INTRODUCTION TO COMPLEX ANALYSIS

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Chapter 13

MÖBIUS TRANSFORMATIONS

13.1. Linear Functions

EXAMPLE 13.1.1. Consider the square $\{z = x + iy \in \mathbb{C} : 0 \le x \le 1 \text{ and } 0 \le y \le 1\}$. The pictures below show the images of this square under the functions $f(z) = z + 1 + i$, $f(z) = e^{i\phi}z$ and $f(z) = 2z$. Note that the image of the square in each case is also a square.

original square

image under $f(z) = z + 1 + i$

image under $f(z) = e^{i\phi}z$

image under $f(z) = 2z$
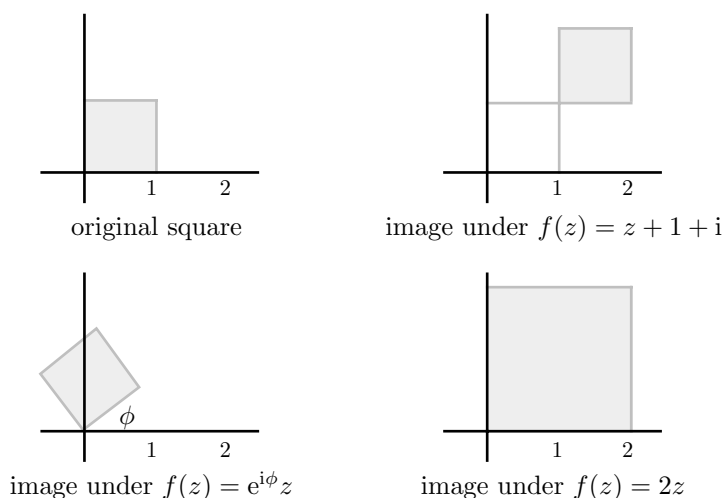
The function $f(z) = z + 1 + i$ is an example of a function of the type $f(z) = z + c$, where $c \in \mathbb{C}$ is fixed. This function describes a translation on the complex plane $\mathbb{C}$, where every point is shifted by a vector corresponding to the complex number c. The function $f(z) = e^{i\phi}z$, where $\phi \in \mathbb{R}$ is fixed, describes a rotation on the complex plane $\mathbb{C}$, where every point is rotated in the anticlockwise direction by an angle ϕ about the origin. The function $f(z) = 2z$ is an example of a function of the type $f(z) = \rho z$, where

$\rho \in \mathbb{R}$ is positive and fixed. This function describes a magnification on the complex plane $\mathbb{C}$, where the distance between points is magnified by a factor ρ, noting that $|\rho z_1 - \rho z_2| = \rho |z_1 - z_2|$ for every $z_1, z_2 \in \mathbb{C}$.

It is easily seen that if we take the domain and codomain of each of the above functions to be the complex plane $\mathbb{C}$, then $f : \mathbb{C} \to \mathbb{C}$ is both one-to-one and onto. Furthermore, any geometric object in $\mathbb{C}$ has an image under f which is similar to itself.

DEFINITION. A linear function is a function $f : \mathbb{C} \to \mathbb{C}$ of the form $f(z) = az + b$, where $a, b \in \mathbb{C}$ are fixed, and $a \neq 0$.

EXAMPLE 13.1.2. Let us return to the three examples earlier. For the function $f(z) = z + c$, we have $a = 1$ and $b = c$. For the function $f(z) = e^{i\phi} z$, we have $a = e^{i\phi}$ and $b = 0$. For the function $f(z) = \rho z$, we have $a = \rho$ and $b = 0$.

THEOREM 13A. *Any linear function $f : \mathbb{C} \to \mathbb{C}$ is the composition of a rotation, a magnification and a translation. Furthermore, it is one-to-one and onto.*

PROOF. Suppose that $f(z) = az + b$ for every $z \in \mathbb{C}$. Write $a = \rho e^{i\phi}$, where $\rho, \phi \in \mathbb{R}$ and $\rho > 0$. Then $f = f_3 \circ f_2 \circ f_1$, where

$$f_1(z) = e^{i\phi} z \qquad \text{and} \qquad f_2(z) = \rho z \qquad \text{and} \qquad f_3(z) = z + b.$$

We have the picture below:
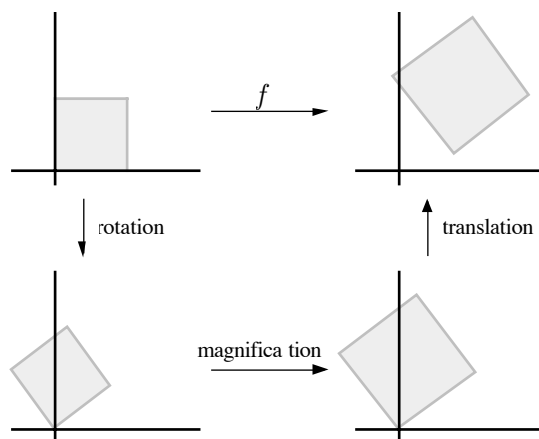
The last assertion follows from the observation that composition of functions preserves the one-to-one and onto properties. ◯

THEOREM 13B. *The composition of any two linear functions is also a linear function.*

PROOF. Suppose that $f_1(z) = a_1 z + b_1$ and $f_2(z) = a_2 z + b_2$, where $a_1, b_1, a_2, b_2 \in \mathbb{C}$ and $a_1, a_2 \neq 0$. Then $(f_2 \circ f_1)(z) = a_2(a_1 z + b_1) + b_2 = a_1 a_2 z + (a_2 b_1 + b_2)$. Clearly $a_1 a_2, a_2 b_1 + b_2 \in \mathbb{C}$ and $a_1 a_2 \neq 0$. ◯

EXAMPLE 13.1.3. Suppose that $z_0 \in \mathbb{C}$ is fixed. Consider the linear function $f : \mathbb{C} \to \mathbb{C}$ which rotates the complex plane $\mathbb{C}$ in the anticlockwise direction by an angle θ about the point z_0. We may adopt the following strategy: Translate the point z_0 to the origin, then rotate in the anticlockwise direction by an angle θ about the origin, and then translate the origin back to the point z_0. Then $f = f_3 \circ f_2 \circ f_1$, where

$$f_1(z) = z - z_0 \qquad \text{and} \qquad f_2(z) = e^{i\theta} z \qquad \text{and} \qquad f_3(z) = z + z_0.$$

We have the picture below:
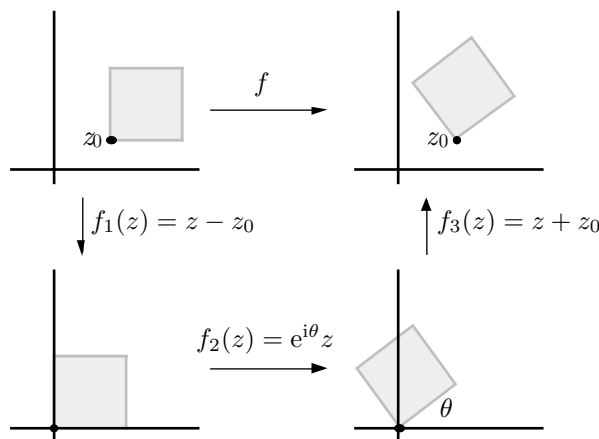
Hence

$$f(z) = e^{i\theta}(z - z_0) + z_0 = e^{i\theta}z + z_0(1 - e^{i\theta}).$$
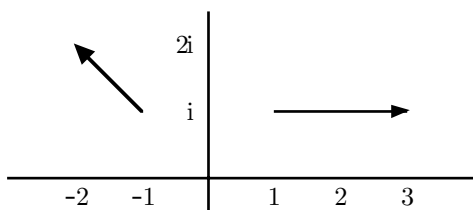
Alternatively, we may adopt the following strategy: Rotate in the anticlockwise direction by an angle θ about the origin, and then translate the image of z_0 under this rotation back to z_0. Then $f = g_2 \circ g_1$, where

$$g_1(z) = e^{i\theta}z \qquad \text{and} \qquad g_2(z) = z + (z_0 - e^{i\theta}z_0).$$

Hence

$$f(z) = e^{i\theta}z + (z_0 - e^{i\theta}z_0) = e^{i\theta}z + z_0(1 - e^{i\theta}).$$

EXAMPLE 13.1.4.   Consider the linear function $f : \mathbb{C} \to \mathbb{C}$ which maps the horizontal arrow shown to the other arrow shown.

We may adopt the following strategy: Translate the tip of the arrow from $3 + i$ to the origin, magnify the arrow by a factor $1/\sqrt{2}$, rotate it about its tip (now at the origin) in the anticlockwise direction by an angle $3\pi/4$, and finally translate its tip from the origin to the point $-2 + 2i$. Then $f = f_4 \circ f_3 \circ f_2 \circ f_1$, where

$$f_1(z) = z - (3 + i) \quad \text{and} \quad f_2(z) = \frac{1}{\sqrt{2}}z \quad \text{and} \quad f_3(z) = e^{3\pi i/4}z \quad \text{and} \quad f_4(z) = z + (-2 + 2i).$$

Hence

$$f(z) = \frac{e^{3\pi i/4}}{\sqrt{2}}(z - 3 - i) - 2 + 2i = \left(-\frac{1}{2} + \frac{i}{2}\right)(z - 3 - i) - 2 + 2i = \left(-\frac{1}{2} + \frac{i}{2}\right)z + i.$$

13.2. The Inversion Function

Consider the inversion function

$$w = f(z) = \frac{1}{z}.$$

This function can be considered a function of the type $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, where $\overline{\mathbb{C}}$ denotes the extended complex plane, so that $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. We write formally $f(0) = \infty$ and $f(\infty) = 0$.

Let us first study some geometric properties of this function. For our purposes, the point at ∞ is considered to belong to every line on the extended complex plane.

REMARKS. (1) A line passing through the origin contains all points of the form $z = r\mathrm{e}^{\mathrm{i}\theta}$, where $\theta \in \mathbb{R}$ is fixed and $r \in \mathbb{R}$. The images of these points under the inversion function are of the form

$$w = \frac{1}{z} = \frac{1}{r}\mathrm{e}^{-\mathrm{i}\theta}.$$

They form a line through the origin. Note that the point at ∞ and the origin change roles under the inversion function.

(2) A line not passing through the origin consists of all points of the form $z = x + \mathrm{i}y$, where $x, y \in \mathbb{R}$ and $Ax + By = C$, where $A, B, C \in \mathbb{R}$ are fixed and $C \neq 0$. The images of these points under the inversion function are of the form $w = u + \mathrm{i}v$, where $u, v \in \mathbb{R}$ and $w = 1/z$. It is easy to see that

$$z = \frac{1}{w} = \frac{1}{u + \mathrm{i}v} = \frac{u - \mathrm{i}v}{u^2 + v^2},$$

so that

(1) $$x = \frac{u}{u^2 + v^2} \qquad \text{and} \qquad y = -\frac{v}{u^2 + v^2}.$$

It follows that

$$\frac{Au}{u^2 + v^2} - \frac{Bv}{u^2 + v^2} = C.$$

This can be rewritten in the form

$$u^2 + v^2 - \frac{A}{C}u + \frac{B}{C}v = 0,$$

the equation of a circle passing through the origin.

(3) Note now that the inverse of the inversion function is the inversion function itself. It follows from the previous observation that a circle passing through the origin becomes a line not passing through the origin under the inversion function.

(4) A circle not passing through the origin consists of all points of the form $z = x + \mathrm{i}y$, where $x, y \in \mathbb{R}$ and $x^2 + y^2 + Ax + By = C$, where $A, B, C \in \mathbb{R}$ are fixed and $C \neq 0$. The images of these points under the inversion function are of the form $w = u + \mathrm{i}v$, where $u, v \in \mathbb{R}$ and $w = 1/z$. In view of (1), we have

$$\frac{u^2}{(u^2 + v^2)^2} + \frac{v^2}{(u^2 + v^2)^2} + \frac{Au}{u^2 + v^2} - \frac{Bv}{u^2 + v^2} = C.$$

This can be rewritten in the form

$$u^2 + v^2 - \frac{A}{C}u + \frac{B}{C}v = \frac{1}{C},$$

the equation of a circle not passing through the origin.

We now state a result which includes these four remarks.

THEOREM 13C. *The inversion function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, given by $f(z) = 1/z$ for every non-zero $z \in \mathbb{C}$, and $f(0) = \infty$ and $f(\infty) = 0$, is one-to-one and onto. On the other hand, its inverse function is itself. Furthermore, the image under this function of a line or a circle in $\overline{\mathbb{C}}$ is also a line or a circle in $\overline{\mathbb{C}}$.*

REMARKS. (1) We have in fact shown the following: Under the inversion function, the image of a line through the origin is a line through the origin, the image of a line not through the origin is a circle through the origin, the image of a circle through the origin is a line not through the origin, and the image of a circle not through the origin is a circle not through the origin.

(2) If we think of a line as a circle of infinite radius, then we can think of circles and lines as belonging to the "same" class. The inversion function therefore maps members of this class to members of this class.

13.3. A Generalization

If we extend any linear function discussed in §13.1 to a function of the type $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ by writing $f(\infty) = \infty$, then it is easy to see that this extended function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is also one-to-one and onto, and that its inverse function $f^{-1} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is also a linear function.

Note also that the class of all circles and lines in $\overline{\mathbb{C}}$ is carried to itself by all linear functions as well as the inversion function. We now try to generalize these two types of functions.

DEFINITION. A Möbius transformation, or a bilinear transformation, is a rational function $T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of the form

$$(2) \qquad T(z) = \frac{az + b}{cz + d},$$

where $a, b, c, d \in \mathbb{C}$ are fixed and $ad - bc \neq 0$. We write formally

$$(3) \qquad T\left(-\frac{d}{c}\right) = \infty \qquad \text{and} \qquad T(\infty) = \frac{a}{c}.$$

REMARKS. (1) Note that if $ad - bc = 0$, then

$$T'(z) = \frac{ad - bc}{(cz + d)^2} = 0,$$

so that $T(z)$ is constant. Hence the requirement $ad - bc \neq 0$ is essential.

(2) To justify (3), note that the function $T(z)$ has a simple pole at $z = -d/c$, and that

$$\lim_{|z| \to \infty} T(z) = \frac{a}{c}.$$

(3)   Since

$$T'(z) = \frac{ad - bc}{(cz + d)^2} \neq 0$$

for every $z \in \mathbb{C}$ satisfying $z \neq -d/c$, it follows that a Möbius transformation is conformal at every point in $\mathbb{C}$ where it is analytic.

(4)   The case $c = 0$ and $d = 1$ reduces to $T(z) = az + b$, a linear function.

(5)   The case $a = d = 0$ and $b = c = 1$ reduces to $T(z) = 1/z$, the inversion function.

(6)   If $c \neq 0$, then it is easy to check that

(4)
$$\frac{az + b}{cz + d} = \frac{a}{c} + \left(b - \frac{ad}{c} \right) \frac{1}{cz + d}.$$

(7)   Writing $w = T(z)$, then (2) can be written in the form $cwz - az + dw - b = 0$, and this is linear in both z and w. This is the reason for calling such a function a bilinear transformation.

The following result is a generalization of Theorems 13A and 13C.

THEOREM 13D.   *Suppose that $T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a Möbius transformation. Then*
(a) T is the composition of a sequence of translations, magnifications, rotations and inversions;
(b) $T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is one-to-one and onto;
(c) the inverse function $T^{-1} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is also a Möbius transformation;
(d) T maps the class of circles and lines in $\overline{\mathbb{C}}$ to itself; and
(e) for every Möbius transformation $S : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, $S \circ T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is also a Möbius transformation.

PROOF.   (a)   Suppose that

$$T(z) = \frac{az + b}{cz + d},$$

where $ad - bc \neq 0$. If $c = 0$, then we must have $ad \neq 0$, so that

$$T(z) = \frac{az + b}{d} = \frac{a}{d} z + \frac{b}{d}.$$

In this case, T is a linear function, and the result follows from Theorem 13A. On the other hand, if $c \neq 0$, then we use the identity (4). We can write $T = T_3 \circ T_2 \circ T_1$, where

$$T_1(z) = cz + d \qquad \text{and} \qquad T_2(z) = \frac{1}{z} \qquad \text{and} \qquad T_3(z) = \left(b - \frac{ad}{c} \right) z + \frac{a}{c}.$$

It is easy to check that T_1 and T_3 are linear functions, while T_2 is the inversion function. The result now follows from Theorem 13A.

(b) and (d) follow from (a) on noting that translations, magnifications, rotations and inversions all have the properties in question, and that composition of functions preserves these properties.

(c) and (e) are left as exercises.   ◯

EXAMPLE 13.3.1. Suppose that $a \in \mathbb{C}$ is fixed and $|a| < 1$. Consider the Möbius transformation $T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, given by

$$T(z) = \frac{a - z}{1 - \overline{a}z},$$

where $\overline{a} \in \mathbb{C}$ denotes the complex conjugate of a. Note that

$$|T(z)|^2 = \frac{|a - z|^2}{|1 - \overline{a}z|^2} = \frac{|a|^2 - 2\mathfrak{Re}(\overline{a}z) + |z|^2}{1 - 2\mathfrak{Re}(\overline{a}z) + |a|^2|z|^2}.$$

It is easy to see that if $|z| = 1$, then $|T(z)| = 1$. It follows from Theorem 13D(d) that the image under T of the unit circle $\{z : |z| = 1\}$ is the unit circle itself. On the other hand, the inequality $|T(z)| < 1$ is equivalent to the inequality

$$|a|^2 + |z|^2 < 1 + |a|^2|z|^2,$$

which is equivalent to the inequality

$$(1 - |a|^2)(1 - |z|^2) > 0,$$

which is equivalent to the inequality $|z| < 1$, in view of the assumption $|a| < 1$. It now follows from this observation and Theorem 13D(b) that the interior $\{z : |z| < 1\}$ of the unit circle must be mapped onto itself by T.

DEFINITION. A fixed point $z \in \overline{\mathbb{C}}$ of a Möbius transformation $T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a solution of the equation $T(z) = z$.

THEOREM 13E. *A Möbius transformation $T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ has at most two distinct fixed points in $\overline{\mathbb{C}}$ unless $T(z) = z$ identically.*

PROOF. Suppose that

$$T(z) = \frac{az + b}{cz + d},$$

where $ad - bc \neq 0$. If $c = 0$, then $ad \neq 0$, so that T is a linear function. In this case, the equation $T(z) = z$ becomes $az + b = dz$. If $T(z)$ is not identically equal to z, then $a \neq d$ or $b \neq 0$, so that this equation has at most one solution in $\mathbb{C}$. Suppose next that $c \neq 0$. Then clearly ∞ is not a fixed point. The equation $T(z) = z$ is now a quadratic equation, and so has at most two distinct roots in $\mathbb{C}$. $\bigcirc$

It follows from Theorem 13E that a Möbius transformation must be the identity function if it has three fixed points. Suppose now that S and T are Möbius transformations, and that there exist distinct $z_1, z_2, z_3 \in \overline{\mathbb{C}}$ such that $S(z_j) = T(z_j)$ for $j = 1, 2, 3$. By Theorem 13D(c)(e), the composition $S^{-1} \circ T$ is also a Möbius transformation. Clearly $(S^{-1} \circ T)(z_j) = z_j$ for $j = 1, 2, 3$, so that $S^{-1} \circ T$ has three fixed points, and so must be the identity function. In other words, $(S^{-1} \circ T)(z) = z$, and so $S(z) = T(z)$, for every $z \in \overline{\mathbb{C}}$. We summarize this observation below.

THEOREM 13F. *Suppose that two Möbius transformations $S : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ and $T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ are equal at three distinct points in $\overline{\mathbb{C}}$. Then $S(z) = T(z)$ for every $z \in \overline{\mathbb{C}}$.*

Next, we show that three points determine uniquely a Möbius transfomation.

THEOREM 13G. *Suppose that $z_1, z_2, z_3 \in \overline{\mathbb{C}}$ are distinct, and that $w_1, w_2, w_3 \in \overline{\mathbb{C}}$ are also distinct. Then there exists a unique Möbius transformation $T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that $T(z_j) = w_j$ for $j = 1, 2, 3$.*

PROOF. To establish the existence of such a function, note that $T_1 : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, given by

$$T_1(z) = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)},$$

is a Möbius transformation, with $T_1(z_1) = 0$, $T_1(z_2) = 1$ and $T_1(z_3) = \infty$. Similarly, $T_2 : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, given by

$$T_2(w) = \frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)},$$

is a Möbius transformation, with $T_2(w_1) = 0$, $T_2(w_2) = 1$ and $T_2(w_3) = \infty$. Clearly $T = T_2^{-1} \circ T_1$ is a Möbius transformation such that $T(z_j) = w_j$ for $j = 1, 2, 3$. The uniqueness follows from Theorem 13F. $\bigcirc$

EXAMPLE 13.3.2. To find a Möbius transformation $T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that $T(0) = 2$, $T(1) = 3$ and $T(6) = 4$, note that $T_1 : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, given by

$$T_1(z) = \frac{(z - 0)(1 - 6)}{(z - 6)(1 - 0)} = \frac{-5z}{z - 6},$$

is a Möbius transformation, with $T_1(0) = 0$, $T_1(1) = 1$ and $T_1(6) = \infty$. Similarly, $T_2 : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, given by

$$T_2(w) = \frac{(w - 2)(3 - 4)}{(w - 4)(3 - 2)} = \frac{-w + 2}{w - 4},$$

is a Möbius transformation, with $T_2(2) = 0$, $T_2(3) = 1$ and $T_2(4) = \infty$. We now have to calculate $T = T_2^{-1} \circ T_1$. Note that

$$T_2^{-1}(z) = \frac{4z + 2}{z + 1},$$

so that

$$T(z) = \frac{4\left(\dfrac{-5z}{z - 6}\right) + 2}{\left(\dfrac{-5z}{z - 6}\right) + 1} = \frac{-20z + 2(z - 6)}{-5z + (z - 6)} = \frac{-18z - 12}{-4z - 6} = \frac{9z + 6}{2z + 3}.$$

13.4. Finding Particular Möbius Transformations

Recall that a Möbius transformation $T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, given by

$$T(z) = \frac{az + b}{cz + d},$$

where $ad - bc \neq 0$, maps the class of circles and lines in $\overline{\mathbb{C}}$ to itself. Suppose that a circle or line contains the pole $z = -d/c$ of T, then its image under T is unbounded, and is therefore a line rather than a circle. Suppose, on the other hand, that a circle or line does not contain the pole $z = -d/c$ of T, then its image under T cannot contain the point at ∞, and is therefore a circle rather than a line.

Note next that a circle or line splits the extended complex plane into two domains. Here we adopt the convention that a line contains the point at ∞, whereas a half plane not including its boundary line does not contain the point at ∞. Since $T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is one-to-one and onto, and since an analytic function

maps domains to domains, it follows that the image of any domain arising from a circle or line must be mapped onto a domain arising from the image of this circle or line under T.

REMARK. Strictly speaking, the function

$$T(z) = \frac{az+b}{cz+d}$$

is one-to-one, onto and analytic if we take the domain of T to be $\mathbb{C} \setminus \{-d/c\}$ and the codomain of T to be $\mathbb{C} \setminus \{a/c\}$.

EXAMPLE 13.4.1. Suppose that $z_0, w_0 \in \mathbb{C}$ and $r_1, r_2 > 0$ are fixed, and that we are required to find a Möbius transformation $T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ which maps the disc $\{z : |z - z_0| < r_1\}$ to the annulus $\{w : |w - w_0| > r_2\}$. This can be achieved by taking $T = T_4 \circ T_3 \circ T_2 \circ T_1$, where

$$T_1(z) = z - z_0 \quad \text{and} \quad T_2(z) = \frac{1}{z} \quad \text{and} \quad T_3(z) = r_1 r_2 z \quad \text{and} \quad T_4(z) = z + w_0.$$

We have the picture below:
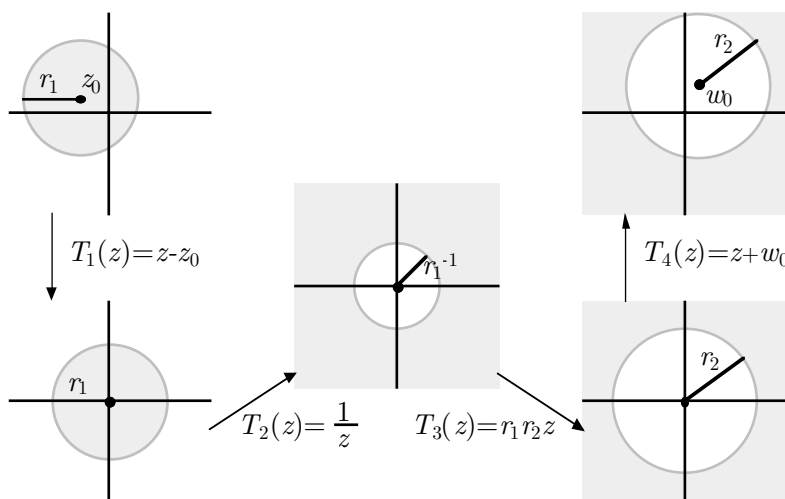
Note that T_1 is a translation which takes the centre of the disc $\{z : |z - z_0| < r_1\}$ to the origin. Then the inversion T_2 turns a disc into an annulus. We now apply a magnification T_3 and then use the translation T_4 to position the disc so that its centre is at w_0. It is easy to see that

$$T(z) = \frac{r_1 r_2}{z - z_0} + w_0 = \frac{w_0 z + (r_1 r_2 - w_0 z_0)}{z - z_0}.$$

EXAMPLE 13.4.2. Suppose that we are required to find a Möbius transformation T which maps the unit disc $\{z : |z| < 1\}$ to the right half plane $\{w : \mathfrak{Re}\, w > 0\}$.
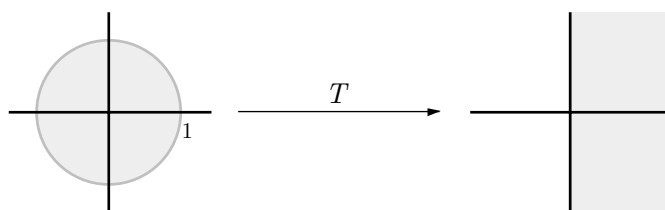
Our first step is to find a Möbius transformation $S : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ which maps the unit circle $\{z : |z| = 1\}$ to the imaginary axis $\{w : \mathfrak{Re}\, w = 0\}$. For

$$S(z) = \frac{az + b}{cz + d}$$

to map the unit circle to a line, the unit circle must contain the pole $z = -d/c$ of S. Suppose that we choose the point $z = 1$ to be this pole. In this case, we may take, for example, $c = 1$ and $d = -1$. Next, some point on the unit circle must have image 0 under S. Suppose that we choose $z = -1$ to be this point. In this case, we may take, for example, $a = 1$ and $b = 1$. Note that $ad - bc \neq 0$, and these choices give

$$S(z) = \frac{z + 1}{z - 1}.$$

Note that $S(1) = \infty$ and $S(-1) = 0$. We also know that the image of the unit circle under S is a line through the origin, but at this point, we do not know whether this line is the imaginary axis. To check what this line is, we use a third point on the unit circle, the point $z = \mathrm{i}$, say. It is easy to check that $S(\mathrm{i}) = -\mathrm{i}$, on the imaginary axis. We therefore conclude that S maps the unit circle to the imaginary axis. It follows that S maps the unit disc $\{z : |z| < 1\}$ to one of the half planes arising from the imaginary axis, but at this point, we do not know whether it is $\{w : \mathfrak{Re}\, w < 0\}$ or $\{w : \mathfrak{Re}\, w > 0\}$. To check which half plane this is, we can use the point $z = 0$. It is easy to check that $S(0) = -1$. Unfortunately, this is in $\{w : \mathfrak{Re}\, w < 0\}$ instead of $\{w : \mathfrak{Re}\, w > 0\}$. This little problem can be eradicated by rotating $S(z)$ about the origin by an angle π. In other words, the Möbius transformation

$$T(z) = \mathrm{e}^{\mathrm{i}\pi} S(z) = -S(z) = \frac{-z - 1}{z - 1}$$

satisfies our requirements.

EXAMPLE 13.4.3. Suppose that we are required to find a Möbius transformation S which maps the half plane $\{z = x + \mathrm{i}y : y < x\}$ to the annulus $\{w : |w - 3| > 5\}$.
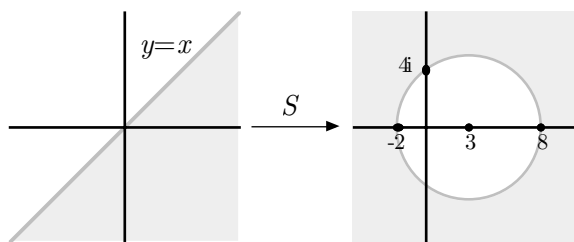
Our first step is to find a Möbius transformation $S : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ which maps the line $\{z = x + \mathrm{i}y : y = x\}$ to the circle $\{w : |w - 3| = 5\}$. Consider first of all the transformation

$$S_1(z) = \sqrt{2}\mathrm{e}^{-\mathrm{i}\pi/4} z = (1 - \mathrm{i})z$$

(the magnification here by $\sqrt{2}$ serves only to simplify the arithmetic), where we attempt to map the line $\{z = x + \mathrm{i}y : y = x\}$ to the real axis $\{z : \mathfrak{Im}\, z = 0\}$. Next, we shall find a Möbius transformation S_2 which maps the real axis $\{z : \mathfrak{Im}\, z = 0\}$ to the circle $\{w : |w - 3| = 5\}$. To do this, we shall use the Möbius transformation S_2 which maps the points $0, 1, \infty$, say, on the real axis to the points $8, 4\mathrm{i}, -2$, say, on the circle. From the proof of Theorem 13G, the inverse Möbius transformation S_2^{-1} is given by

$$z = S_2^{-1}(w) = \frac{(w - 8)(4\mathrm{i} + 2)}{(w + 2)(4\mathrm{i} - 8)} = \frac{w - 8}{2\mathrm{i}(w + 2)}.$$

Simple calculation gives

$$w = S_2(z) = \frac{4z - 8\mathrm{i}}{-2z - \mathrm{i}},$$

and so the Möbius transformation $S = S_2 \circ S_1$, given by

$$S(z) = (S_2 \circ S_1)(z) = \frac{4(1 - \mathrm{i})z - 8\mathrm{i}}{-2(1 - \mathrm{i})z - \mathrm{i}},$$

maps the line $\{z = x + \mathrm{i}y : y = x\}$ to the circle $\{w : |w - 3| = 5\}$. It follows that S maps the half plane $\{z = x + \mathrm{i}y : y < x\}$ to the disc $\{w : |w - 3| < 5\}$ or the annulus $\{w : |w - 3| > 5\}$. To check which this is, we can use the point $z = 1$. It is easy to check that $S(1) = -4 + 4\mathrm{i}$. This is in the annulus $\{w : |w - 3| > 5\}$. It follows that

$$S(z) = \frac{4(1 - \mathrm{i})z - 8\mathrm{i}}{-2(1 - \mathrm{i})z - \mathrm{i}}$$

satisfies our requirements.

13.5. Symmetry and Möbius Transformations

DEFINITION. We say that two points $z_1, z_2 \in \mathbb{C}$ are symmetric with respect to a line L if L is the perpendicular bisector of the line segment joining z_1 and z_2.

Suppose that $z_1, z_2 \in \mathbb{C}$ are symmetric with respect to a line L. Then it is easy to see that every circle or line passing through both z_1 and z_2 intersects L at right angles.
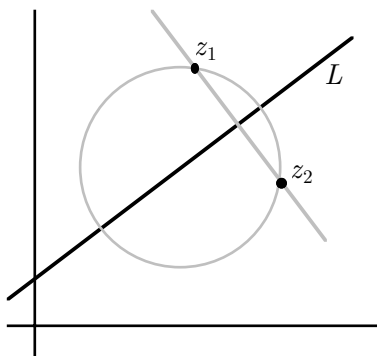
Using this observation, we make the following definition.

DEFINITION. We say that two points $z_1, z_2 \in \mathbb{C}$ are symmetric with respect to a circle C if every circle or line passing through both z_1 and z_2 intersects C at right angles.
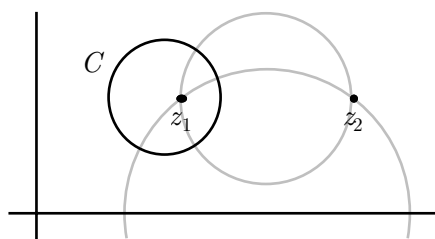
REMARKS. (1) Consider the circle $C = \{z : |z - z_0| = r\}$ with centre z_0 and radius r. Then it can be shown that two points z_1 inside C and z_2 outside C are symmetric with respect to C if and only if there exist $\rho, \theta \in \mathbb{R}$ satisfying $0 < \rho < r$ and such that

$$z_1 = z_0 + \rho e^{i\theta} \qquad \text{and} \qquad z_2 = z_0 + \frac{r^2}{\rho} e^{i\theta};$$

in other words, if and only if $(z_1 - z_0)(\overline{z_2 - z_0}) = r^2$.

(2) We also say that the centre of a circle C and the point at ∞ are symmetric with respect to the circle C.

(3) Note that a line can be interpreted as a circle of infinite radius. It follows that our definition covers symmetry with respect to both lines and circles.

THEOREM 13H. (SYMMETRY PRINCIPLE) *Suppose that $T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a Möbius transformation. Suppose further that C is a circle or line in $\overline{\mathbb{C}}$. Then two points $z_1, z_2 \in \overline{\mathbb{C}}$ are symmetric with respect to C if and only if $T(z_1)$ and $T(z_2)$ are symmetric with respect to $T(C)$.*

PROOF. Note that T maps the class of lines and circles in $\overline{\mathbb{C}}$ to itself. Note also that T is conformal at all points where it is analytic, and so preserves orthogonality. ◯

EXAMPLE 13.5.1. Let us return to Example 13.3.1. Suppose that $a \in \mathbb{C}$ is fixed and $|a| < 1$. Suppose also that $\lambda \in \mathbb{C}$ is fixed and $|\lambda| = 1$. Then the Möbius transformation $T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, given by

$$T(z) = \lambda \frac{z - a}{\overline{a}z - 1},$$

maps the unit disc $D = \{z : |z| < 1\}$ onto itself. Note here that we have introduced an extra rotation λ about the origin. We shall now attempt to show that any Möbius transformation $T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ which maps the unit disc D onto itself must be of this form. Clearly T maps the unit circle $C = \{z : |z| = 1\}$ onto itself. Next, let $a \in \mathbb{C}$ be the unique point satisfying $T(a) = 0$. Then $|a| < 1$. Suppose now that a and a^* are symmetric with respect to the unit circle C. Then by the Symmetry principle, $T(a)$ and $T(a^*)$ are symmetric with respect to the circle $T(C) = C$. Since $T(a) = 0$, we must have $T(a^*) = \infty$. It follows that $T(z)$ must have a zero at $z = a$ and a pole at $z = a^*$. Note now that $a\overline{a^*} = 1$, so that $a^* = 1/\overline{a}$. Hence

$$T(z) = \lambda \frac{z - a}{\overline{a}z - 1}$$

for some $\lambda \in \mathbb{C}$. Recall now that $|T(z)| = 1$ whenever $|z| = 1$. In particular, we require

$$|T(1)| = \left| \lambda \frac{1 - a}{\overline{a} - 1} \right| = 1.$$

It follows that $|\lambda| = 1$.

<center>PROBLEMS FOR CHAPTER 13</center>

1. Find a Möbius transformation that takes the points $0, 2, -2$ to the points $-2, 0, 2$ respectively.

2. Show that the Möbius transformation $w = \dfrac{z - i}{z + i}$ maps the upper half plane $\{z : \mathfrak{Im}\, z > 0\}$ onto the disc $\{w : |w| < 1\}$.

3. Suppose that C is a given circle or line, and that C' is also a given circle or line. Does there exist a Möbius transformation that maps C onto C'? If so, is this Möbius transformation unique? Justify your assertions.

4. The cross ratio of four distinct points $z_1, z_2, z_3, z_4 \in \overline{\mathbb{C}}$ is defined by

$$X(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$$

and by the obvious limit if one of the points is ∞. Show that the cross ratio is invariant under Möbius transformation; in other words, for every Möbius transformation $T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, we have

$$X(T(z_1), T(z_2), T(z_3), T(z_4)) = X(z_1, z_2, z_3, z_4).$$

[HINT: Note that every Möbius transformation is a composition of translations, rotations, magnifications and inversions.]

5. Use the invariance of the cross ratio to find a Möbius transformation that takes the points $0, 1, \infty$ to the points $-i, 1, i$ respectively.
[HINT: Suppose that the Möbius transformation takes z to w.]

6. Show that a Möbius transformation $w = f(z)$ maps the upper half plane $\{z : \mathfrak{Im} z > 0\}$ onto the disc $\{w : |w| < 1\}$ if and only if it is of the form

$$w = \lambda \frac{z - a}{z - \overline{a}},$$

where $a, \lambda \in \mathbb{C}$ satisfy $|\lambda| = 1$ and $\mathfrak{Im} a > 0$.

7. a) Construct a one-parameter family of Möbius transformations that map the real axis onto the unit circle by mapping the points $0, \lambda, \infty$ to the points $-i, 1, i$ respectively, where λ is a non-zero real parameter.
   b) What point of the upper half plane gets mapped to the centre of the circle?
   c) For what values of λ is the upper half plane $\{z : \mathfrak{Im} z > 0\}$ mapped onto the disc $\{w : |w| < 1\}$? Onto the annulus $\{w : |w| > 1\}$?
   d) Taking note of Problem 6, comment whether the family includes all Möbius transformations that map the real axis onto the unit circle.

8. Find all Möbius transformations that map the disc $\{z : |z - 1| < 2\}$ onto the upper half plane $\{w : \mathfrak{Im} w > 0\}$ and takes $z = 1$ to $w = i$.

9. Show that if either of the transformations $w = a + \dfrac{bz}{1 - cz}$ and $w = c + \dfrac{bz}{1 - az}$ maps the unit disc onto the unit disc, then they both do.

10. Find a transformation that maps $A = \{z = x + iy : |z| < 1 \text{ and } y > 0\}$, the upper half of the unit disc, onto the first quadrant.