CHAPTER

8 Techniques of Integration

8.1 Concepts Review

1. $\int u^5 du$

2. e^x

3. $\int_1^2 u^3 du$

4. complete the square.

Problem Set 8.1

1. $\int (x-2)^5 dx = \frac{1}{6}(x-2)^6 + C$

2. $\int \sqrt{3x}\, dx = \frac{1}{3}\int \sqrt{3x}\, 3dx = \frac{2}{9}(3x)^{3/2} + C$

3. $u = x^2 + 1, du = 2x\,dx$
When $x = 0, u = 1$ and when $x = 1, u = 5$.
$\int_0^2 x(x^2+1)^5 dx = \frac{1}{2}\int_0^2 (x^2+1)(2x\,dx)$
$= \frac{1}{2}\int_1^5 u^5\, du$
$= \left[\frac{u^6}{12}\right]_1^5 = \frac{5^5 - 1^5}{12} = \frac{15624}{12} = 1302$

4. $u = 1 - x^2, du = -2x\,dx$
When $x = 0, u = 1$ and when $x = 1, u = 0$.
$\int_0^1 x\sqrt{1-x^2}\, dx = -\frac{1}{2}\int_0^1 \sqrt{1-x^2}\,(-2x\,dx)$
$= -\frac{1}{2}\int_1^0 u^{1/2}\, du = \frac{1}{2}\int_0^1 u^{1/2}\, du$
$= \left[\frac{1}{3}u^{3/2}\right]_0^1 = \frac{1}{3}$

5. $\int \frac{dx}{x^2+4} = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$

6. $u = 2 + e^x, du = e^x dx$
$\int \frac{e^x}{2+e^x} dx = \int \frac{du}{u}$
$= \ln|u| + C$
$= \ln\left|2+e^x\right| + C = \ln(2+e^x) + C$

7. $u = x^2 + 4,\ du = 2x\, dx$
$\int \frac{x}{x^2+4} dx = \frac{1}{2}\int \frac{du}{u}$
$= \frac{1}{2}\ln|u| + C$
$= \frac{1}{2}\ln\left|x^2+4\right| + C$
$= \frac{1}{2}\ln(x^2+4) + C$

8. $\int \frac{2t^2}{2t^2+1} dt = \int \frac{2t^2+1-1}{2t^2+1} dt$
$= \int dt - \int \frac{1}{2t^2+1} dt$
$u = \sqrt{2}t, du = \sqrt{2}dt$
$t - \int \frac{1}{2t^2+1} dt = t - \frac{1}{\sqrt{2}}\int \frac{du}{1+u^2}$
$= t - \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}t) + C$

9. $u = 4 + z^2,\ du = 2z\, dz$
$\int 6z\sqrt{4+z^2}\, dz = 3\int \sqrt{u}\, du$
$= 2u^{3/2} + C$
$= 2(4+z^2)^{3/2} + C$

10. $u = 2t + 1, du = 2dt$
$\int \frac{5}{\sqrt{2t+1}} dt = \frac{5}{2}\int \frac{du}{\sqrt{u}}$
$= 5\sqrt{u} + C$
$= 5\sqrt{2t+1} + C$

11. $\int \frac{\tan z}{\cos^2 z} dz = \int \tan z \sec^2 z\, dz$

$u = \tan z,\ du = \sec^2 z\, dz$

$\int \tan z \sec^2 z\, dz = \int u\, du = \frac{1}{2}u^2 + C$

$= \frac{1}{2}\tan^2 z + C$

12. $u = \cos z,\ du = -\sin z\, dz$

$\int e^{\cos z} \sin z\, dz = -\int e^{\cos z}(-\sin z\, dz)$

$= -\int e^u du = -e^u + C$

$= -e^{\cos z} + C$

13. $u = \sqrt{t}, du = \frac{1}{2\sqrt{t}} dt$

$\int \frac{\sin\sqrt{t}}{\sqrt{t}} dt = 2\int \sin u\, du$

$= -2\cos u + C$

$= -2\cos\sqrt{t} + C$

14. $u = x^2,\ du = 2x\, dx$

$\int \frac{2x\, dx}{\sqrt{1-x^4}} = \int \frac{du}{\sqrt{1-u^2}}$

$= \sin^{-1} u + C$

$= \sin^{-1}(x^2) + C$

15. $u = \sin x, du = \cos x\, dx$

$\int_0^{\pi/4} \frac{\cos x}{1+\sin^2 x} dx = \int_0^{\sqrt{2}/2} \frac{du}{1+u^2}$

$= [\tan^{-1} u]_0^{\sqrt{2}/2} = \tan^{-1}\frac{\sqrt{2}}{2}$

≈ 0.6155

16. $u = \sqrt{1-x}, du = -\frac{1}{2\sqrt{1-x}} dx$

$\int_0^{3/4} \frac{\sin\sqrt{1-x}}{\sqrt{1-x}} dx = -2\int_1^{1/2} \sin u\, du$

$= 2\int_{1/2}^{1} \sin u\, du$

$= [-2\cos u]_{1/2}^{1} = -2\left(\cos 1 - \cos\frac{1}{2}\right)$

≈ 0.6746

17. $\int \frac{3x^2+2x}{x+1} dx = \int (3x-1)dx + \int \frac{1}{x+1} dx$

$= \frac{3}{2}x^2 - x + \ln|x+1| + C$

18. $\int \frac{x^3+7x}{x-1} dx = \int (x^2+x+8)dx + 8\int \frac{1}{x-1} dx$

$= \frac{1}{3}x^3 + \frac{1}{2}x^2 + 8x + 8\ln|x-1| + C$

19. $u = \ln 4x^2, du = \frac{2}{x} dx$

$\int \frac{\sin(\ln 4x^2)}{x} dx = \frac{1}{2}\int \sin u\, du$

$= -\frac{1}{2}\cos u + C$

$= -\frac{1}{2}\cos(\ln 4x^2) + C$

20. $u = \ln x,\ du = \frac{1}{x} dx$

$\int \frac{\sec^2(\ln x)}{2x} dx = \frac{1}{2}\int \sec^2 u\, du$

$= \frac{1}{2}\tan u + C$

$= \frac{1}{2}\tan(\ln x) + C$

21. $\int \frac{6e^x}{\sqrt{1-e^{2x}}} dx = 6\sin^{-1}(e^x) + C$

22. $u = x^2,\ du = 2x\, dx$

$\int \frac{x}{x^4+4} dx = \frac{1}{2}\int \frac{du}{4+u^2} = \frac{1}{4}\tan^{-1}\frac{u}{2} + C$

$= \frac{1}{4}\tan^{-1}\left(\frac{x^2}{2}\right) + C$

23. $u = 1 - e^{2x}, du = -2e^{2x} dx$

$\int \frac{3e^{2x}}{\sqrt{1-e^{2x}}} dx = -\frac{3}{2}\int \frac{du}{\sqrt{u}}$

$= -3\sqrt{u} + C$

$= -3\sqrt{1-e^{2x}} + C$

24. $\int \frac{x^3}{x^4+4} dx = \frac{1}{4}\int \frac{4x^3}{x^4+4} dx$

$= \frac{1}{4}\ln|x^4+4| + C$

$= \frac{1}{4}\ln(x^4+4) + C$

25. $\int_0^1 t3^{t^2}\,dt = \frac{1}{2}\int_0^1 2t3^{t^2}\,dt$

$= \left[\frac{3^{t^2}}{2\ln 3}\right]_0^1 = \frac{3}{2\ln 3} - \frac{1}{2\ln 3}$

$= \frac{1}{\ln 3} \approx 0.9102$

26. $\int_0^{\pi/6} 2^{\cos x}\sin x\,dx = -\int_0^{\pi/6} 2^{\cos x}(-\sin x\,dx)$

$= \left[-\frac{2^{\cos x}}{\ln 2}\right]_0^{\pi/6}$

$= -\frac{1}{\ln 2}(2^{\sqrt{3}/2} - 2)$

$= \frac{2 - 2^{\sqrt{3}/2}}{\ln 2} \approx 0.2559$

27. $\int \frac{\sin x - \cos x}{\sin x}dx = \int\left(1 - \frac{\cos x}{\sin x}\right)dx$

$u = \sin x,\ du = \cos x\,dx$

$\int \frac{\sin x - \cos x}{\sin x}dx = x - \int\frac{du}{u}$

$= x - \ln|u| + C$

$= x - \ln|\sin x| + C$

28. $u = \cos(4t - 1),\ du = -4\sin(4t - 1)dt$

$\int \frac{\sin(4t-1)}{1-\sin^2(4t-1)}dt = \int \frac{\sin(4t-1)}{\cos^2(4t-1)}dt$

$= -\frac{1}{4}\int\frac{1}{u^2}du$

$= \frac{1}{4}u^{-1} + C = \frac{1}{4}\sec(4t-1) + C$

29. $u = e^x,\ du = e^x dx$

$\int e^x \sec e^x dx = \int \sec u\,du$

$= \ln|\sec u + \tan u| + C$

$= \ln\left|\sec e^x + \tan e^x\right| + C$

30. $u = e^x,\ du = e^x dx$

$\int e^x \sec^2(e^x)dx = \int \sec^2 u\,du = \tan u + C$

$= \tan(e^x) + C$

31. $\int \frac{\sec^3 x + e^{\sin x}}{\sec x}dx = \int(\sec^2 x + e^{\sin x}\cos x)\,dx$

$= \tan x + \int e^{\sin x}\cos x\,dx$

$u = \sin x,\ du = \cos x\,dx$

$\tan x + \int e^{\sin x}\cos x\,dx = \tan x + \int e^u\,du$

$= \tan x + e^u + C = \tan x + e^{\sin x} + C$

32. $u = \sqrt{3t^2 - t - 1}$,

$du = \frac{1}{2}(3t^2 - t - 1)^{-1/2}(6t - 1)dt$

$\int \frac{(6t-1)\sin\sqrt{3t^2-t-1}}{\sqrt{3t^2-t-1}}dt = 2\int \sin u\,du$

$= -2\cos u + C$

$= -2\cos\sqrt{3t^2 - t - 1} + C$

33. $u = t^3 - 2,\ du = 3t^2 dt$

$\int \frac{t^2\cos(t^3-2)}{\sin^2(t^3-2)}dt = \frac{1}{3}\int\frac{\cos u}{\sin^2 u}du$

$v = \sin u,\ dv = \cos u\,du$

$\frac{1}{3}\int\frac{\cos u}{\sin^2 u}du = \frac{1}{3}\int v^{-2}dv = -\frac{1}{3}v^{-1} + C$

$= -\frac{1}{3\sin u} + C$

$= -\frac{1}{3\sin(t^3-2)} + C.$

34. $\int\frac{1+\cos 2x}{\sin^2 2x}dx = \int\frac{1}{\sin^2 2x}dx + \int\frac{\cos 2x}{\sin^2 2x}dx$

$= \int \csc^2 2x\,dx + \int \cot 2x\csc 2x\,dx$

$= -\frac{1}{2}\cot 2x - \frac{1}{2}\csc 2x + C$

35. $u = t^3 - 2,\ du = 3t^2 dt$

$\int\frac{t^2\cos^2(t^3-2)}{\sin^2(t^3-2)}dt = \frac{1}{3}\int\frac{\cos^2 u}{\sin^2 u}du$

$= \frac{1}{3}\int\cot^2 u\,du = \frac{1}{3}\int(\csc^2 u - 1)du$

$= \frac{1}{3}[-\cot u - u] + C_1$

$= \frac{1}{3}[-\cot(t^3-2) - (t^3-2)] + C_1$

$= -\frac{1}{3}[\cot(t^3-2) + t^3] + C$

36. $u = 1 + \cot 2t,\ du = -2\csc^2 2t$

$\int\frac{\csc^2 2t}{\sqrt{1+\cot 2t}}dt = -\frac{1}{2}\int\frac{1}{\sqrt{u}}du$

$= -\sqrt{u} + C$

$= -\sqrt{1+\cot 2t} + C$

37. $u = \tan^{-1} 2t$, $du = \frac{2}{1+4t^2}dt$

$$\int \frac{e^{\tan^{-1} 2t}}{1+4t^2}dt = \frac{1}{2}\int e^u du$$

$$= \frac{1}{2}e^u + C = \frac{1}{2}e^{\tan^{-1} 2t} + C$$

38. $u = -t^2 - 2t - 5$,
$du = (-2t - 2)dt = -2(t + 1)dt$

$$\int (t+1)e^{-t^2-2t-5} = -\frac{1}{2}\int e^u du$$

$$= -\frac{1}{2}e^u + C \quad = -\frac{1}{2}e^{-t^2-2t-5} + C$$

39. $u = 3y^2$, $du = 6y\,dy$

$$\int \frac{y}{\sqrt{16-9y^4}}dy = \frac{1}{6}\int \frac{1}{\sqrt{4^2-u^2}}du$$

$$= \frac{1}{6}\sin^{-1}\left(\frac{u}{4}\right) + C$$

$$= \frac{1}{6}\sin^{-1}\left(\frac{3y^2}{4}\right) + C$$

40. $u = 3x$, $du = 3\,dx$

$$\int \cosh 3x\, dx$$

$$= \frac{1}{3}\int (\cosh u)du = \frac{1}{3}\sinh u + C$$

$$= \frac{1}{3}\sinh 3x + C$$

41. $u = x^3$, $du = 3x^2dx$

$$\int x^2 \sinh x^3 dx = \frac{1}{3}\int \sinh u\, du$$

$$= \frac{1}{3}\cosh u + C$$

$$= \frac{1}{3}\cosh x^3 + C$$

42. $u = 2x$, $du = 2\,dx$

$$\int \frac{5}{\sqrt{9-4x^2}}dx = \frac{5}{2}\int \frac{1}{\sqrt{3^2-u^2}}du$$

$$= \frac{5}{2}\sin^{-1}\left(\frac{u}{3}\right) + C$$

$$= \frac{5}{2}\sin^{-1}\left(\frac{2x}{3}\right) + C$$

43. $u = e^{3t}$, $du = 3e^{3t}dt$

$$\int \frac{e^{3t}}{\sqrt{4-e^{6t}}}dt = \frac{1}{3}\int \frac{1}{\sqrt{2^2-u^2}}du$$

$$= \frac{1}{3}\sin^{-1}\left(\frac{u}{2}\right) + C$$

$$= \frac{1}{3}\sin^{-1}\left(\frac{e^{3t}}{2}\right) + C$$

44. $u = 2t$, $du = 2dt$

$$\int \frac{dt}{2t\sqrt{4t^2-1}} = \frac{1}{2}\int \frac{1}{u\sqrt{u^2-1}}du$$

$$= \frac{1}{2}\left[\sec^{-1}|u|\right] + C$$

$$= \frac{1}{2}\sec^{-1}(2t) + C$$

45. $u = \cos x$, $du = -\sin x\, dx$

$$\int_0^{\pi/2} \frac{\sin x}{16+\cos^2 x}dx = -\int_1^0 \frac{1}{16+u^2}du$$

$$= \int_0^1 \frac{1}{16+u^2}du$$

$$= \left[\frac{1}{4}\tan^{-1}\left(\frac{u}{4}\right)\right]_0^1$$

$$= \left[\frac{1}{4}\tan^{-1}\left(\frac{1}{4}\right) - \frac{1}{4}\tan^{-1} 0\right]$$

$$= \frac{1}{4}\tan^{-1}\left(\frac{1}{4}\right) \approx 0.0612$$

46. $u = e^{2x} + e^{-2x}$, $du = (2e^{2x} - 2e^{-2x})dx$
$= 2(e^{2x} - e^{-2x})dx$

$$\int_0^1 \frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}dx = \frac{1}{2}\int_2^{e^2+e^{-2}} \frac{1}{u}du$$

$$= \frac{1}{2}\left[\ln|u|\right]_2^{e^2+e^{-2}}$$

$$= \frac{1}{2}\ln\left|e^2+e^{-2}\right| - \frac{1}{2}\ln 2$$

$$= \frac{1}{2}\ln\left|\frac{e^4+1}{e^2}\right| - \frac{1}{2}\ln 2$$

$$= \frac{1}{2}\ln(e^4+1) - \frac{1}{2}\ln(e^2) - \frac{1}{2}\ln 2$$

$$= \frac{1}{2}\left(\ln\left(\frac{e^4+1}{2}\right) - 2\right) \approx 0.6625$$

47. $\int \frac{1}{x^2+2x+5}dx = \int \frac{1}{x^2+2x+1+4}dx$

$$= \int \frac{1}{(x+1)^2+2^2}d(x+1)$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C$$

48. $\int \frac{1}{x^2-4x+9}dx = \int \frac{1}{x^2-4x+4+5}dx$

$= \int \frac{1}{(x-2)^2+(\sqrt{5})^2}d(x-2)$

$= \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{x-2}{\sqrt{5}}\right)+C$

49. $\int \frac{dx}{9x^2+18x+10} = \int \frac{dx}{9x^2+18x+9+1}$

$= \int \frac{dx}{(3x+3)^2+1^2}$

$u = 3x+3$, $du = 3\,dx$

$\int \frac{dx}{(3x+3)^2+1^2} = \frac{1}{3}\int \frac{du}{u^2+1^2}$

$= \frac{1}{3}\tan^{-1}(3x+3)+C$

50. $\int \frac{dx}{\sqrt{16+6x-x^2}} = \int \frac{dx}{\sqrt{-(x^2-6x+9-25)}}$

$= \int \frac{dx}{\sqrt{-(x-3)^2+5^2}}$

$= \int \frac{dx}{\sqrt{5^2-(x-3)^2}}$

$= \sin^{-1}\left(\frac{x-3}{5}\right)+C$

51. $\int \frac{x+1}{9x^2+18x+10}dx = \frac{1}{18}\int \frac{18x+18}{9x^2+18x+10}dx$

$= \frac{1}{18}\ln\left|9x^2+18x+10\right|+C$

52. $\int \frac{3-x}{\sqrt{16+6x-x^2}}dx = \frac{1}{2}\int \frac{6-2x}{\sqrt{16+6x-x^2}}dx$

$= \sqrt{16+6x-x^2}+C$

53. $u = \sqrt{2}t$, $du = \sqrt{2}dt$

$\int \frac{dt}{t\sqrt{2t^2-9}} = \int \frac{du}{u\sqrt{u^2-3^2}}$

$= \frac{1}{3}\sec^{-1}\left(\frac{\left|\sqrt{2}t\right|}{3}\right)+C$

54. $\int \frac{\tan x}{\sqrt{\sec^2 x-4}}dx = \int \frac{\cos x}{\cos x}\frac{\tan x}{\sqrt{\sec^2 x-4}}dx$

$= \int \frac{\sin x}{\sqrt{1-4\cos^2 x}}dx$

$u = 2\cos x$, $du = -2\sin x\,dx$

$\int \frac{\sin x}{\sqrt{1-4\cos^2 x}}dx$

$= -\frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}du = -\frac{1}{2}\sin^{-1}u+C$

$= -\frac{1}{2}\sin^{-1}(2\cos x)+C$

55. $\int x\sqrt{3x+2}dx$

$= \frac{2}{15\cdot 3^2}(3\cdot 3x-2\cdot 2)(3x+2)^{3/2}+C$

$= \frac{2}{135}(9x-4)(3x+2)^{3/2}+C$

Use Formula 96 with $a = 3$, $b = 2$, and $u = x$ for $\int u\sqrt{3u+2}\,du$.

56. $\int 2t\sqrt{3-4t}\,dt = 2\int t\sqrt{-4t+3}\,dt$

$= 2\left[\frac{2}{15(-4)^2}(3(-4)t-2\cdot 3)(-4t+3)^{3/2}+C\right]$

$= \frac{1}{60}(-12t-6)(3-4t)^{3/2}+C$

$= -\frac{1}{10}(2t+1)(3-4t)^{3/2}+C$

Use Formula 96 with $a = -4$, $b = 3$, and $u = t$ for $\int u\sqrt{-4u+3}\,du$.

57. $u = 4x$, $du = 4\,dx$

$\int \frac{dx}{9-16x^2} = \frac{1}{4}\int \frac{du}{3^2-u^2}du$

$= \frac{1}{4}\left[\frac{1}{2(3)}\ln\left|\frac{u+3}{u-3}\right|\right]+C$

$= \frac{1}{24}\ln\left|\frac{4x+3}{4x-3}\right|+C$

Use Formula 18 with $a = 3$ for $\int \frac{du}{3^2-u^2}$.

58. $\int \frac{dx}{5x^2-11} = -\int \frac{dx}{11-5x^2}$

$u = \sqrt{5}x$, $du = \sqrt{5}\,dx$

$-\int \frac{dx}{11-5x^2} = -\frac{1}{\sqrt{5}}\int \frac{du}{\left(\sqrt{11}\right)^2-u^2}$

$= -\frac{1}{\sqrt{5}}\left[\frac{1}{2\sqrt{11}}\ln\left|\frac{u+\sqrt{11}}{u-\sqrt{11}}\right|\right]+C$

$= -\frac{1}{2\sqrt{55}}\ln\left|\frac{\sqrt{5}x+\sqrt{11}}{\sqrt{5}x-\sqrt{11}}\right|+C$

Use Formula 18 with $a=\sqrt{11}$ for

$$\int \frac{du}{\left(\sqrt{11}\right)^2-u^2}.$$

59. $\int x^2\sqrt{9-2x^2}\,dx = \int \sqrt{2}x^2\sqrt{\frac{9-2x^2}{2}}\,dx$

$$=\sqrt{2}\int x^2\sqrt{\frac{9}{2}-x^2}\,dx=\sqrt{2}\int x^2\sqrt{\left(\frac{3}{\sqrt{2}}\right)^2-x^2}\,dx$$

$$=\sqrt{2}\left[\frac{x}{8}\left(2x^2-\frac{9}{2}\right)\sqrt{\frac{9}{2}-x^2}+\frac{\left(\frac{81}{4}\right)}{8}\sin^{-1}\left(\frac{x}{\frac{3}{\sqrt{2}}}\right)+C\right]$$

$$=\frac{x}{16}(4x^2-9)\sqrt{9-2x^2}+\frac{81\sqrt{2}}{32}\sin^{-1}\left(\frac{\sqrt{2}\,x}{3}\right)+C$$

Use Formula 57 with $a=\frac{3}{\sqrt{2}}$ and $u=x$ for

$$\int x^2\sqrt{\left(\frac{3}{\sqrt{2}}\right)^2-x^2}\,dx.$$

60. $u=\sqrt{3}t$, $du=\sqrt{3}dt$

$$\int\frac{\sqrt{16-3t^2}}{t}dt=\int\frac{\sqrt{4^2-u^2}}{u}du$$

$$=\sqrt{16-u^2}-4\ln\left|\frac{4+\sqrt{16-u^2}}{u}\right|+C$$

$$=\sqrt{16-3t^2}-4\ln\left|\frac{4+\sqrt{16-3t^2}}{\sqrt{3}t}\right|+C$$

Use Formula 55 with $a=4$ for $\int\frac{\sqrt{4^2-u^2}}{u}du$.

61. $u=\sqrt{3}x, du=\sqrt{3}\,dx$

$$\int\frac{dx}{\sqrt{5+3x^2}}=\frac{1}{\sqrt{3}}\int\frac{du}{\sqrt{\left(\sqrt{5}\right)^2+u^2}}$$

$$=\frac{1}{\sqrt{3}}\ln\left|u+\sqrt{u^2+5}\right|+C$$

$$=\frac{1}{\sqrt{3}}\ln\left|\sqrt{3}\,x+\sqrt{3x^2+5}\right|+C$$

Use Formula 45 with $a=\sqrt{5}$ for

$$\int\frac{du}{\sqrt{\left(\sqrt{5}\right)^2+u^2}}.$$

62. $u=\sqrt{5}t$, $du=\sqrt{5}dt$

$$\int t^2\sqrt{3+5t^2}\,dt=\frac{1}{\sqrt{5}}\int\frac{u^2}{5}\sqrt{\left(\sqrt{3}\right)^2+u^2}\,du$$

$$=\frac{1}{5\sqrt{5}}\int u^2\sqrt{\left(\sqrt{3}\right)^2+u^2}\,du$$

$$=\frac{1}{5\sqrt{5}}\cdot\left[\frac{u}{8}(3+2u^2)\sqrt{3+u^2}-\frac{9}{8}\ln\left(u+\sqrt{3+u^2}\right)\right]+C$$

$$=\frac{1}{5\sqrt{5}}\left[\frac{\sqrt{5}t}{8}(3+10t^2)\sqrt{3+5t^2}-\frac{9}{8}\ln\left(\sqrt{5}t+\sqrt{3+5t^2}\right)\right]+C$$

Use Formula 48 with $a=\sqrt{3}$ for

$$\int u^2\sqrt{\left(\sqrt{3}\right)^2+u^2}\,du.$$

63. $u=t+1$, $du=dt$

$$\int\frac{dt}{\sqrt{t^2+2t-3}}=\int\frac{dt}{\sqrt{t^2+2t+1-4}}$$

$$=\int\frac{dt}{\sqrt{(t+1)^2-4}}$$

$$=\int\frac{du}{\sqrt{u^2-2^2}}=\ln\left|u+\sqrt{u^2-4}\right|+C$$

$$=\ln\left|t+1+\sqrt{t^2+2t-3}\right|+C$$

Use Formula 45 with $a=2$ for $\int\frac{du}{\sqrt{u^2-2^2}}$.

64. $u=x+1$, $du=dx$

$$\int\frac{\sqrt{x^2+2x-3}}{x+1}dx=\int\frac{\sqrt{x^2+2x+1-4}}{x+1}dx$$

$$=\int\frac{\sqrt{(x+1)^2-4}}{x+1}dx=\int\frac{\sqrt{u^2-2^2}}{u}du$$

$$=\sqrt{u^2-4}-2\sec^{-1}\frac{u}{2}+C$$

$$=\sqrt{x^2+2x-3}-2\sec^{-1}\left(\frac{x+1}{2}\right)+C$$

Use Formula 47 with $a=2$ for $\int\frac{\sqrt{u^2-2^2}}{u}du$

65. $u=\sin t$, $du=\cos t\,dt$

$$\int\frac{\sin t\cos t}{\sqrt{3\sin t+5}}dt$$

$$=\int\frac{u}{\sqrt{3u+5}}du=\frac{2}{3\cdot 3^2}(3u-2\cdot 5)\sqrt{3u+5}+C$$

$$=\frac{2}{27}(3\sin t-10)\sqrt{3\sin t+5}+C$$

Use Formula 98 with $a = 3$, and $b = 5$ for $\int \frac{u}{\sqrt{3u+5}} du$.

66. $u = \cos x$, $du = -\sin x\, dx$

$$\int \frac{\sin x}{\cos x\sqrt{5-4\cos x}} dx = -\int \frac{du}{u\sqrt{5-4u}}$$

$$= -\int \frac{du}{u\sqrt{-4u+5}}$$

$$= -\frac{1}{\sqrt{5}} \ln\left|\frac{\sqrt{5-4u}-\sqrt{5}}{\sqrt{5-4u}+\sqrt{5}}\right| + C$$

$$= -\frac{1}{\sqrt{5}} \ln\left|\frac{\sqrt{5-4\cos x}-\sqrt{5}}{\sqrt{5-4\cos x}+\sqrt{5}}\right| + C$$

Use Formula 100a with $a = -4$, and $b = 5$ for $\int \frac{du}{u\sqrt{-4u+5}}$.

67. The length is given by

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^{\pi/4} \sqrt{1+\left[\frac{1}{\cos x}(-\sin x)\right]^2}\, dx$$

$$= \int_0^{\pi/4} \sqrt{1+\tan^2 x}\, dx$$

$$= \int_0^{\pi/4} \sqrt{\sec^2 x}\, dx$$

$$\int_0^{\pi/4} \sec x\, dx$$

$$= \left[\ln|\sec x + \tan x|\right]_0^{\pi/4}$$

$$= \ln\left|\sqrt{2}+1\right| - \ln|1|$$

$$= \ln\left|\sqrt{2}+1\right| \approx 0.881$$

68. $\sec x = \frac{1}{\cos x} = \frac{1+\sin x}{\cos x(1+\sin x)}$

$$= \frac{\sin x + \sin^2 x + \cos^2 x}{\cos x(1+\sin x)} = \frac{\sin x(1+\sin x)+\cos^2 x}{\cos x(1+\sin x)}$$

$$= \frac{\sin x}{\cos x} + \frac{\cos x}{1+\sin x}$$

$$\int \sec x = \int\left(\frac{\sin x}{\cos x} + \frac{\cos x}{1+\sin x}\right) dx$$

$$= \int \frac{\sin x}{\cos x} dx + \int \frac{\cos x}{1+\sin x} dx$$

For the first integral use $u = \cos x$, $du = -\sin x\, dx$, and for the second integral use $v = 1 + \sin x$, $dv = \cos x\, dx$.

$$\int \frac{\sin x}{\cos x} dx + \int \frac{\cos x}{1+\sin x} dx = -\int \frac{du}{u} + \int \frac{dv}{v}$$

$$= -\ln|u| + \ln|v| + C$$

$$= -\ln|\cos x| + \ln|1+\sin x| + C$$

$$= \ln\left|\frac{1+\sin x}{\cos x}\right| + C$$

$$= \ln|\sec x + \tan x| + C$$

69. $u = x - \pi$, $du = dx$

$$\int_0^{2\pi} \frac{x|\sin x|}{1+\cos^2 x} dx = \int_{-\pi}^{\pi} \frac{(u+\pi)|\sin(u+\pi)|}{1+\cos^2(u+\pi)} du$$

$$= \int_{-\pi}^{\pi} \frac{(u+\pi)|\sin u|}{1+\cos^2 u} du$$

$$= \int_{-\pi}^{\pi} \frac{u|\sin u|}{1+\cos^2 u} du + \int_{-\pi}^{\pi} \frac{\pi|\sin u|}{1+\cos^2 u} du$$

$\int_{-\pi}^{\pi} \frac{u|\sin u|}{1+\cos^2 u} du = 0$ by symmetry.

$$\int_{-\pi}^{\pi} \frac{\pi|\sin u|}{1+\cos^2 u} du = 2\int_0^{\pi} \frac{\pi \sin u}{1+\cos^2 u} du$$

$v = \cos u$, $dv = -\sin u\, du$

$$-2\int_1^{-1} \frac{\pi}{1+v^2} dv = 2\pi \int_{-1}^{1} \frac{1}{1+v^2} dv$$

$$= 2\pi[\tan^{-1} v]_{-1}^{1} = 2\pi\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right]$$

$$= 2\pi\left(\frac{\pi}{2}\right) = \pi^2$$

70. $V = 2\pi \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \left(x+\frac{\pi}{4}\right)|\sin x - \cos x|\, dx$

$u = x - \frac{\pi}{4}$, $du = dx$

$$V = 2\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(u+\frac{\pi}{2}\right)\left|\sin\left(u+\frac{\pi}{4}\right)-\cos\left(u+\frac{\pi}{4}\right)\right|du$$

$$= 2\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(u+\frac{\pi}{2}\right)\left|\frac{\sqrt{2}}{2}\sin u+\frac{\sqrt{2}}{2}\cos u-\frac{\sqrt{2}}{2}\cos u+\frac{\sqrt{2}}{2}\sin u\right|du$$

$$= 2\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(u+\frac{\pi}{2}\right)\left|\frac{\sqrt{2}}{2}\sin u\right|du = 2\sqrt{2}\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}u|\sin u|du+\sqrt{2}\pi^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}|\sin u|du$$

$$= 2\sqrt{2}\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}u|\sin u|du = 0 \text{ by symmetry.}$$

$$V = \sqrt{2}\pi^2 2\int_0^{\frac{\pi}{2}}\sin u\,du = 2\sqrt{2}\pi^2[-\cos u]_0^{\frac{\pi}{2}} = 2\sqrt{2}\pi^2$$

8.2 Concepts Review

1. $\int\frac{1+\cos 2x}{2}dx$

2. $\int(1-\sin^2 x)\cos x\,dx$

3. $\int\sin^2 x(1-\sin^2 x)\cos x\,dx$

4. $\cos mx\cos nx = \frac{1}{2}[\cos(m+n)x+\cos(m-n)x]$

Problem Set 8.2

1. $\int\sin^2 x\,dx = \int\frac{1-\cos 2x}{2}dx$

$= \frac{1}{2}\int dx-\frac{1}{2}\int\cos 2x\,dx$

$= \frac{1}{2}x-\frac{1}{4}\sin 2x+C$

2. $u = 6x$, $du = 6\,dx$

$\int\sin^4 6x\,dx = \frac{1}{6}\int\sin^4 u\,du$

$= \frac{1}{6}\int\left(\frac{1-\cos 2u}{2}\right)^2 du$

$= \frac{1}{24}\int(1-2\cos 2u+\cos^2 2u)du$

$= \frac{1}{24}\int du-\frac{1}{24}\int 2\cos 2u\,du+\frac{1}{48}\int(1+\cos 4u)du$

$= \frac{3}{48}\int du-\frac{1}{24}\int 2\cos 2u\,du+\frac{1}{192}\int 4\cos 4u\,du$

$= \frac{3}{48}(6x)-\frac{1}{24}\sin 12x+\frac{1}{192}\sin 24x+C$

$= \frac{3}{8}x-\frac{1}{24}\sin 12x+\frac{1}{192}\sin 24x+C$

3. $\int\sin^3 x\,dx = \int\sin x(1-\cos^2 x)dx$

$= \int\sin x\,dx-\int\sin x\cos^2 x\,dx$

$= -\cos x+\frac{1}{3}\cos^3 x+C$

4. $\int\cos^3 x\,dx =$

$= \int\cos x(1-\sin^2 x)dx$

$= \int\cos x\,dx-\int\cos x\sin^2 x\,dx$

$= \sin x-\frac{1}{3}\sin^3 x+C$

5. $\int_0^{\pi/2}\cos^5\theta\,d\theta = \int_0^{\pi/2}(1-\sin^2\theta)^2\cos\theta\,d\theta$

$= \int_0^{\pi/2}(1-2\sin^2\theta+\sin^4\theta)\cos\theta\,d\theta$

$= \left[\sin\theta-\frac{2}{3}\sin^3\theta+\frac{1}{5}\sin^5\theta\right]_0^{\pi/2}$

$= \left(1-\frac{2}{3}+\frac{1}{5}\right)-0 = \frac{8}{15}$

6. $\int_0^{\pi/2}\sin^6\theta\,d\theta = \int_0^{\pi/2}\left(\frac{1-\cos 2\theta}{2}\right)^3 d\theta$

$= \frac{1}{8}\int_0^{\pi/2}(1-3\cos 2\theta+3\cos^2 2\theta-\cos^3 2\theta)d\theta$

$$= \frac{1}{8}\int_0^{\pi/2} d\theta - \frac{3}{16}\int_0^{\pi/2} 2\cos 2\theta\, d\theta + \frac{3}{8}\int_0^{\pi/2} \cos^2 2\theta - \frac{1}{8}\int_0^{\pi/2} \cos^3 2\theta\, d\theta$$

$$= \frac{1}{8}[\theta]_0^{\pi/2} - \frac{3}{16}[\sin 2\theta]_0^{\pi/2} + \frac{3}{8}\int_0^{\pi/2}\left(\frac{1+\cos 4\theta}{2}\right)d\theta - \frac{1}{8}\int_0^{\pi/2}(1-\sin^2 2\theta)\cos 2\theta\, d\theta$$

$$= \frac{1}{8}\cdot\frac{\pi}{2} + \frac{3}{16}\int_0^{\pi/2} d\theta + \frac{3}{64}\int_0^{\pi/2} 4\cos 4\theta\, d\theta - \frac{1}{16}\int_0^{\pi/2} 2\cos 2\theta\, d\theta + \frac{1}{16}\int_0^{\pi/2} \sin^2 2\theta \cdot 2\cos 2\theta\, d\theta$$

$$= \frac{\pi}{16} + \frac{3\pi}{32} + \frac{3}{64}[\sin 4\theta]_0^{\pi/2} - \frac{1}{16}[\sin 2\theta]_0^{\pi/2} + \frac{1}{48}[\sin^3 2\theta]_0^{\pi/2} = \frac{5\pi}{32}$$

7. $\int \sin^5 4x \cos^2 4x\, dx = \int (1-\cos^2 4x)^2 \cos^2 4x \sin 4x\, dx = \int (1 - 2\cos^2 4x + \cos^4 4x)\cos^2 4x \sin 4x\, dx$

$$= -\frac{1}{4}\int (\cos^2 4x - 2\cos^4 4x + \cos^6 4x)(-4\sin 4x)dx = -\frac{1}{12}\cos^3 4x + \frac{1}{10}\cos^5 4x - \frac{1}{28}\cos^7 4x + C$$

8. $\int (\sin^3 2t)\sqrt{\cos 2t}\, dt = \int (1-\cos^2 2t)(\cos 2t)^{1/2}\sin 2t\, dt = -\frac{1}{2}\int [(\cos 2t)^{1/2} - (\cos 2t)^{5/2}](-2\sin 2t)dt$

$$= -\frac{1}{3}(\cos 2t)^{3/2} + \frac{1}{7}(\cos 2t)^{7/2} + C$$

9. $\int \cos^3 3\theta \sin^{-2} 3\theta\, d\theta = \int (1-\sin^2 3\theta)\sin^{-2} 3\theta \cos 3\theta\, d\theta = \frac{1}{3}\int (\sin^{-2} 3\theta - 1)3\cos 3\theta\, d\theta$

$$= -\frac{1}{3}\csc 3\theta - \frac{1}{3}\sin 3\theta + C$$

10. $\int \sin^{1/2} 2z \cos^3 2z\, dz = \int (1-\sin^2 2z)\sin^{1/2} 2z \cos 2z\, dz$

$$= \frac{1}{2}\int (\sin^{1/2} 2z - \sin^{5/2} 2z)2\cos 2z\, dz = \frac{1}{3}\sin^{3/2} 2z - \frac{1}{7}\sin^{7/2} 2z + C$$

11. $\int \sin^4 3t \cos^4 3t\, dt = \int \left(\frac{1-\cos 6t}{2}\right)^2\left(\frac{1+\cos 6t}{2}\right)^2 dt = \frac{1}{16}\int (1 - 2\cos^2 6t + \cos^4 6t)dt$

$$= \frac{1}{16}\int \left[1 - (1+\cos 12t) + \frac{1}{4}(1+\cos 12t)^2\right]dt = -\frac{1}{16}\int \cos 12t\, dt + \frac{1}{64}\int (1 + 2\cos 12t + \cos^2 12t)dt$$

$$= -\frac{1}{192}\int 12\cos 12t\, dt + \frac{1}{64}\int dt + \frac{1}{384}\int 12\cos 12t\, dt + \frac{1}{128}\int (1+\cos 24t)dt$$

$$= -\frac{1}{192}\sin 12t + \frac{1}{64}t + \frac{1}{384}\sin 12t + \frac{1}{128}t + \frac{1}{3072}\sin 24t + C = \frac{3}{128}t - \frac{1}{384}\sin 12t + \frac{1}{3072}\sin 24t + C$$

12. $\int \cos^6\theta \sin^2\theta\, d\theta = \int \left(\frac{1+\cos 2\theta}{2}\right)^3\left(\frac{1-\cos 2\theta}{2}\right)d\theta = \frac{1}{16}\int (1 + 2\cos 2\theta - 2\cos^3 2\theta - \cos^4 2\theta)d\theta$

$$= \frac{1}{16}\int d\theta + \frac{1}{16}\int 2\cos 2\theta\, d\theta - \frac{1}{8}\int (1-\sin^2 2\theta)\cos 2\theta\, d\theta - \frac{1}{64}\int (1+\cos 4\theta)^2 d\theta$$

$$= \frac{1}{16}\int d\theta + \frac{1}{16}\int 2\cos 2\theta\, d\theta - \frac{1}{16}\int 2\cos 2\theta\, d\theta + \frac{1}{16}\int 2\sin^2 2\theta \cos 2\theta\, d\theta - \frac{1}{64}\int (1 + 2\cos 4\theta + \cos^2 4\theta)d\theta$$

$$= \frac{1}{16}\int d\theta + \frac{1}{16}\int \sin^2 2\theta \cdot 2\cos 2\theta\, d\theta - \frac{1}{64}\int d\theta - \frac{1}{128}\int 4\cos 4\theta\, d\theta - \frac{1}{128}\int (1+\cos 8\theta)d\theta$$

$$= \frac{1}{16}\theta + \frac{1}{48}\sin^3 2\theta - \frac{1}{64}\theta - \frac{1}{128}\sin 4\theta - \frac{1}{128}\theta - \frac{1}{1024}\sin 8\theta + C$$

$$= \frac{5}{128}\theta + \frac{1}{48}\sin^3 2\theta - \frac{1}{128}\sin 4\theta - \frac{1}{1024}\sin 8\theta + C$$

13. $\int \sin 4y \cos 5y\, dy = \frac{1}{2}\int [\sin 9y + \sin(-y)]dy = \frac{1}{2}\int (\sin 9y - \sin y)dy$

$= \frac{1}{2}\left(-\frac{1}{9}\cos 9y + \cos y\right) + C = \frac{1}{2}\cos y - \frac{1}{18}\cos 9y + C$

14. $\int \cos y \cos 4y\, dy = \frac{1}{2}\int [\cos 5y + \cos(-3y)]dy = \frac{1}{10}\sin 5y - \frac{1}{6}\sin(-3y) + C = \frac{1}{10}\sin 5y + \frac{1}{6}\sin 3y + C$

15. $\int \sin^4\left(\frac{w}{2}\right)\cos^2\left(\frac{w}{2}\right)dw = \int \left(\frac{1-\cos w}{2}\right)^2\left(\frac{1+\cos w}{2}\right)dw = \frac{1}{8}\int (1 - \cos w - \cos^2 w + \cos^3 w)dw$

$= \frac{1}{8}\int \left[1 - \cos w - \frac{1}{2}(1 + \cos 2w) + (1 - \sin^2 w)\cos w\right]dw = \frac{1}{8}\int \left[\frac{1}{2} - \frac{1}{2}\cos 2w - \sin^2 w \cos w\right]dw$

$= \frac{1}{16}w - \frac{1}{32}\sin 2w - \frac{1}{24}\sin^3 w + C$

16. $\int \sin 3t \sin t\, dt = \int -\frac{1}{2}[\cos 4t - \cos 2t]dt$

$= -\frac{1}{2}\left(\int \cos 4t dt - \int \cos 2t dt\right)$

$= -\frac{1}{2}\left(\frac{1}{4}\sin 4t - \frac{1}{2}\sin 2t\right) + C$

$= -\frac{1}{8}\sin 4t + \frac{1}{4}\sin 2t + C$

17. $\int \tan^4 x\, dx = \int (\tan^2 x)(\tan^2 x)\, dx$

$= \int (\tan^2 x)(\sec^2 x - 1)\, dx$

$= \int (\tan^2 x \sec^2 x - \tan^2 x)dx$

$= \int \tan^2 x \sec^2 x\, dx - \int (\sec^2 x - 1)dx$

$= \frac{1}{3}\tan^3 x - \tan x + x + C$

18. $\int \cot^4 x\, dx = \int (\cot^2 x)(\cot^2 x)\, dx$

$= \int (\cot^2 x)(\csc^2 x - 1)\, dx$

$= \int (\cot^2 x \csc^2 x - \cot^2 x)dx$

$= \int \cot^2 x \csc^2 x\, dx - \int (\csc^2 x - 1)dx$

$= -\frac{1}{3}\cot^3 x + \cot x + x + C$

19. $\tan^3 x = \int (\tan x)(\tan^2 x)dx$

$= \int (\tan x)(\sec^2 x - 1)dx$

$= \frac{1}{2}\tan^2 x + \ln|\cos x| + C$

20. $\int \cot^3 x dx = \int (\cot x)(\cot^2 x)dx$

$= \int (\cot x)(\csc^2 x - 1)dx$

$= \int \cot x \csc^2 x dx - \int \cot x dx$

$= -\frac{1}{2}\cot^2 x - \ln|\sin x| + C$

21. $\int \tan^5\left(\frac{\theta}{2}\right)d\theta$

$u = \left(\frac{\theta}{2}\right);\ du = \frac{d\theta}{2}$

$\int \tan^5\left(\frac{\theta}{2}\right)d\theta = 2\int \tan^5 u\, du$

$= 2\int (\tan^3 u)(\sec^2 u - 1)du$

$= 2\int \tan^3 u \sec^2 u\, du - 2\int \tan^3 u\, du$

$= 2\int \tan^3 u \sec^2 u\, du - 2\int \tan u\left(\sec^2 u - 1\right)du$

$= 2\int \tan^3 u \sec^2 u\, du - 2\int \tan u \sec^2 u\, du + 2\int \tan u\, \text{du}$

$= \frac{1}{2}\tan^4\left(\frac{\theta}{2}\right) - \tan^2\left(\frac{\theta}{2}\right) - 2\ln\left|\cos\frac{\theta}{2}\right| + C$

22. $\int \cot^5 2t\, dt$

$u = 2t; du = 2dt$

$\int \cot^5 2t\, dt = \frac{1}{2}\int \cot^5 u\, du$

$= \frac{1}{2}\int\left(\cot^3 u\right)\left(\cot^2 u\right)du = \frac{1}{2}\int\left(\cot^3 u\right)\left(\csc^2 - 1\right)du$

$= \frac{1}{2}\int\left(\cot^3 u\right)\left(\csc^2 u\right)du - \frac{1}{2}\int \cot^3 u\, du$

$= \frac{1}{2}\int\left(\cot^3 u\right)\left(\csc^2 u\right)du - \frac{1}{2}\int\left(\cot u\right)\left(\csc^2 u - 1\right)du$

$= \frac{1}{2}\int\left(\cot^3 u\right)\left(\csc^2 u\right)du - \frac{1}{2}\int\left(\cot u\right)\left(\csc^2 u\right)du + \frac{1}{2}\int \cot u$

$= -\frac{1}{8}\cot^4 u + \frac{1}{4}\cot^2 u + \frac{1}{2}\ln|\sin u| + C$

$= -\frac{1}{8}\cot^4 2t + \frac{1}{4}\cot^2 2t + \frac{1}{2}\ln|\sin 2t| + C$

23. $\int \tan^{-3} x \sec^4 x dx = \int\left(\tan^{-3} x\right)\left(\sec^2 x\right)\left(\sec^2 x\right)dx$

$= \int\left(\tan^{-3} x\right)\left(1 + \tan^2 x\right)\left(\sec^2 x\right)dx$

$= \int \tan^{-3} x \sec^2 x\, \text{dx} + \int\left(\tan x\right)^{-1}\sec^2 x\, dx$

$= -\frac{1}{2}\tan^{-2} x + \ln|\tan x| + C$

24. $\int \tan^{-3/2} x \sec^4 x\, dx = \int\left(\tan^{-3/2} x\right)\left(\sec^2 x\right)\left(\sec^2 x\right)$

$= \int\left(\tan^{-3/2} x\right)\left(1 + \tan^2 x\right)\left(\sec^2 x\right)dx$

$= \int \tan^{-3/2} x \sec^2 x\, dx + \int \tan^{1/2} x \sec^2 x\, dx$

$= -2\tan^{-1/2} x + \frac{2}{3}\tan^{3/2} x + C$

25. $\int \tan^3 x \sec^2 x\, dx = \int\left(\tan^2 x\right)(\sec x)(\sec x \tan x)dx$

$= \int\left(\sec^2 x - 1\right)(\sec x)(\sec x \tan x)dx$

$= \int \sec^3 x \sec x \tan x\, dx - \int \sec x(\sec x \tan x)dx$

$= \frac{1}{4}\sec^4 x - \frac{1}{2}\sec^2 x + C$

26. $\int \tan^3 x \sec^{-1/2} x\, dx = \int \tan^2 x \sec^{-3/2} x(\sec x \tan x)dx$

$= \int\left(\sec^2 x - 1\right)\left(\sec^{-3/2} x\right)(\sec x \tan x)dx$

$= \int \sec^{1/2} x(\sec x \tan x)dx - \int \sec^{-3/2} x(\sec x \tan x)dx$

$= \frac{2}{3}\sec^{3/2} x + 2\sec^{-1/2} x + C$

27. $\int_{-\pi}^{\pi} \cos mx \cos nx\, dx = \frac{1}{2}\int_{-\pi}^{\pi} (\cos[(m+n)x] + \cos[(m-n)x])dx = \frac{1}{2}\left[\frac{1}{m+n}\sin[(m+n)x] + \frac{1}{m-n}\sin[(m-n)x]\right]_{-\pi}^{\pi}$

$= 0$ for $m \neq n$, since $\sin k\pi = 0$ for all integers k.

28. If we let $u = \frac{\pi x}{L}$ then $du = \frac{\pi}{L}dx$. Making the substitution and changing the limits as necessary, we get

$\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\, dx = \frac{L}{\pi}\int_{-\pi}^{\pi} \cos mu \cos nu\, du = 0$ (See Problem 27)

29. $\int_0^{\pi} \pi(x+\sin x)^2 dx = \pi\int_0^{\pi}(x^2 + 2x\sin x + \sin^2 x)dx = \pi\int_0^{\pi} x^2\, dx + 2\pi\int_0^{\pi} x\sin x\, dx + \frac{\pi}{2}\int_0^{\pi}(1-\cos 2x)dx$

$= \pi\left[\frac{1}{3}x^3\right]_0^{\pi} + 2\pi[\sin x - x\cos x]_0^{\pi} + \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi} = \frac{1}{3}\pi^4 + 2\pi(0+\pi-0) + \frac{\pi}{2}(\pi-0-0) = \frac{1}{3}\pi^4 + \frac{5}{2}\pi^2 \approx 57.1437$

Use Formula 40 with $u = x$ for $\int x\sin x\, dx$

30. $V = 2\pi\int_0^{\sqrt{\pi/2}} x\sin^2(x^2)dx$

$u = x^2,\ du = 2x\, dx$

$V = \pi\int_0^{\pi/2}\sin^2 u\, du = \pi\int_0^{\pi/2}\frac{1-\cos 2u}{2}du = \pi\left[\frac{1}{2}u - \frac{1}{4}\sin 2u\right]_0^{\pi/2} = \frac{\pi^2}{4} \approx 2.4674$

31. a. $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\sum_{n=1}^{N} a_n\sin(nx)\right)\sin(mx)dx = \frac{1}{\pi}\sum_{n=1}^{N} a_n\int_{-\pi}^{\pi}\sin(nx)\sin(mx)dx$

From Example 6,

$\int_{-\pi}^{\pi}\sin(nx)\sin(mx)dx = \begin{cases} 0 \text{ if } n \neq m \\ \pi \text{ if } n = m \end{cases}$ so every term in the sum is 0 except for when $n = m$.

If $m > N$, there is no term where $n = m$, while if $m \le N$, then $n = m$ occurs. When $n = m$

$a_n\int_{-\pi}^{\pi}\sin(nx)\sin(mx)dx = a_m\pi$ so when $m \le N$,

$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)dx = \frac{1}{\pi}\cdot a_m\cdot\pi = a_m$.

b. $\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\sum_{n=1}^{N} a_n\sin(nx)\right)\left(\sum_{m=1}^{N} a_m\sin(mx)\right)dx = \frac{1}{\pi}\sum_{n=1}^{N} a_n\sum_{m=1}^{N} a_m\int_{-\pi}^{\pi}\sin(nx)\sin(mx)\, dx$

From Example 6, the integral is 0 except when $m = n$. When $m = n$, we obtain

$\frac{1}{\pi}\sum_{n=1}^{N} a_n(a_n\pi) = \sum_{n=1}^{N} a_n^2$.

32. a. Proof by induction

$n = 1$: $\cos\frac{x}{2} = \cos\frac{x}{2}$

Assume true for $k \le n$.

$\cos\frac{x}{2}\cos\frac{x}{4}\cdots\cos\frac{x}{2^n}\cdot\cos\frac{x}{2^{n+1}} = \left[\cos\frac{1}{2^n}x + \cos\frac{3}{2^n}x + \cdots + \cos\frac{2^n-1}{2^n}x\right]\frac{1}{2^{n-1}}\cos\frac{x}{2^{n+1}}$

Note that

$$\left(\cos\frac{k}{2^n}x\right)\left(\cos\frac{1}{2^{n+1}}x\right)=\frac{1}{2}\left[\cos\frac{2k+1}{2^{n+1}}x+\cos\frac{2k-1}{2^{n+1}}x\right],\text{ so}$$

$$\left[\cos\frac{1}{2^n}x+\cos\frac{3}{2^n}x+\cdots+\cos\frac{2^n-1}{2^n}x\right]\left(\cos\frac{1}{2^{n+1}}x\right)\frac{1}{2^{n-1}}=\left[\cos\frac{1}{2^{n+1}}x+\cos\frac{3}{2^{n+1}}x+\cdots+\cos\frac{2^{n+1}-1}{2^{n+1}}x\right]\frac{1}{2^n}$$

b. $$\lim_{n\to\infty}\left[\cos\frac{1}{2^n}x+\cos\frac{3}{2^n}x+\cdots+\cos\frac{2^n-1}{2^n}x\right]\frac{1}{2^{n-1}}=\frac{1}{x}\lim_{n\to\infty}\left[\cos\frac{1}{2^n}x+\cos\frac{3}{2^n}x+\cdots+\cos\frac{2^n-1}{2^n}x\right]\frac{x}{2^{n-1}}$$

$$=\frac{1}{x}\int_0^x\cos t\,dt$$

c. $$\frac{1}{x}\int_0^x\cos t\,dt=\frac{1}{x}[\sin t]_0^x=\frac{\sin x}{x}$$

33. Using the half-angle identity $\cos\frac{x}{2}=\sqrt{\frac{1+\cos x}{2}}$, we see that since

$$\cos\frac{\pi}{4}=\cos\frac{\frac{\pi}{2}}{2}=\frac{\sqrt{2}}{2}$$

$$\cos\frac{\pi}{8}=\cos\frac{\frac{\pi}{2}}{4}=\sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}}=\sqrt{\frac{2+\sqrt{2}}{2}},$$

$$\cos\frac{\pi}{16}=\cos\frac{\frac{\pi}{2}}{8}=\sqrt{\frac{1+\frac{\sqrt{2+\sqrt{2}}}{2}}{2}}=\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2},\text{ etc.}$$

Thus, $$\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2+\sqrt{2}}}{2}\cdot\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots=\cos\left(\frac{\frac{\pi}{2}}{2}\right)\cos\left(\frac{\frac{\pi}{2}}{4}\right)\cos\left(\frac{\frac{\pi}{2}}{8}\right)\cdots$$

$$=\lim_{n\to\infty}\cos\left(\frac{\frac{\pi}{2}}{2}\right)\cos\left(\frac{\frac{\pi}{2}}{4}\right)\cdots\cos\left(\frac{\frac{\pi}{2}}{2^n}\right)=\frac{\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}}=\frac{2}{\pi}$$

34. Since $(k-\sin x)^2=(\sin x-k)^2$, the volume of S is $\int_0^\pi\pi(k-\sin x)^2=\pi\int_0^\pi(k^2-2k\sin x+\sin^2 x)dx$

$$=\pi k^2\int_0^\pi dx-2k\pi\int_0^\pi\sin x\,dx+\frac{\pi}{2}\int_0^\pi(1-\cos 2x)dx=\pi k^2[x]_0^\pi+2k\pi[\cos x]_0^\pi+\frac{\pi}{2}\left[x-\frac{1}{2}\sin 2x\right]_0^\pi$$

$$=\pi^2k^2+2k\pi(-1-1)+\frac{\pi}{2}(\pi-0)=\pi^2k^2-4k\pi+\frac{\pi^2}{2}$$

Let $f(k)=\pi^2k^2-4k\pi+\frac{\pi^2}{2}$, then $f'(k)=2\pi^2k-4\pi$ and $f'(k)=0$ when $k=\frac{2}{\pi}$.

The critical points of $f(k)$ on $0\le k\le 1$ are 0, $\frac{2}{\pi}$, 1.

$$f(0)=\frac{\pi^2}{2}\approx 4.93,\ f\left(\frac{2}{\pi}\right)=4-8+\frac{\pi^2}{2}\approx 0.93,\ f(1)=\pi^2-4\pi+\frac{\pi^2}{2}\approx 2.24$$

a. S has minimum volume when $k=\frac{2}{\pi}$.

b. S has maximum volume when $k=0$.

8.3 Concepts Review

1. $\sqrt{x-3}$

2. $2 \sin t$

3. $2 \tan t$

4. $2 \sec t$

Problem Set 8.3

1. $u = \sqrt{x+1}, u^2 = x+1, 2u\,du = dx$
$$\int x\sqrt{x+1}dx = \int (u^2-1)u(2u\,du)$$
$$= \int (2u^4 - 2u^2)du = \frac{2}{5}u^5 - \frac{2}{3}u^3 + C$$
$$= \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C$$

2. $u = \sqrt[3]{x+\pi}, u^3 = x+\pi, 3u^2du = dx$
$$\int x\sqrt[3]{x+\pi}dx = \int (u^3-\pi)u(3u^2du)$$
$$= \int (3u^6 - 3\pi u^3)du = \frac{3}{7}u^7 - \frac{3\pi}{4}u^4 + C$$
$$= \frac{3}{7}(x+\pi)^{7/3} - \frac{3\pi}{4}(x+\pi)^{4/3} + C$$

3. $u = \sqrt{3t+4}, u^2 = 3t+4,\ 2u\,du = 3\,dt$
$$\int \frac{t\,dt}{\sqrt{3t+4}} = \int \frac{\frac{1}{3}(u^2-4)\frac{2}{3}u\,du}{u} = \frac{2}{9}\int (u^2-4)du$$
$$= \frac{2}{27}u^3 - \frac{8}{9}u + C$$
$$= \frac{2}{27}(3t+4)^{3/2} - \frac{8}{9}(3t+4)^{1/2} + C$$

4. $u = \sqrt{x+4}, u^2 = x+4,\ 2u\,du = dx$
$$\int \frac{x^2+3x}{\sqrt{x+4}}dx = \int \frac{(u^2-4)^2 + 3(u^2-4)}{u}2u\,du$$
$$= 2\int (u^4 - 5u^2 + 4)du = \frac{2}{5}u^5 - \frac{10}{3}u^3 + 8u + C$$
$$= \frac{2}{5}(x+4)^{5/2} - \frac{10}{3}(x+4)^{3/2} + 8(x+4)^{1/2} + C$$

5. $u = \sqrt{t}, u^2 = t,\ 2u\,du = dt$
$$\int_1^2 \frac{dt}{\sqrt{t}+e} = \int_1^{\sqrt{2}} \frac{2u\,du}{u+e} = 2\int_1^{\sqrt{2}} \frac{u+e-e}{u+e}du$$
$$= 2\int_1^{\sqrt{2}} du - 2\int_1^{\sqrt{2}} \frac{e}{u+e}du$$
$$= 2[u]_1^{\sqrt{2}} - 2e\left[\ln|u+e|\right]_1^{\sqrt{2}}$$
$$= 2(\sqrt{2}-1) - 2e[\ln(\sqrt{2}+e) - \ln(1+e)]$$
$$= 2\sqrt{2} - 2 - 2e\ln\left(\frac{\sqrt{2}+e}{1+e}\right)$$

6. $u = \sqrt{t}, u^2 = t,\ 2u\,du = dt$
$$\int_0^1 \frac{\sqrt{t}}{t+1}dt = \int_0^1 \frac{u}{u^2+1}(2u\,du)$$
$$= 2\int_0^1 \frac{u^2}{u^2+1}du = 2\int_0^1 \frac{u^2+1-1}{u^2+1}du$$
$$= 2\int_0^1 du - 2\int_0^1 \frac{1}{u^2+1}du = 2[u]_0^1 - 2[\tan^{-1}u]_0^1$$
$$= 2 - 2\tan^{-1}1 = 2 - \frac{\pi}{2} \approx 0.4292$$

7. $u = (3t+2)^{1/2}, u^2 = 3t+2, 2u\,du = 3dt$
$$\int t(3t+2)^{3/2}dt = \int \frac{1}{3}(u^2-2)u^3\left(\frac{2}{3}u\,du\right)$$
$$= \frac{2}{9}\int (u^6 - 2u^4)du = \frac{2}{63}u^7 - \frac{4}{45}u^5 + C$$
$$= \frac{2}{63}(3t+2)^{7/2} - \frac{4}{45}(3t+2)^{5/2} + C$$

8. $u = (1-x)^{1/3}, u^3 = 1-x, 3u^2du = -dx$
$$\int x(1-x)^{2/3}dx = \int (1-u^3)u^2(-3u^2)du$$
$$= 3\int (u^7 - u^4)du = \frac{3}{8}u^8 - \frac{3}{5}u^5 + C$$
$$= \frac{3}{8}(1-x)^{8/3} - \frac{3}{5}(1-x)^{5/3} + C$$

9. $x = 2\sin t, dx = 2\cos t\,dt$
$$\int \frac{\sqrt{4-x^2}}{x}dx = \int \frac{2\cos t}{2\sin t}(2\cos t\,dt)$$
$$= 2\int \frac{1-\sin^2 t}{\sin t}dt = 2\int \csc t\,dt - 2\int \sin t\,dt$$
$$= 2\ln|\csc t - \cot t| + 2\cos t + C$$
$$= 2\ln\left|\frac{2-\sqrt{4-x^2}}{x}\right| + \sqrt{4-x^2} + C$$

10. $x = 4\sin t, dx = 4\cos t\,dt$
$$\int \frac{x^2dx}{\sqrt{16-x^2}} = 16\int \frac{\sin^2 t\cos t}{\cos t}dt$$
$$= 16\int \sin^2 t\,dt = 8\int (1-\cos 2t)dt$$
$$= 8t - 4\sin 2t + C = 8t - 8\sin t\cos t + C$$
$$= 8\sin^{-1}\left(\frac{x}{4}\right) - \frac{x\sqrt{16-x^2}}{2} + C$$

11. $x = 2\tan t, dx = 2\sec^2 t\, dt$

$$\int \frac{dx}{(x^2+4)^{3/2}} = \int \frac{2\sec^2 t\, dt}{(4\sec^2 t)^{3/2}} = \frac{1}{4}\int \cos t\, dt$$

$$= \frac{1}{4}\sin t + C = \frac{x}{4\sqrt{x^2+4}} + C$$

12. $t = \sec x, dt = \sec x \tan x\, dx$

Note that $0 \le x < \frac{\pi}{2}$.

$\sqrt{t^2-1} = |\tan x| = \tan x$

$$\int_2^3 \frac{dt}{t^2\sqrt{t^2-1}} = \int_{\pi/3}^{\sec^{-1}(3)} \frac{\sec x \tan x}{\sec^2 x \tan x} dx$$

$$= \int_{\pi/3}^{\sec^{-1}(3)} \cos x\, dx$$

$$= [\sin x]_{\pi/3}^{\sec^{-1}(3)} = \sin[\sec^{-1}(3)] - \sin\frac{\pi}{3}$$

$$= \sin\left[\cos^{-1}\left(\frac{1}{3}\right)\right] - \frac{\sqrt{3}}{2} = \frac{2\sqrt{2}}{3} - \frac{\sqrt{3}}{2} \approx 0.0768$$

13. $t = \sec x, dt = \sec x \tan x\, dx$

Note that $\frac{\pi}{2} < x \le \pi$.

$\sqrt{t^2-1} = |\tan x| = -\tan x$

$$\int_{-2}^{-3} \frac{\sqrt{t^2-1}}{t^3} dt = \int_{2\pi/3}^{\sec^{-1}(-3)} \frac{-\tan x}{\sec^3 x} \sec x \tan x\, dx$$

$$= \int_{2\pi/3}^{\sec^{-1}(-3)} -\sin^2 x\, dx = \int_{2\pi/3}^{\sec^{-1}(-3)} \left(\frac{1}{2}\cos 2x - \frac{1}{2}\right) dx$$

$$= \left[\frac{1}{4}\sin 2x - \frac{1}{2}x\right]_{2\pi/3}^{\sec^{-1}(-3)}$$

$$= \left[\frac{1}{2}\sin x \cos x - \frac{1}{2}x\right]_{2\pi/3}^{\sec^{-1}(-3)}$$

$$= -\frac{\sqrt{2}}{9} - \frac{1}{2}\sec^{-1}(-3) + \frac{\sqrt{3}}{8} + \frac{\pi}{3} \approx 0.151252$$

14. $t = \sin x, dt = \cos x\, dx$

$$\int \frac{t}{\sqrt{1-t^2}} dt = \int \sin x\, dx = -\cos x + C$$

$$= -\sqrt{1-t^2} + C$$

15. $z = \sin t, dz = \cos t\, dt$

$$\int \frac{2z-3}{\sqrt{1-z^2}} dz = \int (2\sin t - 3) dt$$

$$= -2\cos t - 3t + C$$

$$= -2\sqrt{1-z^2} - 3\sin^{-1} z + C$$

16. $x = \pi \tan t,\ dx = \pi \sec^2 t\, dt$

$$\int \frac{\pi x - 1}{\sqrt{x^2+\pi^2}} dx = \int (\pi^2 \tan t - 1)\sec t\, dt$$

$$= \pi^2 \int \tan t \sec t\, dt - \int \sec t\, dt$$

$$= \pi^2 \sec t - \ln|\sec t + \tan t| + C$$

$$= \pi\sqrt{x^2+\pi^2} - \ln\left|\frac{1}{\pi}\sqrt{x^2+\pi^2} + \frac{x}{\pi}\right| + C$$

$$\int_0^{\pi} \frac{\pi x - 1}{\sqrt{x^2+\pi^2}} dx$$

$$= \left[\pi\sqrt{x^2+\pi^2} - \ln\left|\frac{\sqrt{x^2+\pi^2}}{\pi} + \frac{x}{\pi}\right|\right]_0^{\pi}$$

$$= [\sqrt{2}\pi^2 - \ln(\sqrt{2}+1)] - [\pi^2 - \ln 1]$$

$$= (\sqrt{2}-1)\pi^2 - \ln(\sqrt{2}+1) \approx 3.207$$

17. $x^2 + 2x + 5 = x^2 + 2x + 1 + 4 = (x+1)^2 + 4$

$u = x + 1, du = dx$

$$\int \frac{dx}{\sqrt{x^2+2x+5}} = \int \frac{du}{\sqrt{u^2+4}}$$

$u = 2\tan t,\ du = 2\sec^2 t\, dt$

$$\int \frac{du}{\sqrt{u^2+4}} = \int \sec t\, dt = \ln|\sec t + \tan t| + C$$

$$\ln\left|\frac{\sqrt{u^2+4}}{2} + \frac{u}{2}\right| + C_1$$

$$= \ln\left|\frac{\sqrt{x^2+2x+5} + x + 1}{2}\right| + C_1$$

$$= \ln\left|\sqrt{x^2+2x+5} + x + 1\right| + C$$

18. $x^2 + 4x + 5 = x^2 + 4x + 4 + 1 = (x+2)^2 + 1$

$u = x + 2, du = dx$

$$\int \frac{dx}{\sqrt{x^2+4x+5}} = \int \frac{du}{\sqrt{u^2+1}}$$

$u = \tan t, du = \sec^2 t\, dt$

$$\int \frac{du}{\sqrt{u^2+1}} = \int \sec t\, dt = \ln|\sec t + \tan t| + C$$

$$\int \frac{dx}{\sqrt{x^2+4x+5}} = \ln\left|\sqrt{u^2+1}+u\right|+C$$
$$= \ln\left|\sqrt{x^2+4x+5}+x+2\right|+C$$

19. $x^2+2x+5 = x^2+2x+1+4 = (x+1)^2+4$

$u = x+1,\ du = dx$

$$\int \frac{3x}{\sqrt{x^2+2x+5}}dx = \int \frac{3u-3}{\sqrt{u^2+4}}du$$
$$= 3\int \frac{u}{\sqrt{u^2+4}}du - 3\int \frac{du}{\sqrt{u^2+4}}$$

(Use the result of Problem 17.)

$$= 3\sqrt{u^2+4} - 3\ln\left|\sqrt{u^2+4}+u\right|+C$$
$$= 3\sqrt{x^2+2x+5} - 3\ln\left|\sqrt{x^2+2x+5}+x+1\right|+C$$

20. $x^2+4x+5 = x^2+4x+4+1 = (x+2)^2+1$

$u = x+2,\ du = dx$

$$\int \frac{2x-1}{\sqrt{x^2+4x+5}}dx = \int \frac{2u-5}{\sqrt{u^2+1}}du$$
$$= \int \frac{2u\,du}{\sqrt{u^2+1}} - 5\int \frac{du}{\sqrt{u^2+1}}$$

(Use the result of Problem 18.)

$$= 2\sqrt{u^2+1} - 5\ln\left|\sqrt{u^2+1}+u\right|+C$$
$$= 2\sqrt{x^2+4x+5} - 5\ln\left|\sqrt{x^2+4x+5}+x+2\right|+C$$

21. $5-4x-x^2 = 9-(4+4x+4x^2) = 9-(x+2)^2$

$u = x+2,\ du = dx$

$$\int \sqrt{5-4x-x^2}\,dx = \int \sqrt{9-u^2}\,du$$

$u = 3\sin t,\ du = 3\cos t\,dt$

$$\int \sqrt{9-u^2}\,du = 9\int \cos^2 t\,dt = \frac{9}{2}\int (1+\cos 2t)dt$$
$$= \frac{9}{2}\left(t+\frac{1}{2}\sin 2t\right)+C = \frac{9}{2}(t+\sin t\cos t)+C$$
$$= \frac{9}{2}\sin^{-1}\left(\frac{u}{3}\right)+\frac{1}{2}u\sqrt{9-u^2}+C$$
$$= \frac{9}{2}\sin^{-1}\left(\frac{x+2}{3}\right)+\frac{x+2}{2}\sqrt{5-4x-x^2}+C$$

22. $16+6x-x^2 = 25-(9-6x+x^2) = 25-(x-3)^2$

$u = x-3,\ du = dx$

$$\int \frac{dx}{\sqrt{16+6x-x^2}} = \int \frac{du}{\sqrt{25-u^2}}$$

$u = 5\sin t,\ du = 5\cos t$

$$\int \frac{du}{\sqrt{25-u^2}} = \int dt = t+C = \sin^{-1}\left(\frac{u}{5}\right)+C$$
$$= \sin^{-1}\left(\frac{x-3}{5}\right)+C$$

23. $4x-x^2 = 4-(4-4x+x^2) = 4-(x-2)^2$

$u = x-2,\ du = dx$

$$\int \frac{dx}{\sqrt{4x-x^2}} = \int \frac{du}{\sqrt{4-u^2}}$$

$u = 2\sin t,\ du = 2\cos t\,dt$

$$\int \frac{du}{\sqrt{4-u^2}} = \int dt = t+C = \sin^{-1}\left(\frac{u}{2}\right)+C$$
$$= \sin^{-1}\left(\frac{x-2}{2}\right)+C$$

24. $4x-x^2 = 4-(4-4x+x^2) = 4-(x-2)^2$

$u = x-2,\ du = dx$

$$\int \frac{x}{\sqrt{4x-x^2}}dx = \int \frac{u+2}{\sqrt{4-u^2}}du$$
$$= -\int \frac{-u\,du}{\sqrt{4-u^2}} + 2\int \frac{du}{\sqrt{4-u^2}}$$

(Use the result of Problem 23.)

$$= -\sqrt{4-u^2}+2\sin^{-1}\left(\frac{u}{2}\right)+C$$
$$= -\sqrt{4x-x^2}+2\sin^{-1}\left(\frac{x-2}{2}\right)+C$$

25. $x^2+2x+2 = x^2+2x+1+1 = (x+1)^2+1$

$u = x+1,\ du = dx$

$$\int \frac{2x+1}{x^2+2x+2}dx = \int \frac{2u-1}{u^2+1}du$$
$$= \int \frac{2u}{u^2+1}du - \int \frac{du}{u^2+1}$$
$$= \ln\left|u^2+1\right| - \tan^{-1}u + C$$
$$= \ln\left|x^2+2x+2\right| - \tan^{-1}(x+1)+C$$

26. $x^2-6x+18 = x^2-6x+9+9 = (x-3)^2+9$

$u = x-3,\ du = dx$

$$\int \frac{2x-1}{x^2-6x+18}dx = \int \frac{2u+5}{u^2+9}du$$
$$= \int \frac{2u\,du}{u^2+9} + 5\int \frac{du}{u^2+9}$$
$$= \ln\left|u^2+9\right| + \frac{5}{3}\tan^{-1}\left(\frac{u}{3}\right)+C$$
$$= \ln\left|x^2-6x+18\right| + \frac{5}{3}\tan^{-1}\left(\frac{x-3}{3}\right)+C$$

27. $V = \pi \int_0^1 \left(\frac{1}{x^2+2x+5} \right)^2 dx$

$= \pi \int_0^1 \left[\frac{1}{(x+1)^2+4} \right]^2 dx$

$x + 1 = 2 \tan t,\ dx = 2\sec^2 t\, dt$

$V = \pi \int_{\tan^{-1}(1/2)}^{\pi/4} \left(\frac{1}{4\sec^2 t} \right)^2 2\sec^2 t\, dt$

$= \frac{\pi}{8} \int_{\tan^{-1}(1/2)}^{\pi/4} \frac{1}{\sec^2 t} dt = \frac{\pi}{8} \int_{\tan^{-1}(1/2)}^{\pi/4} \cos^2 t\, dt$

$= \frac{\pi}{8} \int_{\tan^{-1}(1/2)}^{\pi/4} \left(\frac{1}{2} + \frac{1}{2}\cos 2t \right) dt$

$= \frac{\pi}{8} \left[\frac{1}{2}t + \frac{1}{4}\sin 2t \right]_{\tan^{-1}(1/2)}^{\pi/4}$

$= \frac{\pi}{8} \left[\frac{1}{2}t + \frac{1}{2}\sin t \cos t \right]_{\tan^{-1}(1/2)}^{\pi/4}$

$= \frac{\pi}{8} \left[\left(\frac{\pi}{8} + \frac{1}{4} \right) - \left(\frac{1}{2}\tan^{-1}\frac{1}{2} + \frac{1}{5} \right) \right]$

$= \frac{\pi}{16} \left(\frac{1}{10} + \frac{\pi}{4} - \tan^{-1}\frac{1}{2} \right) \approx 0.082811$

28. $V = 2\pi \int_0^1 \frac{1}{x^2+2x+5} x\, dx$

$= 2\pi \int_0^1 \frac{x}{(x+1)^2+4} dx$

$= 2\pi \int_0^1 \frac{x+1}{(x+1)^2+4} dx - 2\pi \int_0^1 \frac{1}{(x+1)^2+4} dx$

$= 2\pi \left[\frac{1}{2}\ln[(x+1)^2+4] \right]_0^1 - 2\pi \left[\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2} \right) \right]_0^1$

$= \pi[\ln 8 - \ln 5] - \pi \left[\tan^{-1} 1 - \tan^{-1}\frac{1}{2} \right]$

$= \pi \left(\ln\frac{8}{5} - \frac{\pi}{4} + \tan^{-1}\frac{1}{2} \right) \approx 0.465751$

29. a. $u = x^2 + 9,\ du = 2x\, dx$

$\int \frac{x\, dx}{x^2+9} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C$

$= \frac{1}{2}\ln\left|x^2+9\right| + C$

b. $x = 3 \tan t,\ dx = 3\sec^2 t\, dt$

$\int \frac{x\, dx}{x^2+9} = \int \tan t\, dt = -\ln|\cos t| + C$

$= -\ln\left| \frac{3}{\sqrt{x^2+9}} \right| + C_1$

$= \ln\left|\sqrt{x^2+9}\right| - \ln|3| + C_1$

$= \ln\left|(x^2+9)^{1/2}\right| + C = \frac{1}{2}\ln\left|x^2+9\right| + C$

30. $u = \sqrt{9+x^2},\ u^2 = 9 + x^2,\ 2u\, du = 2x\, dx$

$\int_0^3 \frac{x^3 dx}{\sqrt{9+x^2}} = \int_0^3 \frac{x^2}{\sqrt{9+x^2}} x\, dx = \int_3^{3\sqrt{2}} \frac{u^2-9}{u} u\, du$

$= \int_3^{3\sqrt{2}} (u^2-9)\, du = \left[\frac{u^3}{3} - 9u \right]_3^{3\sqrt{2}} = 18 - 9\sqrt{2}$

≈ 5.272

31. a. $u = \sqrt{4-x^2},\ u^2 = 4 - x^2,\ 2u\, du = -2x\, dx$

$\int \frac{\sqrt{4-x^2}}{x} dx = \int \frac{\sqrt{4-x^2}}{x^2} x\, dx = -\int \frac{u^2 du}{4-u^2}$

$= \int \frac{-4+4-u^2}{4-u^2} du = -4\int \frac{1}{4-u^2} du + \int du$

$= -4 \cdot \frac{1}{4}\ln\left| \frac{u+2}{u-2} \right| + u + C$

$= -\ln\left| \frac{\sqrt{4-x^2}+2}{\sqrt{4-x^2}-2} \right| + \sqrt{4-x^2} + C$

b. $x = 2 \sin t,\ dx = 2 \cos t\, dt$

$\int \frac{\sqrt{4-x^2}}{x} dx = 2\int \frac{\cos^2 t}{\sin t} dt$

$= 2\int \frac{(1-\sin^2 t)}{\sin t} dt$

$= 2\int \csc t\, dt - 2\int \sin t\, dt$

$= 2\ln|\csc t - \cot t| + 2\cos t + C$

$= 2\ln\left| \frac{2}{x} - \frac{\sqrt{4-x^2}}{x} \right| + \sqrt{4-x^2} + C$

$= 2\ln\left| \frac{2-\sqrt{4-x^2}}{x} \right| + \sqrt{4-x^2} + C$

To reconcile the answers, note that

$-\ln\left| \frac{\sqrt{4-x^2}+2}{\sqrt{4-x^2}-2} \right| = \ln\left| \frac{\sqrt{4-x^2}-2}{\sqrt{4-x^2}+2} \right|$

$= \ln\left| \frac{(\sqrt{4-x^2}-2)^2}{(\sqrt{4-x^2}+2)(\sqrt{4-x^2}-2)} \right|$

$= \ln\left|\frac{(2-\sqrt{4-x^2})^2}{4-x^2-4}\right| = \ln\left|\frac{(2-\sqrt{4-x^2})^2}{-x^2}\right|$

$= \ln\left|\left(\frac{2-\sqrt{4-x^2}}{x}\right)^2\right| = 2\ln\left|\frac{2-\sqrt{4-x^2}}{x}\right|$

32. The equation of the circle with center $(-a, 0)$ is $(x+a)^2+y^2=b^2$, so $y=\pm\sqrt{b^2-(x+a)^2}$. By symmetry, the area of the overlap is four times the area of the region bounded by $x = 0$, $y = 0$, and $y=\sqrt{b^2-(x+a)^2}dx$.

$A=4\int_0^{b-a}\sqrt{b^2-(x+a)^2}dx$

$x + a = b \sin t$, $dx = b \cos t\, dt$

$A=4\int_{\sin^{-1}(a/b)}^{\pi/2} b^2\cos^2 t\, dt$

$=2b^2\int_{\sin^{-1}(a/b)}^{\pi/2}(1+\cos 2t)dt$

$=2b^2\left[t+\frac{1}{2}\sin 2t\right]_{\sin^{-1}(a/b)}^{\pi/2}$

$=2b^2[t+\sin t\cos t]_{\sin^{-1}(a/b)}^{\pi/2}$

$=2b^2\left[\frac{\pi}{2}-\left(\sin^{-1}\left(\frac{a}{b}\right)+\frac{a}{b}\frac{\sqrt{b^2-a^2}}{b}\right)\right]$

$=\pi b^2-2b^2\sin^{-1}\left(\frac{a}{b}\right)-2a\sqrt{b^2-a^2}$

33. a. The coordinate of C is $(0, -a)$. The lower arc of the lune lies on the circle given by the equation $x^2+(y+a)^2=2a^2$ or $y=\pm\sqrt{2a^2-x^2}-a$. The upper arc of the lune lies on the circle given by the equation $x^2+y^2=a^2$ or $y=\pm\sqrt{a^2-x^2}$.

$A=\int_{-a}^{a}\sqrt{a^2-x^2}dx-\int_{-a}^{a}\left(\sqrt{2a^2-x^2}-a\right)dx$

$=\int_{-a}^{a}\sqrt{a^2-x^2}dx-\int_{-a}^{a}\sqrt{2a^2-x^2}dx+2a^2$

Note that $\int_{-a}^{a}\sqrt{a^2-x^2}dx$ is the area of a semicircle with radius a, so

$\int_{-a}^{a}\sqrt{a^2-x^2}dx=\frac{\pi a^2}{2}$.

For $\int_{-a}^{a}\sqrt{2a^2-x^2}dx$, let

$x=\sqrt{2}a\sin t$, $dx=\sqrt{2}a\cos t\,dt$

$\int_{-a}^{a}\sqrt{2a^2-x^2}dx=\int_{-\pi/4}^{\pi/4}2a^2\cos^2 t\,dt$

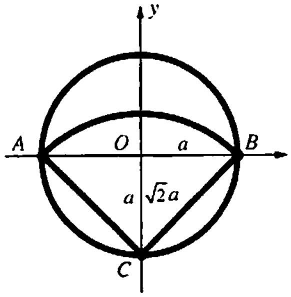
$=a^2\int_{-\pi/4}^{\pi/4}(1+\cos 2t)dt=a^2\left[t+\frac{1}{2}\sin 2t\right]_{-\pi/4}^{\pi/4}$

$=\frac{\pi a^2}{2}+a^2$

$A=\frac{\pi a^2}{2}-\left(\frac{\pi a^2}{2}+a^2\right)+2a^2=a^2$

Thus, the area of the lune is equal to the area of the square.

b. Without using calculus, consider the following labels on the figure.



Area of the lune = Area of the semicircle of radius a at O + Area (ΔABC) – Area of the sector ABC.

$A=\frac{1}{2}\pi a^2+a^2-\frac{1}{2}\left(\frac{\pi}{2}\right)(\sqrt{2}a)^2$

$=\frac{1}{2}\pi a^2+a^2-\frac{1}{2}\pi a^2=a^2$

Note that since BC has length $\sqrt{2}a$, the measure of angle OCB is $\frac{\pi}{4}$, so the measure of angle ACB is $\frac{\pi}{2}$.

34. Using reasoning similar to Problem 33 b, the area is

$\frac{1}{2}\pi a^2+\frac{1}{2}(2a)\sqrt{b^2-a^2}-\frac{1}{2}\left(2\sin^{-1}\frac{a}{b}\right)b^2$

$=\frac{1}{2}\pi a^2+a\sqrt{b^2-a^2}-b^2\sin^{-1}\frac{a}{b}$.

35. $\frac{dy}{dx}=-\frac{\sqrt{a^2-x^2}}{x}$

$y=\int-\frac{\sqrt{a^2-x^2}}{x}dx$

$x = a \sin t$, $dx = a \cos t\, dt$

$y=\int-\frac{a\cos t}{a\sin t}a\cos t\,dt=-a\int\frac{\cos^2 t}{\sin t}dt$

$=-a\int\frac{1-\sin^2 t}{\sin t}dt=a\int(\sin t-\csc t)dt$

$=a\left(-\cos t-\ln|\csc t-\cot t|\right)+C$

$\cos t = \frac{\sqrt{a^2 - x^2}}{a}, \csc t = \frac{a}{x}, \cot t = \frac{\sqrt{a^2 - x^2}}{x}$

$y = a\left(-\frac{\sqrt{a^2 - x^2}}{a} - \ln\left|\frac{a}{x} - \frac{\sqrt{a^2 - x^2}}{x}\right|\right) + C$

$= -\sqrt{a^2 - x^2} - a\ln\left|\frac{a - \sqrt{a^2 - x^2}}{x}\right| + C$

Since $y = 0$ when $x = a$,
$0 = 0 - a \ln 1 + C$, so $C = 0$.

$y = -\sqrt{a^2 - x^2} - a\ln\left|\frac{a - \sqrt{a^2 - x^2}}{x}\right|$

8.4 Concepts Review

1. $uv - \int v\,du$

2. x; $\sin x\,dx$

3. 1

4. reduction

Problem Set 8.4

1. $u = x \qquad dv = e^x dx$
   $du = dx \qquad v = e^x$
   $\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$

2. $u = x \qquad dv = e^{3x} dx$
   $du = dx \qquad v = \frac{1}{3}e^{3x}$
   $\int xe^{3x} dx = \frac{1}{3}xe^{3x} - \int \frac{1}{3}e^{3x} dx$
   $= \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$

3. $u = t \qquad dv = e^{5t+\pi} dt$
   $du = dt \qquad v = \frac{1}{5}e^{5t+\pi}$
   $\int te^{5t+\pi} dt = \frac{1}{5}te^{5t+\pi} - \int \frac{1}{5}e^{5t+\pi} dt$
   $= \frac{1}{5}te^{5t+\pi} - \frac{1}{25}e^{5t+\pi} + C$

4. $u = t + 7 \qquad dv = e^{2t+3} dt$
   $du = dt \qquad v = \frac{1}{2}e^{2t+3}$
   $\int (t+7)e^{2t+3} dt = \frac{1}{2}(t+7)e^{2t+3} - \int \frac{1}{2}e^{2t+3} dt$
   $= \frac{1}{2}(t+7)e^{2t+3} - \frac{1}{4}e^{2t+3} + C$
   $= \frac{t}{2}e^{2t+3} + \frac{13}{4}e^{2t+3} + C$

5. $u = x \qquad dv = \cos x\,dx$
   $du = dx \qquad v = \sin x$
   $\int x\cos x\,dx = x\sin x - \int \sin x\,dx$
   $= x \sin x + \cos x + C$

6. $u = x \qquad dv = \sin 2x\,dx$
   $du = dx \qquad v = -\frac{1}{2}\cos 2x$
   $\int x\sin 2x\,dx = -\frac{1}{2}x\cos 2x - \int -\frac{1}{2}\cos 2x\,dx$
   $= -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C$

7. $u = t - 3 \qquad dv = \cos (t - 3)dt$
   $du = dt \qquad v = \sin (t - 3)$
   $\int (t-3)\cos(t-3)dt = (t-3)\sin(t-3) - \int \sin(t-3)dt$
   $= (t - 3) \sin (t - 3) + \cos (t - 3) + C$

8. $u = x - \pi \qquad dv = \sin(x)dx$
   $du = dx \qquad v = -\cos x$
   $\int (x-\pi)\sin(x)dx = -(x-\pi)\cos x + \int \cos x\,dx$
   $= (\pi - x) \cos x + \sin x + C$

9. $u = t \qquad dv = \sqrt{t+1}\,dt$
   $du = dt \qquad v = \frac{2}{3}(t+1)^{3/2}$
   $\int t\sqrt{t+1}\,dt = \frac{2}{3}t(t+1)^{3/2} - \int \frac{2}{3}(t+1)^{3/2} dt$
   $= \frac{2}{3}t(t+1)^{3/2} - \frac{4}{15}(t+1)^{5/2} + C$

10. $u = t \qquad dv = \sqrt[3]{2t+7}dt$
    $du = dt \qquad v = \frac{3}{8}(2t+7)^{4/3}$
    $\int t\sqrt[3]{2t+7}dt = \frac{3}{8}t(2t+7)^{4/3} - \int \frac{3}{8}(2t+7)^{4/3} dt$
    $= \frac{3}{8}t(2t+7)^{4/3} - \frac{9}{112}(2t+7)^{7/3} + C$

11. $u = \ln 3x \quad dv = dx$

$du = \frac{1}{x}dx \quad v = x$

$\int \ln 3x\,dx = x\ln 3x - \int x\frac{1}{x}dx = x\ln 3x - x + C$

12. $u = \ln(7x^5) \quad dv = dx$

$du = \frac{5}{x}dx \quad v = x$

$\int \ln(7x^5)dx = x\ln(7x^5) - \int x\frac{5}{x}dx$

$= x\ln(7x^5) - 5x + C$

13. $u = \arctan x \quad dv = dx$

$du = \frac{1}{1+x^2}dx \quad v = x$

$\int \arctan x = x\arctan x - \int \frac{x}{1+x^2}dx$

$= x\arctan x - \frac{1}{2}\int \frac{2x}{1+x^2}dx$

$= x\arctan x - \frac{1}{2}\ln(1+x^2) + C$

14. $u = \arctan 5x \quad dv = dx$

$du = \frac{5}{1+25x^2}dx \quad v = x$

$\int \arctan 5x\,dx = x\arctan 5x - \int \frac{5x}{1+25x^2}dx$

$= x\arctan 5x - \frac{1}{10}\int \frac{50x\,dx}{1+25x^2}$

$= x\arctan 5x - \frac{1}{10}\ln(1+25x^2) + C$

15. $u = \ln x \quad dv = \frac{dx}{x^2}$

$du = \frac{1}{x}dx \quad v = -\frac{1}{x}$

$\int \frac{\ln x}{x^2}dx = -\frac{\ln x}{x} - \int -\frac{1}{x}\left(\frac{1}{x}\right)dx$

$= -\frac{\ln x}{x} - \frac{1}{x} + C$

16. $u = \ln 2x^5 \quad dv = \frac{1}{x^2}dx$

$du = \frac{5}{x}dx \quad v = -\frac{1}{x}$

$\int_2^3 \frac{\ln 2x^5}{x^2}dx = -\frac{1}{x}\ln 2x^5 + 5\int_2^3 \frac{1}{x^2}dx$

$= \left[-\frac{1}{x}\ln 2x^5 - \frac{5}{x}\right]_2^3$

$= \left(-\frac{1}{3}\ln(2\cdot 3^5) - \frac{5}{3}\right) - \left(-\frac{1}{2}\ln(2\cdot 2^5) - \frac{5}{2}\right)$

$= \frac{8}{3}\ln 2 - \frac{5}{3}\ln 3 + \frac{5}{6}\ln 0.85 \approx -0.11806$

17. $u = \ln t \quad dv = \sqrt{t}\,dt$

$du = \frac{1}{t}dt \quad v = \frac{2}{3}t^{3/2}$

$\int_1^e \sqrt{t}\ln t\,dt = \left[\frac{2}{3}t^{3/2}\ln t\right]_1^e - \int_1^e \frac{2}{3}t^{1/2}dt$

$= \frac{2}{3}e^{3/2}\ln e - \frac{2}{3}\cdot 1\ln 1 - \left[\frac{4}{9}t^{3/2}\right]_1^e$

$= \frac{2}{3}e^{3/2} - 0 - \frac{4}{9}e^{3/2} + \frac{4}{9} = \frac{2}{9}e^{3/2} + \frac{4}{9}$

18. $u = \ln x^3 \quad dv = \sqrt{2x}dx$

$du = \frac{3}{x}dx \quad v = \frac{1}{3}(2x)^{3/2}$

$\int_5^1 \sqrt{2x}\ln x^3\,dx = \left[\frac{1}{3}(2x)^{3/2}\ln x^3\right]_5^1 - \int_5^1 2^{3/2}\sqrt{x}\,dx$

$= \left[\frac{1}{3}(2x)^{3/2}\ln x^3 - \frac{2^{5/2}}{3}x^{3/2}\right]_5^1$

$= \frac{1}{3}(2)^{3/2}\ln 1 - \frac{2^{5/2}}{3} - \left(\frac{1}{3}(10)^{3/2}\ln 5^3 - \frac{2^{5/2}}{3}5^{3/2}\right)$

$= \frac{4\sqrt{2}}{3}5^{3/2} - \frac{4\sqrt{2}}{3} - 10^{3/2}\ln 5 \approx -31.699$

19. $u = \ln z \quad dv = z^3dz$

$du = \frac{1}{z}dz \quad v = \frac{1}{4}z^4$

$\int z^3\ln z\,dz = \frac{1}{4}z^4\ln z - \int \frac{1}{4}z^4\cdot\frac{1}{z}dz$

$= \frac{1}{4}z^4\ln z - \frac{1}{4}\int z^3dz$

$= \frac{1}{4}z^4\ln z - \frac{1}{16}z^4 + C$

20. $u = \arctan t \quad dv = t\,dt$

$du = \frac{1}{1+t^2}dt \quad v = \frac{1}{2}t^2$

$\int t\arctan t\,dt = \frac{1}{2}t^2\arctan t - \frac{1}{2}\int \frac{t^2}{1+t^2}dt$

$= \frac{1}{2}t^2\arctan t - \frac{1}{2}\int \frac{1+t^2-1}{1+t^2}dt$

$$= \frac{1}{2}t^2 \arctan t - \frac{1}{2}\int dt + \frac{1}{2}\int \frac{1}{1+t^2}dt$$

$$= \frac{1}{2}t^2 \arctan t - \frac{1}{2}t + \frac{1}{2}\arctan t + C$$

21. $u = \arctan\left(\frac{1}{t}\right)$ $\qquad dv = dt$

$du = -\frac{1}{1+t^2}dt \qquad v = t$

$$\int \arctan\left(\frac{1}{t}\right)dt = t\arctan\left(\frac{1}{t}\right) + \int \frac{t}{1+t^2}dt$$

$$= t\arctan\left(\frac{1}{t}\right) + \frac{1}{2}\ln(1+t^2) + C$$

22. $u = \ln(t^7) \qquad dv = t^5 dt$

$du = \frac{7}{t}dt \qquad v = \frac{1}{6}t^6$

$$\int t^5 \ln(t^7)dt = \frac{1}{6}t^6 \ln(t^7) - \frac{7}{6}\int t^5 dt$$

$$= \frac{1}{6}t^6 \ln(t^7) - \frac{7}{36}t^6 + C$$

23. $u = x\cos^2 x \qquad dv = \sin x\, dx$

$du = (\cos^2 x - 2x\cos x \sin x)dx \qquad v = -\cos x$

$$\int x\cos^2 x \sin x\, dx = -x\cos^3 x + \int(\cos^3 x - 2x\cos^2 x \sin x)dx = -x\cos^3 x + \int \cos^3 x\, dx - 2\int x\cos^2 x \sin x\, dx$$

$$3\int x\cos^2 x \sin x\, dx = -x\cos^3 x + \int \cos x(1-\sin^2 x)dx = -x\cos^3 x + \sin x - \frac{1}{3}\sin^3 x + C$$

$$\int x\cos^2 x \sin x\, dx = -\frac{x}{3}\cos^3 x + \frac{1}{3}\sin x - \frac{1}{9}\sin^3 x + C$$

24. $u = x\sin^3 \pi x \qquad v = \frac{1}{\pi}\sin \pi x \qquad dv = \cos \pi x\, dx$

$du = (3\pi x \sin^2 \pi x \cos \pi x + \sin^3 \pi x)dx$

$$\int x\sin^3 \pi x \cos \pi x\, dx = \frac{x}{\pi}\sin^4 \pi x - \frac{1}{\pi}\int(3\pi x \sin^3 \pi x \cos \pi x + \sin^4 \pi x)dx$$

$$= \frac{x}{\pi}\sin^4 \pi x - 3\int x\sin^3 \pi x \cos \pi x\, dx - \frac{1}{\pi}\int \sin^4 \pi x\, dx$$

$$4\int x\sin^3 \pi x \cos \pi x\, dx = \frac{x}{\pi}\sin^4 \pi x - \frac{1}{\pi}\int\left(\frac{1-\cos 2\pi x}{2}\right)^2 dx = \frac{x}{\pi}\sin^4 \pi x - \frac{1}{4\pi}\int(1 - 2\cos 2\pi x + \cos^2 2\pi x)dx$$

$$= \frac{x}{\pi}\sin^4 \pi x - \frac{x}{4\pi} + \frac{1}{4\pi^2}\sin 2\pi x - \frac{1}{4\pi}\int\left(\frac{1+\cos 4\pi x}{2}\right)dx = \frac{x}{\pi}\sin^4 \pi x - \frac{3x}{8\pi} + \frac{1}{4\pi^2}\sin 2\pi x - \frac{1}{32\pi^2}\sin 4\pi x + C$$

$$\int x\sin^3 \pi x \cos \pi x\, dx = \frac{x}{4\pi}\sin^4 \pi x - \frac{3x}{32\pi} + \frac{1}{16\pi^2}\sin 2\pi x - \frac{1}{128\pi^2}\sin 4\pi x + C$$

25. $u = x \qquad dv = \csc^2 x\, dx$

$du = dx \qquad v = -\cot x$

$$\int_{\pi/6}^{\pi/2} x\csc^2 x\, dx = \left[-x\cot x\right]_{\pi/6}^{\pi/2} + \int_{\pi/6}^{\pi/2} \cot x\, dx = \left[-x\cot x + \ln|\sin x|\right]_{\pi/6}^{\pi/2}$$

$$= -\frac{\pi}{2}\cdot 0 + \ln 1 + \frac{\pi}{6}\sqrt{3} - \ln\frac{1}{2} = \frac{\pi}{2\sqrt{3}} + \ln 2 \approx 1.60$$

26. $\int_{\pi/4}^{\pi/2} \csc^2 x\, dx = [-\cot x]_{\pi/4}^{\pi/2} = -\cot\frac{\pi}{2} + \cot\frac{\pi}{4} = 0 + 1 = 1$

27. $u = x \qquad dv = \sec^2 x\, dx$

$du = dx \qquad v = \tan x$

$$\int_{\pi/6}^{\pi/4} x\sec^2 x\,dx = \left[x\tan x\right]_{\pi/6}^{\pi/4} - \int_{\pi/6}^{\pi/4}\tan x\,dx = \left[x\tan x + \ln|\cos x|\right]_{\pi/6}^{\pi/4} = \frac{\pi}{4} + \ln\frac{\sqrt{2}}{2} - \left(\frac{\pi}{6\sqrt{3}} + \ln\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\pi}{4} + \ln\frac{\sqrt{2}}{2} - \frac{\pi}{6\sqrt{3}} - \ln\frac{\sqrt{3}}{2} = \frac{\pi}{4} - \frac{\pi}{6\sqrt{3}} + \frac{1}{2}\ln\frac{2}{3} \approx 0.28$$

28. $u = \sec^{-1}\sqrt{x}$ $\quad dv = dx$

$du = \frac{1}{2x\sqrt{x-1}}dx \quad v = x$

$$\int \sec^{-1}\sqrt{x}dx = x\sec^{-1}\sqrt{x} - \frac{1}{2}\int\frac{dx}{\sqrt{x-1}} = x\sec^{-1}\sqrt{x} - \sqrt{x-1} + C$$

29. $u = x^3$ $\quad dv = x^2\sqrt{x^3+4}dx$

$du = 3x^2dx \quad v = \frac{2}{9}(x^3+4)^{3/2}$

$$\int x^5\sqrt{x^3+4}dx = \frac{2}{9}x^3(x^3+4)^{3/2} - \int\frac{2}{3}x^2(x^3+4)^{3/2}dx = \frac{2}{9}x^3(x^3+4)^{3/2} - \frac{4}{45}(x^3+4)^{5/2} + C$$

30. $u = x^7$ $\quad dv = x^6\sqrt{x^7+1}\,dx$

$du = 7x^6dx \quad v = \frac{2}{21}(x^7+1)^{3/2}$

$$\int x^{13}\sqrt{x^7+1}dx = \frac{2}{21}x^7(x^7+1)^{3/2} - \int\frac{2}{3}x^6(x^7+1)^{3/2}dx = \frac{2}{21}x^7(x^7+1)^{3/2} - \frac{4}{105}(x^7+1)^{5/2} + C$$

31. $u = t^4$ $\quad dv = \frac{t^3}{(7-3t^4)^{3/2}}dt$

$du = 4t^3\,dt \quad v = \frac{1}{6(7-3t^4)^{1/2}}$

$$\int\frac{t^7}{(7-3t^4)^{3/2}}dt = \frac{t^4}{6(7-3t^4)^{1/2}} - \frac{2}{3}\int\frac{t^3}{(7-3t^4)^{1/2}}dt = \frac{t^4}{6(7-3t^4)^{1/2}} + \frac{1}{9}(7-3t^4)^{1/2} + C$$

32. $u = x^2$ $\quad dv = x\sqrt{4-x^2}dx$

$du = 2x\,dx \quad v = -\frac{1}{3}(4-x^2)^{3/2}$

$$\int x^3\sqrt{4-x^2}dx = -\frac{1}{3}x^2(4-x^2)^{3/2} + \frac{2}{3}\int x(4-x^2)^{3/2}dx = -\frac{1}{3}x^2(4-x^2)^{3/2} - \frac{2}{15}(4-x^2)^{5/2} + C$$

33. $u = z^4$ $\quad dv = \frac{z^3}{(4-z^4)^2}dz$

$du = 4z^3dz \quad v = \frac{1}{4(4-z^4)}$

$$\int\frac{z^7}{(4-z^4)^2}dz = \frac{z^4}{4(4-z^4)} - \int\frac{z^3}{4-z^4}dz = \frac{z^4}{4(4-z^4)} + \frac{1}{4}\ln\left|4-z^4\right| + C$$

34. $u = x$ $\quad dv = \cosh x\,dx$

$du = dx \quad v = \sinh x$

$$\int x\cosh x\,dx = x\sinh x - \int\sinh x\,dx = x\sinh x - \cosh x + C$$

35. $u = x \quad dv = \sinh x\, dx$

$du = dx \quad v = \cosh x$

$\int x \sinh x\, dx = x\cosh x - \int \cosh x\, dx = x \cosh x - \sinh x + C$

36. $u = \sec x \quad dv = \sec^2 x\, dx$

$du = \sec x \tan x\, dx \quad v = \tan x$

$\int \sec^3 x\, dx = \sec x \tan x - \int \sec x \tan^2 x\, dx = \sec x \tan x - \int \sec x(\sec^2 x - 1)dx = \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$

$2\int \sec^3 x\, dx = \sec x \tan x + \int \sec x\, dx = \sec x \tan x + \ln|\sec x + \tan x| + C$

$\int \sec^3 x\, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$

37. $u = \ln x \quad dv = x^{-1/2}dx$

$du = \frac{1}{x}dx \quad v = 2x^{1/2}$

$\int \frac{\ln x}{\sqrt{x}}dx = 2\sqrt{x}\ln x - 2\int \frac{1}{x^{1/2}}dx = 2\sqrt{x}\ln x - 4\sqrt{x} + C$

38. $u = x \quad dv = (3x+10)^{49}dx$

$du = dx \quad v = \frac{1}{150}(3x+10)^{50}$

$\int x(3x+10)^{49}dx = \frac{x}{150}(3x+10)^{50} - \frac{1}{150}\int (3x+10)^{50}dx = \frac{x}{150}(3x+10)^{50} - \frac{1}{22,950}(3x+10)^{51} + C$

39. $u = x \quad dv = 2^x dx$

$du = dx \quad v = \frac{1}{\ln 2}2^x$

$\int x2^x dx = \frac{x}{\ln 2}2^x - \frac{1}{\ln 2}\int 2^x dx$

$= \frac{x}{\ln 2}2^x - \frac{1}{(\ln 2)^2}2^x + C$

40. $u = z \quad dv = a^z dz$

$du = dz \quad v = \frac{1}{\ln a}a^z$

$\int za^z dz = \frac{z}{\ln a}a^z - \frac{1}{\ln a}\int a^z dz$

$= \frac{z}{\ln a}a^z - \frac{1}{(\ln a)^2}a^z + C$

41. $u = x^2 \quad dv = e^x dx$

$du = 2x\, dx \quad v = e^x$

$\int x^2 e^x dx = x^2 e^x - \int 2xe^x dx$

$u = x \quad dv = e^x dx$

$du = dx \quad v = e^x$

$\int x^2 e^x\, dx = x^2 e^x - 2\left(xe^x - \int e^x\, dx\right)$

$= x^2 e^x - 2xe^x + 2e^x + C$

42. $u = x^4 \quad dv = xe^{x^2}dx$

$du = 4x^3 dx \quad v = \frac{1}{2}e^{x^2}$

$\int x^5 e^{x^2}dx = \frac{1}{2}x^4 e^{x^2} - \int 2x^3 e^{x^2}dx$

$u = x^2 \quad dv = 2xe^{x^2}dx$

$du = 2x\, dx \quad v = e^{x^2}$

$\int x^5 e^{x^2}dx = \frac{1}{2}x^4 e^{x^2} - \left(x^2 e^{x^2} - \int 2xe^{x^2}dx\right)$

$= \frac{1}{2}x^4 e^{x^2} - x^2 e^{x^2} + e^{x^2} + C$

43. $u = \ln^2 z \quad dv = dz$

$du = \frac{2\ln z}{z}dz \quad v = z$

$\int \ln^2 z\, dz = z\ln^2 z - 2\int \ln z\, dz$

$u = \ln z \quad dv = dz$

$du = \frac{1}{z}dz \quad v = z$

$$\int \ln^2 z\,dz = z\ln^2 z - 2\left(z\ln z - \int dz\right)$$
$$= z\ln^2 z - 2z\ln z + 2z + C$$

44. $u = \ln^2 x^{20}$ $dv = dx$

$du = \frac{40\ln x^{20}}{x}dx$ $v = x$

$$\int \ln^2 x^{20}dx = x\ln^2 x^{20} - 40\int \ln x^{20}dx$$

$u = \ln x^{20}$ $dv = dx$

$du = \frac{20}{x}dx$ $v = x$

$$\int \ln^2 x^{20}dx = x\ln^2 x^{20} - 40\left(x\ln x^{20} - 20\int dx\right)$$
$$= x\ln^2 x^{20} - 40x\ln x^{20} + 800x + C$$

45. $u = e^t$ $dv = \cos t\,dt$

$du = e^t dt$ $v = \sin t$

$$\int e^t \cos t\,dt = e^t \sin t - \int e^t \sin t\,dt$$

$u = e^t$ $dv = \sin t\,dt$

$du = e^t dt$ $v = -\cos t$

$$\int e^t \cos t\,dt = e^t \sin t - \left[-e^t \cos t + \int e^t \cos t\,dt\right]$$
$$\int e^t \cos t\,dt = e^t \sin t + e^t \cos t - \int e^t \cos t\,dt$$
$$2\int e^t \cos t\,dt = e^t \sin t + e^t \cos t + C$$
$$\int e^t \cos t\,dt = \frac{1}{2}e^t(\sin t + \cos t) + C$$

46. $u = e^{at}$ $dv = \sin t\,dt$

$du = ae^{at}dt$ $v = -\cos t$

$$\int e^{at}\sin t\,dt = -e^{at}\cos t + a\int e^{at}\cos t\,dt$$

$u = e^{at}$ $dv = \cos t\,dt$

$du = ae^{at}dt$ $v = \sin t$

$$\int e^{at}\sin t\,dt = -e^{at}\cos t + a\left(e^{at}\sin t - a\int e^{at}\sin t\,dt\right)$$
$$\int e^{at}\sin t\,dt = -e^{at}\cos t + ae^{at}\sin t - a^2\int e^{at}\sin t\,dt$$
$$(1+a^2)\int e^{at}\sin t\,dt = -e^{at}\cos t + ae^{at}\sin t + C$$
$$\int e^{at}\sin t\,dt = \frac{-e^{at}\cos t}{a^2+1} + \frac{ae^{at}\sin t}{a^2+1} + C$$

47. $u = x^2$ $dv = \cos x\,dx$

$du = 2x\,dx$ $v = \sin x$

$$\int x^2 \cos x\,dx = x^2 \sin x - \int 2x\sin x\,dx$$

$u = 2x$ $dv = \sin x\,dx$

$du = 2dx$ $v = -\cos x$

$$\int x^2 \cos x\,dx = x^2 \sin x - \left(-2x\cos x + \int 2\cos x\,dx\right)$$
$$= x^2 \sin x + 2x\cos x - 2\sin x + C$$

48. $u = r^2$ $dv = \sin r\,dr$

$du = 2r\,dr$ $v = -\cos r$

$$\int r^2 \sin r\,dr = -r^2\cos r + 2\int r\cos r\,dr$$

$u = r$ $dv = \cos r\,dr$

$du = dr$ $v = \sin r$

$$\int r^2 \sin r\,dr = -r^2\cos r + 2\left(r\sin r - \int \sin r\,dr\right) = -r^2\cos r + 2r\sin r + 2\cos r + C$$

49. $u = \sin(\ln x)$ $dv = dx$

$du = \cos(\ln x)\cdot\frac{1}{x}dx$ $v = x$

$$\int \sin(\ln x)dx = x\sin(\ln x) - \int \cos(\ln x)\,dx$$

$u = \cos(\ln x)$ $dv = dx$

$du = -\sin(\ln x)\cdot\frac{1}{x}dx$ $v = x$

$$\int \sin(\ln x)dx = x\sin(\ln x) - \left[x\cos(\ln x) - \int -\sin(\ln x)dx\right]$$
$$\int \sin(\ln x)dx = x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)dx$$

$2\int \sin(\ln x)dx = x\sin(\ln x) - x\cos(\ln x) + C$

$\int \sin(\ln x)dx = \frac{x}{2}[\sin(\ln x) - \cos(\ln x)] + C$

50. $u = \cos(\ln x) \qquad dv = dx$

$du = -\sin(\ln x)\frac{1}{x}dx \qquad v = x$

$\int \cos(\ln x)dx = x\cos(\ln x) + \int \sin(\ln x)dx$

$u = \sin(\ln x) \qquad dv = dx$

$du = \cos(\ln x)\frac{1}{x}dx \qquad v = x$

$\int \cos(\ln x)dx = x\cos(\ln x) + \left[x\sin(\ln x) - \int \cos(\ln x)dx\right]$

$2\int \cos(\ln x)dx = x[\cos(\ln x) + \sin(\ln x)] + C$

$\int \cos(\ln x)dx = \frac{x}{2}[\cos(\ln x) + \sin(\ln x)] + C$

51. $u = (\ln x)^3 \qquad dv = dx$

$du = \frac{3\ln^2 x}{x}dx \qquad v = x$

$\int (\ln x)^3 dx = x(\ln x)^3 - 3\int \ln^2 x\,dx$

$= x\ln^3 x - 3(x\ln^2 x - 2x\ln x + 2x + C)$

$= x\ln^3 x - 3x\ln^2 x + 6x\ln x - 6x + C$

52. $u = (\ln x)^4 \qquad dv = dx$

$du = \frac{4\ln^3 x}{x}dx \qquad v = x$

$\int (\ln x)^4 dx = x(\ln x)^4 - 4\int \ln^3 x\,dx = x\ln^4 x - 4(x\ln^3 x - 3x\ln^2 x + 6x\ln x - 6x + C)$

$= x\ln^4 x - 4x\ln^3 x + 12x\ln^2 x - 24x\ln x + 24x + C$

53. $u = \sin x \qquad dv = \sin(3x)dx$

$du = \cos x\,dx \qquad v = -\frac{1}{3}\cos(3x)$

$\int \sin x\sin(3x)dx = -\frac{1}{3}\sin x\cos(3x) + \frac{1}{3}\int \cos x\cos(3x)dx$

$u = \cos x \qquad dv = \cos(3x)dx$

$du = -\sin x\,dx \qquad v = \frac{1}{3}\sin(3x)$

$\int \sin x\sin(3x)dx = -\frac{1}{3}\sin x\cos(3x) + \frac{1}{3}\left[\frac{1}{3}\cos x\sin(3x) + \frac{1}{3}\int \sin x\sin(3x)dx\right]$

$= -\frac{1}{3}\sin x\cos(3x) + \frac{1}{9}\cos x\sin(3x) \div \frac{1}{9}\int \sin x\sin(3x)dx$

$\frac{8}{9}\int \sin x\sin(3x)dx = -\frac{1}{3}\sin x\cos(3x) + \frac{1}{9}\cos x\sin(3x) + C$

$\int \sin x\sin(3x)dx = -\frac{3}{8}\sin x\cos(3x) + \frac{1}{8}\cos x\sin(3x) + C$

54. $u = \cos(5x) \qquad dv = \sin(7x)dx$

$du = -5\sin(5x)dx \qquad v = -\frac{1}{7}\cos(7x)$

$$\int \cos(5x)\sin(7x)dx = -\frac{1}{7}\cos(5x)\cos(7x) - \frac{5}{7}\int \sin(5x)\cos(7x)dx$$

$u = \sin(5x) \qquad dv = \cos(7x)dx$

$du = 5\cos(5x)dx \qquad v = \frac{1}{7}\sin(7x)$

$$\int \cos(5x)\sin(7x)dx = -\frac{1}{7}\cos(5x)\cos(7x) - \frac{5}{7}\left[\frac{1}{7}\sin(5x)\sin(7x) - \frac{5}{7}\int \cos(5x)\sin(7x)dx\right]$$

$$= -\frac{1}{7}\cos(5x)\cos(7x) - \frac{5}{49}\sin(5x)\sin(7x) + \frac{25}{49}\int \cos(5x)\sin(7x)dx$$

$$\frac{24}{49}\int \cos(5x)\sin(7x)dx = -\frac{1}{7}\cos(5x)\cos(7x) - \frac{5}{49}\sin(5x)\sin(7x) + C$$

$$\int \cos(5x)\sin(7x)dx = -\frac{7}{24}\cos(5x)\cos(7x) - \frac{5}{24}\sin(5x)\sin(7x) + C$$

55. $u = e^{\alpha z} \qquad dv = \sin\beta z\, dz$

$du = \alpha e^{\alpha z}dz \qquad v = -\frac{1}{\beta}\cos\beta z$

$$\int e^{\alpha z}\sin\beta z\, dz = -\frac{1}{\beta}e^{\alpha z}\cos\beta z + \frac{\alpha}{\beta}\int e^{\alpha z}\cos\beta z\, dz$$

$u = e^{\alpha z} \qquad dv = \cos\beta z\, dz$

$du = \alpha e^{\alpha z}dz \qquad v = \frac{1}{\beta}\sin\beta z$

$$\int e^{\alpha z}\sin\beta z\, dz = -\frac{1}{\beta}e^{\alpha z}\cos\beta z + \frac{\alpha}{\beta}\left[\frac{1}{\beta}e^{\alpha z}\sin\beta z - \frac{\alpha}{\beta}\int e^{\alpha z}\sin\beta z\, dz\right]$$

$$= -\frac{1}{\beta}e^{\alpha z}\cos\beta z + \frac{\alpha}{\beta^2}e^{\alpha z}\sin\beta z - \frac{\alpha^2}{\beta^2}\int e^{\alpha z}\sin\beta z\, dz$$

$$\frac{\beta^2+\alpha^2}{\beta^2}\int e^{\alpha z}\sin\beta z\, dz = -\frac{1}{\beta}e^{\alpha z}\cos\beta z + \frac{\alpha}{\beta^2}e^{\alpha z}\sin\beta z + C$$

$$\int e^{\alpha z}\sin\beta z\, dz = \frac{-\beta}{\alpha^2+\beta^2}e^{\alpha z}\cos\beta z + \frac{\alpha}{\alpha^2+\beta^2}e^{\alpha z}\sin\beta z + C = \frac{e^{\alpha z}(\alpha\sin\beta z - \beta\cos\beta z)}{\alpha^2+\beta^2} + C$$

56. $u = e^{\alpha z} \qquad dv = \cos\beta z\, dz$

$du = \alpha e^{\alpha z}dz \qquad v = \frac{1}{\beta}\sin\beta z$

$$\int e^{\alpha z}\cos\beta z\, dz = \frac{1}{\beta}e^{\alpha z}\sin\beta z - \frac{\alpha}{\beta}\int e^{\alpha z}\sin\beta z\, dz$$

$u = e^{\alpha z} \qquad dv = \sin\beta z\, dz$

$du = \alpha e^{\alpha z}dz \qquad v = -\frac{1}{\beta}\cos\beta z$

$$\int e^{\alpha z}\cos\beta z\, dz = \frac{1}{\beta}e^{\alpha z}\sin\beta z - \frac{\alpha}{\beta}\left[-\frac{1}{\beta}e^{\alpha z}\cos\beta z + \frac{\alpha}{\beta}\int e^{\alpha z}\cos\beta z\, dz\right]$$

$$= \frac{1}{\beta}e^{\alpha z}\sin\beta z + \frac{\alpha}{\beta^2}e^{\alpha z}\cos\beta z - \frac{\alpha^2}{\beta^2}\int e^{\alpha z}\cos\beta z\, dz$$

$$\frac{\alpha^2+\beta^2}{\beta^2}\int e^{\alpha z}\cos\beta z\,dz=\frac{\alpha}{\beta^2}e^{\alpha z}\cos\beta z+\frac{1}{\beta}e^{\alpha z}\sin\beta z+C$$

$$\int e^{\alpha z}\cos\beta z\,dz=\frac{e^{\alpha z}(\alpha\cos\beta z+\beta\sin\beta z)}{\alpha^2+\beta^2}+C$$

57. $u=\ln x \qquad dv=x^\alpha dx$

$du=\frac{1}{x}dx \qquad v=\frac{x^{\alpha+1}}{\alpha+1},\ \alpha\neq-1$

$$\int x^\alpha \ln x\,dx=\frac{x^{\alpha+1}}{\alpha+1}\ln x-\frac{1}{\alpha+1}\int x^\alpha dx=\frac{x^{\alpha+1}}{\alpha+1}\ln x-\frac{x^{\alpha+1}}{(\alpha+1)^2}+C,\alpha\neq-1$$

58. $u=(\ln x)^2 \qquad dv=x^\alpha dx$

$du=\frac{2\ln x}{x}dx \qquad v=\frac{x^{\alpha+1}}{\alpha+1},\ \alpha\neq-1$

$$\int x^\alpha(\ln x)^2dx=\frac{x^{\alpha+1}}{\alpha+1}(\ln x)^2-\frac{2}{\alpha+1}\int x^\alpha\ln x\,dx=\frac{x^{\alpha+1}}{\alpha+1}(\ln x)^2-\frac{2}{\alpha+1}\left[\frac{x^{\alpha+1}}{\alpha+1}\ln x-\frac{x^{\alpha+1}}{(\alpha+1)^2}\right]+C$$

$$=\frac{x^{\alpha+1}}{\alpha+1}(\ln x)^2-2\frac{x^{\alpha+1}}{(\alpha+1)^2}\ln x+2\frac{x^{\alpha+1}}{(\alpha+1)^3}+C,\alpha\neq-1$$

Problem 57 was used for $\int x^\alpha \ln x\,dx$.

59. $u=x^\alpha \qquad dv=e^{\beta x}dx$

$du=\alpha x^{\alpha-1}dx \qquad v=\frac{1}{\beta}e^{\beta x}$

$$\int x^\alpha e^{\beta x}dx=\frac{x^\alpha e^{\beta x}}{\beta}-\frac{\alpha}{\beta}\int x^{\alpha-1}e^{\beta x}dx$$

60. $u=x^\alpha \qquad dv=\sin\beta x\,dx$

$du=\alpha x^{\alpha-1}dx \qquad v=-\frac{1}{\beta}\cos\beta x$

$$\int x^\alpha\sin\beta x\,dx=-\frac{x^\alpha\cos\beta x}{\beta}+\frac{\alpha}{\beta}\int x^{\alpha-1}\cos\beta x\,dx$$

61. $u=x^\alpha \qquad dv=\cos\beta x\,dx$

$du=\alpha x^{\alpha-1}dx \qquad v=\frac{1}{\beta}\sin\beta x$

$$\int x^\alpha\cos\beta x\,dx=\frac{x^\alpha\sin\beta x}{\beta}-\frac{\alpha}{\beta}\int x^{\alpha-1}\sin\beta x\,dx$$

62. $u=(\ln x)^\alpha \qquad dv=dx$

$du=\frac{\alpha(\ln x)^{\alpha-1}}{x}dx \qquad v=x$

$$\int(\ln x)^\alpha dx=x(\ln x)^\alpha-\alpha\int(\ln x)^{\alpha-1}dx$$

63. $u = (a^2 - x^2)^\alpha \qquad dv = dx$

$du = -2\alpha x(a^2 - x^2)^{\alpha-1}dx \quad v = x$

$\int (a^2 - x^2)^\alpha dx = x(a^2 - x^2)^\alpha + 2\alpha \int x^2(a^2 - x^2)^{\alpha-1} dx$

64. $u = \cos^{\alpha-1} x \qquad dv = \cos x\, dx$

$du = -(\alpha - 1)\cos^{\alpha-2} x \sin x\, dx \qquad v = \sin x$

$\int \cos^\alpha x\, dx = \cos^{\alpha-1} x \sin x + (\alpha - 1)\int \cos^{\alpha-2} x \sin^2 x\, dx$

$= \cos^{\alpha-1} x \sin x + (\alpha - 1)\int \cos^{\alpha-2} x(1 - \cos^2 x)\, dx \;= \cos^{\alpha-1} x \sin x + (\alpha - 1)\int \cos^{\alpha-2} x\, dx - (\alpha - 1)\int \cos^\alpha x\, dx$

$\alpha \int \cos^\alpha x\, dx = \cos^{\alpha-1} x \sin x + (\alpha - 1)\int \cos^{\alpha-2} x\, dx$

$\int \cos^\alpha x\, dx = \frac{\cos^{\alpha-1} x \sin x}{\alpha} + \frac{\alpha - 1}{\alpha}\int \cos^{\alpha-2} x\, dx$

65. $u = \cos^{\alpha-1} \beta x \qquad dv = \cos \beta x\, dx$

$du = -\beta(\alpha - 1)\cos^{\alpha-2} \beta x \sin \beta x\, dx \quad v = \frac{1}{\beta}\sin \beta x$

$\int \cos^\alpha \beta x\, dx = \frac{\cos^{\alpha-1} \beta x \sin \beta x}{\beta} + (\alpha - 1)\int \cos^{\alpha-2} \beta x \sin^2 \beta x\, dx$

$= \frac{\cos^{\alpha-1} \beta x \sin \beta x}{\beta} + (\alpha - 1)\int \cos^{\alpha-2} \beta x(1 - \cos^2 \beta x)\, dx$

$= \frac{\cos^{\alpha-1} \beta x \sin \beta x}{\beta} + (\alpha - 1)\int \cos^{\alpha-2} \beta x\, dx - (\alpha - 1)\int \cos^\alpha \beta x\, dx$

$\alpha \int \cos^\alpha \beta x = \frac{\cos^{\alpha-1} \beta x \sin \beta x}{\beta} + (\alpha - 1)\int \cos^{\alpha-2} \beta x dx$

$\int \cos^\alpha \beta x\, dx = \frac{\cos^{\alpha-1} \beta x \sin \beta x}{\alpha\beta} + \frac{\alpha - 1}{\alpha}\int \cos^{\alpha-2} \beta x dx$

66. $\int x^4 e^{3x} dx = \frac{1}{3}x^4 e^{3x} - \frac{4}{3}\int x^3 e^{3x} dx = \frac{1}{3}x^4 e^{3x} - \frac{4}{3}\left[\frac{1}{3}x^3 e^{3x} - \int x^2 e^{3x} dx\right]$

$= \frac{1}{3}x^4 e^{3x} - \frac{4}{9}x^3 e^{3x} + \frac{4}{3}\left[\frac{1}{3}x^2 e^{3x} - \frac{2}{3}\int x e^{3x} dx\right] = \frac{1}{3}x^4 e^{3x} - \frac{4}{9}x^3 e^{3x} + \frac{4}{9}x^2 e^{3x} - \frac{8}{9}\left[\frac{1}{3}x e^{3x} - \frac{1}{3}\int e^{3x} dx\right]$

$= \frac{1}{3}x^4 e^{3x} - \frac{4}{9}x^3 e^{3x} + \frac{4}{9}x^2 e^{3x} - \frac{8}{27}x e^{3x} + \frac{8}{81}e^{3x} + C$

67. $\int x^4 \cos 3x\, dx = \frac{1}{3}x^4 \sin 3x - \frac{4}{3}\int x^3 \sin 3x\, dx \;= \frac{1}{3}x^4 \sin 3x - \frac{4}{3}\left[-\frac{1}{3}x^3 \cos 3x + \int x^2 \cos 3x\, dx\right]$

$= \frac{1}{3}x^4 \sin 3x + \frac{4}{9}x^3 \cos 3x - \frac{4}{3}\left[\frac{1}{3}x^2 \sin 3x - \frac{2}{3}\int x \sin 3x\, dx\right]$

$= \frac{1}{3}x^4 \sin 3x + \frac{4}{9}x^3 \cos 3x - \frac{4}{9}x^2 \sin 3x + \frac{8}{9}\left[-\frac{1}{3}x \cos 3x + \frac{1}{3}\int \cos 3x\, dx\right]$

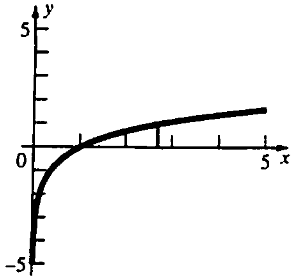
$= \frac{1}{3}x^4 \sin 3x + \frac{4}{9}x^3 \cos 3x - \frac{4}{9}x^2 \sin 3x - \frac{8}{27}x \cos 3x + \frac{8}{81}\sin 3x + C$

68. $\int \cos^6 3x\,dx = \frac{1}{18}\cos^5 3x \sin 3x + \frac{5}{6}\int \cos^4 3x\,dx = \frac{1}{18}\cos^5 3x \sin 3x + \frac{5}{6}\left[\frac{1}{12}\cos^3 3x \sin 3x + \frac{3}{4}\int \cos^2 3x\,dx\right]$

$= \frac{1}{18}\cos^5 3x \sin 3x + \frac{5}{72}\cos^3 3x \sin 3x + \frac{5}{8}\left[\frac{1}{6}\cos 3x \sin 3x + \frac{1}{2}\int dx\right]$

$= \frac{1}{18}\cos^5 3x \sin 3x + \frac{5}{72}\cos^3 3x \sin 3x + \frac{5}{48}\cos 3x \sin 3x + \frac{5x}{16} + C$

69. First make a sketch.



From the sketch, the area is given by

$\int_1^e \ln x\,dx$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x}dx \qquad v = x$

$\int_1^e \ln x\,dx = [x\ln x]_1^e - \int_1^e dx = [x\ln x - x]_1^e = (e-e) - (1\cdot 0 - 1) = 1$

70. $V = \int_1^e \pi(\ln x)^2\,dx$

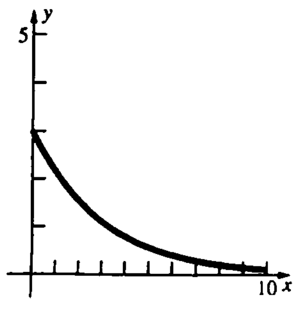
$u = (\ln x)^2 \qquad dv = dx$

$du = \frac{2\ln x}{x}dx \qquad v = x$

$\pi\int_1^e (\ln x)^2\,dx = \pi\left(\left[x(\ln x)^2\right]_1^e - 2\int_1^e \ln x\,dx\right) = \pi\left[x(\ln x)^2 - 2(x\ln x - x)\right]_1^e = \pi[x(\ln x)^2 - 2x\ln x + 2x]_1^e$

$= \pi[(e - 2e + 2e) - (0 - 0 + 2)] = \pi(e-2) \approx 2.26$

71.



$\int_0^9 3e^{-x/3}\,dx = -9\int_0^9 e^{-x/3}\left(-\frac{1}{3}dx\right) = -9[e^{-x/3}]_0^9 = -\frac{9}{e^3} + 9 \approx 8.55$

72. $V = \int_0^9 \pi(3e^{-x/3})^2\,dx = 9\pi\int_0^9 e^{-2x/3}\,dx$

$= 9\pi\left(-\frac{3}{2}\right)\int_0^9 e^{-2x/3}\left(-\frac{2}{3}dx\right) = -\frac{27\pi}{2}[e^{-2x/3}]_0^9 = -\frac{27\pi}{2e^6} + \frac{27\pi}{2} \approx 42.31$

73. $\int_0^{\pi/4}(x\cos x - x\sin x)dx = \int_0^{\pi/4} x\cos x\,dx - \int_0^{\pi/4} x\sin x\,dx$

$= \left(\left[x\sin x\right]_0^{\pi/4} - \int_0^{\pi/4}\sin x\,dx\right) - \left(\left[-x\cos x\right]_0^{\pi/4} + \int_0^{\pi/4}\cos x\,dx\right)$

$= [x\sin x + \cos x + x\cos x - \sin x]_0^{\pi/4} = \frac{\sqrt{2}\pi}{4} - 1 \approx 0.11$

Use Problems 60 and 61 for $\int x\sin x\,dx$ and $\int x\cos x\,dx$.

74. $V = 2\pi\int_0^{2\pi} x\sin\left(\frac{x}{2}\right)dx$

$u = x \qquad dv = \sin\frac{x}{2}dx$

$du = dx \qquad v = -2\cos\frac{x}{2}$

$V = 2\pi\left(\left[-2x\cos\frac{x}{2}\right]_0^{2\pi} + \int_0^{2\pi} 2\cos\frac{x}{2}dx\right) = 2\pi\left(4\pi + \left[4\sin\frac{x}{2}\right]_0^{2\pi}\right) = 8\pi^2$

75. $\int_1^e \ln x^2 dx = 2\int_1^e \ln x\,dx$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x}dx \qquad v = x$

$2\int_1^e \ln x\,dx = 2\left([x\ln x]_1^e - \int_1^e dx\right) = 2\left(e - [x]_1^e\right) = 2$

$\int_1^e x\ln x^2 dx = 2\int_1^e x\ln x\,dx$

$u = \ln x \qquad dv = x\,dx$

$du = \frac{1}{x}dx \qquad v = \frac{1}{2}x^2$

$2\int_1^e x\ln x\,dx = 2\left(\left[\frac{1}{2}x^2\ln x\right]_1^e - \int_1^e \frac{1}{2}x\,dx\right) = 2\left(\frac{1}{2}e^2 - \left[\frac{1}{4}x^2\right]_1^e\right) = \frac{1}{2}(e^2+1)$

$\frac{1}{2}\int_1^e (\ln x)^2 dx$

$u = (\ln x)^2 \qquad dv = dx$

$du = \frac{2\ln x}{x}dx \qquad v = x$

$\frac{1}{2}\int_1^e (\ln x)^2 dx = \frac{1}{2}\left([x(\ln x)^2]_1^e - 2\int_1^e \ln x\,dx\right) = \frac{1}{2}(e-2)$

$\bar{x} = \frac{\frac{1}{2}(e^2+1)}{2} = \frac{e^2+1}{4}$

$\bar{y} = \frac{\frac{1}{2}(e-2)}{2} = \frac{e-2}{4}$

76. a. $u = \cot x \qquad dv = \csc^2 x\,dx$

$du = -\csc^2 x\,dx \qquad v = -\cot x$

$\int \cot x\csc^2 x\,dx = -\cot^2 x - \int \cot x\csc^2 x\,dx = -\frac{1}{2}\cot^2 x + C$

b. $u = \csc x$ $\quad dv = \cot x \csc x\, dx$
$du = -\cot x \csc x\, dx$ $\quad v = -\csc x$

$$\int \cot x \csc^2 x\, dx = -\csc^2 x - \int \cot x \csc^2 x\, dx = -\frac{1}{2}\csc^2 x + C$$

c. $-\frac{1}{2}\cot^2 x = -\frac{1}{2}(\csc^2 x - 1) = -\frac{1}{2}\csc^2 x + \frac{1}{2}$

77. a. $p(x) = x^3 - 2x$

$g(x) = e^x$

All antiderivatives of $g(x) = e^x$

$$\int (x^3 - 2x)e^x dx = (x^3 - 2x)e^x - (3x^2 - 2)e^x + 6xe^x - 6e^x + C$$

b. $p(x) = x^2 - 3x + 1$
$g(x) = \sin x$
$G_1(x) = -\cos x$
$G_2(x) = -\sin x$
$G_3(x) = \cos x$

$$\int (x^2 - 3x + 1)\sin x\, dx = (x^2 - 3x + 1)(-\cos x) - (2x - 3)(-\sin x) + 2\cos x + C$$

78. $2\sec^3 x\, dx = [\sec^3 x + \sec x(1 + \tan^2 x)]dx$
$= (\sec x + \sec^3 x + \sec x \tan^2 x)dx = \sec x\, dx + (\sec^3 x + \sec x \tan^2 x)dx$
$= \sec x\, dx + d(\sec x \tan x)$

$$\int \sec^3 x\, dx = \frac{1}{2}\int [\sec x\, dx + d(\sec x \tan x)] = \frac{1}{2}\int \sec x\, dx + \frac{1}{2}\int d(\sec x \tan x)$$

$$= \frac{1}{2}\ln|\sec x + \tan x| + \frac{1}{2}(\sec x \tan x) + C$$

79. $$\int_0^{2\pi} \sin^n x\, dx = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 4\int_0^{\pi/2} \sin^n x\, dx & \text{if } n \text{ is even} \end{cases}$$

From Formula 113,

$$4\int_0^{\pi/2} \sin^n x\, dx = 4\left(\frac{1\cdot 3\cdot 5\cdot \ldots \cdot (n-1)}{2\cdot 4\cdot 6\cdot \ldots \cdot n}\frac{\pi}{2}\right) \text{ if } n \text{ is even}$$

$$\int_0^{2\pi} \sin^n x\, dx = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \dfrac{1\cdot 3\cdot 5\cdot \ldots \cdot (n-1)}{2\cdot 4\cdot 6\cdot \ldots \cdot n}2\pi & \text{if } n \text{ is even} \end{cases}$$

80. a. $$\int_{2\pi n - 2\pi}^{2\pi n - \pi} x \sin x\, dx$$

b. $$V = 2\pi \int_{2\pi}^{3\pi} x^2 \sin x\, dx$$

$u = x^2$ $\quad dv = \sin x\, dx$
$du = 2x\, dx$ $\quad v = -\cos x$

$$V = 2\pi\left(\left[-x^2 \cos x\right]_{2\pi}^{3\pi} + \int_{2\pi}^{3\pi} 2x \cos x\, dx\right) = 2\pi\left(9\pi^2 + 4\pi^2 + \int_{2\pi}^{3\pi} 2x \cos x\, dx\right)$$

$u = 2x$ $\quad dv = \cos x\, dx$
$du = 2\, dx$ $\quad v = \sin x$

$$V = 2\pi\left(13\pi^2 + [2x\sin x]_{2\pi}^{3\pi} - \int_{2\pi}^{3\pi} 2\sin x\right)$$

$$= 2\pi\left(13\pi^2 + [2\cos x]_{2\pi}^{3\pi}\right) = 2\pi(13\pi^2 - 4) \approx 781$$

81. $u = f(x)$ $\qquad dv = \sin nx\,dx$

$du = f'(x)dx \qquad v = -\frac{1}{n}\cos nx$

$$a_n = \frac{1}{\pi}\left[\underbrace{\left[-\frac{1}{n}\cos(nx)f(x)\right]_{-\pi}^{\pi}}_{\text{Term 1}} + \underbrace{\frac{1}{n}\int_{-\pi}^{\pi}\cos(nx)f'(x)dx}_{\text{Term 2}}\right]$$

$$\text{Term 1} = \frac{1}{n}\cos(n\pi)(f(-\pi) - f(\pi)) = \pm\frac{1}{n}(f(-\pi) - f(\pi))$$

Since $f'(x)$ is continuous on $[-\infty, \infty]$, it is bounded. Thus, $\int_{-\pi}^{\pi}\cos(nx)f'(x)dx$ is bounded so

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{\pi n}\left[\pm(f(-\pi) - f(\pi)) + \int_{-\pi}^{\pi}\cos(nx)f'(x)\,dx\right] = 0.$$

82. $$\frac{G_n}{n} = \frac{[(n+1)(n+2)\cdots(n+n)]^{1/n}}{[n^n]^{1/n}} = \left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\left(1+\frac{n}{n}\right)\right]^{1/n}$$

$$\ln\left(\frac{G_n}{n}\right) = \frac{1}{n}\ln\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\left(1+\frac{n}{n}\right)\right]$$

$$= \frac{1}{n}\left[\ln\left(1+\frac{1}{n}\right) + \ln\left(1+\frac{2}{n}\right) + \cdots + \ln\left(1+\frac{n}{n}\right)\right]$$

$$\lim_{n\to\infty}\ln\left(\frac{G_n}{n}\right) = \int_1^2 \ln x\,dx = 2\ln 2 - 1$$

$$\lim_{n\to\infty}\left(\frac{G_n}{n}\right) = e^{2\ln 2 - 1} = 4e^{-1} = \frac{4}{e}$$

83. The proof fails to consider the constants when integrating $\frac{1}{t}$.

$$\int\frac{1}{t}dt = 1 + \int\frac{1}{t}dt$$

$$\ln t + C_1 = 1 + \ln t + C_2$$

Thus $C_1 - C_2 = 1$.

84. $$\frac{d}{dx}[e^{5x}(C_1\cos 7x + C_2\sin 7x) + C_3] = 5e^{5x}(C_1\cos 7x + C_2\sin 7x) + e^{5x}(-7C_1\sin 7x + 7C_2\cos 7x)$$

$$= e^{5x}[(5C_1 + 7C_2)\cos 7x + (5C_2 - 7C_1)\sin 7x]$$

Thus, $5C_1 + 7C_2 = 4$ and $5C_2 - 7C_1 = 6$.

Solving, $C_1 = -\frac{11}{37}; C_2 = \frac{29}{37}$

85. $u = f(x)$ $\qquad dv = dx$

$du = f'(x)dx \qquad v = x$

$$\int_a^b f(x)dx = [xf(x)]_a^b - \int_a^b xf'(x)dx$$

Starting with the same integral,

$u = f(x) \qquad dv = dx$

$du = f'(x)dx \qquad v = x - a$

$$\int_a^b f(x)\,dx = [(x-a)f(x)]_a^b - \int_a^b (x-a)f'(x)dx$$

86. $u = f'(x) \qquad dv = dx$

$du = f''(x)dx \qquad v = x - a$

$$f(b) - f(a) = \int_a^b f'(x)dx = [(x-a)f'(x)]_a^b - \int_a^b (x-a)f''(x)dx = f'(b)(b-a) - \int_a^b (x-a)f''(x)dx$$

Starting with the same integral,

$u = f'(x) \qquad dv = dx$

$du = f''(x)dx \qquad v = x - b$

$$f(b) - f(a) = \int_a^b f'(x)dx = [(x-b)f'(x)]_a^b - \int_a^b (x-b)f''(x)dx = f'(a)(b-a) - \int_a^b (x-b)f''(x)dx$$

87. Use proof by induction.

$$n = 1:\ f(a) + f'(a)(t-a) + \int_a^t (t-x)f''(x)dx = f(a) + f'(a)(t-a) + [f'(x)(t-x)]_a^t + \int_a^t f'(x)dx$$

$$= f(a) + f'(a)(t-a) - f'(a)(t-a) + [f(x)]_a^t = f(t)$$

Thus, the statement is true for $n = 1$. Note that integration by parts was used with $u = (t-x),\ dv = f''(x)dx$.

Suppose the statement is true for n.

$$f(t) = f(a) + \sum_{i=1}^{n} \frac{f^{(i)}(a)}{i!}(t-a)^i + \int_a^t \frac{(t-x)^n}{n!} f^{(n+1)}(x)dx$$

Integrate $\int_a^t \frac{(t-x)^n}{n!} f^{(n+1)}(x)dx$ by parts.

$u = f^{(n+1)}(x) \qquad dv = \frac{(t-x)^n}{n!}dx$

$du = f^{(n+2)}(x) \qquad v = -\frac{(t-x)^{n+1}}{(n+1)!}$

$$\int_a^t \frac{(t-x)^n}{n!} f^{(n+1)}(x)dx = \left[-\frac{(t-x)^{n+1}}{(n+1)!} f^{(n+1)}(x)\right]_a^t + \int_a^t \frac{(t-x)^{n+1}}{(n+1)!} f^{(n+2)}(x)dx$$

$$= \frac{(t-a)^{n+1}}{(n+1)!} f^{(n+1)}(a) + \int_a^t \frac{(t-x)^{n+1}}{(n+1)!} f^{(n+2)}(x)dx$$

Thus $f(t) = f(a) + \sum_{i=1}^{n} \frac{f^{(i)}(a)}{i!}(t-a)^i + \frac{(t-a)^{n+1}}{(n+1)!} f^{(n+1)}(a) + \int_a^t \frac{(t-x)^{n+1}}{(n+1)!} f^{(n+2)}(x)dx$

$$= f(a) + \sum_{i=1}^{n+1} \frac{f^{(i)}(a)}{i!}(t-a)^i + \int_a^t \frac{(t-x)^{n+1}}{(n+1)!} f^{(n+2)}(x)dx$$

Thus, the statement is true for $n + 1$.

88. a. $B(\alpha, \beta) = \int_0^1 x^\alpha (1-x)^\beta dx$

$x = 1 - u,\ dx = -du$

$$\int_0^1 x^\alpha (1-x)^\beta dx = \int_1^0 (1-u)^\alpha (u)^\beta (-du) = \int_0^1 (1-u)^\alpha u^\beta du = B(\beta, \alpha)$$

Thus, $B(\alpha, \beta) = B(\beta, \alpha)$.

b. $B(\alpha, \beta) = \int_0^1 x^\alpha (1-x)^\beta dx$

$u = x^\alpha \qquad dv = (1-x)^\beta dx$

$du = \alpha x^{\alpha-1} dx \qquad v = -\frac{1}{\beta+1}(1-x)^{\beta+1}$

$$B(\alpha, \beta) = \left[-\frac{1}{\beta+1}x^\alpha (1-x)^{\beta+1}\right]_0^1 + \frac{\alpha}{\beta+1}\int_0^1 x^{\alpha-1}(1-x)^{\beta+1} dx$$

$$= \frac{\alpha}{\beta+1}\int_0^1 x^{\alpha-1}(1-x)^{\beta+1} dx = \frac{\alpha}{\beta+1}B(\alpha-1, \beta+1)$$

$B(\alpha, \beta) = \int_0^1 x^\alpha (1-x)^\beta dx$

$u = (1-x)^\beta \qquad dv = x^\alpha dx$

$du = -\beta(1-x)^{\beta-1} dx \qquad v = \frac{1}{\alpha+1}x^{\alpha+1}$

$$B(\alpha, \beta) = \left[\frac{1}{\alpha+1}x^{\alpha+1}(1-x)^\beta\right]_0^1 + \frac{\beta}{\alpha+1}\int_0^1 x^{\alpha+1}(1-x)^{\beta-1} dx = \frac{\beta}{\alpha+1}\int_0^1 x^{\alpha+1}(1-x)^{\beta-1} dx = \frac{\beta}{\alpha+1}B(\alpha+1, \beta-1)$$

c. Assume that $n \le m$. Using part b. n times,

$$B(n, m) = \frac{n}{m+1}B(n-1, m+1) = \frac{n(n-1)}{(m+1)(m+2)}B(n-2, m+2) = \ldots = \frac{n(n-1)\ldots\cdot 2\cdot 1}{(m+1)(m+2)\ldots(m+n)}B(0, m+n).$$

$$B(0, m+n) = \int_0^1 (1-x)^{m+n} dx = -\frac{1}{m+n+1}[(1-x)^{m+n+1}]_0^1 = \frac{1}{m+n+1}$$

Thus, $B(n, m) = \frac{n(n-1)\ldots\cdot 2\cdot 1}{(m+1)(m+2)\ldots(m+n)(m+n+1)} = \frac{n!m!}{(m+n+1)!}$

If $n > m$, then $B(n, m) = B(m, n) = \frac{n!m!}{(m+n+1)!}$ by the above reasoning.

89. $u = f(t) \qquad dv = f''(t)dt$

$du = f'(t)dt \qquad v = f'(t)$

$$\int_a^b f''(t)f(t)dt = [f(t)f'(t)]_a^b - \int_a^b [f'(t)]^2 dt$$

$$= f(b)f'(b) - f(a)f'(a) - \int_a^b [f'(t)]^2 dt = -\int_a^b [f'(t)]^2 dt$$

$[f'(t)]^2 \ge 0$, so $-\int_a^b [f'(t)]^2 \le 0$.

90. $\int_0^x \left(\int_0^t f(z)dz\right)dt$

$u = \int_0^t f(z)dz \qquad dv = dt$

$du = f(t)dt \qquad v = t$

$$\int_0^x \left(\int_0^t f(z)dz\right)dt = \left[t\int_0^t f(z)dz\right]_0^x - \int_0^x t\, f(t)dt = \int_0^x x\, f(z)dz - \int_0^x t\, f(t)dt$$

By letting $z = t$, $\int_0^x x\, f(z)dz = \int_0^x x\, f(t)dt$, so

$$\int_0^x \left(\int_0^t f(z)dz\right)dt = \int_0^x x\, f(t)dt - \int_0^x t\, f(t)dt = \int_0^x (x-t) f(t)dt$$

91. Let $I = \int_0^x \int_0^{t_1} \cdots \int_0^{t_{n-1}} f(t_n)\,dt_n \ldots dt_2 dt_1$ be the iterated integral. Note that for $i \geq 2$, the limits of integration of the integral with respect to t_i are 0 to t_{i-1} so that any change of variables in an outer integral affects the limits, and hence the variables in all interior integrals.

$$I = \int_0^x \left[\int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{n-1}} f(t_n)\,dt_n \ldots dt_3 dt_2 \right] dt_1$$

$$u = \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{n-1}} f(t_n)\,dt_n \ldots dt_3 dt_2 \qquad dv = dt_1$$

$$du = \left(\int_0^{t_2} \cdots \int_0^{t_{n-1}} f(t_n)\,dt_n \ldots dt_3 \right) dt_1 \qquad v = t_1$$

$$I = \left[t_1 \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{n-1}} f(t_n)\,dt_n \ldots dt_3 dt_2 \right]_{t_1=0}^{t_1=x} - \int_0^x t_1 \left(\int_0^{t_2} \cdots \int_0^{t_{n-1}} f(t_n)\,dt_n \ldots dt_3 \right) dt_1$$

$$= x \int_0^x \int_0^{t_2} \cdots \int_0^{t_{n-1}} f(t_n)\,dt_n \ldots dt_3 dt_2 - \int_0^x t_1 \left(\int_0^{t_2} \cdots \int_0^{t_{n-1}} f(t_n)\,dt_n \ldots dt_3 \right) dt_1$$

Make the change of variables $t_i = t_{i-1}$, so $dt_i = dt_{i-1}$ for $i = 2$ to n.

Hence, $I = \int_0^x x \int_0^{t_1} \cdots \int_0^{t_{n-2}} f(t_{n-1})\,dt_{n-1} \ldots dt_2 dt_1 - \int_0^x t_1 \left(\int_0^{t_1} \cdots \int_0^{t_{n-2}} f(t_{n-1})\,dt_{n-1} \ldots dt_2 \right) dt_1$

$$= \int_0^x (x - t_1) \left(\int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{n-2}} f(t_{n-1})\,dt_{n-1} \ldots dt_3 dt_2 \right) dt_1$$

Use integration by parts again.

$$u = \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{n-2}} f(t_{n-1})\,dt_{n-1} \ldots dt_3 dt_2 \qquad dv = (x - t_1)\,dt_1$$

$$du = \left(\int_0^{t_2} \cdots \int_0^{t_{n-2}} f(t_{n-1})\,dt_{n-1} \ldots dt_3 \right) dt_1 \qquad v = -\frac{1}{2}(x - t_1)^2$$

$$I = \left[-\frac{1}{2}(x - t_1)^2 \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{n-2}} f(t_{n-1})\,dt_{n-1} \ldots dt_3 dt_2 \right]_{t_1=0}^{t_1=x} + \int_0^x \frac{1}{2}(x - t_1)^2 \left(\int_0^{t_2} \cdots \int_0^{t_{n-2}} f(t_{n-1})\,dt_{n-1} \ldots dt_3 \right) dt_1$$

$$= \int_0^x \frac{1}{2}(x - t_1)^2 \left(\int_0^{t_2} \cdots \int_0^{t_{n-2}} f(t_{n-1})\,dt_{n-1} \ldots dt_3 \right) dt_1$$

Since $(x - t_1)^2 = 0$ when $t_1 = x$ while $\int_0^{t_1} g(t_2)\,dt_2 = 0$ when $t_1 = 0$ for any integrand $g(t_2)$.

Again change variables so that $t_i = t_{i-1}$ for $i = 2$ to $n - 1$.

$$I = \int_0^x \frac{1}{2}(x - t_1)^2 \left(\int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{n-3}} f(t_{n-2})\,dt_{n-2} \ldots dt_3 dt_2 \right) dt_1$$

$$u = \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{n-3}} f(t_{n-2})\,dt_{n-2} \ldots dt_3 dt_2 \qquad dv = \frac{1}{2}(x - t_1)^2\,dt_1$$

$$du = \left(\int_0^{t_2} \cdots \int_0^{t_{n-3}} f(t_{n-2})\,dt_{n-2} \ldots dt_3 \right) dt_1 \qquad v = -\frac{1}{3!}(x - t_1)^3$$

$$I = \left[-\frac{1}{3!}(x - t_1)^3 \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{n-3}} f(t_{n-2})\,dt_{n-2} \ldots dt_3 dt_2 \right]_{t_1=0}^{t_1=x} + \int_0^x \frac{1}{3!}(x - t_1)^3 \left(\int_0^{t_2} \cdots \int_0^{t_{n-3}} f(t_{n-2})\,dt_{n-2} \ldots dt_3 \right) dt_1$$

Changing variables and using integration by parts as before, then changing variables again yields

$$I = \int_0^x \frac{1}{4!}(x - t_1)^4 \left(\int_0^{t_1} \cdots \int_0^{t_{n-5}} f(t_{n-4})\,dt_{n-4} \ldots dt_2 \right) dt_1$$

Reducing the n-fold iterated integral to a single integral requires $n - 1$ integrations by parts, each one contributing a factor of $x - t_1$. The integral is multiplied by $\frac{1}{k!}$ after k integrations by parts, hence

$$I = \frac{1}{(n-1)!} \int_0^x f(t_1)(x - t_1)^{n-1}\,dt_1 .$$

92. Proof by induction.
$n = 1$:

$u = P_1(x) \qquad dv = e^x dx$

$du = \frac{dP_1(x)}{dx}dx \qquad v = e^x$

$$\int e^x P_1(x)dx = e^x P_1(x) - \int e^x \frac{dP_1(x)}{dx}dx = e^x P_1(x) - \frac{dP_1(x)}{dx}\int e^x dx = e^x P_1(x) - e^x \frac{dP_1(x)}{dx}$$

Note that $\frac{dP_1(x)}{dx}$ is a constant.

Suppose the formula is true for n. By using integration by parts with $u = P_{n+1}(x)$ and $dv = e^x dx$,

$$\int e^x P_{n+1}(x)dx = e^x P_{n+1}(x) - \int e^x \frac{dP_{n+1}(x)}{dx}dx$$

Note that $\frac{dP_{n+1}(x)}{dx}$ is a polynomial of degree n, so

$$\int e^x P_{n+1}(x)dx = e^x P_{n+1}(x) - \left[e^x \sum_{j=0}^{n}(-1)^j \frac{d^j}{dx^j}\left(\frac{dP_{n+1}(x)}{dx}\right)\right]$$

$$= e^x P_{n+1}(x) + e^x \sum_{j=1}^{n+1}(-1)^j \frac{d^j P_{n+1}(x)}{dx^{j+1}} = e^x \sum_{j=0}^{n+1}(-1)^j \frac{d^j P_{n+1}(x)}{dx^{j+1}}$$

93. $$\int(3x^4 + 2x^2)e^x dx = e^x \sum_{j=0}^{4}(-1)^j \frac{d^j(3x^4 + 2x^2)}{dx^j}$$

$$= e^x[3x^4 + 2x^2 - 12x^3 - 4x + 36x^2 + 4 - 72x + 72]$$

$$= e^x(3x^4 - 12x^3 + 38x^2 - 76x + 76)$$

8.5 Concepts Review

1. proper

2. $x - 1 + \frac{5}{x+1}$

3. $2x^2 + 3x - 1 = ax^2 + bx + c$
$a = 2; b = 3; c = -1$

4. $\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$

Problem Set 8.5

1. $\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$
$1 = A(x+1) + Bx$
$A = 1, B = -1$
$$\int \frac{1}{x(x+1)}dx = \int \frac{1}{x}dx - \int \frac{1}{x+1}dx$$
$$= \ln|x| - \ln|x+1| + C$$

2. $\frac{2}{x^2+3x} = \frac{2}{x(x+3)} = \frac{A}{x} + \frac{B}{x+3}$
$2 = A(x+3) + Bx$
$A = \frac{2}{3}, B = -\frac{2}{3}$
$$\int \frac{2}{x^2+3x}dx = \frac{2}{3}\int \frac{1}{x}dx - \frac{2}{3}\int \frac{B}{x+3}dx$$
$$= \frac{2}{3}\ln|x| - \frac{2}{3}\ln|x+3| + C$$

3. $\frac{3}{x^2-1} = \frac{3}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$
$3 = A(x-1) + B(x+1)$
$A = -\frac{3}{2}, B = \frac{3}{2}$
$$\int \frac{3}{x^2-1}dx = -\frac{3}{2}\int \frac{1}{x+1}dx + \frac{3}{2}\int \frac{1}{x-1}dx$$
$$= -\frac{3}{2}\ln|x+1| + \frac{3}{2}\ln|x-1| + C$$

4. $\frac{5x}{2x^3+6x^2}=\frac{5x}{2x^2(x+3)}=\frac{5}{2x(x+3)}$
$=\frac{A}{x}+\frac{B}{x+3}$
$\frac{5}{2}=A(x+3)+Bx$
$A=\frac{5}{6}, B=-\frac{5}{6}$
$\int\frac{5x}{2x^3+6x^2}=\frac{5}{6}\int\frac{1}{x}dx-\frac{5}{6}\int\frac{1}{x+3}dx$
$=\frac{5}{6}\ln|x|-\frac{5}{6}\ln|x+3|+C$

5. $\frac{x-11}{x^2+3x-4}=\frac{x-11}{(x+4)(x-1)}=\frac{A}{x+4}+\frac{B}{x-1}$
$x-11=A(x-1)+B(x+4)$
$A=3, B=-2$
$\int\frac{x-11}{x^2+3x-4}dx=3\int\frac{1}{x+4}dx-2\int\frac{1}{x-1}dx$
$=3\ln|x+4|-2\ln|x-1|+C$

6. $\frac{x-7}{x^2-x-12}=\frac{x-7}{(x-4)(x+3)}=\frac{A}{x-4}+\frac{B}{x+3}$
$x-7=A(x+3)+B(x-4)$
$A=-\frac{3}{7}, B=\frac{10}{7}$
$\int\frac{x-7}{x^2-x-12}dx=-\frac{3}{7}\int\frac{1}{x-4}dx+\frac{10}{7}\int\frac{1}{x+3}dx$
$=-\frac{3}{7}\ln|x-4|+\frac{10}{7}\ln|x+3|+C$

7. $\frac{3x-13}{x^2+3x-10}=\frac{3x-13}{(x+5)(x-2)}=\frac{A}{x+5}+\frac{B}{x-2}$
$3x-13=A(x-2)+B(x+5)$
$A=4, B=-1$
$\int\frac{3x-13}{x^2+3x-10}dx=4\int\frac{1}{x+5}dx-\int\frac{1}{x-2}dx$
$=4\ln|x+5|-\ln|x-2|+C$

8. $\frac{x+\pi}{x^2-3\pi x+2\pi^2}=\frac{x+\pi}{(x-2\pi)(x-\pi)}=\frac{A}{x-2\pi}+\frac{B}{x-\pi}$
$x+\pi=A(x-\pi)+B(x-2\pi)$
$A=3, B=-2$
$\int\frac{x+\pi}{x^2-3\pi x+2\pi^2}dx=\int\frac{3}{x-2\pi}dx-\int\frac{2}{x-\pi}dx$
$=3\ln|x-2\pi|-2\ln|x-\pi|+C$

9. $\frac{2x+21}{2x^2+9x-5}=\frac{2x+21}{(2x-1)(x+5)}=\frac{A}{2x-1}+\frac{B}{x+5}$
$2x+21=A(x+5)+B(2x-1)$
$A=4, B=-1$
$\int\frac{2x+21}{2x^2+9x-5}dx=\int\frac{4}{2x-1}dx-\int\frac{1}{x+5}dx$
$=2\ln|2x-1|-\ln|x+5|+C$

10. $\frac{2x^2-x-20}{x^2+x-6}=\frac{2(x^2+x-6)-3x-8}{x^2+x-6}$
$=2-\frac{3x+8}{x^2+x-6}$
$\frac{3x+8}{x^2+x-6}=\frac{3x+8}{(x+3)(x-2)}=\frac{A}{x+3}+\frac{B}{x-2}$
$3x+8=A(x-2)+B(x+3)$
$A=\frac{1}{5}, B=\frac{14}{5}$
$\int\frac{2x^2-x-20}{x^2+x-6}dx$
$=\int 2\,dx-\frac{1}{5}\int\frac{1}{x+3}dx-\frac{14}{5}\int\frac{1}{x-2}dx$
$=2x-\frac{1}{5}\ln|x+3|-\frac{14}{5}\ln|x-2|+C$

11. $\frac{17x-3}{3x^2+x-2}=\frac{17x-3}{(3x-2)(x+1)}=\frac{A}{3x-2}+\frac{B}{x+1}$
$17x-3=A(x+1)+B(3x-2)$
$A=5, B=4$
$\int\frac{17x-3}{3x^2+x-2}dx=\int\frac{5}{3x-2}dx+\int\frac{4}{x+1}dx=\frac{5}{3}\ln|3x-2|+4\ln|x+1|+C$

12. $\frac{5-x}{x^2-x(\pi+4)+4\pi}=\frac{5-x}{(x-\pi)(x-4)}=\frac{A}{x-\pi}+\frac{B}{x-4}$

$5-x=A(x-4)+B(x-\pi)$

$A=\frac{5-\pi}{\pi-4}, B=\frac{1}{4-\pi}$

$\int\frac{5-x}{x^2-x(\pi+4)+4\pi}dx=\frac{5-\pi}{\pi-4}\int\frac{1}{x-\pi}dx+\frac{1}{4-\pi}\int\frac{1}{x-4}dx=\frac{5-\pi}{\pi-4}\ln|x-\pi|+\frac{1}{4-\pi}\ln|x-4|+C$

13. $\frac{2x^2+x-4}{x^3-x^2-2x}=\frac{2x^2+x-4}{x(x+1)(x-2)}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-2}$

$2x^2+x-4=A(x+1)(x-2)+Bx(x-2)+Cx(x+1)$

$A=2, B=-1, C=1$

$\int\frac{2x^2+x-4}{x^3-x^2-2x}dx=\int\frac{2}{x}dx-\int\frac{1}{x+1}dx+\int\frac{1}{x-2}dx=2\ln|x|-\ln|x+1|+\ln|x-2|+C$

14. $\frac{7x^2+2x-3}{(2x-1)(3x+2)(x-3)}=\frac{A}{2x-1}+\frac{B}{3x+2}+\frac{C}{x-3}$

$7x^2+2x-3=A(3x+2)(x-3)+B(2x-1)(x-3)+C(2x-1)(3x+2)$

$A=\frac{1}{35}, B=-\frac{1}{7}, C=\frac{6}{5}$

$\int\frac{7x^2+2x-3}{(2x-1)(3x+2)(x-3)}dx=\frac{1}{35}\int\frac{1}{2x-1}dx-\frac{1}{7}\int\frac{1}{3x+2}dx+\frac{6}{5}\int\frac{1}{x-3}dx$

$=\frac{1}{70}\ln|2x-1|-\frac{1}{21}\ln|3x+2|+\frac{6}{5}\ln|x-3|+C$

15. $\frac{6x^2+22x-23}{(2x-1)(x^2+x-6)}=\frac{6x^2+22x-23}{(2x-1)(x+3)(x-2)}=\frac{A}{2x-1}+\frac{B}{x+3}+\frac{C}{x-2}$

$6x^2+22x-23=A(x+3)(x-2)+B(2x-1)(x-2)+C(2x-1)(x+3)$ $A=2, B=-1, C=3$

$\int\frac{6x^2+22x-23}{(2x-1)(x^2+x-6)}dx=\int\frac{2}{2x-1}dx-\int\frac{1}{x+3}dx+\int\frac{3}{x-2}dx=\ln|2x-1|-\ln|x+3|+3\ln|x-2|+C$

16. $\frac{x^3-6x^2+11x-6}{4x^3-28x^2+56x-32}=\frac{1}{4}\left(\frac{x^3-6x^2+11x-6}{x^3-7x^2+14x-8}\right)=\frac{1}{4}\left(1+\frac{x^2-3x+2}{x^3-7x^2+14x-8}\right)$

$=\frac{1}{4}\left(1+\frac{(x-1)(x-2)}{(x-1)(x-2)(x-4)}\right)=\frac{1}{4}\left(1+\frac{1}{x-4}\right)$

$\int\frac{x^3-6x^2+11x-6}{4x^3-28x^2+56x-32}dx=\int\frac{1}{4}dx+\frac{1}{4}\int\frac{1}{x-4}dx=\frac{1}{4}x+\frac{1}{4}\ln|x-4|+C$

17. $\frac{x^3}{x^2+x-2}=x-1+\frac{3x-2}{x^2+x-2}$

$\frac{3x-2}{x^2+x-2}=\frac{3x-2}{(x+2)(x-1)}=\frac{A}{x+2}+\frac{B}{x-1}$

$3x-2=A(x-1)+B(x+2)$

$A = \frac{8}{3}, B = \frac{1}{3}$

$$\int \frac{x^3}{x^2+x-2}dx = \int (x-1)\,dx + \frac{8}{3}\int \frac{1}{x+2}dx + \frac{1}{3}\int \frac{1}{x-1}dx = \frac{1}{2}x^2 - x + \frac{8}{3}\ln|x+2| + \frac{1}{3}\ln|x-1| + C$$

18. $\frac{x^3+x^2}{x^2+5x+6} = x - 4 + \frac{14x+24}{(x+3)(x+2)}$

$\frac{14x+24}{(x+3)(x+2)} = \frac{A}{x+3} + \frac{B}{x+2}$

$14x + 24 = A(x + 2) + B(x + 3)$

$A = 18, B = -4$

$$\int \frac{x^3+x^2}{x^2+5x+6}dx = \int (x-4)\,dx + \int \frac{18}{x+3}dx - \int \frac{4}{x+2}dx = \frac{1}{2}x^2 - 4x + 18\ln|x+3| - 4\ln|x+2| + C$$

19. $\frac{x^4+8x^2+8}{x^3-4x} = x + \frac{12x^2+8}{x(x+2)(x-2)}$

$\frac{12x^2+8}{x(x+2)(x-2)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2}$

$12x^2 + 8 = A(x+2)(x-2) + Bx(x-2) + Cx(x+2)$

$A = -2, B = 7, C = 7$

$$\int \frac{x^4+8x^2+8}{x^3-4x}dx = \int x\,dx - 2\int \frac{1}{x}dx + 7\int \frac{1}{x+2}dx + 7\int \frac{1}{x-2}dx = \frac{1}{2}x^2 - 2\ln|x| + 7\ln|x+2| + 7\ln|x-2| + C$$

20. $\frac{x^6+4x^3+4}{x^3-4x^2} = x^3 + 4x^2 + 16x + 68 + \frac{272x^2+4}{x^3-4x^2}$

$\frac{272x^2+4}{x^2(x-4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-4}$

$272x^2 + 4 = Ax(x-4) + B(x-4) + Cx^2$

$A = -\frac{1}{4}, B = -1, C = \frac{1089}{4}$

$$\int \frac{x^6+4x^3+4}{x^3-4x^2}dx = \int (x^3+4x^2+16x+68)\,dx - \frac{1}{4}\int \frac{1}{x}dx - \int \frac{1}{x^2}dx + \frac{1089}{4}\int \frac{1}{x-4}dx$$

$$= \frac{1}{4}x^4 + \frac{4}{3}x^3 + 8x^2 + 68x - \frac{1}{4}\ln|x| + \frac{1}{x} + \frac{1089}{4}\ln|x-4| + C$$

21. $\frac{x+1}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2}$

$x + 1 = A(x - 3) + B$

$A = 1, B = 4$

$$\int \frac{x+1}{(x-3)^2}dx = \int \frac{1}{x-3}dx + \int \frac{4}{(x-3)^2}dx = \ln|x-3| - \frac{4}{x-3} + C$$

22. $\frac{5x+7}{x^2+4x+4} = \frac{5x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$

$5x + 7 = A(x + 2) + B$

$A = 5, B = -3$

$$\int \frac{5x+7}{x^2+4x+4}dx = \int \frac{5}{x+2}dx - \int \frac{3}{(x+2)^2}dx = 5\ln|x+2| + \frac{3}{x+2} + C$$

23. $\dfrac{3x+2}{x^3+3x^2+3x+1}=\dfrac{3x+2}{(x+1)^3}=\dfrac{A}{x+1}+\dfrac{B}{(x+1)^2}+\dfrac{C}{(x+1)^3}$

$3x+2=A(x+1)^2+B(x+1)+C$

$A=0, B=3, C=-1$

$\int\dfrac{3x+2}{x^3+3x^2+3x+1}dx=\int\dfrac{3}{(x+1)^2}dx-\int\dfrac{1}{(x+1)^3}dx=-\dfrac{3}{x+1}+\dfrac{1}{2(x+1)^2}+C$

24. $\dfrac{x^6}{(x-2)^2(1-x)^5}=\dfrac{A}{x-2}+\dfrac{B}{(x-2)^2}+\dfrac{C}{1-x}+\dfrac{D}{(1-x)^2}+\dfrac{E}{(1-x)^3}+\dfrac{F}{(1-x)^4}+\dfrac{G}{(1-x)^5}$

$A=128, B=-64, C=129, D=-72, E=30, F=-8, G=1$

$\int\dfrac{x^6}{(x-2)^2(1-x)^5}dx=\int\left[\dfrac{128}{x-2}-\dfrac{64}{(x-2)^2}+\dfrac{129}{1-x}-\dfrac{72}{(1-x)^2}+\dfrac{30}{(1-x)^3}-\dfrac{8}{(1-x)^4}+\dfrac{1}{(1-x)^5}\right]dx$

$=128\ln|x-2|+\dfrac{64}{x-2}-129\ln|1-x|+\dfrac{72}{1-x}-\dfrac{15}{(1-x)^2}+\dfrac{8}{3(1-x)^3}-\dfrac{1}{4(1-x)^4}+C$

25. $\dfrac{3x^2-21x+32}{x^3-8x^2+16x}=\dfrac{3x^2-21x+32}{x(x-4)^2}=\dfrac{A}{x}+\dfrac{B}{x-4}+\dfrac{C}{(x-4)^2}$

$3x^2-21x+32=A(x-4)^2+Bx(x-4)+Cx$

$A=2, B=1, C=-1$

$\int\dfrac{3x^2-21x+32}{x^3-8x^2+16}dx=\int\dfrac{2}{x}dx+\int\dfrac{1}{x-4}dx-\int\dfrac{1}{(x-4)^2}dx=2\ln|x|+\ln|x-4|+\dfrac{1}{x-4}+C$

26. $\dfrac{x^2+19x+10}{2x^4+5x^3}=\dfrac{x^2+19x+10}{x^3(2x+5)}=\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x^3}+\dfrac{D}{2x+5}$

$A=-1, B=3, C=2, D=2$

$\int\dfrac{x^2+19x+10}{2x^4+5x^3}dx=\int\left(-\dfrac{1}{x}+\dfrac{3}{x^2}+\dfrac{2}{x^3}+\dfrac{2}{2x+5}\right)dx=-\ln|x|-\dfrac{3}{x}-\dfrac{1}{x^2}+\ln|2x+5|+C$

27. $\dfrac{2x^2+x-8}{x^3+4x}=\dfrac{2x^2+x-8}{x(x^2+4)}=\dfrac{A}{x}+\dfrac{Bx+C}{x^2+4}$

$A=-2, B=4, C=1$

$\int\dfrac{2x^2+x-8}{x^3+4x}dx=-2\int\dfrac{1}{x}dx+\int\dfrac{4x+1}{x^2+4}dx=-2\int\dfrac{1}{x}dx+2\int\dfrac{2x}{x^2+4}dx+\int\dfrac{1}{x^2+4}dx$

$=-2\ln|x|+2\ln|x^2+4|+\dfrac{1}{2}\tan^{-1}\left(\dfrac{x}{2}\right)+C$

28. $\dfrac{3x+2}{x(x+2)^2+16x}=\dfrac{3x+2}{x(x^2+4x+20)}=\dfrac{A}{x}+\dfrac{Bx+C}{x^2+4x+20}$

$A=\dfrac{1}{10}, B=-\dfrac{1}{10}, C=\dfrac{13}{5}$

$\int\dfrac{3x+2}{x(x+2)^2+16x}dx=\dfrac{1}{10}\int\dfrac{1}{x}dx+\int\dfrac{-\frac{1}{10}x+\frac{13}{5}}{x^2+4x+20}dx=\dfrac{1}{10}\int\dfrac{1}{x}dx+\dfrac{14}{5}\int\dfrac{1}{(x+2)^2+16}dx-\dfrac{1}{20}\int\dfrac{2x+4}{x^2+4x+20}dx$

$=\dfrac{1}{10}\ln|x|+\dfrac{7}{10}\tan^{-1}\left(\dfrac{x+2}{4}\right)-\dfrac{1}{20}\ln|x^2+4x+20|+C$

29. $\frac{2x^2-3x-36}{(2x-1)(x^2+9)}=\frac{A}{2x-1}+\frac{Bx+C}{x^2+9}$

$A=-4, B=3, C=0$

$\int\frac{2x^2-3x-36}{(2x-1)(x^2+9)}dx=-4\int\frac{1}{2x-1}dx+\int\frac{3x}{x^2+9}dx=-2\ln|2x-1|+\frac{3}{2}\ln\left|x^2+9\right|+C$

30. $\frac{1}{x^4-16}=\frac{1}{(x-2)(x+2)(x^2+4)}$

$=\frac{A}{x-2}+\frac{B}{x+2}+\frac{Cx+D}{x^2+4}$

$A=\frac{1}{32}, B=-\frac{1}{32}, C=0, D=-\frac{1}{8}$

$\int\frac{1}{x^4-16}dx=\frac{1}{32}\int\frac{1}{x-2}dx-\frac{1}{32}\int\frac{1}{x+2}dx-\frac{1}{8}\int\frac{1}{x^2+4}dx=\frac{1}{32}\ln|x-2|-\frac{1}{32}\ln|x+2|-\frac{1}{16}\tan^{-1}\left(\frac{x}{2}\right)+C$

31. $\frac{1}{(x-1)^2(x+4)^2}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+4}+\frac{D}{(x+4)^2}$

$A=-\frac{2}{125}, B=\frac{1}{25}, C=\frac{2}{125}, D=\frac{1}{25}$

$\int\frac{1}{(x-1)^2(x+4)^2}dx=-\frac{2}{125}\int\frac{1}{x-1}dx+\frac{1}{25}\int\frac{1}{(x-1)^2}dx+\frac{2}{125}\int\frac{1}{x+4}dx+\frac{1}{25}\int\frac{1}{(x+4)^2}dx$

$=-\frac{2}{125}\ln|x-1|-\frac{1}{25(x-1)}+\frac{2}{125}\ln|x+4|-\frac{1}{25(x+4)}+C$

32. $\frac{x^3-8x^2-1}{(x+3)(x^2-4x+5)}=1+\frac{-7x^2+7x-16}{(x+3)(x^2-4x+5)}$

$\frac{-7x^2+7x-16}{(x+3)(x^2-4x+5)}=\frac{A}{x+3}+\frac{Bx+C}{x^2-4x+5}$

$A=-\frac{50}{13}, B=-\frac{41}{13}, C=\frac{14}{13}$

$\int\frac{x^3-8x^2-1}{(x+3)(x^2-4x+5)}dx=\int\left[1-\frac{50}{13}\left(\frac{1}{x+3}\right)+\frac{-\frac{41}{13}x+\frac{14}{13}}{x^2-4x+5}\right]dx$

$=\int dx-\frac{50}{13}\int\frac{1}{x+3}dx-\frac{68}{13}\int\frac{1}{(x-2)^2+1}dx-\frac{41}{26}\int\frac{2x-4}{x^2-4x+5}dx$

$=x-\frac{50}{13}\ln|x+3|-\frac{68}{13}\tan^{-1}(x-2)-\frac{41}{26}\ln\left|x^2-4x+5\right|+C$

33. $x=\sin t$, $dx=\cos t\,dt$

$\int\frac{(\sin^3 t-8\sin^2 t-1)\cos t}{(\sin t+3)(\sin^2 t-4\sin t+5)}dt=\int\frac{x^3-8x^2-1}{(x+3)(x^2-4x+5)}dx$

$=x-\frac{50}{13}\ln|x+3|-\frac{68}{13}\tan^{-1}(x-2)-\frac{41}{26}\ln\left|x^2-4x+5\right|+C$

which is the result of Problem 32.

$\int\frac{(\sin^3 t-8\sin^2 t-1)\cos t}{(\sin t+3)(\sin^2 t-4\sin t+5)}dt=\sin t-\frac{50}{13}\ln|\sin t+3|-\frac{68}{13}\tan^{-1}(\sin t-2)-\frac{41}{26}\ln\left|\sin^2 t-4\sin t+5\right|+C$

34. $x = \sin t,\ dx = \cos t\,dt$

$$\int \frac{\cos t}{\sin^4 t - 16}\,dt = \int \frac{1}{x^4 - 16}\,dx = \frac{1}{32}\ln|x-2| - \frac{1}{32}\ln|x+2| - \frac{1}{16}\tan^{-1}\left(\frac{x}{2}\right) + C$$

which is the result of Problem 30.

$$\int \frac{\cos t}{\sin^4 t - 16}\,dt = \frac{1}{32}\ln|\sin t - 2| - \frac{1}{32}\ln|\sin t + 2| - \frac{1}{16}\tan^{-1}\left(\frac{\sin t}{2}\right) + C$$

35. $\dfrac{x^3 - 4x}{(x^2+1)^2} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{(x^2+1)^2}$

$A = 1, B = 0, C = -5, D = 0$

$$\int \frac{x^3 - 4x}{(x^2+1)^2}\,dx = \int \frac{x}{x^2+1}\,dx - 5\int \frac{x}{(x^2+1)^2}\,dx = \frac{1}{2}\ln\left|x^2+1\right| + \frac{5}{2(x^2+1)} + C$$

36. $x = \cos t,\ dx = -\sin t\,dt$

$$\int \frac{(\sin t)(4\cos^2 t - 1)}{(\cos t)(1 + 2\cos^2 t + \cos^4 t)}\,dt = -\int \frac{4x^2 - 1}{x(1 + 2x^2 + x^4)}\,dx$$

$$\frac{4x^2 - 1}{x(1 + 2x^2 + x^4)} = \frac{4x^2 - 1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$A = -1, B = 1, C = 0, D = 5, E = 0$

$$-\int\left[-\frac{1}{x} + \frac{x}{x^2+1} + \frac{5x}{(x^2+1)^2}\right]dx = \ln|x| - \frac{1}{2}\ln\left|x^2+1\right| + \frac{5}{2(x^2+1)} + C = \ln|\cos t| - \frac{1}{2}\ln\left|\cos^2 t + 1\right| + \frac{5}{2(\cos^2 t + 1)} + C$$

37. $\dfrac{2x^3 + 5x^2 + 16x}{x^5 + 8x^3 + 16x} = \dfrac{x(2x^2 + 5x + 16)}{x(x^4 + 8x^2 + 16)} = \dfrac{2x^2 + 5x + 16}{(x^2+4)^2} = \dfrac{Ax+B}{x^2+4} + \dfrac{Cx+D}{(x^2+4)^2}$

$A = 0, B = 2, C = 5, D = 8$

$$\int \frac{2x^3 + 5x^2 + 16x}{x^5 + 8x^3 + 16x}\,dx = \int \frac{2}{x^2+4}\,dx + \int \frac{5x+8}{(x^2+4)^2}\,dx = \int \frac{2}{x^2+4}\,dx + \int \frac{5x}{(x^2+4)^2}\,dx + \int \frac{8}{(x^2+4)^2}\,dx$$

To integrate $\int \frac{8}{(x^2+4)^2}\,dx$, let $x = 2\tan\theta,\ dx = 2\sec^2\theta\,d\theta$.

$$\int \frac{8}{(x^2+4)^2}\,dx = \int \frac{16\sec^2\theta}{16\sec^4\theta}\,d\theta = \int \cos^2\theta\,d\theta = \int\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right)d\theta$$

$$= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C = \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C = \frac{1}{2}\tan^{-1}\frac{x}{2} + \frac{x}{x^2+4} + C$$

$$\int \frac{2x^3 + 5x^2 + 16x}{x^5 + 8x^3 + 16x}\,dx = \tan^{-1}\frac{x}{2} - \frac{5}{2(x^2+4)} + \frac{1}{2}\tan^{-1}\frac{x}{2} + \frac{x}{x^2+4} + C = \frac{3}{2}\tan^{-1}\frac{x}{2} + \frac{2x-5}{2(x^2+4)} + C$$

38. $\dfrac{x-17}{x^2 + x - 12} = \dfrac{x-17}{(x+4)(x-3)} = \dfrac{A}{x+4} + \dfrac{B}{x-3}$

$A = 3, B = -2$

$$\int_4^6 \frac{x-17}{x^2+x-12}\,dx = \int_4^6\left(\frac{3}{x+4} - \frac{2}{x-3}\right)dx = \left[3\ln|x+4| - 2\ln|x-3|\right]_4^6 = (3\ln 10 - 2\ln 3) - (3\ln 8 - 2\ln 1)$$

$= 3\ln 10 - 2\ln 3 - 3\ln 8 \approx -1.53$

39. $u = \sin\theta,\ du = \cos\theta\,d\theta$

$$\int_0^{\pi/4} \frac{\cos\theta}{(1 - \sin^2\theta)(\sin^2\theta + 1)^2}\,d\theta = \int_0^{1/\sqrt{2}} \frac{1}{(1-u^2)(u^2+1)^2}\,du = \int_0^{1/\sqrt{2}} \frac{1}{(1-u)(1+u)(u^2+1)^2}\,du$$

$$\frac{1}{(1-u^2)(u^2+1)^2}=\frac{A}{1-u}+\frac{B}{1+u}+\frac{Cu+D}{u^2+1}+\frac{Eu+F}{(u^2+1)^2}$$

$$A=\frac{1}{8}, B=\frac{1}{8}, C=0, D=\frac{1}{4}, E=0, F=\frac{1}{2}$$

$$\int_0^{1/\sqrt{2}}\frac{1}{(1-u^2)(u^2+1)^2}du=\frac{1}{8}\int_0^{1/\sqrt{2}}\frac{1}{1-u}du+\frac{1}{8}\int_0^{1/\sqrt{2}}\frac{1}{1+u}du+\frac{1}{4}\int_0^{1/\sqrt{2}}\frac{1}{u^2+1}du+\frac{1}{2}\int_0^{1/\sqrt{2}}\frac{1}{(u^2+1)^2}du$$

$$=\left[-\frac{1}{8}\ln|1-u|+\frac{1}{8}\ln|1+u|+\frac{1}{4}\tan^{-1}u+\frac{1}{4}\left(\tan^{-1}u+\frac{u}{u^2+1}\right)\right]_0^{1/\sqrt{2}}=\left[\frac{1}{8}\ln\left|\frac{1+u}{1-u}\right|+\frac{1}{2}\tan^{-1}u+\frac{u}{4(u^2+1)}\right]_0^{1/\sqrt{2}}$$

$$=\frac{1}{8}\ln\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|+\frac{1}{2}\tan^{-1}\frac{1}{\sqrt{2}}+\frac{1}{6\sqrt{2}}\approx 0.65$$

(To integrate $\int\frac{1}{(u^2+1)^2}du$, let $u=\tan t$.)

40. $\frac{3x+13}{x^2+4x+3}=\frac{3x+13}{(x+3)(x+1)}=\frac{A}{x+3}+\frac{B}{x+1}$

$A=-2, B=5$

$$\int_1^5\frac{3x+13}{x^2+4x+3}dx=\left[-2\ln|x+3|+5\ln|x+1|\right]_1^5$$

$=-2\ln 8+5\ln 6+2\ln 4-5\ln 2\approx 4.11$

41. a. Separating variables, we obtain

$$\frac{dx}{(a-x)(b-x)}=k\,dt$$

$$\frac{1}{(a-x)(b-x)}=\frac{A}{a-x}+\frac{B}{b-x}$$

$$A=-\frac{1}{a-b}, B=\frac{1}{a-b}$$

$$\int\frac{dx}{(a-x)(b-x)}$$

$$=\frac{1}{a-b}\int\left(-\frac{1}{a-x}+\frac{1}{b-x}\right)dx=\int k\,dt$$

$$\frac{\ln|a-x|-\ln|b-x|}{a-b}=kt+C$$

$$\frac{1}{a-b}\ln\left|\frac{a-x}{b-x}\right|=kt+C$$

$$\frac{a-x}{b-x}=Ce^{(a-b)kt}$$

Since $x=0$ when $t=0$, $C=\frac{a}{b}$, so

$$a-x=(b-x)\frac{a}{b}e^{(a-b)kt}$$

$$a\left(1-e^{(a-b)kt}\right)=x\left(1-\frac{a}{b}e^{(a-b)kt}\right)$$

$$x(t)=\frac{a(1-e^{(a-b)kt})}{1-\frac{a}{b}e^{(a-b)kt}}=\frac{ab(1-e^{(a-b)kt})}{b-ae^{(a-b)kt}}$$

b. Since $b>a$ and $k>0$, $e^{(a-b)kt}\to 0$ as $t\to\infty$. Thus,

$$x\to\frac{ab(1)}{b-0}=a.$$

c. $x(t)=\frac{8(1-e^{-2kt})}{4-2e^{-2kt}}$

$x(20)=1$, so $4-2e^{-40k}=8-8e^{-40k}$

$6e^{-40k}=4$

$k=-\frac{1}{40}\ln\frac{2}{3}$

$$e^{-2kt}=e^{t/20\ln 2/3}=e^{\ln(2/3)^{t/20}}=\left(\frac{2}{3}\right)^{t/20}$$

$$x(t)=\frac{4\left(1-\left(\frac{2}{3}\right)^{t/20}\right)}{2-\left(\frac{2}{3}\right)^{t/20}}$$

$$x(60)=\frac{4\left(1-\left(\frac{2}{3}\right)^3\right)}{2-\left(\frac{2}{3}\right)^3}=\frac{38}{23}\approx 1.65 \text{ grams}$$

d. If $a=b$, the differential equation is, after separating variables

$$\frac{dx}{(a-x)^2}=k\,dt$$

$$\int\frac{dx}{(a-x)^2}=\int k\,dt$$

$$\frac{1}{a-x}=kt+C$$

$$\frac{1}{kt+C}=a-x$$

$$x(t)=a-\frac{1}{kt+C}$$

Since $x = 0$ when $t = 0$, $C = \frac{1}{a}$, so

$$x(t) = a - \frac{1}{kt + \frac{1}{a}} = a - \frac{a}{akt+1}$$

$$= a\left(1 - \frac{1}{akt+1}\right) = a\left(\frac{akt}{akt+1}\right).$$

42. a. $\frac{dy}{dt} = ky(16-y)$

$$\frac{dy}{y(16-y)} = kdt$$

$$\int \frac{dy}{y(16-y)} = \int kdt$$

$$\frac{1}{16}\int\left(\frac{1}{y} + \frac{1}{16-y}\right)dy = kt + C$$

$$\frac{1}{16}(\ln|y| - \ln|16-y|) = kt + C$$

$$\ln\left|\frac{y}{16-y}\right| = 16kt + C$$

$$\frac{y}{16-y} = Ce^{16kt}$$

$$y(0) = 2:\ \frac{1}{7} = C; \frac{y}{16-y} = \frac{1}{7}e^{16kt}$$

$$y(50) = 4:\ \frac{1}{3} = \frac{1}{7}e^{800k},\ \text{so}\ k = \frac{1}{800}\ln\frac{7}{3}$$

$$\frac{y}{16-y} = \frac{1}{7}e^{\left(\frac{1}{50}\ln\frac{7}{3}\right)t}$$

$$7y = 16e^{\left(\frac{1}{50}\ln\frac{7}{3}\right)t} - ye^{\left(\frac{1}{50}\ln\frac{7}{3}\right)t}$$

$$y = \frac{16e^{\left(\frac{1}{50}\ln\frac{7}{3}\right)t}}{7 + e^{\left(\frac{1}{50}\ln\frac{7}{3}\right)t}} = \frac{16}{1 + 7e^{-\left(\frac{1}{50}\ln\frac{7}{3}\right)t}}$$

b. $y(90) = \frac{16}{1 + 7e^{-\left(\frac{1}{50}\ln\frac{7}{3}\right)90}} \approx 6.34$ billion

c. $9 = \frac{16}{1 + 7e^{-\left(\frac{1}{50}\ln\frac{7}{3}\right)t}}$

$$7e^{-\left(\frac{1}{50}\ln\frac{7}{3}\right)t} = \frac{16}{9} - 1$$

$$e^{-\left(\frac{1}{50}\ln\frac{7}{3}\right)t} = \frac{1}{9}$$

$$-\left(\frac{1}{50}\ln\frac{7}{3}\right)t = \ln\frac{1}{9}$$

$$t = -50\left(\frac{\ln\frac{1}{9}}{\ln\frac{7}{3}}\right) \approx 129.66$$

The population will be 9 billion in 2055.

43. a. $\frac{dy}{dt} = ky(10-y)$

$$\frac{dy}{y(10-y)} = kdt$$

$$\frac{1}{10}\int\left(\frac{1}{y} + \frac{1}{10-y}\right)dy = \int kdt$$

$$\ln\left|\frac{y}{10-y}\right| = 10kt + C$$

$$\frac{y}{10-y} = Ce^{10kt}$$

$$y(0) = 2:\ \frac{1}{4} = C; \frac{y}{10-y} = \frac{1}{4}e^{10kt}$$

$$y(50) = 4:\ \frac{2}{3} = \frac{1}{4}e^{500k},\ k = \frac{1}{500}\ln\frac{8}{3}$$

$$\frac{y}{10-y} = \frac{1}{4}e^{\left(\frac{1}{50}\ln\frac{8}{3}\right)t}$$

$$4y = 10e^{\left(\frac{1}{50}\ln\frac{8}{3}\right)t} - ye^{\left(\frac{1}{50}\ln\frac{8}{3}\right)t}$$

$$y = \frac{10e^{\left(\frac{1}{50}\ln\frac{8}{3}\right)t}}{4 + e^{\left(\frac{1}{50}\ln\frac{8}{3}\right)t}} = \frac{10}{1 + 4e^{-\left(\frac{1}{50}\ln\frac{8}{3}\right)t}}$$

b. $y(90) = \frac{10}{1 + 4e^{-\left(\frac{1}{50}\ln\frac{8}{3}\right)90}} \approx 5.94$ billion

c. $9 = \frac{10}{1 + 4e^{-\left(\frac{1}{50}\ln\frac{8}{3}\right)t}}$

$$4e^{-\left(\frac{1}{50}\ln\frac{8}{3}\right)t} = \frac{10}{9} - 1$$

$$e^{-\left(\frac{1}{50}\ln\frac{8}{3}\right)t} = \frac{1}{36}$$

$$-\left(\frac{1}{50}\ln\frac{8}{3}\right)t = \ln\frac{1}{36}$$

$$t = -50\left(\frac{\ln\frac{1}{36}}{\ln\frac{8}{3}}\right) \approx 182.68$$

The population will be 9 billion in 2108.

44. Separating variables, we obtain

$$\frac{dy}{(y-m)(M-y)} = k\,dt.$$

$$\frac{1}{(y-m)(M-y)}=\frac{A}{y-m}+\frac{B}{M-y}$$

$$A=\frac{1}{M-m}, B=\frac{1}{M-m}$$

$$\int\frac{dy}{(y-m)(M-y)}=\frac{1}{M-m}\int\left(\frac{1}{y-m}+\frac{1}{M-y}\right)dy$$

$$=\int k\,dt$$

$$\frac{\ln|y-m|-\ln|M-y|}{M-m}=kt+C$$

$$\frac{1}{M-m}\ln\left|\frac{y-m}{M-y}\right|=kt+C$$

$$\frac{y-m}{M-y}=Ce^{(M-m)kt}$$

$$y-m=(M-y)Ce^{(M-m)kt}$$

$$y(1+Ce^{(M-m)kt}=m+MCe^{(M-m)kt}$$

$$y=\frac{m+MCe^{(M-m)kt}}{1+Ce^{(M-m)kt}}=\frac{me^{-(M-m)kt}+MC}{e^{-(M-m)kt}+C}$$

as $t\to\infty, e^{-(M-m)kt}\to 0$ since $M>m$.

Thus $y\to\frac{MC}{C}=M$ as $t\to\infty$.

45. Separating variables, we obtain

$$\frac{dy}{(A-y)(B+y)}=k\,dt$$

$$\frac{1}{(A-y)(B+y)}=\frac{C}{A-y}+\frac{D}{B+y}$$

$$C=\frac{1}{A+B}, D=\frac{1}{A+B}$$

$$\int\frac{dy}{(A-y)(B+y)}=\frac{1}{A+B}\int\left(\frac{1}{A-y}+\frac{1}{B+y}\right)dy$$

$$=\int k\,dt$$

$$\frac{-\ln(A-y)+\ln(B+y)}{A+B}=kt+C$$

$$\frac{1}{A+B}\ln\left|\frac{B+y}{A-y}\right|=kt+C$$

$$\frac{B+y}{A-y}=Ce^{(A+B)kt}$$

$$B+y=(A-y)Ce^{(A+B)kt}$$

$$y(1+Ce^{(A+B)kt})=ACe^{(A+B)kt}-B$$

$$y(t)=\frac{ACe^{(A+B)kt}-B}{1+Ce^{(A+B)kt}}$$

46. $u=\sin x, du=\cos x\,dx$

$$\int_{\pi/6}^{\pi/2}\frac{\cos x}{\sin x(\sin^2 x+1)^2}dx=\int_{\frac{1}{2}}^{1}\frac{1}{x(x^2+1)}dx$$

$$\frac{1}{x(x^2+1)^2}=\frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}$$

$A=1, B=-1, C=0, D=-1, E=0$

$$\int_{\frac{1}{2}}^{1}\frac{1}{x(x^2+1)^2}dx$$

$$=\int_{\frac{1}{2}}^{1}\frac{1}{x}dx-\int_{\frac{1}{2}}^{1}\frac{x}{x^2+1}dx-\int_{\frac{1}{2}}^{1}\frac{x}{(x^2+1)^2}dx$$

$$=\left[\ln x-\frac{1}{2}\ln(x^2+1)+\frac{1}{2(x^2+1)}\right]_{\frac{1}{2}}^{1}$$

$$=0-\frac{1}{2}\ln 2+\frac{1}{4}-\left(\ln\frac{1}{2}-\frac{1}{2}\ln\frac{5}{4}+\frac{2}{5}\right)\approx 0.308$$

8.6 Chapter Review

Concepts Test

1. True: The resulting integrand will be of the form $\sin u$.

2. True: The resulting integrand will be of the form $\frac{1}{a^2+u^2}$.

3. False: Try the substitution $u=x^4, du=4x^3\,dx$

4. False: Use the substitution $u=x^2-3x+5$, $du=(2x-3)dx$.

5. True: The resulting integrand will be of the form $\frac{1}{a^2+u^2}$.

6. True: The resulting integrand will be of the form $\frac{1}{\sqrt{a^2-x^2}}$.

7. True: This integral is most easily solved with a partial fraction decomposition.

8. False: This improper fraction should be reduced first, then a partial fraction decomposition can be used.

9. True: Because both exponents are even positive integers, half-angle formulas are used.

10. False: Use the substitution $u = 1 + e^x,\ du = e^x dx$

11. False: Use the substitution $u = -x^2 - 4x, du = (-2x - 4)dx$

12. True: This substitution eliminates the radical.

13. True: Then expand and use the substitution $u = \sin x$, $du = \cos x\, dx$

14. True: The trigonometric substitution $x = 3\sin t$ will eliminate the radical.

15. True: Let $u = \ln x$ $\quad dv = x^2 dx$

$du = \frac{1}{x}dx \quad v = \frac{1}{3}x^3$

16. False: Use a product identity.

17. False: $\frac{x^2}{x^2-1} = 1 + \frac{1}{2(x-1)} - \frac{1}{2(x+1)}$

18. True: $\frac{x^2+2}{x(x^2-1)} = -\frac{2}{x} + \frac{3}{2(x+1)} + \frac{3}{2(x-1)}$

19. True: $\frac{x^2+2}{x(x^2+1)} = \frac{2}{x} + \frac{-x}{x^2+1}$

20. False: $\frac{x+2}{x^2(x^2-1)}$

$= -\frac{1}{x} - \frac{2}{x^2} + \frac{3}{2(x-1)} - \frac{1}{2(x+1)}$

21. False: To complete the square, add $\left(\frac{b}{2a}\right)^2$.

22. False: Polynomials can be factored into products of linear and quadratic polynomials with real coefficients.

23. True: Polynomials with the same values for all x will have identical coefficients for like degree terms.

Sample Test Problems

1. $\int_0^4 \frac{t}{\sqrt{9+t^2}}dt = \left[\sqrt{9+t^2}\right]_0^4 = 5 - 3 = 2$

2. $\int \cot^2(2\theta)d\theta = \int \frac{\cos^2 2\theta}{\sin^2 2\theta}d\theta$

$= \int \frac{1-\sin^2 2\theta}{\sin^2 2\theta}d\theta = \int(\csc^2 2\theta - 1)d\theta$

$= -\frac{1}{2}\cot 2\theta - \theta + C$

3. $\int_0^{\pi/2} e^{\cos x}\sin x\,dx = \left[-e^{\cos x}\right]_0^{\pi/2} = e - 1 \approx 1.718$

4. $\int_0^{\pi/4} x\sin 2x\,dx = \left[\frac{\sin 2x}{4} - \frac{x}{2}\cos 2x\right]_0^{\pi/4} = \frac{1}{4}$

(Use integration by parts with $u = x$, $dv = \sin 2x\,dx$.)

5. $\int \frac{y^3+y}{y+1}dy = \int\left(y^2 - y + 2 - \frac{2}{1+y}\right)dy$

$= \frac{1}{3}y^3 - \frac{1}{2}y^2 + 2y - 2\ln|1+y| + C$

6. $\int \sin^3(2t)dt = \int[1-\cos^2(2t)]\sin(2t)dt$

$= -\frac{1}{2}\cos(2t) + \frac{1}{6}\cos^3(2t) + C$

7. $\int \frac{y-2}{y^2-4y+2}dy = \frac{1}{2}\int\frac{2y-4}{y^2-4y+2}dy$

$= \frac{1}{2}\ln\left|y^2-4y+2\right| + C$

8. $\int_0^{3/2} \frac{dy}{\sqrt{2y+1}} = \left[\sqrt{2y+1}\right]_0^{3/2} = 2 - 1 = 1$

9. $\int \frac{e^{2t}}{e^t - 2}dt = e^t + 2\ln\left|e^t - 2\right| + C$

(Use the substitution $u = e^t - 2$, $du = e^t dt$ which gives the integral $\int\frac{u+2}{u}du$.)

10. $\int \frac{\sin x + \cos x}{\tan x}dx = \int\left(\cos x + \frac{\cos^2 x}{\sin x}\right)dx$

$= \int\left(\cos x + \frac{1-\sin^2 x}{\sin x}\right)dx$

$= \int (\cos x + \csc x - \sin x)dx$

$= \sin x + \ln|\csc x - \cot x| + \cos x + C$

(Use Formula 15 for $\int \csc x\, dx$.)

11. $\int \frac{dx}{\sqrt{16+4x-2x^2}} = \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{x-1}{3}\right) + C$

(Complete the square.)

12. $\int x^2 e^x dx = e^x(2-2x+x^2)+C$

Use integration by parts twice.

13. $y = \sqrt{\frac{2}{3}} \tan t,\ dy = \sqrt{\frac{2}{3}} \sec^2 t\, dt$

$\int \frac{dy}{\sqrt{2+3y^2}} = \int \frac{\sqrt{\frac{2}{3}} \sec^2 t}{\sqrt{2} \sec t} dt$

$= \frac{1}{\sqrt{3}} \int \sec t\, dt = \frac{1}{\sqrt{3}} \ln|\sec t + \tan t| + C_1$

$= \frac{1}{\sqrt{3}} \ln\left|\frac{\sqrt{y^2+\frac{2}{3}}}{\sqrt{\frac{2}{3}}} + \frac{y}{\sqrt{\frac{2}{3}}}\right| + C_1$

$= \frac{1}{\sqrt{3}} \ln\left|\frac{\sqrt{y^2+\frac{2}{3}}+y}{\sqrt{\frac{2}{3}}}\right| + C_1$

$= \frac{1}{\sqrt{3}} \ln\left|\sqrt{y^2+\frac{2}{3}}+y\right| + C$

Note that $\tan t = \frac{y}{\sqrt{\frac{2}{3}}}$, so $\sec t = \frac{\sqrt{y^2+\frac{2}{3}}}{\sqrt{\frac{2}{3}}}$.

14. $\int \frac{w^3}{1-w^2} dw = -\frac{1}{2}w^2 - \frac{1}{2}\ln\left|1-w^2\right| + C$

Divide the numerator by the denominator.

15. $\int \frac{\tan x}{\ln|\cos x|} dx = -\ln\left|\ln|\cos x|\right| + C$

Use the substitution $u = \ln|\cos x|$.

16. $\int \frac{3dt}{t^3-1} = \int \frac{1}{t-1} dt - \int \frac{t+2}{t^2+t+1} dt$

$= \int \frac{1}{t-1} dt - \frac{1}{2} \int \frac{2t+4}{t^2+t+1} dt$

$= \int \frac{1}{t-1} dt - \frac{1}{2} \int \frac{2t+1+3}{t^2+t+1} dt$

$= \int \frac{1}{t-1} dt - \frac{1}{2} \int \frac{2t+1}{t^2+t+1} dt - \frac{3}{2} \int \frac{1}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}} dt$

$= \ln|t-1| - \frac{1}{2}\ln\left|t^2+t+1\right| - \sqrt{3}\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right) + C$

17. $\int \sinh x\, dx = \cosh x + C$

18. $u = \ln y,\ du = \frac{1}{y} dy$

$\int \frac{(\ln y)^5}{y} dy = \int u^5 du = \frac{1}{6}(\ln y)^6 + C$

19. $u = x \qquad dv = \cot^2 x\, dx$

$du = dx \qquad v = -\cot x - x$

$\int x\cot^2 x\, dx = -x\cot x - x^2 - \int (-\cot x - x)dx$

$= -x\cot x - \frac{1}{2}x^2 + \ln|\sin x| + C$

Use $\cot^2 x = \csc^2 x - 1$ for $\int \cot^2 x\, dx$.

20. $u = \sqrt{x},\ du = \frac{1}{2}x^{-1/2}dx$

$\int \frac{\sin\sqrt{x}}{\sqrt{x}} dx = 2\int \sin u\, du$

$= -2\cos\sqrt{x} + C$

21. $u = \ln t^2, du = \frac{2}{t} dt$

$\int \frac{\ln t^2}{t} dt = \frac{[\ln(t^2)]^2}{4} + C$

22. $u = \ln(y^2+9) \qquad dv = dy$

$du = \frac{2y}{y^2+9} dy \qquad v = y$

$\int \ln(y^2+9)dy = y\ln(y^2+9) - \int \frac{2y^2}{y^2+9} dy$

$= y\ln(y^2+9) - \int \left(2 - \frac{18}{y^2+9}\right) dy$

$= y\ln(y^2+9) - 2y + 6\tan^{-1}\left(\frac{y}{3}\right) + C$

23. $\int e^{t/3} \sin 3t\, dt = \frac{-3e^{t/3}(9\cos 3t - \sin 3t)}{82} + C$

Use integration by parts twice.

24. $\int \frac{t+9}{t^3+9t}dt = \int \frac{1}{t}dt + \int \frac{-t+1}{t^2+9}dt$

$= \int \frac{1}{t}dt - \int \frac{t}{t^2+9}dt + \int \frac{1}{t^2+9}dt$

$= \ln|t| - \frac{1}{2}\ln\left|t^2+9\right| + \frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right) + C$

25. $\int \sin\frac{3x}{2}\cos\frac{x}{2}dx = -\frac{\cos x}{2} - \frac{\cos 2x}{4} + C$

Use a product identity.

26. $\int \cos^4\left(\frac{x}{2}\right)dx = \int\left(\frac{1+\cos x}{2}\right)^2 dx$

$= \frac{1}{4}\int dx + \frac{1}{4}\int 2\cos x\,dx + \frac{1}{4}\int \cos^2 x\,dx$

$= \frac{1}{4}\int dx + \frac{1}{2}\int \cos x\,dx + \frac{1}{8}\int(1+\cos 2x)dx$

$= \frac{3}{8}x + \frac{1}{2}\sin x + \frac{1}{16}\sin 2x + C$

27. $\int \tan^3 2x\sec 2x\,dx = \frac{1}{2}\int(\sec^2 2x - 1)d(\sec 2x)$

$= \frac{1}{6}\sec^3(2x) - \frac{1}{2}\sec(2x) + C$

28. $u = \sqrt{x}, du = \frac{1}{2\sqrt{x}}dx$

$\int \frac{\sqrt{x}}{1+\sqrt{x}}dx = \int \frac{2x}{1+\sqrt{x}}\left(\frac{1}{2\sqrt{x}}dx\right) = 2\int \frac{u^2}{1+u}du$

$= 2\int \frac{(u+1)(u-1)+1}{u+1}du = 2\int\left(u-1+\frac{1}{u+1}\right)du$

$= 2\left(\frac{u^2}{2} - u + \ln|u+1|\right) + C$

$= x - 2\sqrt{x} + 2\ln\left|1+\sqrt{x}\right| + C$

29. $\int \tan^{3/2}x\sec^4 x\,dx = \int \tan^{3/2}x(1+\tan^2 x)\sec^2 x\,dx = \int \tan^{3/2}x\sec^2 x\,dx + \int \tan^{7/2}x\sec^2 x\,dx$

$= \frac{2}{5}\tan^{5/2}x + \frac{2}{9}\tan^{9/2}x + C$

30. $u = t^{1/6}+1, (u-1)^6 = t, 6(u-1)^5 du = dt$

$\int \frac{dt}{t(t^{1/6}+1)} = \int \frac{6(u-1)^5 du}{(u-1)^6 u} = \int \frac{6du}{u(u-1)} = -6\int \frac{1}{u}du + 6\int \frac{1}{u-1}du = -6\ln\left|t^{1/6}+1\right| + 6\ln\left|t^{1/6}\right| + C$

31. $u = 9 - e^{2y}$, $du = -2e^{2y}dy$

$\int \frac{e^{2y}}{\sqrt{9-e^{2y}}}dy = -\frac{1}{2}\int u^{-1/2}du = -\sqrt{u} + C = -\sqrt{9-e^{2y}} + C$

32. $\int \cos^5 x\sqrt{\sin x}dx = \int(1-\sin^2 x)^2(\sin^{1/2}x)\cos x\,dx = \int \sin^{1/2}x\cos x\,dx - 2\int \sin^{5/2}x\cos x\,dx + \int \sin^{9/2}x\cos x\,dx$

$= \frac{2}{3}\sin^{3/2}x - \frac{4}{7}\sin^{7/2}x + \frac{2}{11}\sin^{11/2}x + C$

33. $\int e^{\ln(3\cos x)}dx = \int 3\cos x\,dx = 3\sin x + C$

34. $y = 3\sin t$, $dy = 3\cos t\,dt$

$\int \frac{\sqrt{9-y^2}}{y}dy = \int \frac{3\cos t}{3\sin t}\cdot 3\cos t\,dt$

$= 3\int \frac{1-\sin^2 t}{\sin t} = 3\int(\csc t - \sin t)dt$

$= 3\left[\ln\left|\csc t - \cot t\right| + \cos t\right] + C$

$= 3\ln\left|\frac{3}{y} - \frac{\sqrt{9-y^2}}{y}\right| + \sqrt{9-y^2} + C$

Note that $\sin t = \frac{y}{3}$, so $\csc t = \frac{3}{y}$ and $\cot t = \frac{\sqrt{9-y^2}}{y}$.

35. $u = e^{4x}$, $du = 4e^{4x}dx$

$$\int \frac{e^{4x}}{1+e^{8x}}dx = \frac{1}{4}\int \frac{du}{1+u^2} = \frac{1}{4}\tan^{-1}(e^{4x}) + C$$

36. $x = a \tan t$, $dx = a\sec^2 t\, dt$

$$\int \frac{\sqrt{x^2+a^2}}{x^4}dx = \int \frac{a\sec t}{a^4 \tan^4 t} a\sec^2 t\, dt$$

$$= \frac{1}{a^2}\int \frac{\sec^3 t}{\tan^4 t}dt = \frac{1}{a^2}\int \frac{\cos t}{\sin^4 t}dt$$

$$= \frac{1}{a^2}\left(-\frac{1}{3}\frac{1}{\sin^3 t}\right) + C = -\frac{1}{3a^2}\csc^3 t + C$$

$$= -\frac{1}{3a^2}\frac{(x^2+a^2)^{3/2}}{x^3} + C$$

Note that $\tan t = \frac{x}{a}$, so $\csc t = \frac{\sqrt{x^2+a^2}}{x}$.

37. $u = \sqrt{w+5}, u^2 = w+5$, $2u\, du = dw$

$$\int \frac{w}{\sqrt{w+5}}dw = 2\int (u^2-5)du = \frac{2}{3}u^3 - 10u + C$$

$$= \frac{2}{3}(w+5)^{3/2} - 10(w+5)^{1/2} + C$$

38. $u = 1 + \cos t$, $du = -\sin t\, dt$

$$\int \frac{\sin t\, dt}{\sqrt{1+\cos t}} = -\int \frac{du}{\sqrt{u}} = -2\sqrt{1+\cos t} + C$$

39. $u = \cos^2 y$, $du = -2\cos y \sin y\, dy$

$$\int \frac{\sin y \cos y}{9+\cos^4 y}dy = -\frac{1}{2}\int \frac{du}{9+u^2}$$

$$= -\frac{1}{6}\tan^{-1}\left(\frac{\cos^2 y}{3}\right) + C$$

40. $$\int \frac{dx}{\sqrt{1-6x-x^2}} = \int \frac{dx}{\sqrt{10-(x+3)^2}}$$

$$= \sin^{-1}\left(\frac{x+3}{\sqrt{10}}\right) + C$$

41. $$\frac{4x^2+3x+6}{x^2(x^2+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+3}$$

$A = 1, B = 2, C = -1, D = 2$

$$\int \frac{4x^2+3x+6}{x^2(x^2+3)}dx = \int \frac{1}{x}dx + 2\int \frac{1}{x^2}dx + \int \frac{-x+2}{x^2+3}dx$$

$$= \int \frac{1}{x}dx + 2\int \frac{1}{x^2}dx - \frac{1}{2}\int \frac{2x}{x^2+3}dx + 2\int \frac{1}{x^2+3}dx$$

$$= \ln|x| - \frac{2}{x} - \frac{1}{2}\ln\left|x^2+3\right| + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$

42. $x = 4\tan t$, $dx = 4\sec^2 t\, dt$

$$\int \frac{dx}{(16+x^2)^{3/2}} = \frac{1}{16}\int \cos t\, dt = \frac{1}{16}\sin t + C = \frac{1}{16}\left(\frac{x}{\sqrt{x^2+16}}\right) + C = \frac{x}{16\sqrt{x^2+16}} + C$$

43. a. $$\frac{3-4x^2}{(2x+1)^3} = \frac{A}{2x+1} + \frac{B}{(2x+1)^2} + \frac{C}{(2x+1)^3}$$

b. $$\frac{7x-41}{(x-1)^2(2-x)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{2-x} + \frac{D}{(2-x)^2} + \frac{E}{(2-x)^3}$$

c. $$\frac{3x+1}{(x^2+x+10)^2} = \frac{Ax+B}{x^2+x+10} + \frac{Cx+D}{(x^2+x+10)^2}$$

d. $$\frac{(x+1)^2}{(x^2-x+10)^2(1-x^2)^2} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1+x} + \frac{D}{(1+x)^2} + \frac{Ex+F}{x^2-x+10} + \frac{Gx+H}{(x^2-x+10)^2}$$

e. $$\frac{x^5}{(x+3)^4(x^2+2x+10)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{(x+3)^3} + \frac{D}{(x+3)^4} + \frac{Ex+F}{x^2+2x+10} + \frac{Gx+H}{(x^2+2x+10)^2}$$

f. $\dfrac{(3x^2+2x-1)^2}{(2x^2+x+10)^3} = \dfrac{Ax+B}{2x^2+x+10} + \dfrac{Cx+D}{(2x^2+x+10)^2} + \dfrac{Ex+F}{(2x^2+x+10)^3}$

44. a. $V = \pi\int_1^2 \left[\dfrac{1}{\sqrt{3x-x^2}}\right]^2 dx = \pi\int_1^2 \dfrac{1}{3x-x^2}dx$

$\dfrac{1}{3x-x^2} = \dfrac{A}{x} + \dfrac{B}{3-x}$

$A = \dfrac{1}{3}, B = \dfrac{1}{3}$

$V = \pi\int_1^2 \dfrac{1}{3}\left(\dfrac{1}{x} + \dfrac{1}{3-x}\right)dx = \dfrac{\pi}{3}\Big[\ln|x| - \ln|3-x|\Big]_1^2 = \dfrac{\pi}{3}(\ln 2 + \ln 2) = \dfrac{2\pi}{3}\ln 2 \approx 1.4517$

b. $V = 2\pi\int_1^2 \dfrac{x}{\sqrt{3x-x^2}}dx = -\pi\int_1^2 \dfrac{-2x+3-3}{\sqrt{3x-x^2}}dx = -\pi\int_1^2 \dfrac{3-2x}{\sqrt{3x-x^2}}dx + 3\pi\int_1^2 \dfrac{1}{\sqrt{3x-x^2}}dx$

$= -\pi\Big[2\sqrt{3x-x^2}\Big]_1^2 + 3\pi\int_1^2 \dfrac{1}{\sqrt{\frac{9}{4}-\left(x-\frac{3}{2}\right)^2}}dx = \left[-2\pi\sqrt{3x-x^2} + 3\pi\sin^{-1}\left(\dfrac{2x-3}{3}\right)\right]_1^2$

$= -2\pi\sqrt{2} + 3\pi\sin^{-1}\dfrac{1}{3} + 2\pi\sqrt{2} - 3\pi\sin^{-1}\left(-\dfrac{1}{3}\right) = 6\pi\sin^{-1}\dfrac{1}{3} \approx 6.4058$

45. $y = \dfrac{x^2}{16}, y' = \dfrac{x}{8}$

$L = \int_0^4 \sqrt{1+\left(\dfrac{x}{8}\right)^2}dx = \int_0^4 \sqrt{1+\dfrac{x^2}{64}}dx$

$x = 8\tan t,\ dx = 8\sec^2 t$

$L = \int_0^{\tan^{-1}\frac{1}{2}} \sec t \cdot 8\sec^2 t\,dt = 8\int_0^{\tan^{-1}\frac{1}{2}} \sec^3 t\,dt = 4\Big[\sec t\tan t + \ln|\sec t + \tan t|\Big]_0^{\tan^{-1}\frac{1}{2}}$

$= 4\left[\left(\dfrac{\sqrt{5}}{2}\right)\left(\dfrac{1}{2}\right) + \ln\left|\dfrac{1}{2} + \dfrac{\sqrt{5}}{2}\right|\right] = \sqrt{5} + 4\ln\left(\dfrac{1+\sqrt{5}}{2}\right) \approx 4.1609$

Use Formula 28 for $\int \sec^3 t\,dt$.

46. $V = \pi\int_0^3 \dfrac{1}{(x^2+5x+6)^2}dx = \pi\int_0^3 \dfrac{1}{(x+3)^2(x+2)^2}dx$

$\dfrac{1}{(x+3)^2(x+2)^2} = \dfrac{A}{x+3} + \dfrac{B}{(x+3)^2} + \dfrac{C}{x+2} + \dfrac{D}{(x+2)^2}$

$A = 2, B = 1, C = -2, D = 1$

$V = \pi\int_0^3 \left[\dfrac{2}{x+3} + \dfrac{1}{(x+3)^2} - \dfrac{2}{x+2} + \dfrac{1}{(x+2)^2}\right]dx = \pi\left[2\ln|x+3| - \dfrac{1}{x+3} - 2\ln|x+2| - \dfrac{1}{x+2}\right]_0^3$

$= \pi\left[\left(2\ln 6 - \dfrac{1}{6} - 2\ln 5 - \dfrac{1}{5}\right) - \left(2\ln 3 - \dfrac{1}{3} - 2\ln 2 - \dfrac{1}{2}\right)\right] = \pi\left(\dfrac{7}{15} + 2\ln\dfrac{4}{5}\right) \approx 0.06402$

47. $V = 2\pi\int_0^3 \dfrac{x}{x^2+5x+6}dx$

$\dfrac{x}{x^2+5x+6} = \dfrac{A}{x+2} + \dfrac{B}{x+3}$

$A = -2, B = 3$

$$V = 2\pi \int_0^3 \left[-\frac{2}{x+2} + \frac{3}{x+3} \right] dx = 2\pi[-2\ln(x+2) + 3\ln(x+3)]_0^3$$

$$= 2\pi[(-2\ln 5 + 3\ln 6) - (-2\ln 2 + 3\ln 3)] = 2\pi\left(3\ln 2 + 2\ln\frac{2}{5}\right) = 2\pi \ln\frac{32}{25} \approx 1.5511$$

48. $V = 2\pi \int_0^2 4x^2\sqrt{2-x}\,dx$

$u = 2 - x \qquad du = -dx$
$x = 2 - u \qquad dx = -du$

$$V = 2\pi \int_2^0 4(2-u)^2\sqrt{u}(-du) = 8\pi \int_0^2 (4u^{1/2} - 4u^{3/2} + u^{5/2})du = 8\pi\left[\frac{8}{3}u^{3/2} - \frac{8}{5}u^{5/2} + \frac{2}{7}u^{7/2}\right]_0^2$$

$$= 8\pi\left(\frac{16\sqrt{2}}{3} - \frac{32\sqrt{2}}{5} + \frac{16\sqrt{2}}{7}\right) = 8\pi\left(\frac{128\sqrt{2}}{105}\right) = \frac{1024\sqrt{2}\pi}{105} \approx 43.3287$$

49. $V = 2\pi \int_0^{\ln 3} 2(e^x - 1)(\ln 3 - x)dx = 4\pi \int_0^{\ln 3} [(\ln 3)e^x - xe^x - \ln 3 + x]dx$

Note that $\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$ by using integration by parts.

$$V = 4\pi\left[(\ln 3)e^x - xe^x + e^x - (\ln 3)x + \frac{1}{2}x^2\right]_0^{\ln 3} = 4\pi\left[\left(3\ln 3 - 3\ln 3 + 3 - (\ln 3)^2 + \frac{1}{2}(\ln 3)^2\right) - (\ln 3 + 1)\right]$$

$$= 4\pi\left[2 - \ln 3 - \frac{1}{2}(\ln 3)^2\right] \approx 3.7437$$

50. $A = \int_{\sqrt{3}}^{3\sqrt{3}} \frac{18}{x^2\sqrt{x^2+9}}dx$

$x = 3\tan t,\ dx = 3\sec^2 t\,dt$

$$A = \int_{\pi/6}^{\pi/3} \frac{18}{27\tan^2 t\sec t} 3\sec^2 t\,dt = 2\int_{\pi/6}^{\pi/3} \frac{\cos t}{\sin^2 t}dt = 2\left[-\frac{1}{\sin t}\right]_{\pi/6}^{\pi/3} = 2\left(-\frac{2}{\sqrt{3}} + 2\right) = 4\left(1 - \frac{1}{\sqrt{3}}\right) \approx 1.6906$$

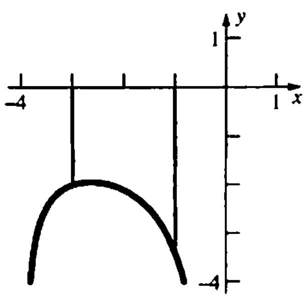
51. $A = -\int_{-6}^0 \frac{t}{(t-1)^2}dt$

$$\frac{t}{(t-1)^2} = \frac{A}{(t-1)} + \frac{B}{(t-1)^2}$$

$A = 1, B = 1$

$$A = -\int_{-6}^0 \left[\frac{1}{t-1} + \frac{1}{(t-1)^2}\right]dt = -\left[\ln|t-1| - \frac{1}{t-1}\right]_{-6}^0 = -\left[(0+1) - \left(\ln 7 + \frac{1}{7}\right)\right] = \ln 7 - \frac{6}{7} \approx 1.0888$$

52.



$$V = \pi\int_{-3}^{-1}\left(\frac{6}{x\sqrt{x+4}}\right)^2 dx = \pi\int_{-3}^{-1}\frac{36}{x^2(x+4)}dx$$

$$\frac{36}{x^2(x+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+4}$$

$$A = -\frac{9}{4}, B = 9, C = \frac{9}{4}$$

$$V = \pi\int_{-3}^{-1}\left[-\frac{9}{4x} + \frac{9}{x^2} + \frac{9}{4(x+4)}\right]dx = \frac{9\pi}{4}\int_{-3}^{-1}\left(-\frac{1}{x} + \frac{4}{x^2} + \frac{1}{x+4}\right)dx = \frac{9\pi}{4}\left[-\ln|x| - \frac{4}{x} + \ln|x+4|\right]_{-3}^{-1}$$

$$= \frac{9\pi}{4}\left[(4+\ln 3) - \left(-\ln 3 + \frac{4}{3}\right)\right] = \frac{9\pi}{4}\left(\frac{8}{3} + 2\ln 3\right) = \frac{3\pi}{2}(4 + 3\ln 3) \approx 34.3808$$

53. The length is given by

$$\int_{\pi/6}^{\pi/3}\sqrt{1+[f'(x)]^2}\,dx = \int_{\pi/6}^{\pi/3}\sqrt{1+\frac{\cos^2 x}{\sin^2 x}}\,dx = \int_{\pi/6}^{\pi/3}\sqrt{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}}\,dx = \int_{\pi/6}^{\pi/3}\frac{1}{\sin x}dx = \int_{\pi/6}^{\pi/3}\csc x\,dx$$

$$= \left[\ln|\csc x - \cot x|\right]_{\pi/6}^{\pi/3} = \ln\left|\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right| - \ln\left|2-\sqrt{3}\right| = \ln\left(\frac{1}{\sqrt{3}}\right) - \ln(2-\sqrt{3}) = \ln\left(\frac{2\sqrt{3}+3}{3}\right) \approx 0.768$$