INTRODUCTION TO COMPLEX ANALYSIS

W W L CHEN

Chapter 9

CAUCHY'S INTEGRAL THEOREM REVISITED

9.1. Simply Connected Domains

Cauchy's integral theorem states that in a simply connected domain, the integral of an analytic function over a closed contour is zero. In Chapter 5, we have studied a special case of the theorem in order to develop the basic properties of analytic functions. More precisely, we have studied the special case when the domain in question is a star domain. For most purposes in pure and applied mathematics, this special case is adequate. However, by introducing further line segments, one may be able to extend the results to curves which may not lie in star domains.

EXAMPLE 9.1.1.   It is easy to see that the domain $D = \{z : 2 < |z| < 6 \text{ and } \mathfrak{Im}\, z > 0\}$ is the part of the annulus $\{z : 2 < |z| < 6\}$ in the upper half plane. It is not difficult to check that D is not a star domain.
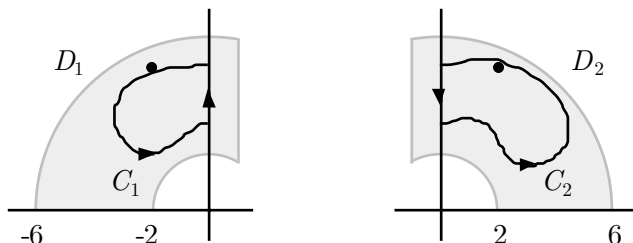

Let

$$D_1 = \{z : 2 < |z| < 6 \text{ and } \mathfrak{Im}\, z > 0 \text{ and } \mathfrak{Re}\, z < 1\}$$

and
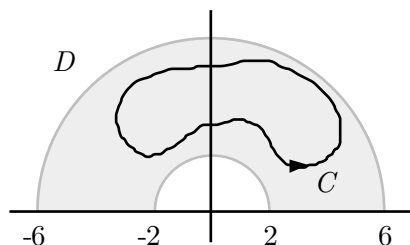
$$D_2 = \{z : 2 < |z| < 6 \text{ and } \mathfrak{Im}\, z > 0 \text{ and } \mathfrak{Re}\, z > -1\}.$$

It is not difficult to see that $D = D_1 \cup D_2$, and that D_1 and D_2 are star domains with star centres $-2 + 5i$ and $2 + 5i$ respectively.

If C is a simple closed contour lying in D, then by introducing line segments along the imaginary axis, it is not difficult to see that there is a simple closed contour C_1 in D_1 and a simple closed contour C_2 in D_2 such that for any function f analytic in D, we have

$$\int_C f(z)\,\mathrm{d}z = \int_{C_1} f(z)\,\mathrm{d}z + \int_{C_2} f(z)\,\mathrm{d}z.$$

Note now that we can apply Theorem 5D to the two integrals on the right hand side.

However, it is of great theoretical interest to formulate results that are less restricting. Here we introduce the idea of a simply connected domain. This can be done in a number of ways. Here we use the Jordan curve theorem for simple closed polygons.

DEFINITION. By a polygonal curve, we mean a curve $\zeta : [A, B] \to \mathbb{C}$ which is continuous and piecewise linear. In other words, there exists a dissection

$$A = A_1 < B_1 = A_2 < B_2 = \ldots = A_k < B_k = B$$

such that for every $j = 1, \ldots, k$, the edge $\zeta : [A_j, B_j] \to \mathbb{C}$ is of the form $\zeta(t) = \alpha_j t + \beta_j$, where $\alpha_j, \beta_j \in \mathbb{C}$ and we assume further that $\alpha_j \neq 0$.
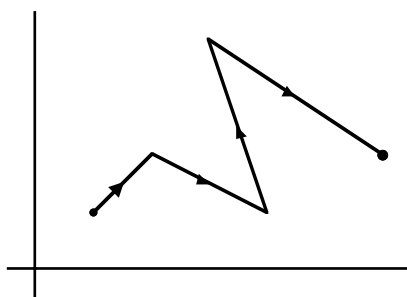

DEFINITION. By a simple closed polygon, we mean a polygonal curve that is closed and does not intersect itself; in other words, if $\zeta(t_1) \neq \zeta(t_2)$ whenever $t_1 \neq t_2$, with the one exception that $\zeta(A) = \zeta(B)$.
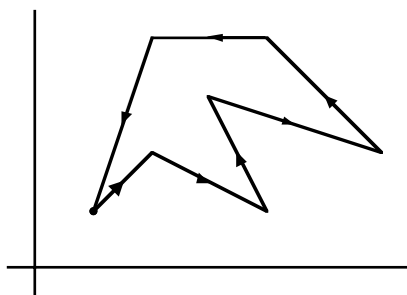
It is not hard to prove the Jordan curve theorem for a simple closed polygon, that such a polygon divides the plane $\mathbb{C}$ into two domains, the bounded one interior to the polygon and the unbounded one exterior to the polygon. Here we shall not give the proof.

DEFINITION. A domain D is said to be simply connected if the interior of every simple closed polygon in D is contained in D.

REMARK. Recall that a domain is an open connected set. A simply connected domain is one which is free of holes or cuts in its interior.

simply connected   not simply connected
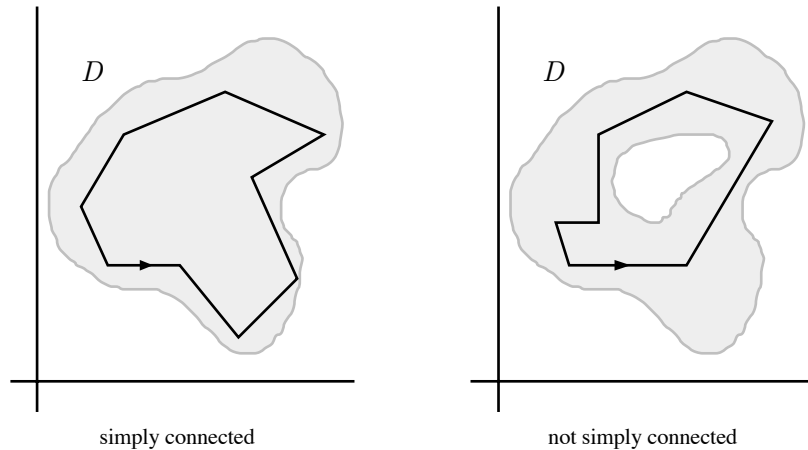
9.2. Cauchy's Integral Theorem

In this section, we indicate the proof of the following generalization of Theorem 5B.

THEOREM 9A. *Suppose that a function f is analytic in a simply connected domain D. Then there exists a function F, analytic in D and such that $F'(z) = f(z)$ for every $z \in D$.*

In view of Remark (1) immediately after Theorem 4A, Theorem 9A immediately leads to the following generalization of Theorem 5D.

THEOREM 9B. *Suppose that a function f is analytic in a simply connected domain D. Suppose further that C is a closed contour lying in D. Then*

$$\int_C f(z)\,\mathrm{d}z = 0.$$

EXAMPLE 9.2.1. The punctured plane $D = \{z : z \neq 0\}$ is not simply connected. Although the function $f(z) = 1/z$ is analytic in D,

$$\int_C \frac{\mathrm{d}z}{z} = 2\pi\mathrm{i} \neq 0$$

if C is the positive (anticlockwise) oriented unit circle centred at 0. This also shows that there is no single valued branch of $\log z$ in D, and confirms that the condition that D is simply connected in Theorems 9A and 9B is essential.

The proof of Theorem 9A depends on a process known sometimes as "triangulation". It can be shown by induction that a simple closed polygon of $k + 2$ sides can be decomposed by diagonals into a set of k triangles.

We shall first prove the following generalization of Theorem 5A.

THEOREM 9C. *Suppose that a function f is analytic in a simply connected domain D. Suppose further that C is a simple closed polygon in D. Then*
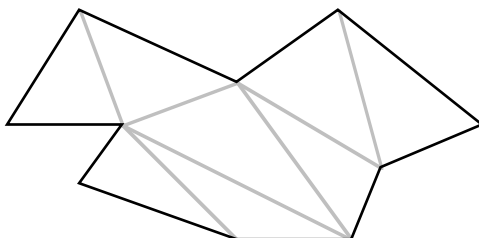
$$\int_C f(z)\,\mathrm{d}z = 0.$$

PROOF. We may assume, without loss of generality, that C is in the positive (anticlockwise) direction. Suppose that the triangulation process gives rise to triangles $T_1, \ldots, T_k$, with boundaries $C_1, \ldots, C_k$ in the positive (anticlockwise) direction. Then it is easy to see that

$$\int_C f(z)\,\mathrm{d}z = \int_{C_1} f(z)\,\mathrm{d}z + \ldots + \int_{C_k} f(z)\,\mathrm{d}z.$$

The result follows on applying Theorem 5A to each of the integrals on the right hand side. ◯

We now sketch a proof of Theorem 9A. Suppose that $z_0 \in D$ is fixed. For every $z \in D$, let C_1 and C_2 denote polygonal curves from z_0 to z that lie entirely in D.
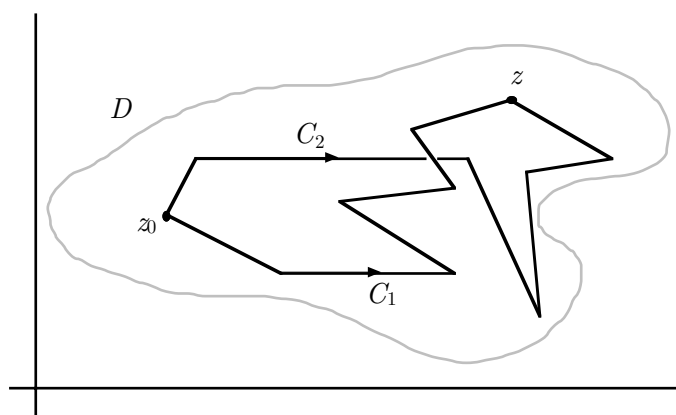
We shall first of all indicate that

(1) $$\int_{C_1} f(z)\,\mathrm{d}z = \int_{C_2} f(z)\,\mathrm{d}z.$$

Let C be the closed polygonal curve obtained by C_1 followed by $-C_2$. To show (1), it suffices to show that

(2) $$\int_C f(z)\,\mathrm{d}z = 0.$$

It can be shown by induction that the closed polygonal curve C consists of a number of line segments followed in opposite directions and a number of simple closed polygonal curves. (2) now follows in view of Theorem 9C. Note that (1) shows that the integral is independent of the polygonal curve chosen. We can therefore define

$$F(z) = \int_{z_0}^{z} f(\zeta)\, d\zeta,$$

where the integral is taken along any polygonal curve from z_0 to z. It follows that if $|h|$ is sufficiently small, then the segment $[z, z+h]$ lies entirely in D, and that

$$F(z+h) - F(z) = \int_{[z,z+h]} f(\zeta)\, d\zeta.$$

The proof of Theorem 9A is now completed in the same way as in the proof of Theorem 5B.

9.3. Cauchy's Integral Formula

Suppose that f is analytic in a simply connected domain D. Suppose further that C is a closed contour in D, and that the point z does not lie on C. If $z \in D$, then the function

$$g(\zeta) = \frac{f(\zeta) - f(z)}{\zeta - z}$$

is analytic in D, apart from a removable singularity at $\zeta = z$. Furthermore, this singularity is removed by defining $g(z) = f'(z)$. It now follows from Theorem 9B that

$$\int_{C} \frac{f(\zeta) - f(z)}{\zeta - z}\, dz = 0, \qquad \text{and so} \qquad \int_{C} \frac{f(\zeta)}{\zeta - z}\, dz = f(z) \int_{C} \frac{dz}{\zeta - z}.$$

Here

$$n(C, z) = \frac{1}{2\pi i} \int_{C} \frac{dz}{\zeta - z} = \frac{\operatorname{var}(i \arg(\zeta - z), C)}{2\pi i}$$

is the winding number, and counts the number of times the contour C winds round the point z in the positive (anticlockwise) direction. Hence

(3) $$\frac{1}{2\pi i} \int_{C} \frac{f(\zeta)}{\zeta - z}\, dz = n(C, z) f(z).$$

Note also that if $z \notin D$, then both sides of (3) are 0. It follows that (3) holds whenever $z \in C$.

9.4. Analytic Logarithm

Suppose that f is analytic and non-zero in a simply connected domain D. Let $z_0 \in D$ be fixed. For every $z \in D$, we can define

(4) $$g(z) = \int_{z_0}^{z} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta + \operatorname{Log} f(z_0),$$

where the integral is over any contour in D from z_0 to z, and is independent of the choice of the contour, in view of Theorem 9B. Differentiating (4) with respect to z, we obtain $g'(z) f(z) = f'(z)$. It follows that

$$\frac{d}{dz}(e^{-g(z)} f(z)) = e^{-g(z)} f'(z) - g'(z) e^{-g(z)} f(z) = 0,$$

so that $\mathrm{e}^{-g(z)}f(z)$ is contant in D. Since $\mathrm{e}^{-g(z_0)}f(z_0) = 1$, it follows that $\mathrm{e}^{-g(z)}f(z) = 1$ for every $z \in D$. In other words, $f(z) = \mathrm{e}^{g(z)}$ for every $z \in D$, so that f has an analytic logarithm in D.

PROBLEMS FOR CHAPTER 9

1. Suppose that T is a triangle, followed in the positive (anticlockwise) direction. Suppose further that for any $a \in \mathbb{C} \setminus T$, we write

$$n(T, a) = \frac{1}{2\pi\mathrm{i}} \int_T \frac{\mathrm{d}z}{z - a}.$$

   a) Show that $n(T, a) = 0$ for every $a \in \mathbb{C}$ outside T.
   b) Suppose now that $a \in \mathbb{C}$ is inside T. By relating T to a suitably small circular path centred at the point a, show that $n(T, a) = 1$.

2. Suppose that C is simple closed polygon, followed in the positive (anticlockwise) direction. Suppose further that for any $a \in \mathbb{C} \setminus C$, we write

$$n(C, a) = \frac{1}{2\pi\mathrm{i}} \int_C \frac{\mathrm{d}z}{z - a}.$$

   a) Show that $n(C, a) = 0$ for every $a \in \mathbb{C}$ outside C.
   b) Suppose now that $a \in \mathbb{C}$ is inside C. Apply the "triangulation" process to C and show that $n(C, a) = 1$ if the point a does not lie on the boundary of any of the triangles that arise from the process.
   c) Suppose now that $a \in \mathbb{C}$ is inside C and lies on the boundary of some of the triangles that arise from the "triangulation" process. Explain why we also have $n(C, a) = 1$.

3. Suppose that $D \subseteq \mathbb{C}$ is a domain. For every contour C lying in D and for every $a \notin D$, write

$$n(C, a) = \frac{1}{2\pi\mathrm{i}} \int_C \frac{\mathrm{d}z}{z - a}.$$

   a) Suppose that D is simply connected. Explain why $n(C, a) = 0$.
   b) Suppose that D is not simply connected. Show that there exists a contour C lying in D and a point $a \notin D$ such that $n(C, a) \neq 0$.

4. Deduce from the conclusion of Problem 3 that every star domain is simply connected.

5. Suppose that $D \subseteq \mathbb{C}$ is a domain. Suppose further that every function which is analytic and non-zero in D has an analytic logarithm in D, so that in particular, for every $a \notin D$, there exists a function $g(z)$ analytic in D and such that $z - a = \mathrm{e}^{g(z)}$ for every $z \in D$. Use the conclusion of Problem 3 to show that D is simply connected.

6. Suppose D is a bounded domain. For any $\epsilon > 0$, let D_ϵ denote the larger domain containing D and every point in $\mathbb{C}$ whose distance from D is less than ϵ. Give an example of a simply connected domain D such that D_ϵ is not simply connected for any $\epsilon > 0$.

7. Suppose that $f(z)$ is an entire function with a finite number of zeros. Show that there exist a polynomial $P(z)$ and an entire function $g(z)$ such that $f(z) = P(z)\mathrm{e}^{g(z)}$.

8. Suppose that $f(z)$ is analytic and non-zero in the disc $\{z : |z| \leq R\}$, where $R > 0$ is fixed. Prove the following special case of Jensen's formula, that

$$\log|f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log|f(R\mathrm{e}^{\mathrm{i}\theta}|\,\mathrm{d}\theta.$$