Chapter Eleven

Argument Principle

11.1. Argument principle. Let C be a simple closed curve, and suppose f is analytic on C. Suppose moreover that the only singularities of f inside C are poles. If $f(z) \neq 0$ for all $z \epsilon C$, then $\Gamma = f(C)$ is a closed curve which does not pass through the origin. If

$$\gamma(t),\ \alpha \leq t \leq \beta$$

is a complex description of C, then

$$\zeta(t) = f(\gamma(t)),\ \ \alpha \leq t \leq \beta$$

is a complex description of Γ. Now, let's compute

$$\int_C \frac{f'(z)}{f(z)} dz = \int_\alpha^\beta \frac{f'(\gamma(t))}{f(\gamma(t))} \gamma'(t) dt.$$

But notice that $\zeta'(t) = f'(\gamma(t))\gamma'(t)$. Hence,

$$\int_C \frac{f'(z)}{f(z)} dz = \int_\alpha^\beta \frac{f'(\gamma(t))}{f(\gamma(t))} \gamma'(t) dt = \int_\alpha^\beta \frac{\zeta'(t)}{\zeta(t)} dt$$
$$= \int_\Gamma \frac{1}{z} dz = n2\pi i,$$

where $|n|$ is the number of times Γ "winds around" the origin. The integer n is positive in case Γ is traversed in the positive direction, and negative in case the traversal is in the negative direction.

Next, we shall use the Residue Theorem to evaluate the integral $\int_C \frac{f'(z)}{f(z)} dz$. The singularities of the integrand $\frac{f'(z)}{f(z)}$ are the poles of f together with the zeros of f. Let's find the residues at these points. First, let $Z = \{z_1, z_2, \ldots, z_K\}$ be set of all zeros of f. Suppose the order of the zero z_j is n_j. Then $f(z) = (z - z_j)^{n_j} h(z)$ and $h(z_j) \neq 0$. Thus,

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{(z-z_j)^{n_j}h'(z)+n_j(z-z_j)^{n_j-1}h(z)}{(z-z_j)^{n_j}h(z)} \\ &= \frac{h'(z)}{h(z)} + \frac{n_j}{(z-z_j)}.\end{aligned}$$

Then

$$\phi(z) = (z-z_j)\frac{f'(z)}{f(z)} = (z-z_j)\frac{h'(z)}{h(z)} + n_{j,}$$

and

$$\operatorname*{Res}_{z=z_j} \frac{f'}{f} = n_j.$$

The sum of all these residues is thus

$$N = n_1 + n_2 + \ldots + n_K.$$

Next, we go after the residues at the poles of f. Let the set of poles of f be $P = \{p_1, p_2, \ldots, p_J\}$. Suppose p_j is a pole of order m_j. Then

$$h(z) = (z-p_j)^{m_j} f(z)$$

is analytic at p_j. In other words,

$$f(z) = \frac{h(z)}{(z-p_j)^{m_j}}.$$

Hence,

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{(z-p_j)^{m_j}h'(z)-m_j(z-p_j)^{m_j-1}h(z)}{(z-p_j)^{2m_j}} \bullet \frac{(z-p_j)^{m_j}}{h(z)} \\ &= \frac{h'(z)}{h(z)} - \frac{m_j}{(z-p_j)^{m_j}}.\end{aligned}$$

Now then,

$$\phi(z) = (z-p_j)^{m_j}\frac{f'(z)}{f(z)} = (z-p_j)^{m_j}\frac{h'(z)}{h(z)} - m_j,$$

and so

$$\operatorname*{Res}_{z=p_j} \frac{f'}{f} = \phi(p_j) = -m_j.$$

The sum of all these residues is

$$-P = -m_1 - m_2 - \ldots - m_J$$

Then,

$$\int_C \frac{f'(z)}{f(z)} dz = 2\pi i(N-P);$$

and we already found that

$$\int_C \frac{f'(z)}{f(z)} dz = n2\pi i,$$

where n is the ”winding number”, or the number of times Γ winds around the origin—$n > 0$ means Γ winds in the positive sense, and n negative means it winds in the negative sense. Finally, we have

$$n = N-P,$$

where $N = n_1 + n_2 + \ldots + n_K$ is the number of zeros inside C, counting multiplicity, or the order of the zeros, and $P = m_1 + m_2 + \ldots + m_J$ is the number of poles, counting the order. This result is the celebrated **argument principle.**

Exercises

1. Let C be the unit circle $|z| = 1$ positively oriented, and let f be given by

$$f(z) = z^3.$$

How many times does the curve $f(C)$ wind around the origin? Explain.

2. Let C be the unit circle $|z| = 1$ positively oriented, and let f be given by

$$f(z) = \frac{z^2 + 2}{z^3}.$$

How many times does the curve $f(C)$ wind around the origin? Explain.

3. Let $p(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$, with $a_n \neq 0$. Prove there is an $R > 0$ so that if C is the circle $|z| = R$ positively oriented, then

$$\int_C \frac{p'(z)}{p(z)} dz = 2n\pi i.$$

4. How many solutions of $3e^z - z = 0$ are in the disk $|z| \leq 1$? Explain.

5. Suppose f is entire and $f(z)$ is real if and only if z is real. Explain how you know that f has at most one zero.

11.2 Rouche's Theorem. Suppose f and g are analytic on and inside a simple closed contour C. Suppose moreover that $|f(z)| > |g(z)|$ for all $z \epsilon C$. Then we shall see that f and $f + g$ have the same number of zeros inside C. This result is **Rouche's Theorem.** To see why it is so, start by defining the function $\Psi(t)$ on the interval $0 \leq t \leq 1$:

$$\Psi(t) = \frac{1}{2\pi i} \int_C \frac{f'(z) + tg'(t)}{f(z) + tg(z)} dz.$$

Observe that this is okay—that is, the denominator of the integrand is never zero:

$$|f(z) + tg(z)| \geq ||f(t)| - t|g(t)|| \geq ||f(t)| - |g(t)|| > 0.$$

Observe that Ψ is continuous on the interval $[0, 1]$ and is integer-valued—$\Psi(t)$ is the number of zeros of $f + tg$ inside C. Being continuous and integer-valued on the connected set $[0,1]$, it must be constant. In particular, $\Psi(0) = \Psi(1)$. This does the job!

$$\Psi(0) = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz$$

is the number of zeros of f inside C, and

$$\Psi(1) = \frac{1}{2\pi i}\int_C \frac{f'(z)+g'(z)}{f(z)+g(z)}dz$$

is the number of zeros of $f + g$ inside C.

Example

How many solutions of the equation $z^6 - 5z^5 + z^3 - 2 = 0$ are inside the circle $|z| = 1$? Rouche's Theorem makes it quite easy to answer this. Simply let $f(z) = -5z^5$ and let $g(z) = z^6 + z^3 - 2$. Then $|f(z)| = 5$ and $|g(z)| \le |z|^6 + |z|^3 + 2 = 4$ for all $|z| = 1$. Hence $|f(z)| > |g(z)|$ on the unit circle. From Rouche's Theorem we know then that f and $f + g$ have the same number of zeros inside $|z| = 1$. Thus, there are 5 such solutions.

The following nice result follows easily from Rouche's Theorem. Suppose U is an open set (*i.e.*, every point of U is an interior point) and suppose that a sequence (f_n) of functions analytic on U converges uniformly to the function f. Suppose further that f is not zero on the circle $C = \{z : |z - z_0| = R\} \subset U$. Then there is an integer N so that for all $n \ge N$, the functions f_n and f have the same number of zeros inside C.

This result, called **Hurwitz's Theorem**, is an easy consequence of Rouche's Theorem. Simply observe that for $z \epsilon C$, we have $|f(z)| > \varepsilon > 0$ for some ε. Now let N be large enough to insure that $|f_n(z) - f(z)| < \varepsilon$ on C. It follows from Rouche's Theorem that f and $f + (f_n - f) = f_n$ have the same number of zeros inside C.

Example

On any bounded set, the sequence (f_n), where $f_n(z) = 1 + z + \frac{z^2}{2} + \ldots + \frac{z^n}{n!}$, converges uniformly to $f(z) = e^z$, and $f(z) \neq 0$ for all z. Thus for any R, there is an N so that for $n > N$, every zero of $1 + z + \frac{z^2}{2} + \ldots + \frac{z^n}{n!}$ has modulus $> R$. Or to put it another way, given an R there is an N so that for $n > N$ no polynomial $1 + z + \frac{z^2}{2} + \ldots + \frac{z^n}{n!}$ has a zero inside the circle of radius R.

Exercises

6. Show that the polynomial $z^6 + 4z^2 - 1$ has exactly two zeros inside the circle $|z| = 1$.

7. How many solutions of $2z^4 - 2z^3 + 2z^2 - 2z + 9 = 0$ lie inside the circle $|z| = 1$?

8. Use Rouche's Theorem to prove that every polynomial of degree n has exactly n zeros (counting multiplicity, of course).

9. Let C be the closed unit disk $|z| \leq 1$. Suppose the function f analytic on C maps C into the open unit disk $|z| < 1$—that is, $|f(z)| < 1$ for all $z \epsilon C$. Prove there is exactly one $w \epsilon C$ such that $f(w) = w$. (The point w is called a **fixed point** of f.)