Chapter Four

Integration

4.1. Introduction. If $\gamma : D \to \mathbf{C}$ is simply a function on a real interval $D = [\alpha, \beta]$, then the integral $\int_\alpha^\beta \gamma(t)dt$ is, of course, simply an ordered pair of everyday 3^{rd} grade calculus integrals:

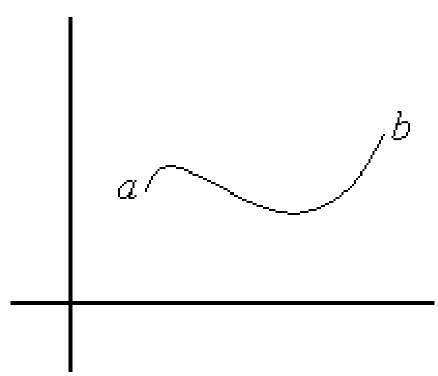
$$\int_\alpha^\beta \gamma(t)dt = \int_\alpha^\beta x(t)dt + i\int_\alpha^\beta y(t)dt,$$

where $\gamma(t) = x(t) + iy(t)$. Thus, for example,

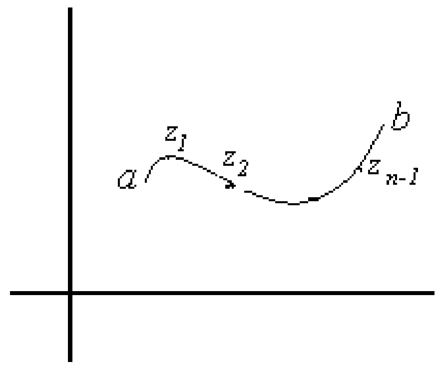
$$\int_0^1 [(t^2+1) + it^3]dt = \frac{4}{3} + \frac{i}{4}.$$

Nothing really new here. The excitement begins when we consider the idea of an integral of an honest-to-goodness complex function $f : D \to \mathbf{C}$, where D is a subset of the complex plane. Let's define the integral of such things; it is pretty much a straight-forward extension to two dimensions of what we did in one dimension back in Mrs. Turner's class.

Suppose f is a complex-valued function on a subset of the complex plane and suppose a and b are *complex* numbers in the domain of f. In one dimension, there is just one way to get from one number to the other; here we must also specify a path from a to b. Let C be a path from a to b, and we must also require that C be a subset of the domain of f.



(Note we do not even require that $a \neq b$; but in case $a = b$, we must specify an *orientation* for the closed path C.) Next, let P be a **partition** of the curve; that is, $P = \{z_0, z_1, z_2, \ldots, z_n\}$ is a finite subset of C, such that $a = z_0$, $b = z_n$, and such that z_j comes immediately after z_{j-1} as we travel along C from a to b.



A Riemann sum associated with the partition P is just what it is in the real case:

$$S(P) = \sum_{j=1}^{n} f(z_j^*)\Delta z_j,$$

where z_j^* is a point on the arc between z_{j-1} and z_j , and $\Delta z_j = z_j - z_{j-1}$. (Note that for a given partition P, there are many $S(P)$—depending on how the points z_j^* are chosen.) If there is a number L so that given any $\varepsilon > 0$, there is a partition P_ε of C such that

$$|S(P) - L| < \varepsilon$$

whenever $P \supset P_\varepsilon$, then f is said to be integrable on C and the number L is called the **integral of** f on C. This number L is usually written $\int_C f(z)dz$.

Some properties of integrals are more or less evident from looking at Riemann sums:

$$\int_C cf(z)dz = c\int_C f(z)dz$$

for any complex constant c.

$$\int_C (f(z)+g(z))dz = \int_C f(z)dz + \int_C g(z)dz$$

4.2 Evaluating integrals. Now, how on Earth do we ever find such an integral? Let $\gamma : [\alpha,\beta] \to \mathbf{C}$ be a complex description of the curve C. We partition C by partitioning the interval $[\alpha,\beta]$ in the usual way: $\alpha = t_0 < t_1 < t_2 <\ldots< t_n = \beta$. Then $\{a = \gamma(\alpha), \gamma(t_1), \gamma(t_2), \ldots, \gamma(\beta) = b\}$ is partition of C. (Recall we assume that $\gamma'(t) \neq 0$ for a complex description of a curve C.) A corresponding Riemann sum looks like

$$S(P) = \sum_{j=1}^{n} f(\gamma(t_j^*))(\gamma(t_j) - \gamma(t_{j-1})).$$

We have chosen the points $z_j^* = \gamma(t_j^*)$, where $t_{j-1} \leq t_j^* \leq t_j$. Next, multiply each term in the sum by 1 in disguise:

$$S(P) = \sum_{j=1}^{n} f(\gamma(t_j^*))(\frac{\gamma(t_j) - \gamma(t_{j-1})}{t_j - t_{j-1}})(t_j - t_{j-1}).$$

I hope it is now reasonably convincing that "in the limit", we have

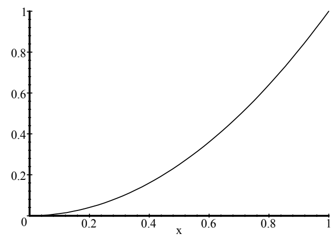
$$\int_C f(z)dz = \int_\alpha^\beta f(\gamma(t))\gamma'(t)dt.$$

(We are, of course, assuming that the derivative γ' exists.)

Example

We shall find the integral of $f(z) = (x^2+y) + i(xy)$ from $a = 0$ to $b = 1+i$ along three different paths, or **contours**, as some call them.

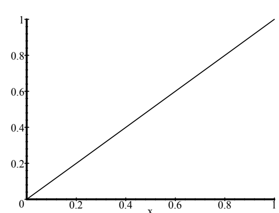
First, let C_1 be the part of the parabola $y = x^2$ connecting the two points. A complex description of C_1 is $\gamma_1(t) = t + it^2, 0 \leq t \leq 1$:



Now, $\gamma_1'(t) = 1 + 2ti$, and $f(\gamma_1(t)) = (t^2 + t^2) + itt^2 = 2t^2 + it^3$. Hence,

$$\begin{aligned}\int_{C_1} f(z)dz &= \int_0^1 f(\gamma_1(t))\gamma_1'(t)dt \\ &= \int_0^1 (2t^2 + it^3)(1 + 2ti)dt \\ &= \int_0^1 (2t^2 - 2t^4 + 5t^3 i)dt \\ &= \frac{4}{15} + \frac{5}{4}i\end{aligned}$$

Next, let's integrate along the straight line segment C_2 joining 0 and $1 + i$.



Here we have $\gamma_2(t) = t + it$, $0 \le t \le 1$. Thus, $\gamma_2'(t) = 1 + i$, and our integral looks like

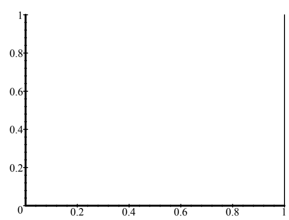
$$\int_{C_2} f(z)dz = \int_0^1 f(\gamma_2(t))\gamma_2'(t)dt$$

$$= \int_0^1 [(t^2+t)+it^2](1+i)dt$$

$$= \int_0^1 [t+i(t+2t^2)]dt$$

$$= \frac{1}{2} + \frac{7}{6}i$$

Finally, let's integrate along C_3, the path consisting of the line segment from 0 to 1 together with the segment from 1 to $1+i$.



We shall do this in two parts: C_{31}, the line from 0 to 1 ; and C_{32}, the line from 1 to $1+i$. Then we have

$$\int_{C_3} f(z)dz = \int_{C_{31}} f(z)dz + \int_{C_{32}} f(z)dz.$$

For C_{31} we have $\gamma(t) = t$, $0 \leq t \leq 1$. Hence,

$$\int_{C_{31}} f(z)dz = \int_0^1 dt = \frac{1}{3}.$$

For C_{32} we have $\gamma(t) = 1+it$, $0 \leq t \leq 1$. Hence,

$$\int_{C_{32}} f(z)dz = \int_0^1 (1+t+it)itdt = -\frac{1}{3} + \frac{5}{6}i.$$

Thus,

$$\begin{aligned}\int_{C_3} f(z)dz &= \int_{C_{31}} f(z)dz + \int_{C_{32}} f(z)dz \\ &= \frac{5}{6}i.\end{aligned}$$

Suppose there is a number M so that $|f(z)| \leq M$ for all $z \epsilon C$. Then

$$\begin{aligned}\left|\int_C f(z)dz\right| &= \left|\int_\alpha^\beta f(\gamma(t))\gamma'(t)dt\right| \\ &\leq \int_\alpha^\beta |f(\gamma(t))\gamma'(t)|dt \\ &\leq M\int_\alpha^\beta |\gamma'(t)|dt = ML,\end{aligned}$$

where $L = \int_\alpha^\beta |\gamma'(t)|dt$ is the length of C.

Exercises

1. Evaluate the integral $\int_C \overline{z}\,dz$, where C is the parabola $y = x^2$ from 0 to $1 + i$.

2. Evaluate $\int_C \frac{1}{z}dz$, where C is the circle of radius 2 centered at 0 oriented counterclockwise.

4. Evaluate $\int_C f(z)dz$, where C is the curve $y = x^3$ from $-1 - i$ to $1 + i$, and

$$f(z) = \begin{cases} 1 & \text{for } y < 0 \\ 4y & \text{for } y \geq 0 \end{cases}.$$

5. Let C be the part of the circle $\gamma(t) = e^{it}$ in the first quadrant from $a = 1$ to $b = i$. Find as small an upper bound as you can for $\left|\int_C (z^2 - \overline{z}^4 + 5)dz\right|$.

6. Evaluate $\int_C f(z)dz$ where $f(z) = z + 2\overline{z}$ and C is the path from $z = 0$ to $z = 1 + 2i$ consisting of the line segment from 0 to 1 together with the segment from 1 to $1 + 2i$.

4.3 Antiderivatives. Suppose D is a subset of the reals and $\gamma : D \to \mathbf{C}$ is differentiable at t. Suppose further that g is differentiable at $\gamma(t)$. Then let's see about the derivative of the composition $g(\gamma(t))$. It is, in fact, exactly what one would guess. First,

$$g(\gamma(t)) = u(x(t),y(t)) + iv(x(t),y(t)),$$

where $g(z) = u(x,y) + iv(x,y)$ and $\gamma(t) = x(t) + iy(t)$. Then,

$$\frac{d}{dt}g(\gamma(t)) = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + i\left(\frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial v}{\partial y}\frac{dy}{dt}\right).$$

The places at which the functions on the right-hand side of the equation are evaluated are obvious. Now, apply the Cauchy-Riemann equations:

$$\begin{aligned}\frac{d}{dt}g(\gamma(t)) &= \frac{\partial u}{\partial x}\frac{dx}{dt} - \frac{\partial v}{\partial x}\frac{dy}{dt} + i\left(\frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial x}\frac{dy}{dt}\right)\\ &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\left(\frac{dx}{dt} + i\frac{dy}{dt}\right)\\ &= g'(\gamma(t))\gamma'(t).\end{aligned}$$

The nicest result in the world!

Now, back to integrals. Let $F : D \to \mathbf{C}$ and suppose $F'(z) = f(z)$ in D. Suppose moreover that a and b are in D and that $C \subset D$ is a contour from a to b. Then

$$\int_C f(z)dz = \int_\alpha^\beta f(\gamma(t))\gamma'(t)dt,$$

where $\gamma : [\alpha,\beta] \to C$ describes C. From our introductory discussion, we know that $\frac{d}{dt}F(\gamma(t)) = F'(\gamma(t))\gamma'(t) = f(\gamma(t))\gamma'(t)$. Hence,

$$\begin{aligned}\int_C f(z)dz &= \int_\alpha^\beta f(\gamma(t))\gamma'(t)dt \\ &= \int_\alpha^\beta \frac{d}{dt}F(\gamma(t))dt = F(\gamma(\beta)) - F(\gamma(\alpha)) \\ &= F(b) - F(a).\end{aligned}$$

This is very pleasing. Note that integral depends only on the points a and b and not at all on the path C. We say the integral is **path independent**. Observe that this is equivalent to saying that the integral of f around any closed path is 0. We have thus shown that if in D the integrand f is the derivative of a function F, then any integral $\int_C f(z)dz$ for $C \subset D$ is path independent.

Example

Let C be the curve $y = \frac{1}{x^2}$ from the point $z = 1 + i$ to the point $z = 3 + \frac{i}{9}$. Let's find

$$\int_C z^2 dz.$$

This is easy—we know that $F'(z) = z^2$, where $F(z) = \frac{1}{3}z^3$. Thus,

$$\begin{aligned}\int_C z^2 dz &= \frac{1}{3}\left[(1+i)^3 - \left(3 + \frac{i}{9}\right)^3\right] \\ &= -\frac{260}{27} - \frac{728}{2187}i\end{aligned}$$

Now, instead of assuming f has an antiderivative, let us suppose that the integral of f between any two points in the domain is independent of path and that f is continuous. Assume also that every point in the domain D is an interior point of D and that D is connected. We shall see that in this case, f has an antiderivative. To do so, let z_0 be any point in D, and define the function F by

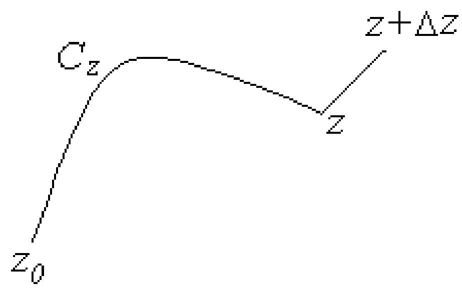
$$F(z) = \int_{C_z} f(z)dz,$$

where C_z is any path in D from z_0 to z. Here is important that the integral is path independent, otherwise $F(z)$ would not be well-defined. Note also we need the assumption that D is connected in order to be sure there always is at least one such path.

Now, for the computation of the derivative of F:

$$F(z+\Delta z) - F(z) = \int_{L_{\Delta z}} f(s)ds,$$

where $L_{\Delta z}$ is the line segment from z to $z + \Delta z$.



Next, observe that $\int_{L_{\Delta z}} ds = \Delta z$. Thus, $f(z) = \frac{1}{\Delta z}\int_{L_{\Delta z}} f(z)ds$, and we have

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int_{L_{\Delta z}} (f(s) - f(z))ds.$$

Now then,

$$\left|\frac{1}{\Delta z}\int_{L_{\Delta z}} (f(s) - f(z))ds\right| \leq \left|\frac{1}{\Delta z}\right||\Delta z| \max\{|f(s) - f(z)| : s\epsilon L_{\Delta z}\}$$
$$\leq \max\{|f(s) - f(z)| : s\epsilon L_{\Delta z}\}.$$

We know f is continuous at z, and so $\lim_{\Delta z \to 0} \max\{|f(s) - f(z)| : s\epsilon L_{\Delta z}\} = 0$. Hence,

$$\lim_{\Delta z\to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \lim_{\Delta z\to 0}\left(\frac{1}{\Delta z}\int_{L_{\Delta z}} (f(s) - f(z))ds\right)$$
$$= 0.$$

In other words, $F'(z) = f(z)$, and so, just as promised, f has an antiderivative! Let's summarize what we have shown in this section:

Suppose $f : D \to \mathbf{C}$ is continuous, where D is connected and every point of D is an interior point. Then f has an antiderivative if and only if the integral between any two points of D is path independent.

Exercises

7. Suppose C is any curve from 0 to $\pi + 2i$. Evaluate the integral

$$\int_C \cos\left(\frac{z}{2}\right) dz.$$

8. a)Let $F(z) = \log z$, $0 < \arg z < 2\pi$. Show that the derivative $F'(z) = \frac{1}{z}$.
b)Let $G(z) = \log z, -\frac{\pi}{4} < \arg z < \frac{7\pi}{4}$. Show that the derivative $G'(z) = \frac{1}{z}$.
c)Let C_1 be a curve in the right-half plane $D_1 = \{z : \operatorname{Re} z \geq 0\}$ from $-i$ to i that does not pass through the origin. Find the integral

$$\int_{C_1} \frac{1}{z} dz.$$

d)Let C_2 be a curve in the left-half plane $D_2 = \{z : \operatorname{Re} z \leq 0\}$ from $-i$ to i that does not pass through the origin. Find the integral.

$$\int_{C_2} \frac{1}{z} dz.$$

9. Let C be the circle of radius 1 centered at 0 with the *clockwise* orientation. Find

$$\int_C \frac{1}{z} dz.$$

10. a)Let $H(z) = z^c, -\pi < \arg z < \pi$. Find the derivative $H'(z)$.
b)Let $K(z) = z^c, -\frac{\pi}{4} < \arg z < \frac{7\pi}{4}$. What is the largest subset of the plane on which $H(z) = K(z)$?
c)Let C be any path from -1 to 1 that lies completely in the upper half-plane. (Upper

half-plane = $\{z : \operatorname{Im} z \geq 0\}$.) Find

$$\int_C F(z)dz,$$

where $F(z) = z^i, -\pi < \arg z \leq \pi$.

11. Suppose P is a polynomial and C is a closed curve. Explain how you know that $\int_C P(z)dz = 0$.