CHAPTER

9 Indeterminate Forms and Improper Integrals

9.1 Concepts Review

1. $\lim_{x\to a} f(x)$; $\lim_{x\to a} g(x)$

2. $\dfrac{f'(x)}{g'(x)}$

3. $\sec^2 x$; 1; $\lim_{x\to 0} \cos x \neq 0$

4. Cauchy's Mean Value

Problem Set 9.1

1. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0} \frac{2x - \sin x}{x} = \lim_{x\to 0} \frac{2 - \cos x}{1} = 1$$

2. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to \pi/2} \frac{\cos x}{\pi/2 - x} = \lim_{x\to \pi/2} \frac{-\sin x}{-1} = 1$$

3. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0} \frac{x - \sin 2x}{\tan x} = \lim_{x\to 0} \frac{1 - 2\cos 2x}{\sec^2 x} = \frac{1-2}{1} = -1$$

4. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0} \frac{\tan^{-1} 3x}{\sin^{-1} x} = \lim_{x\to 0} \frac{\frac{3}{1+9x^2}}{\frac{1}{\sqrt{1-x^2}}} = \frac{3}{1} = 3$$

5. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to -2} \frac{x^2 + 6x + 8}{x^2 - 3x - 10} = \lim_{x\to -2} \frac{2x+6}{2x-3}$$

$$= \frac{2}{-7} = -\frac{2}{7}$$

6. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0} \frac{x^3 - 3x^2 + x}{x^3 - 2x} = \lim_{x\to 0} \frac{3x^2 + 6x + 1}{3x^2 - 2} = \frac{1}{-2} = -\frac{1}{2}$$

7. The limit is not of the form $\frac{0}{0}$.

As $x \to 1^-$, $x^2 - 2x + 2 \to 1$, and $x^2 - 1 \to 0^-$ so

$$\lim_{x\to 1^-} \frac{x^2 - 2x + 2}{x^2 + 1} = -\infty$$

8. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 1} \frac{\ln x^2}{x^2 - 1} = \lim_{x\to 1} \frac{\frac{1}{x^2} 2x}{2x} = \lim_{x\to 1} \frac{1}{x^2} = 1$$

9. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to \pi/2} \frac{\ln(\sin x)^3}{\pi/2 - x} = \lim_{x\to \pi/2} \frac{\frac{1}{\sin^3 x} 3\sin^2 x \cos x}{-1}$$

$$= \frac{0}{-1} = 0$$

10. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0} \frac{e^x - e^{-x}}{2\sin x} = \lim_{x\to 0} \frac{e^x + e^{-x}}{2\cos x} = \frac{2}{2} = 1$$

11. The limit is of the form $\frac{0}{0}$.

$$\lim_{t\to 1} \frac{\sqrt{t} - t^2}{\ln t} = \lim_{t\to 1} \frac{\frac{1}{2\sqrt{t}} - 2t}{\frac{1}{t}} = \frac{-\frac{3}{2}}{1} = -\frac{3}{2}$$

12. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0^+}\frac{7^{\sqrt{x}}-1}{2^{\sqrt{x}}-1}=\lim_{x\to 0^+}\frac{\frac{7^{\sqrt{x}}\ln 7}{2\sqrt{x}}}{\frac{2^{\sqrt{x}}\ln 2}{2\sqrt{x}}}=\lim_{x\to 0^+}\frac{7^{\sqrt{x}}\ln 7}{2^{\sqrt{x}}\ln 2}$$

$$=\frac{\ln 7}{\ln 2}\approx 2.81$$

13. The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule twice.)

$$\lim_{x\to 0}\frac{\ln\cos 2x}{7x^2}=\lim_{x\to 0}\frac{\frac{-2\sin 2x}{\cos 2x}}{14x}=\lim_{x\to 0}\frac{-2\sin 2x}{14x\cos 2x}$$

$$=\lim_{x\to 0}\frac{-4\cos 2x}{14\cos 2x-28x\sin 2x}=\frac{-4}{14-0}=-\frac{2}{7}$$

14. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0^-}\frac{3\sin x}{\sqrt{-x}}=\lim_{x\to 0^-}\frac{3\cos x}{-\frac{1}{2\sqrt{-x}}}$$

$$=\lim_{x\to 0^-}-6\sqrt{-x}\cos x=0$$

15. The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule three times.)

$$\lim_{x\to 0}\frac{\tan x-x}{\sin 2x-2x}=\lim_{x\to 0}\frac{\sec^2 x-1}{2\cos 2x-2}$$

$$=\lim_{x\to 0}\frac{2\sec^2 x\tan x}{-4\sin 2x}=\lim_{x\to 0}\frac{2\sec^4 x+4\sec^2 x\tan^2 x}{-8\cos 2x}$$

$$=\frac{2+0}{-8}=-\frac{1}{4}$$

16. The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule three times.)

$$\lim_{x\to 0}\frac{\sin x-\tan x}{x^2\sin x}=\lim_{x\to 0}\frac{\cos x-\sec^2 x}{2x\sin x+x^2\cos x}$$

$$=\lim_{x\to 0}\frac{-\sin x-2\sec^2 x\tan x}{2\sin x+4x\cos x-x^2\sin x}$$

$$=\lim_{x\to 0}\frac{-\cos x-2\sec^4 x-4\sec^2 x\tan^2 x}{6\cos x-x^2\cos x-6x\sin x}$$

$$=\frac{-1-2-0}{6-0-0}=-\frac{1}{2}$$

17. The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule twice.)

$$\lim_{x\to 0^+}\frac{x^2}{\sin x-x}=\lim_{x\to 0^+}\frac{2x}{\cos x-1}=\lim_{x\to 0^+}\frac{2}{-\sin x}$$

This limit is not of the form $\frac{0}{0}$. As $x\to 0^+, 2\to 2$, and $-\sin x\to 0^-$, so $\lim_{x\to 0^+}\frac{2}{\sin x}=-\infty$.

18. The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule twice.)

$$\lim_{x\to 0}\frac{e^x-\ln(1+x)-1}{x^2}=\lim_{x\to 0}\frac{e^x-\frac{1}{1+x}}{2x}$$

$$=\lim_{x\to 0}\frac{e^x+\frac{1}{(1+x)^2}}{2}=\frac{1+1}{2}=1$$

19. The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule twice.)

$$\lim_{x\to 0}\frac{\tan^{-1}x-x}{8x^3}=\lim_{x\to 0}\frac{\frac{1}{1+x^2}-1}{24x^2}=\lim_{x\to 0}\frac{\frac{-2x}{(1+x^2)^2}}{48x}$$

$$=\lim_{x\to 0}-\frac{1}{24(1+x^2)^2}=-\frac{1}{24}$$

20. The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule twice.)

$$\lim_{x\to 0}\frac{\cosh x-1}{x^2}=\lim_{x\to 0}\frac{\sinh x}{2x}=\lim_{x\to 0}\frac{\cosh x}{2}=\frac{1}{2}$$

21. The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule twice.)

$$\lim_{x\to 0^+}\frac{1-\cos x-x\sin x}{2-2\cos x-\sin^2 x}=\lim_{x\to 0^+}\frac{-x\cos x}{2\sin x-2\cos x\sin x}$$

$$=\lim_{x\to 0^+}\frac{x\sin x-\cos x}{2\cos x-2\cos^2 x+2\sin^2 x}$$

This limit is not of the form $\frac{0}{0}$.

As $x\to 0^+$, $x\sin x-\cos x\to -1$ and $2\cos x-2\cos^2 x+2\sin^2 x\to 0^+$, so

$$\lim_{x\to 0^+}\frac{x\sin x-\cos x}{2\cos x-2\cos^2 x+2\sin^2 x}=-\infty$$

22. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0^-}\frac{\sin x+\tan x}{e^x+e^{-x}-2}=\lim_{x\to 0^-}\frac{\cos x+\sec^2 x}{e^x-e^{-x}}$$

This limit is not of the form $\frac{0}{0}$.

As $x \to 0^-, \cos x + \sec^2 x \to 2$, and $e^x - e^{-x} \to 0^-$, so $\lim_{x\to 0^-} \frac{\cos x + \sec^2 x}{e^x - e^{-x}} = -\infty$.

23. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0} \frac{\int_0^x \sqrt{1+\sin t}\, dt}{x} = \lim_{x\to 0} \sqrt{1+\sin x} = 1$$

24. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0^+} \frac{\int_0^x \sqrt{t}\cos t\, dt}{x^2} = \lim_{x\to 0^+} \frac{\sqrt{x}\cos x}{2x}$$
$$= \lim_{x\to 0^+} \frac{\cos x}{2\sqrt{x}} = \infty$$

25. It would not have helped us because we proved $\lim_{x\to 0} \frac{\sin x}{x} = 1$ in order to find the derivative of $\sin x$.

26. Note that $\sin(1/0)$ is undefined (not zero), so l'Hôpital's Rule cannot be used.

As $x \to 0, \frac{1}{x} \to \infty$ and $\sin\left(\frac{1}{x}\right)$ oscillates rapidly between –1 and 1, so

$$\lim_{x\to 0} \left| \frac{x^2 \sin\left(\frac{1}{x}\right)}{\tan x} \right| \le \lim_{x\to 0} \frac{x^2}{\tan x}.$$

$$\frac{x^2}{\tan x} = \frac{x^2 \cos x}{\sin x}$$

$$\lim_{x\to 0} \frac{x^2 \cos x}{\sin x} = \lim_{x\to 0}\left[\left(\frac{x}{\sin x}\right) x\cos x\right] = 0.$$

Thus, $\lim_{x\to 0} \frac{x^2 \sin\left(\frac{1}{x}\right)}{\tan x} = 0$.

A table of values or graphing utility confirms this.

27. **a.** $\overline{OB} = \cos t, \overline{BC} = \sin t$ and $\overline{AB} = 1 - \cos t$, so the area of triangle ABC is $\frac{1}{2}\sin t(1-\cos t)$.

The area of the sector COA is $\frac{1}{2}t$ while the area of triangle COB is $\frac{1}{2}\cos t \sin t$, thus the area of the curved region ABC is $\frac{1}{2}(t - \cos t \sin t)$.

$$\lim_{t\to 0^+} \frac{\text{area of triangle } ABC}{\text{area of curved region } ABC} = \lim_{t\to 0^+} \frac{\frac{1}{2}\sin t(1-\cos t)}{\frac{1}{2}(t-\cos t \sin t)}$$

$$= \lim_{t\to 0^+} \frac{\sin t(1-\cos t)}{t - \cos t \sin t} = \lim_{t\to 0^+} \frac{\cos t - \cos^2 t + \sin^2 t}{1 - \cos^2 t + \sin^2 t} = \lim_{t\to 0^+} \frac{4\sin t \cos t - \sin t}{4\cos t \sin t} = \lim_{t\to 0^+} \frac{4\cos t - 1}{4\cos t} = \frac{3}{4}$$

(L'Hôpital's Rule was applied twice.)

b. The area of the sector BOD is $\frac{1}{2}t\cos^2 t$, so the area of the curved region BCD is $\frac{1}{2}\cos t \sin t - \frac{1}{2}t\cos^2 t$.

$$\lim_{t\to 0^+} \frac{\text{area of curved region } BCD}{\text{area of curved region } ABC} = \lim_{t\to 0^+} \frac{\frac{1}{2}\cos t(\sin t - t\cos t)}{\frac{1}{2}(t - \cos t \sin t)}$$

$$= \lim_{t\to 0^+} \frac{\cos t(\sin t - t\cos t)}{t - \sin t \cos t} = \lim_{t\to 0^+} \frac{\sin t(2t\cos t - \sin t)}{1 - \cos^2 t + \sin^2 t} = \lim_{t\to 0^+} \frac{2t(\cos^2 t - \sin^2 t)}{4\cos t \sin t} = \lim_{t\to 0^+} \frac{t(\cos^2 t - \sin^2 t)}{2\cos t \sin t}$$

$$= \lim_{t\to 0^+} \frac{\cos^2 t - 4t\cos t \sin t - \sin^2 t}{2\cos^2 t - 2\sin^2 t} = \frac{1-0-0}{2-0} = \frac{1}{2}$$

(L'Hôpital's Rule was applied three times.)

28. a. Note that $\angle DOE$ has measure t radians. Thus the coordinates of E are $(\cos t, \sin t)$.

Also, slope $\overline{BC}$ = slope $\overline{CE}$. Thus,

$$\frac{0-y}{(1-t)-0} = \frac{\sin t - 0}{\cos t - (1-t)}$$

$$-y = \frac{(1-t)\sin t}{\cos t + t - 1}$$

$$y = \frac{(t-1)\sin t}{\cos t + t - 1}$$

$$\lim_{t\to 0^+} y = \lim_{t\to 0^+} \frac{(t-1)\sin t}{\cos t + t - 1}$$

This limit is of the form $\frac{0}{0}$.

$$\lim_{t\to 0^+} \frac{(t-1)\sin t}{\cos t + t - 1} = \lim_{t\to 0^+} \frac{\sin t + (t-1)\cos t}{-\sin t + 1} = \frac{0+(-1)(1)}{-0+1} = -1$$

b. Slope $\overline{AF}$ = slope $\overline{EF}$. Thus,

$$\frac{t}{1-x} = \frac{t - \sin t}{1 - \cos t}$$

$$\frac{t(1-\cos t)}{t - \sin t} = 1 - x$$

$$x = 1 - \frac{t(1+\cos t)}{t - \sin t}$$

$$x = \frac{t\cos t - \sin t}{t - \sin t}$$

$$\lim_{t\to 0^+} x = \lim_{t\to 0^+} \frac{t\cos t - \sin t}{t - \sin t}$$

The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule three times.)

$$\lim_{t\to 0^+} \frac{t\cos t - \sin t}{t - \sin t} = \lim_{t\to 0^+} \frac{-t\sin t}{1-\cos t}$$

$$= \lim_{t\to 0^+} \frac{-\sin t - t\cos t}{\sin t} = \lim_{t\to 0^+} \frac{t\sin t - 2\cos t}{\cos t} = \frac{0-2}{1} = -2$$

29. A should approach $4\pi b^2$, the surface area of a sphere of radius b.

$$\lim_{a\to b^+}\left[2\pi b^2 + \frac{2\pi a^2 b \arcsin\frac{\sqrt{a^2-b^2}}{a}}{\sqrt{a^2-b^2}}\right] = 2\pi b^2 + 2\pi b \lim_{a\to b^+} \frac{a^2 \arcsin\frac{\sqrt{a^2-b^2}}{a}}{\sqrt{a^2-b^2}}$$

Focusing on the limit, we have

$$\lim_{a\to b^+} \frac{a^2 \arcsin\frac{\sqrt{a^2-b^2}}{a}}{\sqrt{a^2-b^2}} = \lim_{a\to b^+} \frac{2a\arcsin\frac{\sqrt{a^2-b^2}}{a} + a^2\left(\frac{b}{a\sqrt{a^2-b^2}}\right)}{\frac{a}{\sqrt{a^2-b^2}}} = \lim_{a\to b^+}\left(2\sqrt{a^2-b^2}\arcsin\frac{\sqrt{a^2-b^2}}{a} + b\right) = b.$$

Thus, $\lim_{a\to b^+} A = 2\pi b^2 + 2\pi b(b) = 4\pi b^2$.

30. In order for l'Hôpital's Rule to be of any use, $a(1)^4 + b(1)^3 + 1 = 0$, so $b = -1 - a$.

Using l'Hôpital's Rule,

$$\lim_{x\to 1}\frac{ax^4+bx^3+1}{(x-1)\sin\pi x}=\lim_{x\to 1}\frac{4ax^3+3bx^2}{\sin\pi x+\pi(x-1)\cos\pi x}$$

To use l'Hôpital's Rule here,

$4a(1)^3+3b(1)^2=0$, so $4a+3b=0$, hence $a=3$, $b=-4$.

$$\lim_{x\to 1}\frac{3x^4-4x^3+1}{(x-1)\sin\pi x}=\lim_{x\to 1}\frac{12x^3-12x^2}{\sin\pi x+\pi(x-1)\cos\pi x}=\lim_{x\to 1}\frac{36x^2-24x}{2\pi\cos\pi x-\pi^2(x-1)\sin\pi x}=\frac{12}{-2\pi}=-\frac{6}{\pi}$$

$a=3,\ b=-4,\ c=-\frac{6}{\pi}$

31. If $f'(a)$ and $g'(a)$ both exist, then f and g are both continuous at a. Thus, $\lim_{x\to a} f(x)=0=f(a)$ and $\lim_{x\to a} g(x)=0=g(a)$.

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}$$

$$\lim_{x\to a}\frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}=\frac{\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}{\lim_{x\to a}\frac{g(x)-g(a)}{x-a}}=\frac{f'(a)}{g'(a)}$$

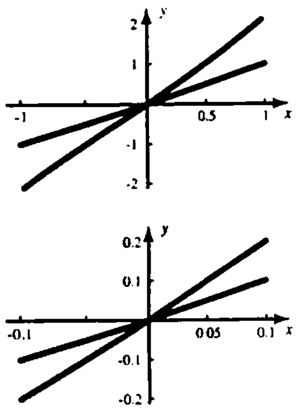
32. $\lim_{x\to 0}\frac{\cos x-1+\frac{x^2}{2}}{x^4}=\frac{1}{24}$

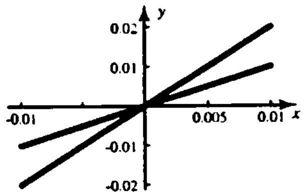
33. $\lim_{x\to 0}\frac{e^x-1-x-\frac{x^2}{2}-\frac{x^3}{6}}{x^4}=\frac{1}{24}$

34. $\lim_{x\to 0}\frac{1-\cos(x^2)}{x^3\sin x}=\frac{1}{2}$

35. $\lim_{x\to 0}\frac{\tan x-x}{\arcsin x-x}=2$

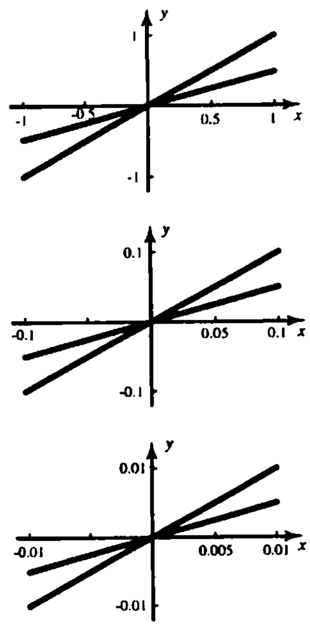
36.



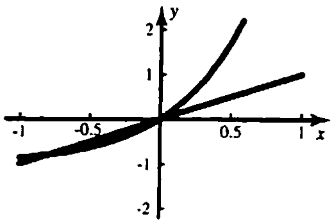


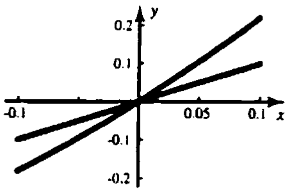
The slopes are approximately $0.02/0.01=2$ and $0.01/0.01=1$. The ratio of the slopes is therefore $2/1=2$, indicating that the limit of the ratio should be about 2. An application of l'Hopital's Rule confirms this.

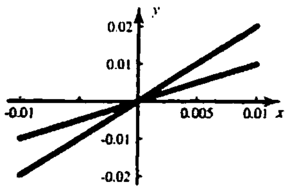
37.



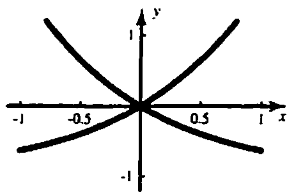
The slopes are approximately $0.005/0.01=1/2$ and $0.01/0.01=1$. The ratio of the slopes is therefore $1/2$, indicating that the limit of the ratio should be about $1/2$. An application of l'Hopital's Rule confirms this.

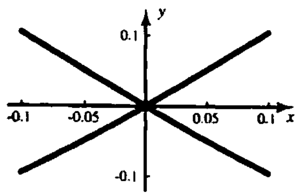
38. 

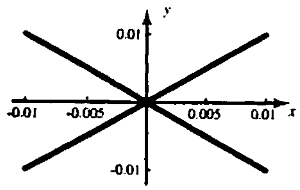




The slopes are approximately $0.01/0.01 = 1$ and $0.02/0.01 = 2$. The ratio of the slopes is therefore $1/2$, indicating that the limit of the ratio should be about $1/2$. An application of l'Hopital's Rule confirms this.

39. 





The slopes are approximately $0.01/0.01 = 1$ and $-0.01/0.01 = 1$. The ratio of the slopes is therefore $-1/1 = -1$, indicating that the limit of the ratio should be about -1. An application of l'Hopital's Rule confirms this.

40. If f and g are locally linear at zero, then, since $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$, $f(x) \approx px$ and $g(x) \approx qx$, where $p = f'(0)$ and $q = g'(0)$. Then $f(x)/g(x) \approx px/px = p/q$ when x is near 0.

9.2 Concepts Review

1. $\dfrac{f'(x)}{g'(x)}$

2. $\lim_{x\to a} \dfrac{f(x)}{\frac{1}{g(x)}}$ or $\lim_{x\to a} \dfrac{g(x)}{\frac{1}{f(x)}}$

3. $\infty - \infty, 0^0, \infty^0, 1^\infty$

4. $\ln x$

Problem Set 9.2

1. The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x\to\infty} \frac{\ln x^{1000}}{x} = \lim_{x\to\infty} \frac{\frac{1}{x^{1000}} 1000x^{999}}{1}$$
$$= \lim_{x\to\infty} \frac{1000}{x} = 0$$

2. The limit is of the form $\frac{\infty}{\infty}$. (Apply l'Hôpital's Rule twice.)

$$\lim_{x\to\infty} \frac{(\ln x)^2}{2^x} = \lim_{x\to\infty} \frac{2(\ln x)\frac{1}{x}}{2^x \ln 2}$$

$$= \lim_{x\to\infty} \frac{2\ln x}{x\cdot 2^x \ln 2} = \lim_{x\to\infty} \frac{2\left(\frac{1}{x}\right)}{2^x \ln 2(1+x\ln 2)}$$

$$= \lim_{x\to\infty} \frac{2}{x\cdot 2^x \ln 2(1+x\ln 2)} = 0$$

3. $\lim_{x\to\infty} \frac{x^{10000}}{e^x} = 0$ (See Example 2).

4. The limit is of the form $\frac{\infty}{\infty}$. (Apply l'Hôpital's Rule three times.)

$$\lim_{x\to\infty} = \frac{3x}{\ln(100x+e^x)} = \lim_{x\to\infty} \frac{3}{\frac{1}{100x+e^x}(100+e^x)}$$

$$= \lim_{x\to\infty} \frac{300x+3e^x}{100+e^x} = \lim_{x\to\infty} \frac{300+3e^x}{e^x}$$

$$= \lim_{x\to\infty} \frac{3e^x}{e^x} = 3$$

5. The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x\to\frac{\pi}{2}} \frac{3\sec x+5}{\tan x} = \lim_{x\to\frac{\pi}{2}} \frac{3\sec x\tan x}{\sec^2 x}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{3\tan x}{\sec x} = \lim_{x\to\frac{\pi}{2}} 3\sin x = 3$$

6. The limit is of the form $\frac{-\infty}{-\infty}$.

$$\lim_{x\to 0^+} \frac{\ln\sin^2 x}{3\ln\tan x} = \lim_{x\to 0^+} \frac{\frac{1}{\sin^2 x}2\sin x\cos x}{\frac{3}{\tan x}\sec^2 x}$$

$$= \lim_{x\to 0^+} \frac{2\cos^2 x}{3} = \frac{2}{3}$$

7. The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x\to\infty} \frac{\ln(\ln x^{1000})}{\ln x} = \lim_{x\to\infty} \frac{\frac{1}{\ln x^{1000}}\left(\frac{1}{x^{1000}}1000x^{999}\right)}{\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{1000}{x\ln x^{1000}} = 0$$

8. The limit is of the form $\frac{-\infty}{\infty}$. (Apply l'Hôpital's Rule twice.)

$$\lim_{x\to\left(\frac{1}{2}\right)^-} \frac{\ln(4-8x)^2}{\tan\pi x} = \lim_{x\to\left(\frac{1}{2}\right)^-} \frac{\frac{1}{(4-8x)^2}2(4-8x)(-8)}{\pi\sec^2\pi x}$$

$$= \lim_{x\to\left(\frac{1}{2}\right)^-} \frac{-16\cos^2\pi x}{\pi(4-8x)} = \lim_{x\to\left(\frac{1}{2}\right)^-} \frac{32\pi\cos\pi x\sin\pi x}{-8\pi}$$

$$= \lim_{x\to\left(\frac{1}{2}\right)^-} -4\cos\pi x\sin\pi x = 0$$

9. The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x\to 0^+} \frac{\cot x}{\sqrt{-\ln x}} = \lim_{x\to 0^+} \frac{-\csc^2 x}{-\frac{1}{2x\sqrt{-\ln x}}}$$

$$= \lim_{x\to 0^+} \frac{2x\sqrt{-\ln x}}{\sin^2 x}$$

$$= \lim_{x\to 0^+} \left[\frac{2x}{\sin x}\csc x\sqrt{-\ln x}\right] = \infty$$

since $\lim_{x\to 0^+} \frac{x}{\sin x} = 1$ while $\lim_{x\to 0^+} \csc x = \infty$ and $\lim_{x\to 0^+} \sqrt{-\ln x} = \infty$.

10. The limit is of the form $\frac{\infty}{\infty}$, but the fraction can be simplified.

$$\lim_{x\to 0} \frac{2\csc^2 x}{\cot^2 x} = \lim_{x\to 0} \frac{2}{\cos^2 x} = \frac{2}{1^2} = 2$$

11. $\lim_{x\to 0} (x\ln x^{1000}) = \lim_{x\to 0} \frac{\ln x^{1000}}{\frac{1}{x}}$

The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x\to 0} \frac{\ln x^{1000}}{\frac{1}{x}} = \lim_{x\to 0} \frac{\frac{1}{x^{1000}}1000x^{999}}{-\frac{1}{x^2}}$$

$$= \lim_{x\to 0} -1000x = 0$$

12. $\lim_{x\to 0} 3x^2\csc^2 x = \lim_{x\to 0} 3\left(\frac{x}{\sin x}\right)^2 = 3$ since $\lim_{x\to 0} \frac{x}{\sin x} = 1$

13. $\lim_{x\to 0} (\csc^2 x - \cot^2 x) = \lim_{x\to 0} \frac{1-\cos^2 x}{\sin^2 x}$

$$= \lim_{x\to 0} \frac{\sin^2 x}{\sin^2 x} = 1$$

14. $\lim_{x\to\frac{\pi}{2}}(\tan x-\sec x)=\lim_{x\to\frac{\pi}{2}}\frac{\sin x-1}{\cos x}$

The limit is of the form $\frac{0}{0}$.

$\lim_{x\to\frac{\pi}{2}}\frac{\sin x-1}{\cos x}=\lim_{x\to\frac{\pi}{2}}\frac{\cos x}{-\sin x}=\frac{0}{-1}=0$

15. The limit is of the form 0^0.

Let $y=(3x)^{x^2}$, then $\ln y=x^2\ln 3x$

$\lim_{x\to 0^+}x^2\ln 3x=\lim_{x\to 0^+}\frac{\ln 3x}{\frac{1}{x^2}}$

The limit is of the form $\frac{\infty}{\infty}$.

$\lim_{x\to 0^+}\frac{\ln 3x}{\frac{1}{x^2}}=\lim_{x\to 0^+}\frac{\frac{1}{3x}\cdot 3}{-\frac{2}{x^3}}=\lim_{x\to 0^+}-\frac{x^2}{2}=0$

$\lim_{x\to 0^+}(3x)^{x^2}=\lim_{x\to 0^+}e^{\ln y}=1$

16. The limit is of the form 1^∞.

Let $y=(\cos x)^{\csc x}$, then $\ln y=\csc x(\ln(\cos x))$

$\lim_{x\to 0}\csc x(\ln(\cos x))=\lim_{x\to 0}\frac{\ln(\cos x)}{\sin x}$

The limit is of the form $\frac{0}{0}$.

$\lim_{x\to 0}\frac{\ln(\cos x)}{\sin x}=\lim_{x\to 0}\frac{\frac{1}{\cos x}(-\sin x)}{\cos x}$

$=\lim_{x\to 0}-\frac{\sin x}{\cos^2 x}=-\frac{0}{1}=0$

$\lim_{x\to 0}(\cos x)^{\csc x}=\lim_{x\to 0}e^{\ln y}=1$

17. The limit is of the form 0^∞, which is not an indeterminate form. $\lim_{x\to(\pi/2)^-}(5\cos x)^{\tan x}=0$

18. $\lim_{x\to 0}\left(\csc^2 x-\frac{1}{x^2}\right)^2=\lim_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right)^2=\lim_{x\to 0}\left(\frac{x^2-\sin^2 x}{x^2\sin^2 x}\right)^2$

Consider $\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^2\sin^2 x}$. The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule four times.)

$\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^2\sin^2 x}=\lim_{x\to 0}\frac{2x-2\sin x\cos x}{2x\sin^2 x+2x^2\sin x\cos x}=\lim_{x\to 0}\frac{x-\sin x\cos x}{x\sin^2 x+x^2\sin x\cos x}$

$=\lim_{x\to 0}\frac{1-\cos^2 x+\sin^2 x}{\sin^2 x+4x\sin x\cos x+x^2\cos^2 x-x^2\sin^2 x}=\frac{4\sin x\cos x}{6x\cos^2 x+6\cos x\sin x-4x^2\cos x\sin x-6x\sin^2 x}$

$=\lim_{x\to 0}\frac{4\cos^2 x-4\sin^2 x}{12\cos^2 x-4x^2\cos^2 x-32x\cos x\sin x-12\sin^2 x+4x^2\sin^2 x}=\frac{4}{12}=\frac{1}{3}$

Thus, $\lim_{x\to 0}\left(\frac{x^2-\sin^2 x}{x^2\sin^2 x}\right)^2=\left(\frac{1}{3}\right)^2=\frac{1}{9}$

19. The limit is of the form 1^∞.

Let $y=(x+e^{x/3})^{3/x}$, then $\ln y=\frac{3}{x}\ln(x+e^{x/3})$.

$\lim_{x\to 0}\frac{3}{x}\ln(x+e^{x/3})=\lim_{x\to 0}\frac{3\ln(x+e^{x/3})}{x}$

The limit is of the form $\frac{0}{0}$.

$\lim_{x\to 0}\frac{3\ln(x+e^{x/3})}{x}=\lim_{x\to 0}\frac{\frac{3}{x+e^{x/3}}\left(1+\frac{1}{3}e^{x/3}\right)}{1}$

$=\lim_{x\to 0}\frac{3+e^{x/3}}{x+e^{x/3}}=\frac{4}{1}=4$

$\lim_{x\to 0}(x+e^{x/3})^{3/x}=\lim_{x\to 0}e^{\ln y}=e^4$

20. The limit is of the form $(-1)^0$.

The limit does not exist.

21. The limit is of the form 1^0, which is not an indeterminate form.

$$\lim_{x\to\frac{\pi}{2}} (\sin x)^{\cos x} = 1$$

22. The limit is of the form ∞^∞, which is not an indeterminate form.

$$\lim_{x\to\infty} x^x = \infty$$

23. The limit is of the form ∞^0. Let

$y = x^{1/x}$, then $\ln y = \frac{1}{x}\ln x$.

$$\lim_{x\to\infty} \frac{1}{x}\ln x = \lim_{x\to\infty} \frac{\ln x}{x}$$

The limit is of the form $\frac{-\infty}{\infty}$.

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$$

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\ln y} = 1$$

24. The limit is of the form 1^∞.

Let $y = (\cos x)^{1/x^2}$, then $\ln y = \frac{1}{x^2}\ln(\cos x)$.

$$\lim_{x\to 0} \frac{1}{x^2}\ln(\cos x) = \lim_{x\to 0} \frac{\ln(\cos x)}{x^2}$$

The limit is of the form $\frac{0}{0}$.

(Apply l'Hôpital's rule twice.)

$$\lim_{x\to 0} \frac{\ln(\cos x)}{x^2} = \lim_{x\to 0} \frac{\frac{1}{\cos x}(-\sin x)}{2x} = \lim_{x\to 0} \frac{-\tan x}{2x}$$

$$= \lim_{x\to 0} \frac{-\sec^2 x}{2} = \frac{-1}{2} = -\frac{1}{2}$$

$$\lim_{x\to 0} (\cos x)^{1/x^2} = \lim_{x\to 0} e^{\ln y} = e^{-1/2} = \frac{1}{\sqrt{e}}$$

25. The limit is of the form 0^∞, which is not an indeterminate form.

$$\lim_{x\to 0^+} (\tan x)^{2/x} = 0$$

26. The limit is of the form $\infty + \infty$, which is not an indeterminate form.

$$\lim_{x\to -\infty} (e^{-x} - x) = \lim_{x\to\infty} (e^x + x) = \infty$$

27. The limit is of the form 0^0. Let

$y = (\sin x)^x$, then $\ln y = x\ln(\sin x)$.

$$\lim_{x\to 0^+} x\ln(\sin x) = \lim_{x\to 0^+} \frac{\ln(\sin x)}{\frac{1}{x}}$$

The limit is of the form $\frac{-\infty}{\infty}$.

$$\lim_{x\to 0^+} \frac{\ln(\sin x)}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{\sin x}\cos x}{-\frac{1}{x^2}}$$

$$= \lim_{x\to 0^+} \left[\frac{x}{\sin x}(-x\cos x)\right] = 1\cdot 0 = 0$$

$$\lim_{x\to 0^+} (\sin x)^x = \lim_{x\to 0^+} e^{\ln y} = 1$$

28. The limit is of the form 1^∞. Let

$y = (\cos x - \sin x)^{1/x}$, then $\ln y = \frac{1}{x}\ln(\cos x - \sin x)$.

$$\lim_{x\to 0} \frac{1}{x}\ln(\cos x - \sin x) = \lim_{x\to 0} \frac{\ln(\cos x - \sin x)}{x}$$

$$= \lim_{x\to 0} \frac{\frac{1}{\cos x - \sin x}(-\sin x - \cos x)}{1}$$

$$= \lim_{x\to 0} \frac{-\sin x - \cos x}{\cos x - \sin x} = -1$$

$$\lim_{x\to 0} (\cos x - \sin x)^{1/x} = \lim_{x\to 0} e^{\ln y} = e^{-1}$$

29. The limit is of the form $\infty - \infty$.

$$\lim_{x\to 0}\left(\csc x - \frac{1}{x}\right) = \lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0} \frac{x - \sin x}{x\sin x}$$

The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule twice.)

$$\lim_{x\to 0} \frac{x - \sin x}{x\sin x} = \lim_{x\to 0} \frac{1 - \cos x}{\sin x + x\cos x}$$

$$= \lim_{x\to 0} \frac{\sin x}{2\cos x - x\sin x} = \frac{0}{2} = 0$$

30. The limit is of the form 1^∞.

Let $y = \left(1 + \frac{1}{x}\right)^x$, then $\ln y = x\ln\left(1 + \frac{1}{x}\right)$.

$$\lim_{x\to\infty} x\ln\left(1 + \frac{1}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1 + \frac{1}{x}\right)}{\frac{1}{x}}$$

The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to\infty} \frac{\ln\left(1 + \frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} e^{\ln y} = e^1 = e$$

31. The limit is of the form 3^∞, which is not an indeterminate form.

$$\lim_{x\to 0^+}(1+2e^x)^{1/x} = \infty$$

32. The limit is of the form $\infty - \infty$.

$$\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{x}{\ln x}\right) = \lim_{x\to 1}\frac{\ln x - x^2 + x}{(x-1)\ln x}$$

The limit is of the form $\frac{0}{0}$.

Apply l'Hôpital's Rule twice.

$$\lim_{x\to 1}\frac{\ln x - x^2 + x}{(x-1)\ln x} = \lim_{x\to 1}\frac{\frac{1}{x}-2x+1}{\ln x + \frac{x-1}{x}}$$

$$= \lim_{x\to 1}\frac{1-2x^2+x}{x\ln x + x - 1} = \lim_{x\to 1}\frac{-4x+1}{\ln x + 2} = \frac{-3}{2} = -\frac{3}{2}$$

33. The limit is of the form 1^∞.

Let $y = (\cos x)^{1/x}$, then $\ln y = \frac{1}{x}\ln(\cos x)$.

$$\lim_{x\to 0}\frac{1}{x}\ln(\cos x) = \lim_{x\to 0}\frac{\ln(\cos x)}{x}$$

The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{\ln(\cos x)}{x} = \lim_{x\to 0}\frac{\frac{1}{\cos x}(-\sin x)}{1} = \lim_{x\to 0} -\frac{\sin x}{\cos x} = 0$$

$$\lim_{x\to 0}(\cos x)^{1/x} = \lim_{x\to 0} e^{\ln y} = 1$$

34. The limit is of the form $0 \cdot -\infty$.

$$\lim_{x\to 0^+}(x^{1/2}\ln x) = \lim_{x\to 0^+}\frac{\ln x}{\frac{1}{\sqrt{x}}}$$

The limit is of the form $\frac{-\infty}{\infty}$.

$$\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{\sqrt{x}}} = \lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{2x^{3/2}}} = \lim_{x\to 0^+} -2\sqrt{x} = 0$$

35. Since $\cos x$ oscillates between -1 and 1 as $x\to\infty$, this limit is not of an indeterminate form previously seen.

Let $y = e^{\cos x}$, then $\ln y = (\cos x)\ln e = \cos x$

$\lim_{x\to\infty}\cos x$ does not exist, so $\lim_{x\to\infty} e^{\cos x}$ does not exist.

36. The limit is of the form $\infty - \infty$.

$$\lim_{x\to\infty}[\ln(x+1)-\ln(x-1)] = \lim_{x\to\infty}\ln\frac{x+1}{x-1}$$

$\lim_{x\to\infty}\frac{x+1}{x-1} = \lim_{x\to\infty}\frac{1+\frac{1}{x}}{1-\frac{1}{x}} = 1$, so $\lim_{x\to\infty}\ln\frac{x+1}{x-1} = 0$

37. The limit is of the form $\frac{0}{-\infty}$, which is not an indeterminate form.

$$\lim_{x\to 0^+}\frac{x}{\ln x} = 0$$

38. The limit is of the form $-\infty\cdot\infty$, which is not an indeterminate form.

$$\lim_{x\to 0^+}(\ln x \cot x) = -\infty$$

39. $\sqrt{1+e^{-t}} > 1$ for all t, so

$$\int_1^x \sqrt{1+e^{-t}}\,dt > \int_1^x dt = x - 1.$$

The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x\to\infty}\frac{\int_1^x \sqrt{1+e^{-t}}\,dt}{x} = \lim_{x\to\infty}\frac{\sqrt{1+e^{-x}}}{1} = 1$$

40. This limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 1^+}\frac{\int_1^x \sin t\,dt}{x-1} = \lim_{x\to 1^+}\frac{\sin x}{1} = \sin(1)$$

41. **a.** Let $y = \sqrt[n]{a}$, then $\ln y = \frac{1}{n}\ln a$.

$$\lim_{n\to\infty}\frac{1}{n}\ln a = 0$$

$$\lim_{n\to\infty}\sqrt[n]{a} = \lim_{n\to\infty} e^{\ln y} = 1$$

b. The limit is of the form ∞^0.

Let $y = \sqrt[n]{n}$, then $\ln y = \frac{1}{n}\ln n$.

$$\lim_{n\to\infty}\frac{1}{n}\ln n = \lim_{n\to\infty}\frac{\ln n}{n}$$

This limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{n\to\infty}\frac{\ln n}{n} = \lim_{n\to\infty}\frac{\frac{1}{n}}{1} = 0$$

$$\lim_{n\to\infty}\sqrt[n]{n} = \lim_{n\to\infty} e^{\ln y} = 1$$

c. $\lim_{n\to\infty} n\left(\sqrt[n]{a}-1\right) = \lim_{n\to\infty} \frac{\sqrt[n]{a}-1}{\frac{1}{n}}$

This limit is of the form $\frac{0}{0}$,

since $\lim_{n\to\infty} \sqrt[n]{a} = 1$ by part a.

$$\lim_{n\to\infty} \frac{\sqrt[n]{a}-1}{\frac{1}{n}} = \lim_{n\to\infty} \frac{-\frac{1}{n^2}\sqrt[n]{a}\ln a}{-\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \sqrt[n]{a}\ln a = \ln a$$

d. $\lim_{n\to\infty} n\left(\sqrt[n]{n}-1\right) = \lim_{n\to\infty} \frac{\sqrt[n]{n}-1}{\frac{1}{n}}$

This limit is of the form $\frac{0}{0}$,

since $\lim_{n\to\infty} \sqrt[n]{n} = 1$ by part b.

$$\lim_{n\to\infty} \frac{\sqrt[n]{n}-1}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}\left(\frac{1}{n^2}\right)(1-\ln n)}{-\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \sqrt[n]{n}(\ln n - 1) = \infty$$

42. a. The limit is of the form 0^0.

Let $y = x^x$, then $\ln y = x\ln x$.

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}$$

The limit is of the form $\frac{-\infty}{\infty}$.

$$\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0$$

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{\ln y} = 1$$

b. The limit is of the form 1^0, since

$\lim_{x\to 0^+} x^x = 1$ by part a.

Let $y = (x^x)^x$, then $\ln y = x\ln(x^x)$.

$$\lim_{x\to 0^+} x\ln(x^x) = 0$$

$$\lim_{x\to 0^+} (x^x)^x = \lim_{x\to 0^+} e^{\ln y} = 1$$

Note that 1^0 is not an indeterminate form.

c. The limit is of the form 0^1, since

$\lim_{x\to 0^+} x^x = 1$ by part a.

Let $y = x^{(x^x)}$, then $\ln y = x^x \ln x$

$$\lim_{x\to 0^+} x^x \ln x = -\infty$$

$$\lim_{x\to 0^+} x^{(x^x)} = \lim_{x\to 0^+} e^{\ln y} = 0$$

Note that 0^1 is not an indeterminate form.

d. The limit is of the form 1^0, since

$\lim_{x\to 0^+} (x^x)^x = 1$ by part b.

Let $y = ((x^x)^x)^x$, then $\ln y = x\ln((x^x)^x)$.

$$\lim_{x\to 0^+} x\ln((x^x)^x) = 0$$

$$\lim_{x\to 0^+} ((x^x)^x)^x = \lim_{x\to 0^+} e^{\ln y} = 1$$

Note that 1^0 is not an indeterminate form.

e. The limit is of the form 0^0, since

$\lim_{x\to 0^+} (x^{(x^x)}) = 0$ by part c.

Let $y = x^{(x^{(x^x)})}$, then $\ln y = x^{(x^x)}\ln x$.

$$\lim_{x\to 0^+} x^{(x^x)}\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x^{(x^x)}}}$$

The limit is of the form $\frac{-\infty}{\infty}$.

$$\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x^{(x^x)}}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-x^{(x^x)}\left[x^x(\ln x+1)\ln x+\frac{x^x}{x}\right]}{(x^{(x^x)})^2}}$$

$$= \lim_{x\to 0^+} \frac{-x^{(x^x)}}{x^x x(\ln x)^2 + x^x x\ln x + x^x}$$

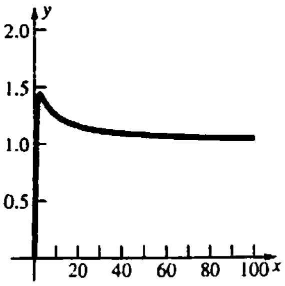
$$= \frac{0}{1\cdot 0 + 1\cdot 0 + 1} = 0$$

Note: $\lim_{x\to 0^+} x(\ln x)^2 = \lim_{x\to 0^+} \frac{(\ln x)^2}{\frac{1}{x}}$

$$= \lim_{x\to 0^+} \frac{\frac{2}{x}\ln x}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -2x\ln x = 0$$

$$\lim_{x\to 0^+} x^{(x^{(x^x)})} = \lim_{x\to 0^+} e^{\ln y} = 1$$

43.



$\ln y = \frac{\ln x}{x}$

$\lim_{x\to 0^+} \frac{\ln x}{x} = -\infty$, so $\lim_{x\to 0^+} x^{1/x} = \lim_{x\to 0^+} e^{\ln y} = 0$

$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$, so $\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\ln y} = 1$

$y = x^{1/x} = e^{\frac{1}{x}\ln x}$

$y' = \left(\frac{1}{x^2} - \frac{\ln x}{x^2}\right)e^{\frac{1}{x}\ln x}$

$y' = 0$ when $x = e$.

y is maximum at $x = e$ since $y' > 0$ on $(0, e)$ and $y' < 0$ on (e, ∞). When $x = e$, $y = e^{1/e}$.

44. a. The limit is of the form $(1+1)^\infty = 2^\infty$, which is not an indeterminate form.

$\lim_{x\to 0^+} (1^x + 2^x)^{1/x} = \infty$

b. The limit is of the form $(1+1)^{-\infty} = 2^{-\infty}$, which is not an indeterminate form.

$\lim_{x\to 0^-} (1^x + 2^x)^{1/x} = 0$

c. The limit is of the form ∞^0.

Let $y = (1^x + 2^x)^{1/x}$, then

$\ln y = \frac{1}{x}\ln(1^x + 2^x)$

$\lim_{x\to\infty} \frac{1}{x}\ln(1^x + 2^x) = \lim_{x\to\infty} \frac{\ln(1^x + 2^x)}{x}$

The limit is of the form $\frac{\infty}{\infty}$. (Apply l'Hôpital's Rule twice.)

$\lim_{x\to\infty} \frac{\ln(1^x + 2^x)}{x} = \lim_{x\to\infty} \frac{\frac{1}{1^x+2^x}(1^x \ln 1 + 2^x \ln 2)}{1}$

$= \lim_{x\to\infty} \frac{2^x \ln 2}{1^x + 2^x} = \lim_{x\to\infty} \frac{2^x (\ln 2)^2}{1^x \ln 1 + 2^x \ln 2} = \ln 2$

$\lim_{x\to\infty} (1^x + 2^x)^{1/x} = \lim_{x\to\infty} e^{\ln y} = e^{\ln 2} = 2$

d. The limit is of the form 1^0, since $1^x = 1$ for all x. This is not an indeterminate form.

$\lim_{x\to-\infty} (1^x + 2^x)^{1/x} = 1$

45. $\lim_{n\to\infty} \frac{1^k + 2^k + \ldots + n^k}{n^{k+1}}$

$= \lim_{n\to\infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^k + \left(\frac{2}{n}\right)^k + \ldots + \left(\frac{n}{n}\right)^k\right]$

$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{n^k} + \frac{2^k}{n^k} + \ldots + 1\right] = 0$

46. Let $y = \left(\sum_{i=1}^n c_i x_i^t\right)^{1/t}$, then $\ln y = \frac{1}{t}\ln\left(\sum_{i=1}^n c_i x_i^t\right)$.

$\lim_{t\to 0^+} \frac{1}{t}\ln\left(\sum_{i=1}^n c_i x_i^t\right) = \lim_{t\to 0^+} \frac{\ln\left(\sum_{i=1}^n c_i x_i^t\right)}{t}$

The limit is of the form $\frac{0}{0}$, since $\sum_{i=1}^n c_i = 1$.

$\lim_{t\to 0^+} \frac{\ln\left(\sum_{i=1}^n c_i x_i^t\right)}{t} = \lim_{t\to 0^+} \frac{1}{\sum_{i=1}^n c_i x_i^t} \sum_{i=1}^n c_i x_i^t \ln x_i$

$= \sum_{i=1}^n c_i \ln x_i = \sum_{i=1}^n \ln x_i^{c_i}$

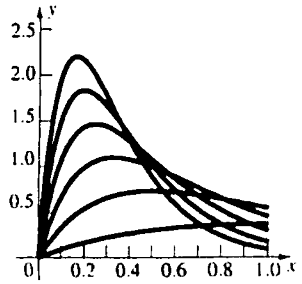
$\lim_{t\to 0^+} \left(\sum_{i=1}^n c_i x_i^t\right)^{1/t} = \lim_{t\to 0^+} e^{\ln y}$

$= e^{\sum_{i=1}^n \ln x_i^{c_i}} = x_1^{c_1} x_2^{c_2} \ldots x_n^{c_n} = \prod_{i=1}^n x_i^{c_i}$

47. a. $\lim_{t\to 0^+} \left(\frac{1}{2}2^t + \frac{1}{2}5^t\right)^{1/t} = \sqrt{2}\sqrt{5} \approx 3.162$

b. $\lim_{t\to 0^+} \left(\frac{1}{5}2^t + \frac{4}{5}5^t\right)^{1/t} = \sqrt[5]{2}\cdot\sqrt[5]{5^4} \approx 4.163$

c. $\lim_{t\to 0^+} \left(\frac{1}{10}2^t + \frac{9}{10}5^t\right)^{1/t} = \sqrt[10]{2}\cdot\sqrt[10]{5^9} \approx 4.562$

48. a. 

b. $n^2xe^{-nx} = \frac{n^2x}{e^{nx}}$, so the limit is of the form $\frac{\infty}{\infty}$.

$\lim_{n\to\infty} \frac{n^2x}{e^{nx}} = \lim_{n\to\infty} \frac{2nx}{xe^{nx}}$

This limit is of the form $\frac{\infty}{\infty}$.

$\lim_{n\to\infty} \frac{2nx}{xe^{nx}} = \lim_{n\to\infty} \frac{2x}{x^2e^{nx}} = 0$

c. $\int_0^1 xe^{-x}dx = \left[-xe^{-x} - e^{-x}\right]_0^1 = 1 - \frac{2}{e}$

$\int_0^1 4xe^{-2x}dx = \left[-2xe^{-2x} - e^{-2x}\right]_0^1 = 1 - \frac{3}{e^2}$

$\int_0^1 9xe^{-3x}dx = \left[-3xe^{-3x} - e^{-3x}\right]_0^1 = 1 - \frac{4}{e^3}$

$\int_0^1 16xe^{-4x}dx = \left[-4xe^{-4x} - e^{-4x}\right]_0^1 = 1 - \frac{5}{e^4}$

$\int_0^1 25xe^{-5x} = \left[-5xe^{-5x} - e^{-5x}\right]_0^1 = 1 - \frac{6}{e^5}$

$\int_0^1 36e^{-6x}dx = \left[-6xe^{-6x} - e^{-6x}\right]_0^1 = 1 - \frac{7}{e^6}$

d. Guess: $\lim_{n\to\infty} \int_0^1 n^2xe^{-nx}dx = 1$

$\int_0^1 n^2xe^{-nx}dx = \left[-nxe^{-nx} - e^{-nx}\right]_0^1$

$= -(n+1)e^{-n} + 1 = 1 - \frac{n+1}{e^n}$

$\lim_{n\to\infty} \int_0^1 n^2xe^{-nx}dx = \lim_{n\to\infty}\left(1 - \frac{n+1}{e^n}\right)$

$= 1 - \lim_{n\to\infty} \frac{n+1}{e^n}$ if this last limit exists. The limit is of the form $\frac{\infty}{\infty}$.

$\lim_{n\to\infty} \frac{n+1}{e^n} = \lim_{n\to\infty} \frac{1}{e^n} = 0$, so

$\lim_{n\to\infty} \int_0^1 n^2xe^{-nx}dx = 1$.

49. Note $f(x) > 0$ on $[0, \infty)$.

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\frac{x^{25}}{e^x} + \frac{x^3}{e^x} + \left(\frac{2}{e}\right)^x\right) = 0$

Therefore there is no absolute minimum.

$f'(x) = (25x^{24} + 3x^2 + 2^x \ln 2)e^{-x}$
$\quad - (x^{25} + x^3 + 2^x)e^{-x}$
$= (-x^{25} + 25x^{24} - x^3 + 3x^2 - 2^x + 2^x \ln 2)e^{-x}$

Solve for x when $f'(x) = 0$. Using a numerical method, $x \approx 25$.

A graph using a computer algebra system verifies that an absolute maximum occurs at about $x = 25$.

9.3 Concepts Review

1. converge

2. $\lim_{b\to\infty} \int_0^b \cos x\, dx$

3. $\int_{-\infty}^0 f(x)dx;\ \int_0^\infty f(x)dx$

4. $p > 1$

Problem Set 9.3

In this section and the chapter review, it is understood that $[g(x)]_a^\infty$ means $\lim_{b\to\infty} [g(x)]_a^b$ and likewise for similar expressions.

1. $\int_{100}^\infty e^x dx = \left[e^x\right]_{100}^\infty = \infty - e^{100} = \infty$

The integral diverges.

2. $\int_{-\infty}^{-5} \frac{dx}{x^4} = \left[-\frac{1}{3x^3}\right]_{-\infty}^{-5} = -\frac{1}{3(-125)} - 0 = \frac{1}{375}$

3. $\int_1^\infty 2xe^{-x^2} dx = \left[-e^{-x^2}\right]_1^\infty = 0 - (-e^{-1}) = \frac{1}{e}$

4. $\int_{-\infty}^1 e^{4x}dx = \left[\frac{1}{4}e^{4x}\right]_{-\infty}^1 = \frac{1}{4}e^4 - 0 = \frac{1}{4}e^4$

5. $\int_9^\infty \frac{x\,dx}{\sqrt{1+x^2}} = \left[\sqrt{1+x^2}\right]_9^\infty = \infty - \sqrt{82} = \infty$

The integral diverges.

6. $\int_1^\infty \frac{dx}{\sqrt{\pi x}} = \left[2\sqrt{\frac{x}{\pi}}\right]_1^\infty = \infty - \frac{2}{\sqrt{\pi}} = \infty$

The integral diverges.

7. $\int_1^\infty \frac{dx}{x^{1.00001}} = \left[-\frac{1}{0.00001x^{0.00001}}\right]_1^\infty$

$= 0 - \left(-\frac{1}{0.00001}\right) = \frac{1}{0.00001} = 100,000$

8. $\int_{10}^\infty \frac{x}{1+x^2}dx = \frac{1}{2}\left[\ln(1+x^2)\right]_{10}^\infty$

$= \infty - \frac{1}{2}\ln|101| = \infty$

The integral diverges.

9. $\int_1^\infty \frac{dx}{x^{0.99999}} = \left[\frac{x^{0.00001}}{0.00001}\right]_1^\infty = \infty - 100,000 = \infty$

The integral diverges.

10. $\int_1^\infty \frac{x}{(1+x^2)^2}dx = \left[-\frac{1}{2(1+x^2)}\right]_1^\infty$

$= 0 - \left(-\frac{1}{4}\right) = \frac{1}{4}$

11. $\int_e^\infty \frac{1}{x\ln x}dx = [\ln(\ln x)]_e^\infty = \infty - 0 = \infty$

The integral diverges.

12. $\int_e^\infty \frac{\ln x}{x}dx = \left[\frac{1}{2}(\ln x)^2\right]_e^\infty = \infty - \frac{1}{2} = \infty$

The integral diverges.

13. Let $u = \ln x,\ du = \frac{1}{x}dx, dv = \frac{1}{x^2}dx, v = -\frac{1}{x}.$

$\int_2^\infty \frac{\ln x}{x^2}dx = \left[-\frac{\ln x}{x}\right]_2^\infty + \int_2^\infty \frac{1}{x^2}dx$

$= \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_2^\infty = \lim_{b\to\infty}\left(-\frac{\ln x + 1}{x}\right) + \frac{\ln 2 + 1}{2}$

$= \frac{\ln 2 + 1}{2}$

14. $\int_1^\infty xe^{-x}dx$

$u = x,\ du = dx$

$dv = e^{-x}dx,\ v = -e^{-x}$

$\int_1^\infty xe^{-x}d = \left[-xe^{-x}\right]_1^\infty + \int_1^\infty e^{-x}dx$

$= \left[-xe^{-x} - e^{-x}\right]_1^\infty = 0 - 0 - (-e^{-1} - e^{-1}) = \frac{2}{e}$

15. $\int_{-\infty}^1 \frac{dx}{(2x-3)^3} = \left[-\frac{1}{4(2x-3)^2}\right]_{-\infty}^1$

$= -\frac{1}{4} - (-0) = -\frac{1}{4}$

16. $\int_4^\infty \frac{dx}{(\pi - x)^{2/3}} = \left[-3(\pi - x)^{1/3}\right]_4^\infty$

$= \infty + 3\sqrt[3]{\pi - 4} = \infty$

The integral diverges.

17. $\int_{-\infty}^\infty \frac{x}{\sqrt{x^2+9}}dx = \int_{-\infty}^0 \frac{x}{\sqrt{x^2+9}}dx + \int_0^\infty \frac{x}{\sqrt{x^2+9}}dx = \left[\sqrt{x^2+9}\right]_{-\infty}^0 + \left[\sqrt{x^2+9}\right]_0^\infty = (3-\infty) + (\infty - 3)$

The integral diverges since both $\int_{-\infty}^0 \frac{x}{\sqrt{x^2+9}}dx$ and $\int_0^\infty \frac{x}{\sqrt{x^2+9}}dx$ diverge.

18. $\int_{-\infty}^\infty \frac{dx}{(x^2+16)^2} = \int_{-\infty}^0 \frac{dx}{(x^2+16)^2} + \int_0^\infty \frac{dx}{(x^2+16)^2}$

$\int \frac{dx}{(x^2+16)^2} = \frac{1}{128}\tan^{-1}\frac{x}{4} + \frac{x}{32(x^2+16)}$ by using the substitution $x = 4\tan\theta$.

$\int_{-\infty}^0 \frac{dx}{(x^2+16)^2} = \left[\frac{1}{128}\tan^{-1}\frac{x}{4} + \frac{x}{32(x^2+16)}\right]_{-\infty}^0 = 0 - \left[\frac{1}{128}\left(-\frac{\pi}{2}\right) + 0\right] = \frac{\pi}{256}$

$$\int_0^{\infty}\frac{dx}{(x^2+16)^2}=\left[\frac{1}{128}\tan^{-1}\frac{x}{4}+\frac{x}{32(x^2+16)}\right]_0^{\infty}=\frac{1}{128}\left(\frac{\pi}{2}\right)+0-(0)=\frac{\pi}{256}$$

$$\int_{-\infty}^{\infty}\frac{dx}{(x^2+16)^2}=\frac{\pi}{256}+\frac{\pi}{256}=\frac{\pi}{128}$$

19. $\int_{-\infty}^{\infty}\frac{1}{x^2+2x+10}dx=\int_{-\infty}^{\infty}\frac{1}{(x+1)^2+9}dx=\int_{-\infty}^{0}\frac{1}{(x+1)^2+9}dx+\int_0^{\infty}\frac{1}{(x+1)^2+9}dx$

$\int\frac{1}{(x+1)^2+9}dx=\frac{1}{3}\tan^{-1}\frac{x+1}{3}$ by using the substitution $x + 1 = 3\tan\theta$.

$$\int_{-\infty}^{0}\frac{1}{(x+1)^2+9}dx=\left[\frac{1}{3}\tan^{-1}\frac{x+1}{3}\right]_{-\infty}^{0}=\frac{1}{3}\tan^{-1}\frac{1}{3}-\frac{1}{3}\left(-\frac{\pi}{2}\right)=\frac{1}{6}\left(\pi+2\tan^{-1}\frac{1}{3}\right)$$

$$\int_0^{\infty}\frac{1}{(x+1)^2+9}dx=\left[\frac{1}{3}\tan^{-1}\frac{x+1}{3}\right]_0^{\infty}=\frac{1}{3}\left(\frac{\pi}{2}\right)-\frac{1}{3}\tan^{-1}\frac{1}{3}=\frac{1}{6}\left(\pi-2\tan^{-1}\frac{1}{3}\right)$$

$$\int_{-\infty}^{\infty}\frac{1}{x^2+2x+10}dx=\frac{1}{6}\left(\pi+2\tan^{-1}\frac{1}{3}\right)+\frac{1}{6}\left(\pi-2\tan^{-1}\frac{1}{3}\right)=\frac{\pi}{3}$$

20. $\int_{-\infty}^{\infty}\frac{x}{e^{2|x|}}dx=\int_{-\infty}^{0}\frac{x}{e^{-2x}}dx+\int_0^{\infty}\frac{x}{e^{2x}}dx$

For $\int_{-\infty}^{0}\frac{x}{e^{-2x}}dx=\int_{-\infty}^{0}xe^{2x}dx,$ use $u=x,\ du=dx,\ dv=e^{2x}dx,\ v=\frac{1}{2}e^{2x}$.

$$\int_{-\infty}^{0}xe^{2x}dx=\left[\frac{1}{2}xe^{2x}\right]_{-\infty}^{0}-\frac{1}{2}\int_{-\infty}^{0}e^{2x}dx=\left[\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}\right]_{-\infty}^{0}=0-\frac{1}{4}-(0)=-\frac{1}{4}$$

For $\int_0^{\infty}\frac{x}{e^{2x}}dx=\int_0^{\infty}xe^{-2x}dx,$ use $u=x,\ du=dx,\ dv=e^{-2x}dx,\ v=-\frac{1}{2}e^{-2x}$.

$$\int_0^{\infty}xe^{-2x}dx=\left[-\frac{1}{2}xe^{-2x}\right]_0^{\infty}+\frac{1}{2}\int_0^{\infty}e^{-2x}dx=\left[-\frac{1}{2}xe^{-2x}-\frac{1}{4}e^{-2x}\right]_0^{\infty}=0-\left(0-\frac{1}{4}\right)=\frac{1}{4}$$

$$\int_{-\infty}^{\infty}\frac{x}{e^{2|x|}}dx=-\frac{1}{4}+\frac{1}{4}=0$$

21. $\int_{-\infty}^{\infty}\operatorname{sech}x\,dx=\int_{-\infty}^{0}\operatorname{sech}x\,dx=\int_0^{\infty}\operatorname{sech}x\,dx$

$=[\tan^{-1}(\sinh x)]_{-\infty}^{0}+[\tan^{-1}(\sinh x)]_0^{\infty}$

$=\left[0-\left(-\frac{\pi}{2}\right)\right]+\left[\frac{\pi}{2}-0\right]=\pi$

22. $\int_1^{\infty}\operatorname{csch}x\,dx=\int_1^{\infty}\frac{1}{\sinh x}dx=\int_1^{\infty}\frac{2}{e^x-e^{-x}}dx$

$=\int_1^{\infty}\frac{2e^x}{e^{2x}-1}dx$

Let $u=e^x, du=e^x dx$.

$\int_1^{\infty}\frac{2e^x}{e^{2x}-1}dx=\int_e^{\infty}\frac{2}{u^2-1}du=\int_e^{\infty}\left(\frac{1}{u-1}-\frac{1}{u+1}\right)du$

$=[\ln(u-1)-\ln(u+1)]_e^{\infty}=\left[\ln\frac{u-1}{u+1}\right]_e^{\infty}$

$=0-\ln\frac{e-1}{e+1}\approx 0.7719$

$\left(\lim_{b\to\infty}\ln\frac{b-1}{b+1}=0\text{ since }\lim_{b\to\infty}\frac{b-1}{b+1}=1\right)$

23. $\int_0^{\infty}e^{-x}\cos x\,dx=\left[\frac{1}{2e^x}(\sin x-\cos x)\right]_0^{\infty}$

$=0-\frac{1}{2}(0-1)=\frac{1}{2}$

(Use Formula 68 with $a=-1$ and $b=1$.)

24. $\int_0^{\infty}e^{-x}\sin x\,dx=\left[-\frac{1}{2e^x}(\cos x+\sin x)\right]_0^{\infty}$

$=0+\frac{1}{2}(1+0)=\frac{1}{2}$

(Use Formula 67 with $a=-1$ and $b=1$.)

25. The area is given by

$$\int_1^\infty \frac{2}{4x^2-1}dx = \int_1^\infty \left(\frac{1}{2x-1} - \frac{1}{2x+1}\right)dx$$

$$= \frac{1}{2}\left[\ln|2x-1| - \ln|2x+1|\right]_1^\infty = \frac{1}{2}\left[\ln\left|\frac{2x-1}{2x+1}\right|\right]_1^\infty$$

$$= \frac{1}{2}\left(0 - \ln\left(\frac{1}{3}\right)\right) = \frac{1}{2}\ln 3$$

Note:. $\lim_{x\to\infty} \ln = \left|\frac{2x-1}{2x+1}\right| = 0$ since

$\lim_{x\to\infty}\left(\frac{2x-1}{2x+1}\right) = 1$.

26. The area is

$$\int_1^\infty \frac{1}{x^2+x}dx = \int_1^\infty\left(\frac{1}{x} - \frac{1}{x+1}\right)dx$$

$$= \left[\ln|x| - \ln|x+1|\right]_1^\infty = \left[\ln\left|\frac{x}{x+1}\right|\right]_1^\infty = 0 - \ln\frac{1}{2} = \ln 2$$

27. The integral would take the form

$$k\int_{3960}^\infty \frac{1}{x}dx = [k\ln x]_{3960}^\infty = \infty$$

which would make it impossible to send anything out of the earth's gravitational field.

28. At $x = 1080$ mi, $F = 165$, so $k = 165(1080)^2 \approx 1.925\times 10^8$. So the work done in mi-lb is

$$1.925\times 10^8 \int_{1080}^\infty \frac{1}{x^2}dx = 1.925\times 10^8\left[-x^{-1}\right]_{1080}^\infty$$

$$= \frac{1.925\times 10^8}{1080} \approx 1.782\times 10^5 \text{ mi-lb.}$$

29. $FP = \int_0^\infty e^{-rt} f(t)\,dt = \int_0^\infty 100{,}000e^{-0.08t}$

$$= \left[-\frac{1}{0.08}100{,}000e^{-0.08t}\right]_0^\infty = 1{,}250{,}000$$

The present value is \$1,250,000.

30. $FP = \int_0^\infty e^{-0.08t}(100{,}000 + 1000t)dt$

$$= \left[-1{,}250{,}000e^{-0.08t} - 12{,}500te^{-0.08t} - 156{,}250e^{-0.08t}\right]_0^\infty = 1{,}406{,}250$$

The present value is \$1,406,250.

31. **a.**

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a 0\,dx + \int_a^b \frac{1}{b-a}dx + \int_b^\infty 0\,dx$$

$$= 0 + \frac{1}{b-a}[x]_a^b + 0$$

$$= \frac{1}{b-a}(b-a)$$

b.

$$\mu = \int_{-\infty}^\infty x f(x)\,dx$$

$$= \int_{-\infty}^a x\cdot 0\,dx + \int_a^b x\frac{1}{b-a}dx + \int_b^\infty x\cdot 0\,dx$$

$$= 0 + \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b + 0$$

$$= \frac{b^2-a^2}{2(b-a)}$$

$$= \frac{(b+a)(b-a)}{2(b-a)}$$

$$= \frac{a+b}{2}$$

$$\sigma^2 = \int_{-\infty}^\infty (x-\mu)^2\,dx$$

$$= \int_{-\infty}^a (x-\mu)^2\cdot 0\,dx + \int_a^b (x-\mu)^2\frac{1}{b-a}dx + \int_b^\infty (x-\mu)^2\cdot 0\,dx$$

$$= 0 + \frac{1}{b-a}\left[\frac{(x-\mu)^3}{3}\right]_a^b + 0$$

$$= \frac{1}{b-a}\frac{(b-\mu)^3 - (a-\mu)^3}{3}$$

$$= \frac{1}{b-a}\frac{b^3 - 3b^2\mu + 3b\mu^2 - a^3 + 3a^2\mu - 3a\mu^2}{3}$$

Next, substitute $\mu = (a+b)/2$ to obtain

$$\sigma^2 = \frac{1}{3(b-a)}\left[\tfrac{1}{4}b^3 - \tfrac{3}{4}b^2 a + \tfrac{3}{4}ba^2 - \tfrac{1}{4}a^3\right]$$

$$= \frac{1}{12(b-a)}(b-a)^3$$

$$= \frac{(b-a)^2}{12}$$

c. $P(X < 2) = \int_{-\infty}^{2} f(x)\,dx$

$$= \int_{-\infty}^{0} 0\,dx + \int_0^2 \frac{1}{10-0}\,dx$$

$$= \frac{2}{10} = \frac{1}{5}$$

32. a. $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_0^{\infty} \frac{\beta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-(x/\theta)^\beta}\,dx$

In the second integral, let $u = (x/\theta)^\beta$. Then, $du = (\beta/\theta)(t/\theta)^{\beta-1}dt$. When $x = 0, u = 0$ and when $x \to \infty, u \to \infty$. Thus,

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} \frac{\beta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-(x/\theta)^\beta}\,dx$$

$$= \int_0^{\infty} e^{-u}\,du$$

$$= \left[-e^{-u}\right]_0^{\infty}$$

$$= -0 + e^0 = 1$$

b. $\mu = \int_{-\infty}^{\infty} xf(x)\,dx = \int_{-\infty}^{0} x \cdot 0\,dx + \int_0^{\infty} \frac{\beta}{\theta}x\left(\frac{x}{\theta}\right)^{\beta-1} e^{-(x/\theta)}\,dx\partial$

$$= \frac{2}{3}\int_0^{\infty} x^2 e^{-(x/3)^2}\,dx = \frac{3}{2}\sqrt{\pi}$$

$$\sigma^2 = \int_{-\infty}^{\infty}(x-\mu)^2 f(x)\,dx = \int_{-\infty}^{0}(x-\mu)^2 \cdot 0\,dx + \frac{2}{9}\int_0^{\infty}(x-\mu)^2 xe^{-(x^2/9)}\,dx$$

$$= \frac{3}{2}\sqrt{\pi} - \mu = \frac{3}{2}\sqrt{\pi} - \frac{3}{2}\sqrt{\pi} = 0$$

c. The probability of being less than 2 is

$$\int_{-\infty}^{2} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_0^2 \frac{\beta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-(x/\theta)^\beta}\,dx = 0 + \left[-e^{-(x/\theta)^\beta}\right]_0^2$$

$$= 1 - e^{-(2/\theta)^\beta} = 1 - e^{-(2/3)^2} \approx 0.359$$

33. a. $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} \alpha e^{-\alpha x}\,dx$

$= 0 + \left[-e^{-\alpha x}\right]_0^{\infty} = 1$

b. $\mu = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{0} x \cdot 0\,dx + \int_{0}^{\infty} x\alpha e^{-\alpha x}\,dx$

$= \int_{0}^{\infty} x\alpha e^{-\alpha x}\,dx$

Integrate by parts: Let $u = x,\ dv = \alpha e^{-\alpha x}$. Then $du = dx,\ v = -e^{\alpha x}$. Thus,

$\mu = \left[-xe^{-\alpha x}\right]_0^{\infty} - \int_0^{\infty}\left(-e^{-\alpha x}\right)dx$

$= [0-0] + \int_0^{\infty} e^{-\alpha x}\,dx$

$= \left[-\frac{1}{\alpha}e^{-\alpha x}\right]_0^{\infty} = -0 + \frac{1}{\alpha} = \frac{1}{\alpha}$

$\sigma^2 = \int_{-\infty}^{\infty}(x-\mu)^2 f(x)\,dx$

$= \int_{-\infty}^{0}(x-\mu)^2 \cdot 0\,dx + \int_0^{\infty}(x-\mu)^2\alpha e^{-\alpha x}\,dx$

$= 0 + \int_0^{\infty}\left(x^2 - 2x\mu + \mu^2\right)\alpha e^{-\alpha x}dx$

$= \int_0^{\infty} x^2\alpha e^{-\alpha x}dx - 2\mu\int_0^{\infty} x\alpha e^{-\alpha x}dx$

$+ \mu^2\int_0^{\infty}\alpha e^{-\alpha x}\,dx$

$= \int_0^{\infty} x^2\alpha e^{-\alpha x}dx - 2\frac{1}{\alpha}\left(\frac{1}{\alpha}\right) + \left(\frac{1}{\alpha}\right)^2$

On this last line, we have used the results that $\mu = 1/\alpha$, $\int_0^{\infty}\alpha e^{-\alpha x}dx = 1$, and $\int_0^{\infty} x\alpha e^{-\alpha x}dx = 1/\alpha$. To evaluate the integral, use integration by parts. Let $u = x^2, dv = \alpha e^{-\alpha x}$. Then, $du = 2x\,dx$ and $v = -e^{-\alpha x}$. Thus

$\sigma^2 = \left[x^2e^{-\alpha x}\right]_0^{\infty} - \int_0^{\infty}\left(-e^{-\alpha x}\right)2x\,dx$

$-2\frac{1}{\alpha}\left(\frac{1}{\alpha}\right) + \left(\frac{1}{\alpha}\right)^2$

$= 0 + \frac{2}{\alpha}\int_0^{\infty} x\alpha e^{-\alpha x}\,dx - 2\frac{1}{\alpha^2} + \frac{1}{\alpha^2}$

$= 2\frac{1}{\alpha^2} - 2\frac{1}{\alpha^2} + \frac{1}{\alpha^2} = \frac{1}{\alpha^2}$

34. $u = Ar\int_a^{\infty}\frac{dx}{(r^2+x^2)^{3/2}}$

$= \frac{A}{r}\left[\frac{x}{\sqrt{r^2+x^2}}\right]_a^{\infty} = \frac{A}{r}\left(1 - \frac{a}{\sqrt{r^2+a^2}}\right)$

Note that $\int\frac{dx}{(r^2+x^2)^{3/2}} = \frac{x}{r^2\sqrt{r^2+x^2}}$ by using the substitution $x = r\tan\theta$.

35. a. $\int_{-\infty}^{\infty} \sin x\,dx = \int_{-\infty}^{0} \sin x\,dx + \int_{0}^{\infty} \sin x\,dx$

$= \lim_{a\to\infty} [-\cos x]_0^a + \lim_{a\to-\infty} [-\cos x]_a^0$

Both do not converge since $-\cos x$ is oscillating between -1 and 1, so the integral diverges.

b. $\lim_{a\to\infty} \int_{-a}^{a} \sin x\,dx = \lim_{a\to\infty} [-\cos x]_{-a}^{a}$

$= \lim_{a\to\infty} [-\cos a + \cos(-a)]$

$= \lim_{a\to\infty} [-\cos a + \cos a] = \lim_{a\to\infty} 0 = 0$

36. a. The total mass of the wire is $\int_0^{\infty} \frac{1}{1+x^2}dx = \frac{\pi}{2}$ from Example 4.

b. $\int_0^{\infty} \frac{x}{1+x^2}dx = \left[\frac{1}{2}\ln\left|1+x^2\right|\right]_0^{\infty}$ which diverges. Thus, the wire does not have a center of mass.

37. For example, the region under the curve $y = \frac{1}{x}$ to the right of $x = 1$.
Rotated about the x-axis the volume is $\pi\int_1^{\infty} \frac{1}{x^2}dx = \pi$. Rotated about the y-axis, the volume is $2\pi\int_1^{\infty} \frac{1}{x}dx$ which diverges.

38. a. Suppose $\lim_{x\to\infty} f(x) = M \neq 0$, so the limit exists but is non-zero. Since $\lim_{x\to\infty} f(x) = M$, there is some $N > 0$ such that when $x \geq N$,

$|f(x) - M| \leq \frac{M}{2}$, or

$M - \frac{M}{2} \leq f(x) \leq M + \frac{M}{2}$

Since $f(x)$ is nonnegative, $M > 0$, thus $\frac{M}{2} > 0$ and

$\int_0^{\infty} f(x)dx = \int_0^N f(x)dx + \int_N^{\infty} f(x)dx$

$\geq \int_0^N f(x)dx + \int_N^{\infty} \frac{M}{2}dx = \int_0^N f(x)dx + \left[\frac{Mx}{2}\right]_N^{\infty} = \infty$

so the integral diverges. Thus, if the limit exists, it must be 0.

b. For example, let $f(x)$ be given by

$$f(x) = \begin{cases} 2n^2x - 2n^3 + 1 & \text{if } n - \frac{1}{2n^2} \leq x \leq n \\ -2n^2x + 2n^3 + 1 & \text{if } n < x \leq n + \frac{1}{2n^2} \\ 0 & \text{otherwise} \end{cases}$$

for every positive integer n.

$f\left(n - \frac{1}{2n^2}\right) = 2n^2\left(n - \frac{1}{2n^2}\right) - 2n^3 + 1$

$= 2n^3 - 1 - 2n^3 + 1 = 0$

$f(n) = 2n^2(n) - 2n^3 + 1 = 1$

$\lim_{x\to n^+} f(n) = \lim_{x\to n^+} (-2n^2x + 2n^3 + 1) = 1 = f(n)$

$f\left(n + \frac{1}{2n^2}\right) = -2n^2\left(n + \frac{1}{2n^2}\right) + 2n^3 + 1$

$= -2n^3 - 1 + 2n^3 + 1 = 0$

Thus, f is continuous at $n - \frac{1}{2n^2}$, n, and $n + \frac{1}{2n^2}$.

Note that the intervals $\left[n, n + \frac{1}{2n^2}\right]$ and $\left[n + 1 - \frac{1}{2(n+1)^2}, n + 1\right]$ will never overlap since $\frac{1}{2n^2} \leq \frac{1}{2}$ and $\frac{1}{2(n+1)^2} \leq \frac{1}{8}$.

The graph of f consists of a series of isosceles triangles, each of height 1, vertices at $\left(n - \frac{1}{2n^2}, 0\right)$, $(n, 1)$, and $\left(n + \frac{1}{2n^2}, 0\right)$, based on the x-axis, and centered over each integer n.

$\lim_{x\to\infty} f(x)$ does not exist, since $f(x)$ will be 1 at each integer, but 0 between the triangles.

Each triangle has area

$\frac{1}{2}bh = \frac{1}{2}\left[n + \frac{1}{2n^2} - \left(n - \frac{1}{2n^2}\right)\right](1)$

$= \frac{1}{2}\left(\frac{1}{n^2}\right) = \frac{1}{2n^2}$

$\int_0^{\infty} f(x)dx$ is the area in all of the triangles, thus

$\int_0^{\infty} f(x)dx = \sum_{n=1}^{\infty} \frac{1}{2n^2} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$= \frac{1}{2} + \frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{n^2} \le \frac{1}{2} + \frac{1}{2}\int_1^{\infty}\frac{1}{x^2}dx$$

$$= \frac{1}{2} + \frac{1}{2}\left[-\frac{1}{x}\right]_1^{\infty} = \frac{1}{2} + \frac{1}{2}(-0+1) = 1$$

(By viewing $\sum_{n=2}^{\infty}\frac{1}{n^2}$ as a lower Riemann sum for $\frac{1}{x^2}$)

Thus, $\int_0^{\infty} f(x)dx$ converges, although $\lim_{x\to\infty} f(x)$ does not exist.

39. $\int_1^{100}\frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_1^{100} = 0.99$

$\int_1^{100}\frac{1}{x^{1.1}}dx = \left[-\frac{1}{0.1x^{0.1}}\right]_1^{100} \approx 3.69$

$\int_1^{100}\frac{1}{x^{1.01}}dx = \left[-\frac{1}{0.01x^{0.01}}\right]_1^{100} \approx 4.50$

$\int_1^{100}\frac{1}{x}dx = [\ln x]_1^{100} = \ln 100 \approx 4.61$

$\int_1^{100}\frac{1}{x^{0.99}}dx = \left[\frac{x^{0.01}}{0.01}\right]_1^{100} \approx 4.71$

40. $\int_{-10}^{10}\frac{1}{\pi(1+x^2)}dx = \frac{1}{\pi}\left[\tan^{-1}x\right]_{-10}^{10}$

$\approx \frac{2.9423}{\pi} \approx 0.937$

$\int_{-50}^{50}\frac{1}{\pi(1+x^2)}dx = \frac{1}{\pi}\left[\tan^{-1}x\right]_{-50}^{50}$

$\approx \frac{3.1016}{\pi} \approx 0.987$

$\int_{-100}^{100}\frac{1}{\pi(1+x^2)}dx = \frac{1}{\pi}\left[\tan^{-1}x\right]_{-100}^{100}$

$\approx \frac{3.1216}{\pi} \approx 0.994$

41. $\int_{-1}^{1}\frac{1}{\sqrt{2\pi}}\exp(-0.5x^2)dx \approx 0.6827$

$\int_{-2}^{2}\frac{1}{\sqrt{2\pi}}\exp(-0.5x^2)dx \approx 0.9545$

$\int_{-3}^{3}\frac{1}{\sqrt{2\pi}}\exp(-0.5x^2)dx \approx 0.9973$

$\int_{-4}^{4}\frac{1}{\sqrt{2\pi}}\exp(-0.5x^2)dx \approx 0.9999$

9.4 Concepts Review

1. unbounded
2. 2
3. $\lim_{x\to 4^-}\int_0^b \frac{1}{\sqrt{4-x}}dx$
4. $p < 1$

Problem Set 9.4

1. $\int_1^3 \frac{dx}{(x-1)^{1/3}} = \lim_{b\to 1^+}\left[\frac{3(x-1)^{2/3}}{2}\right]_b^3$

$= \frac{3}{2}\sqrt[3]{2^2} - \lim_{b\to 1^+}\frac{3(b-1)^{2/3}}{2} = \frac{3}{\sqrt[3]{2}} - 0 = \frac{3}{\sqrt[3]{2}}$

2. $\int_1^3 \frac{dx}{(x-1)^{4/3}} = \lim_{b\to 1^+}\left[-\frac{3}{(x-1)^{1/3}}\right]_b^3$

$= -\frac{3}{\sqrt[3]{2}} + \lim_{b\to 1^+}\frac{3}{(x-1)^{1/3}} = -\frac{3}{\sqrt[3]{2}} + \infty$

The integral diverges.

3. $\int_3^{10}\frac{dx}{\sqrt{x-3}} = \lim_{b\to 3^+}\left[2\sqrt{x-3}\right]_b^{10}$

$= 2\sqrt{7} - \lim_{b\to 3^+} 2\sqrt{b-3} = 2\sqrt{7}$

4. $\int_0^9 \frac{dx}{\sqrt{9-x}} = \lim_{b\to 9^-}\left[-2\sqrt{9-x}\right]_0^b$

$= \lim_{b\to 9^-} -2\sqrt{9-b} + 2\sqrt{9} = 6$

5. $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{b\to 1^-}\left[\sin^{-1}x\right]_0^b$

$= \lim_{b\to 1^-}\sin^{-1}b - \sin^{-1}0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

6. $\int_{100}^{\infty} \frac{x}{\sqrt{1+x^2}}dx = \lim_{b\to\infty}\left[\sqrt{1+x^2}\right]_{100}^{b}$

$= \lim_{b\to\infty}\sqrt{1+b^2} + \sqrt{10,001} = \infty$

The integral diverges.

$= \left(\lim_{b\to 0^-} -\frac{1}{2b^2}+\frac{1}{2}\right)+\left(-\frac{1}{18}+\lim_{b\to 0^+}\frac{1}{2b^2}\right)$

$= \left(-\infty+\frac{1}{2}\right)+\left(-\frac{1}{8}+\infty\right)$

The integral diverges.

7. $\int_{-1}^{3}\frac{1}{x^3}dx = \lim_{b\to 0^-}\int_{-1}^{b}\frac{1}{x^3}dx + \lim_{b\to 0^+}\int_{b}^{3}\frac{1}{x^3}dx$

$= \lim_{b\to 0^-}\left[-\frac{1}{2x^2}\right]_{-1}^{b} + \lim_{b\to 0^+}\left[-\frac{1}{2x^2}\right]_{b}^{3}$

8. $\int_{5}^{-5}\frac{1}{x^{2/3}}dx = \lim_{b\to 0^+}\int_{5}^{b}\frac{1}{x^{2/3}}dx + \lim_{b\to 0^-}\int_{b}^{-5}\frac{1}{x^{2/3}}dx = \lim_{b\to 0^+}\left[3x^{1/3}\right]_{5}^{b} + \lim_{b\to 0^-}\left[3x^{1/3}\right]_{b}^{-5}$

$= \lim_{b\to 0^+} 3b^{1/3} - 3\sqrt[3]{5} + 3\sqrt[3]{-5} - \lim_{b\to 0^-} 3b^{1/3} = 0 - 3\sqrt[3]{5} + 3\sqrt[3]{5} - 0 = 3\sqrt[3]{-5} - 3\sqrt[3]{5}$

9. $\int_{-1}^{128} x^{-5/7}dx = \lim_{b\to 0^-}\int_{-1}^{b} x^{-5/7}dx + \lim_{b\to 0^+}\int_{b}^{128} x^{-5/7}dx = \lim_{b\to 0^-}\left[\frac{7}{2}x^{2/7}\right]_{-1}^{b} + \lim_{b\to 0^+}\left[\frac{7}{2}x^{2/7}\right]_{b}^{128}$

$= \lim_{b\to 0^-}\frac{7}{2}b^{2/7} - \frac{7}{2}(-1)^{2/7} + \frac{7}{2}(128)^{2/7} - \lim_{b\to 0^+}\frac{7}{2}b^{2/7} = 0 - \frac{7}{2} + \frac{7}{2}(4) - 0 = \frac{21}{2}$

10. $\int_{0}^{1}\frac{x}{\sqrt[3]{1-x^2}}dx = \lim_{b\to 1^-}\int_{0}^{b}\frac{x}{\sqrt[3]{1-x^2}}dx = \lim_{b\to 1^-}\left[-\frac{3}{4}(1-x^2)^{2/3}\right]_{0}^{b} = \lim_{b\to 1^-} -\frac{3}{4}(1-b^2)^{2/3} + \frac{3}{4} = -0 + \frac{3}{4} = \frac{3}{4}$

11. $\int_{0}^{4}\frac{dx}{(2-3x)^{1/3}} = \lim_{b\to \frac{2}{3}^-}\int_{0}^{b}\frac{dx}{(2-3x)^{1/3}} + \lim_{b\to \frac{2}{3}^+}\int_{b}^{4}\frac{dx}{(2-3x)^{1/3}} = \lim_{b\to \frac{2}{3}^-}\left[-\frac{1}{2}(2-3x)^{2/3}\right]_{0}^{b} + \lim_{b\to \frac{2}{3}^+}\left[-\frac{1}{2}(2-3x)^{2/3}\right]_{b}^{4}$

$= \lim_{b\to \frac{2}{3}^-} -\frac{1}{2}(2-3b)^{2/3} + \frac{1}{2}(2)^{2/3} - \frac{1}{2}(-10)^{2/3} + \lim_{b\to \frac{2}{3}^+}\frac{1}{2}(2-3b)^{2/3}$

$= 0 + \frac{1}{2}2^{2/3} - \frac{1}{2}10^{2/3} + 0 = \frac{1}{2}(2^{2/3} - 10^{2/3})$

12. $\int_{\sqrt{5}}^{\sqrt{8}}\frac{x}{(16-2x^2)^{2/3}}dx = \lim_{b\to\sqrt{8}^-}\left[-\frac{3}{4}(16-2x^2)^{1/3}\right]_{\sqrt{5}}^{b} = \lim_{b\to\sqrt{8}^-} -\frac{3}{4}(16-2b^2)^{1/3} + \frac{3}{4}\sqrt[3]{6} = \frac{3}{4}\sqrt[3]{6}$

13. $\int_{0}^{-4}\frac{x}{16-2x^2}dx = \lim_{b\to -\sqrt{8}^+}\int_{0}^{b}\frac{x}{16-2x^2}dx + \lim_{b\to -\sqrt{8}^-}\int_{b}^{-4}\frac{x}{16-2x^2}dx$

$= \lim_{b\to -\sqrt{8}^+}\left[-\frac{1}{4}\ln\left|16-2x^2\right|\right]_{0}^{b} + \lim_{b\to -\sqrt{8}^-}\left[-\frac{1}{4}\ln\left|16-2x^2\right|\right]_{b}^{-4}$

$= \lim_{b\to -\sqrt{8}^+} -\frac{1}{4}\ln\left|16-2b^2\right| + \frac{1}{4}\ln 16 - \frac{1}{4}\ln 16 + \lim_{b\to -\sqrt{8}^-}\frac{1}{4}\ln\left|16-2b^2\right|$

$= \left[-(-\infty) + \frac{1}{4}\ln 16\right] + \left[-\frac{1}{4}\ln 16 + (-\infty)\right]$

The integral diverges.

14. $\int_0^3 \frac{x}{\sqrt{9-x^2}}dx = \lim_{b\to 3^-}\left[-\sqrt{9-x^2}\right]_0^b = \lim_{b\to 3^-} -\sqrt{9-b^2}+\sqrt{9} = 3$

15. $\int_{-2}^{-1} \frac{dx}{(x+1)^{4/3}} = \lim_{b\to -1^-}\left[-\frac{3}{(x+1)^{1/3}}\right]_{-2}^b = \lim_{b\to -1^-} -\frac{3}{(b+1)^{1/3}}+\frac{3}{(-1)^{1/3}} = -(-\infty)-3$

The integral diverges.

16. Note that $\int \frac{dx}{x^2+x-2} = \int \frac{dx}{(x-1)(x+2)} = \int\left[\frac{1}{3(x-1)}-\frac{1}{3(x+2)}\right]dx$ by using a partial fraction decomposition.

$\int_0^3 \frac{dx}{x^2+x-2} = \lim_{b\to 1^-}\int_0^b \frac{dx}{x^2+x-2} + \lim_{b\to 1^+}\int_b^3 \frac{dx}{x^2+x-2}$

$= \lim_{b\to 1^-}\left[\frac{1}{3}\ln|x-1|-\frac{1}{3}\ln|x+2|\right]_0^b + \lim_{b\to 1^+}\left[\frac{1}{3}\ln|x-1|-\frac{1}{3}\ln|x+2|\right]_b^3$

$= \lim_{b\to 1^-}\left[\frac{1}{3}\ln\left|\frac{x-1}{x+2}\right|\right]_0^b + \lim_{b\to 1^+}\left[\frac{1}{3}\ln\left|\frac{x-1}{x+2}\right|\right]_b^3 = \lim_{b\to 1^-}\frac{1}{3}\ln\left|\frac{b-1}{b+2}\right|-\frac{1}{3}\ln\frac{1}{2}+\frac{1}{3}\ln\frac{2}{5}-\lim_{b\to 1^+}\frac{1}{3}\ln\left|\frac{b-1}{b+2}\right|$

$=\left(-\infty-\frac{1}{3}\ln\frac{1}{2}\right)+\left(\frac{1}{3}\ln\frac{2}{5}+\infty\right)$

The integral diverges.

17. Note that $\frac{1}{x^3-x^2-x+1} = \frac{1}{2(x-1)^2}-\frac{1}{4(x-1)}+\frac{1}{4(x+1)}$

$\int_0^3 \frac{dx}{x^3-x^2-x+1} = \lim_{b\to 1^-}\int_0^b \frac{dx}{x^3-x^2-x+1} + \lim_{b\to 1^+}\int_b^3 \frac{dx}{x^3-x^2-x+1}$

$= \lim_{b\to 1^-}\left[-\frac{1}{2(x-1)}-\frac{1}{4}\ln|x-1|+\frac{1}{4}\ln|x+1|\right]_0^b + \lim_{b\to 1^+}\left[-\frac{1}{2(x-1)}-\frac{1}{4}\ln|x-1|+\frac{1}{4}\ln|x+1|\right]_b^3$

$\lim_{b\to 1^-}\left\{\left(-\frac{1}{2(b-1)}+\frac{1}{4}\ln\left|\frac{b+1}{b-1}\right|\right)+\left(-\frac{1}{2}+0\right)+\left[-\frac{1}{4}+\frac{1}{4}\ln 2-\left(-\frac{1}{2(b-1)}+\frac{1}{4}\ln\left|\frac{b+1}{b-1}\right|\right)\right]\right\}$

$=\left(\infty+\infty-\frac{1}{2}\right)+\left(-\frac{1}{4}+\frac{1}{4}\ln 2+\infty-\infty\right)$

The integral diverges.

18. Note that $\frac{x^{1/3}}{x^{2/3}-9} = \frac{1}{x^{1/3}}-\frac{9}{x^{1/3}(x^{2/3}-9)}$.

$\int_0^{27} \frac{x^{1/3}}{x^{2/3}-9}dx = \lim_{b\to 27^-}\left[\frac{3}{2}x^{2/3}+\frac{27}{2}\ln\left|x^{2/3}-9\right|\right]_0^b = \lim_{b\to 27^-}\left(\frac{3}{2}b^{2/3}+\frac{27}{2}\ln\left|b^{2/3}-9\right|\right)-\left(0+\frac{27}{2}\ln 9\right)$

$=\frac{27}{2}-\infty-\frac{27}{2}\ln 9$

The integral diverges.

19. $\int_0^{\pi/4}\tan 2x\,dx = \lim_{b\to \frac{\pi}{4}^-}\left[-\frac{1}{2}\ln|\cos 2x|\right]_0^b$

$= \lim_{b\to \frac{\pi}{4}^-} -\frac{1}{2}\ln|\cos 2b|+\frac{1}{2}\ln 1 = -(-\infty)+0$

The integral diverges.

20. $\int_0^{\pi/2}\csc x\,dx = \lim_{b\to 0^+}\left[\ln|\csc x-\cot x|\right]_b^{\pi/2}$

$= \ln|1-0| - \lim_{b\to 0^+}\ln|\csc b-\cot b|$

$= 0-\lim_{b\to 0^+}\ln\left|\frac{1-\cos b}{\sin b}\right|$

$\lim_{b\to 0^+}\frac{1-\cos b}{\sin b}$ is of the form $\frac{0}{0}$.

$$\lim_{b\to 0^+}\frac{1-\cos b}{\sin b}=\lim_{b\to 0^+}\frac{\sin b}{\cos b}=\frac{0}{1}=0$$

Thus, $\lim_{b\to 0^+}\ln\left|\frac{1-\cos b}{\sin b}\right|=-\infty$ and the integral diverges.

21. $\int_0^{\pi/2}\frac{\sin x}{1-\cos x}dx=\lim_{b\to 0^+}\left[\ln|1-\cos x|\right]_b^{\pi/2}$

$=\ln 1-\lim_{b\to 0^+}\ln|1-\cos b|=0-(-\infty)$

The integral diverges.

22. $\int_0^{\pi/2}\frac{\cos x}{\sqrt[3]{\sin x}}dx=\lim_{b\to 0^+}\left[\frac{3}{2}\sin^{2/3}x\right]_b^{\pi/2}$

$=\frac{3}{2}(1)^{2/3}-\frac{3}{2}(0)^{2/3}=\frac{3}{2}$

23. $\int_0^{\pi/2}\tan^2 x\sec^2 x\,dx=\lim_{b\to\frac{\pi}{2}^-}\left[\frac{1}{3}\tan^3 x\right]_0^b$

$=\lim_{b\to\frac{\pi}{2}^-}\frac{1}{3}\tan^3 b-\frac{1}{3}(0)^3=\infty$

The integral diverges.

24. $\int_0^{\pi/4}\frac{\sec^2 x}{(\tan x-1)^2}dx=\lim_{b\to\frac{\pi}{4}^-}\left[-\frac{1}{\tan x-1}\right]_0^b$

$=\lim_{b\to\frac{\pi}{4}^-}-\frac{1}{\tan b-1}+\frac{1}{0-1}=-(-\infty)-1$

The integral diverges.

25. Since $\frac{1-\cos x}{2}=\sin^2\frac{x}{2}$,

$\frac{1}{\cos x-1}=-\frac{1}{2}\csc^2\frac{x}{2}$.

$\int_0^{\pi}\frac{dx}{\cos x-1}=\lim_{b\to 0^+}\left[\cot\frac{x}{2}\right]_b^{\pi}$

$=\cot\frac{\pi}{2}-\lim_{b\to 0^+}\cot\frac{b}{2}=0-\infty$

The integral diverges.

26. $\int_{-3}^{-1}\frac{dx}{x\sqrt{\ln(-x)}}=\lim_{b\to -1^-}\left[2\sqrt{\ln(-x)}\right]_{-3}^b$

$=\lim_{b\to -1^-}2\sqrt{\ln(-b)}-2\sqrt{\ln 3}=0-2\sqrt{\ln 3}$

$=-2\sqrt{\ln 3}$

27. $\int_0^{\ln 3}\frac{e^x dx}{\sqrt{e^x-1}}=\lim_{b\to 0^+}\left[2\sqrt{e^x-1}\right]_b^{\ln 3}$

$=2\sqrt{3-1}-\lim_{b\to 0^+}2\sqrt{e^b-1}=2\sqrt{2}-0=2\sqrt{2}$

28. Note that $\sqrt{4x-x^2}=\sqrt{4-(x^2-4x+4)}=\sqrt{2^2-(x-2)^2}$.

$$\int_2^4\frac{dx}{\sqrt{4x-x^2}}=\lim_{b\to 4^-}\int_2^b\frac{dx}{\sqrt{4x-x^2}}=\lim_{b\to 4^-}\left[\sin^{-1}\frac{x-2}{2}\right]_2^b=\lim_{b\to 4^-}\sin^{-1}\frac{b-2}{2}-\sin^{-1}0=\frac{\pi}{2}-0=\frac{\pi}{2}$$

29. $\int_1^e\frac{dx}{x\ln x}=\lim_{b\to 1^+}[\ln(\ln x)]_b^e=\ln(\ln e)-\lim_{b\to 1^+}\ln(\ln b)=\ln 1-\ln 0=0+\infty$

The integral diverges.

30. $\int_1^{10}\frac{dx}{x\ln^{100}x}=\lim_{b\to 1^+}\left[-\frac{1}{99\ln^{99}x}\right]_b^{10}=-\frac{1}{99\ln^{99}10}+\lim_{b\to 1^+}\frac{1}{99\ln^{99}b}=-\frac{1}{99\ln^{99}10}+\infty$

The integral diverges.

31. $\int_{2c}^{4c}\frac{dx}{\sqrt{x^2-4c^2}}=\lim_{b\to 2c^+}\left[\ln\left|x+\sqrt{x^2-4c^2}\right|\right]_b^{4c}=\ln\left[(4+2\sqrt{3})c\right]-\lim_{b\to 2c^+}\ln\left|b+\sqrt{b^2-4c^2}\right|$

$=\ln\left[(4+2\sqrt{3})c\right]-\ln 2c=\ln(2+\sqrt{3})$

32. $\int_c^{2c} \frac{x\,dx}{\sqrt{x^2+xc-2c^2}} = \int_c^{2c} \frac{x\,dx}{\sqrt{\left(x+\frac{c}{2}\right)^2-\frac{9}{4}c^2}} = \int_c^{2c} \frac{\left(x+\frac{c}{2}\right)dx}{\sqrt{\left(x+\frac{c}{2}\right)^2-\frac{9}{4}c^2}} - \frac{c}{2}\int_0^{2c} \frac{dx}{\sqrt{\left(x+\frac{c}{2}\right)^2-\frac{9}{4}c^2}}$

$= \lim_{b\to c^+}\left[\sqrt{x^2+xc-2c^2} - \frac{c}{2}\ln\left|x+\frac{c}{2}+\sqrt{x^2+xc-2c^2}\right|\right]_b^{2c}$

$= \sqrt{4c^2} - \frac{c}{2}\ln\left|\frac{5c}{2}+\sqrt{4c^2}\right| - \lim_{b\to c^+}\left[\sqrt{b^2+bc-2c^2} - \frac{c}{2}\ln\left|b+\frac{c}{2}+\sqrt{b^2+bc-2c^2}\right|\right]$

$= 2c - \frac{c}{2}\ln\frac{9c}{2} - \left(0 - \frac{c}{2}\ln\left|\frac{3c}{2}+0\right|\right) = 2c - \frac{c}{2}\ln\frac{9c}{2} + \frac{c}{2}\ln\frac{3c}{2} = 2c - \frac{c}{2}\ln 3$

33. For $0 < c < 1$, $\frac{1}{\sqrt{x}(1+x)}$ is continuous. Let $u = \frac{1}{1+x}$, $du = -\frac{1}{(1+x)^2}dx$.

$dv = \frac{1}{\sqrt{x}}dx$, $v = 2\sqrt{x}$.

$\int_c^1 \frac{1}{\sqrt{x}(1+x)}dx = \left[\frac{2\sqrt{x}}{1+x}\right]_c^1 + 2\int_c^1 \frac{\sqrt{x}dx}{(1+x)^2} = \frac{2}{2} - \frac{2\sqrt{c}}{1+c} + 2\int_c^1 \frac{\sqrt{x}dx}{(1+x)^2} = 1 - \frac{2\sqrt{c}}{1+c} + 2\int_c^1 \frac{\sqrt{x}dx}{(1+x)^2}$

Thus, $\lim_{c\to 0}\int_c^1 \frac{1}{\sqrt{x}(1+x)}dx = \lim_{c\to 0}\left[1 - \frac{2\sqrt{c}}{1+c} + 2\int_c^1 \frac{\sqrt{x}dx}{(1+x)^2}\right] = 1 - 0 + 2\int_0^1 \frac{\sqrt{x}dx}{(1+x)^2}$

This last integral is a proper integral.

34. Let $u = \frac{1}{\sqrt{1+x}}$, $du = -\frac{1}{2(1+x)^{3/2}}dx$

$dv = \frac{1}{\sqrt{x}}dx$, $v = 2\sqrt{x}$.

For $0 < c < 1$, $\int_c^1 \frac{dx}{\sqrt{x(1+x)}} = \left[\frac{2\sqrt{x}}{\sqrt{1+x}}\right]_c^1 + \int_c^1 \frac{\sqrt{x}}{(1+x)^{3/2}}dx = \frac{2\sqrt{1}}{\sqrt{2}} - \frac{2\sqrt{c}}{\sqrt{1+c}} + \int_c^1 \frac{\sqrt{x}}{(1+x)^{3/2}}dx$

Thus, $\int_0^1 \frac{dx}{\sqrt{x(1+x)}} = \lim_{c\to 0}\int_c^1 \frac{dx}{\sqrt{x(1+x)}} = \lim_{c\to 0}\left[\sqrt{2} - \frac{2\sqrt{c}}{\sqrt{1+c}} + \int_c^1 \frac{\sqrt{x}}{(1+x)^{3/2}}dx\right] = \sqrt{2} - 0 + \int_0^1 \frac{\sqrt{x}}{(1+x)^{3/2}}dx$

This is a proper integral.

35. $\int_{-3}^3 \frac{x}{\sqrt{9-x^2}}dx = \int_{-3}^0 \frac{x}{\sqrt{9-x^2}}dx + \int_0^3 \frac{x}{\sqrt{9-x^2}}dx = \lim_{b\to -3^+}\left[-\sqrt{9-x^2}\right]_b^0 + \lim_{b\to 3^-}\left[-\sqrt{9-x^2}\right]_0^b$

$= -\sqrt{9} + \lim_{b\to -3^+}\sqrt{9-b^2} - \lim_{b\to 3^-}\sqrt{9-b^2} + \sqrt{9} = -3 + 0 - 0 + 3 = 0$

36. $\int_{-3}^3 \frac{x}{9-x^2}dx = \int_{-3}^0 \frac{x}{9-x^2}dx + \int_0^3 \frac{x}{9-x^2}dx = \lim_{b\to 3^+}\left[-\frac{1}{2}\ln\left|9-x^2\right|\right]_b^0 + \lim_{b\to 3^-}\left[-\frac{1}{2}\ln\left|9-x^2\right|\right]_0^b$

$= -\ln 3 + \lim_{b\to -3^+}\frac{1}{2}\ln\left|9-b^2\right| - \lim_{b\to 3^-}\frac{1}{2}\ln\left|9-b^2\right| + \ln 3 = (-\ln 3 - \infty) + (\infty + \ln 3)$

The integral diverges.

37. $\int_{-4}^4 \frac{1}{16-x^2}dx = \int_{-4}^0 \frac{1}{16-x^2}dx + \int_0^4 \frac{1}{16-x^2}dx = \lim_{b\to -4^+}\left[\frac{1}{8}\ln\left|\frac{x+4}{x-4}\right|\right]_b^0 + \lim_{b\to 4^-}\left[\frac{1}{8}\ln\left|\frac{x+4}{x-4}\right|\right]_0^b$

$$= \frac{1}{8}\ln 1 - \lim_{b\to -4^+} \frac{1}{8}\ln\left|\frac{b+4}{b-4}\right| + \lim_{b\to 4^-}\frac{1}{8}\ln\left|\frac{b+4}{b-4}\right| - \frac{1}{8}\ln 1 = (0+\infty) + (\infty - 0)$$

The integral diverges.

38. $$\int_{-1}^{1} \frac{1}{x\sqrt{-\ln|x|}}dx = \int_{-1}^{-1/2} \frac{1}{x\sqrt{-\ln|x|}}dx + \int_{-1/2}^{0} \frac{1}{x\sqrt{-\ln|x|}}dx + \int_{0}^{1/2} \frac{1}{x\sqrt{-\ln|x|}}dx + \int_{1/2}^{1} \frac{1}{x\sqrt{-\ln|x|}}dx$$

$$= \lim_{b\to -1^+}\left[-2\sqrt{-\ln|x|}\right]_b^{-1/2} + \lim_{b\to 0^-}\left[-2\sqrt{-\ln|x|}\right]_{-1/2}^{b} + \lim_{b\to 0^+}\left[-2\sqrt{-\ln|x|}\right]_b^{1/2} + \lim_{b\to 1^-}\left[-2\sqrt{-\ln|x|}\right]_{1/2}^{b}$$

$$= (-2\sqrt{\ln 2}+0) + (-\infty + 2\sqrt{\ln 2}) + (-2\sqrt{\ln 2}+\infty) + (0 + 2\sqrt{\ln 2})$$

The integral diverges.

39. $\int_0^\infty \frac{1}{x^p}dx = \int_0^1 \frac{1}{x^p}dx + \int_1^\infty \frac{1}{x^p}dx$

If $p > 1$, $\int_0^1 \frac{1}{x^p}dx = \left[\frac{1}{-p+1}x^{-p+1}\right]_0^1$ diverges since $\lim_{x\to 0^+} x^{-p+1} = \infty$.

If $p < 1$ and $p \neq 0$, $\int_1^\infty \frac{1}{x^p}dx = \left[\frac{1}{-p+1}x^{-p+1}\right]_1^\infty$ diverges since $\lim_{x\to\infty} x^{-p+1} = \infty$.

If $p = 0$, $\int_0^\infty dx = \infty$.

If $p = 1$, both $\int_0^1 \frac{1}{x}dx$ and $\int_1^\infty \frac{1}{x}dx$ diverge.

40. $\int_0^\infty f(x)dx$

$= \lim_{b\to 1^-}\int_0^b f(x)dx + \lim_{b\to 1^+}\int_b^c f(x)dx + \lim_{b\to\infty}\int_c^b f(x)dx$

where $1 < c < \infty$.

41. $\int_0^8 (x-8)^{-2/3}dx = \lim_{b\to 8^-}\left[3(x-8)^{1/3}\right]_0^b$

$= 3(0) - 3(-2) = 6$

42. $\int_0^1 \left(\frac{1}{x} - \frac{1}{x^3+x}\right)dx$

$= \lim_{b\to 0^-}\int_b^1 \frac{x}{x^2+1}dx = \lim_{b\to 0^-}\left[\frac{1}{2}\ln\left|x^2+1\right|\right]_b^1$

$= \frac{1}{2}\ln 2 - \lim_{b\to 0^-}\frac{1}{2}\ln\left|b^2+1\right| = \frac{1}{2}\ln 2$

43. a. $\int_0^1 x^{-2/3}dx = \lim_{b\to 0^+}\left[3x^{1/3}\right]_b^1 = 3$

b. $V = \pi\int_0^1 x^{-4/3}dx = \lim_{b\to 0^+}\pi\left[-3x^{-1/3}\right]_b^1$

$= -3\pi + 3\pi\lim_{b\to 0} b^{-1/3}$

The limit tends to infinity as $b \to 0$, so the volume is infinite.

44. Since $\ln x < 0$ for $0 < x < 1$, $b > 1$

$\int_0^b \ln x\, dx = \lim_{c\to 0^-}\int_c^1 \ln x\, dx + \int_1^b \ln x\, dx$

$= \lim_{c\to 0^+}[x\ln x - x]_c^1 + [x\ln x - x]_1^b$

$= -1 - \lim_{c\to 0^+}(c\ln c - c) + b\ln b - b + 1$

$= b\ln b - b$

Thus, $b\ln b - b = 0$ when $b = e$.

45. $\int_0^1 \frac{\sin x}{x}dx$ is not an improper integral since $\frac{\sin x}{x}$ is bounded in the interval $0 \le x \le 1$.

46. For $x \ge 1$, $\frac{1}{1+x^4} < 1$ so $\frac{1}{x^4(1+x^4)} < \frac{1}{x^4}$.

$\int_1^\infty \frac{1}{x^4}dx = \lim_{b\to\infty}\left[-\frac{1}{3x^3}\right]_1^b = -\lim_{b\to\infty}\frac{1}{3b^3} + \frac{1}{3}$

$= -0 + \frac{1}{3} = \frac{1}{3}$

Thus, by the Comparison Test $\int_1^\infty \frac{1}{x^4(1+x^4)}dx$ converges.

47. For $x \ge 1$, $x^2 \ge x$ so $-x^2 \le -x$, thus $e^{-x^2} \le e^{-x}$.

$\int_1^\infty e^{-x}dx = \lim_{b\to\infty}[-e^{-x}]_1^b = -\lim_{b\to\infty}\frac{1}{e^b} + e^{-1}$

$= -0 + \frac{1}{e} = \frac{1}{e}$

Thus, by the Comparison Test, $\int_1^{\infty} e^{-x^2} dx$ converges.

48. Since $\sqrt{x+2} - 1 \le \sqrt{x+2}$ we know that $\frac{1}{\sqrt{x+2}-1} \ge \frac{1}{\sqrt{x+2}}$. Consider $\int_0^{\infty} \frac{1}{\sqrt{x+2}} dx$

$\int_2^{\infty} \frac{1}{\sqrt{x+2}} dx = \lim_{b\to\infty} \int_2^{b} \frac{1}{\sqrt{x+2}} dx$

$= \lim_{b\to\infty} \left[2\sqrt{x+2}\right]_2^{\infty} = \lim_{b\to\infty} 2\left(\sqrt{b+2} - 2\right) = \infty$

Thus, by the Comparison Test of Problem 46, we conclude that $\int_0^{\infty} \frac{1}{\sqrt{x+2}} dx$ diverges.

49. Since $x^2 \ln(x+1) \ge x^2$, we know that $\frac{1}{x^2 \ln(x+1)} \le \frac{1}{x^2}$. Since $\int_1^{\infty} \frac{1}{x^2} dx = \left[-\frac{1}{x}\right]_1^{\infty} = 1$ we can apply the Comparison Test of Problem 46 to conclude that $\int_1^{\infty} \frac{1}{x^2 \ln(x+1)} dx$ converges.

50. If $0 \le f(x) \le g(x)$ on $[a, b]$ and either $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$ or $\lim_{x\to b} f(x) = \lim_{x\to b} g(x) = \infty$, then the convergence of $\int_a^b g(x)dx$ implies the convergence of $\int_a^b f(x)dx$ and the divergence of $\int_a^b f(x)dx$ implies the divergence of $\int_a^b g(x)dx$.

51. **a.** From Example 2 of Section 9.2, $\lim_{x\to\infty} \frac{x^a}{e^x} = 0$ for a any positive real number.

Thus $\lim_{x\to\infty} \frac{x^{n+1}}{e^x} = 0$ for any positive real number n, hence there is a number M such that $0 < \frac{x^{n+1}}{e^x} \le 1$ for $x \ge M$. Divide the inequality by x^2 to get that $0 < \frac{x^{n-1}}{e^x} \le \frac{1}{x^2}$ for $x \ge M$.

b. $\int_1^{\infty} \frac{1}{x^2} dx = \lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b = -\lim_{b\to\infty} \frac{1}{b} + \frac{1}{1}$

$= -0 + 1 = 1$

$\int_1^{\infty} x^{n-1}e^{-x} dx = \int_1^{M} x^{n-1}e^{-x} dx + \int_M^{\infty} x^{n-1}e^{-x} dx$

$\le \int_1^{M} x^{n-1}e^{-x} dx + \int_1^{\infty} \frac{1}{x^2} dx$

$= 1 + \int_1^{M} x^{n-1}e^{-x} dx$

by part a and Problem 46. The remaining integral is finite, so $\int_1^{\infty} x^{n-1}e^{-x} dx$ converges.

52. $\int_0^1 e^{-x} dx = \left[-e^{-x}\right]_0^1 = -e^{-1} + 1 = 1 - \frac{1}{e}$, so the integral converges when $n = 1$. For $0 \le x \le 1$, $0 \le x^{n-1} \le 1$ for $n > 1$. Thus, $\frac{x^{n-1}}{e^x} = x^{n-1}e^{-x} \le e^{-x}$. By the comparison test from Problem 50, $\int_0^1 x^{n-1}e^{-x} dx$ converges.

53. **a.** $\Gamma(1) = \int_0^{\infty} x^0 e^{-x} dx = \left[-e^{-x}\right]_0^{\infty} = 1$

b. $\Gamma(n+1) = \int_0^{\infty} x^n e^{-x} dx$

Let $u = x^n, dv = e^{-x} dx$,

$du = nx^{n-1}dx, v = -e^{-x}$.

$\Gamma(n+1) = [-x^n e^{-x}]_0^{\infty} + \int_0^{\infty} nx^{n-1}e^{-x} dx$

$= 0 + n\int_0^{\infty} x^{n-1}e^{-x} dx = n\Gamma(n)$

c. From parts a and b,

$\Gamma(1) = 1, \Gamma(2) = 1 \cdot \Gamma(1) = 1,$

$\Gamma(3) = 2 \cdot \Gamma(2) = 2 \cdot 1 = 2!.$

Suppose $\Gamma(n) = (n-1)!$, then by part b,

$\Gamma(n+1) = n\Gamma(n) = n[(n-1)!] = n!.$

54. $n = 1$, $\int_0^{\infty} e^{-x} dx = 1 = 0! = (1-1)!$

$n = 2$, $\int_0^{\infty} xe^{-x} dx = 1 = 1! = (2-1)!$

$n = 3$, $\int_0^{\infty} x^2e^{-x} dx = 2 = 2! = (3-1)!$

$n = 4$, $\int_0^{\infty} x^3e^{-x} dx = 6 = 3! = (4-1)!$

$n = 5$, $\int_0^{\infty} x^4e^{-x} dx = 24 = 4! = (5-1)!$

55. a. The integral is the area between the curve $y^2 = \frac{1-x}{x}$ and the x-axis from $x = 0$ to $x = 1$.

$y^2 = \frac{1-x}{x}$; $xy^2 = 1-x$; $x(y^2+1) = 1$

$x = \frac{1}{y^2+1}$

As $x \to 0$, $y = \sqrt{\frac{1-x}{x}} \to \infty$, while

when $x = 1$, $y = \sqrt{\frac{1-1}{1}} = 0$, thus the area is

$\int_0^{\infty} \frac{1}{y^2+1} dy = \lim_{b\to\infty} [\tan^{-1} y]_0^b$

$= \lim_{b\to\infty} \tan^{-1} b - \tan^{-1} 0 = \frac{\pi}{2}$

b. The integral is the area between the curve $y^2 = \frac{1+x}{1-x}$ and the x-axis from $x = -1$ to $x = 1$.

$y^2 = \frac{1+x}{1-x}$; $y^2 - xy^2 = 1+x$; $y^2 - 1 = x(y^2+1)$;

$x = \frac{y^2-1}{y^2+1}$

When $x = -1$, $y = \sqrt{\frac{1+(-1)}{1-(-1)}} = \sqrt{\frac{0}{2}} = 0$, while

as $x \to 1$, $y = \sqrt{\frac{1+x}{1-x}} \to \infty$.

The area in question is the area to the right of the curve $y = \sqrt{\frac{1+x}{1-x}}$ and to the left of the line $x = 1$. Thus, the area is

$\int_0^{\infty}\left(1 - \frac{y^2-1}{y^2+1}\right)dy = \int_0^{\infty} \frac{2}{y^2+1} dy$

$= \lim_{b\to\infty}\left[2\tan^{-1} y\right]_0^b$

$\lim_{b\to\infty} 2\tan^{-1} b - 2\tan^{-1} 0 = 2\left(\frac{\pi}{2}\right) = \pi$

56. For $0 < x < 1$, $x^p > x^q$ so $2x^p > x^p + x^q$ and $\frac{1}{x^p+x^q} > \frac{1}{2x^p}$. For $1 < x$, $x^q > x^p$ so $2x^q > x^p + x^q$ and $\frac{1}{x^p+x^q} > \frac{1}{2x^q}$.

$\int_0^{\infty} \frac{1}{x^p+x^q} dx = \int_0^1 \frac{1}{x^p+x^q} dx + \int_1^{\infty} \frac{1}{x^p+x^q} dx$

Both of these integrals must converge.

$\int_0^1 \frac{1}{x^p+x^q} dx > \int_0^1 \frac{1}{2x^p} dx = \frac{1}{2}\int_0^1 \frac{1}{x^p} dx$ which converges if and only if $p < 1$.

$\int_1^{\infty} \frac{1}{x^p+x^q} dx > \int_1^{\infty} \frac{1}{2x^q} dx = \frac{1}{2}\int_1^{\infty} \frac{1}{x^q} dx$ which converges if and only if $q > 1$. Thus, $0 < p < 1$ and $1 < q$.

9.5 Chapter Review

Concepts Test

1. True: See Example 2 of Section 9.2.

2. True: Use l'Hôpital's Rule.

3. False: $\lim_{x\to\infty} \frac{1000x^4+1000}{0.001x^4+1} = \frac{1000}{0.001} = 10^6$

4. False: $\lim_{x\to\infty} xe^{-1/x} = \infty$ since $e^{-1/x} \to 1$ and $x \to \infty$ as $x \to \infty$.

5. False: For example, if $f(x) = x$ and $g(x) = e^x$, $\lim_{x\to\infty} \frac{x}{e^x} = 0$.

6. False: See Example 7 of Section 9.2.

7. True: Take the inner limit first.

8. True: Raising a small number to a large exponent results in an even smaller number.

9. True: Since $\lim_{x\to a} f(x) = -1 \neq 0$, it serves only to affect the sign of the limit of the product.

10. False: Consider $f(x) = (x-a)^2$ and $g(x) = \frac{1}{(x-a)^2}$, then $\lim_{x\to a} f(x) = 0$

and $\lim_{x\to a} g(x) = \infty$, while $\lim_{x\to a}[f(x)g(x)] = 1$.

11. False: Consider $f(x) = 3x^2$ and $g(x) = x^2 + 1$, then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{3x^2}{x^2+1} = \lim_{x\to\infty} \frac{3}{1+\frac{1}{x^2}} = 3, \text{ but}$$

$$\lim_{x\to\infty}[f(x) - 3g(x)] = \lim_{x\to\infty}[3x^2 - 3(x^2+1)] = \lim_{x\to\infty}[-3] = -3$$

12. True: As $x \to a, f(x) \to 2$ while $\frac{1}{|g(x)|} \to \infty$.

13. True: See Example 7 of Section 9.2.

14. True: Let $y = [1+f(x)]^{1/f(x)}$, then

$$\ln y = \frac{1}{f(x)}\ln[1+f(x)].$$

$$\lim_{x\to a} \frac{1}{f(x)}\ln[1+f(x)] = \lim_{x\to a} \frac{\ln[1+f(x)]}{f(x)}$$

This limit is of the form $\frac{0}{0}$.

$$\lim_{x\to a} \frac{\ln[1+f(x)]}{f(x)} = \lim_{x\to a} \frac{\frac{1}{1+f(x)}f'(x)}{f'(x)} = \lim_{x\to a} \frac{1}{1+f(x)} = 1$$

$$\lim_{x\to a}[1+f(x)]^{1/f(x)} = \lim_{x\to a} e^{\ln y} = e^1 = e$$

15. True: Use repeated applications of l'Hôpital's Rule.

16. True: $e^0 = 1$ and $p(0)$ is the constant term.

17. False: Consider $f(x) = 3x^2 + x + 1$ and $g(x) = 4x^3 + 2x + 3$; $f'(x) = 6x + 1$, $g'(x) = 12x^2 + 2$, and so

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{6x+1}{12x^2+2} = \frac{1}{2} \text{ while}$$

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{3x^2+x+1}{4x^3+2x+3} = \frac{1}{3}$$

18. False: $p > 1$. See Example 4 of Section 9.4.

19. True: $\int_0^\infty \frac{1}{x^p}dx = \int_0^1 \frac{1}{x^p}dx + \int_1^\infty \frac{1}{x^p}dx$;

$\int_0^1 \frac{1}{x^p}dx$ diverges for $p \geq 1$ and

$\int_1^\infty \frac{1}{x^p}dx$ diverges for $p \leq 1$.

20. False: Consider $\int_0^\infty \frac{1}{x+1}dx$.

21. True: $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^0 f(x)dx + \int_0^\infty f(x)dx$

If f is an even function, then $f(-x) = f(x)$ so

$\int_{-\infty}^0 f(x)dx = \int_0^\infty f(x)dx.$

Thus, both integrals making up $\int_{-\infty}^\infty f(x)dx$ converge so their sum converges.

22. False: See Problem 33 of Section 9.3.

23. True: $\int_0^\infty f'(x)dx = \lim_{b\to\infty}\int_0^b f'(x)dx$

$= \lim_{b\to\infty}[f(x)]_0^b = \lim_{b\to\infty} f(b) - f(0)$

$= 0 - f(0) = -f(0)$.

$f(0)$ must exist and be finite since $f'(x)$ is continuous on $[0, \infty)$.

24. True: $\int_0^\infty f(x)dx \leq \int_0^\infty e^{-x}dx = \lim_{b\to\infty}[-e^{-x}]_0^b$

$= \lim_{b\to\infty} -e^{-b} + 1 = 1$, so $\int_0^\infty f(x)dx$ must converge.

25. False: The integrand is bounded on the interval $\left[0, \frac{\pi}{4}\right]$.

Sample Test Problems

1. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0} \frac{4x}{\tan x} = \lim_{x\to 0} \frac{4}{\sec^2 x} = 4$$

2. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{\tan 2x}{\sin 3x}=\lim_{x\to 0}\frac{2\sec^2 2x}{3\cos 3x}=\frac{2}{3}$$

3. The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule twice.)

$$\lim_{x\to 0}\frac{\sin x-\tan x}{\frac{1}{3}x^2}=\lim_{x\to 0}\frac{\cos x-\sec^2 x}{\frac{2}{3}x}$$

$$=\lim_{x\to 0}\frac{-\sin x-2\sec x(\sec x\tan x)}{\frac{2}{3}}=0$$

4. $\lim_{x\to 0}\frac{\cos x}{x^2}=\infty$ (L'Hôpital's Rule does not apply since cos(0) = 1.)

5. $\lim_{x\to 0}2x\cot x=\lim_{x\to 0}\frac{2x\cos x}{\sin x}$

The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{2x\cos x}{\sin x}=\lim_{x\to 0}\frac{2\cos x-2x\sin x}{\cos x}$$

$$=\frac{2-0}{1}=2$$

6. The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x\to 1^-}\frac{\ln(1-x)}{\cot \pi x}=\lim_{x\to 1^-}\frac{-\frac{1}{1-x}}{-\pi\csc^2 \pi x}$$

$$=\lim_{x\to 1^-}\frac{\sin^2 \pi x}{\pi(1-x)}$$

The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 1^-}\frac{\sin^2 \pi x}{\pi(1-x)}=\lim_{x\to 1^-}\frac{2\pi\sin\pi x\cos\pi x}{-\pi}=0$$

7. The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{t\to\infty}\frac{\ln t}{t^2}=\lim_{t\to\infty}\frac{\frac{1}{t}}{2t}=\lim_{t\to\infty}\frac{1}{2t^2}=0$$

8. The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x\to\infty}\frac{2x^3}{\ln x}=\lim_{x\to\infty}\frac{6x^2}{\frac{1}{x}}=\lim_{x\to\infty}6x^3=\infty$$

9. As $x\to 0, \sin x\to 0$, and $\frac{1}{x}\to\infty$. A number less than 1, raised to a large power, is a very small number $\left(\left(\frac{1}{2}\right)^{32}=2.328\times 10^{-10}\right)$ so

$$\lim_{x\to 0^+}(\sin x)^{1/x}=0.$$

10. $\lim_{x\to 0^+}x\ln x=\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}$

The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}=\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}=\lim_{x\to 0^+}-x=0$$

11. The limit is of the form 0^0.

Let $y=x^x$, then $\ln y=x\ln x$.

$$\lim_{x\to 0^+}x\ln x=\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}$$

The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}=\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}=\lim_{x\to 0^+}-x=0$$

$$\lim_{x\to 0^+}x^x=\lim_{x\to 0^+}e^{\ln y}=1$$

12. The limit is of the form 1^∞.

Let $y=(1+\sin x)^{2/x}$, then $\ln y=\frac{2}{x}\ln(1+\sin x)$.

$$\lim_{x\to 0}\frac{2}{x}\ln(1+\sin x)=\lim_{x\to 0}\frac{2\ln(1+\sin x)}{x}$$

The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{2\ln(1+\sin x)}{x}=\lim_{x\to 0}\frac{\frac{2}{1+\sin x}\cos x}{1}$$

$$=\lim_{x\to 0}\frac{2\cos x}{1+\sin x}=\frac{2}{1}=2$$

$$\lim_{x\to 0}(1+\sin x)^{2/x}=\lim_{x\to 0}e^{\ln y}=e^2$$

13. $\lim_{x\to 0^+}\sqrt{x}\ln x=\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{\sqrt{x}}}$

The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{\sqrt{x}}}=\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{2x^{3/2}}}=\lim_{x\to 0^+}-2\sqrt{x}=0$$

14. The limit is of the form ∞^0.

Let $y = t^{1/t}$, then $\ln y = \frac{1}{t}\ln t$.

$$\lim_{t\to\infty} \frac{1}{t}\ln t = \lim_{t\to\infty} \frac{\ln t}{t}$$

The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{t\to\infty} \frac{\ln t}{t} = \lim_{t\to\infty} \frac{\frac{1}{t}}{1} = \lim_{t\to\infty} \frac{1}{t} = 0$$

$$\lim_{t\to\infty} t^{1/t} = \lim_{t\to\infty} e^{\ln y} = 1$$

15. $\lim_{x\to 0^+} \left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0^+} \frac{x-\sin x}{x\sin x}$

The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule twice.)

$$\lim_{x\to 0^+} \frac{x-\sin x}{x\sin x} = \lim_{x\to 0^+} \frac{1-\cos x}{\sin x + x\cos x}$$

$$= \lim_{x\to 0^+} \frac{\sin x}{2\cos x - x\sin x} = \frac{0}{2} = 0$$

16. The limit is of the form $\frac{\infty}{\infty}$. (Apply l'Hôpital's Rule three times.)

$$\lim_{x\to\frac{\pi}{2}} \frac{\tan 3x}{\tan x} = \lim_{x\to\frac{\pi}{2}} \frac{3\sec^2 3x}{\sec^2 x}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{3\cos^2 x}{\cos^2 3x} = \lim_{x\to\frac{\pi}{2}} \frac{\cos x\sin x}{\cos 3x\sin 3x}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\cos^2 x - \sin^2 x}{3(\cos^2 3x - \sin^2 3x)} = -\frac{1}{3(0-1)} = \frac{1}{3}$$

17. The limit is of the form 1^∞.

Let $y = (\sin x)^{\tan x}$, then $\ln y = \tan x \ln(\sin x)$.

$$\lim_{x\to\frac{\pi}{2}} \tan x\ln(\sin x) = \lim_{x\to\frac{\pi}{2}} \frac{\sin x\ln(\sin x)}{\cos x}$$

The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to\frac{\pi}{2}} \frac{\sin x\ln(\sin x)}{\cos x} = \lim_{x\to\frac{\pi}{2}} \frac{\cos x\ln(\sin x) + \frac{\sin x}{\sin x}\cos x}{\sin x}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\cos x(1+\ln(\sin x))}{\sin x} = \frac{0}{1} = 0$$

$$\lim_{x\to\frac{\pi}{2}} (\sin x)^{\tan x} = \lim_{x\to\frac{\pi}{2}} e^{\ln y} = 1$$

18. $\lim_{x\to\frac{\pi}{2}} \left(x\tan x - \frac{\pi}{2}\sec x\right) = \lim_{x\to\frac{\pi}{2}} \frac{x\sin x - \frac{\pi}{2}}{\cos x}$

The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to\frac{\pi}{2}} \frac{x\sin x - \frac{\pi}{2}}{\cos x} = \lim_{x\to\frac{\pi}{2}} \frac{\sin x + x\cos x}{\sin x} = \frac{1}{1} = 1$$

19. $\int_0^\infty \frac{dx}{(x+1)^2} = \left[-\frac{1}{x+1}\right]_0^\infty = 0+1 = 1$

20. $\int_0^\infty \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^\infty = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

21. $\int_{-\infty}^1 e^{2x}dx = \left[\frac{1}{2}e^{2x}\right]_{-\infty}^1 = \frac{1}{2}e^2 - 0 = \frac{1}{2}e^2$

22. $\int_{-1}^1 \frac{dx}{1-x} = \lim_{b\to 1}[-\ln(1-x)]_{-1}^b$

$= -\lim_{b\to 1}\ln(1-b) + \ln 2 = \infty$

The integral diverges.

23. $\int_0^\infty \frac{dx}{x+1} = [\ln(x+1)]_0^\infty = \infty - 0 = \infty$

The integral diverges.

24. $$\int_{\frac{1}{2}}^2 \frac{dx}{x(\ln x)^{1/5}} = \lim_{b\to 1^-}\int_{\frac{1}{2}}^b \frac{dx}{x(\ln x)^{1/5}} + \lim_{b\to 1^+}\int_b^2 \frac{dx}{x(\ln x)^{1/5}} = \lim_{b\to 1^-}\left[\frac{5}{4}(\ln x)^{4/5}\right]_{\frac{1}{2}}^b + \lim_{b\to 1^+}\left[\frac{5}{4}(\ln x)^{4/5}\right]_b^2$$

$$= \left(\frac{5}{4}(0) - \frac{5}{4}\left(\ln\frac{1}{2}\right)^{4/5}\right) + \left(\frac{5}{4}(\ln 2)^{4/5} - \frac{5}{4}(0)\right) = \frac{5}{4}(\ln 2)^{4/5} - \frac{5}{4}\left(\ln\frac{1}{2}\right)^{4/5} = \frac{5}{4}[(\ln 2)^{4/5} - (-\ln 2)^{4/5}]$$

$$= \frac{5}{4}[(\ln 2)^{4/5} - (\ln 2)^{4/5}] = 0$$

25. $\int_1^{\infty} \frac{dx}{x^2+x^4} = \int_1^{\infty}\left(\frac{1}{x^2} - \frac{1}{1+x^2}\right)dx = \left[-\frac{1}{x} - \tan^{-1} x\right]_1^{\infty} = 0 - \frac{\pi}{2} + 1 + \tan^{-1} 1 = 1 + \frac{\pi}{4} - \frac{\pi}{2} = 1 - \frac{\pi}{4}$

26. $\int_{-\infty}^{1} \frac{dx}{(2-x)^2} = \left[\frac{1}{2-x}\right]_{-\infty}^{1} = \frac{1}{1} - 0 = 1$

27. $\int_{-2}^{0} \frac{dx}{2x+3} = \lim_{b\to -\frac{3}{2}^-} \int_{-2}^{b} \frac{dx}{2x+3} + \lim_{b\to -\frac{3}{2}^+} \int_{b}^{0} \frac{dx}{2x+3} = \lim_{b\to -\frac{3}{2}^-} \left[\frac{1}{2}\ln|2x+3|\right]_{-2}^{b} + \lim_{b\to -\frac{3}{2}^+} \left[\frac{1}{2}\ln|2x+3|\right]_{b}^{0}$

$= \left(\lim_{b\to -\frac{3}{2}^-} \frac{1}{2}\ln|2b+3| - \frac{1}{2}(0)\right) + \left(\frac{1}{2}\ln 3 - \lim_{b\to -\frac{3}{2}^+} \frac{1}{2}\ln|2b+3|\right) = (-\infty) + \left(\frac{1}{2}\ln 3 + \infty\right)$

The integral diverges.

28. $\int_1^{4} \frac{dx}{\sqrt{x-1}} = \lim_{b\to 1^+} [2\sqrt{x-1}]_b^4 = 2\sqrt{3} - \lim_{b\to 1^+} 2\sqrt{x-1} = 2\sqrt{3} - 0 = 2\sqrt{3}$

29. $\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \left[-\frac{1}{\ln x}\right]_2^{\infty} = -0 + \frac{1}{\ln 2} = \frac{1}{\ln 2}$

30. $\int_0^{\infty} \frac{dx}{e^{x/2}} = \left[-\frac{2}{e^{x/2}}\right]_0^{\infty} = -0 + \frac{2}{1} = 2$

31. $\int_3^{5} \frac{dx}{(4-x)^{2/3}} = \lim_{b\to 4^-} \int_3^{b} \frac{dx}{(4-x)^{2/3}} + \lim_{b\to 4^+} \int_b^{5} \frac{dx}{(4-x)^{2/3}} = \lim_{b\to 4^-} \left[-3(4-x)^{1/3}\right]_3^{b} + \lim_{b\to 4^+} \left[-3(4-x)^{1/3}\right]_b^{5}$

$= \lim_{b\to 4^-} -3(4-b)^{1/3} + 3(1)^{1/3} - 3(-1)^{1/3} + \lim_{b\to 4^+} 3(4-b)^{1/3} = 0 + 3 + 3 + 0 = 6$

32. $\int_2^{\infty} xe^{-x^2} dx = \left[-\frac{1}{2}e^{-x^2}\right]_2^{\infty} = 0 + \frac{1}{2}e^{-4} = \frac{1}{2}e^{-4}$

33. $\int_{-\infty}^{\infty} \frac{x}{x^2+1} dx = \int_{-\infty}^{0} \frac{x}{x^2+1} dx + \int_0^{\infty} \frac{x}{x^2+1} dx$

$= \frac{1}{2}\left[\ln(x^2+1)\right]_{-\infty}^{0} + \frac{1}{2}\left[\ln(x^2+1)\right]_0^{\infty} = (0+\infty) + (\infty - 0)$

The integral diverges.

34. $\int_{-\infty}^{\infty} \frac{x}{1+x^4} dx = \int_{-\infty}^{0} \frac{x}{1+x^4} dx + \int_0^{\infty} \frac{x}{1+x^4} dx$

$= \left[\frac{1}{2}\tan^{-1} x^2\right]_{-\infty}^{0} + \left[\frac{1}{2}\tan^{-1} x^2\right]_0^{\infty}$

$= \frac{1}{2}\tan^{-1} 0 - \frac{1}{2}\left(\frac{\pi}{2}\right) + \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2}\tan^{-1} 0$

$= 0 - \frac{\pi}{4} + \frac{\pi}{4} - 0 = 0$

35. $\frac{e^x}{e^{2x}+1} = \frac{e^x}{(e^x)^2+1}$

Let $u = e^x, du = e^x dx$

$\int_0^{\infty} \frac{e^x}{e^{2x}+1} dx = \int_1^{\infty} \frac{1}{u^2+1} du = \left[\tan^{-1} u\right]_1^{\infty}$

$= \frac{\pi}{2} - \tan^{-1} 1 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

36. Let $u = x^3, du = 3x^2 dx$

$\int_{-\infty}^{\infty} x^2 e^{-x^3} dx = \int_{-\infty}^{\infty} \frac{1}{3} e^{-u} du$

$= \frac{1}{3}\int_{-\infty}^{0} e^{-u} du + \frac{1}{3}\int_0^{\infty} e^{-u} du$

$= \frac{1}{3}\left[-e^{-u}\right]_{-\infty}^{0} + \frac{1}{3}\left[-e^{-u}\right]_0^{\infty}$

$= \frac{1}{3}(-1+\infty) + \frac{1}{3}(-0+1)$

The integral diverges.

37. $\int_{-3}^{3} \frac{x}{\sqrt{9-x^2}} dx = 0$

See Problem 35 in Section 9.4.

38. let $u = \ln(\cos x)$, then

$du = \frac{1}{\cos x} \cdot -\sin x\, dx = -\tan x\, dx$

$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\tan x}{(\ln \cos x)^2} dx = \int_{\ln\frac{1}{2}}^{-\infty} -\frac{1}{u^2} du = \int_{-\infty}^{\ln\frac{1}{2}} \frac{1}{u^2} du$

$= \left[-\frac{1}{u}\right]_{-\infty}^{\ln\frac{1}{2}} = -\frac{1}{\ln\frac{1}{2}} + 0 = \frac{1}{\ln 2}$

39. For $p \neq 1, p \neq 0$, $\int_1^{\infty} \frac{1}{x^p} dx = \left[-\frac{1}{(p-1)x^{p-1}}\right]_1^{\infty}$

$= \lim_{b\to\infty} \frac{1}{(1-p)b^{p-1}} + \frac{1}{p-1}$

$\lim_{b\to\infty} \frac{1}{b^{p-1}} = 0$ when $p - 1 > 0$ or $p > 1$.

and $\lim_{b\to\infty} \frac{1}{b^{p-1}} = \infty$ when $p < 1, p \neq 0$.

When $p = 1$, $\int_1^{\infty} \frac{1}{x} dx = [\ln x]_1^{\infty} = \infty - 0$

The integral diverges.

When $p = 0$, $\int_1^{\infty} 1 dx = [x]_1^{\infty} = \infty - 1$. The integral diverges.

$\int_1^{\infty} \frac{1}{x^p} dx$ converges when $p > 1$ and diverges when $p \leq 1$.

40. For $p \neq 1, p \neq 0$, $\int_0^1 \frac{1}{x^p} dx = \left[-\frac{1}{(p-1)x^{p-1}}\right]_0^1$

$= \frac{1}{1-p} + \lim_{b\to 0} \frac{1}{(p-1)b^{p-1}}$

$\lim_{b\to 0} \frac{1}{b^{p-1}}$ converges when $p - 1 < 0$ or $p < 1$.

When $p = 1$, $\int_0^1 \frac{1}{x} dx = [\ln x]_0^1 = 0 - \lim_{b\to 0^+} \ln b = \infty$

The integral diverges.

When $p = 0$, $\int_0^1 1 dx = [x]_0^1 = 1 - 0 = 1$

$\int_0^1 \frac{1}{x^p} dx$ converges when $p < 1$ and diverges when $1 \leq p$.

41. For $x \geq 1$, $x^6 + x > x^6$, so $\sqrt{x^6 + x} > \sqrt{x^6} = x^3$ and $\frac{1}{\sqrt{x^6+x}} < \frac{1}{x^3}$. Hence,

$\int_1^{\infty} \frac{1}{\sqrt{x^6+x}} dx < \int_1^{\infty} \frac{1}{x^3} dx$ which converges since $3 > 1$ (see Problem 39). Thus

$\int_1^{\infty} \frac{1}{\sqrt{x^6+x}} dx$ converges.

42. For $x > 1$, $\ln x < e^x$, so $\frac{\ln x}{e^x} < 1$ and

$\frac{\ln x}{e^{2x}} = \frac{\ln x}{(e^x)^2} < \frac{1}{e^x}$.

Hence, $\int_1^{\infty} \frac{\ln x}{e^{2x}} dx < \int_1^{\infty} e^{-x} dx = [-e^{-x}]_1^{\infty}$

$= -0 + e^{-1} = \frac{1}{e}$.

Thus, $\int_1^{\infty} \frac{\ln x}{e^{2x}} dx$ converges.

43. For $x > 3$, $\ln x > 1$, so $\frac{\ln x}{x} > \frac{1}{x}$. Hence,

$\int_3^{\infty} \frac{\ln x}{x} dx > \int_3^{\infty} \frac{1}{x} dx = [\ln x]_3^{\infty} = \infty - \ln 3$.

The integral diverges, thus $\int_3^{\infty} \frac{\ln x}{x} dx$ also diverges.

44. For $x \geq 1$, $\ln x < x$, so $\frac{\ln x}{x} < 1$ and $\frac{\ln x}{x^3} < \frac{1}{x^2}$.

Hence,

$\int_1^{\infty} \frac{\ln x}{x^3} dx < \int_1^{\infty} \frac{1}{x^2} dx = \left[-\frac{1}{x}\right]_1^{\infty} = -0 + 1 = 1$.

Thus, $\int_1^{\infty} \frac{\ln x}{x^3} dx$ converges.

9.6 Additional Problem Set

1. a. For $1 < x < c$, $\frac{1}{x}$ is a differentiable function.

Let $u = \frac{1}{x}$, so $x = \frac{1}{u}$; $du = -\frac{1}{x^2}dx$ so

$dx = -x^2 du = -\frac{1}{u^2}du$.

When $x = 1$, $u = 1$, while when $x = c$, $u = \frac{1}{c}$.

$$\int_1^c \frac{1}{1+x^2}dx = \int_1^{\frac{1}{c}} \frac{1}{1+\frac{1}{u^2}}\left(-\frac{1}{u^2}du\right)$$

$$= -\int_1^{\frac{1}{c}} \frac{1}{u^2+1}du = \int_{\frac{1}{c}}^1 \frac{1}{u^2+1}du$$

b. $\int_1^\infty \frac{1}{1+x^2}dx = \lim_{c\to\infty} \int_1^c \frac{1}{1+x^2}dx$

$= \lim_{c\to\infty} \int_{\frac{1}{c}}^1 \frac{1}{u^2+1}du = \int_0^1 \frac{1}{u^2+1}du$ by using part a and $\lim_{c\to\infty} \frac{1}{c} = 0$.

$$\int_0^1 \frac{1}{u^2+1}du = [\tan^{-1} u]_0^1 = \tan^{-1} 1 - \tan^{-1} 0$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

2. For $1 < x < c$, $\frac{1}{x}$ is a differentiable function.

Let $u = \frac{1}{x}$, so $x = \frac{1}{u}$; $du = -\frac{1}{x^2}dx$, so

$dx = -x^2 du = -\frac{1}{u^2}du.$

When $x = 1$, $u = 1$, while when $x = c$, $u = \frac{1}{c}$.

$$\int_1^c \frac{x}{x^3+1}dx = \int_1^{\frac{1}{c}} \frac{\frac{1}{u}}{\frac{1}{u^3}+1}\left(-\frac{1}{u^2}du\right)$$

$$= -\int_1^{\frac{1}{c}} \frac{1}{1+u^3}du = \int_{\frac{1}{c}}^1 \frac{1}{u^3+1}du$$

Thus, $\int_1^\infty \frac{x}{x^3+1}dx = \lim_{c\to\infty} \int_1^c \frac{x}{x^3+1}dx$

$= \lim_{c\to\infty} \int_{\frac{1}{c}}^1 \frac{1}{u^3+1}du = \int_0^1 \frac{1}{u^3+1}du$

3. a. $f(x)$ is an even function, so $\int_{-\infty}^0 f(x)dx = \int_0^\infty f(x)dx$, thus $\int_{-\infty}^\infty f(x)dx = 2\int_0^\infty f(x)dx.$

$$2\int_0^\infty C|x|e^{-kx^2}dx = \left[-\frac{C}{k}e^{-kx^2}\right]_0^\infty = -0 + \frac{C}{k}$$

$\frac{C}{k} = 1$ when $C = k$

b. $\int_{-\infty}^\infty kx|x|e^{-kx^2}dx = \int_{-\infty}^0 -kx^2e^{-kx^2}dx + \int_0^\infty kx^2e^{-kx^2}dx$

In both integrals, use $u = x$, $du = dx$, $dv = kxe^{-kx^2}dx$, $v = -\frac{1}{2}e^{-kx^2}$

$$\int_{-\infty}^0 -kx^2e^{-kx^2}dx + \int_0^\infty kx^2e^{-kx^2}dx = -\left(\left[-\frac{1}{2}xe^{-kx^2}\right]_{-\infty}^0 + \int_{-\infty}^0 \frac{1}{2}e^{-kx^2}dx\right) + \left(\left[-\frac{1}{2}xe^{-kx^2}\right]_0^\infty + \int_0^\infty \frac{1}{2}e^{-kx^2}dx\right)$$

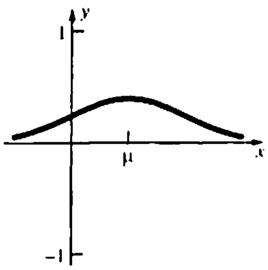
$$= -(0+0) - \frac{1}{2}\int_{-\infty}^0 e^{-kx^2}dx + (0-0) + \frac{1}{2}\int_0^\infty e^{-kx^2}dx = -\frac{1}{2}\int_{-\infty}^0 e^{-kx^2}dx + \frac{1}{2}\int_0^\infty e^{-kx^2}dx$$

Now let $u = \sqrt{2k}x$, so that $kx^2 = \frac{u^2}{2}$ and $du = \sqrt{2k}dx$. Then the integrals become

$$-\frac{1}{2\sqrt{2k}}\int_{-\infty}^0 e^{-u^2/2}du + \frac{1}{2\sqrt{2k}}\int_0^\infty e^{-u^2/2}du = -\frac{1}{2\sqrt{2k}}\left(\frac{1}{2}\sqrt{2\pi}\right) + \frac{1}{2\sqrt{2k}}\left(\frac{1}{2}\sqrt{2\pi}\right) = 0.$$

$\int_0^\infty e^{-u^2/2}du = \int_{-\infty}^0 e^{-u^2/2}du = \frac{1}{2}\sqrt{2\pi}$ from Example 5 in Section 9.3.

4.



$f'(x) = -\frac{x-\mu}{\sigma^3\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}$

$f''(x) = -\frac{1}{\sigma^3\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2} + \frac{(x-\mu)^2}{\sigma^5\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}$

$= \left(\frac{(x-\mu)^2}{\sigma^5\sqrt{2\pi}} - \frac{1}{\sigma^3\sqrt{2\pi}}\right)e^{-(x-\mu)^2/2\sigma^2} = \frac{1}{\sigma^5\sqrt{2\pi}}[(x-\mu)^2 - \sigma^2]e^{-(x-\mu)^2/2\sigma^2}$

$f''(x) = 0$ when $(x-\mu)^2 = \sigma^2$ so $x = \mu \pm \sigma$ and the distance from μ to each inflection point is σ.

5. a. $\int_{-\infty}^{\infty} f(x)dx = \int_M^\infty \frac{CM^k}{x^{k+1}}dx = CM^k\left[-\frac{1}{kx^k}\right]_M^\infty = CM^k\left(0+\frac{1}{kM^k}\right) = \frac{C}{k}$. Thus, $\frac{C}{k} = 1$ when $C = k$.

b. $\mu = \int_{-\infty}^{\infty} xf(x)dx = \int_M^\infty x\frac{kM^k}{x^{k+1}}dx = kM^k\int_M^\infty \frac{1}{x^k}dx = kM^k\left(\lim_{b\to\infty}\int_M^b \frac{1}{x^k}dx\right)$

This integral converges when $k > 1$.

When $k > 1$, $\mu = kM^k\left(\lim_{b\to\infty}\left[-\frac{1}{(k-1)x^{k-1}}\right]_M^b\right) = kM^k\left(-0+\frac{1}{(k-1)M^{k-1}}\right) = \frac{kM}{k-1}$

The mean is finite only when $k > 1$.

c. Since the mean is finite only when $k > 1$, the variance is only defined when $k > 1$.

$\sigma^2 = \int_{-\infty}^{\infty}(x-\mu)^2 f(x)dx = \int_M^\infty\left(x-\frac{kM}{k-1}\right)^2\frac{kM^k}{x^{k+1}}dx = kM^k\int_M^\infty\left(x^2 - \frac{2kM}{k-1}x + \frac{k^2M^2}{(k-1)^2}\right)\frac{1}{x^{k+1}}dx$

$= kM^k\int_M^\infty \frac{1}{x^{k-1}}dx - \frac{2k^2M^{k+1}}{k-1}\int_M^\infty\frac{1}{x^k}dx + \frac{k^3M^{k+2}}{(k-1)^2}\int_M^\infty\frac{1}{x^{k+1}}dx$

The first integral converges only when $k-1>1$ or $k>2$. The second integral converges only when $k > 1$, which is taken care of by requiring $k > 2$.

$\sigma^2 = kM^k\left[-\frac{1}{(k-2)x^{k-2}}\right]_M^\infty - \frac{2k^2M^{k+1}}{k-1}\left[-\frac{1}{(k-1)x^{k-1}}\right]_M^\infty + \frac{k^3M^{k+2}}{(k-1)^2}\left[-\frac{1}{kx^k}\right]_M^\infty$

$= kM^k\left(-0+\frac{1}{(k-2)M^{k-2}}\right) - \frac{2k^2M^{k+1}}{k-1}\left(-0+\frac{1}{(k-1)M^{k-1}}\right) + \frac{k^3M^{k+2}}{(k-1)^2}\left(-0+\frac{1}{kM^k}\right)$

$= \frac{kM^2}{k-2} - \frac{2k^2M^2}{(k-1)^2} + \frac{k^2M^2}{(k-1)^2}$

$= kM^2\left(\frac{1}{k-2} - \frac{k}{(k-1)^2}\right) = kM^2\left(\frac{k^2-2k+1-k^2+2k}{(k-2)(k-1)^2}\right) = \frac{kM^2}{(k-2)(k-1)^2}$

6. a. $\int_{-\infty}^{\infty} f(x)dx = \int_0^{\infty} Cx^{\alpha-1}e^{-\beta x}dx$

Let $y = \beta x$, so $x = \frac{y}{\beta}$ and $dx = \frac{1}{\beta}dy$.

$$\int_0^{\infty} Cx^{\alpha-1}e^{-\beta x}dx = \int_0^{\infty} C\left(\frac{y}{\beta}\right)^{\alpha-1} e^{-y}\frac{1}{\beta}dy = \frac{C}{\beta^{\alpha}}\int_0^{\infty} y^{\alpha-1}e^{-y}dy = C\beta^{-\alpha}\Gamma(\alpha)$$

$C\beta^{-\alpha}\Gamma(\alpha) = 1$ when $C = \frac{1}{\beta^{-\alpha}\Gamma(\alpha)} = \frac{\beta^{\alpha}}{\Gamma(\alpha)}$.

b. $\mu = \int_{-\infty}^{\infty} xf(x)dx = \int_0^{\infty} x\frac{\beta^{\alpha}}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}dx = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\int_0^{\infty} x^{\alpha}e^{-\beta x}dx$

Let $y = \beta x$, so $x = \frac{y}{\beta}$ and $dx = \frac{1}{\beta}dy$.

$$\mu = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\int_0^{\infty}\left(\frac{y}{\beta}\right)^{\alpha} e^{-y}\frac{1}{\beta}dy = \frac{1}{\beta\Gamma(\alpha)}\int_0^{\infty} y^{\alpha}e^{-y}dy = \frac{1}{\beta\Gamma(\alpha)}\Gamma(\alpha+1) = \frac{1}{\beta\Gamma(\alpha)}\alpha\Gamma(\alpha) = \frac{\alpha}{\beta}$$

(Recall that $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$ for $\alpha > 0$.)

c. $\sigma^2 = \int_{-\infty}^{\infty}(x-\mu)^2 f(x)dx = \int_0^{\infty}\left(x - \frac{\alpha}{\beta}\right)^2 \frac{\beta^{\alpha}}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}dx = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\int_0^{\infty}\left(x^2 - \frac{2\alpha}{\beta}x + \frac{\alpha^2}{\beta^2}\right)x^{\alpha-1}e^{-\beta x}dx$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)}\int_0^{\infty} x^{\alpha+1}e^{-\beta x}dx - \frac{2\alpha\beta^{\alpha-1}}{\Gamma(\alpha)}\int_0^{\infty} x^{\alpha}e^{-\beta x}dx + \frac{\alpha^2\beta^{\alpha-2}}{\Gamma(\alpha)}\int_0^{\infty} x^{\alpha-1}e^{-\beta x}dx$$

In all three integrals, let $y = \beta x$, so $x = \frac{y}{\beta}$ and $dx = \frac{1}{\beta}dy$.

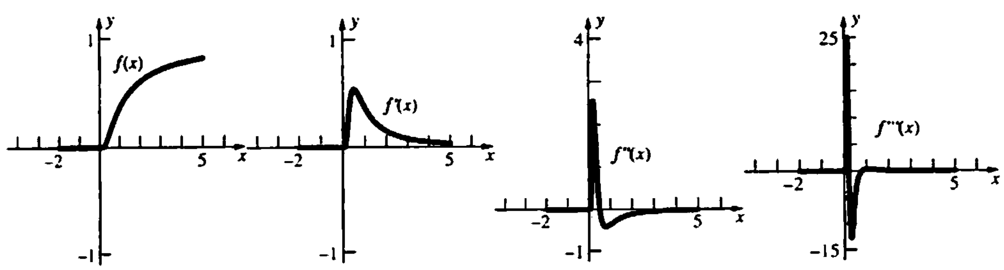
$$\sigma^2 = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\int_0^{\infty}\left(\frac{y}{\beta}\right)^{\alpha+1} e^{-y}\frac{1}{\beta}dy - \frac{2\alpha\beta^{\alpha-1}}{\Gamma(\alpha)}\int_0^{\infty}\left(\frac{y}{\beta}\right)^{\alpha} e^{-y}\frac{1}{\beta}dy + \frac{\alpha^2\beta^{\alpha-2}}{\Gamma(\alpha)}\int_0^{\infty}\left(\frac{y}{\beta}\right)^{\alpha-1} e^{-y}\frac{1}{\beta}dy$$

$$= \frac{1}{\beta^2\Gamma(\alpha)}\int_0^{\infty} y^{\alpha+1}e^{-y}dy - \frac{2\alpha}{\beta^2\Gamma(\alpha)}\int_0^{\infty} y^{\alpha}e^{-y}dy + \frac{\alpha^2}{\beta^2\Gamma(\alpha)}\int_0^{\infty} y^{\alpha-1}e^{-y}dy$$

$$= \frac{1}{\beta^2\Gamma(\alpha)}\Gamma(\alpha+2) - \frac{2\alpha}{\beta^2\Gamma(\alpha)}\Gamma(\alpha+1) + \frac{\alpha^2}{\beta^2\Gamma(\alpha)}\Gamma(\alpha) = \frac{1}{\beta^2\Gamma(\alpha)}(\alpha+1)\alpha\Gamma(\alpha) - \frac{2\alpha}{\beta^2\Gamma(\alpha)}\alpha\Gamma(\alpha) + \frac{\alpha^2}{\beta^2}$$

$$= \frac{\alpha^2+\alpha}{\beta^2} - \frac{2\alpha^2}{\beta^2} + \frac{\alpha^2}{\beta^2} = \frac{\alpha}{\beta^2}$$

7. a.



b. $$f^{(1)}(x) = \begin{cases} \frac{1}{x^2}e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$$

$$f^{(2)}(x) = \begin{cases} \left(\frac{1}{x^4} - \frac{2}{x^3}\right)e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$$

$$f^{(3)}(x)=\begin{cases}\left(\frac{1}{x^6}-\frac{6}{x^5}+\frac{6}{x^4}\right)e^{-1/x} & \text{if } x>0\\ 0 & \text{if } x\le 0\end{cases}$$

$$\lim_{x\to 0^+} f^{(0)}(x)=\lim_{x\to 0^+} e^{-1/x}=\lim_{x\to 0^+}\frac{1}{e^{1/x}}$$

Let $v=\frac{1}{x}$, then as $x\to 0^+, v\to\infty$ and

$$\lim_{x\to 0^+}\frac{1}{e^{1/x}}=\lim_{v\to\infty}\frac{1}{e^v}=0.$$

Use the same change of variables.

$$\lim_{x\to 0^+} f^{(1)}(x)=\lim_{x\to 0^+}\frac{1}{x^2}e^{-1/x}=\lim_{v\to\infty} v^2e^{-v}=\lim_{v\to\infty}\frac{v^2}{e^v}=0$$

$$\lim_{x\to 0^+} f^{(2)}(x)=\lim_{x\to 0^+}\left[\left(\frac{1}{x^4}-\frac{2}{x^3}\right)e^{-1/x}\right]=\lim_{v\to\infty}[(v^4-2v^3)e^{-v}]=\lim_{v\to\infty}\left(\frac{v^4}{e^v}-\frac{2v^3}{e^v}\right)=0$$

$$\lim_{x\to 0^+} f^{(3)}(x)=\lim_{x\to 0^+}\left[\left(\frac{1}{x^6}-\frac{6}{x^5}+\frac{6}{x^4}\right)e^{-1/x}\right]=\lim_{v\to\infty}[(v^6-6v^5+6x^4)e^{-v}]=\lim_{v\to\infty}\left(\frac{v^6}{e^v}-\frac{6v^5}{e^v}+\frac{6v^4}{e^v}\right)=0$$

(Recall from Example 2 of Section 9.2 that $\lim_{x\to\infty}\frac{x^a}{e^x}=0$ when a is any positive real number.)

For $x > 0$ and $x < 0$, $f^{(i)}(x)$ is continuous and since $\lim_{x\to 0^+} f^{(i)}(x)=0=f^{(i)}(0)$ for $i = 0, 1, 2, 3$, these are continuous for all x.

c. For $x\le 0$, $f^{(n)}(x)=0$ for all n, while for $x > 0$, $f^{(n)}(x)$ will be a sum of terms of the form $\frac{b}{x^a}e^{-1/x}$. Using the same change of variables as in part b, this is $\frac{bv^a}{e^v}$ which approaches 0 as $v\to\infty$. Thus $\lim_{x\to 0^+} f^{(n)}(x)=0=f^{(n)}(0)$ so $f^{(n)}(x)$ will be continuous.

8. a. $L\{t^\alpha\}(s)=\int_0^\infty t^\alpha e^{-st}\,dt$

Let $t=\frac{x}{s}$, so $dt=\frac{1}{s}dx$, then

$$\int_0^\infty t^\alpha e^{-st}\,dt=\int_0^\infty\left(\frac{x}{s}\right)^\alpha e^{-x}\frac{1}{s}dx=\int_0^\infty\frac{1}{s^{\alpha+1}}x^\alpha e^{-x}dx=\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}.$$

If $s\le 0$, $t^\alpha e^{-st}\to\infty$ as $t\to\infty$, so the integral does not converge. Thus, the transform is defined only when $s > 0$.

b. $L\{e^{\alpha t}\}(s)=\int_0^\infty e^{\alpha t}e^{-st}\,dt=\int_0^\infty e^{(\alpha-s)t}\,dt=\left[\frac{1}{\alpha-s}e^{(\alpha-s)t}\right]_0^\infty=\frac{1}{\alpha-s}\left[\lim_{b\to\infty}e^{(\alpha-s)b}-1\right]$

$$\lim_{b\to\infty}e^{(\alpha-s)b}=\begin{cases}\infty & \text{if } \alpha>s\\ 0 & \text{if } s>\alpha\end{cases}$$

Thus, $L\{e^{\alpha t}\}(s)=\frac{-1}{\alpha-s}=\frac{1}{s-\alpha}$ when $s > \alpha$. (When $s\le\alpha$, the integral does not converge.)

c. $L\{\sin(\alpha t)\}(s) = \int_0^\infty \sin(\alpha t)e^{-st}dt$

Let $I = \int_0^\infty \sin(\alpha t)e^{-st}dt$ and use integration by parts with $u = \sin(\alpha t)$, $du = \alpha\cos(\alpha t)dt$,

$dv = e^{-st}dt$, and $v = -\frac{1}{s}e^{-st}$.

Then $I = \left[-\frac{1}{s}\sin(\alpha t)e^{-st}\right]_0^\infty + \frac{\alpha}{s}\int_0^\infty \cos(\alpha t)e^{-st}dt$

Use integration by parts on this integral with

$u = \cos(\alpha t)$, $du = -\alpha\sin(\alpha t)dt$, $dv = e^{-st}dt$, and $v = -\frac{1}{s}e^{-st}$.

$$I = \left[-\frac{1}{s}\sin(\alpha t)e^{-st}\right]_0^\infty + \frac{\alpha}{s}\left(\left[-\frac{1}{s}\cos(\alpha t)e^{-st}\right]_0^\infty - \frac{\alpha}{s}\int_0^\infty \sin(\alpha t)e^{-st}dt\right)$$

$$= -\frac{1}{s}\left[e^{-st}\left(\sin(\alpha t) + \frac{\alpha}{s}\cos(\alpha t)\right)\right]_0^\infty - \frac{\alpha^2}{s^2}I$$

Thus,

$$I\left(1 + \frac{\alpha^2}{s^2}\right) = -\frac{1}{s}\left[e^{-st}\left(\sin(\alpha t) + \frac{\alpha}{s}\cos(\alpha t)\right)\right]_0^\infty$$

$$I = -\frac{1}{s\left(1+\frac{\alpha^2}{s^2}\right)}\left[e^{-st}\left(\sin(\alpha t) + \frac{\alpha}{s}\cos(\alpha t)\right)\right]_0^\infty = -\frac{s}{s^2+\alpha^2}\left[\lim_{b\to\infty} e^{-sb}\left(\sin(\alpha b) + \frac{\alpha}{s}\cos(\alpha b)\right) - \frac{\alpha}{s}\right]$$

$$\lim_{b\to\infty} e^{-sb}\left(\sin(\alpha b) + \frac{\alpha}{s}\cos(\alpha b)\right) = \begin{cases} 0 & \text{if } s > 0 \\ \infty & \text{if } s \le 0 \end{cases}$$

Thus, $I = \frac{\alpha}{s^2+\alpha^2}$ when $s > 0$.

d. $L\{\cos(\alpha t)\}(s) = \int_0^\infty \cos(\alpha t)e^{-st}dt$

Let $I = \int_0^\infty \cos(\alpha t)e^{-st}dt$ and use integration by parts with $u = \cos(\alpha t)$, $du = -\alpha\sin(\alpha t)dt$,

$dv = e^{-st}dt$, $v = -\frac{1}{s}e^{-st}$. Then,

$$I = \left[-\frac{1}{s}\cos(\alpha t)e^{-st}\right]_0^\infty - \frac{\alpha}{s}\int_0^\infty \sin(\alpha t)e^{-st}dt$$

Note that this last integral is the same as in part c. Thus,

$$I = \left[-\frac{1}{s}\cos(\alpha t)e^{-st}\right]_0^\infty - \frac{\alpha}{s}\left(\frac{\alpha}{s^2+\alpha^2}\right) = -\frac{1}{s}\lim_{b\to\infty}\cos(\alpha t)e^{-sb} + \frac{1}{s} - \frac{\alpha^2}{s(s^2+\alpha^2)}$$

The integral from part c converges only when $s > 0$ and when this is the case $\lim_{b\to\infty}\cos(\alpha t)e^{-sb} = 0$.

Thus, when $s > 0$,

$$I = \frac{1}{s} - \frac{\alpha^2}{s(s^2+\alpha^2)} = \frac{s^2+\alpha^2-\alpha^2}{s(s^2+\alpha^2)} = \frac{s}{s^2+\alpha^2}$$