INTRODUCTION TO COMPLEX ANALYSIS

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Chapter 8

ISOLATED SINGULARITIES AND LAURENT SERIES

8.1. Removable Singularities

Suppose that a function f is analytic in the punctured disc $\{z : 0 < |z - z_0| < R\}$. Observe that it is not necessary for f to be defined at the point z_0. We say that the function f has an isolated singularity at z_0. Our purpose is to show that there are only three possible ways in which $f(z)$ can behave in a punctured neighbourhood of z_0. To illustrate the first of these, let us first consider the following examples.

Example 8.1.1. The function

$$f(z) = \frac{\sin z}{z}$$

is analytic in the punctured disc $\{z : 0 < |z| < R\}$. However, the quotient is not defined at $z = 0$. However, note that the function $\sin z$ is entire. By Theorem 7C, we can write

$$\sin z = z + z^3 g(z),$$

where g is an entire function. It follows that for $z \neq 0$, we have

$$f(z) = \frac{\sin z}{z} = 1 + z^2 g(z).$$

Note that the function $1 + z^2 g(z)$ is entire. It follows that if we make the further definition $f(0) = 1$, then f is now analytic at $z = 0$, and we have removed the isolated singularity.

EXAMPLE 8.1.2. Suppose that a function f is analytic in a domain D, and that $z_0 \in D$. We define the function g in D by writing

$$g(z_0) = f'(z_0), \tag{1}$$

and writing

$$g(z) = \frac{f(z) - f(z_0)}{z - z_0} \tag{2}$$

if $z \neq z_0$. It is easily seen from Theorem 7C that g is analytic in D. However, note that the function g, defined by (2), is analytic in the domain $D \setminus \{z_0\}$. It also has an isolated singularity at z_0, which is removed by the definition (1).

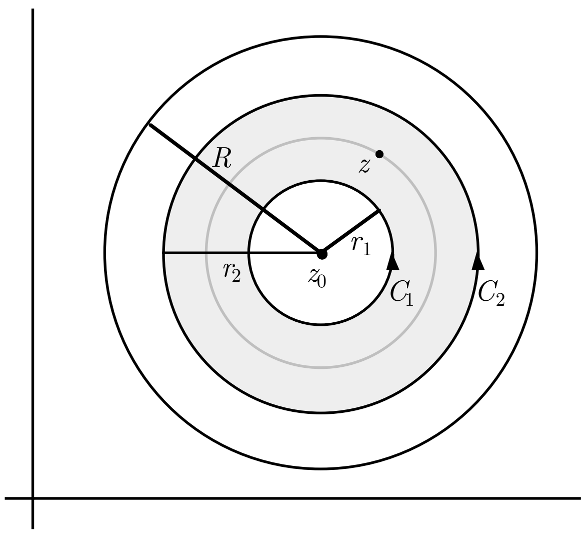
DEFINITION. Suppose that a function f is analytic in the punctured disc $\{z : 0 < |z - z_0| < R\}$. Suppose further that by assigning a suitable value for $f(z_0)$, the function f can be made to be analytic in the disc $\{z : |z - z_0| < R\}$. Then we say that f has a removable singularity at z_0.

THEOREM 8A. (RIEMANN'S THEOREM ON REMOVABLE SINGULARITIES) *Suppose that a function f is analytic in the punctured disc $\{z : 0 < |z - z_0| < R\}$. Suppose further that*

$$\lim_{z \to z_0} (z - z_0) f(z) = 0. \tag{3}$$

Then f has a removable singularity at z_0.

PROOF. Suppose that z is a point in the punctured disc $\{z : 0 < |z - z_0| < R\}$. Then $0 < |z - z_0| < R$. Let r_1 and r_2 satisfy $0 < r_1 < |z - z_0| < r_2 < R$, and let C_1 and C_2 denote two circles in the positive (anticlockwise) direction, centred at z_0, and of radius r_1 and r_2 respectively.



The function g, defined by $g(z) = f'(z)$ and for $\zeta \neq z$ by

$$g(\zeta) = \frac{f(\zeta) - f(z)}{\zeta - z}, \tag{4}$$

is clearly analytic in the punctured disk $\{\zeta : 0 < |\zeta - z_0| < R\}$. Then it can be shown, as in the proof of Theorem 6A, that

$$\int_{C_1} g(\zeta) \,\mathrm{d}\zeta = \int_{C_2} g(\zeta) \,\mathrm{d}\zeta.$$

Combining this with (4), we have

$$\int_{C_1} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta - f(z)\int_{C_1} \frac{\mathrm{d}\zeta}{\zeta - z} = \int_{C_2} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta - f(z)\int_{C_2} \frac{\mathrm{d}\zeta}{\zeta - z}. \tag{5}$$

Note now that the function

$$\frac{1}{\zeta - z}$$

is analytic in the star domain $\{\zeta : |\zeta - z_0| < |z - z_0|\}$ which contains the contour C_1. It follows that

$$\int_{C_1} \frac{\mathrm{d}\zeta}{\zeta - z} = 0. \tag{6}$$

On the other hand, by Cauchy's integral formula as given by Theorem 6A, we have

$$\int_{C_2} \frac{\mathrm{d}\zeta}{\zeta - z} = 2\pi\mathrm{i}. \tag{7}$$

Furthermore, in view of the condition (3), we have, given any $\epsilon > 0$, there exists $\delta > 0$ such that $|(\zeta - z_0)f(\zeta)| < \epsilon$ whenever $|\zeta - z_0| < \delta$. Without loss of generality, we may assume that

$$\delta < \frac{1}{2}|z - z_0|. \tag{8}$$

If we now take $r_1 = \delta$, then

$$\left|\int_{C_1} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta\right| = \left|\int_{C_1} \frac{(\zeta - z_0)f(\zeta)}{(\zeta - z_0)(\zeta - z)}\,\mathrm{d}\zeta\right| \le \frac{\epsilon}{\delta(|z - z_0| - \delta)}2\pi\delta = \frac{2\pi\epsilon}{|z - z_0| - \delta} \le \frac{4\pi\epsilon}{|z - z_0|},$$

in view of Theorem 4B and (8). Since $\epsilon > 0$ is arbitrary, we conclude that

$$\int_{C_1} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = 0. \tag{9}$$

Combining (5)–(7) and (9), we obtain

$$f(z) = \frac{1}{2\pi\mathrm{i}}\int_{C_2} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta. \tag{10}$$

Note now that (10) holds for every z in the punctured disc $\{z : 0 < |z - z_0| < r_2\}$. Note also that the integral on the right hand side of (10) represents an analytic function in the disc $\{z : |z - z_0| < r_2\}$ (see the proof of Theorem 6B). It follows that if we define

$$f(z_0) = \frac{1}{2\pi\mathrm{i}}\int_{C_2} \frac{f(\zeta)}{\zeta - z_0}\,\mathrm{d}\zeta,$$

then the function f is analytic in the disc $\{z : |z - z_0| < r_2\}$. ○

REMARKS. (1) Note that condition (3) will be satisfied if $f(z)$ is continuous at z_0, or if $|f(z)|$ is bounded.

(2) Since an analytic function is continuous, it follows that removable singularities at z_0 can be overcome by defining

$$f(z_0) = \lim_{z \to z_0} f(z).$$

8.2. Poles

Definition. Suppose that a function f is analytic in the punctured disc $\{z : 0 < |z - z_0| < R\}$. Suppose further that

$$f(z) = \frac{g(z)}{(z - z_0)^n}, \tag{11}$$

where $n \in \mathbb{N}$ and the function g is analytic in some neighbourhood of z_0, with $g(z_0) \neq 0$. Then we say that f has a pole of order n at z_0. Furthermore, if $n = 1$, then we say that f has a simple pole at z_0.

THEOREM 8B. *Suppose that a function f is analytic in the punctured disc $\{z : 0 < |z - z_0| < R\}$. Then f has a pole at z_0 if and only if*

$$\lim_{z \to z_0} |f(z)| = \infty; \tag{12}$$

in other words, given any $E > 0$, there exists $\delta > 0$ such that $|f(z)| > E$ whenever $0 < |z - z_0| < \delta$.

Proof. Note first of all that (12) follows immediately from (11), since $g(z_0) \neq 0$. Suppose now that (12) holds. Then $f(z) \neq 0$ in some punctured disc $\{z : 0 < |z - z_0| < r\}$, where $r \leq R$. It follows that the function

$$F(z) = \frac{1}{f(z)}$$

is analytic in $\{z : 0 < |z - z_0| < r\}$, and has an isolated singularity at z_0. On the other hand, it follows from (12) that $F(z) \to 0$ as $z \to z_0$. Hence by Theorem 8A, F has a removable singularity at z_0. If we define $F(z_0) = 0$, then F is now analytic in the disc $\{z : |z - z_0| < r\}$. Clearly $F(z)$ is not identically zero in $\{z : |z - z_0| < r\}$. It follows from Theorem 7F that there exists $n \in \mathbb{N}$ such that

$$F(z) = (z - z_0)^n h(z),$$

where the function h is analytic in $\{z : |z - z_0| < r\}$, with $h(z_0) \neq 0$. Hence

$$g(z) = \frac{1}{h(z)}$$

is analytic in some neighbourhood of z_0, and (11) holds. Clearly $g(z_0) \neq 0$. ○

Remark. Note that a function f has a pole of order n at z_0 if and only if the function $1/f$ has a zero of order n at z_0.

8.3. Essential Singularities

Definition. Suppose that a function f is analytic in the punctured disc $\{z : 0 < |z - z_0| < R\}$. Suppose further that the isolated singularity at z_0 is neither removable nor a pole. Then we say that f has an essential singularity at z_0.

Example 8.3.1. The function $e^{1/z}$ is analytic at every $z \neq 0$. It has an isolated singularity at $z = 0$. Let us restrict z to be real numbers, and consider $e^{1/x}$, where $x > 0$. Clearly

$$\lim_{x \to 0+} e^{1/x} = \lim_{y \to +\infty} e^y = \infty,$$

so that the singularity is not removable. On the other hand, for every $n \in \mathbb{N}$,

$$\lim_{x \to 0+} x^n e^{1/x} = \lim_{y \to +\infty} \frac{e^y}{y^n} = \infty,$$

so that the singularity is not a pole of order n. Hence $e^{1/z}$ has an essential singularity at $z = 0$.

To illustrate the wild behaviour of an analytic function near an essential singularity, we mention Picard's theorem that such a function assumes all values except possibly one in any neighbourhood of an essential singularity. The following result is somewhat weaker, and shows that such a function comes arbitrarily close to any given complex number in any neighbourhood of an essential singularity.

THEOREM 8C. (CASORATI-WEIERSTRASS) *Suppose that a function f is analytic in the punctured disc $\{z : 0 < |z - z_0| < R\}$, with an essential singularity at z_0. Then given any $w \in \mathbb{C}$ and any real numbers $\epsilon > 0$ and $\delta > 0$, there exists z in the punctured disc satisfying*

$$0 < |z - z_0| < \delta \qquad \textit{and} \qquad |f(z) - w| < \epsilon.$$

PROOF. Suppose on the contrary that the conclusion does not hold. Then there exist $w \in \mathbb{C}$ and real numbers $\epsilon > 0$ and $\delta > 0$ such that $|f(z) - w| \geq \epsilon$ whenever $0 < |z - z_0| < \delta$. It follows that the function

$$g(z) = \frac{1}{f(z) - w}$$

is analytic and bounded in the punctured disc $\{z : 0 < |z - z_0| < \delta\}$, with an isolated singularity at z_0 which is removable, in view of Theorem 8A. It follows that by defining $g(z_0)$ appropriately, the function g is analytic in the disc $\{z : |z - z_0| < \delta\}$. On the other hand, the function g is clearly not identically zero in $\{z : |z - z_0| < \delta\}$. Furthermore, note that

$$f(z) = w + \frac{1}{g(z)}.$$

If $g(z_0) \neq 0$, then f is analytic at z_0. If $g(z_0) = 0$, then f has a pole at z_0. In either case, the conclusion contradicts the assumption that f has an essential singularity at z_0, and this completes the proof. ○

8.4. Isolated Singularities at Infinity

The behaviour of a function $f(z)$ at $z = \infty$ can be studied via the behaviour of the function $f(1/\zeta)$ at $\zeta = 0$. A punctured neighbourhood $\{\zeta : 0 < |\zeta| < R^{-1}\}$ of 0 then plays the same role as the "punctured" neighbourhood $\{z : R < |z| < \infty\}$ of ∞.

Suppose now that a function $f(z)$ is analytic in the domain $\{z : R < |z| < \infty\}$. Then by using $z = 1/\zeta$ and considering $\zeta = 0$, we see that the function $f(z)$ has an isolated singularity at $z = \infty$. This may be a removable singularity, a pole or an essential singularity.

Corresponding to Theorem 8A, suppose that $|f(z)/z| \to 0$ as $|z| \to \infty$. Then the singularity is removable by defining $f(\infty)$ suitably to make $f(z)$ continuous at $z = \infty$. In other words, we need to define

$$f(\infty) = \lim_{\zeta \to 0} f\left(\frac{1}{\zeta}\right).$$

In the special case that $f(\infty) = 0$, then we say that f has a zero at $z = \infty$. Furthermore, if f is not identically zero, then, corresponding to Theorem 7F, there exists $n \in \mathbb{N}$ such that

$$f(z) = \frac{h(z)}{z^n},$$

where $h(z)$ is analytic in $\{z : R < |z| < \infty\}$, and $h(\infty) \neq 0$. In this case, we say that f has a zero of order n at $z = \infty$.

Corresponding to Theorem 8B, suppose that $|f(z)| \to \infty$ as $|z| \to \infty$. Then f has a pole at $z = \infty$, and there exists $n \in \mathbb{N}$ such that

$$f(z) = z^n h(z),$$

where $h(z)$ is analytic in $\{z : R < |z| < \infty\}$, and $h(\infty) \neq 0$. In this case, we say that f has a pole of order n at $z = \infty$.

Corresponding to Theorem 8C, suppose that the isolated singularity at $z = \infty$ is neither removable nor a pole. Then it is an essential singularity. In this case, given any $w \in \mathbb{C}$ and any real numbers $\epsilon > 0$ and $N > 0$, there exists z in the domain $\{z : R < |z| < \infty\}$ satisfying

$$|z| > N \qquad \text{and} \qquad |f(z) - w| < \epsilon.$$

In other words, the function $f(z)$ comes arbitrarily close to any given complex number in any neighbourhood of $z = \infty$.

8.5. Further Examples

EXAMPLE 8.5.1. The function

$$f(z) = \frac{e^z - 1}{z(z-1)}$$

is analytic at every $z \in \mathbb{C}$ except for isolated singularities at $z = 0, 1$. At $z = 1$, it has a simple pole; note that we can write

$$f(z) = \frac{g(z)}{z-1} \qquad \text{with} \qquad g(z) = \frac{e^z - 1}{z},$$

and $g(1) \neq 0$. At $z = 0$, it has a removable singularity, since

$$\lim_{z\to 0} \frac{e^z - 1}{z(z-1)} = \lim_{z\to 0} \frac{e^z}{2z-1} = -1$$

by l'Hopital's rule. It follows that if we define $f(0) = -1$, then f is analytic at $z = 0$. The function $f(z)$ also has an isolated singularity at $z = \infty$. To study the isolated singularity at $z = \infty$, note first of all that

$$\lim_{|z|\to\infty} \frac{e^z - 1}{z(z-1)}$$

does not exist. To see this, note that

$$\lim_{x\to+\infty} \frac{e^x - 1}{x(x-1)} = +\infty \qquad \text{and} \qquad \lim_{x\to-\infty} \frac{e^x - 1}{x(x-1)} = 0.$$

Hence the singularity is not removable. Suppose next that $n \in \mathbb{N}$ is given and fixed. Then

$$h(z) = \frac{f(z)}{z^n} = \frac{e^z - 1}{z^{n+1}(z-1)}$$

is not analytic at $z = \infty$, since

$$\lim_{|z|\to\infty} \frac{e^z - 1}{z^{n+1}(z-1)}$$

does not exist. To see this, note that

$$\lim_{x\to+\infty} \frac{e^x - 1}{x^{n+1}(x-1)} = +\infty \qquad \text{and} \qquad \lim_{x\to-\infty} \frac{e^x - 1}{x^{n+1}(x-1)} = 0.$$

Hence the singularity is not a pole. It follows that $f(z)$ has an essential singularity at $z = \infty$.

Example 8.5.2. The function

$$f(z) = \frac{(z^2-4)(z-1)^4}{(\sin \pi z)^4}$$

is analytic at every $z \in \mathbb{C}$ except for isolated singularities at $z = 0, \pm 1, \pm 2, \ldots$, where the denominator vanishes. Note also that the numerator vanishes at $z = 1, \pm 2$. Note that the function $\sin \pi z$ has simple zeros at $z = 0, \pm 1, \pm 2, \ldots$. It follows that f has poles of order 4 at $z = 0, -1, \pm 3, \pm 4, \pm 5, \ldots$. Next, note that the function $(z^2-4)(z-1)^4$ has simple zeros at $z = \pm 2$. It follows that f has poles of order 3 at $z = \pm 2$. To study the isolated singularity at $z = 1$, note that by Theorem 7C, we have

$$\sin \pi z = -\pi(z-1) + g(z)(z-1)^2,$$

where g is entire. It follows that

$$\lim_{z\to 1} \frac{(z^2-4)(z-1)^4}{(\sin \pi z)^4} = \lim_{z\to 1} \frac{z^2-4}{(\pi - g(z)(z-1))^4} = -\frac{3}{\pi^4},$$

and so f has a removable singularity at $z = 1$. Finally, the singularity at $z = \infty$ is not isolated, since there does not exist any $R > 0$ such that the function $f(z)$ is analytic in the domain $\{z : R < |z| < \infty\}$.

8.6. Laurent Series

Example 8.6.1. Suppose that the function f is analytic in the punctured disc $\{z : 0 < |z - z_0| < R\}$, with a pole of order m at z_0. Then

$$f(z) = \frac{g(z)}{(z-z_0)^m},$$

where the function g is analytic in $\{z : |z - z_0| < R\}$, with $g(z_0) \neq 0$. By Theorem 7C, we have

$$g(z) = g(z_0) + g'(z_0)(z-z_0) + \frac{g''(z_0)}{2!}(z-z_0)^2 + \ldots + \frac{g^{(m-1)}(z_0)}{(m-1)!}(z-z_0)^{m-1} + g_m(z)(z-z_0)^m,$$

where $g_m(z)$ is analytic in the disc $\{z : |z - z_0| < R\}$. It follows that

$$f(z) = \frac{g(z_0)}{(z-z_0)^m} + \frac{g'(z_0)}{(z-z_0)^{m-1}} + \frac{g''(z_0)}{2!(z-z_0)^{m-2}} + \ldots + \frac{g^{(m-1)}(z_0)}{(m-1)!(z-z_0)} + g_m(z).$$

The expression

$$\frac{g(z_0)}{(z-z_0)^m}+\frac{g'(z_0)}{(z-z_0)^{m-1}}+\frac{g''(z_0)}{2!(z-z_0)^{m-2}}+\ldots+\frac{g^{(m-1)}(z_0)}{(m-1)!(z-z_0)}$$

is called the principal part of f at z_0. If we use Theorem 7A instead, then we can show that

$$f(z)=\sum_{n=-m}^{\infty}a_n(z-z_0)^n$$

for suitable choices of the coefficients a_n.

Our main task in this section is to generalize this example. The first step in this direction can be summarized by the following result.

THEOREM 8D. *Suppose that a function f is analytic in the punctured disc $\{z:0<|z-z_0|<R\}$, with an isolated singularity at z_0. Then there exist unique functions f_1 and f_2 such that*
(a) $f(z)=f_1(z)+f_2(z)$ in $\{z:0<|z-z_0|<R\}$,
(b) f_1 is analytic in $\mathbb{C}$ except possibly at z_0,
(c) $f_1(z)\to 0$ as $|z|\to\infty$, and
(d) f_2 is analytic in the disc $\{z:|z-z_0|<R\}$.

PROOF. We begin the proof in the same way as for Theorem 8A. Suppose that z is a point in the punctured disc $\{z:0<|z-z_0|<R\}$. Let r_1 and r_2 satisfy $0<r_1<|z-z_0|<r_2<R$, and let C_1 and C_2 denote two circles in the positive (anticlockwise) direction, centred at z_0, and of radius r_1 and r_2 respectively. On combining (5)–(7), we obtain

$$f(z)=\frac{1}{2\pi\mathrm{i}}\int_{C_2}\frac{f(\zeta)}{\zeta-z}\,\mathrm{d}\zeta-\frac{1}{2\pi\mathrm{i}}\int_{C_1}\frac{f(\zeta)}{\zeta-z}\,\mathrm{d}\zeta. \tag{13}$$

Write

$$f_1(z)=-\frac{1}{2\pi\mathrm{i}}\int_{C_1}\frac{f(\zeta)}{\zeta-z}\,\mathrm{d}\zeta \qquad\text{and}\qquad f_2(z)=\frac{1}{2\pi\mathrm{i}}\int_{C_2}\frac{f(\zeta)}{\zeta-z}\,\mathrm{d}\zeta. \tag{14}$$

Part (a) follows immediately. For part (d), note that the second integral in (14) represents an analytic function in the disc $\{z:|z-z_0|<r_2\}$ (as in the proof of Theorems 6B and 8A). For part (b), note that the first integral in (14) represents an analytic function in the annulus $\{z:|z-z_0|>r_1\}$ (similar to the proof of Theorem 6B). Note next that $f_2(z)$ and $f(z)$ are independent of the choice of r_1, so that it follows from (a) that $f_1(z)$ is also independent of the choice of r_1. Similarly, $f_2(z)$ is independent of the choice of r_2. It is easy to see that

$$\lim_{|z|\to\infty}\int_{C_1}\frac{f(\zeta)}{\zeta-z}\,\mathrm{d}\zeta=0.$$

Part (c) follows immediately. To show that the functions f_1 and f_2 are unique, suppose that g_1 and g_2 are functions having the same properties as f_1 and f_2 respectively. Then

$$f_1(z)-g_1(z)=g_2(z)-f_2(z)$$

in the punctured disc $\{z:0<|z-z_0|<R\}$. Let

$$F(z)=\begin{cases} g_2(z)-f_2(z) & \text{if } |z-z_0|<R,\\ f_1(z)-g_1(z) & \text{if } |z-z_0|>0.\end{cases}$$

Then F is entire. On the other hand, it follows from part (c) that $F(z)\to 0$ as $|z|\to\infty$. Hence F is bounded. It follows from Liouville's theorem that F is constant in $\mathbb{C}$, and so we must have $F(z)=0$ for every $z\in\mathbb{C}$. This completes the proof. ○

We can now state our generalization of Example 8.6.1.

THEOREM 8E. *Suppose that a function f is analytic in the punctured disc $\{z : 0 < |z - z_0| < R\}$, with an isolated singularity at z_0. For every $n \in \mathbb{Z}$, let*

$$a_n = \frac{1}{2\pi \mathrm{i}} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z, \tag{15}$$

where C is a circle in the positive (anticlockwise) direction centred at z_0 and of radius r, where $0 < r < R$. Then the series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \tag{16}$$

is convergent in the punctured disc $\{z : 0 < |z - z_0| < R\}$. Furthermore, this convergence is uniform in any annulus $\{z : r_1 < |z - z_0| < r_2\}$, where $0 < r_1 < r_2 < R$.

Remark. To say that the series converges uniformly to $f(z)$ in the annulus $\{z : r_1 < |z - z_0| < r_2\}$, we mean given any $\epsilon > 0$, there exists $N_0 = N_0(\epsilon, r_1, r_2)$, independent of the choice of z, such that

$$\left| f(z) - \sum_{n=-N_1}^{N_2} a_n (z - z_0)^n \right| < \epsilon$$

for every z in the annulus $\{z : r_1 < |z - z_0| < r_2\}$ whenever $N_1 > N_0$ and $N_2 > N_0$.

Definition. The series (16) is called the Laurent series for the function f at z_0.

Proof of Theorem 8E. The first step in our proof is to show that if the series in (16) converges to $f(z)$ uniformly on the circle C centred at z_0 and of radius r, where $0 < r < R$, then the coefficients a_n are given by (15). Suppose that $n \in \mathbb{Z}$ is chosen and fixed. For any $\epsilon > 0$, we can choose N_1 and N_2 so large that $-N_1 \le n \le N_2$ and

$$\left| f(z) - \sum_{j=-N_1}^{N_2} a_j (z - z_0)^j \right| < \epsilon$$

for every $z \in C$. Then it follows from Theorem 4B that

$$\left| \frac{1}{2\pi \mathrm{i}} \int_C \left(f(z) - \sum_{j=-N_1}^{N_2} a_j (z - z_0)^j \right) \frac{\mathrm{d}z}{(z - z_0)^{n+1}} \right| \le \frac{\epsilon}{r^n}. \tag{17}$$

Since

$$\frac{1}{2\pi \mathrm{i}} \int_C (z - z_0)^k \, \mathrm{d}z = \begin{cases} 1 & \text{if } k = -1, \\ 0 & \text{if } k \ne -1, \end{cases}$$

we have

$$\frac{1}{2\pi \mathrm{i}} \int_C \left(\sum_{j=-N_1}^{N_2} a_j (z - z_0)^j \right) \frac{\mathrm{d}z}{(z - z_0)^{n+1}} = a_n,$$

so that (17) can be simplified to

$$\left| \frac{1}{2\pi \mathrm{i}} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z - a_n \right| \le \frac{\epsilon}{r^n}.$$

Since $\epsilon > 0$ is arbitrary, (15) follows immediately. It now remains to show that $f(z)$ can be represented in the form (16) in the punctured disc $\{z : 0 < |z - z_0| < R\}$, and that the convergence is uniform in any annulus $\{z : r_1 < |z - z_0| < r_2\}$, where $0 < r_1 < r_2 < R$. Suppose that $0 < r_1 < r < r_2 < R$. Following Theorem 8D, we can write

$$f(z) = f_1(z) + f_2(z), \tag{18}$$

where $f_1(z)$ and $f_2(z)$ are uniquely determined and satisfy conditions (b)–(d) of Theorem 8D. Since f_2 is analytic in the disc $\{z : |z - z_0| < R\}$, it follows from Theorem 7A that the Taylor series

$$f_2(z) = \sum_{n=0}^{\infty} A_n (z - z_0)^n \tag{19}$$

converges in the disc $\{z : |z - z_0| < R\}$, uniformly in the closed disc $\{z : |z - z_0| \le r_2\}$. To study $f_1(z)$, write

$$w = \frac{1}{z - z_0} \qquad \text{or} \qquad z = \frac{1}{w} + z_0.$$

Then

$$f_1(z) = f_1\left(\frac{1}{w} + z_0\right)$$

is an entire function of w, and so it follows from Theorem 7A that the Taylor series

$$f_1\left(\frac{1}{w} + z_0\right) = \sum_{m=1}^{\infty} B_m w^m \tag{20}$$

converges in $\mathbb{C}$, uniformly in the closed disc $\{w : |w| \le 1/r_1\}$. Note that the constant term B_0 in the Taylor series is missing, since B_0 corresponds to the value of the function at $w = 0$, or $z = \infty$, and this is 0 in view of condition (c) in Theorem 8D. However, (20) is equivalent to saying that the series

$$f_1(z) = \sum_{m=1}^{\infty} B_m (z - z_0)^{-m} \tag{21}$$

converges in $\mathbb{C} \setminus \{0\}$, uniformly in $\{z : |z - z_0| \ge r_1\}$. The result now follows on combining (18), (19) and (21). ○

Definition. The series

$$f_1(z) = \sum_{n=-\infty}^{-1} a_n (z - z_0)^n,$$

where a_n is given by (15), is called the principal part of the function f at z_0.

The next result highlights the relationship between the principal part of a function and the nature of the isolated singularity.

THEOREM 8F. *Suppose that a function f is analytic in the punctured disc $\{z : 0 < |z - z_0| < R\}$, with an isolated singularity at z_0. Suppose further that the Laurent coefficients a_n are given by (15).*

(a) The function f either is analytic or has a removable singularity at z_0 if and only if $a_n = 0$ for every $n < 0$.

(b) The function f has a pole at z_0 if and only if a positive but finite number of coefficients a_n with $n < 0$ are non-zero.

(c) The function f has an essential singularity at z_0 if and only if an infinite number of coefficients a_n with $n < 0$ are non-zero.

PROOF. Note first of all that if f has a removable singularity at z_0, then f can be made analytic at z_0 by a suitable choice of $f(z_0)$. Part (a) now follows on observing that an analytic function has a Taylor series, and that a Laurent series with no principal part is a Taylor series. To prove part (b), note first of all that if a positive but finite number of coefficients a_n with $n < 0$ are non-zero, then there exists $m > 0$ such that $a_{-m} \neq 0$ but $a_n = 0$ for every $n < -m$. In this case, we have

$$f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n,$$

so that

$$f(z) = \frac{g(z)}{(z - z_0)^m},$$

where $m \in \mathbb{N}$ and the function g is analytic in some neighbourhood of z_0, with $g(z_0) = a_{-m} \neq 0$. This shows that f has a pole of order m at z_0. The converse is given in Example 8.6.1. Part (b) follows. Part (c) follows immediately from (a) and (b). ○

EXAMPLE 8.6.2. The observation that a Laurent series is unique enables us to use different methods to find the coefficients apart from the formula (15). Consider, for example, the function $e^{1/z}$. Using the substitution $z = 1/w$ on the Taylor series

$$e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!},$$

we obtain the Laurent series

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n! z^n} = \ldots + \frac{1}{3! z^3} + \frac{1}{2! z^2} + \frac{1}{z} + 1.$$

We conclude this chapter by making a remark on various equivalent definitions of analyticity in a domain D. The reader is advised to check the following theorem very carefully.

THEOREM 8G. *For any function f and any domain D, the following statements are equivalent:*
(a) $f(z)$ is analytic in D.
(b) $f(z)$ has continuous derivatives of all orders in D.
(c) $f'(z)$ exists and is continuous in D.
(d) $f'(z)$ exists in D.
(e) $f'(z)$ exists in D except possibly at a finite number of points in D, and $f(z)$ is continuous at these exceptional points.
(f) $f(z)$ can be represented uniformly by its Taylor series in the neighbourhood of every point in D.

PROBLEMS FOR CHAPTER 8

1. For each of the functions below, classify all the singular points in $\mathbb{C}$:

   a) $f(z) = e^z$ b) $f(z) = \dfrac{\cos z}{z}$ c) $f(z) = \dfrac{z^2+1}{z^2-1}$

   d) $f(z) = \dfrac{z^4}{z^3+z}$ e) $f(z) = \dfrac{z}{\cos z}$

2. Show that the principal parts of the function $f(z) = 8z^3(z+1)^{-1}(z-1)^{-2}$ at $z = -1$ and $z = 1$ are respectively $-2(z+1)^{-1}$ and $4(z-1)^{-2} + 10(z-1)^{-1}$.

3. For each of the functions below, find the principal part at the given points:

   a) $f(z) = \dfrac{\mathrm{e}^z}{z^4}$ at the point $z = 0$ b) $f(z) = \dfrac{z^6}{(1-z)^3}$ at the point $z = 1$

   c) $f(z) = \dfrac{\sin z}{(z-2\pi)^2}$ at the point $z = 2\pi$

4. Expand the function $(z-1)/(z+1)$ in powers of $1/z$.

5. For each of the functions below, use partial fractions if appropriate and find the principal part at each of its singular points in $\mathbb{C}$:

   a) $f(z) = \dfrac{12}{z^2(z^2+4)}$ b) $f(z) = \dfrac{z^4+1}{z(z^2+1)^2}$

   c) $f(z) = \dfrac{48z^6}{(z-1)^2(z-2)}$ d) $f(z) = \dfrac{z^9+1}{(z-1)^3(z^2+4)^2}$

6. Suppose that $f(z) = b_{-m}z^{-m} + b_{-m+1}z^{-m+1} + \ldots + b_0 + b_1 z + \ldots + b_k z^k$, where $m, k \in \mathbb{N}$. Suppose further that $f(z)$ has Laurent series

$$\sum_{n=-\infty}^{\infty} a_n z^n$$

   at the point $z = 0$. Show by direct calculation that $a_n = b_n$ whenever $-m \le n \le k$ and $a_n = 0$ otherwise.

7. a) Consider the function $f(z) = \mathrm{e}^{1/z}$. Note that for every $k \in \mathbb{Z}$, the coefficient for the term z^k in the Laurent series of $f(z)$ at $z = 0$ is given by

$$a_k = \frac{1}{2\pi \mathrm{i}} \int_C \frac{\mathrm{e}^{1/\zeta}}{\zeta^{k+1}}\, \mathrm{d}\zeta,$$

   where C is the circle $\{z : |z| = 1\}$ followed in the positive (anticlockwise) direction. Show that

$$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{\cos\theta} \cos(\sin\theta + k\theta)\, \mathrm{d}\theta.$$

   b) Find the Laurent series for the function $f(z) = \mathrm{e}^{1/z}$ at $z = 0$ without using part (a).

   c) Deduce that for every $n \in \mathbb{N} \cup \{0\}$,

$$\frac{1}{\pi} \int_0^{\pi} \mathrm{e}^{\cos\theta} \cos(\sin\theta - n\theta)\, \mathrm{d}\theta = \frac{1}{n!}.$$