INTRODUCTION TO COMPLEX ANALYSIS

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Chapter 7

TAYLOR SERIES, UNIQUENESS AND THE MAXIMUM PRINCIPLE

7.1. Remarks on Series

The purpose of this chapter is to show that every analytic function can be represented by a Taylor series, and to use the Taylor series to study further properties of such functions.

Here we do not propose to have a systematic study of series. Such a study is postponed until Chapter 16. In this section, we shall make a brief review of standard terminology.

We are concerned with power series of the form

$$a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots, \tag{1}$$

where $a_0, a_1, a_2, \ldots \in \mathbb{C}$ and $z_0 \in \mathbb{C}$ are fixed, and where z belongs to some region in the complex plane $\mathbb{C}$. For every $N \in \mathbb{N}$, the N-th partial sum of the series (1) is defined by

$$s_N(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots + a_N(z - z_0)^N. \tag{2}$$

We are interested in the sequence $s_N(z)$ of partial sums.

Suppose that a function $f(z)$ and a sequence of functions $s_N(z)$ are defined in a region G in the complex plane $\mathbb{C}$. The sequence $s_N(z)$ is said to converge uniformly to $f(z)$ in G if, given any $\epsilon > 0$, there exists $N_0 = N_0(\epsilon)$ such that

$$|s_N(z) - f(z)| < \epsilon$$

for every $N > N_0$ and every $z \in G$. Note here that the notion of uniformity implies the independence of N_0 from the choice of z in G. We also write

$$f(z) = \lim_{N\to\infty} s_N(z)$$

uniformly in G.

7.2. Taylor Series

In particular, if $s_N(z)$ is given by (2), then we write

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

uniformly in G.

THEOREM 7A. (TAYLOR'S THEOREM) *Suppose that a function f is analytic in the domain $\{z : |z - z_0| < R\}$, where $z_0 \in \mathbb{C}$ and $R > 0$ are fixed. Suppose further that $0 \le r < R$. Then*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \tag{3}$$

uniformly in $\{z : |z - z_0| \le r\}$.

Definition. The series (3) is called the Taylor series of the function f at z_0.

Theorem 7A follows easily from the following special case.

THEOREM 7B. *Suppose that a function g is analytic in the domain $\{z : |z| < R\}$, where $R > 0$ is fixed. Suppose further that $0 \le r < R$. Then*

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} z^n \tag{4}$$

uniformly in $\{z : |z| \le r\}$.

We shall show that Theorem 7A follows from Theorem 7B. Suppose that $|z - z_0| < R$. If we write $\zeta = z - z_0$, then $|\zeta| < R$. We now use the substitution $f(z) = g(\zeta)$. Suppose that f is analytic in the region $\{z : |z - z_0| < R\}$. Then clearly g is analytic in the region $\{\zeta : |\zeta| < R\}$. It then follows from Theorem 7B that

$$f(z) = g(\zeta) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!}\zeta^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

uniformly in $\{\zeta : |\zeta| \le r\}$, and so uniformly in $\{z : |z - z_0| \le r\}$. It remains to prove Theorem 7B.

Proof of Theorem 7B. Let the real number ρ be chosen to satisfy $r < \rho < R$, and let C denote the circle $\{\zeta : |\zeta| = \rho\}$, followed in the positive (anticlockwise) direction. By Theorem 6A, we have

$$g(z) = \frac{1}{2\pi \mathrm{i}} \int_C \frac{g(\zeta)}{\zeta - z}\,\mathrm{d}\zeta$$

for every z satisfying $|z| \leq r$. Since (see Remark (1) below)

$$\frac{1}{\zeta - z} = \frac{1}{\zeta} + \frac{z}{\zeta^2} + \frac{z^2}{\zeta^3} + \ldots + \frac{z^{n-1}}{\zeta^n} + \frac{z^n}{\zeta^n}\frac{1}{\zeta - z}, \tag{5}$$

we have

$$g(z) = \frac{1}{2\pi \mathrm{i}}\int_C \frac{g(\zeta)}{\zeta}\,\mathrm{d}\zeta + \ldots + \frac{z^{n-1}}{2\pi \mathrm{i}}\int_C \frac{g(\zeta)}{\zeta^n}\,\mathrm{d}\zeta + z^n g_n(z), \tag{6}$$

where

$$g_n(z) = \frac{1}{2\pi \mathrm{i}}\int_C \frac{g(\zeta)}{\zeta^n(\zeta - z)}\,\mathrm{d}\zeta. \tag{7}$$

By Theorems 6A and 6D, we have

$$\frac{1}{2\pi \mathrm{i}}\int_C \frac{g(\zeta)}{\zeta^{k+1}}\,\mathrm{d}\zeta = \frac{g^{(k)}(0)}{k!} \tag{8}$$

for every $k = 0, \ldots, n-1$, so that (6) becomes

$$g(z) = g(0) + g'(0)z + \ldots + \frac{g^{(n-1)}(0)}{(n-1)!}z^{n-1} + z^n g_n(z). \tag{9}$$

To complete the proof of Theorem 7B, it suffices to show that

$$|z^n g_n(z)| \to 0$$

uniformly in $\{z : |z| \leq r\}$ as $n \to \infty$. Clearly, the circle $C = \{\zeta : |\zeta| = \rho\}$ is closed and bounded, and the function g is continuous on C. It follows that there exists a positive real constant M such that $|g(\zeta)| \leq M$ for every $\zeta \in C$. Hence for every $\zeta \in C$ and every $|z| \leq r$, we have

$$\left|\frac{g(\zeta)}{\zeta^n(\zeta - z)}\right| \leq \frac{M}{\rho^n(\rho - r)}.$$

It follows from Theorem 4B that

$$\left|\int_C \frac{g(\zeta)}{\zeta^n(\zeta - z)}\,\mathrm{d}\zeta\right| \leq \frac{M}{\rho^n(\rho - r)}2\pi\rho,$$

and so

$$|z^n g_n(z)| \leq \frac{r^n}{2\pi}\frac{M}{\rho^n(\rho - r)}2\pi\rho = \frac{M\rho}{\rho - r}\left(\frac{r}{\rho}\right)^n.$$

Since $r < \rho$, the right hand side clearly converges to 0 as $n \to \infty$ independently of the choice of z in the set $\{z : |z| \leq r\}$. ◯

Remarks. (1) To derive the identity (5), note that if $w \neq 1$, then the identity

$$\frac{1}{1-w} = 1 + w + w^2 + w^3 + \ldots + w^{n-1} + \frac{w^n}{1-w}$$

is easily verified. Now substitute $w = z/\zeta$ and then divide by ζ to obtain (5).

(2) It is easily seen that the function $g_n(z)$ in (7) is analytic in the domain $\{z : |z| < R\}$. If $z \neq 0$, then the analyticity follows immediately from (9). To show that $g_n(z)$ is analytic at 0, note that the

function $g(\zeta)/\zeta^n$ is continuous on the circle C, and the result follows from Problem 6 in Chapter 6. Note also that (7) and (8) for $k = n$ give

$$g_n(0) = \frac{g^{(n)}(0)}{n!}.$$

These observations, combined with (9), immediately lead to the following finite version of Taylor's theorem.

THEOREM 7C. *Under the hypotheses of Theorem 7A, we have*

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \ldots + \frac{f^{(n-1)}(z_0)}{(n-1)!}(z - z_0)^{n-1} + f_n(z)(z - z_0)^n,$$

where $f_n(z)$ is analytic in the domain $\{z : |z - z_0| < R\}$ and

$$f_n(z_0) = \frac{f^{(n)}(z_0)}{n!}.$$

A very good reason for studying Taylor series is their polynomial-like behaviour, although great care needs to be exercised. An example is the following result.

THEOREM 7D. *Suppose that a function f is analytic in the domain $\{z : |z - z_0| < R\}$, where $z_0 \in \mathbb{C}$ and $R > 0$ are fixed. Then the series obtained through term-by-term differentiation of the Taylor series (3) of $f(z)$ converges uniformly to $f'(z)$ in any closed disc $\{z : |z - z_0| \le r\}$, where $r < R$. Furthermore, the differentiated series is the Taylor series of $f'(z)$.*

PROOF. Since f is analytic in the domain $\{z : |z - z_0| < R\}$, it follows from Theorem 6B that f' is also analytic in $\{z : |z - z_0| < R\}$. By Theorem 7A, the function f' has its Taylor series

$$f'(z) = \sum_{n=0}^{\infty} \frac{f^{(n+1)}(z_0)}{n!}(z - z_0)^n = \sum_{n=1}^{\infty} \frac{f^{(n)}(z_0)}{n!} n(z - z_0)^{n-1},$$

the same series as obtained through term-by-term differentiation of the Taylor series of f. The uniform convergence of this series to $f'(z)$ in $\{z : |z - z_0| \le r\}$ follows from Theorem 7A. ◯

EXAMPLE 7.2.1. The function $g(z) = \mathrm{e}^z$ is entire. Also, for every $n \in \mathbb{N}$, we have $g^{(n)}(z) = \mathrm{e}^z$, so that $g^{(n)}(0) = 1$. It now follows from Theorem 7B that

$$\mathrm{e}^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \ldots$$

uniformly in $\{z : |z| \le r\}$ for every $r > 0$.

EXAMPLE 7.2.2. The function $g(z) = \sin z$ is entire. Also, for every $n \in \mathbb{N}$, we have

$$g^{(n)}(z) = \begin{cases} \cos z & \text{if } n = 1, 5, 9, \ldots, \\ -\sin z & \text{if } n = 2, 6, 10, \ldots, \\ -\cos z & \text{if } n = 3, 7, 11, \ldots, \\ \sin z & \text{if } n = 4, 8, 12, \ldots, \end{cases}$$

so that

$$g^{(n)}(0) = \begin{cases} 1 & \text{if } n = 1, 5, 9, \ldots, \\ 0 & \text{if } n = 2, 6, 10, \ldots, \\ -1 & \text{if } n = 3, 7, 11, \ldots, \\ 0 & \text{if } n = 4, 8, 12, \ldots. \end{cases}$$

It now follows from Theorem 7B that

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots$$

uniformly in $\{z : |z| \leq r\}$ for every $r > 0$. Applying Theorem 7D and differentiating term-by-term, we obtain

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots$$

uniformly in $\{z : |z| \leq r\}$ for every $r > 0$.

Example 7.2.3. Consider the function

$$h(z) = \frac{\sin z}{z}.$$

From Example 7.2.2 and Theorem 7C, we have

$$\sin z = z + z^2 g_2(z),$$

where the function $g_2(z)$ is entire. It follows that for $z \neq 0$, we have

$$\frac{\sin z}{z} = 1 + z g_2(z),$$

and so $h(z) \to 1$ as $z \to 0$. Note that the function $h(z)$ is analytic at any $z \neq 0$. If we define $h(0) = 1$, then the function h is continuous at 0. We say that h has a removable singularity at 0.

Example 7.2.4. Consider the function

$$k(z) = \frac{1 - \cos z}{z^2}.$$

From Example 7.2.2 and Theorem 7C, we have

$$\cos z = 1 - \frac{z^2}{2} + z^3 g_3(z),$$

where the function $g_3(z)$ is entire. It follows that for $z \neq 0$, we have

$$\frac{1 - \cos z}{z^2} = \frac{1}{2} - z g_3(z),$$

and so $k(z) \to 1/2$ as $z \to 0$. Note that the function $k(z)$ is analytic at any $z \neq 0$. If we define $k(0) = 1/2$, then the function k is continuous at 0.

7.3. Uniqueness

Recall the Cauchy integral formula as given by Theorem 6A. To determine the value of an analytic function at interior points of a disc, we need the values of the function on the boundary of the disc.

On the other hand, if we know the values of $f(z_0), f'(z_0), f''(z_0), \ldots$ of an analytic function f at a point z_0 in a domain D, then the Taylor series determines $f(z)$ in some disc $\{z : |z - z_0| < R\}$ centred at z_0. It follows that if $f(z)$ is known in some infinitely differentiable short arc in the disc $\{z : |z - z_0| < R\}$, then $f(z)$ is uniquely determined in the disc $\{z : |z - z_0| < R\}$, since the derivatives of $f(z)$ can be calculated by differentiation of $f(z)$ on this arc.

The purpose of this section is to extend this second observation to show that an analytic function f is uniquely determined in a domain D, and not just a disc centred at z_0, by the values of $f^{(n)}(z_0)$ at some z_0 in D. We shall also show that an analytic function f is determined in a domain D by the values on a short continuous curve C in D.

THEOREM 7E. *Suppose that two functions f and g are analytic in a domain D. Suppose further that $z_0 \in D$, and that $f(z) = g(z)$ in some ϵ-neighbourhood of z_0. Then $f(z) = g(z)$ for every $z \in D$.*

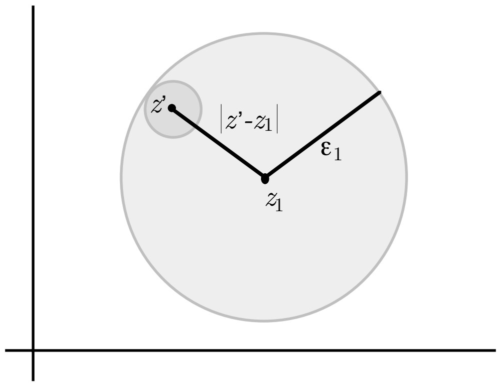
REMARK. The proof is based on the following argument: A domain D is open and connected, and therefore cannot be written as a disjoint union of two non-empty open sets.

PROOF OF THEOREM 7E. For every $z \in D$, write $h(z) = f(z) - g(z)$. Then the function h is analytic in D. Let

$$S_1 = \{z_1 \in D : h(z) = 0 \text{ in some neighbourhood of } z_1\} \qquad \text{and} \qquad S_2 = D \setminus S_1.$$

To prove the theorem, it clearly suffices to show that $S_1 = D$.

Suppose that $z_1 \in S_1$. Then there exists $\epsilon_1 > 0$ such that $h(z) = 0$ in $\{z : |z - z_1| < \epsilon_1\}$. Suppose now that $|z' - z_1| < \epsilon_1$.



Then clearly

$$\{z : |z - z'| < \epsilon_1 - |z' - z_1|\} \subseteq \{z : |z - z_1| < \epsilon_1\},$$

and so it follows that $h(z) = 0$ in the neighbourhood $\{z : |z - z'| < \epsilon_1 - |z' - z_1|\}$ of z'. This shows that $z' \in S_1$ whenever $|z' - z| < \epsilon_1$. It follows that $\{z : |z - z_1| < \epsilon_1\} \subseteq S_1$, so that S_1 is open.

Suppose next that $z_2 \in S_2$. Since $z_2 \in D$, there exists $R > 0$ such that the disc $\{z : |z - z_2| < R\}$ is contained in D and so it follows from Theorem 7A that the Taylor series expansion

$$h(z) = \sum_{n=0}^{\infty} \frac{h^{(n)}(z_2)}{n!}(z - z_2)^n$$

is valid in the disc $\{z : |z - z_2| \le r\}$ for every $r < R$. Since h is not identically zero in this latter disc, there exists a smallest n such that $h^{(n)}(z_2) \ne 0$. By Theorem 7C, we can then write

$$h(z) = h_n(z)(z - z_2)^n,$$

where $h_n(z)$ is analytic in the disc $\{z : |z - z_2| < R\}$ and

$$h_n(z) \to \frac{h^{(n)}(z_2)}{n!} \neq 0$$

as $z \to z_2$. It follows from continuity of h_n that there exists $\epsilon_2 > 0$ such that $h(z) \neq 0$ in the punctured disc $\{z : 0 < |z - z_2| < \epsilon_2\}$, and so $\{z : |z - z_2| < \epsilon_2\} \subseteq S_2$. Hence S_2 is open.

Clearly $S_1 \neq \emptyset$, since $z_0 \in S_1$. Suppose, on the contrary, that $S_1 \neq D$. Then the two open sets S_1 and S_2 are both non-empty. Clearly

$$S_1 \cup S_2 = D \qquad \text{and} \qquad S_1 \cap S_2 = \emptyset.$$

In view of our earlier remark, this is absurd. Hence we must have $S_2 = \emptyset$, and so $S_1 = D$. ○

In fact, a slight elaboration of the ideas in part of the above proof gives the following two results.

THEOREM 7F. *Suppose that a function f is analytic in the domain D. Suppose further that $z_0 \in D$, and that $f(z_0) = 0$. Then either $f(z)$ is identically zero in D or else there exists $n \in \mathbb{N}$ such that*

$$f(z) = (z - z_0)^n g(z),$$

where the function g is analytic in D, and

$$g(z_0) = \frac{f^{(n)}(z_0)}{n!} \neq 0.$$

DEFINITION. If the latter conclusion of Theorem 7F holds, then we say that the function has a zero of order n at z_0. Furthermore, if $n = 1$, then we say that the function f has a simple zero at z_0.

THEOREM 7G. *Suppose that a function f is analytic in the domain D. Suppose further that $f(z)$ is not identically zero in D. Then for every $z_0 \in D$ such that $f(z_0) = 0$, there exists $\epsilon > 0$ such that $f(z) \neq 0$ for every $0 < |z - z_0| < \epsilon$. In other words, the zeros of f are isolated.*

PROOF OF THEOREM 7F. Clearly there exists $R > 0$ such that the disc $\{z : |z - z_0| < R\}$ is contained in D. Suppose that $f^{(n)}(z_0) = 0$ for every $n \in \mathbb{N}$. Then it follows from Theorem 7A that $f(z)$ is identically zero in this disc. Let the function g be identically zero in D. Then $f(z) = g(z)$ in some neighbourhood of z_0. It now follows from Theorem 7E that $f(z) = 0$ for every $z \in D$. Suppose next that $f^{(k)}(z_0) \neq 0$ for some $k \in \mathbb{N}$. Then there exists a smallest $k \in \mathbb{N}$ such that $f^{(k)}(z_0) \neq 0$. Denote this value of k by n. By Theorem 7C, we can then write

$$f(z) = f_n(z)(z - z_0)^n,$$

where $f_n(z)$ is analytic in the disc $\{z : |z - z_0| < R\}$ and

$$f_n(z) \to \frac{f^{(n)}(z_0)}{n!} \neq 0$$

as $z \to z_0$. We now define $g(z) = f_n(z)$ in this disc, and by $g(z) = f(z)(z - z_0)^{-n}$ in the remainder of D to complete the proof. ○

PROOF OF THEOREM 7G. It follows from Theorem 7F that there exists $n \in \mathbb{N}$ and an analytic function g in D such that $f(z) = (z - z_0)^n g(z)$, where $g(z_0) \neq 0$. It follows from the continuity of g that there exists $\epsilon > 0$ such that $g(z) \neq 0$ if $|z - z_0| < \epsilon$. The result follows immediately. ○

Example 7.3.1. Suppose that two functions f and g are analytic in a domain D and not identically zero in D. Suppose further that $z_0 \in D$, and that $f(z_0) = g(z_0) = 0$. Then by Theorem 7F, there exist $m, n \in \mathbb{N}$ and functions F and G analytic in D and satisfying $F(z_0) \neq 0$ and $G(z_0) \neq 0$ such that

$$f(z) = (z - z_0)^m F(z) \qquad \text{and} \qquad g(z) = (z - z_0)^n G(z).$$

Then

$$\frac{f(z)}{g(z)} = (z - z_0)^k \frac{F(z)}{G(z)} \qquad \text{and} \qquad \frac{f'(z)}{g'(z)} = (z - z_0)^k \frac{mF(z) + (z - z_0)F'(z)}{nG(z) + (z - z_0)G'(z)},$$

where $k = m - n$. Consider now the special case $m = n = 1$, so that $k = 0$. We have

$$\lim_{z \to z_0} \frac{f'(z)}{g'(z)} = \lim_{z \to z_0} \frac{F(z) + (z - z_0)F'(z)}{G(z) + (z - z_0)G'(z)} = \lim_{z \to z_0} \frac{F(z)}{G(z)} = \lim_{z \to z_0} \frac{f(z)}{g(z)}.$$

This is l'Hopital's rule.

We complete this section by proving the following result, which shows that an analytic function is determined in a domain D by the values on a short continuous curve in D.

THEOREM 7H. *Suppose that a function f is analytic in the domain D. Suppose further that z_n is a sequence of distinct points having a limit $z_0 \in D$, and that $f(z_n) = g(z_n)$ for every $n \in \mathbb{N}$. Then $f(z) = g(z)$ for every $z \in D$.*

Proof. For every $z \in D$, write $h(z) = f(z) - g(z)$. Then the function h is analytic in D. Furthermore, it follows from continuity that $h(z_0) = 0$. Since $h(z_n) = 0$ for every $n \in \mathbb{N}$ and $z_n \to z_0$ as $n \to \infty$, the zero z_0 of the function h is not isolated. It follows from Theorem 7G that $h(z)$ is identically zero in D. ◯

7.4. The Maximum Principle

Let us return to Cauchy's integral formula, as given by Theorem 6A. If we take z to be the centre α of the circle C, then (1) in Chapter 6 gives

$$f(\alpha) = \frac{1}{2\pi \mathrm{i}} \int_C \frac{f(\zeta)}{\zeta - \alpha} \, \mathrm{d}\zeta, \tag{10}$$

so that

$$|f(\alpha)| \le \max_{\zeta \in C} |f(\zeta)|, \tag{11}$$

in view of Theorem 4B. In other words, the modulus of an analytic function at a point in a domain never exceeds the maximum modulus of the function on the boundary of any disc centred at that point and contained in the domain.

In this section, we shall establish the following stronger result.

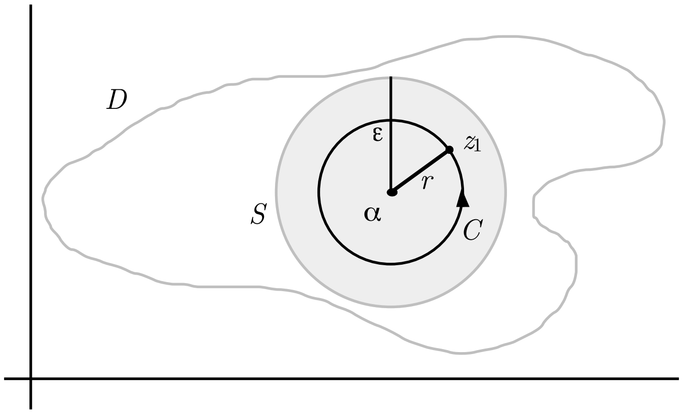
THEOREM 7J. (MAXIMUM PRINCIPLE) *Suppose that a function f is analytic in a domain D. Then $|f(z)|$ cannot have a maximum anywhere in D unless $f(z)$ is constant in D.*

Proof. Suppose on the contrary that there exists $\alpha \in D$ such that

$$|f(\alpha)| \ge |f(z)| \tag{12}$$

for every $z \in D$. Since D is open, there exists an ϵ-neighbourhood S of α which is contained in D. If $|f(z)| = |f(\alpha)|$ for every $z \in S$, then it follows from Example 3.3.5 that $f(z)$ is constant in S, and so constant in D by Theorem 7E. We may therefore assume that there exists $z_1 \in S$ such that

$$|f(z_1)| < |f(\alpha)|. \tag{13}$$



Let $|z_1 - \alpha| = r$. Clearly $r < \epsilon$. If we denote by C the circle in the positive (anticlockwise) direction centred at α and with radius r, then (10) holds. Furthermore, writing $\zeta = \alpha + r\mathrm{e}^{\mathrm{i}t}$, we have

$$f(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} f(\alpha + r\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}t,$$

so that

$$|f(\alpha)| \le \frac{1}{2\pi} \int_0^{2\pi} |f(\alpha + r\mathrm{e}^{\mathrm{i}t})|\,\mathrm{d}t. \tag{14}$$

On the other hand, it clearly follows from (12) that

$$|f(\alpha)| \ge \frac{1}{2\pi} \int_0^{2\pi} |f(\alpha + r\mathrm{e}^{\mathrm{i}t})|\,\mathrm{d}t. \tag{15}$$

However, note that $z_1 = \alpha + r\mathrm{e}^{\mathrm{i}t_1}$ for some $t_1 \in [0, 2\pi]$. It follows from continuity that there exists an interval $I \subseteq [0, 2\pi]$ for which $|f(\alpha + r\mathrm{e}^{\mathrm{i}t})| < |f(\alpha)|$ for every $t \in I$. It follows that equality cannot hold in (15), so that

$$|f(\alpha)| > \frac{1}{2\pi} \int_0^{2\pi} |f(\alpha + r\mathrm{e}^{\mathrm{i}t})|\,\mathrm{d}t. \tag{16}$$

Note now that (14) and (16) contradict each other, and this concludes the proof of the theorem. ◯

The following alternative form of the Maximum principle is perhaps more useful.

THEOREM 7K. *Suppose that a function f is analytic in a bounded domain D, and continuous in the closed region $\overline{D}$. Then $|f(z)|$ attains its maximum on the boundary of D.*

Proof. It is well known from real analysis that $|f(z)|$ assumes its maximum somewhere in the closed bounded region $\overline{D}$. By Theorem 7J, this maximum cannot be attained in D, and so must be attained on the boundary of D. ◯

Problems for Chapter 7

1. Obtain the Taylor series for the function $(1-z)^{-1}$ at $z=0$. Deduce from this the Taylor series for the function $(1-z)^{-2}$ at $z=0$. In what open discs centred at $z=0$ are these series valid?

2. Suppose that a function $f(z)$ is analytic in the disc $\{z : |z| < R\}$, where $R>0$ is fixed, and has Taylor series
$$f(z)=\sum_{n=0}^{\infty} a_n z^n.$$
   a) Show that
$$\int_0^z f(\zeta)\,\mathrm{d}\zeta=\sum_{n=0}^{\infty}\frac{a_n}{n+1}z^{n+1}$$
   uniformly in $\{z : |z| \le r\}$ for every $r<R$.
   b) Denote the integral by $F(z)$. Explain why the series in (a) is the Taylor series for $F(z)$.

3. Deduce from the Taylor series for $(1-z)^{-1}$ at $z=0$ in Problem 1 the Taylor series for the function $\log(1-z)$ at $z=0$, where $\log 1=0$. In what open discs centred at $z=0$ is this series valid?

4. Suppose that $\alpha\in\mathbb{C}$ is fixed. By interpreting the function $(1+z)^\alpha$ as $\mathrm{e}^{\alpha\log(1+z)}$, with $\log 1=0$, show that
$$(1+z)^\alpha=1+\alpha z+\frac{\alpha(\alpha-1)}{2!}z^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}z^3+\ldots$$
   uniformly in $\{z : |z| \le r\}$ for every $r<1$.

5. a) Show that the function $f(z)$, defined by $f(0)=1$ and $f(z)=z^{-1}\sin z$ when $z\neq 0$, is entire.
   b) Obtain the Taylor series for $f(z)$ at $z=0$.
   c) Obtain the Taylor series for the integral $\displaystyle\int_0^z f(\zeta)\,\mathrm{d}\zeta$ at $z=0$.

6. Suppose that $P(z)$ is a polynomial of degree at most 3. Using partial fractions if necessary, find a_0, a_1 and a_2 such that
$$\frac{P(z)}{(z^2+1)(z-1)(z-2)}=a_0+a_1z+a_2z^2+\ldots$$
   valid whenever $|z|<1$.

7. Suppose that a power series
$$\sum_{n=0}^{\infty} a_n z^n$$
   converges uniformly to an analytic function $f(z)$ in the disc $D=\{z : |z|\le R\}$, where $R>0$ is fixed. For every $N\in\mathbb{N}$ and $z\in D$, let
$$s_N(z)=\sum_{n=0}^{N} a_n z^n.$$
   Then given any $\epsilon>0$, there exists N_0 such that $|f(z)-s_N(z)|<\epsilon$ for every $N>N_0$ and $z\in D$.

a) Show that for every $k \in \mathbb{N} \cup \{0\}$ and every $N > N_0$,

$$\left| \int_C (f(z) - S_N(z)) z^{-k-1} \, \mathrm{d}z \right| \leq \frac{2\pi\epsilon}{R^k},$$

where $C = \{z : |z| = R\}$, followed in the positive (anticlockwise) direction.

b) Now let $N > k$. Show that

$$\left| \frac{1}{2\pi \mathrm{i}} \int_C \frac{f(z)}{z^{k+1}} \, \mathrm{d}z - a_k \right| \leq \frac{\epsilon}{R^k}.$$

c) Deduce that $a_k = f^{(k)}(0)/k!$ for every $k \in \mathbb{N} \cup \{0\}$.

[Remark: This shows that if a power series converges uniformly to an analytic function $f(z)$ in D, then it is the Taylor series for $f(z)$.]

8. Suppose that two functions $f(z)$ and $g(z)$ are analytic in the disc $D = \{z : |z| < R\}$, where $R > 0$ is fixed, with Taylor series

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots \qquad \text{and} \qquad g(z) = b_0 + b_1 z + b_2 z^2 + \ldots$$

respectively.

a) Without worrying about convergence problems, multiply the two series together to obtain another power series $c_0 + c_1 z + c_2 z^2 + \ldots$. Check that $c_n = a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0$ for every $n \in \mathbb{N} \cup \{0\}$.

b) Suppose that $f(z)g(z)$ has a Taylor series. Show that the coefficient of the term z^n in the Taylor series is given by the value at $z = 0$ of the function

$$\frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}z^n} (f(z)g(z)).$$

c) Explain why the power series in (a) is the Taylor series for $f(z)g(z)$.

9. Suppose that two functions $f(z)$ and $g(z)$ are analytic in a bounded region D and continuous in $\overline{D}$. Suppose further that $f(z) = g(z)$ for every z on the boundary of D. Show from the Maximum principle that $f(z) = g(z)$ for every $z \in D$.

10. Suppose that a function $f(z)$ is analytic in a closed bounded region D. Suppose further that $f(z) \neq 0$ for any $z \in D$. Show that $|f(z)|$ assumes its minimum value on the boundary of D.

11. Using l'Hopital's rule, or otherwise, evaluate each of the following limits:

a) $\displaystyle\lim_{z \to \pi} \frac{\sin z}{\pi - z}$ b) $\displaystyle\lim_{z \to \mathrm{i}} \frac{\mathrm{e}^{\pi z} + 1}{z^2 + 1}$

12. Suppose that $f(z)$ is analytic in the disc $D = \{z : |z| \leq R\}$, where $R > 0$ is fixed. Suppose further that $f(0) = 0$ and $|f(z)| \leq M$ whenever $|z| = R$.

a) Explain why $f(z) = z g(z)$ for some function $g(z)$ analytic in D.

b) By applying the Maximum principle on the function $g(z)$, prove Schwarz's lemma, that

$$|f(z)| < \frac{M}{R} |z|$$

whenever $0 < |z| < R$, unless $f(z) = cz$ for some constant $c \in \mathbb{C}$.

13. Suppose that C is a contour of length L. Suppose further that a function $f(z)$ is continuous on C, and $|f(z)| \leq M$ for every $z \in C$. Show that, unless $|f(z)| = M$ for every $z \in C$, we have

$$\left| \int_C f(z) \, \mathrm{d}z \right| < ML.$$