INTRODUCTION TO COMPLEX ANALYSIS

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Chapter 1

COMPLEX NUMBERS

1.1. Arithmetic and Conjugates

The purpose of this chapter is to give a review of various properties of the complex numbers that we shall need in the discussion of complex analysis. As the reader is expected to be familiar with the material, all proofs have been omitted.

The equation $x^2+1=0$ has no solution $x \in \mathbb{R}$. To "solve" this equation, we have to introduce extra numbers into our number system. To do this, we define the number i by $\mathrm{i}^2+1=0$, and then extend the field of all real numbers by adjoining the number i, which is then combined with the real numbers by the operations addition and multiplication in accordance with the Field axioms of the real number system. The numbers $a+\mathrm{i}b$, where $a,b \in \mathbb{R}$, of the extended field are then added and multiplied in accordance with the Field axioms, suitably extended, and the restriction $\mathrm{i}^2+1=0$. Note that the number $a+0\mathrm{i}$, where $a \in \mathbb{R}$, behaves like the real number a.

What we have said in the last paragraph basically amounts to the following. Consider two complex numbers $a+\mathrm{i}b$ and $c+\mathrm{i}d$, where $a,b,c,d \in \mathbb{R}$. We have the addition and multiplication rules

$$(a+\mathrm{i}b)+(c+\mathrm{i}d)=(a+c)+\mathrm{i}(b+d) \qquad \text{and} \qquad (a+\mathrm{i}b)(c+\mathrm{i}d)=(ac-bd)+\mathrm{i}(ad+bc).$$

These lead to the subtraction rule

$$(a+\mathrm{i}b)-(c+\mathrm{i}d)=(a-c)+\mathrm{i}(b-d),$$

and the division rule, that if $c+\mathrm{i}d \neq 0$, then

$$\frac{a+\mathrm{i}b}{c+\mathrm{i}d}=\frac{ac+bd}{c^2+d^2}+\mathrm{i}\frac{bc-ad}{c^2+d^2}.$$

Note the special case $a = 1$ and $b = 0$.

Suppose that $z = x + \mathrm{i}y$, where $x, y \in \mathbb{R}$. The real number x is called the real part of z, and denoted by $x = \mathfrak{Re} z$. The real number y is called the imaginary part of z, and denoted by $y = \mathfrak{Im} z$. The set $\mathbb{C} = \{z = x + \mathrm{i}y : x, y \in \mathbb{R}\}$ is called the set of all complex numbers. The complex number $\overline{z} = x - \mathrm{i}y$ is called the conjugate of z.

It is easy to see that for every $z \in \mathbb{C}$, we have

$$\mathfrak{Re} z = \frac{z + \overline{z}}{2} \qquad \text{and} \qquad \mathfrak{Im} z = \frac{z - \overline{z}}{2\mathrm{i}}.$$

Furthermore, if $w \in \mathbb{C}$, then

$$\overline{z + w} = \overline{z} + \overline{w} \qquad \text{and} \qquad \overline{zw} = \overline{z}\,\overline{w}.$$

1.2. Polar Coordinates

Suppose that $z = x + \mathrm{i}y$, where $x, y \in \mathbb{R}$. The real number

$$r = \sqrt{x^2 + y^2}$$

is called the modulus of z, and denoted by $|z|$. On the other hand, if $z \neq 0$, then any number $\theta \in \mathbb{R}$ satisfying the equations

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta \tag{1}$$

is called an argument of z, and denoted by $\arg z$. Hence we can write z in polar form

$$z = r(\cos\theta + \mathrm{i}\sin\theta).$$

Note, however, that for a given $z \in \mathbb{C}$, $\arg z$ is not unique. Clearly we can add any integer multiple of 2π to θ without affecting (1). We sometimes call a real number $\theta \in \mathbb{R}$ the principal argument of z if θ satisfies the equations (1) and $-\pi < \theta \le \pi$. The principal argument of z is usually denoted by $\mathrm{Arg}\, z$.

It is easy to see that for every $z \in \mathbb{C}$, we have $|z|^2 = z\overline{z}$. Also, if $w \in \mathbb{C}$, then

$$|zw| = |z||w| \qquad \text{and} \qquad |z + w| \le |z| + |w|.$$

Furthermore, if

$$z = r(\cos\theta + \mathrm{i}\sin\theta) \qquad \text{and} \qquad w = s(\cos\phi + \mathrm{i}\sin\phi),$$

where $r, s, \theta, \phi \in \mathbb{R}$ and $r, s > 0$, then

$$zw = rs(\cos(\theta + \phi) + \mathrm{i}\sin(\theta + \phi)) \qquad \text{and} \qquad \frac{z}{w} = \frac{r}{s}(\cos(\theta - \phi) + \mathrm{i}\sin(\theta - \phi)).$$

1.3. Rational Powers

De Moivre's theorem, that

$$\cos n\theta + \mathrm{i}\sin n\theta = (\cos\theta + \mathrm{i}\sin\theta)^n \qquad \text{for every } n \in \mathbb{N} \text{ and } \theta \in \mathbb{R}, \tag{2}$$

is useful in finding n-th roots of complex numbers.

Suppose that $c = R(\cos\alpha + \mathrm{i}\sin\alpha)$, where $R, \alpha \in \mathbb{R}$ and $R > 0$. Then the solutions of the equation $z^n = c$ are given by

$$z = \sqrt[n]{R}\left(\cos\frac{\alpha + 2k\pi}{n} + \mathrm{i}\sin\frac{\alpha + 2k\pi}{n}\right), \qquad \text{where } k = 0, 1, \ldots, n-1.$$

Finally, we can define c^b for any $b \in \mathbb{Q}$ and non-zero $c \in \mathbb{C}$ as follows. The rational number b can be written uniquely in the form $b = p/q$, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ have no prime factors in common. Then there are exactly q distinct numbers z satisfying $z^q = c$. We now define $c^b = z^p$, noting that the expression (2) can easily be extended to all $n \in \mathbb{Z}$. It is not too difficult to show that there are q distinct values for the rational power c^b.

Problems for Chapter 1

1. Suppose that $z_0 \in \mathbb{C}$ is fixed. A polynomial $P(z)$ is said to be divisible by $z - z_0$ if there is another polynomial $Q(z)$ such that $P(z) = (z - z_0)Q(z)$.
   a) Show that for every $c \in \mathbb{C}$ and $k \in \mathbb{N}$, the polynomial $c(z^k - z_0^k)$ is divisible by $z - z_0$.
   b) Consider the polynomial $P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$, where $a_0, a_1, a_2, \ldots, a_n \in \mathbb{C}$ are arbitrary. Show that the polynomial $P(z) - P(z_0)$ is divisible by $z - z_0$.
   c) Deduce that $P(z)$ is divisible by $z - z_0$ if $P(z_0) = 0$.
   d) Suppose that a polynomial $P(z)$ of degree n vanishes at n distinct values $z_1, z_2, \ldots, z_n \in \mathbb{C}$, so that $P(z_1) = P(z_2) = \ldots = P(z_n) = 0$. Show that $P(z) = c(z - z_1)(z - z_2)\ldots(z - z_n)$, where $c \in \mathbb{C}$ is a constant.
   e) Suppose that a polynomial $P(z)$ of degree n vanishes at more than n distinct values. Show that $P(z) = 0$ identically.

2. Suppose that $\alpha \in \mathbb{C}$ is fixed and $|\alpha| < 1$. Show that $|z| \le 1$ if and only if $\left|\dfrac{z - \alpha}{1 - \overline{\alpha}z}\right| \le 1$.

3. Suppose that $z = x + \mathrm{i}y$, where $x, y \in \mathbb{R}$. Express each of the following in terms of x and y:
   a) $|z - 1|^3$ b) $\left|\dfrac{z + 1}{z - 1}\right|$ c) $\left|\dfrac{z + \mathrm{i}}{1 - \mathrm{i}z}\right|$

4. Suppose that $c \in \mathbb{R}$ and $\alpha \in \mathbb{C}$ with $\alpha \neq 0$.
   a) Show that $\alpha z + \overline{\alpha z} + c = 0$ is the equation of a straight line on the plane.
   b) What does the equation $z\overline{z} + \alpha z + \overline{\alpha z} + c = 0$ represent if $|\alpha|^2 \ge c$?

5. Suppose that $z, w \in \mathbb{C}$. Show that $|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2)$.

6. Find all the roots of the equation $(z^8 - 1)(z^3 + 8) = 0$.

7. For each of the following, compute all the values and plot them on the plane:
   a) $(1 + \mathrm{i})^{-1/2}$ b) $(-4)^{3/4}$ c) $(1 - \mathrm{i})^{3/8}$