Chapter Nine

Taylor and Laurent Series

9.1. Taylor series. Suppose f is analytic on the open disk $|z - z_0| < r$. Let z be any point in this disk and choose C to be the positively oriented circle of radius ρ, where $|z - z_0| < \rho < r$. Then for $s \epsilon C$ we have

$$\frac{1}{s-z} = \frac{1}{(s-z_0)-(z-z_0)} = \frac{1}{s-z_0}\left[\frac{1}{1-\frac{z-z_0}{s-z_0}}\right] = \sum_{j=0}^{\infty}\frac{(z-z_0)^j}{(s-z_0)^{j+1}}$$

since $|\frac{z-z_0}{s-z_0}| < 1$. The convergence is uniform, so we may integrate

$$\int_C \frac{f(s)}{s-z}ds = \sum_{j=0}^{\infty}\left(\int_C \frac{f(s)}{(s-z_0)^{j+1}}ds\right)(z-z_0)^j, \text{or}$$

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(s)}{s-z}ds = \sum_{j=0}^{\infty}\left(\frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z_0)^{j+1}}ds\right)(z-z_0)^j.$$

We have thus produced a power series having the given analytic function as a limit:

$$f(z) = \sum_{j=0}^{\infty} c_j(z-z_0)^j,\ |z-z_0| < r,$$

where

$$c_j = \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z_0)^{j+1}}ds.$$

This is the celebrated **Taylor Series** for f at $z = z_0$.

We know we may differentiate the series to get

$$f'(z) = \sum_{j=1}^{\infty} jc_j(z-z_0)^{j-1}$$

and this one converges uniformly where the series for f does. We can thus differentiate again and again to obtain

$$f^{(n)}(z) = \sum_{j=n}^{\infty} j(j-1)(j-2)\ldots(j-n+1)c_j(z-z_0)^{j-n}.$$

Hence,

$$f^{(n)}(z_0) = n!c_n, \text{ or}$$
$$c_n = \frac{f^{(n)}(z_0)}{n!}.$$

But we also know that

$$c_n = \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z_0)^{n+1}}ds.$$

This gives us

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(s)}{(s-z_0)^{n+1}}ds, \text{ for } n = 0,1,2,\ldots.$$

This is the famous **Generalized Cauchy Integral Formula.** Recall that we previously derived this formula for $n = 0$ and 1.

What does all this tell us about the radius of convergence of a power series? Suppose we have

$$f(z) = \sum_{j=0}^{\infty} c_j(z-z_0)^j,$$

and the radius of convergence is R. Then we know, of course, that the limit function f is analytic for $|z-z_0| < R$. We showed that if f is analytic in $|z-z_0| < r$, then the series converges for $|z-z_0| < r$. Thus $r \leq R$, and so f cannot be analytic at any point z for which $|z-z_0| > R$. In other words, the circle of convergence is the largest circle centered at z_0 inside of which the limit f is analytic.

Example

Let $f(z) = \exp(z) = e^z$. Then $f(0) = f'(0) = \ldots = f^{(n)}(0) = \ldots = 1$, and the Taylor series for f at $z_0 = 0$ is

$$e^z = \sum_{j=0}^{\infty} \frac{1}{j!} z^j$$

and this is valid for all values of z since f is entire. (We also showed earlier that this particular series has an infinite radius of convergence.)

Exercises

1. Show that for all z,

$$e^z = e \sum_{j=0}^{\infty} \frac{1}{j!}(z-1)^j.$$

2. What is the radius of convergence of the Taylor series $\left(\sum_{j=0}^{n} c_j z^j \right)$ for $\tanh z$?

3. Show that

$$\frac{1}{1-z} = \sum_{j=0}^{\infty} \frac{(z-i)^j}{(1-i)^{j+1}}$$

for $|z - i| < \sqrt{2}$.

4. If $f(z) = \frac{1}{1-z}$, what is $f^{(10)}(i)$?

5. Suppose f is analytic at $z = 0$ and $f(0) = f'(0) = f''(0) = 0$. Prove there is a function g analytic at 0 such that $f(z) = z^3 g(z)$ in a neighborhood of 0.

6. Find the Taylor series for $f(z) = \sin z$ at $z_0 = 0$.

7. Show that the function f defined by

$$f(z) = \begin{cases} \frac{\sin z}{z} & \text{for } z \neq 0 \\ 1 & \text{for } z = 0 \end{cases}$$

is analytic at $z = 0$, and find $f'(0)$.

9.2. Laurent series. Suppose f is analytic in the region $R_1 < |z - z_0| < R_2$, and let C be a positively oriented simple closed curve around z_0 in this region. (Note: we include the possiblites that R_1 can be 0, and $R_2 = \infty$.) We shall show that for $z \notin C$ in this region

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j + \sum_{j=1}^{\infty} \frac{b_j}{(z - z_0)^j},$$

where

$$a_j = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z_0)^{j+1}} ds, \text{ for } j = 0, 1, 2, \ldots$$

and

$$b_j = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z_0)^{-j+1}} ds, \text{ for } j = 1, 2, \ldots.$$

The sum of the limits of these two series is frequently written

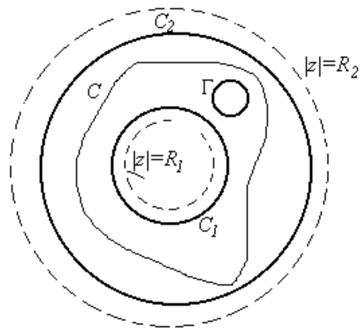
$$f(z) = \sum_{j=-\infty}^{\infty} c_j (z - z_0)^j,$$

where

$$c_j = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z_0)^{j+1}} ds, j = 0, \pm 1, \pm 2, \ldots.$$

This recipe for $f(z)$ is called a **Laurent series**, although it is important to keep in mind that it is really two series.

Okay, now let's derive the above formula. First, let r_1 and r_2 be so that $R_1 < r_1 \le |z - z_0| \le r_2 < R_2$ and so that the point z and the curve C are included in the region $r_1 \le |z - z_0| \le r_2$. Also, let Γ be a circle centered at z and such that Γ is included in this region.



Then $\frac{f(s)}{s-z}$ is an analytic function (of s) on the region bounded by $C_1, C_2,$ and Γ, where C_1 is the circle $|z| = r_1$ and C_2 is the circle $|z| = r_2$. Thus,

$$\int_{C_2} \frac{f(s)}{s-z} ds = \int_{C_1} \frac{f(s)}{s-z} ds + \int_{\Gamma} \frac{f(s)}{s-z} ds.$$

(All three circles are positively oriented, of course.) But $\int_{\Gamma} \frac{f(s)}{s-z} ds = 2\pi i f(z)$, and so we have

$$2\pi i f(z) = \int_{C_2} \frac{f(s)}{s-z} ds - \int_{C_1} \frac{f(s)}{s-z} ds.$$

Look at the first of the two integrals on the right-hand side of this equation. For $s \in C_2$, we have $|z - z_0| < |s - z_0|$, and so

$$\begin{aligned}
\frac{1}{s-z} &= \frac{1}{(s-z_0)-(z-z_0)} \\
&= \frac{1}{s-z_0}\left[\frac{1}{1-\left(\frac{z-z_0}{s-z_0}\right)}\right] \\
&= \frac{1}{s-z_0}\sum_{j=0}^{\infty}\left(\frac{z-z_0}{s-z_0}\right)^j \\
&= \sum_{j=0}^{\infty}\frac{1}{(s-z_0)^{j+1}}(z-z_0)^j.
\end{aligned}$$

Hence,

$$\int_{C_2} \frac{f(s)}{s-z}ds = \sum_{j=0}^{\infty}\left(\int_{C_2} \frac{f(s)}{(s-z_0)^{j+1}}ds \right)(z-z_0)^j$$
$$= . \sum_{j=0}^{\infty}\left(\int_{C} \frac{f(s)}{(s-z_0)^{j+1}}ds \right)(z-z_0)^j$$

For the second of these two integrals, note that for $s \epsilon C_1$ we have $|s - z_0| < |z - z_0|$, and so

$$\frac{1}{s-z} = \frac{-1}{(z-z_0)-(s-z_0)} = \frac{-1}{z-z_0}\left[\frac{1}{1-(\frac{s-z_0}{z-z_0})} \right]$$
$$= \frac{-1}{z-z_0}\sum_{j=0}^{\infty}\left(\frac{s-z_0}{z-z_0}\right)^j = -\sum_{j=0}^{\infty}(s-z_0)^j\frac{1}{(z-z_0)^{j+1}}$$
$$= -\sum_{j=1}^{\infty}(s-z_0)^{j-1}\frac{1}{(z-z_0)^j} = -\sum_{j=1}^{\infty}\left(\frac{1}{(s-z_0)^{-j+1}}\right)\frac{1}{(z-z_0)^j}$$

As before,

$$\int_{C_1} \frac{f(s)}{s-z}ds = -\sum_{j=1}^{\infty}\left(\int_{C_1} \frac{f(s)}{(s-z_0)^{-j+1}}ds \right)\frac{1}{(z-z_0)^j}$$
$$= -\sum_{j=1}^{\infty}\left(\int_{C} \frac{f(s)}{(s-z_0)^{-j+1}}ds \right)\frac{1}{(z-z_0)^j}$$

Putting this altogether, we have the Laurent series:

$$f(z) = \frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{s-z}ds - \frac{1}{2\pi i}\int_{C_1} \frac{f(s)}{s-z}ds$$
$$= \sum_{j=0}^{\infty}\left(\frac{1}{2\pi i}\int_{C} \frac{f(s)}{(s-z_0)^{j+1}}ds \right)(z-z_0)^j + \sum_{j=1}^{\infty}\left(\frac{1}{2\pi i}\int_{C} \frac{f(s)}{(s-z_0)^{-j+1}}ds \right)\frac{1}{(z-z_0)^j}.$$

Example

Let f be defined by

$$f(z) = \frac{1}{z(z-1)}.$$

First, observe that f is analytic in the region $0 < |z| < 1$. Let's find the Laurent series for f valid in this region. First,

$$f(z) = \frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}.$$

From our vast knowledge of the Geometric series, we have

$$f(z) = -\frac{1}{z} - \sum_{j=0}^{\infty} z^j.$$

Now let's find another Laurent series for f, the one valid for the region $1 < |z| < \infty$. First,

$$\frac{1}{z-1} = \frac{1}{z}\left[\frac{1}{1-\frac{1}{z}}\right].$$

Now since $|\frac{1}{z}| < 1$, we have

$$\frac{1}{z-1} = \frac{1}{z}\left[\frac{1}{1-\frac{1}{z}}\right] = \frac{1}{z}\sum_{j=0}^{\infty} z^{-j} = \sum_{j=1}^{\infty} z^{-j},$$

and so

$$f(z) = -\frac{1}{z} + \frac{1}{z-1} = -\frac{1}{z} + \sum_{j=1}^{\infty} z^{-j}$$

$$f(z) = \sum_{j=2}^{\infty} z^{-j}.$$

Exercises

8. Find two Laurent series in powers of z for the function f defined by

$$f(z) = \frac{1}{z^2(1 - z)}$$

and specify the regions in which the series converge to $f(z)$.

9. Find two Laurent series in powers of z for the function f defined by

$$f(z) = \frac{1}{z(1 + z^2)}$$

and specify the regions in which the series converge to $f(z)$.

10. Find the Laurent series in powers of $z - 1$ for $f(z) = \frac{1}{z}$ in the region $1 < |z - 1| < \infty$.