

INTRODUCTION TO COMPLEX ANALYSIS

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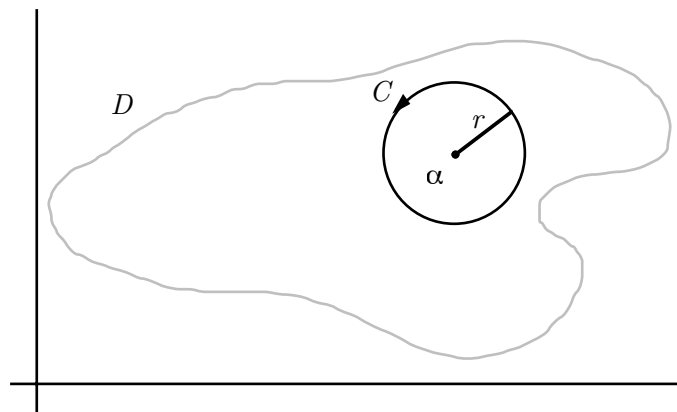
Chapter 6

CAUCHY'S INTEGRAL FORMULA

6.1. Introduction

In this chapter, we study a remarkable formula due to Cauchy and which shows that the values of an analytic function at the interior points of a disc are determined by the values of the function on the boundary of the disc.

THEOREM 6A. *Suppose that a function f is analytic in a domain D . Suppose further that the closed disc $\{z : |z - \alpha| \leq r\}$ is contained in D , and that C denotes the circle $\{z : |z - \alpha| = r\}$ followed in the positive (anticlockwise) direction.*

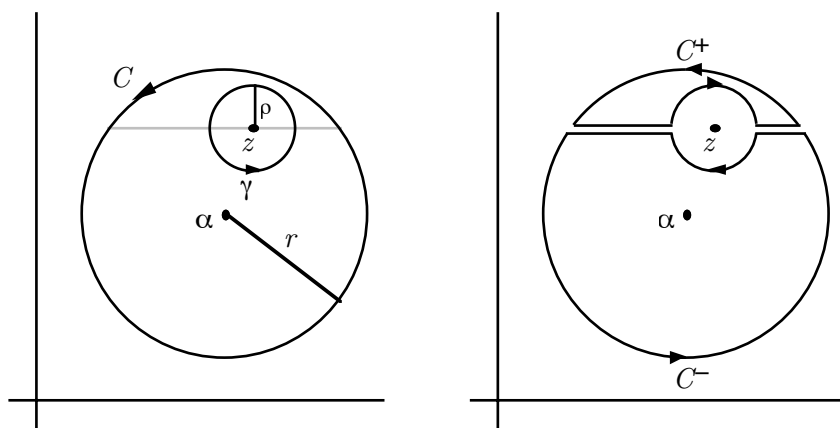


Then for every $z \in D$ satisfying $|z - \alpha| < r$, we have

$$(1) \quad f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta.$$

REMARK. Theorem 6A is a special case of Cauchy's integral formula in a simply connected domain which we shall study in Chapter 9.

PROOF OF THEOREM 6A. Suppose that γ is a circle of radius ρ and centred at z , followed in the positive direction. Suppose further that ρ is sufficiently small so that γ lies in the interior of C . Note that a horizontal line through the point z intersects C at two points and intersects γ at two points and gives rise to two line segments inside C and outside γ .



The part of the two circles above this line and the two line segments give rise to a simple closed contour C^+ followed in the positive direction and which can be shown to lie in a star domain lying in D but not containing z , so that

$$\int_{C^+} \frac{f(\zeta)}{\zeta - z} d\zeta = 0,$$

in view of Theorem 5D. Similarly, the part of the two circles below this line and the two line segments give rise to a simple closed contour C^- followed in the positive direction and which again can be shown to lie in a star domain lying in D but not containing z , so that

$$\int_{C^-} \frac{f(\zeta)}{\zeta - z} d\zeta = 0.$$

It is easily seen that

$$\int_C \frac{f(\zeta)}{\zeta - z} d\zeta - \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta = \int_{C^+} \frac{f(\zeta)}{\zeta - z} d\zeta + \int_{C^-} \frac{f(\zeta)}{\zeta - z} d\zeta,$$

so that

$$\int_C \frac{f(\zeta)}{\zeta - z} d\zeta = \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta.$$

We can write

$$(2) \quad \int_C \frac{f(\zeta)}{\zeta - z} d\zeta = f(z) \int_\gamma \frac{d\zeta}{\zeta - z} + \int_\gamma \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta.$$

The first integral on the right hand side of (2) is studied in a similar way as in Example 4.2.2, and we have

$$(3) \quad \int_{\gamma} \frac{d\zeta}{\zeta - z} = 2\pi i.$$

On the other hand, note that f is continuous at z . It follows that given any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(\zeta) - f(z)| < \epsilon$ whenever $|\zeta - z| < \delta$. If we choose ρ so that $\rho < \delta$, then

$$\left| \frac{f(\zeta) - f(z)}{\zeta - z} \right| \leq \frac{\epsilon}{\rho}$$

for every $\zeta \in \gamma$, and so it follows from Theorem 4B that

$$(4) \quad \left| \int_{\gamma} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta \right| \leq 2\pi\epsilon.$$

Combining (2)–(4), we obtain

$$\left| \int_C \frac{f(\zeta)}{\zeta - z} d\zeta - 2\pi i f(z) \right| \leq 2\pi\epsilon.$$

The result follows immediately, since ϵ is arbitrary. \circ

6.2. Derivatives

An important consequence of Cauchy's integral formula is that we can show that an analytic function possesses derivatives of all orders. This is a rather remarkable result, and much nicer than in real analysis. We shall establish this result by a number of steps.

THEOREM 6B. *Suppose that a function f is analytic in a domain D . Then the derivative f' is analytic in D .*

REMARK. Recall that in our first proof of Theorem 5A, we use the additional assumption that f' is continuous in D . In view of Theorem 6B, it appears that this extra assumption is superfluous. However, our proof of Theorem 6B below will depend on Theorem 6A, whose proof uses Theorem 5D, which follows somewhat from Theorem 5A. Hence we cannot reasonably use our first proof of Theorem 5A without running into the danger of a “circular argument” of deducing two results from each other and possibly establishing neither. Note, however, that our second proof of Theorem 5A in Section 5.3 saves us from this dubious distinction.

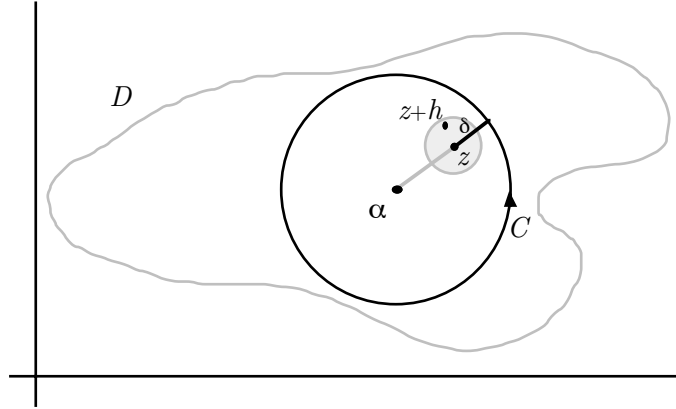
THEOREM 6C. *Suppose that a function f is analytic in a domain D . Then the derivative $f^{(n)}$ exists for every $n \in \mathbb{N}$ and is analytic in D .*

THEOREM 6D. *Suppose that a function f is analytic in a domain D . Then, in the notation of Theorem 6A, we have*

$$(5) \quad f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

for every $n \in \mathbb{N}$.

PROOF OF THEOREM 6B. Note that every $z \in D$ is contained inside some circle C with α as centre, and so (1) is valid. Suppose that $h \in \mathbb{C}$ has sufficiently small modulus so that $z + h$, as well as z , lies inside the circle C .



Then by (1), we have

$$\begin{aligned} \frac{f(z+h) - f(z)}{h} &= \frac{1}{2\pi i h} \int_C \left(\frac{f(\zeta)}{\zeta - z - h} - \frac{f(\zeta)}{\zeta - z} \right) d\zeta = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z - h)(\zeta - z)} d\zeta \\ &= \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta + \frac{h}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^2(\zeta - z - h)} d\zeta. \end{aligned}$$

Since z is inside the circle C , the number

$$\delta = \min_{\zeta \in C} |\zeta - z| > 0.$$

If $|h| < \delta/2$, then for every $\zeta \in C$, we have

$$|\zeta - z - h| \geq |\zeta - z| - |h| > \delta - \frac{\delta}{2} = \frac{\delta}{2}.$$

On the other hand, the circle C is a closed and bounded set. It follows that there exists some real constant M such that $|f(\zeta)| \leq M$ for every $\zeta \in C$. Recall that the circle C has radius r . It now follows from Theorem 4B that

$$\left| \frac{h}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^2(\zeta - z - h)} d\zeta \right| \leq \frac{|h|}{2\pi} \frac{2M}{\delta^3} 2\pi r = \frac{2Mr|h|}{\delta^3} \rightarrow 0$$

as $h \rightarrow 0$. This establishes the existence of f' in D and (5) for $n = 1$. We now repeat the argument, starting with (5) with $n = 1$. This establishes the existence of f'' in D , and so the analyticity of f' in D . \circ

PROOF OF THEOREM 6C. Suppose that $f^{(n)}$ is analytic in D . Applying Theorem 6B to the function $f^{(n)}$, we conclude that $f^{(n+1)}$ is analytic in D . The conclusion now follows by induction. \circ

PROOF OF THEOREM 6D. It follows from Theorem 6C that $f^{(n)}$ is analytic in D . Applying Theorem 6A to the function $f^{(n)}$, we have

$$f^{(n)}(z) = \frac{1}{2\pi i} \int_C \frac{f^{(n)}(\zeta)}{\zeta - z} d\zeta.$$

Integrating this by parts n times gives (5). \circ

6.3. Further Consequences

THEOREM 6E. (CAUCHY'S ESTIMATE) *Suppose that a function f is analytic in a domain D . Suppose further that the closed disc $\{z : |z - \alpha| \leq r\}$ is contained in D , and that there exists a positive constant M such that $|f(z)| \leq M$ in this disc. Then*

$$|f^{(n)}(\alpha)| \leq \frac{n!M}{r^n}.$$

PROOF. It follows from Theorem 6D that

$$f^{(n)}(\alpha) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - \alpha)^{n+1}} d\zeta.$$

Note now that for every $\zeta \in C$, we have $|\zeta - \alpha| = r$, so that

$$\left| \frac{f(\zeta)}{(\zeta - \alpha)^{n+1}} \right| \leq \frac{M}{r^{n+1}}.$$

It now follows from Theorem 4B that

$$|f^{(n)}(\alpha)| \leq \frac{n!}{2\pi} \frac{M}{r^{n+1}} 2\pi r = \frac{n!M}{r^n}$$

as required. \circ

THEOREM 6F. (LIOUVILLE'S THEOREM) *An entire and non-constant function f cannot be bounded in \mathbb{C} .*

PROOF. Suppose on the contrary that f is bounded. Then there exists a positive constant M such that $|f(z)| \leq M$ for every $z \in \mathbb{C}$. For every $\alpha \in \mathbb{C}$, Cauchy's estimate for $n = 1$ gives

$$|f'(\alpha)| \leq \frac{M}{r}.$$

This inequality is valid for every $r > 0$ since the closed disc $\{z : |z - \alpha| \leq r\}$ is clearly contained in \mathbb{C} . It follows that we must have $f'(\alpha) = 0$ for every $\alpha \in \mathbb{C}$. Hence it follows from Theorem 4A that

$$f(z) - f(0) = \int_0^z f'(\zeta) d\zeta = 0,$$

so that f is constant, a contradiction. \circ

THEOREM 6G. (MORERA'S THEOREM) *Suppose that f is continuous in a domain D . Suppose further that*

$$\int_C f(z) dz = 0$$

holds for every closed triangular contour C which together with its interior lies in D . Then f is analytic in D .

PROOF. Suppose that $z \in D$. Then there exists an ϵ -neighbourhood D_z of z lying entirely in D . Clearly D_z is a star domain. It follows from Theorem 5C that there exists a function F , analytic in D_z and such that $F' = f$ in D_z . By Theorem 6B, f is analytic in D_z , and so analytic at z . Since $z \in D$ is arbitrary, the result follows immediately. \circ

EXAMPLE 6.3.1. We can now prove the Fundamental theorem of algebra, that every non-constant polynomial $P(z)$ has at least one root. It is easily checked that every such non-constant polynomial satisfies $|P(z)| \rightarrow \infty$ as $|z| \rightarrow \infty$. Hence the function $1/P(z)$ is bounded outside some circle $\{z : |z| = r\}$. Suppose that $P(z)$ does not vanish. Then $1/P(z)$ is an entire function. Hence it is continuous and so bounded in the closed set $\{z : |z| \leq r\}$. It follows that it is bounded in \mathbb{C} . By Liouville's theorem, it must be constant, a contradiction.

PROBLEMS FOR CHAPTER 6

1. Suppose that a function $f = u + iv$ is analytic in a region D . Show that all partial derivatives of u and v are continuous in D . Show also that uv is harmonic in D .
2. Suppose that $f(z)$ is an entire function. Suppose further that there exist $M \in \mathbb{R}$ and $m \in \mathbb{N}$ such that $|f(z)| \leq M|z|^m$ whenever $|z|$ is large.
 - a) Use Cauchy's estimate to show that $f^{(n)}(0) = 0$ for every integer $n > m$.
 - b) Deduce that $f(z)$ is a polynomial of degree at most m .
3. Suppose that a function $f(z)$ is analytic in the closed disc $\{z : |z| \leq R\}$, where $R > 0$ is fixed.
 - a) Prove Gauss's mean value theorem, that

$$f(0) = \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\theta}) d\theta.$$

- b) Prove also that for every $n \in \mathbb{N} \cup \{0\}$, we have

$$|f^{(n)}(0)| \leq \frac{n!}{2\pi R^n} \int_0^{2\pi} |f(Re^{i\theta})| d\theta.$$

- c) Suppose that L is the length of the image of the circle $\{z : |z| = R\}$ under f , so that

$$L = \int_C |f'(z)| |dz| = R \int_0^{2\pi} |f'(Re^{i\theta})| d\theta.$$

Show that $L \geq 2\pi R |f'(0)|$.

- d) By first expressing the integral in polar coordinates, show that

$$\int_{\{z:|z|\leq R\}} f(x+iy) dx dy = \pi R^2 f(0).$$

4. Suppose that $f(z)$ is an entire function. Suppose further that there exists $M \in \mathbb{R}$ such that $\Re f(z) \leq M$ for every $z \in \mathbb{C}$. By applying Liouville's theorem to the function $e^{f(z)}$, show that $f(z)$ is constant.
5. Suppose that $f(z)$ is an entire function, and that $g(R) \rightarrow 0$ as $R \rightarrow \infty$. Suppose further that for all large R , the inequality $|f(z)| \leq Rg(R)$ holds whenever $|z| = R$. By proceeding along the lines of the proof of Liouville's theorem, show that $f(z)$ is constant.
6. Suppose that a function $f(\zeta)$ is continuous on a contour C . Show that the function

$$F(z) = \int_C \frac{f(\zeta)}{\zeta - z} d\zeta \quad \text{satisfies} \quad F'(z) = \int_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta$$

for every $z \notin C$, so that $F(z)$ is analytic off the contour C .

7. Suppose that a function $f(\zeta)$ is continuous on a contour C . Consider again the function $F(z)$ in Problem 6. Use Morera's theorem to show that $F(z)$ is analytic off the contour C .