0;\ (2, \infty)$

c. $2t - \frac{16}{t^2} < 0;\ (0, 2)$

d. $2 + \frac{32}{t^3} < 0$

$\frac{2t^3 + 32}{t^3} < 0$; The acceleration is not negative for any positive t.

e. 

28. a. $v(t) = \frac{ds}{dt} = 1 - \frac{4}{t^2}$

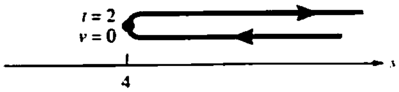
$a(t) = \frac{d^2s}{dt^2} = \frac{8}{t^3}$

b. $1 - \frac{4}{t^2} > 0$

$\frac{t^2 - 4}{t^2} > 0;\ (2, \infty)$

c. $1-\frac{4}{t^2}<0$; (0, 2)

d. $\frac{8}{t^3}<0$; The acceleration is not negative for any positive t.

e. 

29. $v(t)=\frac{ds}{dt}=2t^3-15t^2+24t$

$a(t)=\frac{d^2s}{dt^2}=6t^2-30t+24$

$6t^2-30t+24=0$
$6(t-4)(t-1)=0$
$t=4,\ 1$
$v(4)=-16,\ v(1)=11$

30. $v(t)=\frac{ds}{dt}=\frac{1}{10}(4t^3-42t^2+120t)$

$a(t)=\frac{d^2s}{dt^2}=\frac{1}{10}(12t^2-84t+120)$

$\frac{1}{10}(12t^2-84t+120)=0$

$\frac{12}{10}(t-2)(t-5)=0$
$t=2,\ t=5$
$v(2)=10.4,\ v(5)=5$

31. $v_1(t)=\frac{ds_1}{dt}=4-6t$

$v_2(t)=\frac{ds_2}{dt}=2t-2$

a. $4-6t=2t-2$
$8t=6$
$t=\frac{3}{4}$ sec

b. $|4-6t|=|2t-2|$; $4-6t=-2t+2$
$t=\frac{1}{2}$ sec and $t=\frac{3}{4}$ sec

c. $4t-3t^2=t^2-2t$
$4t^2-6t=0$
$2t(2t-3)=0$
$t=0$ sec and $t=\frac{3}{2}$ sec

32. $v_1(t)=\frac{ds_1}{dt}=9t^2-24t+18$

$v_2(t)=\frac{ds_2}{dt}=-3t^2+18t-12$

$9t^2-24t+18=-3t^2+18t-12$
$12t^2-42t+30=0$
$2t^2-7t+5=0$
$(2t-5)(t-1)=0$
$t=1,\ \frac{5}{2}$

33. a. $v(t)=-32t+48$
initial velocity = $v_0=48$ ft/sec

b. $-32t+48=0$
$t=\frac{3}{2}$ sec

c. $s=-16(1.5)^2+48(1.5)+256=292$ ft

d. $-16t^2+48t+256=0$

$t=\frac{-48\pm\sqrt{48^2-4(-16)(256)}}{-32}\approx-2.77, 5.77$

The object hits the ground at $t=5.77$ sec.

e. $v(5.77)\approx-137$ ft/sec;
speed = $|-137|=137$ ft/sec.

34. $v(t)=48-32t$

a. $48-32t=0$
$t=1.5$
$s=48(1.5)-16(1.5)^2=36$ ft

b. $v(1)=16$ ft/sec upward

c. $48t-16t^2=0$
$-16t(-3+t)=0$
$t=3$ sec

35. $v(t)=v_0-32t$
$v_0-32t=0$

$t=\frac{v_0}{32}$

$v_0\left(\frac{v_0}{32}\right)-16\left(\frac{v_0}{32}\right)^2=5280$

$\frac{v_0^2}{32}-\frac{v_0^2}{64}=5280$

$\frac{v_0^2}{64}=5280$

$v_0=\sqrt{337,920}\approx581$ ft/sec

36. $v(t) = v_0 + 32t$

$v_0 + 32t = 140$

$v_0 + 32(3) = 140$

$v_0 = 44$

$s = 44(3) + 16(3)^2 = 276$ ft

37. $v(t) = 3t^2 - 6t - 24$

$$\frac{d}{dt}\left|3t^2 - 6t - 24\right| = \frac{\left|3t^2 - 6t - 24\right|}{3t^2 - 6t - 24}(6t - 6)$$

$$= \frac{|(t-4)(t+2)|}{(t-4)(t+2)}(6t-6)$$

$$\frac{|(t-4)(t+2)|(6t-6)}{(t-4)(t+2)} < 0$$

$t < -2,\ 1 < t < 4;\ (-\infty, -2) \cup (1, 4)$

38. Point slowing down when

$$\frac{d}{dt}|v(t)| < 0$$

$$\frac{d}{dt}|v(t)| = \frac{|v(t)|a(t)}{v(t)}$$

$\frac{|v(t)|a(t)}{v(t)} < 0$ when $a(t)$ and $v(t)$ have opposite signs.

39. Let s be the distance traveled. Then $\frac{ds}{dt}$ is the speed of the car.

a. $\frac{ds}{dt} = ks$, k a constant

b. $\frac{d^2s}{dt^2} > 0$

c. $\frac{d^3s}{dt^3} < 0, \frac{d^2s}{dt^2} > 0$

d. $\frac{d^2s}{dt^2} = 10$ mph/min

e. $\frac{ds}{dt}$ and $\frac{d^2s}{dt^2}$ are approaching zero.

f. $\frac{ds}{dt}$ is constant.

40. a. $\frac{dV}{dt} = k < 0$, V is the volume of water in the tank, k is a constant.

b. $\frac{dV}{dt} = 3 - \frac{1}{2} = 2\frac{1}{2}$ gal/min

c. $\frac{dV}{dt} = k, \frac{dh}{dt} > 0, \frac{d^2h}{dt^2} < 0$

d. $I(t) = k$ now, but $\frac{dI}{dt}, \frac{d^2I}{dt^2} > 0$ in the future where I is inflation.

e. $\frac{dp}{dt} < 0$, but $\frac{d^2p}{dt^2} > 0$ and at $t = 2$: $\frac{dp}{dt} > 0$. where p is the price of oil.

f. $\frac{dT}{dt} > 0, \frac{d^2T}{dt^2} < 0$, where T is David's temperature.

41. a. $\frac{dC}{dt} > 0, \frac{d^2C}{dt^2} > 0$, where C is the car's cost.

b. $f(t)$ is oil consumption at time t.

$\frac{df}{dt} < 0, \frac{d^2f}{dt^2} > 0$

c. $\frac{dP}{dt} > 0, \frac{d^2P}{dt^2} < 0$, where P is world population.

d. $\frac{d\theta}{dt} > 0, \frac{d^2\theta}{dt^2} > 0$, where θ is the angle that the tower makes with the vertical.

e. $P = f(t)$ is profit at time t.

$\frac{dP}{dt} > 0, \frac{d^2P}{dt^2} < 0$

f. R is revenue at time t.

$R < 0,\ \frac{dR}{dt} > 0$

42. a. $R(t) \approx 0.28,\ t < 1981$

b. On [1981, 1983], $\frac{dR}{dt} > 0, \frac{d^2R}{dt^2} > 0$,

$R(1983) \approx 0.36$

43. $D_x(uv) = uv' + u'v$

$D_x^2(uv) = uv'' + u'v' + u'v' + u''v$

$= uv'' + 2u'v' + u''v$

$D_x^3(uv) = uv''' + u'v'' + 2(u'v'' + u''v') + u''v' + u'''v$

$= uv''' + 3u'v'' + 3u''v' + u'''v$

$D_x^n(uv) = \sum_{k=0}^{n} \binom{n}{k} D_x^{n-k}(u) D_x^k(v)$

where $\binom{n}{k}$ is the binomial coefficient

$\frac{n!}{(n-k)!k!}$.

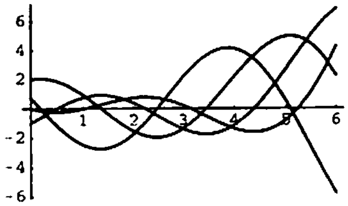
44. $D_x^4(x^4 \sin x) = \binom{4}{0} D_x^4(x^4) D_x^0(\sin x)$

$+\binom{4}{1} D_x^3(x^4) D_x^1(\sin x) + \binom{4}{2} D_x^2(x^4) D_x^2(\sin x)$

$+\binom{4}{3} D_x^1(x^4) D_x^3(\sin x) + \binom{4}{4} D_x^0(x^4) D_x^4(\sin x)$

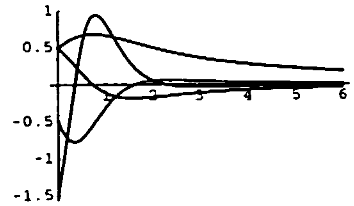
$= 24\sin x + 96x\cos x - 72x^2 \sin x - 16x^3 \cos x + x^4 \sin x$

45. a.



b. $f'''(2.13) \approx -1.2826$

46. a.



b. $f'''(2.13) \approx 0.0271$

3.8 Concepts Review

1. $\frac{9}{x^3 - 3}$

2. $3y^2 \frac{dy}{dx}$

3. $x(2y)\frac{dy}{dx} + y^2 + 3y^2 \frac{dy}{dx} - \frac{dy}{dx} = 3x^2$

4. $\frac{p}{q} x^{p/q-1}; \frac{5}{3}(x^2 - 5x)^{2/3}(2x - 5)$

Problem Set 3.8

1. $2y D_x y - 2x = 0$

$D_x y = \frac{2x}{2y} = \frac{x}{y}$

2. $18x + 8y D_x y = 0$

$D_x y = \frac{-18x}{8y} = -\frac{9x}{4y}$

3. $x D_x y + y = 0$

$D_x y = -\frac{y}{x}$

4. $2x + 2\alpha^2 y D_x y = 0$

$D_x y = -\frac{2x}{2\alpha^2 y} = -\frac{x}{\alpha^2 y}$

5. $x(2y) D_x y + y^2 = 1$

$D_x y = \frac{1 - y^2}{2xy}$

6. $2x + 2x^2 D_x y + 4xy + 3x D_x y + 3y = 0$

$D_x y(2x^2 + 3x) = -2x - 4xy - 3y$

$D_x y = \frac{-2x - 4xy - 3y}{2x^2 + 3x}$

7. $12x^2 + 7x(2y) D_x y + 7y^2 = 6y^2 D_x y$

$12x^2 + 7y^2 = 6y^2 D_x y - 14xy D_x y$

$D_x y = \frac{12x^2 + 7y^2}{6y^2 - 14xy}$

8. $x^2 D_x y + 2xy = y^2 + x(2y) D_x y$

$x^2 D_x y - 2xy D_x y = y^2 - 2xy$

$D_x y = \frac{y^2 - 2xy}{x^2 - 2xy}$

9. $\frac{1}{2\sqrt{5xy}}\cdot(5x\,D_xy+5y)+2D_xy$

$=2y\,D_xy+x(3y^2)D_xy+y^3$

$\frac{5x}{2\sqrt{5xy}}D_xy+2D_xy-2y\,D_xy-3xy^2D_xy$

$=y^3-\frac{5y}{2\sqrt{5xy}}$

$D_xy=\frac{y^3-\frac{5y}{2\sqrt{5xy}}}{\frac{5x}{2\sqrt{5xy}}+2-2y-3xy^2}$

10. $x\frac{1}{2\sqrt{y+1}}D_xy+\sqrt{y+1}=x\,D_xy+y$

$\frac{x}{2\sqrt{y+1}}D_xy-x\,D_xy=y-\sqrt{y+1}$

$D_xy=\frac{y-\sqrt{y+1}}{\frac{x}{2\sqrt{y+1}}-x}$

11. $x\,D_xy+y+\cos(xy)(x\,D_xy+y)=0$

$x\,D_xy+x\cos(xy)D_xy=-y-y\cos(xy)$

$D_xy=\frac{-y-y\cos(xy)}{x+x\cos(xy)}=-\frac{y}{x}$

12. $-\sin(xy^2)(2xy\,D_xy+y^2)=2yD_xy+1$

$-2xy\sin(xy^2)D_xy-2y\,D_xy=1+y^2\sin(xy^2)$

$D_xy=\frac{1+y^2\sin(xy^2)}{-2xy\sin(xy^2)-2y}$

13. $x^3y'+3x^2y+y^3+3xy^2y'=0$

$y'(x^3+3xy^2)=-3x^2y-y^3$

$y'=\frac{-3x^2y-y^3}{x^3+3xy^2}$

At (1, 3), $y'=-\frac{36}{28}=-\frac{9}{7}$

Tangent line: $y-3=-\frac{9}{7}(x-1)$

14. $x^2(2y)y'+2xy^2+4xy'+4y=12y'$

$y'(2x^2y+4x-12)=-2xy^2-4y$

$y'=\frac{-2xy^2-4y}{2x^2y+4x-12}=\frac{-xy^2-2y}{x^2y+2x-6}$

At (2, 1), $y'=-2$

Tangent line: $y-1=-2(x-2)$

15. $\cos(xy)(xy'+y)=y'$

$y'[x\cos(xy)-1]=-y\cos(xy)$

$y'=\frac{-y\cos(xy)}{x\cos(xy)-1}=\frac{y\cos(xy)}{1-x\cos(xy)}$

At $\left(\frac{\pi}{2},1\right)$, $y'=0$

Tangent line: $y-1=0\left(x-\frac{\pi}{2}\right)$

$y=1$

16. $y'+[-\sin(xy^2)][2xyy'+y^2]+6x=0$

$y'[1-2xy\sin(xy^2)]=y^2\sin(xy^2)-6x$

$y'=\frac{y^2\sin(xy^2)-6x}{1-2xy\sin(xy^2)}$

At (1, 0), $y'=-\frac{6}{1}=-6$

Tangent line: $y-0=-6(x-1)$

17. $\frac{2}{3}x^{-1/3}-\frac{2}{3}y^{-1/3}y'-2y'=0$

$\frac{2}{3}x^{-1/3}=y'\left(\frac{2}{3}y^{-1/3}+2\right)$

$y'=\frac{\frac{2}{3}x^{-1/3}}{\frac{2}{3}y^{-1/3}+2}$

At (1, −1), $y'=\frac{\frac{2}{3}}{\frac{4}{3}}=\frac{1}{2}$

Tangent line: $y+1=\frac{1}{2}(x-1)$

18. $\frac{1}{2\sqrt{y}}y'+2xyy'+y^2=0$

$y'\left(\frac{1}{2\sqrt{y}}+2xy\right)=-y^2$

$y'=\frac{-y^2}{\frac{1}{2\sqrt{y}}+2xy}$

At (4, 1), $y'=\frac{-1}{\frac{17}{2}}=-\frac{2}{17}$

Tangent line: $y-1=-\frac{2}{17}(x-4)$

19. $\frac{dy}{dx}=5x^{2/3}+\frac{1}{2\sqrt{x}}$

20. $\frac{dy}{dx}=\frac{1}{3}x^{-2/3}-7x^{5/2}=\frac{1}{3\sqrt[3]{x^2}}-7x^{5/2}$

21. $\frac{dy}{dx}=\frac{1}{3}x^{-2/3}-\frac{1}{3}x^{-4/3}=\frac{1}{3\sqrt[3]{x^2}}-\frac{1}{3\sqrt[3]{x^4}}$

22. $\frac{dy}{dx}=\frac{1}{4}(2x+1)^{-3/4}(2)=\frac{1}{2\sqrt[4]{(2x+1)^3}}$

23. $\frac{dy}{dx}=\frac{1}{4}(3x^2-4x)^{-3/4}(6x-4)$
$=\frac{6x-4}{4\sqrt[4]{(3x^2-4x)^3}}=\frac{3x-2}{2\sqrt[4]{(3x^2-4x)^3}}$

24. $\frac{dy}{dx}=\frac{1}{3}(x^3-2x)^{-2/3}(3x^2-2)$

25. $\frac{dy}{dx}=\frac{d}{dx}[(x^3+2x)^{-2/3}]$
$=-\frac{2}{3}(x^3+2x)^{-5/3}(3x^2+2)=-\frac{6x^2+4}{3\sqrt[3]{(x^3+2x)^5}}$

26. $\frac{dy}{dx}=-\frac{5}{3}(3x-9)^{-8/3}(3)=-5(3x-9)^{-8/3}$

27. $\frac{dy}{dx}=\frac{1}{2\sqrt{x^2+\sin x}}(2x+\cos x)$
$=\frac{2x+\cos x}{2\sqrt{x^2+\sin x}}$

28. $\frac{dy}{dx}=\frac{1}{2\sqrt{x^2\cos x}}[x^2(-\sin x)+2x\cos x]$
$=\frac{2x\cos x-x^2\sin x}{2\sqrt{x^2\cos x}}$

29. $\frac{dy}{dx}=\frac{d}{dx}[(x^2\sin x)^{-1/3}]$
$=-\frac{1}{3}(x^2\sin x)^{-4/3}(x^2\cos x+2x\sin x)$
$=-\frac{x^2\cos x+2x\sin x}{3\sqrt[3]{(x^2\sin x)^4}}$

30. $\frac{dy}{dx}=\frac{1}{4}(1+\sin 5x)^{-3/4}(\cos 5x)(5)$
$=\frac{5\cos 5x}{4\sqrt[4]{(1+\sin 5x)^3}}$

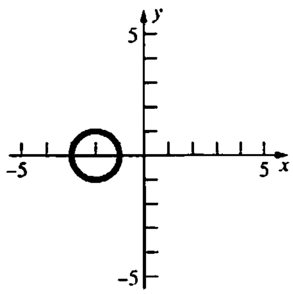
31. $\frac{dy}{dx}=\frac{[1+\cos(x^2+2x)]^{-3/4}[-\sin(x^2+2x)(2x+2)]}{4}$
$=-\frac{(x+1)\sin(x^2+2x)}{2\sqrt[4]{[1+\cos(x^2+2x)]^3}}$

32. $\frac{dy}{dx}=\frac{(\tan^2 x+\sin^2 x)^{-1/2}(2\tan x\sec^2 x+2\sin x\cos x)}{2}$
$=\frac{\tan x\sec^2 x+\sin x\cos x}{\sqrt{\tan^2 x+\sin^2 x}}$

33. $s^2+2st\frac{ds}{dt}+3t^2=0$
$\frac{ds}{dt}=\frac{-s^2-3t^2}{2st}=-\frac{s^2+3t^2}{2st}$
$s^2\frac{dt}{ds}+2st+3t^2\frac{dt}{ds}=0$
$\frac{dt}{ds}(s^2+3t^2)=-2st$
$\frac{dt}{ds}=-\frac{2st}{s^2+3t^2}$

34. $1=\cos(x^2)(2x)\frac{dx}{dy}+6x^2\frac{dx}{dy}$
$\frac{dx}{dy}=\frac{1}{2x\cos(x^2)+6x^2}$

35.



$2x+4+2y\frac{dy}{dx}=0$
$\frac{dy}{dx}=-\frac{2x+4}{2y}=-\frac{x+2}{y}$
The tangent line at (x_0, y_0) has equation
$y-y_0=-\frac{x_0+2}{y_0}(x-x_0)$ which simplifies to
$2x_0-yy_0-2x-xx_0+y_0^2+x_0^2=0$. Since
(x_0, y_0) is on the circle, $x_0^2+y_0^2=-3-4x_0$,
so the equation of the tangent line is
$-yy_0-2x_0-2x-xx_0=3$.

If (0, 0) is on the tangent line, then $x_0 = -\frac{3}{2}$. Solve for y_0 in the equation of the circle to get $y_0 = \pm\frac{\sqrt{3}}{2}$. Put these values into the equation of the tangent line to get that the tangent lines are $\sqrt{3}y + x = 0$ and $\sqrt{3}y - x = 0$.

36. $16(x^2 + y^2)(2x + 2yy') = 100(2x - 2yy')$

$32x^3 + 32x^2yy' + 32xy^2 + 32y^3y' = 200x - 200yy'$

$y'(4x^2y + 4y^3 + 25y) = 25x - 4x^3 - 4xy^2$

$y' = \frac{25x - 4x^3 - 4xy^2}{4x^2y + 4y^3 + 25y}$

The slope of the normal line $= -\frac{1}{y'}$

$= \frac{4x^2y + 4y^3 + 25y}{4x^3 + 4xy^2 - 25x}$

At (3, 1), slope $= \frac{65}{45} = \frac{13}{9}$

Normal line: $y - 1 = \frac{13}{9}(x - 3)$

37. a. $xy' + y + 3y^2y' = 0$

$y'(x + 3y^2) = -y$

$y' = -\frac{y}{x + 3y^2}$

b. $xy'' + \left(\frac{-y}{x + 3y^2}\right) + \left(\frac{-y}{x + 3y^2}\right) + 3y^2y''$

$+6y\left(\frac{-y}{x + 3y^2}\right)^2 = 0$

$xy'' + 3y^2y'' - \frac{2y}{x + 3y^2} + \frac{6y^3}{(x + 3y^2)^2} = 0$

$y''(x + 3y^2) = \frac{2y}{x + 3y^2} - \frac{6y^3}{(x + 3y^2)^2}$

$y''(x + 3y^2) = \frac{2xy}{(x + 3y^2)^2}$

$y'' = \frac{2xy}{(x + 3y^2)^3}$

38. $3x^2 - 8yy' = 0$

$y' = \frac{3x^2}{8y}$

$6x - 8(yy'' + (y')^2) = 0$

$6x - 8yy'' - 8\left(\frac{3x^2}{8y}\right)^2 = 0$

$6x - 8yy'' - \frac{9x^4}{8y^2} = 0$

$\frac{48xy^2 - 9x^4}{8y^2} = 8yy''$

$y'' = \frac{48xy^2 - 9x^4}{64y^3}$

39. $2(x^2y' + 2xy) - 12y^2y' = 0$

$2x^2y' - 12y^2y' = -4xy$

$y' = \frac{2xy}{6y^2 - x^2}$

$2(x^2y'' + 2xy' + 2xy' + 2y) - 12[y^2y'' + 2y(y')^2] = 0$

$2x^2y'' - 12y^2y'' = -8xy' - 4y + 24y(y')^2$

$y''(2x^2 - 12y^2) = -\frac{16x^2y}{6y^2 - x^2} - 4y + \frac{96x^2y^3}{(6y^2 - x^2)^2}$

$y''(2x^2 - 12y^2) = \frac{12x^4y + 48x^2y^3 - 144y^5}{(6y^2 - x^2)^2}$

$y''(6y^2 - x^2) = \frac{72y^5 - 6x^4y - 24x^2y^3}{(6y^2 - x^2)^2}$

$y'' = \frac{72y^5 - 6x^4y - 24x^2y^3}{(6y^2 - x^2)^3}$

At (2, 1), $y'' = \frac{-120}{8} = -15$

40. $2x + 2yy' = 0$

$y' = -\frac{2x}{2y} = -\frac{x}{y}$

$2 + 2[yy'' + (y')^2] = 0$

$2 + 2yy'' + 2\left(-\frac{x}{y}\right)^2 = 0$

$2yy'' = -2 - \frac{2x^2}{y^2}$

$y'' = -\frac{1}{y} - \frac{x^2}{y^3} = -\frac{y^2 + x^2}{y^3}$

At (3, 4), $y'' = -\frac{25}{64}$

41. $3x^2 + 3y^2y' = 3(xy' + y)$

$y'(3y^2 - 3x) = 3y - 3x^2$

$y' = \frac{y - x^2}{y^2 - x}$

At $\left(\frac{3}{2}, \frac{3}{2}\right)$, $y' = -1$

Slope of the normal line is 1.

Normal line: $y - \frac{3}{2} = 1\left(x - \frac{3}{2}\right)$; $y = x$

This line includes the point (0, 0).

42. $xy' + y = 0$

$y' = -\frac{y}{x}$

$2x - 2yy' = 0$

$y' = \frac{x}{y}$

The slopes of the tangents are negative reciprocals, so the hyperbolas intersect at right angles.

43. Implicitly differentiate the first equation.

$4x + 2yy' = 0$

$y' = -\frac{2x}{y}$

Implicitly differentiate the second equation.

$2yy' = 4$

$y' = \frac{2}{y}$

Solve for the points of intersection.

$2x^2 + 4x = 6$

$2(x^2 + 2x - 3) = 0$

$(x + 3)(x - 1) = 0$

$x = -3, x = 1$

$x = -3$ is extraneous, and $y = -2, 2$ when $x = 1$.

The graphs intersect at (1, –2) and (1, 2).

At (1, –2): $m_1 = 1, m_2 = -1$

At (1, 2): $m_1 = -1, m_2 = 1$

44. $(x - 1)^2 + (1 - x^2) = 1$

$x^2 - 2x + 1 + 1 - x^2 = 1$

$x = \frac{1}{2}$

Points of intersection: $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

Implicitly differentiate the first equation.

$2x + 2yy' = 0$

$y' = -\frac{x}{y}$

Implicitly differentiate the second equation.

$2(x - 1) + 2yy' = 0$

$y' = \frac{1 - x}{y}$

At $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$: $m_1 = -\frac{1}{\sqrt{3}}, m_2 = \frac{1}{\sqrt{3}}$

$\tan\theta = \frac{\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}}{1 + \left(\frac{1}{\sqrt{3}}\right)\left(-\frac{1}{\sqrt{3}}\right)} = \frac{\frac{2}{\sqrt{3}}}{\frac{2}{3}} = \sqrt{3}$

$\theta = \frac{\pi}{3}$

At $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$: $m_1 = \frac{1}{\sqrt{3}}, m_2 = -\frac{1}{\sqrt{3}}$

$\tan\theta = \frac{-\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}}}{1 + \left(\frac{1}{\sqrt{3}}\right)\left(-\frac{1}{\sqrt{3}}\right)} = \frac{-\frac{2}{\sqrt{3}}}{\frac{2}{3}} = -\sqrt{3}$

$\theta = \frac{2\pi}{3}$

45. $x^2 - x(2x) + 2(2x)^2 = 28$

$7x^2 = 28$

$x^2 = 4$

$x = -2, 2$

Intersection point in first quadrant: (2, 4)

$y_1' = 2$

$2x - xy_2' - y + 4yy_2' = 0$

$y_2'(4y - x) = y - 2x$

$y_2' = \frac{y - 2x}{4y - x}$

At (2, 4): $m_1 = 2, m_2 = 0$

$\tan\theta = \frac{0 - 2}{1 + (0)(2)} = -2; \theta = \pi + \tan^{-1}(-2) \approx 2.034$

46. The equation is $mv^2 - mv_0^2 = kx_0^2 - kx^2$.

Differentiate implicitly with respect to t to get $2mv\frac{dv}{dt} = -2kx\frac{dx}{dt}$. Since $v = \frac{dx}{dt}$ this simplifies to $2mv\frac{dv}{dt} = -2kxv$ or $m\frac{dv}{dt} = -kx$.

47. $x^2 - xy + y^2 = 16$, when $y = 0$,

$x^2 = 16$

$x = -4, 4$

The ellipse intersects the x-axis at $(-4, 0)$ and $(4, 0)$.

$2x - xy' - y + 2yy' = 0$

$y'(2y - x) = y - 2x$

$y' = \frac{y-2x}{2y-x}$

At $(-4, 0)$, $y' = 2$

At $(4, 0)$, $y' = 2$

Tangent lines: $y = 2(x + 4)$ and $y = 2(x - 4)$

48. $x^2 + 2xy\frac{dx}{dy} - 2xy - y^2\frac{dx}{dy} = 0$

$\frac{dx}{dy}(2xy - y^2) = 2xy - x^2$;

$\frac{dx}{dy} = \frac{2xy - x^2}{2xy - y^2}$

$\frac{2xy - x^2}{2xy - y^2} = 0$ if $x(2y - x) = 0$, which occurs when $x = 0$ or $y = \frac{x}{2}$. There are no points on $x^2y - xy^2 = 2$ where $x = 0$. If $y = \frac{x}{2}$, then

$2 = x^2\left(\frac{x}{2}\right) - x\left(\frac{x}{2}\right)^2 = \frac{x^3}{2} - \frac{x^3}{4} = \frac{x^3}{4}$ so $x = 2$,

$y = \frac{2}{2} = 1$.

The tangent line is vertical at (2, 1).

49. $2x + 2y\frac{dy}{dx} = 0; \frac{dy}{dx} = -\frac{x}{y}$

The tangent line at (x_0, y_0) has slope $-\frac{x_0}{y_0}$, hence the equation of the tangent line is $y - y_0 = -\frac{x_0}{y_0}(x - x_0)$ which simplifies to $yy_0 + xx_0 - (x_0^2 + y_0^2) = 0$ or $yy_0 + xx_0 = 1$ since (x_0, y_0) is on $x^2 + y^2 = 1$. If (1.25, 0) is on the tangent line through (x_0, y_0), $x_0 = 0.8$. Put this into $x^2 + y^2 = 1$ to get $y_0 = 0.6$, since $y_0 > 0$. The line is $6y + 8x = 10$. When $x = -2$, $y = \frac{13}{3}$, so the light bulb must be $\frac{13}{3}$ units high.

3.9 Concepts Review

1. $\frac{du}{dt}; t = 2$

2. 400 mi/hr

3. negative

4. negative; positive

Problem Set 3.9

1. $V = x^3; \frac{dx}{dt} = 3$

$\frac{dV}{dt} = 3x^2\frac{dx}{dt}$

When $x = 12$, $\frac{dV}{dt} = 3(12)^2(3) = 1296$ in.³/s.

2. $V = \frac{4}{3}\pi r^3; \frac{dV}{dt} = 3$

$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$

When $r = 3$, $3 = 4\pi(3)^2\frac{dr}{dt}$.

$\frac{dr}{dt} = \frac{1}{12\pi} \approx 0.027$ in./s

3. $y^2 = x^2 + 1^2; \frac{dx}{dt} = 400$

$2y\frac{dy}{dt} = 2x\frac{dx}{dt}$

$\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt}$ mi/hr

When $x = 5, y = \sqrt{26}, \frac{dy}{dt} = \frac{5}{\sqrt{26}}(400)$

≈ 392 mi/h.

4. $V = \frac{1}{3}\pi r^2 h; \frac{r}{h} = \frac{3}{10}; r = \frac{3h}{10}$

$V = \frac{1}{3}\pi\left(\frac{3h}{10}\right)^2 h = \frac{3\pi h^3}{100}; \frac{dV}{dt} = 3, h = 5$

$\frac{dV}{dt} = \frac{9\pi h^2}{100}\frac{dh}{dt}$

When $h = 5$, $3 = \frac{9\pi(5)^2}{100}\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{4}{3\pi} \approx 0.42$ cm/s

5. $s^2 = (x+300)^2 + y^2; \frac{dx}{dt} = 300, \frac{dy}{dt} = 400,$

$2s\frac{ds}{dt} = 2(x+300)\frac{dx}{dt} + 2y\frac{dy}{dt}$

$s\frac{ds}{dt} = (x+300)\frac{dx}{dt} + y\frac{dy}{dt}$

When $x = 300$, $y = 400$, $s = 200\sqrt{13}$, so

$200\sqrt{13}\frac{ds}{dt} = (300+300)(300) + 400(400)$

$\frac{ds}{dt} \approx 471$ mi/h

6. $y^2 = x^2 + (10)^2; \frac{dy}{dt} = 2$

$2y\frac{dy}{dt} = 2x\frac{dx}{dt}$

When $y = 25$, $x \approx 22.9$, so

$\frac{dx}{dt} = \frac{y}{x}\frac{dy}{dt} \approx \frac{25}{22.9}(2) \approx 2.18$ ft/s

7. $20^2 = x^2 + y^2; \frac{dx}{dt} = 1$

$0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

When $x = 5$, $y = \sqrt{375} = 5\sqrt{15}$, so

$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -\frac{5}{5\sqrt{15}}(1) \approx -0.258$ ft/s

The top of the ladder is moving down at 0.258 ft/s.

8. $\frac{dV}{dt} = -4$ ft³/h; $V = \pi h r^2; \frac{dh}{dt} = -0.0005$ ft/h

$A = \pi r^2 = \frac{V}{h} = Vh^{-1}$, so $\frac{dA}{dt} = h^{-1}\frac{dV}{dt} - \frac{V}{h^2}\frac{dh}{dt}$.

When $h = 0.001$ ft, $V = \pi(0.001)(250)^2 = 62.5\pi$

and $\frac{dA}{dt} = 1000(-4) - 1{,}000{,}000(62.5\pi)(-0.0005)$

$= -4000 + 31{,}250\pi \approx 94{,}175$ ft²/h.

(The height is decreasing due to the spreading of the oil rather than the bacteria.)

9. $V = \frac{1}{3}\pi r^2 h; h = \frac{d}{4} = \frac{r}{2}, r = 2h$

$V = \frac{1}{3}\pi(2h)^2 h = \frac{4}{3}\pi h^3; \frac{dV}{dt} = 16$

$\frac{dV}{dt} = 4\pi h^2\frac{dh}{dt}$

When $h = 4$, $16 = 4\pi(4)^2\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{1}{4\pi} \approx 0.0796$ ft/s

10. $y^2 = x^2 + (90)^2; \frac{dx}{dt} = 5$

$2y\frac{dy}{dt} = 2x\frac{dx}{dt}$

When $y = 150$, $x = 120$, so

$\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt} = \frac{120}{150}(5) = 4$ ft/s

11. $V = \frac{hx}{2}(20); \frac{40}{5} = \frac{x}{h}, x = 8h$

$V = 10h(8h) = 80h^2; \frac{dV}{dt} = 40$

$\frac{dV}{dt} = 160h\frac{dh}{dt}$

When $h = 3$, $40 = 160(3)\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{1}{12}$ ft/s

12. $y = \sqrt{x^2 - 4}; \frac{dx}{dt} = 5$

$\frac{dy}{dt} = \frac{1}{2\sqrt{x^2-4}}(2x)\frac{dx}{dt} = \frac{x}{\sqrt{x^2-4}}\frac{dx}{dt}$

When $x = 3$, $\frac{dy}{dt} = \frac{3}{\sqrt{3^2-4}}(5) = \frac{15}{\sqrt{5}} \approx 6.7$ units/s

13. $A = \pi r^2; \frac{dr}{dt} = 0.02$

$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$

When $r = 8.1$, $\frac{dA}{dt} = 2\pi(0.02)(8.1) = 0.324\pi$

≈ 1.018 in.²/s

14. $s^2 = x^2 + (y+48)^2; \frac{dx}{dt} = 30, \frac{dy}{dt} = 24$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2(y+48)\frac{dy}{dt}$$

$$s\frac{ds}{dt} = x\frac{dx}{dt} + (y+48)\frac{dy}{dt}$$

At 2:00 p.m., $x = 3(30) = 90, y = 3(24) = 72$, so $s = 150$.

$$(150)\frac{ds}{dt} = 90(30) + (72+48)(24)$$

$$\frac{ds}{dt} = \frac{5580}{150} = 37.2 \text{ knots/h}$$

15. Let x be the distance from the beam to the point opposite the lighthouse and θ be the angle between the beam and the line from the lighthouse to the point opposite.

$$\tan\theta = \frac{x}{1}; \frac{d\theta}{dt} = 2(2\pi) = 4\pi \text{ rad/min.}$$

$$\sec^2\theta\frac{d\theta}{dt} = \frac{dx}{dt}$$

At $x = \frac{1}{2}, \theta = \tan^{-1}\frac{1}{2}$ and $\sec^2\theta = \frac{5}{4}$.

$$\frac{dx}{dt} = \frac{5}{4}(4\pi) \approx 15.71 \text{ km/min}$$

16. $\tan\theta = \frac{4000}{x}$

$$\sec^2\theta\frac{d\theta}{dt} = -\frac{4000}{x^2}\frac{dx}{dt}$$

When $\theta = \frac{1}{2}, \frac{d\theta}{dt} = \frac{1}{10}$ and $x = \frac{4000}{\tan\frac{1}{2}} \approx 7322$.

$$\frac{dx}{dt} \approx \sec^2\frac{1}{2}\left(\frac{1}{10}\right)\left[-\frac{(7322)^2}{4000}\right]$$

≈ -1740 ft/s or -1186 mi/h

The plane's ground speed is 1186 mi/h.

17. a. Let x be the distance along the ground from the light pole to Chris, and let s be the distance from Chris to the tip of his shadow. By similar triangles, $\frac{6}{s} = \frac{30}{x+s}$, so $s = \frac{x}{4}$ and $\frac{ds}{dt} = \frac{1}{4}\frac{dx}{dt}$. $\frac{dx}{dt} = 2$ ft/s, hence $\frac{ds}{dt} = \frac{1}{2}$ ft/s no matter how far from the light pole Chris is.

b. Let $l = x + s$, then

$$\frac{dl}{dt} = \frac{dx}{dt} + \frac{ds}{dt} = 2 + \frac{1}{2} = \frac{5}{2} \text{ ft/s.}$$

c. The angular rate at which Chris must lift his head to follow his shadow is the same as the rate at which the angle that the light makes with the ground is decreasing. Let θ be the angle that the light makes with the ground at the tip of Chris' shadow.

$\tan\theta = \frac{6}{s}$ so $\sec^2\theta\frac{d\theta}{dt} = -\frac{6}{s^2}\frac{ds}{dt}$ and

$$\frac{d\theta}{dt} = -\frac{6\cos^2\theta}{s^2}\frac{ds}{dt}. \quad \frac{ds}{dt} = \frac{1}{2} \text{ ft/s}$$

When $s = 6$, $\theta = \frac{\pi}{4}$, so

$$\frac{d\theta}{dt} = -\frac{6\left(\frac{1}{\sqrt{2}}\right)^2}{6^2}\left(\frac{1}{2}\right) = -\frac{1}{24}.$$

Chris must lift his head at the rate of $\frac{1}{24}$ rad/s.

18. Let θ be the measure of the vertex angle, a be the measure of the equal sides, and b be the measure of the base. Observe that $b = 2a\sin\frac{\theta}{2}$ and the height of the triangle is $a\cos\frac{\theta}{2}$.

$$A = \frac{1}{2}\left(2a\sin\frac{\theta}{2}\right)\left(a\cos\frac{\theta}{2}\right) = \frac{1}{2}a^2\sin\theta$$

$$A = \frac{1}{2}(100)^2\sin\theta = 5000\sin\theta; \frac{d\theta}{dt} = \frac{1}{10}$$

$$\frac{dA}{dt} = 5000\cos\theta\frac{d\theta}{dt}$$

When $\theta = \frac{\pi}{6}, \frac{dA}{dt} = 5000\left(\cos\frac{\pi}{6}\right)\left(\frac{1}{10}\right) = 250\sqrt{3}$

$\approx 433 \text{ cm}^2/\text{min}$.

19. Let p be the point on the bridge directly above the railroad tracks. If a is the distance between p and the automobile, then $\frac{da}{dt} = 66$ ft/s. If l is the distance between the train and the point directly below p, then $\frac{dl}{dt} = 88$ ft/s. The distance from the train to p is $\sqrt{100^2 + l^2}$, while the distance from p to the automobile is a. The distance between the train and automobile is

$$D = \sqrt{a^2 + \left(\sqrt{100^2 + l^2}\right)^2} = \sqrt{a^2 + l^2 + 100^2}.$$

$$\frac{dD}{dt} = \frac{1}{2\sqrt{a^2 + l^2 + 100^2}} \cdot \left(2a\frac{da}{dt} + 2l\frac{dl}{dt}\right)$$

$$= \frac{a\frac{da}{dt} + l\frac{dl}{dt}}{\sqrt{a^2 + l^2 + 100^2}}.$$ After 10 seconds, $a = 660$ and $l = 880$, so

$$\frac{dD}{dt} = \frac{660(66) + 880(88)}{\sqrt{660^2 + 880^2 + 100^2}} \approx 110 \text{ ft/s.}$$

20. $V = \frac{1}{3}\pi h \cdot (a^2 + ab + b^2); a = 20, b = \frac{h}{4} + 20,$

$$V = \frac{1}{3}\pi h\left(400 + 5h + 400 + \frac{h^2}{16} + 10h + 400\right)$$

$$= \frac{1}{3}\pi\left(1200h + 15h^2 + \frac{h^3}{16}\right)$$

$$\frac{dV}{dt} = \frac{1}{3}\pi\left(1200 + 30h + \frac{3h^2}{16}\right)\frac{dh}{dt}$$

When $h = 30$ and $\frac{dV}{dt} = 2000$,

$$2000 = \frac{1}{3}\pi\left(1200 + 900 + \frac{675}{4}\right)\frac{dh}{dt} = \frac{3025\pi}{4}\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{320}{121\pi} \approx 0.84 \text{ cm/min.}$$

21. $V = \pi h^2\left[r - \frac{h}{3}\right]; \frac{dV}{dt} = -2, r = 8$

$$V = \pi r h^2 - \frac{\pi h^3}{3} = 8\pi h^2 - \frac{\pi h^3}{3}$$

$$\frac{dV}{dt} = 16\pi h\frac{dh}{dt} - \pi h^2\frac{dh}{dt}$$

When $h = 3$, $-2 = \frac{dh}{dt}[16\pi(3) - \pi(3)^2]$

$$\frac{dh}{dt} = \frac{-2}{39\pi} \approx -0.016 \text{ ft/hr}$$

22. $s^2 = a^2 + b^2 - 2ab\cos\theta;$

$a = 5, b = 4, \frac{d\theta}{dt} = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$ rad/h

$$s^2 = 41 - 40\cos\theta$$

$$2s\frac{ds}{dt} = 40\sin\theta\frac{d\theta}{dt}$$

At 3:00, $\theta = \frac{\pi}{2}$ and $s = \sqrt{41}$, so

$$2\sqrt{41}\frac{ds}{dt} = 40\sin\left(\frac{\pi}{2}\right)\left(\frac{11\pi}{6}\right) = \frac{220\pi}{3}$$

$$\frac{ds}{dt} \approx 18 \text{ in./hr}$$

23. $PV = k$

$$P\frac{dV}{dt} + V\frac{dP}{dt} = 0$$

At $t = 6.5$, $P \approx 67$, $\frac{dP}{dt} \approx -30$, $V = 300$

$$\frac{dV}{dt} = -\frac{V}{P}\frac{dP}{dt} = -\frac{300}{67}(-30) \approx 134 \text{ in.}^3\text{/min}$$

24. $V = \pi h^2\left(r - \frac{h}{3}\right); r = 20$

$$V = \pi h^2\left(20 - \frac{h}{3}\right) = 20\pi h^2 - \frac{\pi}{3}h^3$$

$$\frac{dV}{dt} = (40\pi h - \pi h^2)\frac{dh}{dt}$$

At 7:00 a.m., $h = 15$, $\frac{dh}{dt} \approx -3$, so

$$\frac{dV}{dt} = (40\pi(15) - \pi(15)^2)(-3) \approx -1125\pi \approx -3534.$$

Webster City residents used water at the rate of $2400 + 3534 = 5934$ ft^3/h.

25. Assuming that the tank is now in the shape of an upper hemisphere with radius r, we again let t be the number of hours past midnight and h be the height of the water at time t. The volume, V, of water in the tank at that time is given by

$$V = \frac{2}{3}\pi r^3 - \frac{\pi}{3}(r - h)^2(2r + h)$$

and so $V = \frac{16000}{3}\pi - \frac{\pi}{3}(20 - h)^2(40 + h)$

from which

$$\frac{dV}{dt} = -\frac{\pi}{3}(20 - h)^2\frac{dh}{dt} + \frac{2\pi}{3}(20 - h)(40 + h)\frac{dh}{dt}$$

At $t = 7$, $\frac{dV}{dt} \approx -525\pi \approx -1649$

Thus Webster City residents were using water at the rate of $2400 + 1649 = 4049$ cubic feet per hour at 7:00 A.M.

26. The amount of water used by Webster City can be found by

usage = beginning amount + added amount − remaining amount

Thus the usage is
$\approx \pi(20)^2(9) + 2400(12) - \pi(20)^2(10.5) \approx 26{,}915$ ft^3
over the 12 hour period.

27. a. Let x be the distance from the bottom of the wall to the end of the ladder on the ground, so $\frac{dx}{dt} = 2$ ft/s. Let y be the height of the opposite end of the ladder. By similar triangles, $\frac{y}{12} = \frac{18}{\sqrt{144+x^2}}$, so $y = \frac{216}{\sqrt{144+x^2}}$.

$$\frac{dy}{dt} = -\frac{216}{2(144+x^2)^{3/2}}2x\frac{dx}{dt} = -\frac{216x}{(144+x^2)^{3/2}}\frac{dx}{dt}$$

When the ladder makes an angle of 60° with the ground, $x = 4\sqrt{3}$ and $\frac{dy}{dt} = -\frac{216(4\sqrt{3})}{(144+48)^{3/2}} \cdot 2 = -1.125$ ft/s.

b. $$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(-\frac{216x}{(144+x^2)^{3/2}}\frac{dx}{dt}\right) = \frac{d}{dt}\left(-\frac{216x}{(144+x^2)^{3/2}}\right)\frac{dx}{dt} - \frac{216x}{(144+x^2)^{3/2}} \cdot \frac{d^2x}{dt^2}$$

Since $\frac{dx}{dt} = 2, \frac{d^2x}{dt^2} = 0$, thus

$$\frac{d^2y}{dt^2} = \left[\frac{-216(144+x^2)^{3/2}\frac{dx}{dt} + 216x\left(\frac{3}{2}\right)\sqrt{144+x^2}(2x)\frac{dx}{dt}}{(144+x^2)^3}\right]\frac{dx}{dt}$$

$$= \frac{-216(144+x^2)+648x^2}{(144+x^2)^{5/2}}\left(\frac{dx}{dt}\right)^2 = \frac{432x^2-31{,}104}{(144+x^2)^{5/2}}\left(\frac{dx}{dt}\right)^2$$

When the ladder makes an angle of 60° with the ground,

$$\frac{d^2y}{dt^2} = \frac{432 \cdot 48 - 31{,}104}{(144+48)^{5/2}}(2)^2 \approx -0.08 \text{ ft/s}^2$$

28. a. If the ball has radius 6 in., the volume of the water in the tank is

$$V = 8\pi h^2 - \frac{\pi h^3}{3} - \frac{4}{3}\pi\left(\frac{1}{2}\right)^3$$

$$= 8\pi h^2 - \frac{\pi h^3}{3} - \frac{\pi}{6}$$

$$\frac{dV}{dt} = 16\pi h\frac{dh}{dt} - \pi h^2\frac{dh}{dt}$$

This is the same as in Problem 21, so $\frac{dh}{dt}$ is again –0.016 ft/hr.

b. If the ball has radius 2 ft, and the height of the water in the tank is h feet with $2 \le h \le 3$, the part of the ball in the water has volume

$$\frac{4}{3}\pi(2)^3 - \pi(4-h)^2\left[2 - \frac{4-h}{3}\right] = \frac{(6-h)h^2\pi}{3}.$$

The volume of water in the tank is

$$V = 8\pi h^2 - \frac{\pi h^3}{3} - \frac{(6-h)h^2\pi}{3} = 6h^2\pi$$

$$\frac{dV}{dt} = 12h\pi\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{12h\pi}\frac{dV}{dt}$$

When $h = 3$, $\frac{dh}{dt} = \frac{1}{36\pi}(-2) \approx -0.018$ ft/hr.

29. $\frac{dV}{dt} = k(4\pi r^2)$

a. $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

$$k(4\pi r^2) = 4\pi r^2\frac{dr}{dt}$$

$$\frac{dr}{dt} = k$$

b. If the original volume was V_0, the volume after 1 hour is $\frac{8}{27}V_0$. The original radius was $r_0 = \sqrt[3]{\frac{3}{4\pi}V_0}$ while the radius after 1 hour is $r_1 = \sqrt[3]{\frac{8}{27}V_0 \cdot \frac{3}{4\pi}} = \frac{2}{3}r_0$. Since $\frac{dr}{dt}$ is constant, $\frac{dr}{dt} = -\frac{1}{3}r_0$ unit/hr. The snowball will take 3 hours to melt completely.

30. Let P be the point on the ground where the ball hits. Then the distance from P to the bottom of the light pole is 10 ft. Let s be the distance between P and the shadow of the ball. The height of the ball t seconds after it is dropped is $64 - 16t^2$.

By similar triangles, $\frac{48}{64-16t^2} = \frac{10+s}{s}$

(for $t > 1$), so $s = \frac{10t^2 - 40}{1-t^2}$.

$\frac{ds}{dt} = \frac{20t(1-t^2)-(10t^2-40)(-2t)}{(1-t^2)^2} = -\frac{60t}{(1-t^2)^2}$

The ball hits the ground when $t = 2$, $\frac{ds}{dt} = -\frac{120}{9}$.

The shadow is moving $\frac{120}{9} \approx 13.33$ ft/s.

31. Let l be the distance along the ground from the brother to the tip of the shadow. The shadow is controlled by both siblings when $\frac{3}{l} = \frac{5}{l+4}$ or $l = 6$. Again using similar triangles, this occurs when $\frac{y}{20} = \frac{6}{3}$, so $y = 40$. Thus, the girl controls the tip of the shadow when $y \ge 40$ and the boy controls it when $y < 40$.
Let x be the distance along the ground from the light pole to the girl. $\frac{dx}{dt} = -4$

When $y \ge 40$, $\frac{20}{y} = \frac{5}{y-x}$ or $y = \frac{4}{3}x$.

When $y < 40$, $\frac{20}{y} = \frac{3}{y-(x+4)}$ or $y = \frac{20}{17}(x+4)$.

$x = 30$ when $y = 40$. Thus,

$$y = \begin{cases} \frac{4}{3}x & \text{if } x \ge 30 \\ \frac{20}{17}(x+4) & \text{if } x < 30 \end{cases}$$

and

$$\frac{dy}{dt} = \begin{cases} \frac{4}{3}\frac{dx}{dt} & \text{if } x \ge 30 \\ \frac{20}{17}\frac{dx}{dt} & \text{if } x < 30 \end{cases}$$

Hence, the tip of the shadow is moving at the rate of $\frac{4}{3}(4) = \frac{16}{3}$ ft/s when the girl is at least 30 feet from the light pole, and it is moving $\frac{20}{17}(4) = \frac{80}{17}$ ft/s when the girl is less than 30 ft from the light pole.

3.10 Concepts Review

1. $f'(x)dx$

2. Δy; dy

3. Δx is small.

4. larger ; smaller

Problem Set 3.10

1. $dy = (2x + 1)dx$

2. $dy = (21x^2 + 6x)dx$

3. $dy = -4(2x+3)^{-5}(2)dx = -8(2x+3)^{-5}dx$

4. $dy = -2(3x^2 + x + 1)^{-3}(6x+1)dx$
$= -2(6x+1)(3x^2 + x + 1)^{-3}dx$

5. $dy = 3(\sin x + \cos x)^2(\cos x - \sin x)dx$

6. $dy = 3(\tan x + 1)^2(\sec^2 x)dx$
$= 3\sec^2 x(\tan x + 1)^2 dx$

7. $dy = -\frac{3}{2}(7x^2 + 3x - 1)^{-5/2}(14x+3)dx$
$= -\frac{3}{2}(14x+3)(7x^2 + 3x - 1)^{-5/2}dx$

8. $dy = 2(x^{10} + \sqrt{\sin 2x})[10x^9 + \frac{1}{2\sqrt{\sin 2x}} \cdot (\cos 2x)(2)]dx$

$= 2\left(10x^9 + \frac{\cos 2x}{\sqrt{\sin 2x}}\right)(x^{10} + \sqrt{\sin 2x})dx$

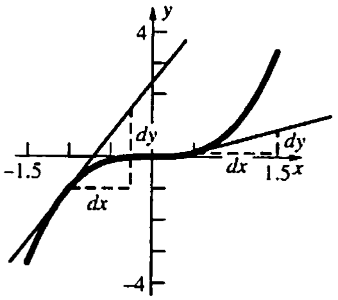
9. $ds = \frac{3}{2}(t^2 - \cot t + 2)^{1/2}(2t + \csc^2 t)dt$

$= \frac{3}{2}(2t + \csc^2 t)\sqrt{t^2 - \cot t + 2}dt$

10. a. $dy = 3x^2dx = 3(0.5)^2(1) = 0.75$

b. $dy = 3x^2dx = 3(-1)^2(0.75) = 2.25$

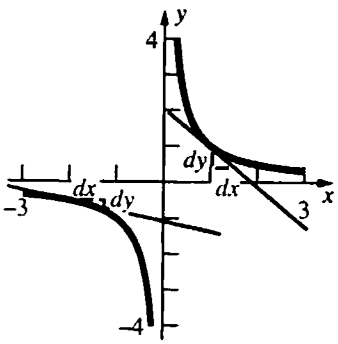
11.



12. a. $dy = -\frac{dx}{x^2} = -\frac{0.5}{(1)^2} = -0.5$

b. $dy = -\frac{dx}{x^2} = -\frac{0.75}{(-2)^2} = -0.1875$

13.



14. a. $\Delta y = (1.5)^3 - (0.5)^3 = 3.25$

b. $\Delta y = (-0.25)^3 - (-1)^3 = 0.984375$

15. a. $\Delta y = \frac{1}{1.5} - \frac{1}{1} = -\frac{1}{3}$

b. $\Delta y = \frac{1}{-1.25} + \frac{1}{2} = -0.3$

16. a. $\Delta y = [(2.5)^2 - 3] - [(2)^2 - 3] = 2.25$

$dy = 2xdx = 2(2)(0.5) = 2$

b. $\Delta y = [(2.88)^2 - 3] - [(3)^2 - 3] = -0.7056$

$dy = 2xdx = 2(3)(-0.12) = -0.72$

17. a. $\Delta y = [(3)^4 + 2(3)] - [(2)^4 + 2(2)] = 67$

$dy = (4x^3 + 2)dx = [4(2)^3 + 2](1) = 34$

b. $\Delta y = [(2.005)^4 + 2(2.005)] - [(2)^4 + 2(2)]$

≈ 0.1706

$dy = (4x^3 + 2)dx = [4(2)^3 + 2](0.005) = 0.17$

18. $y = \sqrt{x}; dy = \frac{1}{2\sqrt{x}}dx; x = 400, dx = 2$

$dy = \frac{1}{2\sqrt{400}}(2) = 0.05$

$\sqrt{402} \approx \sqrt{400} + dy = 20 + 0.05 = 20.05$

19. $y = \sqrt{x}; dy = \frac{1}{2\sqrt{x}}dx; x = 36, dx = -0.1$

$dy = \frac{1}{2\sqrt{36}}(-0.1) \approx -0.0083$

$\sqrt{35.9} \approx \sqrt{36} + dy = 6 - 0.0083 = 5.9917$

20. $y = \sqrt[3]{x}; dy = \frac{1}{3}x^{-2/3}dx = \frac{1}{3\sqrt[3]{x^2}}dx;$

$x = 27, dx = -0.09$

$dy = \frac{1}{3\sqrt[3]{(27)^2}}(-0.09) \approx -0.0033$

$\sqrt[3]{26.91} \approx \sqrt[3]{27} + dy = 3 - 0.0033 = 2.9967$

21. $V = \frac{4}{3}\pi r^3; r = 5, dr = 0.125$

$dV = 4\pi r^2dr = 4\pi(5)^2(0.125) \approx 39.27 \text{ cm}^3$

22. $V = x^3; x = \sqrt[3]{40}, dx = 0.5$

$dV = 3x^2dx = 3(\sqrt[3]{40})^2(0.5) \approx 17.54 \text{ in.}^3$

23. $V = \frac{4}{3}\pi r^3; r = 6 \text{ ft} = 72 \text{ in.}, dr = -0.3$

$dV = 4\pi r^2dr = 4\pi(72)^2(-0.3) \approx -19{,}543$

$V \approx \frac{4}{3}\pi(72)^3 - 19{,}543$

$\approx 1{,}543{,}915 \text{ in}^3 \approx 893 \text{ ft}^3$

24. $V = \pi r^2 h; r = 6\text{ ft} = 72\text{ in.}, dr = -0.05,$
$h = 8\text{ ft} = 96\text{ in.}$

$dV = 2\pi rh\,dr = 2\pi(72)(96)(-0.05) \approx -2171\text{ in.}^3$
About 9.4 gal of paint are needed.

25. $C = 2\pi r$; $r = 4000$ mi = 21,120,000 ft, $dr = 2$
$dC = 2\pi dr = 2\pi(2) = 4\pi \approx 12.6\,ft$

26. $T = 2\pi\sqrt{\frac{L}{32}}; L = 4, dL = -0.03$

$$dT = \frac{2\pi}{2\sqrt{\frac{L}{32}}}\cdot\frac{1}{32}\cdot dL = \frac{\pi}{\sqrt{32L}}dL$$

$$dT = \frac{\pi}{\sqrt{32(4)}}(-0.03) \approx -0.0083$$

The time change in 24 hours is
$(0.0083)(60)(60)(24) \approx 717$ sec

27. $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(10)^3 \approx 4189$

$dV = 4\pi r^2 dr = 4\pi(10)^2(0.05) \approx 62.8$ The volume is 4189 ± 62.8 cm^3.
The absolute error is ≈ 62.8 while the relative error is $62.8/4189 \approx 0.015$ or 1.5%.

28. $V = \pi r^2 h = \pi(3)^2(12) \approx 339$
$dV = 24\pi r\,dr = 24\pi(3)(0.0025) \approx 0.565$
The volume is 339 ± 0.565 in.^{3}
The absolute error is ≈ 0.565 while the relative error is $0.565/339 \approx 0.0017$ or 0.17%.

29. $s = \sqrt{a^2 + b^2 - 2ab\cos\theta}$

$$= \sqrt{151^2 + 151^2 - 2(151)(151)\cos 0.53} \approx 79.097$$

$$s = \sqrt{45{,}602 - 45{,}602\cos\theta}$$

$$ds = \frac{1}{2\sqrt{45{,}602 - 45{,}602\cos\theta}}\cdot 45{,}602\sin\theta\,d\theta$$

$$= \frac{22{,}801\sin\theta}{\sqrt{45{,}602 - 45{,}602\cos\theta}}d\theta$$

$$= \frac{22{,}801\sin 0.53}{\sqrt{45{,}602 - 45{,}602\cos 0.53}}(0.005) \approx 0.729$$

$s \approx 79.097 \pm 0.729$ cm
The absolute error is ≈ 0.729 while the relative error is $0.729/79.097 \approx 0.0092$ or 0.92%.

30. $A = \frac{1}{2}ab\sin\theta = \frac{1}{2}(151)(151)\sin 0.53 \approx 5763.33$

$$A = \frac{22{,}801}{2}\sin\theta; \theta = 0.53, d\theta = 0.005$$

$$dA = \frac{22{,}801}{2}(\cos\theta)d\theta$$

$$= \frac{22{,}801}{2}(\cos 0.53)(0.005) \approx 49.18$$

$A \approx 5763.33 \pm 49.18$ cm^2
The absolute error is ≈ 49.18 while the relative error is $49.18/5763.33 \approx 0.0085$ or 0.85%.

31. $y = 3x^2 - 2x + 11; x = 2, dx = 0.001$
$dy = (6x - 2)dx = [6(2) - 2](0.001) = 0.01$
$\frac{d^2y}{dx^2} = 6$, so with $\Delta x = 0.001$,

$$|\Delta y - dy| \le \frac{1}{2}(6)(0.001)^2 = 0.000003$$

32. Using the approximation
$f(x + \Delta x) \approx f(x) + f'(x)\Delta x$
we let $x = 1.02$ and $\Delta x = -0.02$. We can rewrite the above form as
$f(x) \approx f(x + \Delta x) - f'(x)\Delta x$
which gives
$f(1.02) \approx f(1) - f'(1.02)(-0.02)$
$= 10 + 12(0.02) = 10.24$

33. $V = \pi r^2 h + \frac{4}{3}\pi r^3$

$V = 100\pi r^2 + \frac{4}{3}\pi r^3; r = 10, dr = 0.1$

$dV = (200\pi r + 4\pi r^2)dr$
$= (2000\pi + 400\pi)(0.1) = 240\pi \approx 754$ cm^3

34. From similar triangles, the radius at height h is $\frac{2}{5}h$. Thus, $V = \frac{1}{3}\pi r^2 h = \frac{4}{75}\pi h^3$, so

$dV = \frac{4}{25}\pi h^2 dh$. $h = 10$, $dh = -1$:

$dV = \frac{4}{25}\pi(100)(-1) \approx -50\text{ cm}^3$

The ice cube has volume $3^3 = 27\text{ cm}^3$, so there is room for the ice cube without the cup overflowing.

35. The percent increase in mass is $\frac{dm}{m}$.

$$dm = -\frac{m_0}{2}\left(1 - \frac{v^2}{c^2}\right)^{-3/2}\left(-\frac{2v}{c^2}\right)dv$$

$$= \frac{m_0 v}{c^2}\left(1 - \frac{v^2}{c^2}\right)^{-3/2}dv$$

$$\frac{dm}{m} = \frac{v}{c^2}\left(1 - \frac{v^2}{c^2}\right)^{-1} dv = \frac{v}{c^2}\left(\frac{c^2}{c^2 - v^2}\right)dv$$

$$= \frac{v}{c^2 - v^2}dv$$

$v = 0.9c$, $dv = 0.02c$

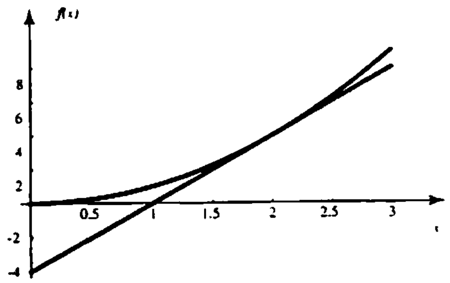
$$\frac{dm}{m} = \frac{0.9c}{c^2 - 0.81c^2}(0.02c) = \frac{0.018}{0.19} \approx 0.095$$

The percent increase in mass is about 9.5.

36. **a.** $f(x) = x^2$; $f'(x) = 2x$; $a = 2$

The linear approximation is then

$$L(x) = f(2) + f'(2)(x - 2)$$
$$= 4 + 4(x - 2) = 4x - 4$$

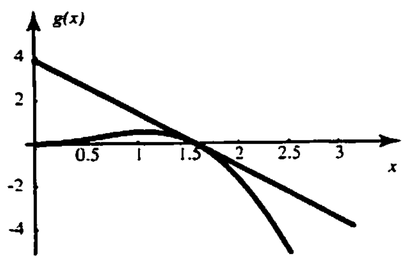


b. $g(x) = x^2 \cos x$; $g'(x) = -x^2 \sin x + 2x \cos x$

$a = \pi / 2$

The linear approximation is then

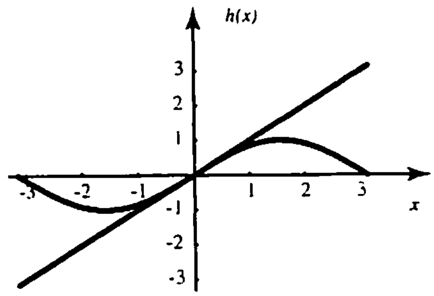
$$L(x) = 0 + -\left(\frac{\pi}{2}\right)^2\left(x - \frac{\pi}{2}\right)$$
$$= -\frac{\pi^2}{4}x + \frac{\pi^3}{8}$$
$$L(x) = 0 + -\frac{\pi^2}{4}\left(x - \frac{\pi}{2}\right)$$
$$= -\frac{\pi^2}{4}x + \frac{\pi^3}{8}$$



37. **a.** $h(x) = \sin x$; $h'(x) = \cos x$; $a = 0$

The linear approximation is then

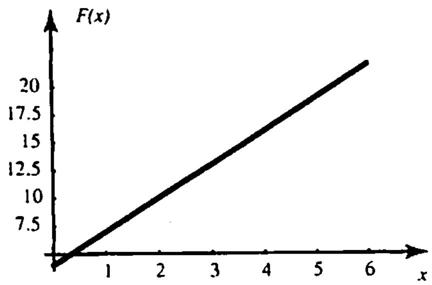
$$L(x) = 0 + 1(x - 0) = x$$



b. $F(x) = 3x + 4$; $F'(x) = 3$; $a = 3$

The linear approximation is then

$$L(x) = 13 + 3(x - 3) = 13 + 3x - 9$$
$$= 3x + 4$$



38. The linear approximation to $f(x)$ at a is

$$L(x) = f(a) + f'(a)(x - a)$$
$$= a^2 + 2a(x - a)$$
$$= 2ax - a^2$$

Thus,

$$f(x) - L(x) = x^2 - \left(2ax - a^2\right)$$
$$= x^2 - 2ax + a^2$$
$$= (x - a)^2$$
$$\geq 0$$

3.11 Chapter Review

Concepts Test

1. False: If $f(x) = x^3$, $f'(x) = 3x^2$ and the tangent line $y = 0$ at $x = 0$ crosses the curve at the point of tangency.

2. False: The tangent line can touch the curve at infinitely many points.

3. True: $m_{\tan} = 4x^3$, which is unique for each value of x.

4. False: $m_{\tan} = -\sin x$, which is periodic.

5. True: If the velocity is negative and increasing, the speed is decreasing.

6. True: If the velocity is negative and decreasing, the speed is increasing.

7. True: If the tangent line is horizontal, the slope must be 0.

8. False: $f(x) = ax^2 + b$, $g(x) = ax^2 + c$, $b \neq c$. Then $f'(x) = 2ax = g'(x)$, but $f(x) \neq g(x)$.

9. True: $D_x f(g(x)) = f'(g(x))g'(x)$; since $g(x) = x$, $g'(x) = 1$, so $D_x f(g(x)) = f'(g(x))$.

10. False: $D_x y = 0$ because π is a constant, not a variable.

11. True: Theorem 3.2.A

12. True: The derivative does not exist when the tangent line is vertical.

13. False: $(f \cdot g)'(x) = f(x)g'(x) + g(x)f'(x)$

14. True: Negative acceleration indicates decreasing velocity.

15. True: If $f(x) = x^3 g(x)$, then
$D_x f(x) = x^3 g'(x) + 3x^2 g(x)$
$= x^2[xg'(x) + 3g(x)]$.

16. False: $D_x y = 3x^2$; At (1, 1):
$m_{\tan} = 3(1)^2 = 3$
Tangent line: $y - 1 = 3(x - 1)$

17. False:
$$D_x y = f(x)g'(x) + g(x)f'(x)$$
$$D_x^2 y = f(x)g''(x) + g'(x)f'(x) + g(x)f''(x) + f'(x)g'(x)$$
$$= f(x)g''(x) + 2f'(x)g'(x) + f''(x)g(x)$$

18. True: The degree of $y = (x^3 + x)^8$ is 24, so $D_x^{25} y = 0$.

19. True: $f(x) = ax^n$; $f'(x) = anx^{n-1}$

20. True: $D_x \dfrac{f(x)}{g(x)} = \dfrac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$

21. True:
$h'(x) = f(x)g'(x) + g(x)f'(x)$
$h'(c) = f(c)g'(c) + g(c)f'(c)$
$= f(c)(0) + g(c)(0) = 0$

22. True:
$$f'\left(\frac{\pi}{2}\right) = \lim_{x \to \frac{\pi}{2}} \frac{\sin x - \sin\left(\frac{\pi}{2}\right)}{x - \frac{\pi}{2}}$$
$$= \lim_{x \to \frac{\pi}{2}} \frac{\sin x - 1}{x - \frac{\pi}{2}}$$

23. True: $D^2(kf) = kD^2 f$ and $D^2(f + g) = D^2 f + D^2 g$

24. True:
$h'(x) = f'(g(x)) \cdot g'(x)$
$h'(c) = f'(g(c)) \cdot g'(c) = 0$

25. True:
$(f \circ g)'(2) = f'(g(2)) \cdot g'(2)$
$= f'(2) \cdot g'(2) = 2 \cdot 2 = 4$

26. False: Consider $f(x) = \sqrt{x}$. The curve always lies below the tangent.

27. False: The rate of volume change depends on the radius of the sphere.

28. True:
$c = 2\pi r$; $\dfrac{dr}{dt} = 4$
$$\frac{dc}{dt} = 2\pi\frac{dr}{dt} = 2\pi(4) = 8\pi$$

29. True:
$D_x(\sin x) = \cos x$;
$D_x^2(\sin x) = -\sin x$;
$D_x^3(\sin x) = -\cos x$;
$D_x^4(\sin x) = \sin x$;
$D_x^5(\sin x) = \cos x$

30. False: $D_x(\cos x) = -\sin x$;
$D_x^2(\cos x) = -\cos x$;
$D_x^3(\cos x) = \sin x$;
$D_x^4(\cos x) = D_x[D_x^3(\cos x)] = D_x(\sin x)$
Since $D_x^{1+3}(\cos x) = D_x^1(\sin x)$,
$D_x^{n+3}(\cos x) = D_x^n(\sin x)$.

31. True: $\lim_{x\to 0}\frac{\tan x}{3x} = \frac{1}{3}\lim_{x\to 0}\frac{\sin x}{x\cos x}$
$= \frac{1}{3}\cdot 1 = \frac{1}{3}$

32. True: $v = \frac{ds}{dt} = 15t^2 + 6$ which is greater than 0 for all t.

33. True: $V = \frac{4}{3}\pi r^3$
$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$
If $\frac{dV}{dt} = 3$, then $\frac{dr}{dt} = \frac{3}{4\pi r^2}$ so
$\frac{dr}{dt} > 0$.
$\frac{d^2r}{dt^2} = -\frac{3}{2\pi r^3}\frac{dr}{dt}$ so $\frac{d^2r}{dt^2} < 0$

34. True: When $h > r$, then $\frac{d^2h}{dt^2} > 0$

35. True: $V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$
$dV = 4\pi r^2 dr = S\cdot dr$
If $\Delta r = dr$, then $dV = S\cdot\Delta r$

36. False: $dy = 5x^4 dx$, so $dy > 0$ when $dx > 0$, but $dy < 0$ when $dx < 0$.

37. False: The slope of the linear approximation is equal to $f'(a) = f'(0) = -\sin(0) = 0$.

Sample Test Problems

1. a. $f'(x) = \lim_{h\to 0}\frac{3(x+h)^3 - 3x^3}{h} = \lim_{h\to 0}\frac{9x^2h + 9xh^2 + 3h^3}{h} = \lim_{h\to 0}(9x^2 + 9xh + 3h^2) = 9x^2$

b. $f'(x) = \lim_{h\to 0}\frac{[2(x+h)^5 + 3(x+h)] - (2x^5 + 3x)}{h} = \lim_{h\to 0}\frac{10x^4h + 20x^3h^2 + 20x^2h^3 + 10xh^4 + 2h^5 + 3h}{h}$
$= \lim_{h\to 0}(10x^4 + 20x^3h + 20x^2h^2 + 10xh^3 + 2h^4 + 3) = 10x^4 + 3$

c. $f'(x) = \lim_{h\to 0}\frac{\frac{1}{3(x+h)} - \frac{1}{3x}}{h} = \lim_{h\to 0}\left[-\frac{h}{3(x+h)x}\right]\frac{1}{h} = \lim_{h\to 0}-\left(\frac{1}{3x(x+h)}\right) = -\frac{1}{3x^2}$

d. $f'(x) = \lim_{h\to 0}\left[\left(\frac{1}{3(x+h)^2+2} - \frac{1}{3x^2+2}\right)\frac{1}{h}\right] = \lim_{h\to 0}\left[\frac{3x^2 + 2 - 3(x+h)^2 - 2}{(3(x+h)^2+2)(3x^2+2)}\cdot\frac{1}{h}\right]$
$= \lim_{h\to 0}\left[\frac{-6xh - 3h^2}{(3(x+h)^2+2)(3x^2+2)}\cdot\frac{1}{h}\right] = \lim_{h\to 0}\frac{-6x - 3h}{(3(x+h)^2+2)(3x^2+2)} = -\frac{6x}{(3x^2+2)^2}$

e. $f'(x) = \lim_{h\to 0}\frac{\sqrt{3(x+h)} - \sqrt{3x}}{h} = \lim_{h\to 0}\frac{(\sqrt{3x+3h} - \sqrt{3x})(\sqrt{3x+3h} + \sqrt{3x})}{h(\sqrt{3x+3h} + \sqrt{3x})}$
$= \lim_{h\to 0}\frac{3h}{h(\sqrt{3x+3h} + \sqrt{3x})} = \lim_{h\to 0}\frac{3}{\sqrt{3x+3h} + \sqrt{3x}} = \frac{3}{2\sqrt{3x}}$

f. $f'(x) = \lim_{h\to 0} \frac{\sin[3(x+h)] - \sin 3x}{h} = \lim_{h\to 0} \frac{\sin(3x+3h) - \sin 3x}{h}$

$= \lim_{h\to 0} \frac{\sin 3x \cos 3h + \sin 3h \cos 3x - \sin 3x}{h} = \lim_{h\to 0} \frac{\sin 3x(\cos 3h - 1)}{h} + \lim_{h\to 0} \frac{\sin 3h \cos 3x}{h}$

$= 3\sin 3x \lim_{h\to 0} \frac{\cos 3h - 1}{3h} + \cos 3x \lim_{h\to 0} \frac{\sin 3h}{h} = (3\sin 3x)(0) + (\cos 3x)3 \lim_{h\to 0} \frac{\sin 3h}{3h} = (\cos 3x)(3)(1) = 3\cos 3x$

g. $f'(x) = \lim_{h\to 0} \frac{\sqrt{(x+h)^2+5} - \sqrt{x^2+5}}{h} = \lim_{h\to 0} \frac{\left(\sqrt{(x+h)^2+5} - \sqrt{x^2+5}\right)\left(\sqrt{(x+h)^2+5} + \sqrt{x^2+5}\right)}{h\left(\sqrt{(x+h)^2+5} + \sqrt{x^2+5}\right)}$

$= \lim_{h\to 0} \frac{2xh + h^2}{h\left(\sqrt{(x+h)^2+5} + \sqrt{x^2+5}\right)} = \lim_{h\to 0} \frac{2x+h}{\sqrt{(x+h)^2+5} + \sqrt{x^2+5}} = \frac{2x}{2\sqrt{x^2+5}} = \frac{x}{\sqrt{x^2+5}}$

h. $f'(x) = \lim_{h\to 0} \frac{\cos[\pi(x+h)] - \cos \pi x}{h} = \lim_{h\to 0} \frac{\cos(\pi x + \pi h) - \cos \pi x}{h} = \lim_{h\to 0} \frac{\cos \pi x \cos \pi h - \sin \pi x \sin \pi h - \cos \pi x}{h}$

$= \lim_{h\to 0}\left(-\pi \cos \pi x \frac{1 - \cos \pi h}{\pi h}\right) - \lim_{h\to 0}\left(\pi \sin \pi x \frac{\sin \pi h}{\pi h}\right) = (-\pi \cos \pi x)(0) - (\pi \sin \pi x) = -\pi \sin \pi x$

2. a. $g'(x) = \lim_{t\to x} \frac{2t^2 - 2x^2}{t - x} = \lim_{t\to x} \frac{2(t-x)(t+x)}{t-x}$

$= 2\lim_{t\to x}(t+x) = 2(2x) = 4x$

b. $g'(x) = \lim_{t\to x} \frac{(t^3+t) - (x^3+x)}{t-x}$

$= \lim_{t\to x} \frac{(t-x)(t^2+tx+x^2) + (t-x)}{t-x}$

$= \lim_{t\to x}(t^2+tx+x^2+1) = 3x^2+1$

c. $g'(x) = \lim_{t\to x} \frac{\frac{1}{t} - \frac{1}{x}}{t-x} = \lim_{t\to x} \frac{x-t}{tx(t-x)}$

$= \lim_{t\to x} \frac{-1}{tx} = -\frac{1}{x^2}$

d. $g'(x) = \lim_{t\to x}\left[\left(\frac{1}{t^2+1} - \frac{1}{x^2+1}\right)\left(\frac{1}{t-x}\right)\right]$

$= \lim_{t\to x} \frac{x^2 - t^2}{(t^2+1)(x^2+1)(t-x)}$

$= \lim_{t\to x} \frac{-(x+t)(t-x)}{(t^2+1)(x^2+1)(t-x)}$

$= \lim_{t\to x} \frac{-(x+t)}{(t^2+1)(x^2+1)} = -\frac{2x}{(x^2+1)^2}$

e. $g'(x) = \lim_{t\to x} \frac{\sqrt{t} - \sqrt{x}}{t-x}$

$= \lim_{t\to x} \frac{(\sqrt{t} - \sqrt{x})(\sqrt{t} + \sqrt{x})}{(t-x)(\sqrt{t} + \sqrt{x})}$

$= \lim_{t\to x} \frac{t-x}{(t-x)(\sqrt{t} + \sqrt{x})} = \lim_{t\to x} \frac{1}{\sqrt{t} + \sqrt{x}}$

$= \frac{1}{2\sqrt{x}}$

f. $g'(x) = \lim_{t\to x} \frac{\sin \pi t - \sin \pi x}{t-x}$

Let $v = t - x$, then $t = v + x$ and as $t \to x, v \to 0$.

$\lim_{t\to x} \frac{\sin \pi t - \sin \pi x}{t-x} = \lim_{v\to 0} \frac{\sin \pi(v+x) - \sin \pi x}{v}$

$= \lim_{v\to 0} \frac{\sin \pi v \cos \pi x + \sin \pi x \cos \pi v - \sin \pi x}{v}$

$= \lim_{v\to 0}\left[\pi \cos \pi x \frac{\sin \pi v}{\pi v} + \pi \sin \pi x \frac{\cos \pi v - 1}{\pi v}\right]$

$= \pi \cos \pi x \cdot 1 + \pi \sin \pi x \cdot 0 = \pi \cos \pi x$

Other method:

Use the subtraction formula

$\sin \pi t - \sin \pi x = 2\cos \frac{\pi(t+x)}{2} \sin \frac{\pi(t-x)}{2}$

g. $g'(x)=\lim_{t\to x}\frac{\sqrt{t^3+C}-\sqrt{x^3+C}}{t-x}$

$=\lim_{t\to x}\frac{\left(\sqrt{t^3+C}-\sqrt{x^3+C}\right)\left(\sqrt{t^3+C}+\sqrt{x^3+C}\right)}{(t-x)\left(\sqrt{t^3+C}+\sqrt{x^3+C}\right)}$

$=\lim_{t\to x}\frac{t^3-x^3}{(t-x)\left(\sqrt{t^3+C}+\sqrt{x^3+C}\right)}$

$=\lim_{t\to x}\frac{t^2+tx+x^2}{\sqrt{t^3+C}+\sqrt{x^3+C}}=\frac{3x^2}{2\sqrt{x^3+C}}$

h. $g'(x)=\lim_{t\to x}\frac{\cos 2t-\cos 2x}{t-x}$

Let $v=t-x$, then $t=v+x$ and as $t\to x, v\to 0$.

$\lim_{t\to x}\frac{\cos 2t-\cos 2x}{t-x}=\lim_{v\to 0}\frac{\cos 2(v+x)-\cos 2x}{v}$

$=\lim_{v\to 0}\frac{\cos 2v\cos 2x-\sin 2v\sin 2x-\cos 2x}{v}$

$=\lim_{v\to 0}\left[2\cos 2x\frac{\cos 2v-1}{2v}-2\sin 2x\frac{\sin 2v}{2v}\right]$

$=2\cos 2x\cdot 0-2\sin 2x\cdot 1=-2\sin 2x$

Other method:

Use the subtraction formula

$\cos 2t-\cos 2x=-2\sin(t+x)\sin(t-x)$.

3. a. $f(x)=3x$ at $x=1$

b. $f(x)=4x^3$ at $x=2$

c. $f(x)=\sqrt{x^3}$ at $x=1$

d. $f(x)=\sin x$ at $x=\pi$

e. $f(x)=\frac{4}{x}$ at x

f. $f(x)=-\sin 3x$ at x

g. $f(x)=\tan x$ at $x=\frac{\pi}{4}$

h. $f(x)=\frac{1}{\sqrt{x}}$ at $x=5$

4. a. $f'(2)\approx -\frac{3}{4}$

b. $f'(6)\approx \frac{3}{2}$

c. $V_{\text{avg}}=\frac{6-\frac{3}{2}}{7-3}=\frac{9}{8}$

d. $\frac{d}{dt}f(t^2)=f'(t^2)(2t)$

At $t=2$, $4f'(4)\approx 4\left(\frac{2}{3}\right)=\frac{8}{3}$

e. $\frac{d}{dt}[f^2(t)]=2f(t)f'(t)$

At $t=2$,

$2f(2)f'(2)\approx 2(2)\left(-\frac{3}{4}\right)=-3$

f. $\frac{d}{dt}(f(f(t)))=f'(f(t))f'(t)$

At $t=2$, $f'(f(2))f'(2)=f'(2)f'(2)$

$\approx\left(-\frac{3}{4}\right)\left(-\frac{3}{4}\right)=\frac{9}{16}$

5. $D_x(3x^5)=15x^4$

6. $D_x(x^3-3x^2+x^{-2})=3x^2-6x+(-2)x^{-3}$

$=3x^2-6x-2x^{-3}$

7. $D_z(z^3+4z^2+2z)=3z^2+8z+2$

8. $D_x\left(\frac{3x-5}{x^2+1}\right)=\frac{(x^2+1)(3)-(3x-5)(2x)}{(x^2+1)^2}$

$=\frac{-3x^2+10x+3}{(x^2+1)^2}$

9. $D_t\left(\frac{4t-5}{6t^2+2t}\right)=\frac{(6t^2+2t)(4)-(4t-5)(12t+2)}{(6t^2+2t)^2}$

$=\frac{-24t^2+60t+10}{(6t^2+2t)^2}$

10. $D_x(3x+2)^{2/3}=\frac{2}{3}(3x+2)^{-1/3}(3)$

$=2(3x+2)^{-1/3}$

$D_x^2(3x+2)^{2/3}=-\frac{2}{3}(3x+2)^{-4/3}(3)$

$=-2(3x+2)^{-4/3}$

11. $\frac{d}{dx}\left(\frac{4x^2-2}{x^3+x}\right)=\frac{(x^3+x)(8x)-(4x^2-2)(3x^2+1)}{(x^3+x)^2}$
$=\frac{-4x^4+10x^2+2}{(x^3+x)^2}$

12. $D_t(t\sqrt{2t+6})=t\frac{1}{2\sqrt{2t+6}}(2)+\sqrt{2t+6}$
$=\frac{t}{\sqrt{2t+6}}+\sqrt{2t+6}$

13. $\frac{d}{dx}\left(\frac{1}{\sqrt{x^2+4}}\right)=\frac{d}{dx}(x^2+4)^{-1/2}$
$=-\frac{1}{2}(x^2+4)^{-3/2}(2x)$
$=-\frac{x}{\sqrt{(x^2+4)^3}}$

14. $\frac{d}{dx}\sqrt{\frac{x^2-1}{x^3-x}}=\frac{d}{dx}\frac{1}{\sqrt{x}}=\frac{d}{dx}x^{-1/2}=-\frac{1}{2x^{3/2}}$

15. $D_\theta(\sin\theta+\cos^3\theta)=\cos\theta+3\cos^2\theta(-\sin\theta)$
$=\cos\theta-3\sin\theta\cos^2\theta$
$D_\theta^2(\sin\theta+\cos^3\theta)$
$=-\sin\theta-3[\sin\theta(2)(\cos\theta)(-\sin\theta)+\cos^3\theta]$
$=-\sin\theta+6\sin^2\theta\cos\theta-3\cos^3\theta$

16. $\frac{d}{dt}[\sin(t^2)-\sin^2(t)]=\cos(t^2)(2t)-(2\sin t)(\cos t)$
$=2t\cos(t^2)-\sin(2t)$

17. $D_\theta[\sin(\theta^2)]=\cos(\theta^2)(2\theta)=2\theta\cos(\theta^2)$

18. $\frac{d}{dx}(\cos^3 5x)=(3\cos^2 5x)(-\sin 5x)(5)$
$=-15\cos^2 5x\sin 5x$

19. $\frac{d}{d\theta}[\sin^2(\sin(\pi\theta))]=2\sin(\sin(\pi\theta))\cos(\sin(\pi\theta))(\cos(\pi\theta))(\pi)=2\pi\sin(\sin(\pi\theta))\cos(\sin(\pi\theta))\cos(\pi\theta)$

20. $\frac{d}{dt}[\sin^2(\cos 4t)]=2\sin(\cos 4t)\big(\cos(\cos 4t)\big)(-\sin 4t)(4)=-8\sin(\cos 4t)\cos(\cos 4t)\sin 4t$

21. $D_\theta\tan 3\theta=(\sec^2 3\theta)(3)=3\sec^2 3\theta$

22. $\frac{d}{dx}\left(\frac{\sin 3x}{\cos 5x^2}\right)=\frac{(\cos 5x^2)(\cos 3x)(3)-(\sin 3x)(-\sin 5x^2)(10x)}{\cos^2 5x^2}=\frac{3\cos 5x^2\cos 3x+10x\sin 3x\sin 5x^2}{\cos^2 5x^2}$

23. $f'(x)=(x^2-1)^2(9x^2-4)+(3x^3-4x)(2)(x^2-1)(2x)=(x^2-1)^2(9x^2-4)+4x(x^2-1)(3x^3-4x)$
$f'(2)=672$

24. $g'(x)=3\cos 3x+2(\sin 3x)(\cos 3x)(3)=3\cos 3x+3\sin 6x$
$g''(x)=-9\sin 3x+18\cos 6x$
$g''(0)=18$

25. $\frac{d}{dx}\left(\frac{\cot x}{\sec x^2}\right)=\frac{(\sec x^2)(-\csc^2 x)-(\cot x)(\sec x^2)(\tan x^2)(2x)}{\sec^2 x^2}=\frac{-\csc^2 x-2x\cot x\tan x^2}{\sec x^2}$

26. $D_t\left(\frac{4t\sin t}{\cos t-\sin t}\right)=\frac{(\cos t-\sin t)(4t\cos t+4\sin t)-(4t\sin t)(-\sin t-\cos t)}{(\cos t-\sin t)^2}$
$=\frac{4t\cos^2 t+2\sin 2t-4\sin^2 t+4t\sin^2 t}{(\cos t-\sin t)^2}=\frac{4t+2\sin 2t-4\sin^2 t}{(\cos t-\sin t)^2}$

27. $f'(x) = (x-1)^3 2(\sin \pi x - x)(\pi \cos \pi x - 1) + (\sin \pi x - x)^2 3(x-1)^2$

$= 2(x-1)^3(\sin \pi x - x)(\pi \cos \pi x - 1) + 3(\sin \pi x - x)^2(x-1)^2$

$f'(2) = 16 - 4\pi \approx 3.43$

28. $h'(t) = 5(\sin(2t) + \cos(3t))^4(2\cos(2t) - 3\sin(3t))$

$h''(t) = 5(\sin(2t) + \cos(3t))^4(-4\sin(2t) - 9\cos(3t)) + 20(\sin(2t) + \cos(3t))^3(2\cos(2t) - 3\sin(3t))^2$

$h''(0) = 5 \cdot 1^4 \cdot (-9) + 20 \cdot 1^3 \cdot 2^2 = 35$

29. $g'(r) = 3(\cos^2 5r)(-\sin 5r)(5) = -15\cos^2 5r \sin 5r$

$g''(r) = -15[(\cos^2 5r)(\cos 5r)(5) + (\sin 5r)2(\cos 5r)(-\sin 5r)(5)] = -15[5\cos^3 5r - 10(\sin^2 5r)(\cos 5r)]$

$g'''(r) = -15[5(3)(\cos^2 5r)(-\sin 5r)(5) - (10\sin^2 5r)(-\sin 5r)(5) - (\cos 5r)(20\sin 5r)(\cos 5r)(5)]$

$= -15[-175(\cos^2 5r)(\sin 5r) + 50\sin^3 5r]$

$g'''(1) \approx 458.8$

30. $f'(t) = h'(g(t))g'(t) + 2g(t)g'(t)$

31. $G'(x) = F'(r(x) + s(x))(r'(x) + s'(x)) + s'(x)$

$G''(x) = F'(r(x) + s(x))(r''(x) + s''(x)) + (r'(x) + s'(x))F''(r(x) + s(x))(r'(x) + s'(x)) + s''(x)$

$= F'(r(x) + s(x))(r''(x) + s''(x)) + (r'(x) + s'(x))^2 F''(r(x) + s(x)) + s''(x)$

32. $F'(x) = Q'(R(x))R'(x) = 3[R(x)]^2(-\sin x)$

$= -3\cos^2 x \sin x$

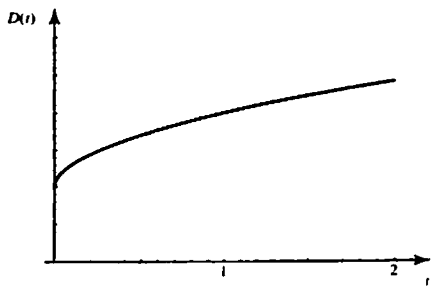
33. $F'(z) = r'(s(z))s'(z) = [3\cos(3s(z))](9z^2)$

$= 27z^2 \cos(9z^3)$

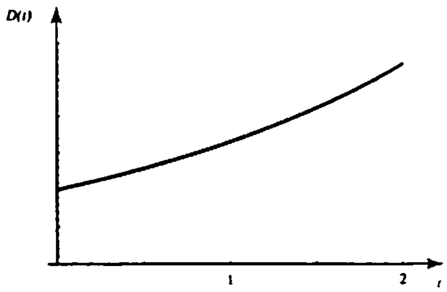
34. $\frac{d}{d\beta}F(\beta) = \Sigma'(\Phi(\beta))\Phi'(\beta)$

$= \frac{3}{2\sqrt{3\Phi(\beta)}}(3\beta^2 - 1) = \frac{3}{2\sqrt{3(\beta^3 - \beta)}}(3\beta^2 - 1)$

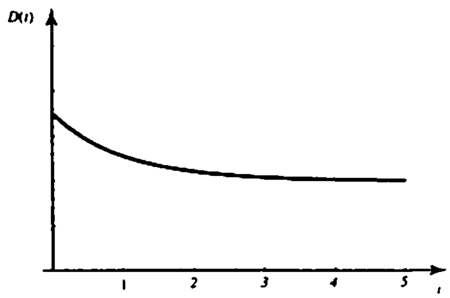
35. One possibility:



36. One possibility:



37. One possibility:



38. $\frac{dy}{dx} = 2(x-2)$

$2x - y + 2 = 0;\ y = 2x + 2;\ m = 2$

$2(x-2) = -\frac{1}{2}$

$x = \frac{7}{4}$

$y = \left(\frac{7}{4} - 2\right)^2 = \frac{1}{16}; \left(\frac{7}{4}, \frac{1}{16}\right)$

39. $V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

When $r = 5$, $\frac{dV}{dr} = 4\pi(5)^2 = 100\pi \approx 314$ m³ per meter of increase in the radius.

40. $V = \frac{4}{3}\pi r^3; \frac{dV}{dt} = 10$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

When $r = 5$, $10 = 4\pi(5)^2 \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{1}{10\pi} \approx 0.0318$ m/h

41. $V = \frac{1}{2}bh(12); \frac{6}{4} = \frac{b}{h}; b = \frac{3h}{2}$

$V = 6\left(\frac{3h}{2}\right)h = 9h^2; \frac{dV}{dt} = 9$

$\frac{dV}{dt} = 18h\frac{dh}{dt}$

When $h = 3$, $9 = 18(3)\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{1}{6} \approx 0.167$ ft/min

42. **a.** $v = 128 - 32t$
$v = 0$, when $t = 4s$
$s = 128(4) - 16(4)^2 = 256$ ft

b. $128t - 16t^2 = 0$
$-16t(t - 8) = 0$
The object hits the ground when $t = 8$s
$v = 128 - 32(8) = -128$ ft/s

43. $s = t^3 - 6t^2 + 9t$

$v(t) = \frac{ds}{dt} = 3t^2 - 12t + 9$

$a(t) = \frac{d^2s}{dt^2} = 6t - 12$

a. $3t^2 - 12t + 9 < 0$
$3(t - 3)(t - 1) < 0$
$1 < t < 3$; $(1, 3)$

b. $3t^2 - 12t + 9 = 0$
$3(t - 3)(t - 1) = 0$
$t = 1, 3$
$a(1) = -6$, $a(3) = 6$

c. $6t - 12 > 0$
$t > 2$; $(2, \infty)$

44. **a.** $D_x^{20}(x^{19} + x^{12} + x^5 + 100) = 0$

b. $D_x^{20}(x^{20} + x^{19} + x^{18}) = 20!$

c. $D_x^{20}(7x^{21} + 3x^{20}) = (7 \cdot 21!)x + (3 \cdot 20!)$

d. $D_x^{20}(\sin x + \cos x) = D_x^4(\sin x + \cos x)$
$= \sin x + \cos x$

e. $D_x^{20}(\sin 2x) = 2^{20} \sin 2x$
$= 1{,}048{,}576 \sin 2x$

f. $D_x^{20}\left(\frac{1}{x}\right) = \frac{(-1)^{20}(20!)}{x^{21}} = \frac{20!}{x^{21}}$

45. **a.** $2(x - 1) + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-(x-1)}{y} = \frac{1-x}{y}$

b. $x(2y)\frac{dy}{dx} + y^2 + y(2x) + x^2\frac{dy}{dx} = 0$

$\frac{dy}{dx}(2xy + x^2) = -(y^2 + 2xy)$

$\frac{dy}{dx} = -\frac{y^2 + 2xy}{x^2 + 2xy}$

c. $3x^2 + 3y^2\frac{dy}{dx} = x^3(3y^2)\frac{dy}{dx} + 3x^2y^3$

$\frac{dy}{dx}(3y^2 - 3x^3y^2) = 3x^2y^3 - 3x^2$

$\frac{dy}{dx} = \frac{3x^2y^3 - 3x^2}{3y^2 - 3x^3y^2} = \frac{x^2y^3 - x^2}{y^2 - x^3y^2}$

d. $x\cos(xy)\left[x\frac{dy}{dx} + y\right] + \sin(xy) = 2x$

$x^2\cos(xy)\frac{dy}{dx} = 2x - \sin(xy) - xy\cos(xy)$

$\frac{dy}{dx} = \frac{2x - \sin(xy) - xy\cos(xy)}{x^2\cos(xy)}$

c. $x\sec^2(xy)\left(x\frac{dy}{dx}+y\right)+\tan(xy)=0$

$x^2\sec^2(xy)\frac{dy}{dx}=-[\tan(xy)+xy\sec^2(xy)]$

$\frac{dy}{dx}=-\frac{\tan(xy)+xy\sec^2(xy)}{x^2\sec^2(xy)}$

46. $2yy_1'=12x^2$

$y_1'=\frac{6x^2}{y}$

At (1, 2): $y_1'=3$

$4x+6yy_2'=0$

$y_2'=-\frac{2x}{3y}$

At (1, 2): $y_2'=-\frac{1}{3}$

Since $(y_1')(y_2')=-1$ at (1, 2), the tangents are perpendicular.

47. $dy=[\pi\cos(\pi x)+2x]dx$; $x=2$, $dx=0.01$

$dy=[\pi\cos(2\pi)+2(2)](0.01)=(4+\pi)(0.01)$

≈ 0.0714

48. $x(2y)\frac{dy}{dx}+y^2+2y[2(x+2)]+(x+2)^2(2)\frac{dy}{dx}=0$

$\frac{dy}{dx}[2xy+2(x+2)^2]=-[y^2+2y(2x+4)]$

$\frac{dy}{dx}=\frac{-(y^2+4xy+8y)}{2xy+2(x+2)^2}$

$dy=-\frac{y^2+4xy+8y}{2xy+2(x+2)^2}dx$

When $x=-2$, $y=\pm 1$

a. $dy=-\frac{(1)^2+4(-2)(1)+8(1)}{2(-2)(1)+2(-2+2)^2}(-0.01)$

$=-0.0025$

b. $dy=-\frac{(-1)^2+4(-2)(-1)+8(-1)}{2(-2)(-1)+2(-2+2)^2}(-0.01)$

$=0.0025$

49. a. $\frac{d}{dx}[f^2(x)+g^3(x)]$

$=2f(x)f'(x)+3g^2(x)g'(x)$

$2f(2)f'(2)+3g^2(2)g'(2)$

$=2(3)(4)+3(2)^2(5)=84$

c.

b. $\frac{d}{dx}[f(x)g(x)]=f(x)g'(x)+g(x)f'(x)$

$f(2)g'(2)+g(2)f'(2)=(3)(5)+(2)(4)=23$

c. $\frac{d}{dx}[f(g(x))]=f'(g(x))g'(x)$

$f'(g(2))g'(2)=f'(2)g'(2)=(4)(5)=20$

d. $D_x[f^2(x)]=2f(x)f'(x)$

$D_x^2[f^2(x)]=2[f(x)f''(x)+f'(x)f'(x)]$

$2f(2)f''(2)+2[f'(2)]^2$

$=2(3)(-1)+2(4)^2=26$

50. $(13)^2=x^2+y^2;\ \frac{dx}{dt}=2$

$0=2x\frac{dx}{dt}+2y\frac{dy}{dt}$

$\frac{dy}{dt}=-\frac{x}{y}\frac{dx}{dt}$

When $y=5$, $x=12$, so

$\frac{dy}{dt}=-\frac{12}{5}(2)=-\frac{24}{5}=-4.8$ ft/s

51. $\sin 15°=\frac{y}{x},\ \frac{dx}{dt}=400$

$y=x\sin 15°$

$\frac{dy}{dt}=\sin 15°\frac{dx}{dt}$

$\frac{dy}{dt}=400\sin 15° \approx 104$ mi/hr

52. a. $D_x(|x|^2)=2|x|\cdot\frac{|x|}{x}=\frac{2(|x|^2)}{x}=\frac{2x^2}{x}=2x$

b. $D_x^2|x|=D_x\left(\frac{|x|}{x}\right)=\frac{x\left(\frac{|x|}{x}\right)-|x|}{x^2}=\frac{|x|-|x|}{x^2}=0$

c. $D_x^3|x|=D_x(D_x^2|x|)=D_x(0)=0$

d. $D_x^2(|x|^2)=D_x(2x)=2$

53. a. $D_\theta|\sin\theta|=\frac{|\sin\theta|}{\sin\theta}\cos\theta=\cot\theta|\sin\theta|$

b. $D_\theta|\cos\theta|=\frac{|\cos\theta|}{\cos\theta}(-\sin\theta)=-\tan\theta|\cos\theta|$

54. a. $f(x)=\sqrt{x+1}; f'(x)=-\frac{1}{2}(x+1)^{-1/2}$; $a=3$

$L(x)=f(3)+f'(3)(x-3)$

$=\sqrt{4}+-\frac{1}{2}(4)^{-1/2}(x-3)$

$=2-\frac{1}{4}x+\frac{3}{4}$

$=-\frac{1}{4}x+\frac{11}{4}$

b. $f(x)=x\cos x; f'(x)=-x\sin x+\cos x$; $a=1$

$L(x)=f(1)+f'(1)(x-1)$

$=\cos 1+(-\sin 1+\cos 1)(x-1)$

$=\cos 1-(\sin 1)x+\sin 1+(\cos 1)x-\cos 1$

$=(\cos 1-\sin 1)x+\sin 1$

$\approx -0.3012x+0.8415$

3.12 Additional Problem Set

1. $g(t)$ is not differentiable at $t=\alpha$ if $\lim_{h\to 0}\frac{g(\alpha+h)-g(\alpha)}{h}$ does not exist.

2. $y=mx+b$; Solve for x:

$x=\frac{y-b}{m}=g(y)$

$f(g(y))=f\left(\frac{y-b}{m}\right)=m\left(\frac{y-b}{m}\right)+b$

$=y-b+b=y$

$g(f(x))=g(mx+b)=\frac{mx+b-b}{m}=\frac{mx}{m}=x$

If we think of a function as a rule which tells us what to do to a value for x, then the inverse of that function should undo what the function does. In other words, if we put the output of the function into the inverse function, we should be back where we started from – at x.

3. $f(f(x))=f((1-x^n)^{1/n})$

$=\left\{1-[(1-x^n)^{1/n}]^n\right\}^{1/n}=\{1-(1-x^n)\}^{1/n}$

$=\{x^n\}^{1/n}=x$

$f'(x)=\frac{1}{n}(1-x^n)^{\frac{1-n}{n}}(-nx^{n-1})$

$=-x^{n-1}(1-x^n)^{\frac{1-n}{n}}$

$f'(f(x))f'(x)=f'\left((1-x^n)^{1/n}\right)f'(x)$

$=-[(1-x^n)^{1/n}]^{n-1}\{1-[(1-x^n)^{1/n}]^n\}^{\frac{1-n}{n}}\cdot$

$(-1)x^{n-1}(1-x^n)^{\frac{1-n}{n}}$

$=(1-x^n)^{\frac{n-1}{n}+\frac{1-n}{n}}\{1-(1-x^n)\}^{\frac{1-n}{n}}x^{n-1}$

$=\{x^n\}^{\frac{1-n}{n}}x^{n-1}=x^{1-n+n-1}=1$

4. a. $G(F(x))=G\left(\sqrt{x^2-7}\right)=\sqrt{\left(\sqrt{x^2-7}\right)^2+7}$

$=\sqrt{x^2-7+7}=\sqrt{x^2}=x$ for $x\ge\sqrt{7}$

$F(G(y))=F\left(\sqrt{y^2+7}\right)=\sqrt{\left(\sqrt{y^2+7}\right)^2-7}$

$=\sqrt{y^2+7-7}=\sqrt{y^2}=y$ for $y\ge 0$

b. $F'(x)=\frac{1}{2\sqrt{x^2-7}}\cdot 2x=\frac{x}{\sqrt{x^2-7}}$

$G'(y)=\frac{1}{2\sqrt{y^2+7}}\cdot 2y=\frac{y}{\sqrt{y^2+7}}$

c. $F'(G(y))G'(y)=F'\left(\sqrt{y^2+7}\right)G'(y)$

$=\frac{\sqrt{y^2+7}}{\sqrt{\left(\sqrt{y^2+7}\right)^2-7}}\cdot\frac{y}{\sqrt{y^2+7}}$

$=\frac{y}{\sqrt{y^2+7-7}}=\frac{y}{\sqrt{y^2}}=1$

$G'(F(x))F'(x)=G'\left(\sqrt{x^2-7}\right)F'(x)$

$=\frac{\sqrt{x^2-7}}{\sqrt{\left(\sqrt{x^2-7}\right)^2+7}}\cdot\frac{x}{\sqrt{x^2-7}}=\frac{x}{\sqrt{x^2+7-7}}$

$=\frac{x}{\sqrt{x^2}}=1$

5. a. Using the chain rule,

$D_r[(r+1)^n]=n(r+1)^{n-1}$

b. $(r+1)^n=C_n^n r^n+C_n^{n-1}r^{n-1}+\ldots+C_n^1 r+C_n^0$

so

$D_r[(r+1)^n]=D_r[C_n^n r^n+C_n^{n-1}r^{n-1}+\ldots$

$+C_n^1 r+C_n^0]$

$=nC_n^n r^{n-1}+(n-1)C_n^{n-1}r^{n-2}+\ldots+C_n^1$

$(r+1)^{n-1} = C_{n-1}^{n-1} r^{n-1} + C_{n-1}^{n-2} r^{n-2} + \dots$
$+ C_{n-1}^{1} r + C_{n-1}^{0}$

so

$D_r[(r+1)^n] = n(r+1)^{n-1}$

$= nC_{n-1}^{n-1} r^{n-1} + nC_{n-1}^{n-2} r^{n-2} + nC_{n-1}^{1} r + nC_{n-1}^{0}$

Equating like powers of r,

$nC_n^n = nC_{n-1}^{n-1}, (n-1)C_n^{n-1} = nC_{n-1}^{n-2},$

$(n-2)C_n^{n-2} = nC_{n-1}^{n-3}, \dots, C_n^1 = nC_{n-1}^0$

or

$mC_n^m = nC_{n-1}^{m-1}$

c. $C_1^1 = 1$, thus $2 \cdot C_2^2 = 2C_1^1, C_2^2 = C_1^1 = 1$

Similarly, $C_3^3 = C_4^4 = C_n^n = 1$

For $m < n$, then using part b repeatedly,

$$C_n^m = \frac{n}{m} C_{n-1}^{m-1} = \frac{n}{m} \cdot \frac{n-1}{m-1} C_{n-2}^{m-2}$$

$$= \frac{n \cdot (n-1)}{m \cdot (m-1)} \cdot \frac{n-2}{m-2} C_{n-3}^{m-3}$$

$$= \dots = \frac{n \cdot (n-1) \dots (n-m+2)}{m \cdot (m-1) \dots 2} C_{n-m+1}^{1}$$

$$= \frac{n \cdot (n-1) \dots (n-m+1)}{m \cdot (m-1) \dots 2 \cdot 1} C_{n-m}^{0}$$

$$= \frac{n \cdot (n-1) \dots (n-m+1)}{1 \cdot 2 \cdot \dots \cdot m}$$

6. a.

x	0.01	0.02	0.03	0.04	0.05
$\sqrt{1+x}$	1.005	1.01	1.0149	1.0198	1.0247
$1+\frac{x}{2}$	1.005	1.01	1.015	1.02	1.025

The values of $\sqrt{1+x}$ and $1+\frac{x}{2}$ are very close.

b. $dy = \frac{1}{2\sqrt{1+x}} dx$ At $x = 0$,

$dy = \frac{1}{2} dx$

dx	0.01	0.02	0.03	0.04	0.05
dy	0.005	0.01	0.015	0.02	0.025

c. $\alpha = \frac{1}{2}$, error is positive.

$\alpha = -\frac{1}{2}$, error is negative.

d. $dy = \frac{1}{2\sqrt{1+x}} dx$; at $x = 4$, $dy = \frac{1}{2\sqrt{5}} dx$ and $y = \sqrt{5}$.

$(1+x)^{1/2} \approx \sqrt{5} + \frac{1}{2\sqrt{5}}(x-4)$ for x near 4.

7. No. The linear approximation is $L(x) = x$ for both $y = \sin x$ and $y = \tan x$.
For $\sin x$, the error is negative when $x > 0$ because the approximation lies above the curve; the error is positive when $x < 0$ because the approximation lies below the curve.
For $\tan x$, the error is positive when $x > 0$ because the approximation lies below the curve; the error is negative when $x < 0$ because the approximation lies above the curve.

8. a. $g'(6.0) \approx \frac{60-45}{1} = 15$

$g'(7.5) \approx \frac{86-68}{1} = 18$

$g'(7.25) \approx \frac{77-68}{0.5} = 18$

b. $g'(7.0) \approx \frac{77-60}{1} = 17$

The rate of change of g' is approximately 2.
($\frac{17-15}{1} = 2$ using the above value and the first part of **a.**)

c. $g'(7.0) \approx 17$ so the tangent line is approximately
$y - 68 = 17(x - 7)$ or
$y = 17x - 51$

d. $g(7.1) \approx 17(7.1) - 51 = 69.7$

e. $g'(x) = 2x + 3$
$g'(6) = 15, g'(7.5) = 18, g'(7.25) = 17.5.$
$g''(x) = 2$, so the rate of change of $g'(x)$ is equal to 2.
$g'(7.0) = 17$, the tangent line is
$y = 17x - 51.$
$g(7.1) \approx 69.7$

9. a. $f(3) = \sqrt{4} - 2.1 = -0.1,$
$f(8) = \sqrt{9} - 2.1 = 0.9$ so there must be at least one zero of $f(x)$ between $x = 3$ and $x = 8$.

b. $f'(x) = \frac{1}{2\sqrt{1+x}}$; $f'(8) = \frac{1}{6}$

$L(x) = 0.9 + \frac{1}{6}(x-8)$ or $L(x) = \frac{1}{6}x - \frac{13}{30}$

$L(x) = 0$ when $x = \frac{13}{5} = 2.6$ which is less than three, so must be discarded.

c. $f'(3) = \frac{1}{4}$

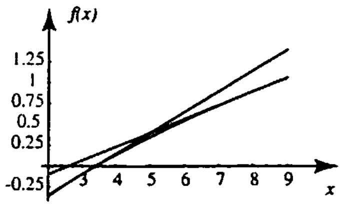
$L(x) = -0.1 + \frac{1}{4}(x-3)$ or

$L(x) = 0.25x - 0.85$

$L(x) = 0$ when $x = 3.4$

d. $f(x) = 0$ at $x = 3.41$

e.



10. a. $Error(\alpha) = f(\alpha) - L(\alpha)$
$= f(\alpha) - [c + m(\alpha - \alpha)] = f(\alpha) - c$
$Error(\alpha) = 0$ implies $f(\alpha) = c$.

b. $|Error(x)| \le |K(x)(x-\alpha)|$ so

$\left|\frac{Error(x)}{x-\alpha}\right| \le |K(x)|$

Thus, since $\lim_{x\to\alpha} K(x) = 0$,

$\lim_{x\to\alpha} \frac{Error(x)}{x-\alpha} = 0$. But

$\frac{Error(x)}{x-\alpha} = \frac{f(x)-L(x)}{x-\alpha}$

$= \frac{f(x)}{x-\alpha} - \frac{c+m(x-\alpha)}{x-\alpha} = \frac{f(x)-c}{x-\alpha} - m$, so

$0 = \lim_{x\to\alpha} \frac{Error(x)}{x-\alpha} = \lim_{x\to\alpha}\left[\frac{f(x)-c}{x-\alpha} - m\right]$

$= \left[\lim_{x\to\alpha} \frac{f(x)-f(\alpha)}{x-\alpha}\right] - m = f'(\alpha) - m$,

hence $f'(\alpha) = m$.

11. a. $\frac{5.04}{93.5} \approx 0.054$ ¢/in.^{2}

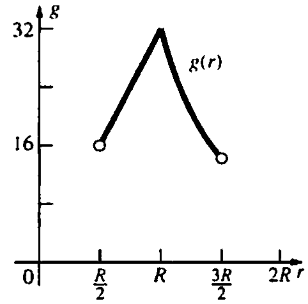
b. $dP = P'(A)dA$
$P'(93.5) = 10.5$ while
$dA = 8.7 \cdot 11.2 - 93.5 = 3.94$ in.^{2}
$dP = 10.5 \cdot 3.94 = 41.37$ ¢/pg

12. a. $\lim_{r\to R^+} g(r) = \lim_{r\to R^+} \frac{GM_1}{r^2} = \frac{GM_1}{R^2} = g$

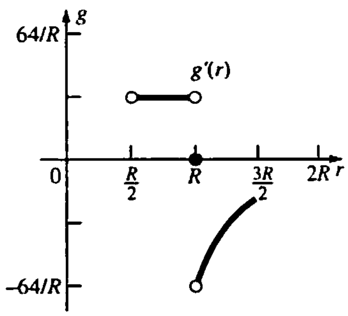
$\lim_{r\to R^-} g(r) = \lim_{r\to R^-} \frac{GM_1 r}{R^3} = \frac{GM_1}{R^2} = g$

Thus, $g(r)$ is continuous.

b.



c. $g'(r) = \begin{cases} \frac{-2GM_1}{r^3} & \text{if } r > R \\ 0 & \text{if } r = R \\ \frac{GM_1}{R^3} & \text{if } r < R \end{cases}$



d. No, the derivative is not continuous at $r = R$.

13. Measure r in feet for consistency.

a. $\frac{GM_1}{R^2} = 32$ ft/s^2

$\Delta g = g(3956 \text{ mi}) - g(R)$

$= \frac{GM_1(20{,}887{,}680)}{R^3} - \frac{GM_1}{R^2}$

$= \frac{GM_1}{R^2}\left(\frac{20{,}887{,}680}{20{,}908{,}800} - 1\right)$

$= 32\left(-\frac{21{,}120}{20{,}908{,}800}\right)$

$$= -\frac{528}{16,355} \approx -0.0323 \text{ ft/s}^2$$

$$\frac{\Delta g}{g} = \frac{-\frac{528}{16,355}}{32} = -\frac{33}{32,710} \approx -0.0010$$

$$\frac{\Delta g}{g} \times 100 = -0.10$$

$$dg \frac{GM_1}{R^3} dr = \frac{32}{R}(-21,120) = -\frac{528}{16,355} = \Delta g$$

The exact and approximate values of the absolute, relative, and percent change in g are the same since $g(r)$ is linear with respect to r for $r < R$.

b. $\Delta g = g(R + 30,000) - g(R)$

$$= \frac{GM_1}{(R+30,000)^2} - \frac{GM_1}{R^2}$$

$$= \frac{GM_1}{R^2}\left[\frac{R^2}{(R+30,000)^2} - 1\right]$$

$$= 32\left[\frac{R^2 - (R+30,000)^2}{(R+30,000)^2}\right]$$

$$= 32\left(\frac{-60,000R - 900,000,000}{(R+30,000)^2}\right) \approx -0.0916$$

$$\frac{\Delta g}{g} = \frac{-60,000R - 900,000,000}{(R+30,000)^2} \approx -0.0029$$

$$\frac{\Delta g}{g} \times 100 \approx -0.29$$

$$dg = -\frac{2GM_1}{R^3} dr = -\frac{2}{R}\left(\frac{GM_1}{R^2}\right)(30,000)$$

$$= \frac{(-60,000)(32)}{20,908,800} \approx -0.0918$$

$$\frac{dg}{g} = \frac{-60,000}{20,908,800} \approx -0.0029$$

$$\frac{dg}{g} \times 100 \approx -0.29$$

c. $\Delta g = g(R + 2000) - g(R)$

$$= \frac{GM_1}{(R+2000)^2} - \frac{GM_1}{R^2}$$

$$= \frac{GM_1}{R^2}\left[\frac{R^2 - (R+2000)^2}{(R+2000)^2}\right]$$

$$= 32\left[\frac{-4000R - 4,000,000}{(R+2000)^2}\right] \approx -0.00612$$

$$\frac{\Delta g}{g} = \frac{-4000R - 4,000,000}{(R+2000)^2} \approx -1.913 \times 10^{-4}$$

$$\frac{\Delta g}{g} \times 100 \approx -1.913 \times 10^{-2} = -0.019$$

$$dg = -\frac{2GM_1}{R^3} dr = -\frac{2}{R}(32)(2000)$$

$$= \frac{(-4000)(32)}{20,908,800} \approx -0.00612$$

$$\frac{dg}{g} = \frac{-4000}{20,908,800} \approx -1.913 \times 10^{-4}$$

$$\frac{dg}{g} \times 100 \approx -1.913 \times 10^{-2} = -0.019$$

14. a. The cross-sectional area of the vase is approximately equal to ΔV and the corresponding radius is $r = \sqrt{\Delta V / \pi}$. The table below gives the approximate values for r. The vase becomes slightly narrower as you move above the base, and then gets wider as you near the top.

Depth	V	$A \approx \Delta V$	$r = \sqrt{\Delta V / \pi}$
1	4	4	1.13
2	8	4	1.13
3	11	3	0.98
4	14	3	0.98
5	20	6	1.38
6	28	8	1.60

b. Near the base, this vase is like the one in part (a), but just above the base it becomes larger. Near the middle of the vase it becomes very narrow. The top of the vase is similar to the one in part (a).

Depth	V	$A \approx \Delta V$	$r = \sqrt{\Delta V / \pi}$
1	4	4	1.13
2	9	5	1.26
3	12	3	0.98
4	14	2	0.80
5	20	6	1.38
6	28	8	1.60

CHAPTER

4 Applications of the Derivative

4.1 Concepts Review

1. continuous; closed

2. extreme

3. endpoints; stationary points; singular points

4. $f'(c) = 0; f'(c)$ does not exist

Problem Set 4.1

1. $f'(x) = 2x + 4; 2x + 4 = 0$ when $x = -2$.
Critical points: –4, –2, 0
$f(-4) = 4, f(-2) = 0, f(0) = 4$
Maximum value = 4, minimum value = 0

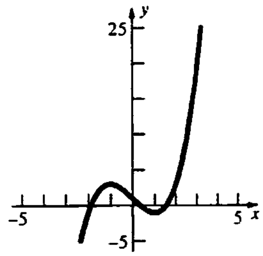
2. $h'(x) = 2x + 1; 2x + 1 = 0$ when $x = -\frac{1}{2}$.
Critical points: $-2, -\frac{1}{2}, 2$
$h(-2) = 2, h\left(-\frac{1}{2}\right) = -\frac{1}{4}, h(2) = 6$
Maximum value = 6, minimum value = $-\frac{1}{4}$

3. $\Psi'(x) = 2x + 3;\ 2x + 3 = 0$ when $x = -\frac{3}{2}$.
Critical points: $-2, -\frac{3}{2}, 1$
$\Psi(-2) = -2, \Psi\left(-\frac{3}{2}\right) = -\frac{9}{4}, \Psi(1) = 4$
Maximum value = 4, minimum value = $-\frac{9}{4}$

4. $G'(x) = \frac{1}{5}(6x^2 + 6x - 12) = \frac{6}{5}(x^2 + x - 2);$
$x^2 + x - 2 = 0$ when $x = -2, 1$
Critical points: –3, –2, 1, 3
$G(-3) = \frac{9}{5}, G(-2) = 4, G(1) = -\frac{7}{5}, G(3) = 9$
Maximum value = 9,
minimum value $= -\frac{7}{5}$

5. $f'(x) = 3x^2 - 3; 3x^2 - 3 = 0$ when $x = -1, 1$.
Critical points: –1, 1
$f(-1) = 3, f(1) = -1$
No maximum value, minimum value = –1

(See graph.)



6. $f'(x) = 3x^2 - 3; 3x^2 - 3 = 0$ when $x = -1, 1$.
Critical points: $-\frac{3}{2}, -1, 1, 3$
$f\left(-\frac{3}{2}\right) = \frac{17}{8}, f(-1) = 3, f(1) = -1, f(3) = 19$
Maximum value = 19, minimum value = –1

7. $h'(r) = -\frac{1}{r^2}; h'(r)$ is never 0; $h'(r)$ is not defined when $r = 0$, but $r = 0$ is not in the domain on $[-1, 3]$ since $h(0)$ is not defined.
Critical points: –1, 3
Note that $\lim_{x\to 0^-} h(r) = -\infty$ and $\lim_{x\to 0^+} h(x) = \infty$.
No maximum value, no minimum value.

8. $g'(x) = -\frac{2x}{(1+x^2)^2};\ -\frac{2x}{(1+x^2)^2} = 0$ when $x = 0$
Critical points: –3, 0, 1
$g(-3) = \frac{1}{10}, g(0) = 1, g(1) = \frac{1}{2}$
Maximum value = 1, minimum value = $\frac{1}{10}$