INTRODUCTION TO COMPLEX ANALYSIS

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Chapter 2

FOUNDATIONS OF COMPLEX ANALYSIS

2.1. Three Approaches

We start by remarking that analysis is sometimes known as the study of the four C's: convergence, continuity, compactness and connectedness. In real analysis, we have studied convergence and continuity to some depth, but the other two concepts have been somewhat disguised. In this course, we shall try to illustrate these two latter concepts a little bit more, particularly connectedness.

Complex analysis is the study of complex valued functions of complex variables. Here we shall restrict the number of variables to one, and study complex valued functions of one complex variable. Unless otherwise stated, all functions in these notes are of the form $f : S \to \mathbb{C}$, where S is a set in $\mathbb{C}$.

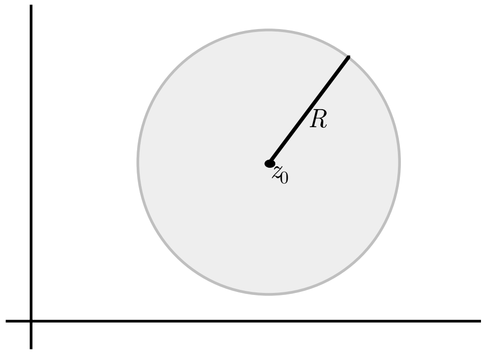
We shall study the behaviour of such functions using three different approaches. The first of these, discussed in Chapter 3 and usually attributed to Riemann, is based on differentiation and involves pairs of partial differential equations called the Cauchy-Riemann equations. The second approach, discussed in Chapters 4–11 and usually attributed to Cauchy, is based on integration and depends on a fundamental theorem known nowadays as Cauchy's integral theorem. The third approach, discussed in Chapter 16 and usually attributed to Weierstrass, is based on the theory of power series.

2.2. Point Sets in the Complex Plane

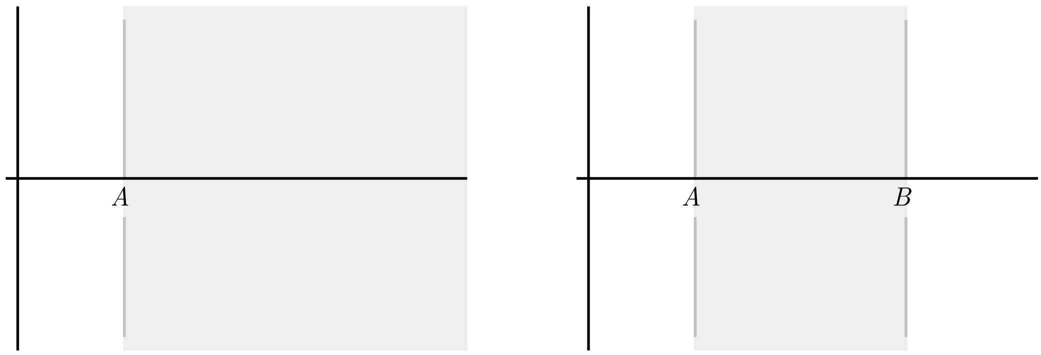
We shall study functions of the form $f : S \to \mathbb{C}$, where S is a set in $\mathbb{C}$. In most situations, various properties of the point sets S play a crucial role in our study. We therefore begin by discussing various types of point sets in the complex plane.

Before making any definitions, let us consider a few examples of sets which frequently occur in our subsequent discussion.

Example 2.2.1. Suppose that $z_0 \in \mathbb{C}$, $r, R \in \mathbb{R}$ and $0 < r < R$. The set $\{z \in \mathbb{C} : |z - z_0| < R\}$ represents a disc, with centre z_0 and radius R, and the set $\{z \in \mathbb{C} : r < |z - z_0| < R\}$ represents an annulus, with centre z_0, inner radius r and outer radius R.



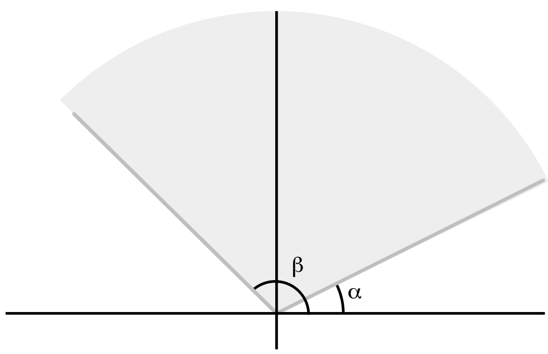
Example 2.2.2. Suppose that $A, B \in \mathbb{R}$ and $A < B$. The set $\{z = x + \mathrm{i}y \in \mathbb{C} : x, y \in \mathbb{R} \text{ and } x > A\}$ represents a half-plane, and the set $\{z = x + \mathrm{i}y \in \mathbb{C} : x, y \in \mathbb{R} \text{ and } A < x < B\}$ represents a strip.



Example 2.2.3. Suppose that $\alpha, \beta \in \mathbb{R}$ and $0 \leq \alpha < \beta < 2\pi$. The set

$$\{z = r(\cos\theta + \mathrm{i}\sin\theta) \in \mathbb{C} : r, \theta \in \mathbb{R} \text{ and } r > 0 \text{ and } \alpha < \theta < \beta\}$$

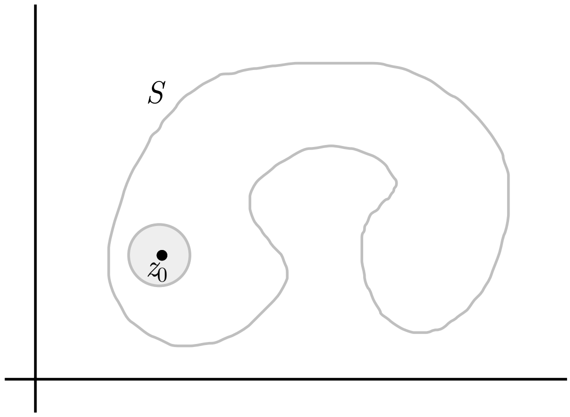
represents a sector.



We now make a number of important definitions. The reader may subsequently need to return to these definitions.

Definition. Suppose that $z_0 \in \mathbb{C}$ and $\epsilon \in \mathbb{R}$, with $\epsilon > 0$. By an ϵ-neighbourhood of z_0, we mean a disc of the form $\{z \in \mathbb{C} : |z - z_0| < \epsilon\}$, with centre z_0 and radius $\epsilon > 0$.

Definition. Suppose that S is a point set in $\mathbb{C}$. A point $z_0 \in S$ is said to be an interior point of S if there exists an ϵ-neighbourhood of z_0 which is contained in S. The set S is said to be open if every point of S is an interior point of S.



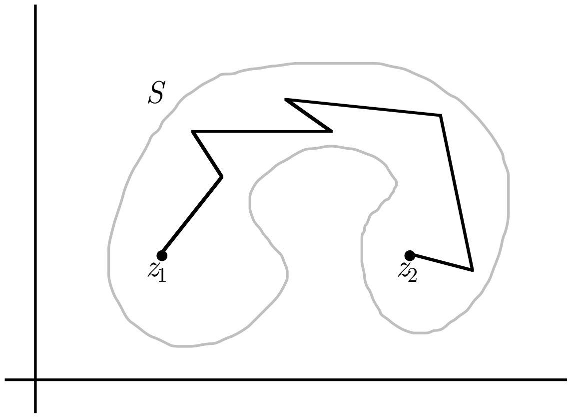
Example 2.2.4. The sets in Examples 2.2.1–2.2.3 are open.

Example 2.2.5. The punctured disc $\{z \in \mathbb{C} : 0 < |z - z_0| < R\}$ is open.

Example 2.2.6. The disc $\{z \in \mathbb{C} : |z - z_0| \leq R\}$ is not open.

Example 2.2.7. The empty set $\emptyset$ is open. Why?

Definition. An open set S is said to be connected if every two points $z_1, z_2 \in S$ can be joined by the union of a finite number of line segments lying in S. An open connected set is called a domain.



Remarks. (1) Sometimes, we say that an open set S is connected if there do not exist non-empty open sets S_1 and S_2 such that $S_1 \cup S_2 = S$ and $S_1 \cap S_2 = \emptyset$. In other words, an open connected set cannot be the disjoint union of two non-empty open sets.

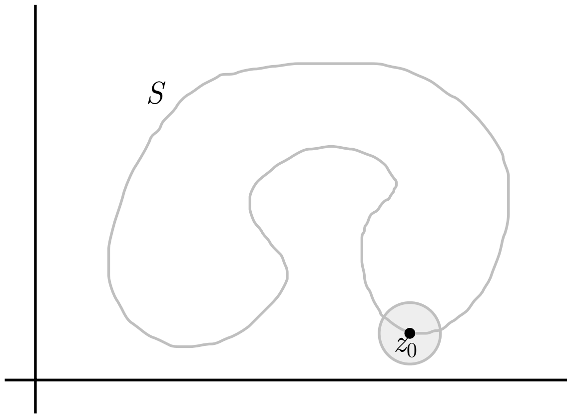
(2) In fact, it can be shown that the two definitions are equivalent.

(3) Note that we have not made any definition of connectedness for sets that are not open. In fact, the definition of connectedness for an open set given by (1) here is a special case of a much more complicated definition of connectedness which applies to all point sets.

EXAMPLE 2.2.8. The sets in Examples 2.2.1–2.2.3 are domains.

EXAMPLE 2.2.9. The punctured disc $\{z \in \mathbb{C} : 0 < |z - z_0| < R\}$ is a domain.

DEFINITION. A point $z_0 \in \mathbb{C}$ is said to be a boundary point of a set S if every ϵ-neighbourhood of z_0 contains a point in S as well as a point not in S. The set of all boundary points of a set S is called the boundary of S.



EXAMPLE 2.2.10. The annulus $\{z \in \mathbb{C} : r < |z - z_0| < R\}$, where $0 < r < R$, has boundary $C_1 \cup C_2$, where

$$C_1 = \{z \in \mathbb{C} : |z - z_0| = r\} \qquad \text{and} \qquad C_2 = \{z \in \mathbb{C} : |z - z_0| = R\}$$

are circles, with centre z_0 and radius r and R respectively. Note that the annulus is connected and hence a domain. However, note that its boundary is made up of two separate pieces.

DEFINITION. A region is a domain together with all, some or none of its boundary points. A region which contains all its boundary points is said to be closed. For any region S, we denote by $\overline{S}$ the closed region containing S and all its boundary points, and call $\overline{S}$ the closure of S.

REMARK. Note that we have not made any definition of closedness for sets that are not regions. In fact, our definition of closedness for a region here is a special case of a much more complicated definition of closedness which applies to all point sets.

DEFINITION. A region S is said to be bounded or finite if there exists a real number M such that $|z| \le M$ for every $z \in S$. A region that is closed and bounded is said to be compact.

EXAMPLE 2.2.11. The region $\{z \in \mathbb{C} : |z - z_0| \le R\}$ is closed and bounded, hence compact. It is called the closed disc with centre z_0 and radius R.

EXAMPLE 2.2.12. The region $\{z = x + \mathrm{i}y \in \mathbb{C} : x, y \in \mathbb{R} \text{ and } 0 \le x \le 1\}$ is closed but not bounded.

EXAMPLE 2.2.13. The square $\{z = x + \mathrm{i}y \in \mathbb{C} : x, y \in \mathbb{R} \text{ and } 0 \le x \le 1 \text{ and } 0 < y < 1\}$ is bounded but not closed.

2.3. Complex Functions

In these lectures, we study complex valued functions of one complex variable. In other words, we study functions of the form $f : S \to \mathbb{C}$, where S is a set in $\mathbb{C}$. Occasionally, we will abuse notation and simply refer to a function by its formula, without explicitly defining the domain S. For instance, when we discuss the function $f(z) = 1/z$, we implicitly choose a set S which will not include the point $z = 0$ where the function is not defined. Also, we may occasionally wish to include the point $z = \infty$ in the domain or codomain.

We may separate the independent variable z as well as the dependent variable $w = f(z)$ into real and imaginary parts. Our usual notation will be to write $z = x + \mathrm{i}y$ and $w = f(z) = u + \mathrm{i}v$, where $x, y, u, v \in \mathbb{R}$. It follows that $u = u(x, y)$ and $v = v(x, y)$ can be interpreted as real valued functions of the two real variables x and y.

Example 2.3.1. Consider the function $f : S \to \mathbb{C}$, given by $f(z) = z^2$ and where $S = \{z \in \mathbb{C} : |z| < 2\}$ is the open disc with radius 2 and centre 0. Using polar coordinates, it is easy to see that the range of the function is the open disc $f(S) = \{w \in \mathbb{C} : |w| < 4\}$ with radius 4 and centre 0.

Example 2.3.2. Consider the function $f : \mathcal{H} \to \mathbb{C}$, where $\mathcal{H} = \{z = x + \mathrm{i}y \in \mathbb{C} : y > 0\}$ is the upper half-plane and $f(z) = z^2$. Using polar coordinates, it is easy to see that the range of the function is the complex plane minus the non-negative real axis.

Example 2.3.3. Consider the function $f : T \to \mathbb{C}$, where $T = \{z = x + \mathrm{i}y \in \mathbb{C} : 1 < x < 2\}$ is a strip and $f(z) = z^2$. Let $x_0 \in (1, 2)$ be fixed, and consider the image of a point (x_0, y) on the vertical line $x = x_0$. Here we have

$$u = x_0^2 - y^2 \qquad \text{and} \qquad v = 2x_0 y.$$

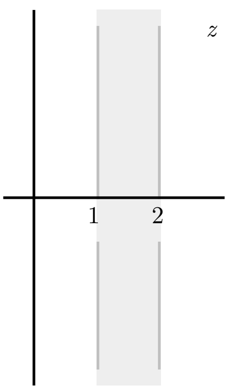
Eliminating y, we obtain the equation of a parabola

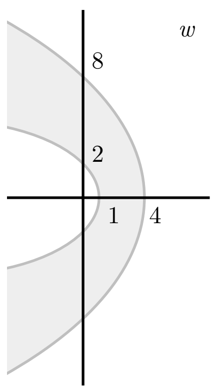
$$u = x_0^2 - \frac{v^2}{4x_0^2}$$

in the w-plane. It follows that the image of the vertical line $x = x_0$ under the function $w = z^2$ is this parabola. Now the boundary of the strip are the two lines $x = 1$ and $x = 2$. Their images under the mapping $w = z^2$ are respectively the parabolas

$$u = 1 - \frac{v^2}{4} \qquad \text{and} \qquad u = 4 - \frac{v^2}{16}.$$

It is easy to see that the range of the function is the part of the w-plane between these two parabolas.





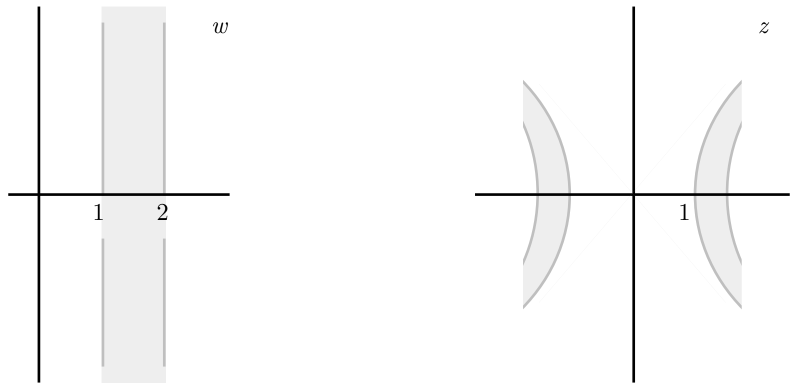
Example 2.3.4. Consider again the function $w = z^2$. We would like to find all $z = x + \mathrm{i}y \in \mathbb{C}$ for which $1 < \mathfrak{Re}\,w < 2$. In other words, we have the restriction $1 < u < 2$, but no rectriction on v. Let $u_0 \in (1,2)$ be fixed, and consider points (x,y) in the z-plane with images on the vertical line $u = u_0$. Here we have the hyperbola

$$x^2 - y^2 = u_0.$$

The boundaries $u = 1$ and $u = 2$ are represented by the hyperbolas

$$x^2 - y^2 = 1 \qquad \text{and} \qquad x^2 - y^2 = 2.$$

It is easy to see that the points in question are precisely those between the two hyperbolas.



2.4. Extended Complex Plane

It is sometimes useful to extend the complex plane $\mathbb{C}$ by the introduction of the point ∞ at infinity. Its connection with finite complex numbers can be established by setting $z + \infty = \infty + z = \infty$ for all $z \in \mathbb{C}$, and setting $z \cdot \infty = \infty \cdot z = \infty$ for all non-zero $z \in \mathbb{C}$. We can also write $\infty \cdot \infty = \infty$.

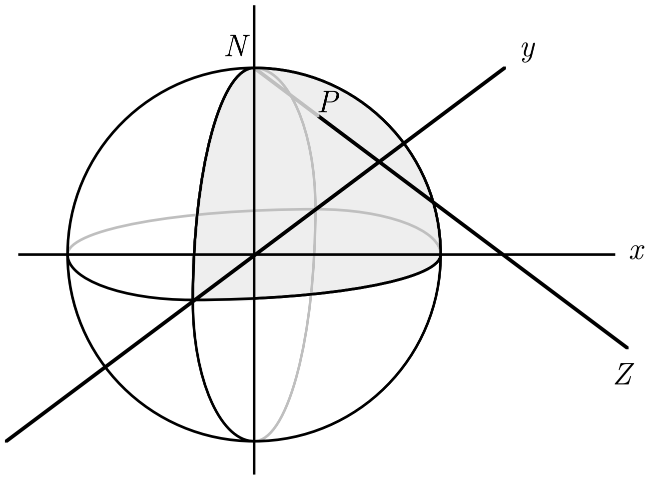
Note that it is not possible to define $\infty + \infty$ and $0 \cdot \infty$ without violating the laws of arithmetic. However, by special convention, we shall write $z/0 = \infty$ for $z \neq 0$ and $z/\infty = 0$ for $z \neq \infty$.

In the complex plane $\mathbb{C}$, there is no room for a point corresponding to ∞. We can, of course, introduce an "ideal" point which we call the point at infinity. The points in $\mathbb{C}$, together with the point at infinity, form the extended complex plane. We decree that every straight line on the complex plane shall pass through the point at infinity, and that no half-plane shall contain the ideal point.

The main purpose of this section is to introduce a geometric model in which each point of the extended complex plane has a concrete representative. To do this, we shall use the idea of stereographic projection.

Consider a sphere of radius 1 in $\mathbb{R}^3$. A typical point on this sphere will be denoted by $P(x_1, x_2, x_3)$. Note that $x_1^2 + x_2^2 + x_3^2 = 1$. Let us call the point $N(0,0,1)$ the north pole. The equator of this sphere is the set of all points of the form $(x_1, x_2, 0)$, where $x_1^2 + x_2^2 = 1$. Consider next the complex plane $\mathbb{C}$. This can be viewed as a plane in $\mathbb{R}^3$. Let us position this plane in such a way that the equator of the sphere lies on this plane; in other words, our copy of the complex plane is "horizontal" and passes through the origin. We can further insist that the x-direction on our complex plane is the same as the x_1-direction in $\mathbb{R}^3$, and that the y-direction on our complex plane is the same as the x_2-direction in $\mathbb{R}^3$. Clearly a typical point $z = x + \mathrm{i}y$ on our complex plane $\mathbb{C}$ can be identified with the point $Z(x, y, 0)$ in $\mathbb{R}^3$.

Suppose that $Z(x, y, 0)$ is on the plane. Consider the straight line that passes through Z and the north pole N. It is not too difficult to see that this straight line intersects the surface of the sphere at precisely one other point $P(x_1, x_2, x_3)$. In fact, if Z is on the equator of the sphere, then $P = Z$. If Z is on the part of the plane outside the sphere, then P is on the northern hemisphere, but is not the north pole N. If Z is on the part of the plane inside the sphere, then P is on the southern hemisphere. Check that for $Z(0, 0, 0)$, the point $P(0, 0, -1)$ is the south pole.



On the other hand, if P is any point on the sphere different from the north pole N, then a straight line passing through P and N intersects the plane at precisely one point Z. It follows that there is a pairing of all the points P on the sphere different from the north pole N and all the points on the plane. This pairing is governed by the requirement that the straight line through any pair must pass through the north pole N.

We can now visualize the north pole N as the point on the sphere corresponding to the point at infinity of the plane. The sphere is called the Riemann sphere.

2.5. Limits and Continuity

The concept of a limit in complex analysis is exactly the same as in real analysis. So, for example, we say that $f(z) \to L$ as $z \to z_0$, or

$$\lim_{z \to z_0} f(z) = L,$$

if, given any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(z) - L| < \epsilon$ whenever $0 < |z - z_0| < \delta$.

This definition will be perfectly in order if the function f is defined in some open set containing z_0, with the possible exception of z_0 itself. It follows that if z_0 is an interior point of the region S of definition of the function, our definition is in order. However, if z_0 is a boundary point of the region S of definition of the function, then we agree that the conclusion $|f(z) - L| < \epsilon$ need only hold for those $z \in S$ satisfying $0 < |z - z_0| < \delta$.

Similarly, we say that a function $f(z)$ is continuous at z_0 if $f(z) \to f(z_0)$ as $z \to z_0$. A similar qualification on z applies if z_0 is a boundary point of the region S of definition of the function. We also say that a function is continuous in a region if it is continuous at every point of the region.

Note that for a function to be continuous in a region, it is enough to have continuity at every point of the region. Hence the choice of δ may depend on a point z_0 in question. If δ can be chosen independently of z_0, then we have some uniformity as well. To be precise, we make the following definition.

DEFINITION. A function $f(z)$ is said to be uniformly continuous in a region S if, given any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(z_1) - f(z_2)| < \epsilon$ for every $z_1, z_2 \in S$ satisfying $|z_1 - z_2| < \delta$.

REMARK. Note that if we fix z_2 to be a point z_0 and write z for z_1, then we require $|f(z) - f(z_0)| < \epsilon$ for every $z \in S$ satisfying $|z - z_0| < \delta$. In other words, δ cannot depend on z_0.

EXAMPLE 2.5.1. Consider the punctured disc $S = \{z \in \mathbb{C} : 0 < |z| < 1\}$. The function $f(z) = 1/z$ is continuous in S but not uniformly continuous in S. To see this, note first of all that continuity follows from the simple observation that the function z is continuous and non-zero in S. To show that the function is not uniformly continuous in S, it suffices to show that there exists $\epsilon > 0$ such that for every $\delta > 0$, there exist $z_1, z_2 \in S$ such that

$$|z_1 - z_2| < \delta \qquad \text{and} \qquad \left|\frac{1}{z_1} - \frac{1}{z_2}\right| \geq \epsilon.$$

Let $\epsilon = 1$. For every $\delta > 0$, choose $n \in \mathbb{N}$ such that $n > \delta^{-1/2}$, and let

$$z_1 = \frac{1}{n} \qquad \text{and} \qquad z_2 = \frac{1}{n+1}.$$

Clearly $z_1, z_2 \in S$. It is easy to see that

$$|z_1 - z_2| = \left|\frac{1}{n} - \frac{1}{n+1}\right| = \frac{1}{n(n+1)} < \delta \qquad \text{and} \qquad \left|\frac{1}{z_1} - \frac{1}{z_2}\right| = 1.$$

PROBLEMS FOR CHAPTER 2

1. For each of the following functions, find $f(z+3)$, $f(1/z)$ and $f(f(z))$:
   a) $f(z) = z - 1$ b) $f(z) = z^2$ c) $f(z) = 1/z$ d) $f(z) = \dfrac{1-z}{3+z}$

2. Which of the sets below are domains?
   a) $\{z : 0 < |z| < 1\}$ b) $\{z : \mathfrak{Im} z < 3|z|\}$ c) $\{z : |z-1| \leq |z+1|\}$
   d) $\{z : |z^2 - 1| < 1\}$ e) $\{z : 0 < \mathfrak{Re} z \leq 1\}$

3. Find the image of the strip $\{z : |\mathfrak{Re} z| < 1\}$ and of the disc $\{z : |z| < 1\}$ under each of the following mappings:
   a) $w = (1+\mathrm{i})z + 1$ b) $w = 2z^2$ c) $w = z^{-1}$ d) $w = \dfrac{z+1}{z-1}$

4. A function $f(z)$ is said to be an isometry if $|f(z_1) - f(z_2)| = |z_1 - z_2|$ for every $z_1, z_2 \in \mathbb{C}$; in other words, if it preserves distance.
   a) Suppose that $f(z)$ is an isometry. Show that for every $a, b \in \mathbb{C}$ with $|a| = 1$, the function $g(z) = af(z) + b$ is also an isometry.
   b) Show that the function

   $$h(z) = \frac{f(z) - f(0)}{f(1) - f(0)}$$

   is an isometry with $h(0) = 0$ and $h(1) = 1$.

c) Suppose that $k(z)$ is an isometry with $k(0) = 0$ and $k(1) = 1$. Show that $\mathfrak{Re}\, k(z) = \mathfrak{Re}\, z$, and that $k(\mathrm{i}) = \pm\mathrm{i}$.
[HINT: Explain first of all why $|k(z)| = |z|$ and $|1 - k(z)| = |1 - z|$.]
d) Suppose that in (c), we have $k(\mathrm{i}) = \mathrm{i}$. Show that $\mathfrak{Im}\, k(z) = \mathfrak{Im}\, z$ and that $k(z) = z$ for all $z \in \mathbb{C}$.
e) Suppose that in (c), we have $k(\mathrm{i}) = -\mathrm{i}$. Show that $\mathfrak{Im}\, k(z) = -\mathfrak{Im}\, z$ and that $k(z) = \overline{z}$ for all $z \in \mathbb{C}$.
f) Deduce that every isometry has the form $f(z) = az + b$ or $f(z) = a\overline{z} + b$, where $a, b \in \mathbb{C}$ with $|a| = 1$.

5. In the notation of Section 2.4, let the point $z = x + \mathrm{i}y$ on the complex plane $\mathbb{C}$ correspond to the point (x_1, x_2, x_3) of the sphere under stereographic projection, so that the three points $(0, 0, 1)$, (x_1, x_2, x_3) and $(x, y, 0)$ are collinear. Note that $(x_1, x_2, x_3 - 1) = \lambda(x, y, -1)$ for some $\lambda \in \mathbb{R}$, and that $x_1^2 + x_2^2 + x_3^2 = 1$.
   a) Show that $(x_1, x_2, x_3) = \left(\frac{2x}{|z|^2 + 1}, \frac{2y}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1}\right)$.
   b) Note that a circle on the sphere is the intersection of the sphere with a plane $ax_1 + bx_2 + cx_3 = d$. By expressing this equation of the plane in terms of x and y, show that a circle on the sphere not containing the pole $(0, 0, 1)$ corresponds to a circle in the complex plane. Show also that a circle on the sphere containing the pole $(0, 0, 1)$ corresponds to a line in the complex plane.
   c) Suppose that (x_1, x_2, x_3) and (x_1', x_2', x_3') are two points on the sphere corresponding to the complex numbers z and z' respectively. Show that the distance between (x_1, x_2, x_3) and (x_1', x_2', x_3') is given by

$$d(z, z') = \frac{2|z - z'|}{\sqrt{1 + |z|^2}\sqrt{1 + |z'|^2}}.$$

   [REMARK: The number $d(z, z')$ is known as the chordal distance.]

6. Each of the following functions is not defined at $z = z_0$. What value must $f(z_0)$ take to ensure continuity at $z = z_0$?
   a) $f(z) = \frac{z - z_0}{z - z_0}$
   b) $f(z) = \frac{z^3 - z_0^3}{z - z_0}$
   c) $f(z) = \frac{1}{z - z_0}\left(\frac{1}{z} - \frac{1}{z_0}\right)$
   d) $f(z) = \frac{1}{z - z_0}\left(\frac{1}{z^3} - \frac{1}{z_0^3}\right)$

7. Suppose that

$$f(z) = \frac{a_0 + a_1 z + a_2 z^2}{b_0 + b_1 z + b_2 z^2},$$

where $a_0, a_1, a_2, b_0, b_1, b_2 \in \mathbb{C}$. Examine the behaviour of $f(z)$ at $z = 0$ and at $z = \infty$.