

INTRODUCTION TO COMPLEX ANALYSIS

W W L CHEN

© W W L Chen, 1996, 2003.

This chapter originates from material used by the author at Imperial College, University of London, between 1981 and 1990.

It is available free to all individuals, on the understanding that it is not to be used for financial gains,
and may be downloaded and/or photocopied, with or without permission from the author.

However, this document may not be kept on any information storage and retrieval system without permission
from the author, unless such system is not accessible to any individuals other than its owners.

Chapter 10

RESIDUE THEORY

10.1. Cauchy's Residue Theorem

If we extend Cauchy's integral theorem to functions having isolated singularities, then the integral is in general not equal to zero. Instead, each singularity contributes a term called the residue. Our principal aim in this section is to show that this residue depends only on the coefficient of $(z - z_0)^{-1}$ in the Laurent expansion of the function near the singularity z_0 , since all the other powers of $z - z_0$ has single valued integrals and so integrate to zero.

DEFINITION. By a simple closed contour or Jordan contour, we mean a contour $\zeta : [A, B] \rightarrow \mathbb{C}$ such that $\zeta(t_1) \neq \zeta(t_2)$ whenever $t_1 \neq t_2$, with the one exception $\zeta(A) = \zeta(B)$.

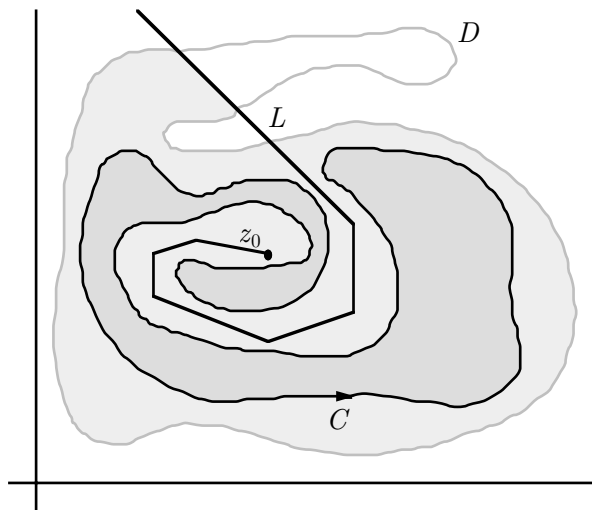
THEOREM 10A. *Suppose that a function f is analytic in a simply connected domain D , except for an isolated singularity at z_0 , and that*

$$f_1(z) = \sum_{n=-\infty}^{-1} a_n(z - z_0)^n$$

is the principal part of f at z_0 . Suppose further that C is a Jordan contour in D followed in the positive (anticlockwise) direction and not passing through z_0 . Then

$$\frac{1}{2\pi i} \int_C f_1(z) dz = \begin{cases} a_{-1} & \text{if } z_0 \text{ lies inside } C, \\ 0 & \text{if } z_0 \text{ lies outside } C. \end{cases}$$

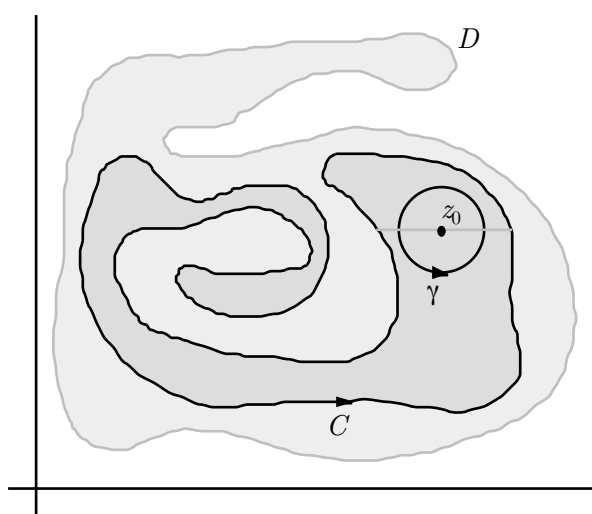
PROOF. Suppose first of all that z_0 is outside C . Then z_0 is in the exterior domain of C which also contains the point at ∞ . It follows that z_0 can be joined to the point at ∞ by a simple polygonal curve L , as shown in the picture below.



The Jordan contour C is clearly contained in the simply connected domain obtained when L is deleted from the complex plane. In fact, it is contained in a simply connected domain which is a subset of $D \setminus L$, as shown by the shaded part in the picture above. Clearly f is analytic in this simply connected domain, so it follows from Theorem 9B that

$$\int_C f(z) dz = 0.$$

Suppose next that z_0 is inside C . Then there exists $r > 0$ such that the closed disc $\{z : |z - z_0| \leq r\}$ is inside C . Let γ denote the boundary of this disc, followed in the positive (anticlockwise) direction.

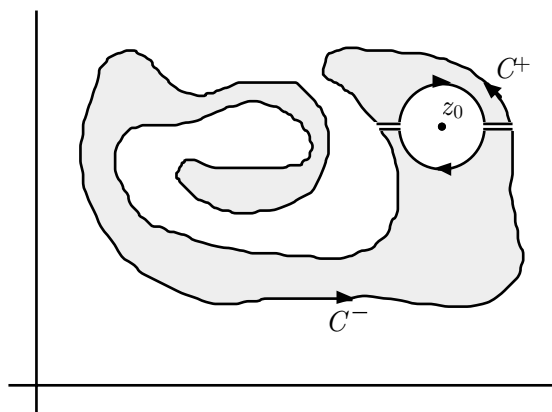


We now draw a horizontal line through the point z_0 . Following this line to the left from z_0 , it first intersects γ and then C (for the first time). Draw a line segment joining these two intersection points. Similarly, following this line to the right from z_0 , it first intersects γ and then C (for the first time). Again draw a line segment joining these two intersection points. Note that these two line segments are inside C and outside γ . We now divide C into two parts by cutting it at the two intersection points

mentioned. It can be shown that one part of this, together with the part of γ above the horizontal line and the two line segments, gives rise to a simple closed contour C^+ followed in the positive direction and which can be shown to lie in a simply connected domain lying in D but not containing z_0 . Clearly f is analytic in this simply connected domain, so that

$$\int_{C^+} f(z) dz = 0,$$

in view of Theorem 9B.



Similarly, the other part of C , together with the part of γ below the horizontal line and the two line segments, gives rise to a simple closed contour C^- followed in the positive direction and which again can be shown to lie in a simply connected domain lying in D but not containing z_0 . Clearly f is analytic in this simply connected domain, so that

$$\int_{C^-} f(z) dz = 0.$$

It is easily seen that

$$\int_C f(z) dz - \int_\gamma f(z) dz = \int_{C^+} f(z) dz + \int_{C^-} f(z) dz,$$

so that

$$\int_C f(z) dz = \int_\gamma f(z) dz.$$

By Theorem 8E, we have

$$\int_\gamma f(z) dz = 2\pi i a_{-1}.$$

It follows that

$$\int_C f(z) dz = 2\pi i a_{-1}.$$

Finally, note that $f_2(z) = f(z) - f_1(z)$ is analytic in D , so that

$$\int_C f_2(z) dz = 0,$$

whence

$$\int_C f(z) dz = \int_C f_1(z) dz.$$

The result follows. \circ

DEFINITION. The value a_{-1} in Theorem 10A is called the residue of the function f at z_0 , and denoted by $\text{res}(f, z_0)$.

We are now in a position to state and prove a simple version of Cauchy's residue theorem.

THEOREM 10B. Suppose that the function f is analytic in a simply connected domain D , except for isolated singularities at z_1, \dots, z_k . Suppose further that C is a Jordan contour in D followed in the positive (anticlockwise) direction and not passing through z_1, \dots, z_k . Then

$$\frac{1}{2\pi i} \int_C f(z) dz = \sum_{\substack{j=1 \\ z_j \text{ inside } C}}^k \text{res}(f, z_j).$$

PROOF. For every $j = 1, \dots, k$, let $f_j(z)$ denote the principal part of $f(z)$ at z_j . By Theorem 8D, f_j is analytic in \mathbb{C} except at z_j . It follows that the function

$$g(z) = f(z) - \sum_{j=1}^k f_j(z)$$

is analytic in D , so that

$$\int_C g(z) dz = 0$$

by Theorem 9B, and so

$$\int_C f(z) dz = \sum_{j=1}^k \int_C f_j(z) dz.$$

The result now follows from Theorem 10A. \circ

10.2. Finding the Residue

In order to use Theorem 10B to evaluate the integral

$$\int_C f(z) dz,$$

we need a technique to evaluate the residues at the isolated singularities.

Suppose that $f(z)$ has a removable singularity at z_0 . Then $f(z)$ has a Taylor series expansion which is valid in a neighbourhood of z_0 . The residue is clearly 0.

Suppose that $f(z)$ has a simple pole at z_0 . Then we can write

$$f(z) = \frac{a_{-1}}{z - z_0} + g(z),$$

where $g(z)$ is analytic at z_0 , so that $(z - z_0)g(z) \rightarrow 0$ as $z \rightarrow z_0$. It follows that the residue is given by

$$a_{-1} = \lim_{z \rightarrow z_0} (z - z_0)f(z).$$

Suppose that $f(z)$ has a pole of order m at z_0 . Then we can write

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-m+1}}{(z - z_0)^{m-1}} + \dots + \frac{a_{-1}}{z - z_0} + g(z),$$

where $g(z)$ is analytic at z_0 , so that

$$(z - z_0)^m f(z) = a_{-m} + a_{-m+1}(z - z_0) + \dots + a_{-1}(z - z_0)^{m-1} + (z - z_0)^m g(z)$$

is analytic at z_0 . Differentiating $m - 1$ times gives

$$\frac{d^{m-1}}{dz^{m-1}}((z - z_0)^m f(z)) = a_{-1}(m - 1)! + \frac{d^{m-1}}{dz^{m-1}}((z - z_0)^m g(z)).$$

Since $g(z)$ is analytic at z_0 , we have

$$\lim_{z \rightarrow z_0} \frac{d^{m-1}}{dz^{m-1}}((z - z_0)^m g(z)) = 0.$$

It follows that the residue is given by

$$a_{-1} = \frac{1}{(m - 1)!} \lim_{z \rightarrow z_0} \frac{d^{m-1}}{dz^{m-1}}((z - z_0)^m f(z)).$$

DEFINITION. A function is said to be meromorphic in a domain D if it is analytic in D except for poles.

EXAMPLE 10.2.1. The function

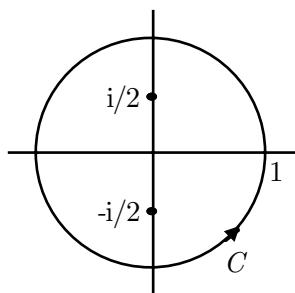
$$f(z) = \frac{e^{2iz}}{1 + 4z^2}$$

has simple poles at $z = \pm i/2$, with residues

$$\operatorname{res}\left(f, \frac{i}{2}\right) = \lim_{z \rightarrow i/2} \left(z - \frac{i}{2}\right) f(z) = \lim_{z \rightarrow i/2} \frac{e^{2iz}}{4(z + i/2)} = \frac{e^{-1}}{4i}$$

and

$$\operatorname{res}\left(f, -\frac{i}{2}\right) = \lim_{z \rightarrow -i/2} \left(z + \frac{i}{2}\right) f(z) = \lim_{z \rightarrow -i/2} \frac{e^{2iz}}{4(z - i/2)} = -\frac{e}{4i}.$$



It follows from Cauchy's residue theorem that if $C = \{z : |z| = 1\}$ is the circle with centre 0 and radius 1, followed in the positive (anticlockwise) direction, then

$$\int_C \frac{e^{2iz}}{1+4z^2} dz = 2\pi i \left(\frac{e^{-1}}{4i} - \frac{e}{4i} \right) = \frac{\pi}{2} \left(\frac{1}{e} - e \right).$$

EXAMPLE 10.2.2. The function

$$f(z) = \frac{e^z}{z^4}$$

has a pole of order 4 at $z = 0$, with residue

$$\text{res}(f, 0) = \frac{1}{3!} \lim_{z \rightarrow 0} \frac{d^3}{dz^3} (z^4 f(z)) = \frac{1}{3!} \lim_{z \rightarrow 0} \frac{d^3}{dz^3} e^z = \frac{1}{6}.$$

It follows from Cauchy's residue theorem that if C is any Jordan contour with 0 inside and followed in the positive (anticlockwise) direction, then

$$\int_C \frac{e^z}{z^4} dz = 2\pi i \left(\frac{1}{6} \right) = \frac{\pi i}{3}.$$

EXAMPLE 10.2.3. Suppose that a function f is analytic in a simply connected domain D , and that $z_0 \in D$. Suppose further that C is a Jordan contour in D , followed in the positive (anticlockwise) direction and with z_0 inside. If $f(z_0) \neq 0$, then the function

$$F(z) = \frac{f(z)}{z - z_0}$$

has a simple pole at z_0 , with residue

$$\lim_{z \rightarrow z_0} (z - z_0)F(z) = f(z_0).$$

Applying Cauchy's residue theorem, we obtain Cauchy's integral formula

$$\frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz = f(z_0).$$

If $f(z_0) = 0$, then $F(z)$ has a removable singularity at z_0 . The same result follows instead from Cauchy's integral theorem.

10.3. Principle of the Argument

In this section, we shall show that the residue theorem, when applied suitably, can be used to find the number of zeros of an analytic function, as well as the number of zeros minus the number of poles of a meromorphic function.

The main idea underpinning our discussion can be summarized by the following two results.

THEOREM 10C. *Suppose that a function f is analytic in a neighbourhood of z_0 . Suppose further that f has a zero of order m at z_0 . Then the function f'/f is analytic in a punctured neighbourhood of z_0 , with a simple pole at z_0 with residue m .*

THEOREM 10D. Suppose that a function f is analytic in a punctured neighbourhood of z_0 . Suppose further that f has a pole of order m at z_0 . Then the function f'/f is analytic in a punctured neighbourhood of z_0 , with a simple pole at z_0 with residue $-m$.

PROOF OF THEOREM 10C. We can write $f(z) = (z - z_0)^m g(z)$, where $g(z)$ is analytic in a neighbourhood of z_0 and $g(z_0) \neq 0$. Then

$$\frac{f'(z)}{f(z)} = \frac{m(z - z_0)^{m-1}g(z) + (z - z_0)^m g'(z)}{(z - z_0)^m g(z)} = \frac{m}{z - z_0} + \frac{g'(z)}{g(z)}.$$

Since $g(z)$ is analytic in a neighbourhood of z_0 and $g(z_0) \neq 0$, the function $g'(z)/g(z)$ is analytic in a neighbourhood of z_0 . The result follows. \circ

PROOF OF THEOREM 10D. We can write $f(z) = (z - z_0)^{-m} g(z)$, where $g(z)$ is analytic in a neighbourhood of z_0 and $g(z_0) \neq 0$. Then

$$\frac{f'(z)}{f(z)} = \frac{-m(z - z_0)^{-m-1}g(z) + (z - z_0)^{-m} g'(z)}{(z - z_0)^{-m} g(z)} = \frac{-m}{z - z_0} + \frac{g'(z)}{g(z)}.$$

Since $g(z)$ is analytic in a neighbourhood of z_0 and $g(z_0) \neq 0$, the function $g'(z)/g(z)$ is analytic in a neighbourhood of z_0 . The result follows. \circ

The main result in this section is the Principle of the argument, as stated below.

THEOREM 10E. Suppose that a function f is meromorphic in a simply connected domain D . Suppose further that C is a Jordan curve in D , followed in the positive (anticlockwise) direction, and that f has no zeros or poles on C . If N denotes the number of zeros of f in the interior of C , counted with multiplicities, and if P denotes the number of poles of f in the interior of C , counted with multiplicities, then

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = N - P.$$

PROOF. Note that by Theorems 10C and 10D, the poles of the function f'/f are precisely at the zeros and poles of f . Furthermore, a zero of f of order m gives rise to a residue m for f'/f , so that the residues of f'/f arising from the zeros of f are equal to the number of zeros of f counted with multiplicities, and this number is N . On the other hand, a pole of f of order m gives rise to a residue $-m$ for f'/f , so that the residues of f'/f arising from the poles of f are equal to minus the number of poles of f counted with multiplicities, and this number is P . It follows that the sum of the residues is equal to $N - P$. The result now follows from Theorem 10B applied to the function f'/f . \circ

REMARKS. (1) Note that

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \text{var}(\log f(z), C) = \frac{1}{2\pi i} \text{var}(i \arg f(z), C) = \frac{1}{2\pi} \text{var}(\arg f(z), C).$$

It follows that the conclusion of Theorem 10E can be expressed in the form

$$N - P = \frac{1}{2\pi} \text{var}(\arg f(z), C),$$

in terms of the variation of the argument of $f(z)$ along the Jordan curve C .

(2) Note also that

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \int_{f(C)} \frac{dw}{w} = n(f(C), 0).$$

(3) Theorem 10E can be generalized in the following way. Suppose that a function f is meromorphic in a simply connected domain D , and that all its zeros and poles in D are simple. Suppose further that C is a Jordan curve in D , followed in the positive (anticlockwise) direction, and that f has no zeros or poles on C . If a_1, \dots, a_N denote the zeros of f in the interior of C , and if b_1, \dots, b_P denote the poles of f in the interior of C , then

$$(1) \quad \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} g(z) dz = \sum_{j=1}^N g(a_j) - \sum_{k=1}^P g(b_k)$$

for every function g analytic in D . To see this, simply note that any simple zero or simple pole z_0 of f , where $g(z_0) \neq 0$, gives rise to a simple pole of $(f'/f)g$ with residue

$$\lim_{z \rightarrow z_0} (z - z_0) \frac{f'(z)}{f(z)} g(z) = g(z_0) \lim_{z \rightarrow z_0} (z - z_0) \frac{f'(z)}{f(z)} = \begin{cases} g(z_0) & \text{if } z_0 \text{ is a simple zero of } f, \\ -g(z_0) & \text{if } z_0 \text{ is a simple pole of } f; \end{cases}$$

on the other hand, if $g(z_0) = 0$, then $(f'/f)g$ has a removable singularity at z_0 . In fact, (1) remains valid if the zeros and poles of f are of higher order, provided that all zeros and poles are counted with multiplicities. Note also that the choice $g(z) = 1$ in D gives Theorem 10E again. A particular useful choice of f is given by the entire function $f(z) = \sin \pi z$, with simple zeros at every $n \in \mathbb{Z}$. Since

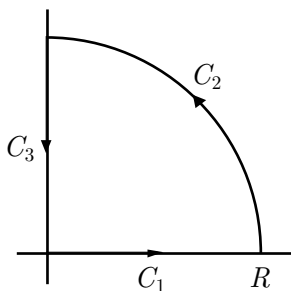
$$\frac{f'(z)}{f(z)} = \frac{\pi \cos \pi z}{\sin \pi z} = \pi \cot \pi z,$$

it follows from (1) that

$$(2) \quad \frac{1}{2\pi i} \int_C g(z) \pi \cot \pi z dz = \sum_{n \text{ inside } C} g(n)$$

for every function g analytic in D . This may be used to obtain a variety of infinite series expansions. See Chapter 16.

EXAMPLE 10.3.1. To find the number of zeros of the function $f(z) = z^4 + z^3 - 2z^2 + 2z + 4$ in the first quadrant of the complex plane, we use the Jordan curve $C = C_1 \cup C_2 \cup C_3$, where $C_1 = [0, R]$ is the straight line segment along the real axis from 0 to R , C_2 is the circular path $\zeta : [0, \pi/2] \rightarrow \mathbb{C}$, given by $\zeta(t) = Re^{it}$, and $C_3 = [iR, 0]$ is the straight line segment along the imaginary axis from iR to 0. Here R is taken to be a large positive real number.



On C_1 , we have $z = x > 0$, so that

$$f(z) = f(x) = x^4 + x^3 - 2x^2 + 2x + 4 \geq \begin{cases} x^4 + x^3 + 4 & \text{if } 0 \leq x \leq 1 \\ 2x + 4 & \text{if } x \geq 1 \end{cases}$$

is clearly positive, so that $\text{var}(\arg f(z), C_1) = 0$. Next, note that

$$f(z) = z^4 \left(1 + \frac{z^3 - 2z^2 + 2z + 4}{z^4} \right).$$

On C_2 , we have $|z| = R$, so that

$$\left| \frac{z^3 - 2z^2 + 2z + 4}{z^4} \right| \leq \frac{R^3 + 2R^2 + 2R + 4}{R^4} < \frac{2R^3}{R^4} = \frac{2}{R}$$

whenever $R > 8$, say. It follows that on C_2 when R is large enough, we have $f(z) = R^4 e^{4it}(1+w)$, where $|w| < 2/R$, so that $\text{var}(\arg f(z), C_2) = 2\pi + \epsilon_1$, where $\epsilon_1 \rightarrow 0$ as $R \rightarrow \infty$. Finally, on C_3 , we have $z = iy$, where $y > 0$, so that

$$f(z) = f(iy) = (y^4 + 2y^2 + 4) + i(2y - y^3) = (y^2 + 1)^2 + 3 + i(2y - y^3).$$

Note that $\Re f(iy) > 0$, so that $f(iy)$ is in the first or fourth quadrant of the complex plane. In fact, when $R > 0$ is large, $f(iR)$ is much nearer the real axis than the imaginary axis, while $f(0) = 4$ is on the positive real axis. It follows that $\text{var}(\arg f(z), C_3) = \epsilon_2$, where $\epsilon_2 \rightarrow 0$ as $R \rightarrow \infty$. We now conclude that $\text{var}(\arg f(z), C) = 2\pi + \epsilon_1 + \epsilon_2$, where $\epsilon_1, \epsilon_2 \rightarrow 0$ as $R \rightarrow \infty$. On the other hand, C is a closed contour, so that $\text{var}(\arg f(z), C)$ must be an integer multiple of 2π . It follows that $\text{var}(\arg f(z), C) = 2\pi$. Note now that the function f has no poles in the first quadrant. It follows from the Argument principle that f has exactly one zero inside the contour C for all large R . Hence f has exactly one zero in the first quadrant of the complex plane.

To find the number of zeros in a region, the following result provides an opportunity to either bypass the Argument principle or at least enable one to apply the Argument principle to a simpler function. Needless to say, the proof is based on an application of the Argument principle.

THEOREM 10F. (ROUCHÉ'S THEOREM) *Suppose that functions f and g are analytic in a simply connected domain D , and that C is a Jordan contour in D . Suppose further that $|f(z)| > |g(z)|$ on C . Then f and $f + g$ have the same number of zeros inside C .*

We shall give two proofs of this result. The first is the one given in most texts.

FIRST PROOF OF THEOREM 10F. Consider the function

$$F(z) = \frac{f(z) + g(z)}{f(z)}.$$

The condition $|f(z)| > |g(z)|$ on C ensures that both f and $f + g$ have no zeros on C . On the other hand, note that

$$(3) \quad |F(z) - 1| = \left| \frac{g(z)}{f(z)} \right| < 1$$

for every $z \in C$. By Remark (2) after Theorem 10E, we have

$$\frac{1}{2\pi i} \int_C \frac{F'(z)}{F(z)} dz = \frac{1}{2\pi i} \int_{F(C)} \frac{dw}{w} = n(F(C), 0).$$

In view of (3), the closed contour $F(C)$ is contained in the open disc $\{w : |w - 1| < 1\}$ with centre 1 and radius 1. This disc does not contain the point 0, so that $n(F(C), 0) = 0$. Hence

$$\frac{1}{2\pi i} \int_C \frac{F'(z)}{F(z)} dz = 0.$$

It follows from the Argument principle that the function F has the same number of zeros and poles inside C . Note now that the poles of F are precisely the zeros of f , and the zeros of F are precisely the zeros of $f + g$. \circ

SECOND PROOF OF THEOREM 10F. For every $\tau \in [0, 1]$, let

$$N(\tau) = \frac{1}{2\pi i} \int_C \frac{f'(z) + \tau g'(z)}{f(z) + \tau g(z)} dz.$$

The condition $|f(z)| > |g(z)|$ on C ensures that

$$|f(z) + \tau g(z)| \geq |f(z)| - \tau |g(z)| \geq |f(z)| - |g(z)| > 0$$

on C , so that $f + \tau g$ does not have any zeros (or poles) on C . In fact, there is a positive lower bound for $|f(z) + \tau g(z)|$ on C independent of τ . It follows easily from this that $N(\tau)$ is continuous in $[0, 1]$. By the Argument principle, $N(\tau)$ is an integer for every $\tau \in [0, 1]$. Hence $N(\tau)$ must be constant in $[0, 1]$. In particular, we must have $N(0) = N(1)$. Clearly, $N(0)$ is the number of zeros of f inside C , and $N(1)$ is the number of zeros of $f + g$ inside C . \circ

EXAMPLE 10.3.2. To determine the number of solutions of $e^z = 2z + 1$ with $|z| < 1$, we write

$$f(z) = -2z \quad \text{and} \quad g(z) = e^z - 1,$$

so that $f(z) + g(z) = e^z - 2z - 1$. We therefore need to find the number of zeros of $f + g$ inside the unit circle $C = \{z : |z| = 1\}$. Clearly, f has precisely one zero, at $z = 0$, inside C . On the other hand, note that

$$e^z - 1 = \int_{[0, z]} e^\zeta d\zeta = \int_0^1 e^{zt} z dt.$$

If $z \in C$, then $|e^{zt}| \leq e^t$, and so

$$|g(z)| = |e^z - 1| \leq \int_0^1 |e^{zt} z| dt \leq \int_0^1 e^t dt = e - 1.$$

Since $|f(z)| = 2$ whenever $z \in C$, it follows that $|f(z)| > |g(z)|$ on C . By Rouché's theorem, $f + g$ has precisely one zero inside C .

PROBLEMS FOR CHAPTER 10

1. a) Write down the Taylor series for e^w about the origin $w = 0$.
 b) Using the substitution $w = 1/z^2$ in (a), find the Laurent series for the function e^{1/z^2} about the origin $z = 0$.
 c) Find the residue of the function e^{1/z^2} at the origin $z = 0$.
 d) What type of singularity does the function e^{1/z^2} have at the origin $z = 0$?
2. Suppose that $f(z) = g(z)/h(z)$, where the functions $g(z)$ and $h(z)$ are analytic at z_0 . Suppose further that $g(z_0) \neq 0$ and $h(z)$ has a simple zero at z_0 . Use l'Hopital's rule to show that

$$\text{res}(f, z_0) = \lim_{z \rightarrow z_0} \frac{g(z)}{h'(z)} = \frac{g(z_0)}{h'(z_0)}.$$

3. For each of the functions $f(z)$ given below, find all the singularities in \mathbb{C} , find the residues at these singularities, and evaluate the integrals

$$\int_{C'} f(z) dz \quad \text{and} \quad \int_{C''} f(z) dz,$$

where C' and C'' are circular paths centred at the origin $z = 0$, of radius $1/2$ and 2 respectively, followed in the positive (anticlockwise) direction:

$$\text{a) } f(z) = \frac{1}{z(z-1)} \qquad \text{b) } f(z) = \frac{z}{z^4+1} \qquad \text{c) } f(z) = \frac{z^3+2}{(z^4-1)(z+1)}$$

4. Suppose that C is a circular path centred at the origin $z = 0$, of radius 1 , followed in the positive (anticlockwise) direction. Show each of the following:

$$\text{a) } \int_C \frac{e^{\pi z}}{4z^2+1} dz = \pi i; \qquad \text{b) } \int_C \frac{e^z}{z^3} dz = \pi i.$$

5. Find the number of zeros of $f(z) = z^4 + z^3 + 5z^2 + 2z + 4$ in the first quadrant of the complex plane. Find also the number of zeros of the function in the fourth quadrant.

6. Consider the equation $2z^5 + 8z - 1 = 0$.

- Writing $f(z) = 2z^5$ and $g(z) = 8z - 1$, use Rouché's theorem to show that all the roots of this equation lie in the open disc $\{z : |z| < 2\}$.
- Writing $f(z) = 8z - 1$ and $g(z) = 2z^5$, use Rouché's theorem to show that this equation has exactly one root in the open disc $\{z : |z| < 1\}$.
- How many roots does this equation have in the open annulus $\{z : 1 < |z| < 2\}$? Justify your assertion.

7. Show that the equation $z^6 + 4z^2 = 1$ has exactly two roots in the open disc $\{z : |z| < 1\}$.

[HINT: Use Rouché's theorem. You will need to make a good choice for $f(z)$ and $g(z)$. Do not give up if your first guess does not work.]