INTRODUCTION TO COMPLEX ANALYSIS

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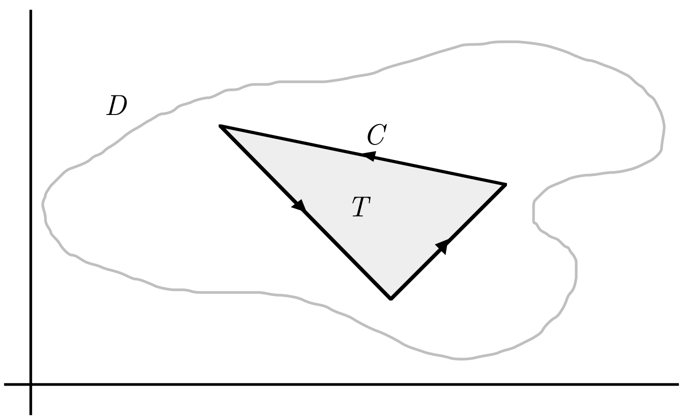
Chapter 5

CAUCHY'S INTEGRAL THEOREM

5.1. A Restricted Case

Cauchy's integral theorem states that in a simply connected domain, the integral of an analytic function over a closed contour is zero. The proof of this general result is rather involved. Here we first study a special case of the theorem in order to develop the basic properties of analytic functions.

THEOREM 5A. *Suppose that a function f is analytic in a domain D. Suppose further that the closed triangular region T lies in D, and that C denotes the boundary of T in the positive (anticlockwise) direction.*



Then

$$\int_C f(z)\,\mathrm{d}z = 0.$$

We shall give two proofs of this result, usually known as Cauchy's integral theorem for a triangular path. The first of these proofs, given next, is based on an additional assumption that the derivative $f'(z)$ is continuous in D.

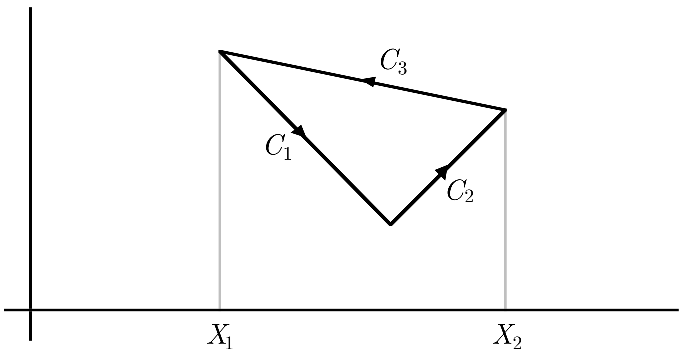
PROOF OF THEOREM 5A. Write $f(z) = u(x, y) + \mathrm{i}v(x, y)$, where u and v are real valued. Since f' exists and is continuous, the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

hold, and the four partial derivatives are continuous. On the other hand, we can write

$$\int_C f(z)\,\mathrm{d}z = \int_C (u + \mathrm{i}v)(\mathrm{d}x + \mathrm{i}\mathrm{d}y) = \int_C (u\,\mathrm{d}x - v\,\mathrm{d}y) + \mathrm{i}\int_C (v\,\mathrm{d}x + u\,\mathrm{d}y).$$

Suppose that $C = C_1 \cup C_2 \cup C_3$, a union of the three straight directed edges.



Consider the integral

$$\int_C u\,\mathrm{d}x.$$

We can write

$$\int_C u\,\mathrm{d}x = \int_{C_1} u\,\mathrm{d}x + \int_{C_2} u\,\mathrm{d}x + \int_{C_3} u\,\mathrm{d}x.$$

For each of the three integrals on the right hand side, y can be represented as a linear function of x, unless the edge is vertical, in which case the integral vanishes. Suppose that the projection of the triangle T on the x-axis is the line segment $X_1 \le x \le X_2$. Suppose also that the vertical line with abscissa x intersects the triangle in $h_1(x)$ and $h_2(x)$, where $h_1(x) \le h_2(x)$ (in the diagram, $h_1(x)$ describes C_1 and C_2, while $h_2(x)$ describes C_3). Then

$$\begin{aligned}\int_C u\,\mathrm{d}x &= \int_{X_1}^{X_2} u(x, h_1(x))\,\mathrm{d}x + \int_{X_2}^{X_1} u(x, h_2(x))\,\mathrm{d}x = -\int_{X_1}^{X_2} (u(x, h_2(x)) - u(x, h_1(x)))\,\mathrm{d}x \\ &= -\int_{X_1}^{X_2} \left(\int_{h_1(x)}^{h_2(x)} \frac{\partial u}{\partial y}(x, y)\,\mathrm{d}y \right) \mathrm{d}x = -\int_T \frac{\partial u}{\partial y}\,\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Note that the third equality above follows from the continuity of $\partial u/\partial y$. Similarly

$$\int_C v\,\mathrm{d}y = \int_T \frac{\partial v}{\partial x}\,\mathrm{d}x\mathrm{d}y.$$

Hence

$$\int_C (u\,\mathrm{d}x - v\,\mathrm{d}y) = -\int_T \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \mathrm{d}x\mathrm{d}y = 0.$$

We can also show that

$$\int_C v\,\mathrm{d}x = -\int_T \frac{\partial v}{\partial y}\,\mathrm{d}x\mathrm{d}y \qquad \text{and} \qquad \int_C u\,\mathrm{d}y = \int_T \frac{\partial u}{\partial x}\,\mathrm{d}x\mathrm{d}y,$$

so that

$$\int_C (v\,\mathrm{d}x + u\,\mathrm{d}y) = \int_T \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) \mathrm{d}x\mathrm{d}y = 0.$$

The result follows. ○

5.2. Analytic Functions in a Star Domain

In this section, we shall use Theorem 5A to establish the existence of an indefinite integral and the Cauchy integral theorem for analytic functions in a certain class of domains.

Definition. A domain $D \subseteq \mathbb{C}$ is called a star domain if there exists a point $z_0 \in D$ such that for every point $z \in D$, the line segment joining z and z_0 also lies in D. In this case, the point z_0 is called a star centre of the domain D.

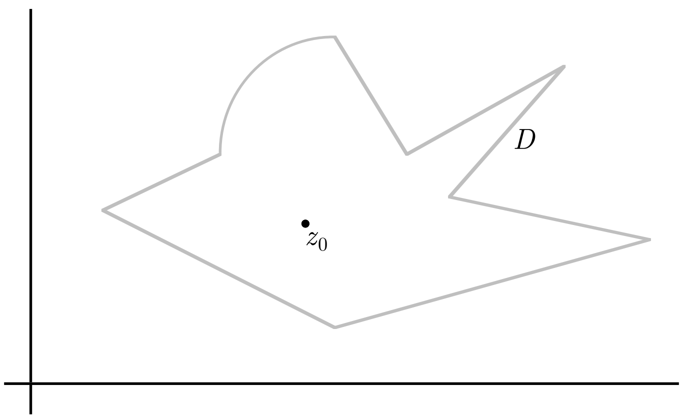
Example 5.2.1. The disc $\{z : |z| < 1\}$ is a star domain. Every point in this domain is a star centre.

Example 5.2.2. The complex plane $\mathbb{C}$ is a star domain. Again, every point in this domain is a star centre.

Example 5.2.3. The complex plane $\mathbb{C}$ with the non-negative real axis $\{x + \mathrm{i}y : x \geq 0, y = 0\}$ deleted is a star domain. Every point on the remaining part of the real axis is a star centre.

Example 5.2.4. The set $\{x + \mathrm{i}y : |xy| < 1\}$ is a star domain. The point 0 is the only star centre.

Example 5.2.5. The interior of the set shown below is a star domain, with a star centre z_0 as shown.

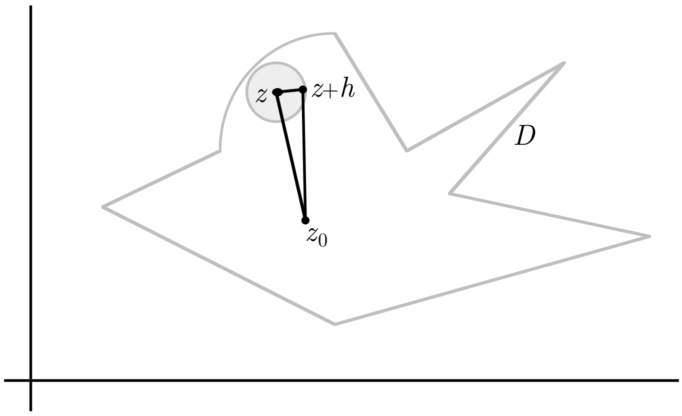


THEOREM 5B. *Suppose that a function f is analytic in a star domain D. Then there exists a function F, analytic in D and such that $F'(z) = f(z)$ for every $z \in D$.*

PROOF. Suppose that $z_0 \in D$ is a star centre. For every $z \in D$, define

$$F(z) = \int_{[z_0,z]} f(\zeta)\,\mathrm{d}\zeta, \tag{1}$$

where, for every $z_1, z_2 \in D$, $[z_1, z_2]$ denotes the directed line segment from z_1 to z_2. Since $z \in D$, there exists an ϵ-neighbourhood of z which is contained in D. Furthermore, for every $h \in \mathbb{C}$ satisfying $|h| < \epsilon$, the point $z+h$ lies in this ϵ-neighbourhood of z. It follows that the closed triangular region with vertices z_0, z and $z+h$ lies in D.



By Theorem 5A, we have

$$\int_{[z_0,z]} f(\zeta)\,\mathrm{d}\zeta + \int_{[z,z+h]} f(\zeta)\,\mathrm{d}\zeta + \int_{[z+h,z_0]} f(\zeta)\,\mathrm{d}\zeta = 0.$$

In other words,

$$\int_{[z_0,z+h]} f(\zeta)\,\mathrm{d}\zeta - \int_{[z_0,z]} f(\zeta)\,\mathrm{d}\zeta = \int_{[z,z+h]} f(\zeta)\,\mathrm{d}\zeta.$$

It follows from (1) that

$$F(z+h) - F(z) = \int_{[z,z+h]} f(\zeta)\,\mathrm{d}\zeta.$$

If $h \neq 0$, then

$$\frac{F(z+h) - F(z)}{h} - f(z) = \frac{1}{h}\int_{[z,z+h]} (f(\zeta) - f(z))\,\mathrm{d}\zeta. \tag{2}$$

Since the function f is continuous at z, it follows that given any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(\zeta) - f(z)| < \epsilon$ whenever $|\zeta - z| < \delta$. This means that if $|h| < \delta$, then $|f(\zeta) - f(z)| < \epsilon$ holds for every $\zeta \in [z, z+h]$. Theorem 4B now gives

$$\left|\int_{[z,z+h]} (f(\zeta) - f(z))\,\mathrm{d}\zeta\right| \le \epsilon|h|. \tag{3}$$

Combining (2) and (3), we have

$$\left|\frac{F(z+h) - F(z)}{h} - f(z)\right| \le \epsilon.$$

This gives

$$\lim_{h\to 0}\frac{F(z+h)-F(z)}{h}=f(z),$$

and completes the proof of the theorem. ◯

If we examine our proof carefully, then it is not difficult to see that we have in fact established the following result.

THEOREM 5C. *Suppose that a function f is continuous in a star domain D. Suppose further that*

$$\int_C f(z)\,\mathrm{d}z=0$$

for every closed triangular contour C lying in D. Then there exists a function F, analytic in D and such that $F'(z)=f(z)$ for every $z\in D$.

We can also deduce the Cauchy integral theorem for a star domain.

THEOREM 5D. *Suppose that a function f is analytic in a star domain D. Suppose further that C is a closed contour lying in D. Then*

$$\int_C f(z)\,\mathrm{d}z=0.$$

PROOF. By Theorem 5B, there exists a function F, analytic in D and such that $F'(z)=f(z)$ for every $z\in D$. The result now follows from Remark (1) immediately after Theorem 4A. ◯

EXAMPLE 5.2.6. Consider the contour integral

$$\int_{|z|=3}\frac{\mathrm{e}^z+\sin z}{z^2-16}\,\mathrm{d}z,$$

where the contour of integration is the circle centred at 0 and with radius 3, followed in the positive (anticlockwise) direction. Note that the function in question is analytic in the disc $D=\{z:|z|<4\}$, clearly a star domain. It follows from Theorem 5D that the integral is 0.

EXAMPLE 5.2.7. Suppose that $0<r<R$. Consider the contour integral

$$\int_{|z|=r}\frac{R+z}{(R-z)z}\,\mathrm{d}z,$$

where the contour of integration is the circle centred at 0 and with radius r, followed in the positive (anticlockwise) direction. For every $z\in\mathbb{C}$, note that using partial fractions, we have

$$\frac{R+z}{(R-z)z}=\frac{1}{z}+\frac{2}{R-z}.$$

It follows that

$$\int_{|z|=r}\frac{R+z}{(R-z)z}\,\mathrm{d}z=\int_{|z|=r}\frac{1}{z}\,\mathrm{d}z+\int_{|z|=r}\frac{2}{R-z}\,\mathrm{d}z.$$

Next, note that the function

$$\frac{2}{R-z}$$

is analytic in the star domain $D = \{z : |z| < R\}$. It follows from Theorem 5D that the last integral is 0, so that

$$\int_{|z|=r} \frac{R+z}{(R-z)z}\,\mathrm{d}z = \int_{|z|=r} \frac{1}{z}\,\mathrm{d}z = 2\pi\mathrm{i}, \tag{4}$$

in view of Example 4.4.2. On the other hand, the contour can be described by $z = r\mathrm{e}^{\mathrm{i}\theta}$, where $\theta \in [0, 2\pi]$. This formal substitution leads to the expression $\mathrm{d}z = \mathrm{i}r\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta = \mathrm{i}z\,\mathrm{d}\theta$ and

$$\int_{|z|=r} \frac{R+z}{(R-z)z}\,\mathrm{d}z = \int_0^{2\pi} \frac{R+r\mathrm{e}^{\mathrm{i}\theta}}{R-r\mathrm{e}^{\mathrm{i}\theta}}\,\mathrm{i}\,\mathrm{d}\theta.$$

Next, note that

$$\frac{R+r\mathrm{e}^{\mathrm{i}\theta}}{R-r\mathrm{e}^{\mathrm{i}\theta}} = \frac{(R+r\mathrm{e}^{\mathrm{i}\theta})(R-r\mathrm{e}^{-\mathrm{i}\theta})}{(R-r\mathrm{e}^{\mathrm{i}\theta})(R-r\mathrm{e}^{-\mathrm{i}\theta})} = \frac{R^2-r^2+2\mathrm{i}Rr\sin\theta}{R^2-2Rr\cos\theta+r^2},$$

so that

$$\int_{|z|=r} \frac{R+z}{(R-z)z}\,\mathrm{d}z = \int_0^{2\pi} \frac{R^2-r^2+2\mathrm{i}Rr\sin\theta}{R^2-2Rr\cos\theta+r^2}\,\mathrm{i}\,\mathrm{d}\theta. \tag{5}$$

Combining (4) and (5) and equating imaginary parts, we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-r^2}{R^2-2Rr\cos\theta+r^2}\,\mathrm{d}\theta = 1.$$

5.3. Nested Triangles

In this section, we shall give a second proof of Theorem 5A, without the additional assumption that the derivative $f'(z)$ is continuous in D. This proof is based on the following well-known result in real analysis: Suppose that

$$a_1 \le a_2 \le a_3 \le \ldots \qquad \text{and} \qquad b_1 \ge b_2 \ge b_3 \ge \ldots.$$

Suppose further that $a_k \le b_k$ for every $k \in \mathbb{N}$, and that $b_k - a_k \to 0$ as $k \to \infty$. Then there exists a unique number $\ell \in \mathbb{R}$ such that $a_k \to \ell$ and $b_k \to \ell$ as $k \to \infty$. This is a special case of the Cantor intersection theorem. In other words, if the intervals

$$[a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3] \supseteq \ldots$$

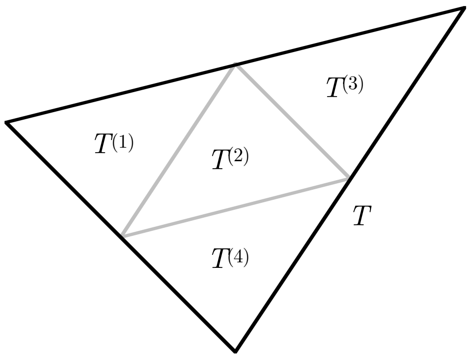
are nested, so that each contains all subsequent ones, and if their lengths decrease to 0, then the intervals collapse to a unique point.

We shall now prove Theorem 5A by the method of bisection.

Suppose that a function f is analytic in a domain D. Suppose further that the closed triangular region T lies in D, and that C denotes the boundary of T in the positive (anticlockwise) direction. Write

$$I(T) = \int_C f(z)\,\mathrm{d}z.$$

We now divide T into four triangular regions by joining the midpoints of the three sides of T as shown in the diagram.



Suppose that the four triangular regions so obtained are denoted by $T^{(j)}$, where $j = 1, 2, 3, 4$, with boundaries $C^{(j)}$ in the positive (anticlockwise) direction. Then since integrals over the common sides cancel each other, we have

$$I(T) = I(T^{(1)}) + I(T^{(2)}) + I(T^{(3)}) + I(T^{(4)}),$$

where for $j = 1, 2, 3, 4$,

$$I(T^{(j)}) = \int_{C^{(j)}} f(z)\,\mathrm{d}z.$$

Since the maximum is never less than the average, at least one of these four triangular regions $T^{(j)}$ must satisfy

(6) $$|I(T^{(j)})| \geq \frac{1}{4}|I(T)|.$$

We denote this triangular region by T_1, with the convention that if more than one of the four triangular regions $T^{(j)}$ satisfies (6), then we choose one under some fixed rule. This process can now be repeated indefinitely, so that we obtain a sequence of nested triangles

$$T = T_0 \supseteq T_1 \supseteq T_2 \supseteq T_3 \supseteq \ldots \supseteq T_k \supseteq \ldots$$

with the property

$$|I(T_k)| \geq \frac{1}{4}|I(T_{k-1})|,$$

so that

(7) $$|I(T_k)| \geq 4^{-k}|I(T)|.$$

Note now that the sequence of nested triangular regions must collapse to a point $z^* \in D$. Suppose now that $\epsilon > 0$ is chosen. Since D is open and the function f is analytic at z^*, there exists a δ-neighbourhood $\{z : |z - z^*| < \delta\}$ of z^*, contained in D and such that

(8) $$\left|\frac{f(z) - f(z^*)}{z - z^*} - f'(z^*)\right| < \epsilon$$

whenever $|z - z^*| < \delta$. Furthermore, we can choose k so large that

(9) $$T_k \subset \{z : |z - z^*| < \delta\}.$$

Note that since

$$\int_{C_k} \mathrm{d}z = 0 \qquad \text{and} \qquad \int_{C_k} z\,\mathrm{d}z = 0,$$

we have

$$I(T_k) = \int_{C_k} f(z)\,\mathrm{d}z = \int_{C_k} (f(z) - f(z^*) - (z - z^*)f'(z^*))\,\mathrm{d}z.$$

In view of (8) and (9), we have

$$|f(z) - f(z^*) - (z - z^*)f'(z^*)| \le \epsilon|z - z^*| \le \epsilon d_k,$$

where d_k denotes the diameter of T_k. It follows from Theorem 4B that

$$|I(T_k)| \le \epsilon d_k L_k, \tag{10}$$

where L_k denotes the perimeter of T_k. Observe now that

$$d_k = 2^{-k}d \qquad \text{and} \qquad L_k = 2^{-k}L, \tag{11}$$

where d and L denote respectively the diameter and perimeter of T. Combining (7), (10) and (11), we obtain

$$|I(T)| \le \epsilon dL.$$

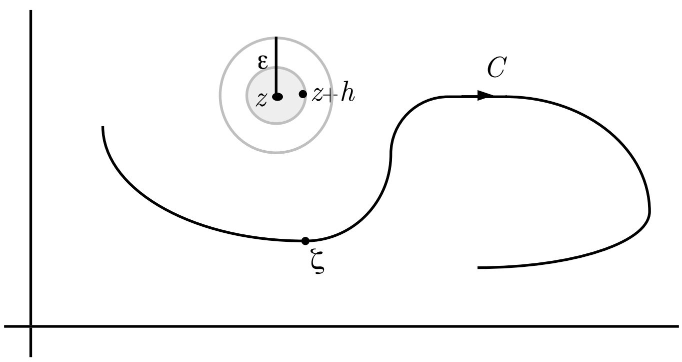
Since $\epsilon > 0$ is arbitrary, we must have $I(T) = 0$. This completes the proof of Theorem 5A.

5.4. Further Examples

EXAMPLE 5.4.1. Suppose that C is any contour. For any $z \in \mathbb{C}$ not lying on C, consider the integral

$$I(z) = \int_C \frac{\mathrm{d}\zeta}{\zeta - z}.$$

We shall show that the function $I(z)$ is continuous at z. Since $z \notin C$, there exists $\epsilon > 0$ such that the ϵ-neighbourhood of z does not meet C. Suppose that $h \in \mathbb{C}$ satisfies $|h| < \epsilon/2$.



Then

$$I(z+h) - I(z) = \int_C \left(\frac{1}{\zeta - z - h} - \frac{1}{\zeta - z} \right) \mathrm{d}\zeta = h \int_C \frac{\mathrm{d}\zeta}{(\zeta - z - h)(\zeta - z)}.$$

Note next that for any $\zeta \in C$, we have

$$|\zeta - z| > \epsilon \qquad \text{and} \qquad |\zeta - z - h| > \frac{\epsilon}{2},$$

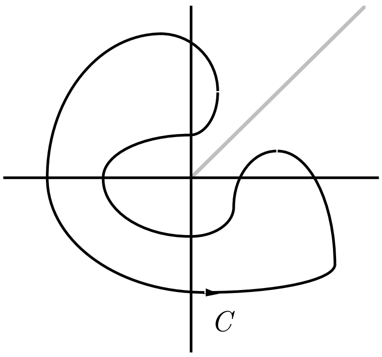
and so it follows from Theorem 4B that

$$|I(z+h) - I(z)| \leq \frac{2L|h|}{\epsilon^2},$$

where L is the length of C. This clearly tends to 0 as $h \to 0$.

The final example in this chapter exhibits the possibility of defining a continuous logarithm.

EXAMPLE 5.4.2. Consider the domain obtained by deleting from $\mathbb{C}$ the origin 0 and a half-line starting from 0. This is a star domain in which the function $1/z$ has a continuous derivative. Suppose that C is a closed contour that does not meet this half-line.



Then

$$\int_C \frac{\mathrm{d}\zeta}{\zeta} = 0.$$

Furthermore, the integral

$$\int_{z_0}^{z} \frac{\mathrm{d}\zeta}{\zeta}$$

is independent of the path joining z_0 to z in this domain, and can therefore be used to define a continuous logarithm.

PROBLEMS FOR CHAPTER 5

1. Give an example to show that the conclusion of Theorem 5D may not hold if D is not a star domain.

2. Suppose that $R > 0$ is fixed. By integrating the function $(R-z)^{-1}$ over the circle $C = \{z : |z| = r\}$, where $0 < r < R$, and referring to Example 5.2.7, show that

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{R\cos\theta}{R^2 - 2Rr\cos\theta + r^2}\,\mathrm{d}\theta = \frac{r}{R^2 - r^2}.$$

3. a) Suppose that C is the rectangle with vertices at $\pm b$ and $\pm b + \mathrm{i}a$, where $a, b > 0$. Explain why

$$\int_C \mathrm{e}^{-z^2} \, \mathrm{d}z = 0.$$

b) Let $C = C_1 \cup C_2 \cup C_3 \cup C_4$, where C_1, C_2, C_3, C_4 represent the four edges of C followed in the positive (anticlockwise) direction, with initial point $z = -b$. Show that

$$\left| \int_{C_2} \mathrm{e}^{-z^2} \, \mathrm{d}z \right| \le \mathrm{e}^{-b^2} \int_0^a \mathrm{e}^{y^2} \, \mathrm{d}y \qquad \text{and} \qquad \left| \int_{C_4} \mathrm{e}^{-z^2} \, \mathrm{d}z \right| \le \mathrm{e}^{-b^2} \int_0^a \mathrm{e}^{y^2} \, \mathrm{d}y.$$

c) Explain why

$$\int_{-b}^{b} \mathrm{e}^{-(x+\mathrm{i}a)^2} \, \mathrm{d}x - \int_{-b}^{b} \mathrm{e}^{-x^2} \, \mathrm{d}x \to 0 \qquad \text{as } b \to \infty.$$

Deduce that the integral

$$\int_{-\infty}^{\infty} \mathrm{e}^{-(x+\mathrm{i}a)^2} \, \mathrm{d}x$$

is independent of the choice of $a > 0$.

4. Suppose that a function $f(z)$ is analytic in $\{z : |z| < R\}$ and continuous in $\{z : |z| \le R\}$, where $R > 0$ is fixed. Suppose further that C denotes the circle $\{z : |z| = R\}$.

a) Suppose that $r < R$. Explain why

$$\int_C f(z) \, \mathrm{d}z = \int_0^{2\pi} f(R\mathrm{e}^{\mathrm{i}\theta}) R\mathrm{e}^{\mathrm{i}\theta} \mathrm{i} \, \mathrm{d}\theta - \int_0^{2\pi} f(r\mathrm{e}^{\mathrm{i}\theta}) r\mathrm{e}^{\mathrm{i}\theta} \mathrm{i} \, \mathrm{d}\theta.$$

b) The function $f(z)z$ is continuous in $\{z : |z| \le R\}$, and so uniformly continuous in $\{z : |z| \le R\}$. This implies that given any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(R\mathrm{e}^{\mathrm{i}\theta})R\mathrm{e}^{\mathrm{i}\theta} - f(r\mathrm{e}^{\mathrm{i}\theta})r\mathrm{e}^{\mathrm{i}\theta}| < \epsilon$ whenever $R - \delta < r < R$. Use this to show that

$$\left| \int_C f(z) \, \mathrm{d}z \right| < 2\pi\epsilon.$$

c) Explain why it follows that

$$\int_C f(z) \, \mathrm{d}z = 0.$$

d) Explain also why this result does not follow directly from Theorem 5D.

5. Suppose that a function $f(z)$ is continuous on a closed contour C. Suppose further that $f(z)$ can be uniformly approximated with arbitrary precision by a polynomial; in other words, given any $\epsilon > 0$, there exists a polynomial $P(z)$ such that $|f(z) - P(z)| < \epsilon$ for every $z \in C$. Prove that

$$\int_C f(z) \, \mathrm{d}z = 0.$$

6. Suppose that a function $f(z)$ is analytic in $\{z : |z| \le 1\}$. By considering a suitable integral over the unit circle $\{z : |z| = 1\}$, show that

$$\max_{|z|=1} \left| \frac{1}{z} - f(z) \right| \ge 1.$$

7. Suppose that C is a closed contour, and that D is a domain not containing any point of C. By noting Examples 4.4.2 and 5.4.1, show that the integral

$$n(C,z_0)=\frac{1}{2\pi\mathrm{i}}\int_C\frac{\mathrm{d}z}{z-z_0}$$

is independent of the choice of $z_0\in D$.
[Remark: The value $n(C,z_0)$ is called the winding number of the contour C round the point z_0, and measures the number of times the contour winds round the point z_0.]

8. Suppose that C is a contour $z=r(\theta)\mathrm{e}^{\mathrm{i}\theta}$ for $\theta\in[0,2\pi]$, where $r(\theta)>0$ for every $\theta\in[0,2\pi]$. Suppose further that $r(0)=r(2\pi)$, so that C is a closed contour. Let D be the domain containing the origin $z=0$ and with boundary C.
   a) Show that D is a star domain with the origin $z=0$ as a star centre.
   b) Suppose that $z_0\notin D\cup C$. Explain why the half line $L=\{\lambda z_0:\lambda\in[1,\infty)\}$ satisfies $L\cap C=\emptyset$. Show also that $\mathbb{C}\setminus L$ is a star domain with star centre $z=0$.
   c) Explain why

$$\frac{1}{2\pi\mathrm{i}}\int_C\frac{\mathrm{d}z}{z-z_0}=\begin{cases}0 & \text{if } z_0\notin D\cup C,\\ 1 & \text{if } z_0\in D.\end{cases}$$

   [Hint: For the case $z_0\in D$, refer to Problem 7 if necessary.]
   d) Suppose that $P(z)$ is a polynomial with no roots on the contour C. By referring to Problem 1 in Chapter 3 if necessary, show that the number of roots of $P(z)$ in D is given by

$$\frac{1}{2\pi\mathrm{i}}\int_C\frac{P'(z)}{P(z)}\,\mathrm{d}z.$$

9. Suppose that $P(z)$ is a polynomial of degree k and with distinct roots $z_1,\ldots,z_k$. Suppose further that C is a closed contour which does not contain any of these roots. By referring to Problem 7 if necessary, show that

$$\frac{1}{2\pi\mathrm{i}}\int_C\frac{P'(z)}{P(z)}\,\mathrm{d}z=n(C,z_1)+\ldots+n(C,z_k).$$

10. Suppose that two star domains D_1 and D_2 both have the point z_0 as star centre. Show that $D_1\cap D_2$ and $D_1\cup D_2$ are both star domains with star centre z_0.