INTRODUCTION TO COMPLEX ANALYSIS

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Chapter 3

COMPLEX DIFFERENTIATION

3.1. Introduction

Suppose that $D \subseteq \mathbb{C}$ is a domain. A function $f : D \to \mathbb{C}$ is said to be differentiable at $z_0 \in D$ if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. In this case, we write

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}, \tag{1}$$

and call $f'(z_0)$ the derivative of f at z_0.

If $z \neq z_0$, then

$$f(z) = \left(\frac{f(z) - f(z_0)}{z - z_0} \right) (z - z_0) + f(z_0).$$

It follows from (1) and the arithmetic of limits that if $f'(z_0)$ exists, then $f(z) \to f(z_0)$ as $z \to z_0$, so that f is continuous at z_0. In other words, differentiability at z_0 implies continuity at z_0.

Note that the argument here is the same as in the case of a real valued function of a real variable. In fact, the similarity in argument extends to the arithmetic of limits. Indeed, if the functions $f : D \to \mathbb{C}$ and $g : D \to \mathbb{C}$ are both differentiable at $z_0 \in D$, then both $f + g$ and fg are differentiable at z_0, and

$$(f + g)'(z_0) = f'(z_0) + g'(z_0) \qquad \text{and} \qquad (fg)'(z_0) = f(z_0)g'(z_0) + f'(z_0)g(z_0).$$

If the extra condition $g'(z_0) \neq 0$ holds, then f/g is differentiable at z_0, and

$$\left(\frac{f}{g}\right)'(z_0) = \frac{g(z_0)f'(z_0) - f(z_0)g'(z_0)}{g^2(z_0)}.$$

One can also establish the Chain rule for differentiation as in real analysis. More precisely, suppose that the function f is differentiable at z_0 and the function g is differentiable at $w_0 = f(z_0)$. Then the function $g \circ f$ is differentiable at $z = z_0$, and

$$(g \circ f)'(z_0) = g'(w_0)f'(z_0).$$

EXAMPLE 3.1.1. Consider the function $f(z) = \overline{z}$, where for every $z \in \mathbb{C}$, $\overline{z}$ denotes the complex conjugate of z. Suppose that $z_0 \in \mathbb{C}$. Then

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{\overline{z} - \overline{z_0}}{z - z_0} = \frac{\overline{z - z_0}}{z - z_0}. \tag{2}$$

If $z - z_0 = h$ is real and non-zero, then (2) takes the value 1. On the other hand, if $z - z_0 = \mathrm{i}k$ is purely imaginary, then (2) takes the value -1. It follows that this function is not differentiable anywhere in $\mathbb{C}$, although its real and imaginary parts are rather well behaved.

3.2. The Cauchy-Riemann Equations

If we use the notation

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h},$$

then in Example 3.1.1, we have examined the behaviour of the ratio

$$\frac{f(z+h) - f(z)}{h}$$

first as $h \to 0$ through real values and then through imaginary values. Indeed, for the derivative to exist, it is essential that these two limiting processes produce the same limit $f'(z)$. Suppose that $f(z) = u(x,y) + \mathrm{i}v(x,y)$, where $z = x + \mathrm{i}y$, and u and v are real valued functions. If h is real, then the two limiting processes above correspond to

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0} \frac{u(x+h,y) - u(x,y)}{h} + \mathrm{i} \lim_{h \to 0} \frac{v(x+h,y) - v(x,y)}{h} = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x}$$

and

$$\lim_{h \to 0} \frac{f(z+\mathrm{i}h) - f(z)}{\mathrm{i}h} = \lim_{h \to 0} \frac{u(x,y+h) - u(x,y)}{\mathrm{i}h} + \mathrm{i} \lim_{h \to 0} \frac{v(x,y+h) - v(x,y)}{\mathrm{i}h} = \frac{\partial v}{\partial y} - \mathrm{i}\frac{\partial u}{\partial y}$$

respectively. Equating real and imaginary parts, we obtain

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{3}$$

Note that while the existence of the derivative in real analysis is a mild smoothness condition, the existence of the derivative in complex analysis leads to a pair of partial differential equations.

Definition. The partial differential equations (3) are called the Cauchy-Riemann equations.

We have proved the following result.

THEOREM 3A. *Suppose that $f(z) = u(x,y) + \mathrm{i}v(x,y)$, where $z = x + \mathrm{i}y$, and u and v are real valued functions. Suppose further that $f'(z)$ exists. Then the four partial derivatives in (3) exist, and the Cauchy-Riemann equations (3) hold. Furthermore, we have*

$$f'(z) = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x} \qquad \text{and} \qquad f'(z) = \frac{\partial v}{\partial y} - \mathrm{i}\frac{\partial u}{\partial y}. \tag{4}$$

A natural question to ask is whether the Cauchy-Riemann equations are sufficient to guarantee the existence of the derivative. We shall show next that we require also the continuity of the partial derivatives in (3).

THEOREM 3B. *Suppose that $f(z) = u(x,y) + \mathrm{i}v(x,y)$, where $z = x + \mathrm{i}y$, and u and v are real valued functions. Suppose further that the four partial derivatives in (3) are continuous and satisfy the Cauchy-Riemann equations (3) at z_0. Then f is differentiable at z_0, and the derivative $f'(z_0)$ is given by the equations (4) evaluated at z_0.*

Proof. Write $z_0 = x_0 + \mathrm{i}y_0$. Then

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{(u(x,y) - u(x_0,y_0)) + \mathrm{i}(v(x,y) - v(x_0,y_0))}{z - z_0}.$$

We can write

$$u(x,y) - u(x_0,y_0) = (x - x_0)\left(\frac{\partial u}{\partial x}\right)_{z_0} + (y - y_0)\left(\frac{\partial u}{\partial y}\right)_{z_0} + |z - z_0|\epsilon_1(z)$$

and

$$v(x,y) - v(x_0,y_0) = (x - x_0)\left(\frac{\partial v}{\partial x}\right)_{z_0} + (y - y_0)\left(\frac{\partial v}{\partial y}\right)_{z_0} + |z - z_0|\epsilon_2(z).$$

If the four partial derivatives in (3) are continuous at z_0, then

$$\lim_{z \to z_0} \epsilon_1(z) = 0 \qquad \text{and} \qquad \lim_{z \to z_0} \epsilon_2(z) = 0.$$

In view of the Cauchy-Riemann equations (3), we have

$$\begin{aligned}
&(u(x,y) - u(x_0,y_0)) + \mathrm{i}(v(x,y) - v(x_0,y_0)) \\
&\quad = (x - x_0)\left(\frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x}\right)_{z_0} + (y - y_0)\left(\frac{\partial u}{\partial y} + \mathrm{i}\frac{\partial v}{\partial y}\right)_{z_0} + |z - z_0|(\epsilon_1(z) + \mathrm{i}\epsilon_2(z)) \\
&\quad = (x - x_0)\left(\frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x}\right)_{z_0} + (y - y_0)\left(-\frac{\partial v}{\partial x} + \mathrm{i}\frac{\partial u}{\partial x}\right)_{z_0} + |z - z_0|(\epsilon_1(z) + \mathrm{i}\epsilon_2(z)) \\
&\quad = (x - x_0)\left(\frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x}\right)_{z_0} + \mathrm{i}(y - y_0)\left(\frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x}\right)_{z_0} + |z - z_0|(\epsilon_1(z) + \mathrm{i}\epsilon_2(z)) \\
&\quad = (z - z_0)\left(\frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x}\right)_{z_0} + |z - z_0|(\epsilon_1(z) + \mathrm{i}\epsilon_2(z)).
\end{aligned}$$

Hence

$$\frac{f(z) - f(z_0)}{z - z_0} = \left(\frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x}\right)_{z_0} + \left(\frac{|z - z_0|}{z - z_0}\right)(\epsilon_1(z) + \mathrm{i}\epsilon_2(z)) \to \left(\frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x}\right)_{z_0}$$

as $z \to z_0$, giving the desired results. ○

3.3. Analytic Functions

In the previous section, we have shown that differentiability in complex analysis leads to a pair of partial differential equations. Now partial differential equations are seldom of interest at a single point, but rather in a region. It therefore seems reasonable to make the following definition.

DEFINITION. A function f is said to be analytic at a point $z_0 \in \mathbb{C}$ if it is differentiable at every z in some ϵ-neighbourhood of the point z_0. The function f is said to be analytic in a region if it is analytic at every point in the region. The function f is said to be entire if it is analytic in $\mathbb{C}$.

EXAMPLE 3.3.1. Consider the function $f(z) = |z|^2$. In our usual notation, we clearly have

$$u = x^2 + y^2 \qquad \text{and} \qquad v = 0.$$

The Cauchy-Riemann equations

$$2x = 0 \qquad \text{and} \qquad 2y = 0$$

can only be satisfied at $z = 0$. It follows that the function is differentiable only at the point $z = 0$, and is therefore analytic nowhere.

EXAMPLE 3.3.2. The function $f(z) = z^2$ is entire.

EXAMPLE 3.3.3. Suppose that the function f is analytic in a domain D. Suppose further that f has constant real part u. Then clearly

$$\frac{\partial u}{\partial x} = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial y} = 0.$$

Since f is analytic in D, it is differentiable at every point in D, and so the Cauchy-Riemann equations hold in D. It follows that

$$\frac{\partial v}{\partial x} = 0 \qquad \text{and} \qquad \frac{\partial v}{\partial y} = 0.$$

Hence f must have constant imaginary part v, and so f must be constant in D.

EXAMPLE 3.3.4. Suppose that the function f is analytic in a domain D. Suppose further that f has constant imaginary part v. A similar argument shows that f must have constant real part u. Hence f must be constant in D.

EXAMPLE 3.3.5. Suppose that the function f is analytic in a domain D. Suppose further that f has constant modulus. In other words, $u^2 + v^2 = C$ for some non-negative real number C. Differentiating this with respect to x and to y, we obtain respectively

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0 \qquad \text{and} \qquad 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0.$$

In view of the Cauchy-Riemann equations, these can be written as

$$2u\frac{\partial u}{\partial x} - 2v\frac{\partial u}{\partial y} = 0 \qquad \text{and} \qquad 2v\frac{\partial u}{\partial x} + 2u\frac{\partial u}{\partial y} = 0.$$

In matrix notation, these become

$$\begin{pmatrix} u & -v \\ v & u \end{pmatrix} \begin{pmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Note now that

$$\det \begin{pmatrix} u & -v \\ v & u \end{pmatrix} = u^2 + v^2 = C.$$

If $C > 0$, then we must have the unique solution

$$\frac{\partial u}{\partial x} = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial y} = 0,$$

so that the real part u is constant. It then follows from Example 3.3.3 that f is constant in D. On the other hand, if $C = 0$, then clearly $u = v = 0$, so that $f = 0$ in D.

3.4. Introduction to Special Functions

In this section, we shall generalize various functions that we have studied in real analysis to the complex domain. Consider first of all the exponential function. It seems reasonable to extend the property $\mathrm{e}^{x_1+x_2} = \mathrm{e}^{x_1}\mathrm{e}^{x_2}$ for real variables to complex values of the variables to obtain

$$\mathrm{e}^z = \mathrm{e}^{x+\mathrm{i}y} = \mathrm{e}^x \mathrm{e}^{\mathrm{i}y}, \qquad \text{where } x, y \in \mathbb{R}.$$

This suggests the following definition.

DEFINITION. Suppose that $z = x + \mathrm{i}y$, where $x, y \in \mathbb{R}$. Then the exponential function e^z is defined for every $z \in \mathbb{C}$ by

$$\mathrm{e}^z = \mathrm{e}^x(\cos y + \mathrm{i}\sin y). \tag{5}$$

If we write $\mathrm{e}^z = u(x, y) + \mathrm{i}v(x, y)$, then

$$u(x, y) = \mathrm{e}^x \cos y \qquad \text{and} \qquad v(x, y) = \mathrm{e}^x \sin y.$$

It is easy to check that the Cauchy-Riemann equations are satisfied for every $z \in \mathbb{C}$, so that e^z is an entire function. Furthermore, it follows from (4) that

$$\frac{\mathrm{d}}{\mathrm{d}z}\mathrm{e}^z = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x} = \mathrm{e}^x \cos y + \mathrm{i}\mathrm{e}^x \sin y = \mathrm{e}^x(\cos y + \mathrm{i}\sin y) = \mathrm{e}^z,$$

so that e^z is its own derivative. On the other hand, note that for every $y_1, y_2 \in \mathbb{R}$, we have

$$\mathrm{e}^{\mathrm{i}(y_1+y_2)} = \cos(y_1 + y_2) + \mathrm{i}\sin(y_1 + y_2) = (\cos y_1 + \mathrm{i}\sin y_1)(\cos y_2 + \mathrm{i}\sin y_2) = \mathrm{e}^{\mathrm{i}y_1}\mathrm{e}^{\mathrm{i}y_2}.$$

Furthermore, if $x_1, x_2 \in \mathbb{R}$, then

$$\mathrm{e}^{x_1+x_2}\mathrm{e}^{\mathrm{i}(y_1+y_2)} = (\mathrm{e}^{x_1}\mathrm{e}^{x_2})(\mathrm{e}^{\mathrm{i}y_1}\mathrm{e}^{\mathrm{i}y_2}) = (\mathrm{e}^{x_1}\mathrm{e}^{\mathrm{i}y_1})(\mathrm{e}^{x_2}\mathrm{e}^{\mathrm{i}y_2}).$$

Writing $z_1 = x_1 + \mathrm{i}y_1$ and $z_2 = x_2 + \mathrm{i}y_2$, we deduce the addition formula

$$\mathrm{e}^{z_1+z_2} = \mathrm{e}^{z_1}\mathrm{e}^{z_2}.$$

Finally, note that

$$|e^z| = |e^x(\cos y + i\sin y)| = e^x|\cos y + i\sin y| = e^x.$$

Since e^x is never zero, it follows that the exponential function e^z is non-zero for every $z \in \mathbb{C}$.

Next, we turn our attention to the trigonometric functions. Note first of all that if $z = x + iy$, where $x, y \in \mathbb{R}$, then $iz = -y + ix$. Replacing z in (5) by iz and by $-iz$ gives respectively

$$e^{iz} = e^{-y}(\cos x + i\sin x) \qquad \text{and} \qquad e^{-iz} = e^{y}(\cos x - i\sin x).$$

The special case $y = 0$ gives respectively

$$e^{ix} = \cos x + i\sin x \qquad \text{and} \qquad e^{-ix} = \cos x - i\sin x.$$

It follows that

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \text{and} \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

This suggests the following definition.

DEFINITION. Suppose that $z \in \mathbb{C}$. Then the trigonometric functions $\cos z$ and $\sin z$ are defined in terms of the exponential function by

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \text{and} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}. \tag{6}$$

Since the exponential function is an entire function, it follows easily from (6) that both $\cos z$ and $\sin z$ are entire functions. Furthermore, it can easily be deduced from (6) that

$$\frac{d}{dz}\cos z = -\sin z \qquad \text{and} \qquad \frac{d}{dz}\sin z = \cos z.$$

We can define the functions $\tan z$, $\cot z$, $\sec z$ and $\operatorname{cosec} z$ in terms of the functions $\cos z$ and $\sin z$ as in real variables. However, note that these four functions are not entire. Also, we can deduce from (6) the formulas

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 \qquad \text{and} \qquad \sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2,$$

and a host of other trigonometric identities that we know hold for real variables.

Finally, we turn our attention to the hyperbolic functions. These are defined as in real analysis.

DEFINITION. Suppose that $z \in \mathbb{C}$. Then the hyperbolic functions $\cosh z$ and $\sinh z$ are defined in terms of the exponential function by

$$\cosh z = \frac{e^{z} + e^{-z}}{2} \qquad \text{and} \qquad \sinh z = \frac{e^{z} - e^{-z}}{2}. \tag{7}$$

Since the exponential function is an entire function, it follows easily from (7) that both $\cosh z$ and $\sinh z$ are entire functions. Furthermore, it can easily be deduced from (7) that

$$\frac{d}{dz}\cosh z = \sinh z \qquad \text{and} \qquad \frac{d}{dz}\sinh z = \cosh z.$$

We can define the functions $\tanh z$, $\coth z$, $\operatorname{sech} z$ and $\operatorname{cosech} z$ in terms of the functions $\cosh z$ and $\sinh z$ as in real variables. However, note that these four functions are not entire. Also, we can deduce from (7) a host of hyperbolic identities that we know hold for real variables. Note also that comparing (6) and (7), we obtain

$$\cosh z = \cos \mathrm{i}z \qquad \text{and} \qquad \sinh z = -\mathrm{i}\sin \mathrm{i}z.$$

3.5. Periodicity and its Consequences

One of the fundamental differences between real and complex analysis is that the exponential function is periodic in $\mathbb{C}$.

DEFINITION. A function f is periodic in $\mathbb{C}$ if there is some fixed non-zero $\omega \in \mathbb{C}$ such that the identity $f(z+\omega) = f(z)$ holds for every $z \in \mathbb{C}$. Any constant $\omega \in \mathbb{C}$ with this property is called a period of f.

THEOREM 3C. *The exponential function* e^z *is periodic in* $\mathbb{C}$ *with period* $2\pi\mathrm{i}$*. Furthermore, any period* $\omega \in \mathbb{C}$ *of* e^z *is of the form* $\omega = 2\pi k\mathrm{i}$*, where* $k \in \mathbb{Z}$ *is non-zero.*

PROOF. The first assertion follows easily from the observation

$$\mathrm{e}^{2\pi\mathrm{i}} = \cos 2\pi + \mathrm{i}\sin 2\pi = 1.$$

Suppose now that $\omega \in \mathbb{C}$. Clearly $\mathrm{e}^{z+\omega} = \mathrm{e}^z$ implies $\mathrm{e}^\omega = 1$. Write $\omega = \alpha + \mathrm{i}\beta$, where $\alpha, \beta \in \mathbb{R}$. Then

$$\mathrm{e}^\alpha(\cos\beta + \mathrm{i}\sin\beta) = 1.$$

Taking modulus, we conclude that $\mathrm{e}^\alpha = 1$, so that $\alpha = 0$. It then follows that $\cos\beta + \mathrm{i}\sin\beta = 1$. Equating real and imaginary parts, we conclude that $\cos\beta = 1$ and $\sin\beta = 0$, so that $\beta = 2\pi k$, where $k \in \mathbb{Z}$. The second assertion follows. ○

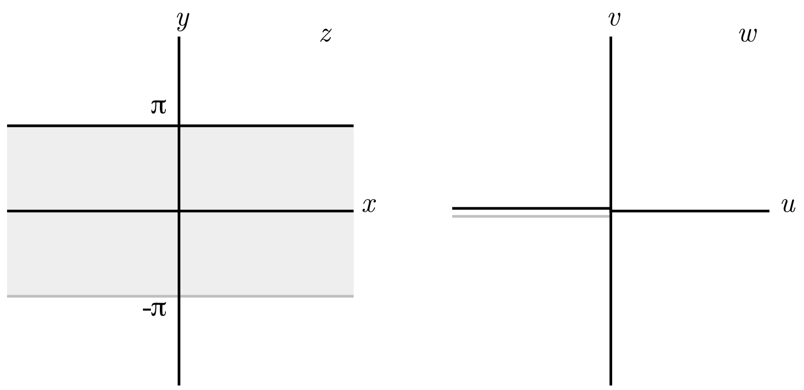
Consider now the mapping $w = \mathrm{e}^z$. By (5), we have $w = \mathrm{e}^x(\cos y + \mathrm{i}\sin y)$, where $x, y \in \mathbb{R}$. It follows that

$$|w| = \mathrm{e}^x \qquad \text{and} \qquad \arg w = y + 2\pi k,$$

where $k \in \mathbb{Z}$. Usually we make the choice $\arg w = y$, with the restriction that $-\pi < y \le \pi$. This restriction means that z lies on the horizontal strip

$$\mathcal{R}_0 = \{z \in \mathbb{C} : -\infty < x < \infty, -\pi < y \le \pi\}. \tag{8}$$

The restriction $-\pi < \arg w \le \pi$ can also be indicated on the complex w-plane by a cut along the negative real axis. The upper edge of the cut, corresponding to $\arg w = \pi$, is regarded as part of the cut w-plane. The lower edge of the cut, corresponding to $\arg w = -\pi$, is not regarded as part of the cut w-plane.



It is easy to check that the function $\exp : \mathcal{R}_0 \to \mathbb{C} \setminus \{0\}$, defined for every $z \in \mathcal{R}_0$ by $\exp(z) = \mathrm{e}^z$, is one-to-one and onto.

REMARK. The region $\mathcal{R}_0$ is usually known as a fundamental region of the exponential function. In fact, it is easy to see that every set of the type

$$\mathcal{R}_k = \{z \in \mathbb{C} : -\infty < x < \infty, (2k-1)\pi < y \le (2k+1)\pi\}, \tag{9}$$

where $k \in \mathbb{Z}$, has this same property as $\mathcal{R}_0$.

Let us return to the function $\exp : \mathcal{R}_0 \to \mathbb{C} \setminus \{0\}$. Since it is one-to-one and onto, there is an inverse function.

DEFINITION. The function $\mathrm{Log} : \mathbb{C} \setminus \{0\} \to \mathcal{R}_0$, defined by $\mathrm{Log}(w) = z \in \mathcal{R}_0$, where $\exp(z) = w$, is called the principal logarithmic function.

Suppose that $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$, where $x, y, u, v \in \mathbb{R}$. Suppose further that we impose the restriction $-\pi < y \le \pi$. If $w = \exp(z)$, then it follows from (5) that $u = \mathrm{e}^x \cos y$ and $v = \mathrm{e}^x \sin y$, and so

$$|w| = (u^2 + v^2)^{1/2} = \mathrm{e}^x \qquad \text{and} \qquad y = \mathrm{Arg}(w),$$

where $\mathrm{Arg}(w)$ denotes the principal argument of w. It follows that

$$x = \log |w| \qquad \text{and} \qquad y = \mathrm{Arg}(w).$$

Hence

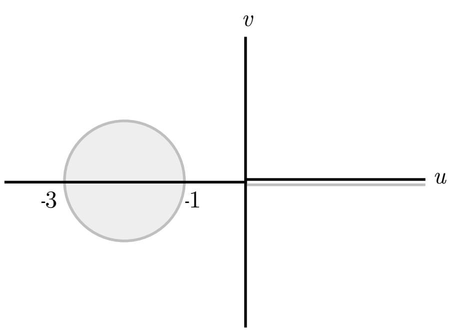
$$\mathrm{Log}(w) = \log |w| + \mathrm{i}\,\mathrm{Arg}(w). \tag{10}$$

In many practical situations, we usually try to define

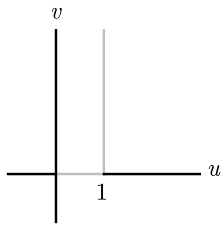
$$\log w = \log |w| + \mathrm{i} \arg w,$$

where the argument is chosen in order to make the logarithmic function continuous in its domain of definition, if this is at all possible. The following three examples show that great care needs to be taken in the study of such "many valued functions".

EXAMPLE 3.5.1. Consider the logarithmic function in the disc $\{w : |w+2| < 1\}$, an open disc of radius 1 and centred at the point $w = -2$. Note that this disc crosses the cut on the w-plane along the negative real axis discussed earlier. In this case, we may restrict the argument to satisfy, for example, $0 \le \arg w < 2\pi$. The logarithmic function defined in this way is then continuous in the disc $\{w : |w + 2| < 1\}$.

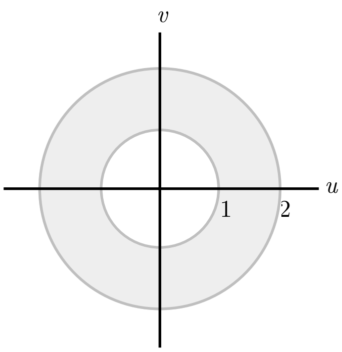


EXAMPLE 3.5.2. Consider the region P obtained from the w-plane by removing both the line segment $\{u + \mathrm{i}v : 0 \le u \le 1, v = 0\}$ and the half-line $\{u + \mathrm{i}v : u = 1, v > 0\}$, as shown below.



Suppose that we wish to define the logarithmic function to be continuous in this region P. One way to do this is to restrict the argument to the range $\pi < \arg w \le 3\pi$ for any $w \in P$ satisfying $u \ge 1$, and to the range $0 < \arg w \le 2\pi$ for any $w \in P$ satisfying $u < 1$.

EXAMPLE 3.5.3. Consider the annulus $\{w : 1 < |w| < 2\}$. It is impossible to define the logarithmic function to be continuous in this annulus. Heuristically, if one goes round the annulus once, the argument has to change by 2π if it varies continuously. If we return to the original starting point after going round once, the argument cannot therefore be the same.



It should now be quite clear that we cannot expect to have

$$\mathrm{Log}(w_1 w_2) = \mathrm{Log}(w_1) + \mathrm{Log}(w_2),$$

or even

$$\log w_1 w_2 = \log w_1 + \log w_2.$$

Instead, we have

$$\log w_1 w_2 = \log w_1 + \log w_2 + 2\pi \mathrm{i} k \qquad \text{for some } k \in \mathbb{Z}.$$

Let us return to the principal logarithmic function $\mathrm{Log} : \mathbb{C} \setminus \{0\} \to \mathcal{R}_0$. Recall (10). We have

$$\mathrm{Log}(z) = \log |z| + \mathrm{i}\,\mathrm{Arg}(z).$$

Recall from real analysis that for any $t \in \mathbb{R}$, the equation $\tan\theta = t$ has a unique solution θ satisfying $-\pi/2 < \theta < \pi/2$. This solution is denoted by $\tan^{-1} t$ and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} \tan^{-1} t = \frac{1}{1+t^2}.$$

It is not difficult to show that if we write

$$(11)\qquad v(x,y)=\begin{cases}-\tan^{-1}\left(\dfrac{x}{y}\right)-\dfrac{\pi}{2} & \text{if } y<0,\\ -\tan^{-1}\left(\dfrac{y}{x}\right) & \text{if } x>0,\\ -\tan^{-1}\left(\dfrac{x}{y}\right)+\dfrac{\pi}{2} & \text{if } y>0,\end{cases}$$

then $\mathrm{Arg}(z)=v(x,y)$. Hence $\mathrm{Log}(z)=u(x,y)+\mathrm{i}v(x,y)$, where

$$(12)\qquad u(x,y)=\frac{1}{2}\log(x^2+y^2).$$

It now follows from (12) that

$$\frac{\partial u}{\partial x}=\frac{x}{x^2+y^2}\qquad\text{and}\qquad\frac{\partial u}{\partial y}=\frac{y}{x^2+y^2},$$

and from (11) that

$$\frac{\partial v}{\partial x}=-\frac{y}{x^2+y^2}\qquad\text{and}\qquad\frac{\partial v}{\partial y}=\frac{x}{x^2+y^2}.$$

Clearly the Cauchy-Riemann equations are satisfied, and so

$$\frac{\mathrm{d}}{\mathrm{d}z}\mathrm{Log}(z)=\frac{\partial u}{\partial x}+\mathrm{i}\frac{\partial v}{\partial x}=\frac{x-\mathrm{i}y}{x^2+y^2}=\frac{1}{x+\mathrm{i}y}=\frac{1}{z}.$$

Power functions are defined in terms of the exponential and logarithmic functions. Given $z,a\in\mathbb{C}$, we write $z^a=\mathrm{e}^{a\log z}$. Naturally, the precise value depends on the logarithmic function that is chosen, and care again must be exercised for these “many valued functions”.

3.6. Laplace’s Equation and Harmonic Conjugates

We have shown that for any function $f=u+\mathrm{i}v$, the existence of the derivative f' leads to the Cauchy-Riemann equations. More precisely, we have

$$(13)\qquad \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\qquad\text{and}\qquad\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.$$

Furthermore,

$$(14)\qquad f'(z)=\frac{\partial u}{\partial x}+\mathrm{i}\frac{\partial v}{\partial x}.$$

Suppose now that the second derivative f'' also exists. Then f' satisfies the Cauchy-Riemann equations. The Cauchy-Riemann equations corresponding to the expression (14) are

$$(15)\qquad \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right)\qquad\text{and}\qquad\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)=-\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right).$$

Substituting (13) into (15), we obtain

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad \text{and} \qquad \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0. \tag{16}$$

We also obtain

$$\frac{\partial^2 v}{\partial y\partial x}=\frac{\partial^2 v}{\partial x\partial y} \qquad \text{and} \qquad \frac{\partial^2 u}{\partial y\partial x}=\frac{\partial^2 u}{\partial x\partial y}.$$

DEFINITION. A continuous function $\phi(x,y)$ that satisfies Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}=0$$

in a domain $D\subseteq\mathbb{C}$ is said to be harmonic in D.

We have in fact proved the following result.

THEOREM 3D. *Suppose that $f=u+\mathrm{i}v$, where u and v are real valued. Suppose further that $f''(z)$ exists in a domain $D\subseteq\mathbb{C}$. Then u and v both satisfy Laplace's equation and are harmonic in D.*

DEFINITION. Two harmonic functions u and v in a domain $D\subseteq\mathbb{C}$ are said to be harmonic conjugates in D if they satisfy the Cauchy-Riemann equations.

The remainder of this chapter is devoted to a discussion on finding harmonic conjugates. We shall illustrate the following theorem by discussing the special case when $D=\mathbb{C}$.

THEOREM 3E. *Suppose that a function u is real valued and harmonic in a domain $D\subseteq\mathbb{C}$. Then there exists a real valued function v which satisfies the following conditions:*
(a) The functions u and v satisfy the Cauchy-Riemann equations in D.
(b) The function $f=u+\mathrm{i}v$ is analytic in D.
(c) The function v is harmonic in D.

Clearly, parts (b) and (c) follow from part (a). We shall now indicate a proof of part (a) in the special case $D=\mathbb{C}$, and shall omit reference to this domain.

Suppose that u is real valued and harmonic. Then we need to find a real valued function v such that

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.$$

Let $X_0+\mathrm{i}Y_0\in D$ be chosen and fixed. Integrating the second of these with respect to x, we obtain

$$v(X,y)=-\int_{X_0}^{X}\frac{\partial u}{\partial y}(x,y)\,\mathrm{d}x+c(y), \tag{17}$$

where $c(y)$ is some function depending at most on y. Differentiating with respect to y, we obtain

$$\frac{\partial v}{\partial y}(X,y)=-\frac{\partial}{\partial y}\int_{X_0}^{X}\frac{\partial u}{\partial y}(x,y)\,\mathrm{d}x+c'(y).$$

Clearly the first of the Cauchy-Riemann equations requires

$$\frac{\partial u}{\partial x}(X,y)=-\frac{\partial}{\partial y}\int_{X_0}^{X}\frac{\partial u}{\partial y}(x,y)\,\mathrm{d}x+c'(y).$$

Changing the order of differentiation and integration, we obtain

$$\frac{\partial u}{\partial x}(X,y)=-\int_{X_0}^{X}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)(x,y)\mathrm{d}x+c'(y)=-\int_{X_0}^{X}\frac{\partial^2 u}{\partial y^2}(x,y)\mathrm{d}x+c'(y).$$

Since u is harmonic, we obtain

$$\frac{\partial u}{\partial x}(X,y)=\int_{X_0}^{X}\frac{\partial^2 u}{\partial x^2}(x,y)\mathrm{d}x+c'(y)=\frac{\partial u}{\partial x}(X,y)-\frac{\partial u}{\partial x}(X_0,y)+c'(y),$$

so that

$$c'(y)=\frac{\partial u}{\partial x}(X_0,y).$$

Integrating with respect to y, we obtain

$$c(Y)=\int_{Y_0}^{Y}\frac{\partial u}{\partial x}(X_0,y)\mathrm{d}y+c, \tag{18}$$

where c is an absolute constant. On the other hand, (17) can be rewritten in the form

$$v(X,Y)=-\int_{X_0}^{X}\frac{\partial u}{\partial y}(x,Y)\mathrm{d}x+c(Y). \tag{19}$$

Combining (18) and (19), we obtain

$$v(X,Y)=-\int_{X_0}^{X}\frac{\partial u}{\partial y}(x,Y)\mathrm{d}x+\int_{Y_0}^{Y}\frac{\partial u}{\partial x}(X_0,y)\mathrm{d}y+c. \tag{20}$$

It is easy to check that this function v satisfies the Cauchy-Riemann equations. Indeed, we have

$$\frac{\partial}{\partial X}v(X,Y)=-\frac{\partial}{\partial X}\int_{X_0}^{X}\frac{\partial u}{\partial y}(x,Y)\mathrm{d}x+\frac{\partial}{\partial X}\int_{Y_0}^{Y}\frac{\partial u}{\partial x}(X_0,y)\mathrm{d}y=-\frac{\partial u}{\partial y}(X,Y).$$

On the other hand, we have

$$\begin{aligned}\frac{\partial}{\partial Y}v(X,Y)&=-\frac{\partial}{\partial Y}\int_{X_0}^{X}\frac{\partial u}{\partial y}(x,Y)\mathrm{d}x+\frac{\partial}{\partial Y}\int_{Y_0}^{Y}\frac{\partial u}{\partial x}(X_0,y)\mathrm{d}y=-\int_{X_0}^{X}\frac{\partial^2 u}{\partial y^2}(x,Y)\mathrm{d}x+\frac{\partial u}{\partial x}(X_0,Y)\\&=\int_{X_0}^{X}\frac{\partial^2 u}{\partial x^2}(x,Y)\mathrm{d}x+\frac{\partial u}{\partial x}(X_0,Y)=\frac{\partial u}{\partial x}(X,Y)-\frac{\partial u}{\partial x}(X_0,Y)+\frac{\partial u}{\partial x}(X_0,Y)=\frac{\partial u}{\partial x}(X,Y).\end{aligned}$$

This completes our sketched proof.

In practice, we may use the following technique. Suppose that u is a real valued harmonic function in a domain D. Write

$$g(z)=\frac{\partial u}{\partial x}-\mathrm{i}\frac{\partial u}{\partial y}. \tag{21}$$

Then the Cauchy-Riemann equations for g are

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=-\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\qquad\text{and}\qquad\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right),$$

which clearly hold. It follows that g is analytic in D. Suppose now that u is the real part of an analytic function f in D. Then $f'(z)$ agrees with the right hand side of (21) in view of (3) and (4). Hence $f'=g$

in D. The question here, of course, is to find this function f. If we are successful, then the imaginary part v of f is a harmonic conjugate of the harmonic function u.

EXAMPLE 3.6.1. Consider the function $u(x,y) = x^3 - 3xy^2$. It is easily checked that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

so that u is harmonic in $\mathbb{C}$. Using $X_0 = Y_0 = 0$ in (20), we obtain

$$v(X,Y) = -\int_0^X \frac{\partial u}{\partial y}(x,Y)\mathrm{d}x + \int_0^Y \frac{\partial u}{\partial x}(0,y)\mathrm{d}y + c = 6\int_0^X xY\mathrm{d}x - 3\int_0^Y y^2\mathrm{d}y + c = 3X^2Y - Y^3 + c,$$

where c is any arbitrary constant. On the other hand, we can write

$$g(z) = \frac{\partial u}{\partial x} - \mathrm{i}\frac{\partial u}{\partial y} = 3(x^2 - y^2) + 6\mathrm{i}xy = 3(x^2 + 2\mathrm{i}xy - y^2) = 3(x + \mathrm{i}y)^2 = 3z^2.$$

It follows that u is the real part of an analytic function f in $\mathbb{C}$ such that $f'(z) = g(z)$ for every $z \in \mathbb{C}$. The function $f(z) = z^3 + C$ satisfies this requirement for any arbitrary constant C. Note that the imaginary part of f is $3x^2y - y^3 + c$, where c is the imaginary part of C.

EXAMPLE 3.6.2. Consider the function $u(x,y) = \mathrm{e}^x \sin y$. It is easily checked that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

so that u is harmonic in $\mathbb{C}$. Using $X_0 = Y_0 = 0$ in (20), we obtain

$$\begin{aligned} v(X,Y) &= -\int_0^X \frac{\partial u}{\partial y}(x,Y)\mathrm{d}x + \int_0^Y \frac{\partial u}{\partial x}(0,y)\mathrm{d}y + c = -\int_0^X \mathrm{e}^x \cos Y\mathrm{d}x + \int_0^Y \sin y\mathrm{d}y + c \\ &= \cos Y - \mathrm{e}^X \cos Y - \cos Y + 1 + c = c' - \mathrm{e}^X \cos Y, \end{aligned}$$

where c' is any arbitrary constant. On the other hand, we can write

$$g(z) = \frac{\partial u}{\partial x} - \mathrm{i}\frac{\partial u}{\partial y} = \mathrm{e}^x \sin y - \mathrm{i}\mathrm{e}^x \cos y = -\mathrm{i}\mathrm{e}^x(\cos y + \mathrm{i}\sin y) = -\mathrm{i}\mathrm{e}^z.$$

It follows that u is the real part of an analytic function f in $\mathbb{C}$ such that $f'(z) = g(z)$ for every $z \in \mathbb{C}$. The function $f(z) = C - \mathrm{i}\mathrm{e}^z$ satisfies this requirement for any arbitrary constant C. Note that the imaginary part of f is $c' - \mathrm{e}^x \cos y$, where c' is the imaginary part of C.

PROBLEMS FOR CHAPTER 3

1. a) Suppose that $P(z) = (z - z_1)(z - z_2)\ldots(z - z_k)$, where $z_1, z_2, \ldots, z_k \in \mathbb{C}$. Show that

$$\frac{P'(z)}{P(z)} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \ldots + \frac{1}{z - z_k} \qquad \text{for every } z \in \mathbb{C} \setminus \{z_1, z_2, \ldots, z_k\}.$$

   b) Suppose further that $\mathfrak{Re} z_j < 0$ for every $j = 1, \ldots, k$, and that $\mathfrak{Re} z \geq 0$. Show in this case that $\mathfrak{Re}(z - z_j)^{-1} > 0$ for every $j = 1, \ldots, k$, and deduce that $P'(z) \neq 0$.

   [REMARK: Polynomials all of whose roots have negative real parts are called Hurwitz polynomials. We have shown here that the derivative of a non-constant Hurwitz polynomial is also a Hurwitz polynomial.]

2. For each of the following functions $f(z)$, determine whether the Cauchy-Riemann equations are satisfied:
   a) $f(z) = x^2 - y^2 - 2\mathrm{i}xy$
   b) $f(z) = \log(x^2 + y^2) + 2\mathrm{i}\cot^{-1}(x/y)$
   c) $f(z) = x^3 - 3y^2 + 2x + \mathrm{i}(3x^2y - y^3 + 2y)$
   d) $f(z) = \log(x^2 - y^2) + 2\mathrm{i}\tan^{-1}(y/x)$

3. Show that a real valued analytic function is constant.

4. We are required to define an analytic function $f(z)$ such that $f(x+\mathrm{i}y) = \mathrm{e}^x f(\mathrm{i}y)$ for every $x, y \in \mathbb{R}$ and $f(0) = 1$. Suppose that for every $y \in \mathbb{R}$, we write $f(\mathrm{i}y) = c(y) + \mathrm{i}s(y)$, where $c(y), s(y) \in \mathbb{R}$ for every $y \in \mathbb{R}$.
   a) Show by the Cauchy-Riemann equations that $c'(y) = -s(y)$ and $s'(y) = c(y)$ for every $y \in \mathbb{R}$.
   b) For every $y \in \mathbb{R}$, write $g(y) = (c(y) - \cos y)^2 + (s(y) - \sin y)^2$. Show that $g'(y) = 0$ for every $y \in \mathbb{R}$. Deduce that $g(y) = 0$ for every $y \in \mathbb{R}$.
   c) Comment on the above.

5. a) Suppose that $P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$, where $a_0, a_1, a_2, \ldots, a_n \in \mathbb{C}$ are constants. Show that for every $k = 0, 1, \ldots, n$, we have

$$a_k = \frac{P^{(k)}(0)}{k!}.$$

   b) Apply the result to the polynomial $(1+z)^n = c_0 + c_1 z + c_2 z^2 + \ldots + c_n z^n$ and show that for every $k = 0, 1, \ldots, n$, we have

$$c_k = \frac{n!}{k!(n-k)!}.$$

6. a) Show that for every $z \in \mathbb{C}$, we have $\mathrm{e}^{\mathrm{i}z} = \cos z + \mathrm{i}\sin z$.
   b) Show that for every $z, w \in \mathbb{C}$, we have

$$\cos(z+w) + \mathrm{i}\sin(z+w) = (\cos z + \mathrm{i}\sin z)(\cos w + \mathrm{i}\sin w)$$

   and

$$\cos(z+w) - \mathrm{i}\sin(z+w) = (\cos z - \mathrm{i}\sin z)(\cos w - \mathrm{i}\sin w).$$

   c) Express $\sin(z+w)$ and $\cos(z+w)$ in terms of $\sin z$, $\sin w$, $\cos z$ and $\cos w$.

7. Suppose that $a_1, a_2, \ldots, a_n \in \mathbb{C}$ are distinct, and consider the polynomial

$$Q(z) = (z - z_1)(z - z_2)\ldots(z - z_n).$$

   Suppose further that $P(z)$ is a polynomial of degree less than n. Follow the steps below to show that there exist $a_1, a_2, \ldots, a_n \in \mathbb{C}$ such that

$$\frac{P(z)}{Q(z)} = \frac{a_1}{z - z_1} + \frac{a_2}{z - z_2} + \ldots + \frac{a_n}{z - z_n}.$$

   a) We shall first of all show that the expression above is possible by multiplying it by $Q(z)$ and then determining $a_1, a_2, \ldots, a_n$ so that the resulting equation between polynomials of degree less than n holds when $z = z_1, z_2, \ldots, z_n$.
   [HINT: Recall Problem 1 in Chapter 1.]
   b) Show that for every $k = 1, \ldots, n$, we have

$$a_k = \lim_{z \to z_k} (z - z_k)\frac{P(z)}{Q(z)} = \frac{P(z_k)}{Q'(z_k)}.$$

   [HINT: Note that $Q(z_k) = 0$ for every $k = 1, \ldots, n$.]

8. Suppose that $a \in \mathbb{C}$ is non-zero. Show that for any fixed choice of value for $\log a$, the function $f(z) = a^z = \mathrm{e}^{z \log a}$ satisfies $f'(z) = f(z) \log a$.

9. For each expression below, compute all possible values and plot their positions in the complex plane:
   a) $\log(-\mathrm{i})$
   b) $\log(1+\mathrm{i})$
   c) $(-\mathrm{i})^{-\mathrm{i}}$
   d) i^2
   e) $2^{\pi \mathrm{i}}$
   f) $(1+\mathrm{i})^{\mathrm{i}}(1+\mathrm{i})^{-\mathrm{i}}$

10. For each of the following equations, find all solutions:
    a) $\mathrm{Log}(z) = \pi\mathrm{i}/3$
    b) $\mathrm{e}^z = 2\mathrm{i}$
    c) $\sin z = \mathrm{i}$
    d) $\sin z = -\cos z$
    e) $\tan^2 z = -1$

11. For each of the functions below, determine whether the function is harmonic. If so, find also its harmonic conjugate:
    a) $x^2 - y^2 + y$
    b) $\mathrm{e}^x \sin y$
    c) $x^3 - y^3$
    d) $x\mathrm{e}^x \cos y - y\mathrm{e}^x \sin y$
    e) $3x^2y - y^3 + xy$
    f) $x^4 - 6x^2y^2 + y^4 + x^3y - xy^3$
    g) $\mathrm{e}^{x^2-y^2} \sin 2xy$

12. a) Suppose that the functions $f(z)$ and $g(z)$ both satisfy the Cauchy-Riemann equations at a particular point $z \in \mathbb{C}$. Show that the functions $f(z) + g(z)$ and $f(z)g(z)$ also satisfy the Cauchy-Riemann equations at the point z.
    b) Show that the constant function and the function $f(z) = z$ both satisfy the Cauchy-Riemann equations everywhere in $\mathbb{C}$.
    c) Deduce that every polynomial $P(z)$ with complex coefficients satisfies the Cauchy-Riemann equations everywhere in $\mathbb{C}$.

13. A real valued function $u(x, y)$ which is continuous and satisfies the inequality $u_{xx} + u_{yy} \geq 0$ in a region D is said to be subharmonic in D. Show that $u = |f(z)|^2$ is subharmonic in any region where $f(z)$ is analytic.