

Chapter Nine

Section 9.1

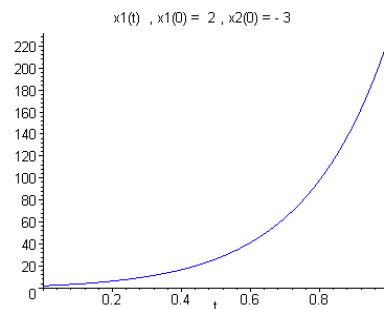
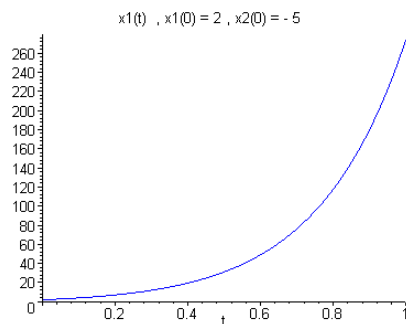
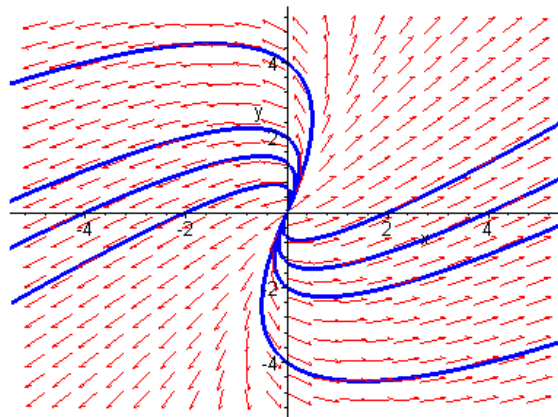
2(a). Setting $\mathbf{x} = \boldsymbol{\xi} e^{rt}$ results in the algebraic equations

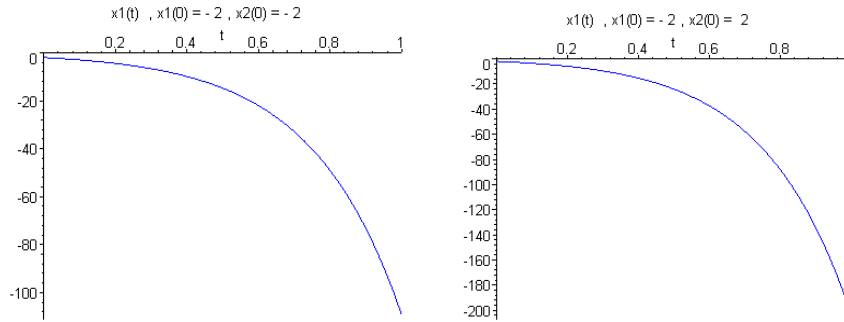
$$\begin{pmatrix} 5 - r & -1 \\ 3 & 1 - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 - 6r + 8 = 0$. The roots of the characteristic equation are $r_1 = 2$ and $r_2 = 4$. For $r = 2$, the system of equations reduces to $3\xi_1 = \xi_2$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1, 3)^T$. Substitution of $r = 4$ results in the single equation $\xi_1 = \xi_2$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (1, 1)^T$.

(b). The eigenvalues are *real* and *positive*, hence the critical point is an *unstable node*.

(c, d).





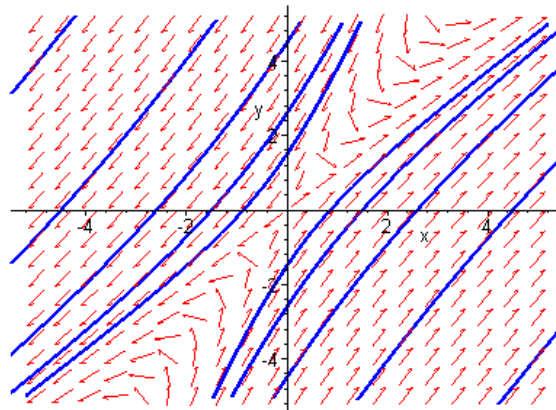
3(a). Solution of the ODE requires analysis of the algebraic equations

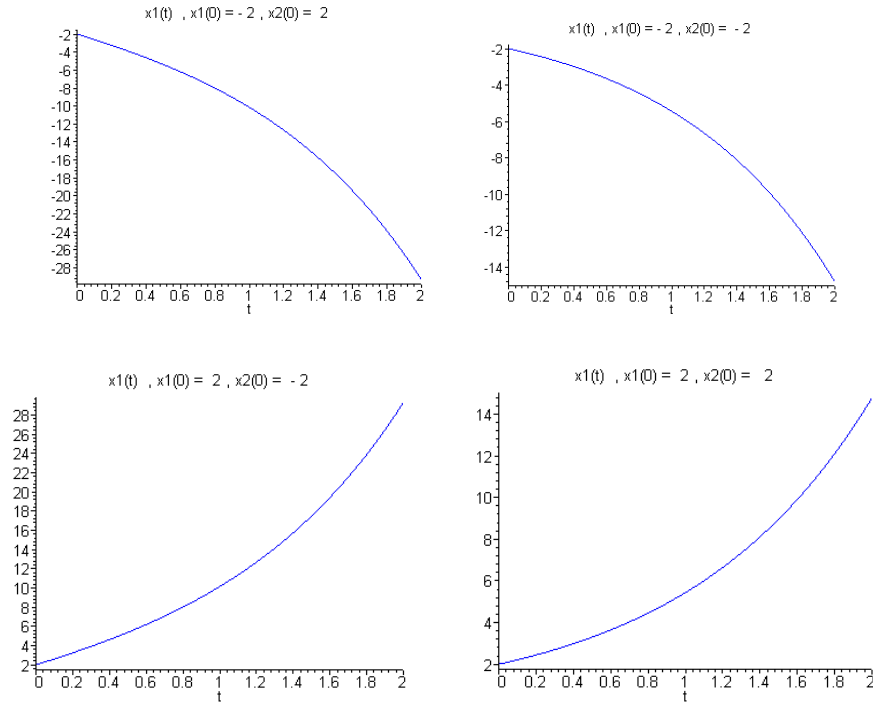
$$\begin{pmatrix} 2 - r & -1 \\ 3 & -2 - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 - 1 = 0$. The roots of the characteristic equation are $r_1 = 1$ and $r_2 = -1$. For $r = 1$, the system of equations reduces to $\xi_1 = \xi_2$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1, 1)^T$. Substitution of $r = -1$ results in the single equation $3\xi_1 - \xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (1, 3)^T$.

(b). The eigenvalues are *real*, with $r_1 r_2 < 0$. Hence the critical point is a *saddle*.

(c, d).





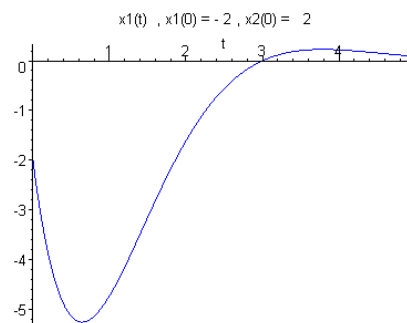
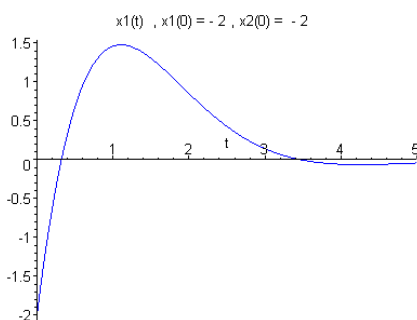
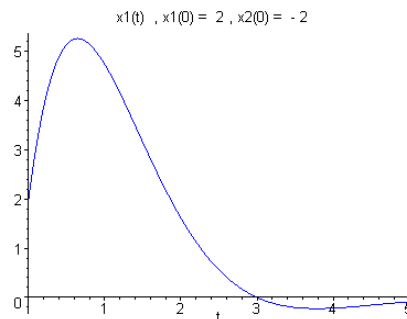
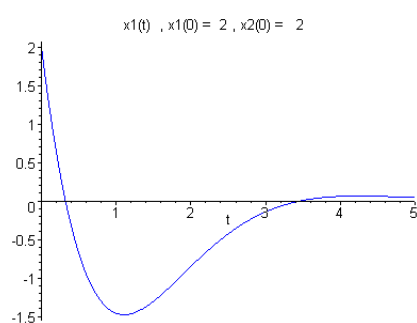
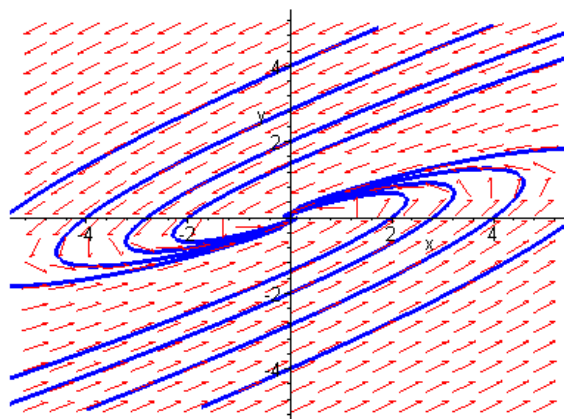
5(a). The characteristic equation is given by

$$\begin{vmatrix} 1 - r & -5 \\ 1 & -3 - r \end{vmatrix} = r^2 + 2r + 2 = 0.$$

The equation has *complex roots* $r_1 = -1 + i$ and $r_2 = -1 - i$. For $r = -1 + i$, the components of the solution vector must satisfy $\xi_1 - (2 + i)\xi_2 = 0$. Thus the corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (2 + i, 1)^T$. Substitution of $r = -1 - i$ results in the single equation $\xi_1 - (2 - i)\xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (2 - i, 1)^T$.

(b). The eigenvalues are *complex conjugates*, with negative real part. Hence the origin is a *stable spiral*.

(c, d).



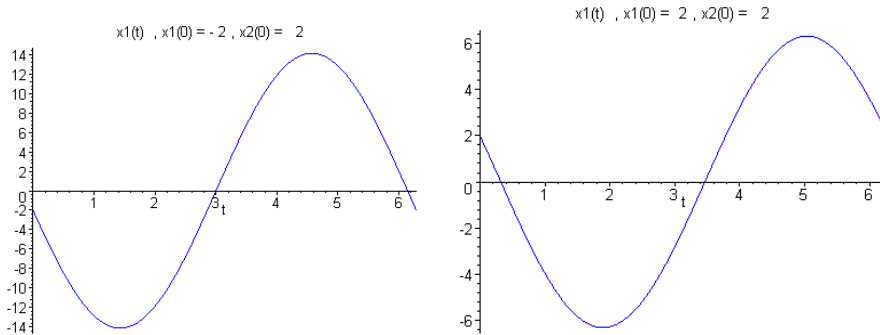
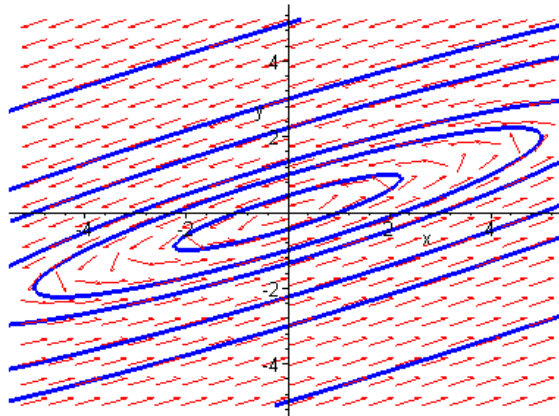
6(a). Solution of the ODEs is based on the analysis of the algebraic equations

$$\begin{pmatrix} 2 - r & -5 \\ 1 & -2 - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we require that $\det(\mathbf{A} - r\mathbf{I}) = r^2 + 1 = 0$. The roots of the characteristic equation are $r = \pm i$. Setting $r = i$, the equations are equivalent to $\xi_1 - (2 + i)\xi_2 = 0$. The eigenvectors are $\boldsymbol{\xi}^{(1)} = (2 + i, 1)^T$ and $\boldsymbol{\xi}^{(2)} = (2 - i, 1)^T$.

(b). The eigenvalues are *purely imaginary*. Hence the critical point is a *center*.

(c, d).



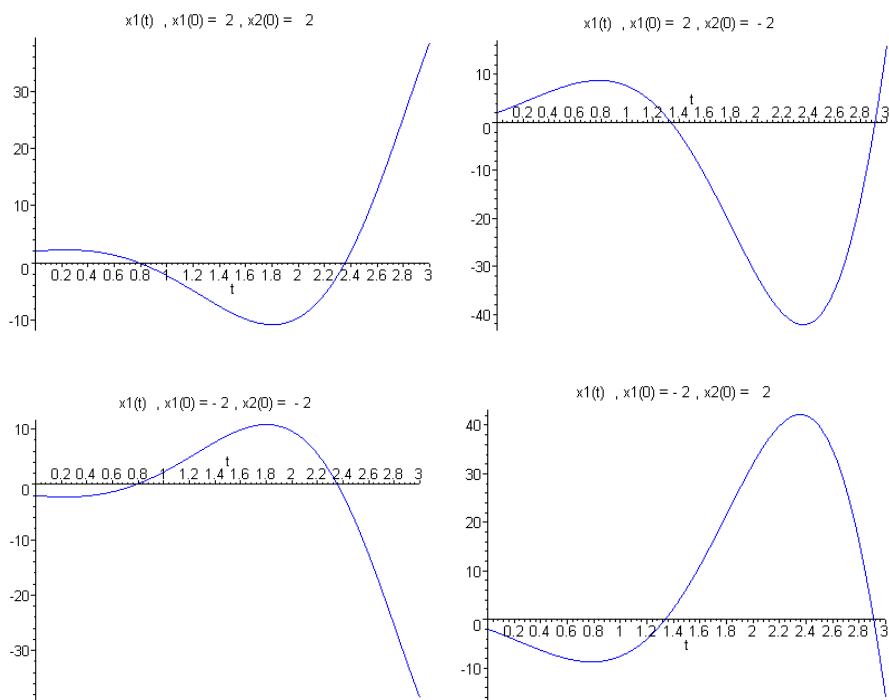
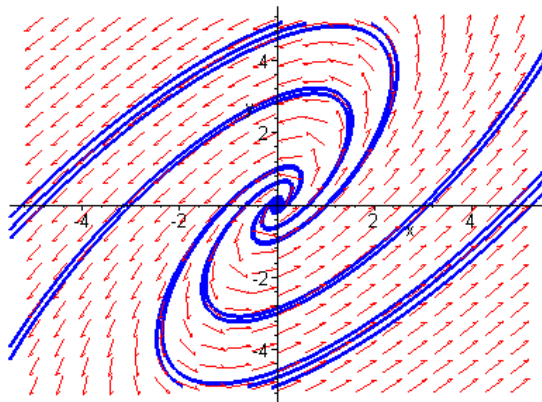
7(a). Setting $\mathbf{x} = \boldsymbol{\xi} e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 3 - r & -2 \\ 4 & -1 - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we require that $\det(\mathbf{A} - r\mathbf{I}) = r^2 - 2r + 5 = 0$. The roots of the characteristic equation are $r = 1 \pm 2i$. Substituting $r = 1 - 2i$, the two equations reduce to $(1 + i)\xi_1 - \xi_2 = 0$. The two eigenvectors are $\boldsymbol{\xi}^{(1)} = (1, 1 + i)^T$ and $\boldsymbol{\xi}^{(2)} = (1, 1 - i)^T$.

(b). The eigenvalues are *complex conjugates*, with positive real part. Hence the origin is an *unstable spiral*.

(c, d).



8(a). The characteristic equation is given by

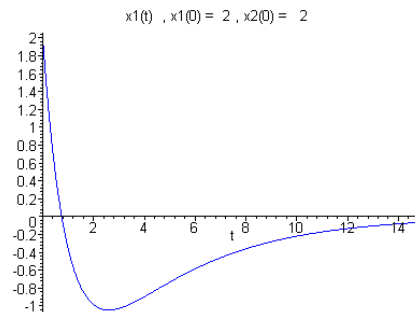
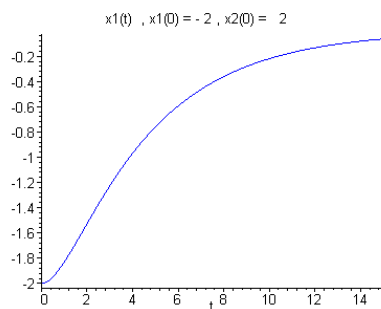
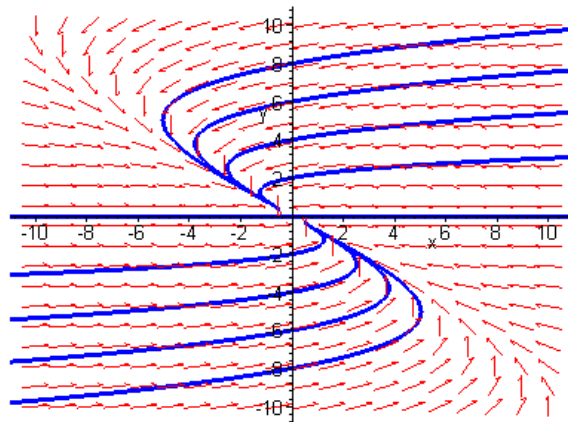
$$\begin{vmatrix} 1 - r & -5 \\ 1 & -3 - r \end{vmatrix} = (r + 1)(r + 0.25) = 0,$$

with roots $r_1 = -1$ and $r_2 = -0.25$. For $r = -1$, the components of the solution vector must satisfy $\xi_2 = 0$. Thus the corresponding eigenvector is $\xi^{(1)} = (1, 0)^T$. Substitution of $r = -0.25$ results in the single equation $0.75 \xi_1 + \xi_2 = 0$. A corresponding eigenvector is $\xi^{(2)} = (4, -3)^T$.

(b). The eigenvalues are *real* and both *negative*. Hence the critical point is a *stable*

node.

(c, d).



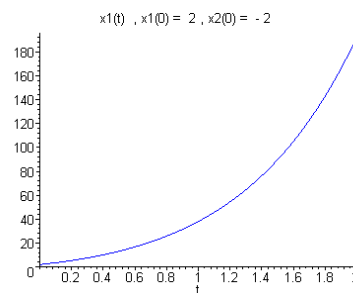
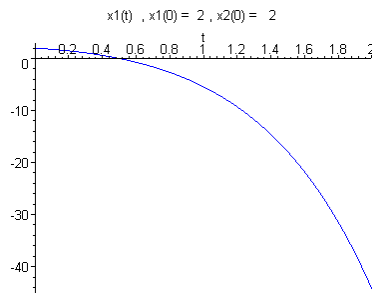
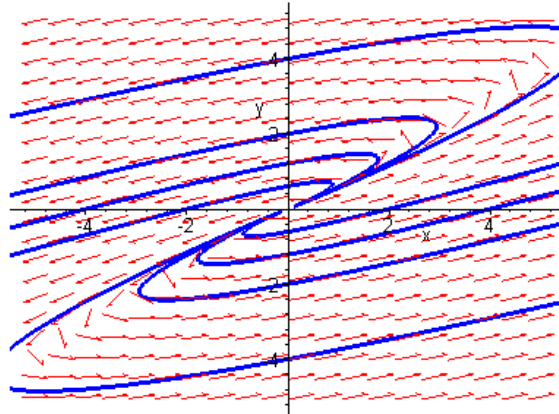
9(a). Solution of the ODEs is based on the analysis of the algebraic equations

$$\begin{pmatrix} 3 - r & -4 \\ 1 & -1 - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we require that $\det(\mathbf{A} - r\mathbf{I}) = r^2 - 2r + 1 = 0$. The single root of the characteristic equation is $r = 1$. Setting $r = 1$, the components of the solution vector must satisfy $\xi_1 - 2\xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi} = (2, 1)^T$.

(b). Since there is only one linearly independent eigenvector, the critical point is an *unstable, improper node*.

(c, d).



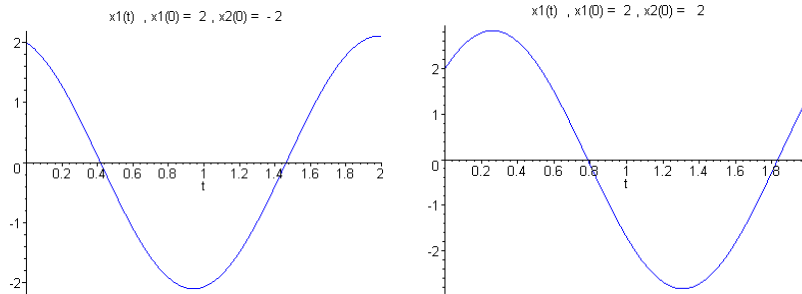
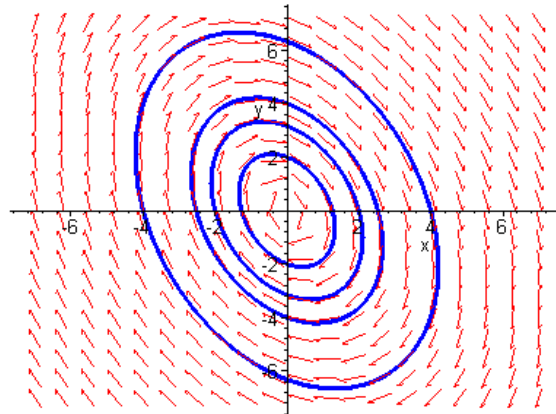
10(a). The characteristic equation is given by

$$\begin{vmatrix} 1 - r & 2 \\ -5 & -1 - r \end{vmatrix} = r^2 + 9 = 0.$$

The equation has *complex* roots $r_{1,2} = \pm 3i$. For $r = -3i$, the components of the solution vector must satisfy $5\xi_1 + (1 - 3i)\xi_2 = 0$. Thus the corresponding eigenvector is $\xi^{(1)} = (1 - 3i, -5)^T$. Substitution of $r = 3i$ results in $5\xi_1 + (1 + 3i)\xi_2 = 0$. A corresponding eigenvector is $\xi^{(2)} = (1 + 3i, -5)^T$.

(b). The eigenvalues are *purely imaginary*, hence the critical point is a *center*.

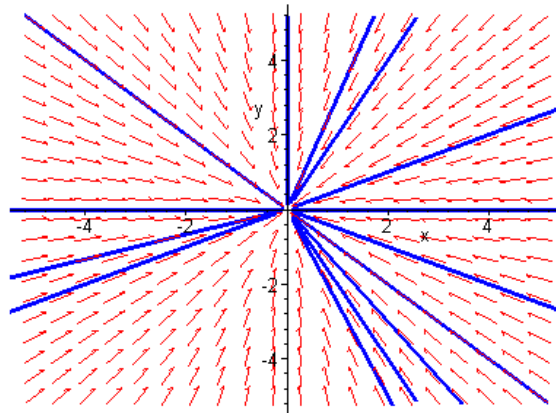
(c, d).

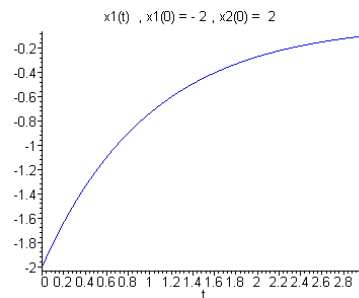
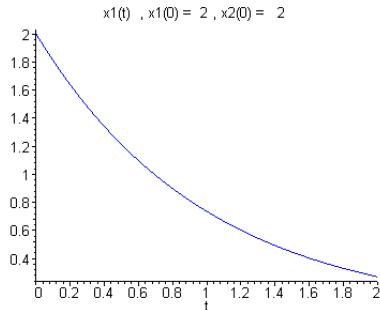


11(a). The characteristic equation is $(r + 1)^2 = 0$, with double root $r = -1$. It is easy to see that the two linearly independent eigenvectors are $\xi^{(1)} = (1, 0)^T$ and $\xi^{(2)} = (0, 1)^T$.

(b). Since there are two linearly independent eigenvectors, the critical point is a *stable proper node*.

(c, d).





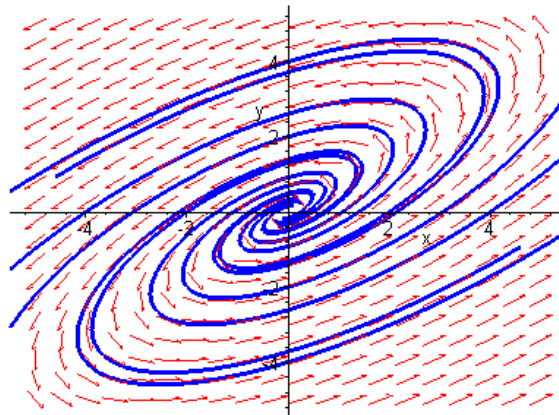
12(a). Setting $\mathbf{x} = \boldsymbol{\xi} e^{rt}$ results in the algebraic equations

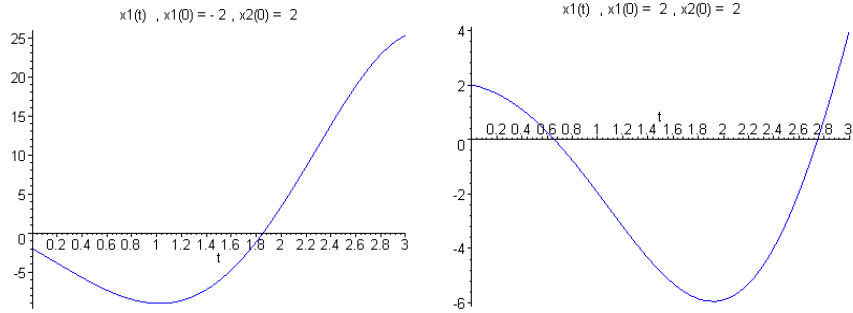
$$\begin{pmatrix} 2 - r & -5/2 \\ 9/5 & -1 - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we require that $\det(\mathbf{A} - r\mathbf{I}) = r^2 - r + 5/2 = 0$. The roots of the characteristic equation are $r = 1/2 \pm 3i/2$. Substituting $r = 1/2 - 3i/2$, the equations reduce to $(3 + 3i)\xi_1 - 5\xi_2 = 0$. Therefore the two eigenvectors are $\boldsymbol{\xi}^{(1)} = (5, 3 + 3i)^T$ and $\boldsymbol{\xi}^{(2)} = (5, 3 - 3i)^T$.

(b). Since the eigenvalues are *complex*, with *positive* real part, the critical point is an *unstable spiral*.

(c, d).





14. Setting $\mathbf{x}' = \mathbf{0}$, that is,

$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 \\ -1 \end{pmatrix},$$

we find that the critical point is $\mathbf{x}^0 = (-1, 0)^T$. With the change of dependent variable, $\mathbf{x} = \mathbf{x}^0 + \mathbf{u}$, the differential equation can be written as

$$\frac{d\mathbf{u}}{dt} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{u}.$$

The critical point for the transformed equation is the origin. Setting $\mathbf{u} = \boldsymbol{\xi} e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} -2-r & 1 \\ 1 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we require that $\det(\mathbf{A} - r\mathbf{I}) = r^2 + 4r + 3 = 0$. The roots of the characteristic equation are $r = -3, -1$. Hence the critical point is a *stable node*.

15. Setting $\mathbf{x}' = \mathbf{0}$, that is,

$$\begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ -5 \end{pmatrix},$$

we find that the critical point is $\mathbf{x}^0 = (-2, 1)^T$. With the change of dependent variable, $\mathbf{x} = \mathbf{x}^0 + \mathbf{u}$, the differential equation can be written as

$$\frac{d\mathbf{u}}{dt} = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix} \mathbf{u}.$$

The characteristic equation is $\det(\mathbf{A} - r\mathbf{I}) = r^2 + 2r + 3 = 0$, with complex conjugate roots $r = -1 \pm i\sqrt{2}$. Since the real parts of the eigenvalues are *negative*, the critical point is a *stable spiral*.

16. The critical point \mathbf{x}^0 satisfies the system of equations

$$\begin{pmatrix} 0 & -\beta \\ \delta & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -\alpha \\ \gamma \end{pmatrix}.$$

It follows that $x^0 = \gamma/\delta$ and $y^0 = \alpha/\beta$. Using the transformation, $\mathbf{x} = \mathbf{x}^0 + \mathbf{u}$, the differential equation can be written as

$$\frac{d\mathbf{u}}{dt} = \begin{pmatrix} 0 & -\beta \\ \delta & 0 \end{pmatrix} \mathbf{u}.$$

The characteristic equation is $\det(\mathbf{A} - r\mathbf{I}) = r^2 + \beta\delta = 0$. Since $\beta\delta > 0$, the roots are purely imaginary, with $r = \pm i\sqrt{\beta\delta}$. Hence the critical point is a *center*.

20. The system of ODEs can be written as

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mathbf{x}.$$

The characteristic equation is $r^2 - pr + q = 0$. The roots are given by

$$r_{1,2} = \frac{p \pm \sqrt{p^2 - 4q}}{2} = \frac{p \pm \sqrt{\Delta}}{2}.$$

The results can be verified using Table 9.1.1.

21(a). If $q > 0$ and $p < 0$, then the roots are either complex conjugates with negative real parts, or both real and negative.

(b). If $q > 0$ and $p = 0$, then the roots are purely imaginary.

(c). If $q < 0$, then the roots are real, with $r_1 \cdot r_2 > 0$. If $p > 0$, then either the roots are real, with $r_1 \cdot r_2 \geq 0$ or the roots are complex conjugates with positive real parts.

Section 9.2

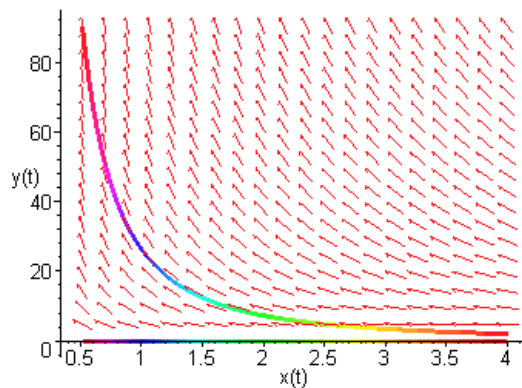
2. The differential equations can be combined to obtain a related ODE

$$\frac{dy}{dx} = -\frac{2y}{x}.$$

The equation is *separable*, with

$$\frac{dy}{y} = -\frac{2 dx}{x}.$$

The solution is given by $y = C x^{-2}$. Note that the system is *uncoupled*, and hence we also have $x = x_0 e^{-t}$ and $y = y_0 e^{2t}$.



In order to determine the direction of motion along the trajectories, observe that for *positive* initial conditions, x will *decrease*, whereas y will *increase*.

4. The trajectories of the system satisfy the ODE

$$\frac{dy}{dx} = -\frac{bx}{ay}.$$

The equation is *separable*, with

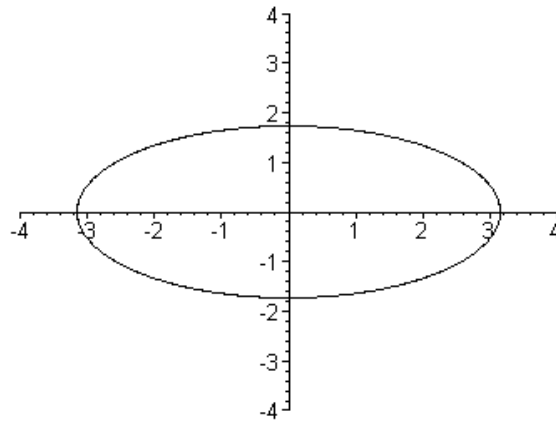
$$ay dy = -bx dx.$$

Hence the trajectories are given by $bx^2 + ay^2 = C^2$, in which C is arbitrary. Evidently, the trajectories are *ellipses*. Invoking the initial condition, we find that $C^2 = ab$. The system of ODEs can also be written as

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix} \mathbf{x}.$$

Using the methods in Chapter 7, it is easy to show that

$$\begin{aligned} x &= \sqrt{a} \cos \sqrt{ab} t \\ y &= -\sqrt{b} \sin \sqrt{ab} t. \end{aligned}$$



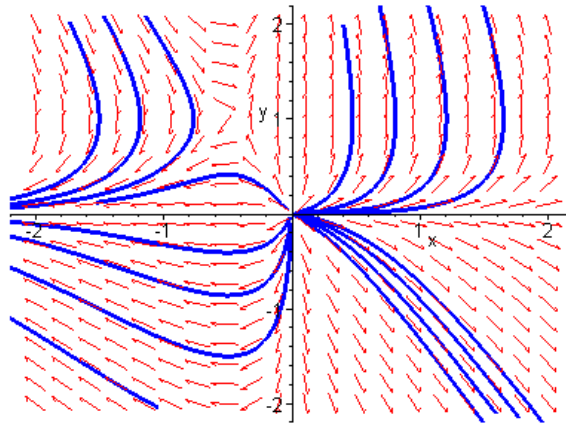
Note that for *positive* initial conditions, x will *increase*, whereas y will *decrease*.

5(a). The critical points are given by the solution set of the equations

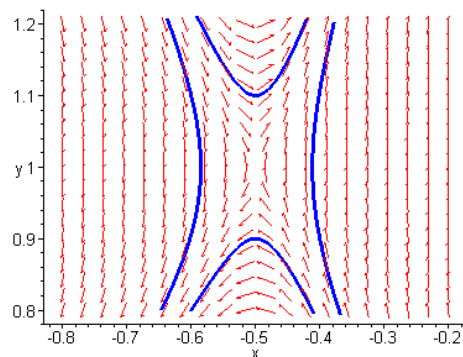
$$\begin{aligned}x(1 - y) &= 0 \\y(1 + 2x) &= 0.\end{aligned}$$

Clearly, $(0, 0)$ is a solution. If $x \neq 0$, then $y = 1$ and $x = -1/2$. Hence the critical points are $(0, 0)$ and $(-1/2, 1)$.

(b).



(c). Based on the phase portrait, all trajectories starting near the origin *diverge*. Hence the critical point $(0, 0)$ is *unstable*. Examining the phase curves near the critical point $(-1/2, 1)$,



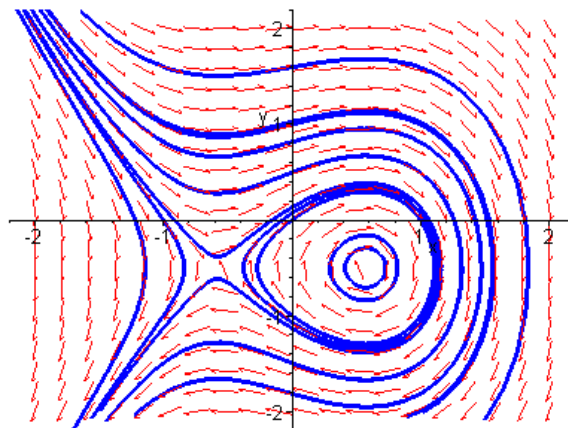
the equilibrium point has the properties of a *saddle*, and hence it is *unstable*.

6(a). The critical points are solutions of the equations

$$\begin{aligned} 1 + 2y &= 0 \\ 1 - 3x^2 &= 0. \end{aligned}$$

There are two equilibrium points, $(-1/\sqrt{3}, -1/2)$ and $(1/\sqrt{3}, -1/2)$.

(b).



(c). Locally, the trajectories near the point $(-1/\sqrt{3}, -1/2)$ resemble the behavior near a *saddle*. Hence the critical point is *unstable*. Near the point $(1/\sqrt{3}, -1/2)$, the solutions are *periodic*. Therefore the second critical point is *stable*.

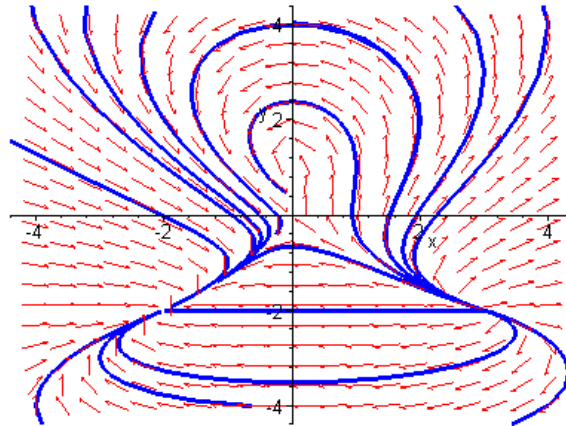
8(a). The critical points are solutions of the equations

$$\begin{aligned} -(x - y)(1 - x - y) &= 0 \\ x(2 + y) &= 0. \end{aligned}$$

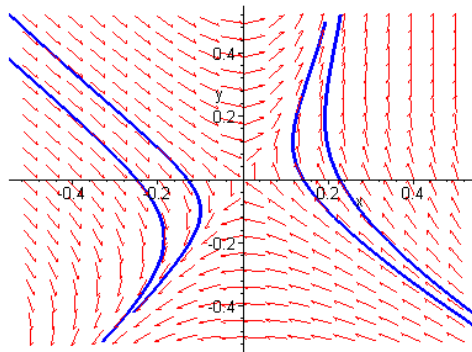
If $x = y$, then $x = y = 0$ or $x = y = -2$. If $x = 1 - y$, then $x = 0$ and $y = 1$, or $x = 3$ and $y = -2$. It follows that the critical points are $(0, 0)$, $(-2, -2)$, $(0, 1)$

and $(3, -2)$.

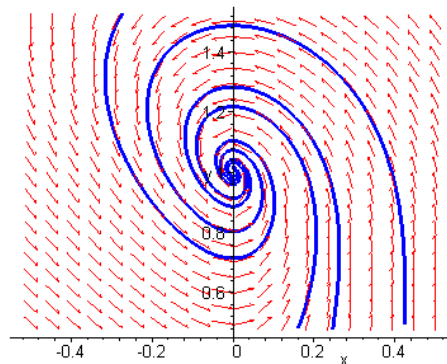
(b).



(c). Near the origin, the trajectories resemble those of a *saddle*, and hence it is *unstable*.



Near the critical point $(0, 1)$, the trajectories resemble those of a *stable spiral*. Hence the equilibrium point is *asymptotically stable*.



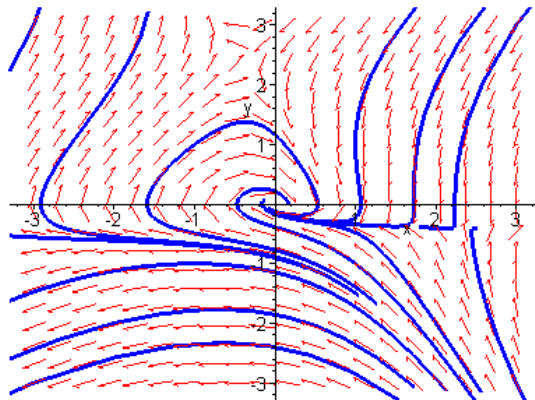
Based on the global phase portrait, it is evident that the other critical points are *nodes*. Closer examination reveals that the point $(-2, -2)$ is *asymptotically stable*, whereas the point $(3, -2)$ is *unstable*.

9(a). The critical points are given by the solution set of the equations

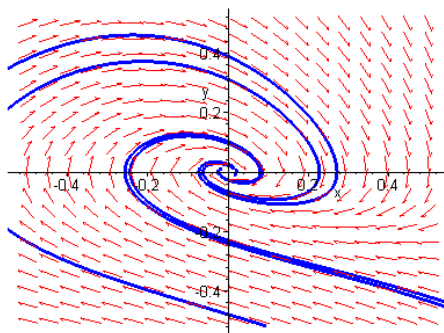
$$\begin{aligned}y(2 - x - y) &= 0 \\ -x - y - 2xy &= 0.\end{aligned}$$

Clearly, $(0, 0)$ is a critical point. If $x = 2 - y$, then it follows that $y(y - 2) = 1$. The additional critical points are $(1 - \sqrt{2}, 1 + \sqrt{2})$ and $(1 + \sqrt{2}, 1 - \sqrt{2})$.

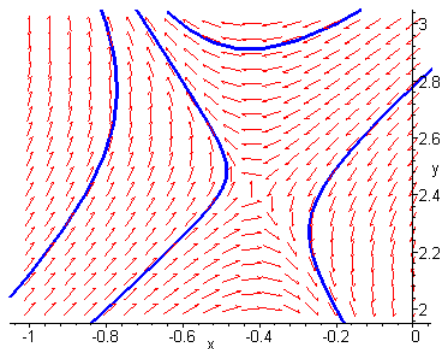
(b).



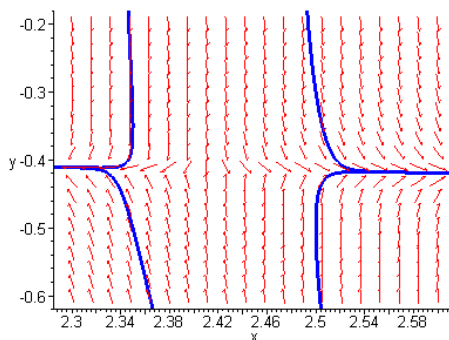
(c). The behavior near the origin is that of a *stable spiral*. Hence the point $(0, 0)$ is *asymptotically stable*.



At the critical point $(1 - \sqrt{2}, 1 + \sqrt{2})$, the trajectories resemble those near a *saddle*. Hence the critical point is *unstable*.



Near the point $(1 + \sqrt{2}, 1 - \sqrt{2})$, the trajectories resemble those near a *saddle*. Hence the critical point is also *unstable*.

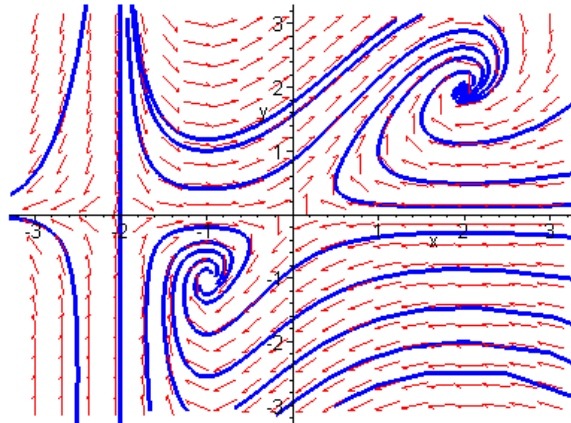


10(a). The critical points are solutions of the equations

$$\begin{aligned} (2 + x)(y - x) &= 0 \\ y(2 + x - x^2) &= 0. \end{aligned}$$

The origin is evidently a critical point. If $x = -2$, then $y = 0$. If $x = y$, then either $y = 0$ or $x = y = -1$ or $x = y = 2$. Hence the other critical points are $(-2, 0)$, $(-1, -1)$ and $(2, 2)$.

(b).



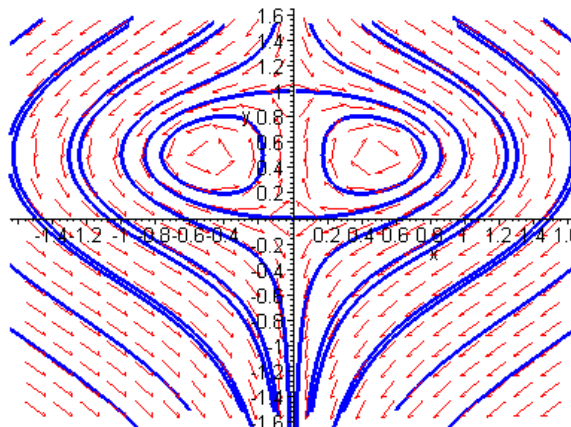
(c). Based on the global phase portrait, the critical points $(0, 0)$ and $(-2, 0)$ have the characteristics of a *saddle*. Hence these points are *unstable*. The behavior near the remaining two critical points resembles those near a *stable spiral*. Hence the critical points $(-1, -1)$ and $(2, 2)$ are *asymptotically stable*.

11(a). The critical points are given by the solution set of the equations

$$\begin{aligned} x(1 - 2y) &= 0 \\ y - x^2 - y^2 &= 0. \end{aligned}$$

If $x = 0$, then either $y = 0$ or $y = 1$. If $y = 1/2$, then $x = \pm 1/2$. Hence the critical points are at $(0, 0)$, $(0, 1)$, $(-1/2, 1/2)$ and $(1/2, 1/2)$.

(b).



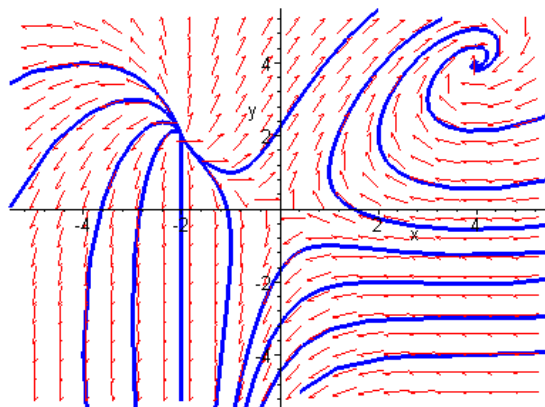
(c). The trajectories near the critical points $(-1/2, 1/2)$ and $(1/2, 1/2)$ are closed curves. Hence the critical points have the characteristics of a *center*, which is *stable*. The trajectories near the critical points $(0, 0)$ and $(0, 1)$ resemble those near a *saddle*. Hence these critical points are *unstable*.

13(a). The critical points are solutions of the equations

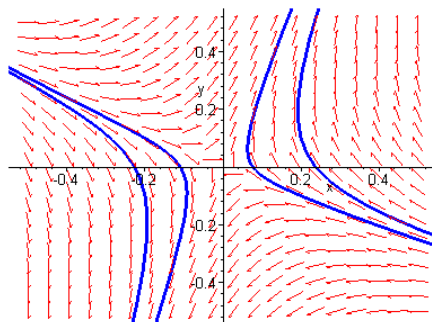
$$\begin{aligned}(2+x)(y-x) &= 0 \\ (4-x)(y+x) &= 0.\end{aligned}$$

If $y = x$, then either $x = y = 0$ or $x = y = 4$. If $x = -2$, then $y = 2$. If $x = -y$, then $y = 2$ or $y = 0$. Hence the critical points are at $(0, 0)$, $(4, 4)$ and $(-2, 2)$.

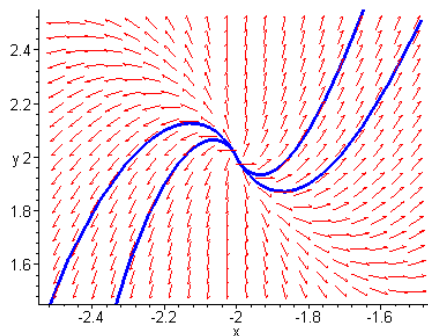
(b).



(c). The critical point at $(4, 4)$ is evidently a *stable spiral*, which is *asymptotically stable*. Closer examination of the critical point at $(0, 0)$ reveals that it is a *saddle*, which is *unstable*.



The trajectories near the critical point $(-2, 2)$ resemble those near an *unstable node*.

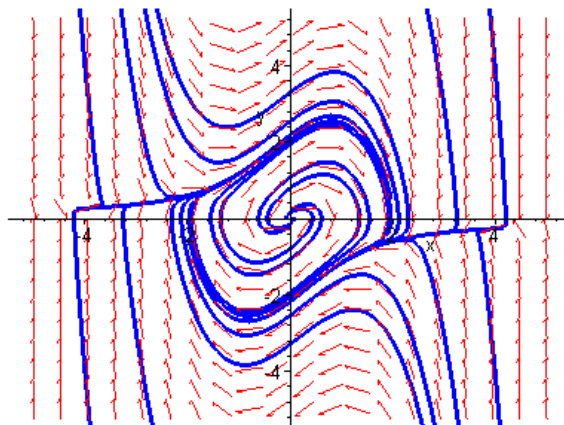


14(a). The critical points consist of the solution set of the equations

$$\begin{aligned} y &= 0 \\ (1 - x^2)y - x &= 0. \end{aligned}$$

It is easy to see that the only critical point is at $(0, 0)$.

(b).



(c). The origin is an *unstable spiral*.

16(a). The trajectories are solutions of the differential equation

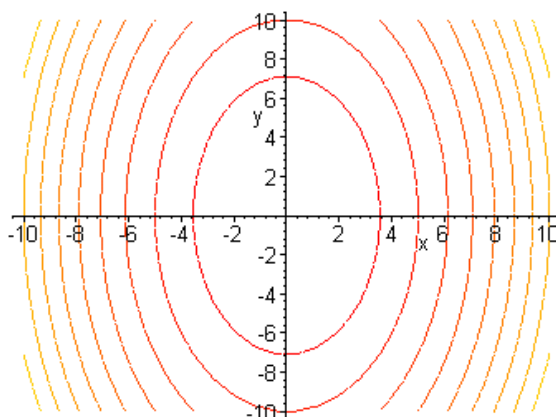
$$\frac{dy}{dx} = -\frac{4x}{y},$$

which can also be written as $4x dx + y dy = 0$. Integrating, we obtain

$$4x^2 + y^2 = C^2.$$

Hence the trajectories are ellipses.

(b).



Based on the differential equations, the direction of motion on each trajectory is *clockwise*.

17(a). The trajectories of the system satisfy the ODE

$$\frac{dy}{dx} = \frac{2x + y}{y},$$

which can also be written as $(2x + y)dx - ydy = 0$. This differential equation is *homogeneous*. Setting $y = xv(x)$, we obtain

$$v + x \frac{dv}{dx} = \frac{2}{v} + 1,$$

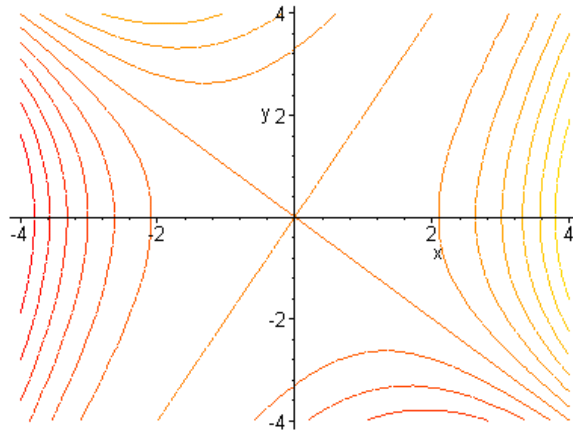
that is,

$$x \frac{dv}{dx} = \frac{2 + v - v^2}{v}.$$

The resulting ODE is *separable*, with solution $x^3(v + 1)(v - 2)^2 = C$. Reverting back to the original variables, the trajectories are level curves of

$$H(x, y) = (x + y)(y - 2x)^2.$$

(b).



The origin is a *saddle*. Along the line $y = 2x$, solutions increase without bound. Along the line $y = -x$, solutions converge toward the origin.

18(a). The trajectories are solutions of the differential equation

$$\frac{dy}{dx} = \frac{x+y}{x-y},$$

which is *homogeneous*. Setting $y = xv(x)$, we obtain

$$v + x \frac{dv}{dx} = \frac{x + xv}{x - xv},$$

that is,

$$x \frac{dv}{dx} = \frac{1 + v^2}{1 - v}.$$

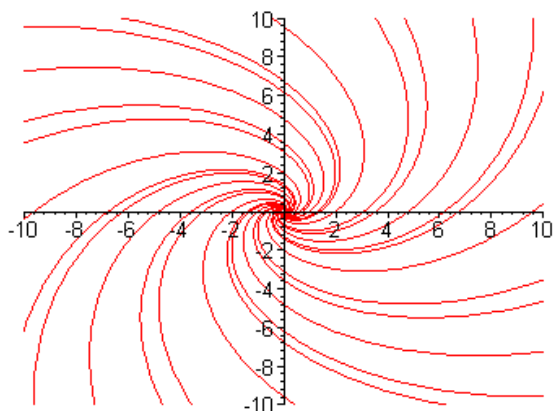
The resulting ODE is *separable*, with solution

$$\arctan(v) = \ln|x| \sqrt{1 + v^2}.$$

Reverting back to the original variables, the trajectories are level curves of

$$H(x, y) = \arctan(y/x) - \ln \sqrt{x^2 + y^2}.$$

(b).



The origin is a *stable spiral*.

20(a). The trajectories are solutions of the differential equation

$$\frac{dy}{dx} = \frac{-2xy^2 + 6xy}{2x^2y - 3x^2 - 4y},$$

which can also be written as $(2xy^2 - 6xy)dx + (2x^2y - 3x^2 - 4y)dy = 0$. The resulting ODE is *exact*, with

$$\frac{\partial H}{\partial x} = 2xy^2 - 6xy \quad \text{and} \quad \frac{\partial H}{\partial y} = 2x^2y - 3x^2 - 4y.$$

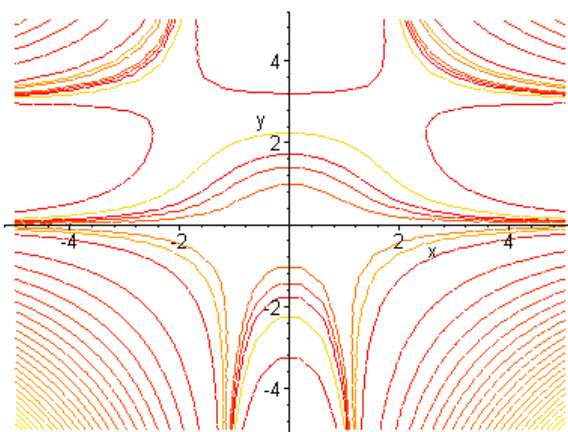
Integrating the first equation, we find that $H(x, y) = x^2y^2 - 3x^2y + f(y)$. It follows that

$$\frac{\partial H}{\partial y} = 2x^2y - 3x^2 + f'(y).$$

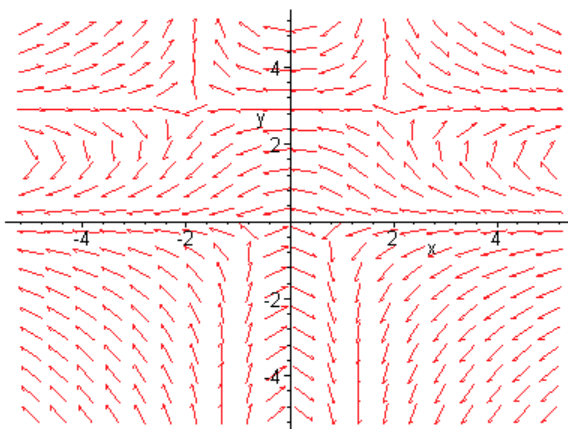
Comparing the two partial derivatives, we obtain $f(y) = -2y^2 + c$. Hence

$$H(x, y) = x^2y^2 - 3x^2y - 2y^2.$$

(b).



The associated direction field shows the direction of motion along the trajectories.



22(a). The trajectories are solutions of the differential equation

$$\frac{dy}{dx} = \frac{-6x + x^3}{6y},$$

which can also be written as $(6x - x^3)dx + 6ydy = 0$. The resulting ODE is *exact*, with

$$\frac{\partial H}{\partial x} = 6x - x^3 \quad \text{and} \quad \frac{\partial H}{\partial y} = 6y.$$

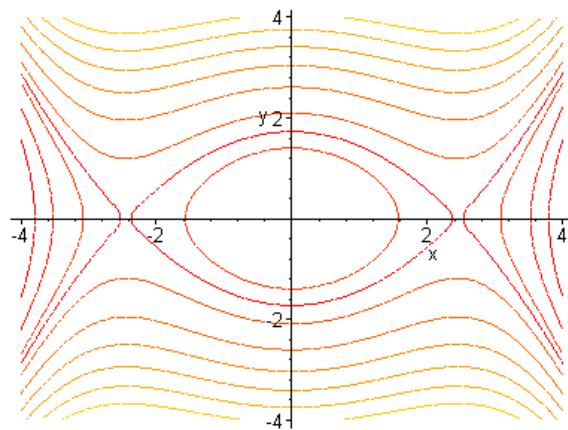
Integrating the first equation, we have $H(x, y) = 3x^2 - x^4/4 + f(y)$. It follows that

$$\frac{\partial H}{\partial y} = f'(y).$$

Comparing the two partial derivatives, we conclude that $f(y) = 3y^2 + c$. Hence

$$H(x, y) = 3x^2 - \frac{x^4}{4} + 3y^2.$$

(b).



Section 9.3

1. Write the system in the form $\mathbf{x}' = \mathbf{Ax} + \mathbf{g}(\mathbf{x})$. In this case, it is evident that

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -y^2 \\ x^2 \end{pmatrix}.$$

That is, $\mathbf{g}(\mathbf{x}) = (-y^2, x^2)^T$. Using polar coordinates, $\|\mathbf{g}(\mathbf{x})\| = r^2 \sqrt{\sin^4 \theta + \cos^4 \theta}$ and $\|\mathbf{x}\| = r$. Hence

$$\lim_{r \rightarrow 0} \frac{\|\mathbf{g}(\mathbf{x})\|}{\|\mathbf{x}\|} = \lim_{r \rightarrow 0} r \sqrt{\sin^4 \theta + \cos^4 \theta} = 0,$$

and the system is *almost linear*. The origin is an isolated critical point of the linear system

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $r^2 + r - 2 = 0$, with roots $r_1 = 1$ and $r_2 = -2$. Hence the critical point is a *saddle*, which is *unstable*.

2. The system can be written as

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -4 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix}.$$

Following the discussion in Example 3, we note that $F(x, y) = -x + y + 2xy$ and $G(x, y) = -4x - y + x^2 - y^2$. Both of the functions F and G are *twice differentiable*, hence the system is *almost linear*. Furthermore,

$$F_x = -1 + 2y, \quad F_y = 1 + 2x, \quad G_x = -4 + 2x, \quad G_y = -1 - 2y.$$

The origin is an isolated critical point, with

$$\begin{pmatrix} F_x(0, 0) & F_y(0, 0) \\ G_x(0, 0) & G_y(0, 0) \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -4 & -1 \end{pmatrix}.$$

The characteristic equation of the associated linear system is $r^2 + 2r + 5 = 0$, with complex conjugate roots $r_{1,2} = -1 \pm 2i$. The origin is a *stable spiral*, which is *asymptotically stable*.

5(a). The critical points consist of the solution set of the equations

$$\begin{aligned} (2+x)(y-x) &= 0 \\ (4-x)(y+x) &= 0. \end{aligned}$$

As shown in Prob. 13 of Section 9.2, the only critical points are at $(0, 0)$, $(4, 4)$ and $(-2, 2)$.

(b, c). First note that $F(x, y) = (2 + x)(y - x)$ and $G(x, y) = (4 - x)(y + x)$. The *Jacobian* matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} F_x(x, y) & F_y(x, y) \\ G_x(x, y) & G_y(x, y) \end{pmatrix} = \begin{pmatrix} -2 - 2x + y & 2 + x \\ 4 - y - 2x & 4 - x \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0, 0) = \begin{pmatrix} -2 & 2 \\ 4 & 4 \end{pmatrix},$$

with eigenvalues $r_1 = 1 - \sqrt{17}$ and $r_2 = 1 + \sqrt{17}$. The eigenvalues are real, with opposite sign. Hence the critical point is a *saddle*, which is *unstable*. At the equilibrium point $(-2, 2)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(-2, 2) = \begin{pmatrix} 4 & 0 \\ 6 & 6 \end{pmatrix},$$

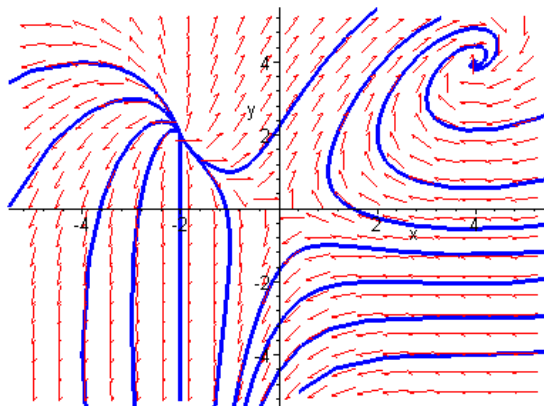
with eigenvalues $r_1 = 4$ and $r_2 = 6$. The eigenvalues are real, unequal and positive, hence the critical point is an *unstable node*. At the point $(4, 4)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(4, 4) = \begin{pmatrix} -6 & 6 \\ -8 & 0 \end{pmatrix},$$

with complex conjugate eigenvalues $r_{1,2} = -3 \pm i\sqrt{39}$. The critical point is a *stable spiral*, which is *asymptotically stable*.

Based on Table 9.3.1, the nonlinear terms do not affect the stability and type of each critical point.

(d).



7(a). The critical points are solutions of the equations

$$\begin{aligned} 1 - y &= 0 \\ (x - y)(x + y) &= 0. \end{aligned}$$

The first equation requires that $y = 1$. Based on the second equation, $x = \pm 1$. Hence the critical points are $(-1, 1)$ and $(1, 1)$.

(b, c). $F(x, y) = 1 - y$ and $G(x, y) = x^2 - y^2$. The *Jacobian* matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} F_x(x, y) & F_y(x, y) \\ G_x(x, y) & G_y(x, y) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 2x & -2y \end{pmatrix}.$$

At the critical point $(-1, 1)$, the coefficient matrix of the linearized system is

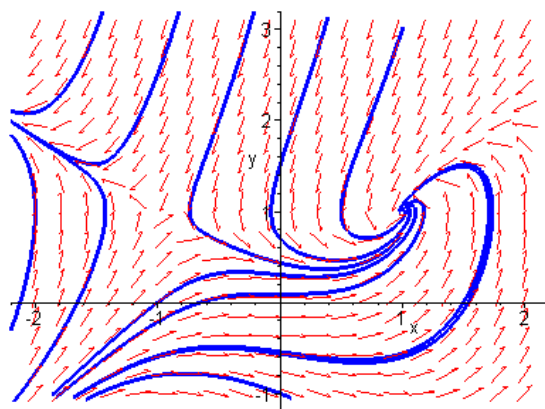
$$\mathbf{J}(-1, 1) = \begin{pmatrix} 0 & -1 \\ -2 & -2 \end{pmatrix},$$

with eigenvalues $r_1 = -1 - \sqrt{3}$ and $r_2 = -1 + \sqrt{3}$. The eigenvalues are real, with opposite sign. Hence the critical point is a *saddle*, which is *unstable*. At the equilibrium point $(1, 1)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(1, 1) = \begin{pmatrix} 0 & -1 \\ 2 & -2 \end{pmatrix},$$

with complex conjugate eigenvalues $r_{1,2} = -1 \pm i$. The critical point is a *stable spiral*, which is *asymptotically stable*.

(d).



Based on Table 9.3.1, the nonlinear terms do not affect the stability and type of each critical point.

8(a). The critical points are given by the solution set of the equations

$$\begin{aligned}x(1 - x - y) &= 0 \\y(2 - y - 3x) &= 0.\end{aligned}$$

If $x = 0$, then either $y = 0$ or $y = 2$. If $y = 0$, then $x = 0$ or $x = 1$. If $y = 1 - x$, then either $x = 1/2$ or $x = 1$. If $y = 2 - 3x$, then $x = 0$ or $x = 1/2$. Hence the critical points are at $(0, 0)$, $(0, 2)$, $(1, 0)$ and $(1/2, 1/2)$.

(b, c) . Note that $F(x, y) = x - x^2 - xy$ and $G(x, y) = (2y - y^2 - 3xy)/4$. The *Jacobian* matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} F_x(x, y) & F_y(x, y) \\ G_x(x, y) & G_y(x, y) \end{pmatrix} = \begin{pmatrix} 1 - 2x - y & -x \\ -3y/4 & 1/2 - y/2 - 3x/4 \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix},$$

with eigenvalues $r_1 = 1$ and $r_2 = 1/2$. The eigenvalues are real and both positive. Hence the critical point is an *unstable node*. At the equilibrium point $(0, 2)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0, 2) = \begin{pmatrix} -1 & 0 \\ -\frac{3}{2} & -\frac{1}{2} \end{pmatrix},$$

with eigenvalues $r_1 = -1$ and $r_2 = -1/2$. The eigenvalues are both negative, hence the critical point is a *stable node*. At the point $(1, 0)$, the coefficient matrix of the linearized system is

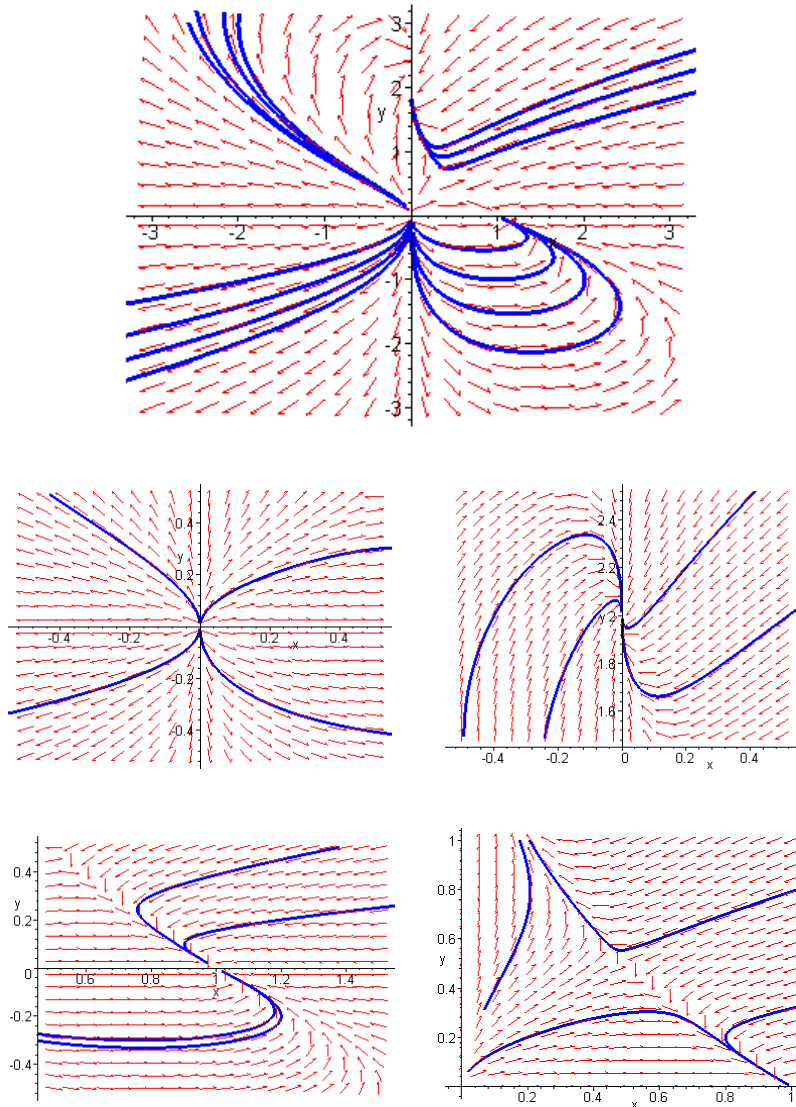
$$\mathbf{J}(1, 0) = \begin{pmatrix} -1 & -1 \\ 0 & -\frac{1}{4} \end{pmatrix},$$

with eigenvalues $r_1 = -1$ and $r_2 = -1/4$. Both of the eigenvalues are negative, and hence the critical point is a *stable node*. At the critical point $(1/2, 1/2)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(1/2, 1/2) = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{3}{8} & -\frac{1}{8} \end{pmatrix},$$

with eigenvalues $r_1 = -5/16 - \sqrt{57}/16$ and $r_2 = -5/16 + \sqrt{57}/16$. The eigenvalues are real, with opposite sign. Hence the critical point is a *saddle*, which is *unstable*.

(d).



Based on Table 9.3.1, the nonlinear terms do not affect the stability and type of each critical point.

9(a). Based on Prob. 8, in Section 9.2, the critical points are at $(0, 0)$, $(-2, -2)$, $(0, 1)$ and $(3, -2)$.

(b, c). First note that $F(x, y) = -(x - y)(1 - x - y)$ and $G(x, y) = x(2 + y)$. The *Jacobian* matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 2x - 1 & 1 - 2y \\ 2 + y & x \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0,0) = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix},$$

with eigenvalues $r_1 = 1$ and $r_2 = -2$. The eigenvalues are real, with opposite sign. Hence the critical point is a *saddle*, which is *unstable*. At the critical point $(0, 1)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0,1) = \begin{pmatrix} -1 & -1 \\ 3 & 0 \end{pmatrix},$$

with complex conjugate eigenvalues $r_{1,2} = -1/2 \pm i\sqrt{11}/2$. The critical point is a *stable spiral*, which is *asymptotically stable*. At the point $(-2, -2)$, the coefficient matrix of the linearized system is

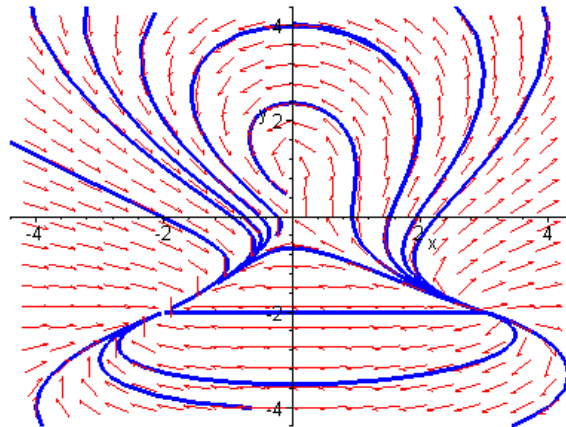
$$\mathbf{J}(-2, -2) = \begin{pmatrix} -5 & 5 \\ 0 & -2 \end{pmatrix},$$

with eigenvalues $r_1 = -2$ and $r_2 = -5$. The eigenvalues are unequal and negative, hence the critical point is a *stable node*. At the point $(3, -2)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(3, -2) = \begin{pmatrix} 5 & 5 \\ 0 & 3 \end{pmatrix},$$

with eigenvalues $r_1 = 3$ and $r_2 = 5$. The eigenvalues are unequal and positive, hence the critical point is an *unstable node*.

(d).



Based on Table 9.3.1, the nonlinear terms do not affect the stability and type of each critical point.

11(a). The critical points are solutions of the equations

$$\begin{aligned}2x + y + xy^3 &= 0 \\ x - 2y - xy &= 0.\end{aligned}$$

Substitution of $y = x/(x + 2)$ into the first equation results in

$$3x^4 + 13x^3 + 28x^2 + 20x = 0.$$

One root of the resulting equation is $x = 0$. The only other real root of the equation is

$$x = \frac{1}{9} \left[\left(287 + 18\sqrt{2019} \right)^{1/3} - 83 \left(287 + 18\sqrt{2019} \right)^{-1/3} - 13 \right].$$

Hence the critical points are $(0, 0)$ and $(-1.19345\dots, 1.4797\dots)$.

(b, c). $F(x, y) = x - x^2 - xy$ and $G(x, y) = (2y - y^2 - 3xy)/4$. The *Jacobian* matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} F_x(x, y) & F_y(x, y) \\ G_x(x, y) & G_y(x, y) \end{pmatrix} = \begin{pmatrix} 2 + y^3 & 1 + 3xy^2 \\ 1 - y & -2 - x \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

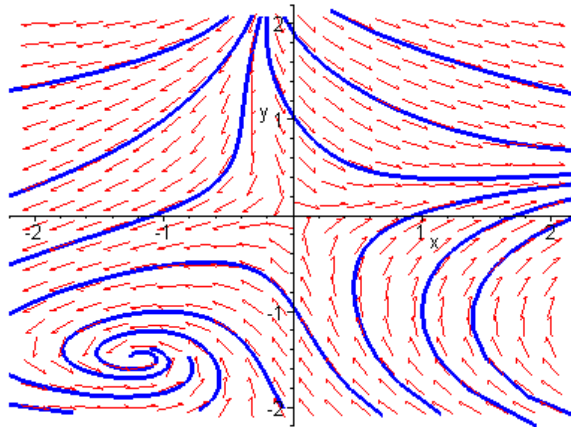
$$\mathbf{J}(0, 0) = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix},$$

with eigenvalues $r_1 = \sqrt{5}$ and $r_2 = -\sqrt{5}$. The eigenvalues are real and of opposite sign. Hence the critical point is a *saddle*, which is *unstable*. At the equilibrium point $(-1.19345\dots, 1.4797\dots)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(-1.19345, 1.4797) = \begin{pmatrix} -1.2399 & -6.8393 \\ -2.4797 & -0.8065 \end{pmatrix},$$

with complex conjugate eigenvalues $r_{1,2} = -1.0232 \pm 4.1125i$. The critical point is a *stable spiral*, which is *asymptotically stable*.

(d).



In both cases, the nonlinear terms do not affect the stability and type of the critical point.

12(a). The critical points are given by the solution set of the equations

$$\begin{aligned}(1+x)\sin y &= 0 \\ 1-x-\cos y &= 0.\end{aligned}$$

If $x = -1$, then we must have $\cos y = 2$, which is impossible. Therefore $\sin y = 0$, which implies that $y = n\pi$, $n = 0, \pm 1, 2, \dots$. Based on the second equation,

$$x = 1 - \cos n\pi.$$

It follows that the critical points are located at $(0, 2k\pi)$ and $(2, (2k+1)\pi)$, where $k = 0, \pm 1, 2, \dots$.

(b, c). Given that $F(x, y) = (1+x)\sin y$ and $G(x, y) = 1-x-\cos y$, the Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} \sin y & (1+x)\cos y \\ -1 & \sin y \end{pmatrix}.$$

At the critical points $(0, 2k\pi)$, the coefficient matrix of the linearized system is

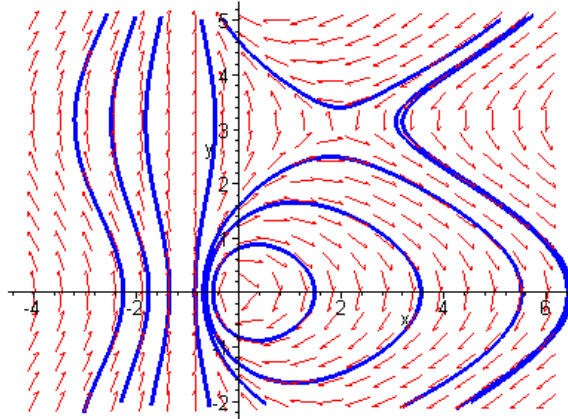
$$\mathbf{J}(0, 2k\pi) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

with purely complex eigenvalues $r_{1,2} = \pm i$. The critical points of the associated linear systems are *centers*, which are *stable*. Note that Theorem 9.3.2 does *not* provide a definite conclusion regarding the relation between the nature of the critical points of the nonlinear systems and their corresponding linearizations. At the points $(2, (2k+1)\pi)$, the coefficient matrix of the linearized system is

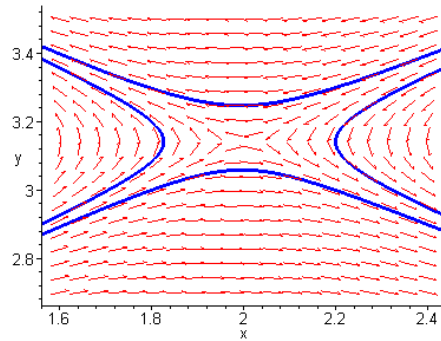
$$\mathbf{J}[2, (2k+1)\pi] = \begin{pmatrix} 0 & -3 \\ -1 & 0 \end{pmatrix},$$

with eigenvalues $r_1 = \sqrt{3}$ and $r_2 = -\sqrt{3}$. The eigenvalues are real, with opposite sign. Hence the critical points of the associated linear systems are *saddles*, which are *unstable*.

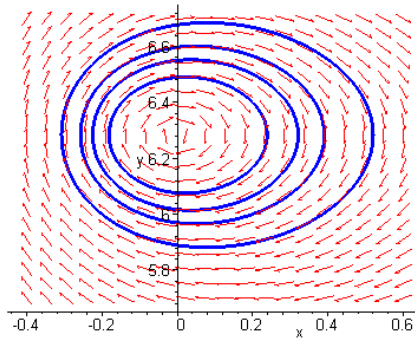
(d).



As asserted in Theorem 9.3.2, the trajectories near the critical points $(2, (2k + 1)\pi)$ resemble those near a saddle.



Upon closer examination, the critical points $(0, 2k\pi)$ are indeed centers.



13(a). The critical points are solutions of the equations

$$\begin{aligned}x - y^2 &= 0 \\y - x^2 &= 0.\end{aligned}$$

Substitution of $y = x^2$ into the first equation results in

$$x - x^4 = 0,$$

with real roots $x = 0, 1$. Hence the critical points are at $(0, 0)$ and $(1, 1)$.

(b, c) . In this problem, $F(x, y) = x - y^2$ and $G(x, y) = y - x^2$. The *Jacobian* matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 1 & -2y \\ -2x & 1 \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

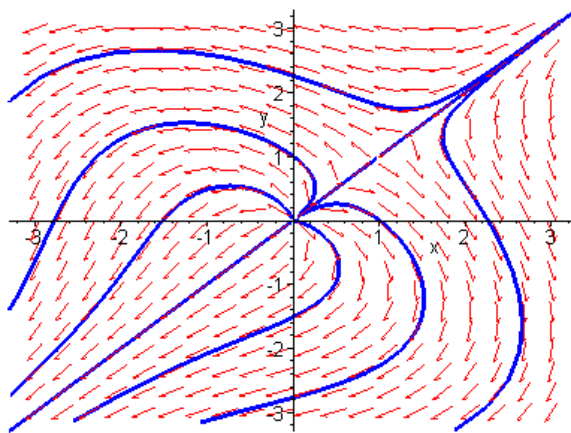
$$\mathbf{J}(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

with *repeated* eigenvalues $r_1 = 1$ and $r_2 = 1$. It is easy to see that the corresponding eigenvectors are linearly independent. Hence the critical point is an *unstable proper node*. Theorem 9.3.2 does *not* provide a definite conclusion regarding the relation between the nature of the critical point of the nonlinear system and the corresponding linearization. At the critical point $(1, 1)$, the coefficient matrix of the linearized system is

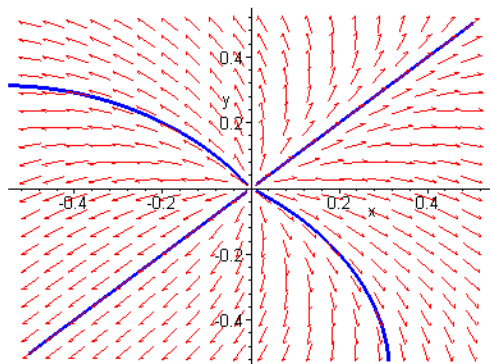
$$\mathbf{J}(1, 1) = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix},$$

with eigenvalues $r_1 = 3$ and $r_2 = -1$. The eigenvalues are real, with opposite sign. Hence the critical point is a *saddle*, which is *unstable*.

(d) .



Closer examination reveals that the critical point at the origin is indeed a proper node.



14(a). The critical points are given by the solution set of the equations

$$\begin{aligned} 1 - xy &= 0 \\ x - y^3 &= 0. \end{aligned}$$

After multiplying the second equation by y , it follows that $y = \pm 1$. Hence the critical points of the system are at $(1, 1)$ and $(-1, -1)$.

(b, c). Note that $F(x, y) = 1 - xy$ and $G(x, y) = x - y^3$. The *Jacobian* matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} -y & -x \\ 1 & -3y^2 \end{pmatrix}.$$

At the critical point $(1, 1)$, the coefficient matrix of the linearized system is

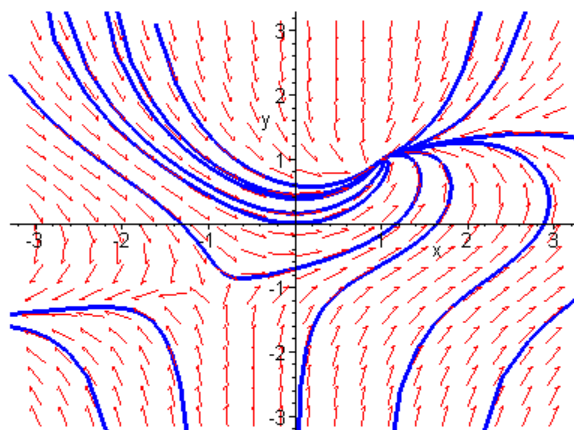
$$\mathbf{J}(1, 1) = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix},$$

with eigenvalues $r_1 = -2$ and $r_2 = -2$. The eigenvalues are real and *equal*. It is easy to show that there is only *one* linearly independent eigenvector. Hence the critical point is a *stable improper node*. Theorem 9.3.2 does *not* provide a definite conclusion regarding the relation between the nature of the critical point of the nonlinear system and the corresponding linearization. At the point $(-1, -1)$, the coefficient matrix of the linearized system is

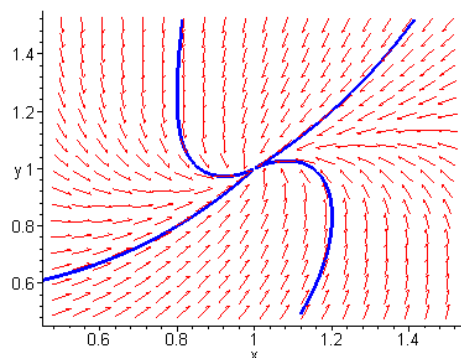
$$\mathbf{J}(-1, -1) = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix},$$

with eigenvalues $r_1 = -1 + \sqrt{5}$ and $r_2 = -1 - \sqrt{5}$. The eigenvalues are real, with opposite sign. Hence the critical point of the associated linear system is a *saddle*, which is *unstable*.

(d).



Closer examination reveals that the critical point at $(1, 1)$ is indeed a *stable* improper node, which is asymptotically stable.



15(a). The critical points are given by the solution set of the equations

$$\begin{aligned} -2x - y - x(x^2 + y^2) &= 0 \\ x - y + y(x^2 + y^2) &= 0. \end{aligned}$$

It is clear that the origin is a critical point. Solving the *first* equation for y , we find that

$$y = \frac{-1 \pm \sqrt{1 - 8x^2 - 4x^4}}{2x}.$$

Substitution of these relations into the *second* equation results in two equations of the form $f_1(x) = 0$ and $f_2(x) = 0$. Plotting these functions, we note that only $f_1(x) = 0$ has real roots given by $x \approx \pm 0.33076$. It follows that the additional critical points are at $(-0.33076, 1.0924)$ and $(0.33076, -1.0924)$.

(b, c). Given that

$$\begin{aligned} F(x, y) &= -2x - y - x(x^2 + y^2) \\ G(x, y) &= x - y + y(x^2 + y^2), \end{aligned}$$

the *Jacobian* matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} -2 - 3x^2 - y^2 & -1 - 2xy \\ 1 + 2xy & -1 + x^2 + 3y^2 \end{pmatrix}.$$

At the critical point $(0, 0)$, the coefficient matrix of the linearized system is

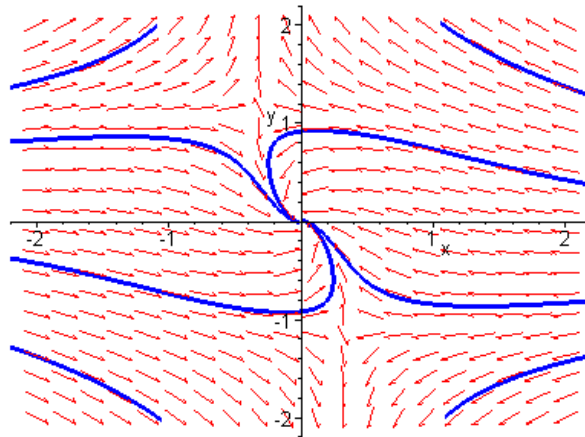
$$\mathbf{J}(0, 0) = \begin{pmatrix} -2 & -1 \\ 1 & -1 \end{pmatrix},$$

with complex conjugate eigenvalues $r_{1,2} = (-3 \pm i\sqrt{3})/2$. Hence the critical point is a *stable spiral*, which is *asymptotically stable*. At the point $(-0.33076, 1.0924)$, the coefficient matrix of the linearized system is

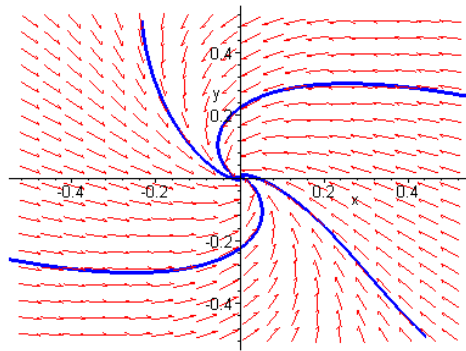
$$\mathbf{J}(-0.33076, 1.0924) = \begin{pmatrix} -3.5216 & -0.27735 \\ 0.27735 & 2.6895 \end{pmatrix},$$

with eigenvalues $r_1 = -3.5092$ and $r_2 = 2.6771$. The eigenvalues are real, with opposite sign. Hence the critical point of the associated linear system is a *saddle*, which is *unstable*. Identical results hold for the point at $(0.33076, -1.0924)$.

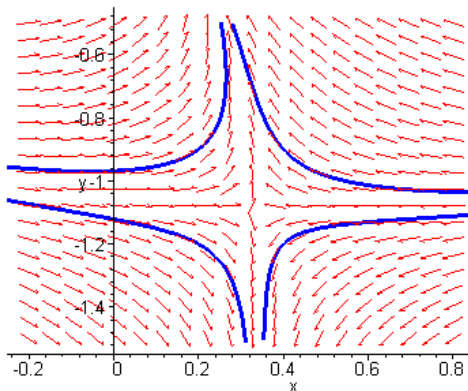
(d).



A closer look at the origin reveals a spiral:



Near the point $(0.33076, -1.0924)$ the nature of the critical point is evident:



Based on Table 9.3.1, the nonlinear terms do not affect the stability and type of each critical point.

16(a). The critical points are solutions of the equations

$$\begin{aligned}y + x(1 - x^2 - y^2) &= 0 \\ -x + y(1 - x^2 - y^2) &= 0.\end{aligned}$$

Multiply the *first* equation by y and the *second* equation by x . The difference of the two equations gives $x^2 + y^2 = 0$. Hence the only critical point is at the origin.

(b, c). With $F(x, y) = y + x(1 - x^2 - y^2)$ and $G(x, y) = -x + y(1 - x^2 - y^2)$, the *Jacobian* matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 1 - 3x^2 - y^2 & 1 - 2xy \\ -1 - 2xy & 1 - x^2 - 3y^2 \end{pmatrix}.$$

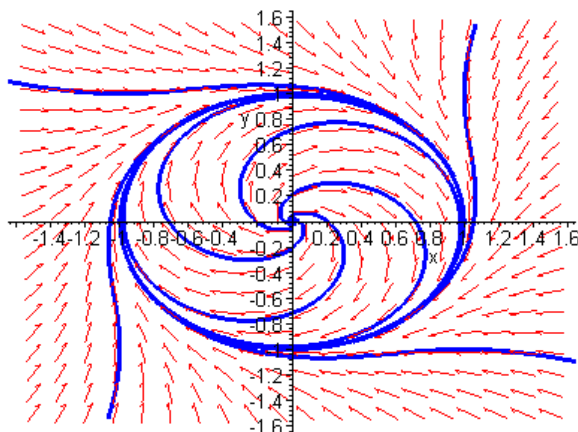
At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0, 0) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

with complex conjugate eigenvalues $r_{1,2} = 1 \pm i$. Hence the origin is an *unstable*

spiral.

(d).



17(a). The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ 1 + 6x^2 & 0 \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

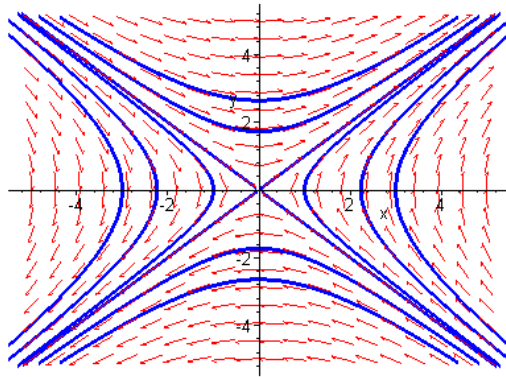
$$\mathbf{J}(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

with eigenvalues $r_1 = 1$ and $r_2 = -1$. The eigenvalues are real, with opposite sign. Hence the critical point is a *saddle point*.

(b). The trajectories of the *linearized* system are solutions of the differential equation

$$\frac{dy}{dx} = \frac{x}{y},$$

which is separable. Integrating both sides of the equation $x dx - y dy = 0$, the solution is $x^2 - y^2 = C$. The trajectories consist of a family of hyperbolas.



It is easy to show that the general solution is given by $x(t) = c_1 e^t + c_2 e^{-t}$ and $y(t) = c_1 e^t - c_2 e^{-t}$. The only *bounded* solutions consist of those for which $c_1 = 0$. In that case, $x(t) = c_2 e^{-t} = -y(t)$.

(c). The trajectories of the given system are solutions of the differential equation

$$\frac{dy}{dx} = \frac{x + 2x^3}{y},$$

which can also be written as $(x + 2x^3)dx - y dy = 0$. The resulting ODE is *exact*, with

$$\frac{\partial H}{\partial x} = x + 2x^3 \quad \text{and} \quad \frac{\partial H}{\partial y} = -y.$$

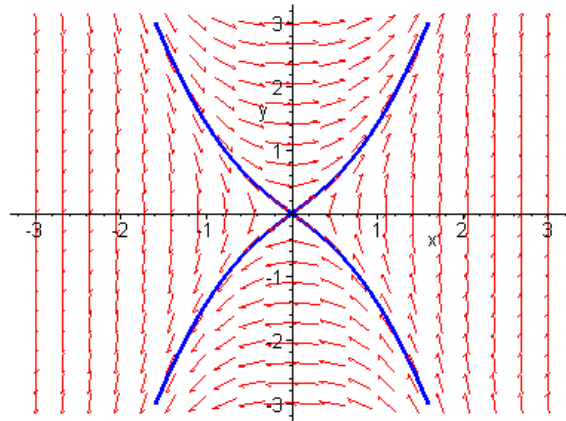
Integrating the first equation, we find that $H(x, y) = x^2/2 + x^4/2 + f(y)$. It follows that

$$\frac{\partial H}{\partial y} = f'(y).$$

Comparing the partial derivatives, we obtain $f(y) = -y^2/2 + c$. Hence the solutions are level curves of the function

$$H(x, y) = x^2/2 + x^4/2 - y^2/2.$$

The trajectories *approaching* to, or *diverging* from, the origin are no longer straight lines.



19(a). The solutions of the system of equations

$$\begin{aligned} y &= 0 \\ -\omega^2 \sin x &= 0 \end{aligned}$$

consist of the points $(\pm n\pi, 0)$, $n = 0, 1, 2, \dots$. The functions $F(x, y) = y$ and $G(x, y) = -\omega^2 \sin x$ are *analytic* on the entire plane. It follows that the system is almost linear near each of the critical points.

(b). The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -\omega^2 \cos x & 0 \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0, 0) = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix},$$

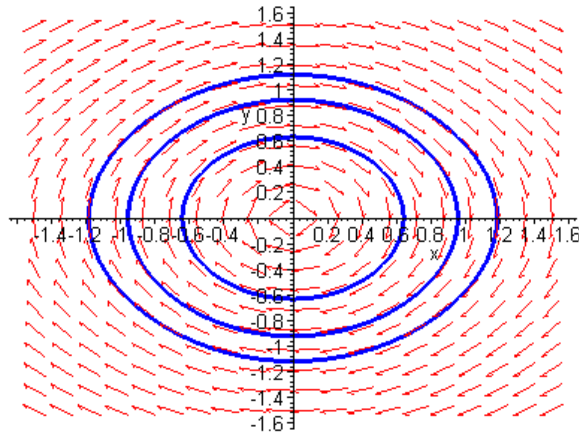
with purely complex eigenvalues $r_{1,2} = \pm i\omega$. Hence the origin is a *center*. Since the eigenvalues are purely complex, Theorem 9.3.2 gives no definite conclusion about the critical point of the nonlinear system. Physically, the critical point corresponds to the state $\theta = 0$, $\theta' = 0$. That is, the rest configuration of the pendulum.

(c). At the critical point $(\pi, 0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(\pi, 0) = \begin{pmatrix} 0 & 1 \\ \omega^2 & 0 \end{pmatrix},$$

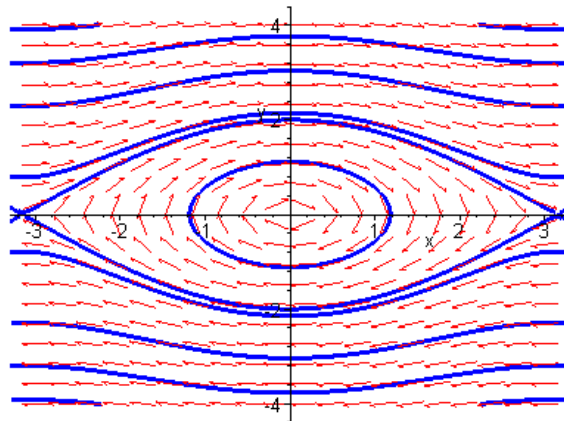
with eigenvalues $r_{1,2} = \pm \omega$. The eigenvalues are real and of opposite sign. Hence the critical point is a *saddle*. Theorem 9.3.2 asserts that the critical point for the nonlinear system is also a saddle, which is unstable. This critical point corresponds to the state $\theta = \pi$, $\theta' = 0$. That is, the *upright* rest configuration.

(d). Let $\omega^2 = 1$. The following is a plot of the phase curves near $(0, 0)$.



The local phase portrait shows that the origin is indeed a center.

(e).



It should be noted that the phase portrait has a periodic pattern, since $\theta = x \bmod 2\pi$.

20(a). The trajectories of the system in Problem 19 are solutions of the differential equation

$$\frac{dy}{dx} = \frac{-\omega^2 \sin x}{y},$$

which can also be written as $\omega^2 \sin x \, dx + y \, dy = 0$. The resulting ODE is *exact*, with

$$\frac{\partial H}{\partial x} = \omega^2 \sin x \quad \text{and} \quad \frac{\partial H}{\partial y} = y.$$

Integrating the first equation, we find that $H(x, y) = -\omega^2 \cos x + f(y)$. It follows that

$$\frac{\partial H}{\partial y} = f'(y).$$

Comparing the partial derivatives, we obtain $f(y) = y^2/2 + C$. Hence the solutions are level curves of the function

$$H(x, y) = -\omega^2 \cos x + y^2/2.$$

Adding an arbitrary constant, say ω^2 , to the function $H(x, y)$ does not change the nature of the level curves. Hence the trajectories are can be written as

$$\frac{1}{2}y^2 + \omega^2(1 - \cos x) = c,$$

in which c is an arbitrary constant.

(b). Multiplying by mL^2 and reverting to the original physical variables, we obtain

$$\frac{1}{2}mL^2 \left(\frac{d\theta}{dt} \right)^2 + mL^2\omega^2(1 - \cos \theta) = mL^2c.$$

Since $\omega^2 = g/L$, the equation can be written as

$$\frac{1}{2}mL^2 \left(\frac{d\theta}{dt} \right)^2 + mgL(1 - \cos \theta) = E,$$

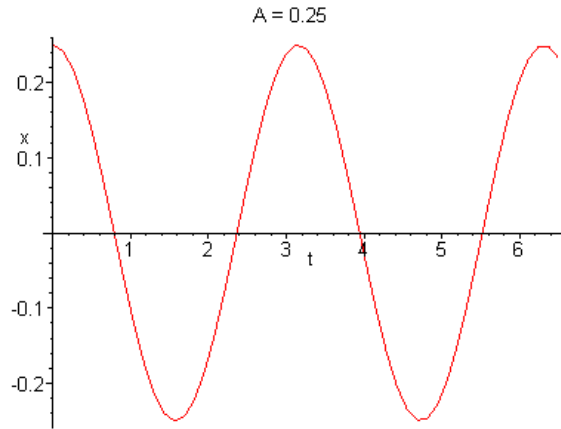
in which $E = mL^2c$.

(c). The *absolute velocity* of the point mass is given by $v = L d\theta/dt$. The kinetic energy of the mass is $T = mv^2/2$. Choosing the rest position as the *datum*, that is, the level of *zero potential energy*, the gravitational potential energy of the point mass is

$$V = mgL(1 - \cos \theta).$$

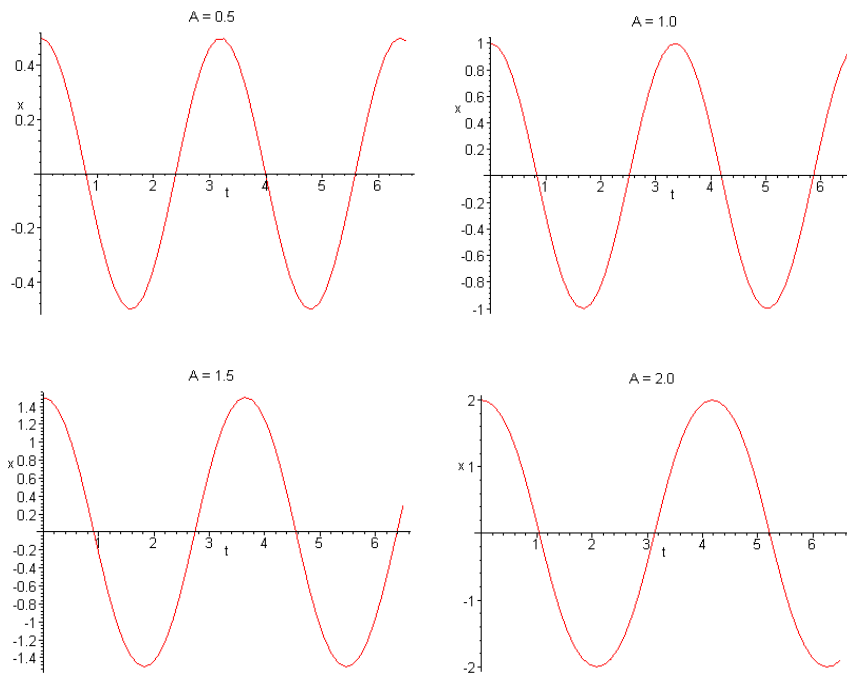
It follows that the total energy, $T + V$, is *constant* along the trajectories.

21(a). $A = 0.25$



Since the system is *undamped*, and $y(0) = 0$, the amplitude is 0.25. The period is estimated at $\tau \approx 3.16$.

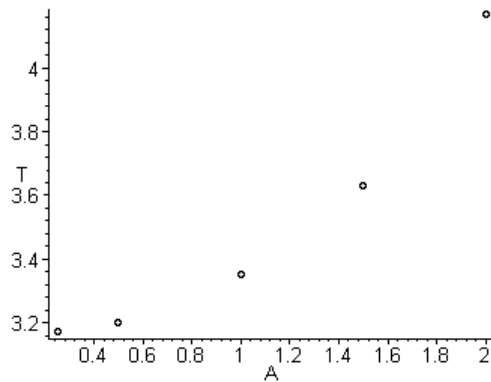
(b).



	R	τ
$A = 0.5$	0.5	3.20
$A = 1.0$	1.0	3.35
$A = 1.5$	1.5	3.63
$A = 2.0$	2.0	4.17

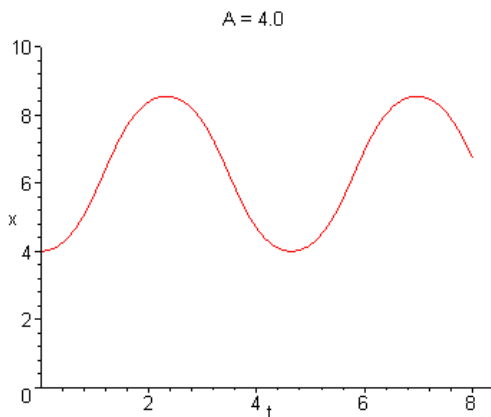
(c). Since the system is conservative, the amplitude is equal to the initial amplitude. On

the other hand, the period of the pendulum is a *monotone increasing* function of the initial position A .



It appears that as $A \rightarrow 0$, the period approaches π , the period of the corresponding *linear* pendulum ($2\pi/\omega$).

(d).



The pendulum is released from rest, at an inclination of $4 - \pi$ radians from the vertical. Based on *conservation of energy*, the pendulum will swing past the lower equilibrium position ($\theta = 2\pi$) and come to rest, momentarily, at a maximum rotational displacement of $\theta_{max} = 3\pi - (4 - \pi) = 4\pi - 4$. The transition between the two dynamics occurs at $A = \pi$, that is, once the pendulum is released *beyond* the upright configuration.

24(a). It is evident that the origin is a critical point of each system. Furthermore, it is easy to see that the corresponding linear system, in each case, is given by

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -x. \end{aligned}$$

The eigenvalues of the coefficient matrix are $r_{1,2} = \pm i$. Hence the critical point of the

linearized system is a *center*.

(b). Using polar coordinates, it is also easy to show that

$$\lim_{r \rightarrow 0} \frac{\|\mathbf{g}(\mathbf{x})\|}{\|\mathbf{x}\|} = 0.$$

Alternatively, the nonlinear terms are *analytic* in the entire plane. Hence both systems are almost linear near the origin.

(c). For system (ii), note that

$$x \frac{dx}{dt} + y \frac{dy}{dt} = xy - x^2(x^2 + y^2) - xy - y^2(x^2 + y^2).$$

Converting to polar coordinates, and differentiating the equation $r^2 = x^2 + y^2$ with respect to t , we find that

$$r \frac{dr}{dt} = x \frac{dx}{dt} + y \frac{dy}{dt} = -(x^2 + y^2)^2 = -r^4.$$

That is, $r' = -r^3$. It follows that $r^2 = 1/(2t + c)$, where $c = 1/r_0^2$. Since $r \rightarrow 0$ as $t \rightarrow \infty$, regardless of the value of r_0 , the origin is an *asymptotically stable* equilibrium point.

On the other hand, for system (i),

$$r \frac{dr}{dt} = x \frac{dx}{dt} + y \frac{dy}{dt} = (x^2 + y^2)^2 = r^4.$$

That is, $r' = r^3$. Solving the differential equation results in

$$r^2 = \frac{c - 2t}{(2t - c)^2}.$$

Imposing the initial condition $r(0) = r_0$, we obtain a specific solution

$$r^2 = -\frac{r_0^2}{2r_0^2 t - 1}.$$

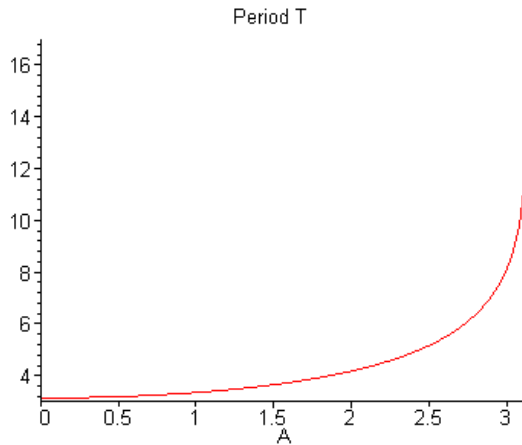
Since the solution becomes *unbounded* as $t \rightarrow 1/2r_0^2$, the critical point is *unstable*.

25. The characteristic equation of the coefficient matrix is $r^2 + 1 = 0$, with complex roots $r_{1,2} = \pm i$. Hence the critical point at the origin is a *center*. The characteristic equation of the perturbed matrix is $r^2 - 2\epsilon r + 1 + \epsilon^2 = 0$, with complex conjugate roots $r_{1,2} = \epsilon \pm i$. As long as $\epsilon \neq 0$, the critical point of the perturbed system is a *spiral point*. Its stability depends on the sign of ϵ .

26. The characteristic equation of the coefficient matrix is $(r + 1)^2 = 0$, with roots

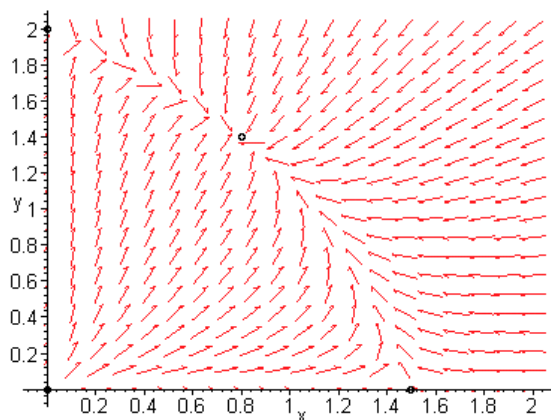
$r_1 = r_2 = -1$. Hence the critical point is an *asymptotically stable node*. On the other hand, the characteristic equation of the perturbed system is $r^2 + 2r + 1 + \epsilon = 0$, with roots $r_{1,2} = -1 \pm \sqrt{-\epsilon}$. If $\epsilon > 0$, then $r_{1,2} = -1 \pm i\sqrt{\epsilon}$ are complex roots. The critical point is a *stable spiral*. If $\epsilon < 0$, then $r_{1,2} = -1 \pm \sqrt{|\epsilon|}$ are real and both negative ($|\epsilon| \ll 1$). The critical point remains a *stable node*.

27(d). Set $k = \sin(\alpha/2) = \sin(A/2)$ and $g/L = 4$.



Section 9.4

1(a).



(b). The critical points are solutions of the system of equations

$$\begin{aligned}x(1.5 - x - 0.5y) &= 0 \\y(2 - y - 0.75x) &= 0.\end{aligned}$$

The four critical points are $(0, 0)$, $(0, 2)$, $(1.5, 0)$ and $(0.8, 1.4)$.

(c). The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 3/2 - 2x - y/2 & -x/2 \\ -3y/4 & 2 - 3x/4 - 2y \end{pmatrix}.$$

At the critical point $(0, 0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0, 0) = \begin{pmatrix} 3/2 & 0 \\ 0 & 2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 3/2, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad r_2 = 2, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are positive, hence the origin is an *unstable node*.At the critical point $(0, 2)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0, 2) = \begin{pmatrix} 1/2 & 0 \\ -3/2 & -2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 1/2, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -0.6 \end{pmatrix}; \quad r_2 = -2, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are of opposite sign. Hence the critical point is a *saddle*, which is *unstable*.

At the critical point $(1.5, 0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(1.5, 0) = \begin{pmatrix} -1.5 & -0.75 \\ 0 & 0.875 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = -1.5, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad r_2 = 0.875, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} -0.31579 \\ 1 \end{pmatrix}.$$

The eigenvalues are of opposite sign. Hence the critical point is also a *saddle*, which is *unstable*.

At the critical point $(0.8, 1.4)$, the coefficient matrix of the linearized system is

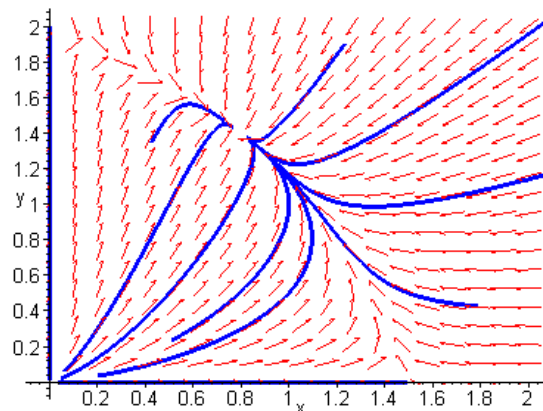
$$\mathbf{J}(0.8, 1.4) = \begin{pmatrix} -0.8 & -0.4 \\ -1.05 & -1.4 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = -\frac{11}{10} + \frac{\sqrt{51}}{10}, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ \frac{3-\sqrt{51}}{4} \end{pmatrix}; \quad r_2 = -\frac{11}{10} - \frac{\sqrt{51}}{10}, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ \frac{3+\sqrt{51}}{4} \end{pmatrix}.$$

The eigenvalues are both negative. Hence the critical point is a *stable node*, which is *asymptotically stable*.

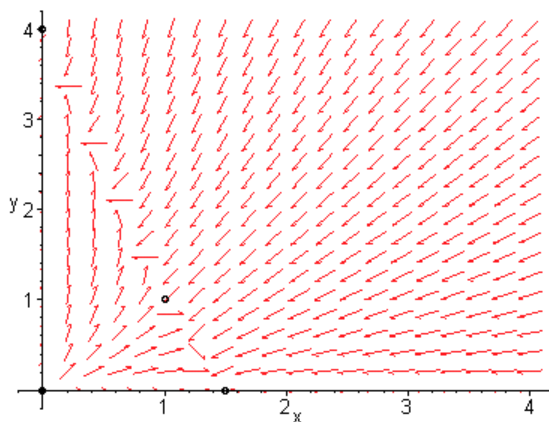
(d, e).



(f). Except for initial conditions lying on the coordinate axes, almost all trajectories

converge to the stable node at $(0.8, 1.4)$.

2(a).



(b). The critical points are the solution set of the system of equations

$$\begin{aligned}x(1.5 - x - 0.5y) &= 0 \\y(2 - 0.5y - 1.5x) &= 0.\end{aligned}$$

The four critical points are $(0, 0)$, $(0, 4)$, $(1.5, 0)$ and $(1, 1)$.

(c). The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 3/2 - 2x - y/2 & -x/2 \\ -3y/2 & 2 - 3x/2 - y \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0, 0) = \begin{pmatrix} 3/2 & 0 \\ 0 & 2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 3/2, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad r_2 = 2, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are positive, hence the origin is an *unstable node*.

At the critical point $(0, 4)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0, 4) = \begin{pmatrix} -1/2 & 0 \\ -6 & -2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = -1/2, \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}; \quad r_2 = -2, \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are both negative, hence the critical point $(0, 4)$ is a *stable node*, which is *asymptotically stable*.

At the critical point $(3/2, 0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(3/2, 0) = \begin{pmatrix} -3/2 & -3/4 \\ 0 & -1/4 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = -3/2, \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad r_2 = -1/4, \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}.$$

The eigenvalues are both negative, hence the critical point is a *stable node*, which is *asymptotically stable*.

At the critical point $(1, 1)$, the coefficient matrix of the linearized system is

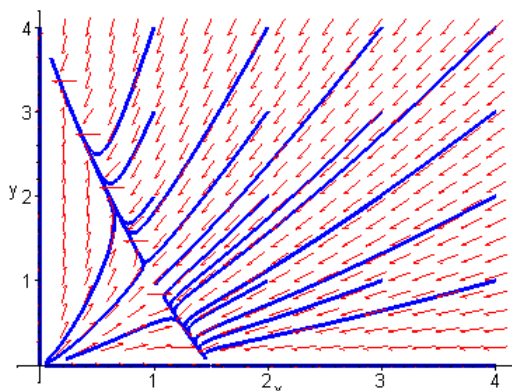
$$\mathbf{J}(1, 1) = \begin{pmatrix} -1 & -1/2 \\ -3/2 & -1/2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = \frac{-3 + \sqrt{13}}{4}, \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -\frac{1 + \sqrt{13}}{2} \end{pmatrix}; \quad r_2 = -\frac{3 + \sqrt{13}}{4}, \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ \frac{-1 + \sqrt{13}}{2} \end{pmatrix}.$$

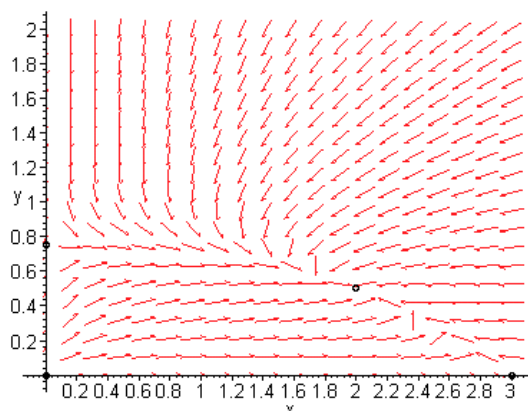
The eigenvalues are of opposite sign, hence $(1, 1)$ is a *saddle*, which is *unstable*.

(d, e).



(f). Trajectories *approaching* the critical point $(1, 1)$ form a *separatrix*. Solutions on either side of the separatrix approach either $(0, 4)$ or $(1.5, 0)$.

4(a).



(b). The critical points are solutions of the system of equations

$$\begin{aligned}x(1.5 - 0.5x - y) &= 0 \\y(0.75 - y - 0.125x) &= 0.\end{aligned}$$

The four critical points are $(0, 0)$, $(0, 3/4)$, $(3, 0)$ and $(2, 1/2)$.

(c). The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 3/2 - x - y & -x \\ -y/8 & 3/4 - x/8 - 2y \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0, 0) = \begin{pmatrix} 3/2 & 0 \\ 0 & 3/4 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 3/2, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad r_2 = 3/4, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are positive, hence the origin is an *unstable node*.At the critical point $(0, 3/4)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0, 3/4) = \begin{pmatrix} 3/4 & 0 \\ -3/32 & -3/4 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 3/4, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} -16 \\ 1 \end{pmatrix}; \quad r_2 = -3/4, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the critical point $(0, 3/4)$ is a *saddle*, which is *unstable*.

At the critical point $(3, 0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(3, 0) = \begin{pmatrix} -3/2 & -3 \\ 0 & 3/8 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = -3/2, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad r_2 = 3/8, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} -8 \\ 5 \end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the critical point $(0, 3/4)$ is a *saddle*, which is *unstable*.

At the critical point $(2, 1/2)$, the coefficient matrix of the linearized system is

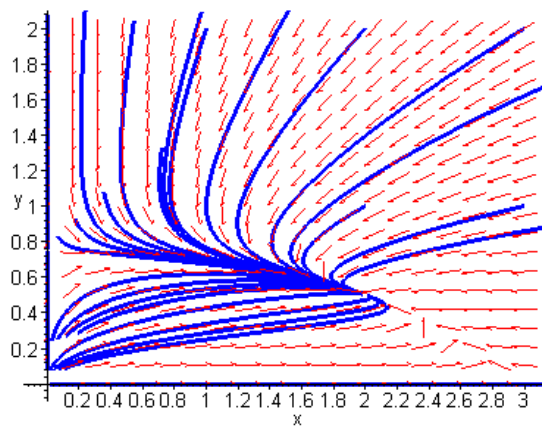
$$\mathbf{J}(2, 1/2) = \begin{pmatrix} -1 & -2 \\ -1/16 & -1/2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = \frac{-3 + \sqrt{3}}{4}, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -\frac{1 + \sqrt{3}}{8} \end{pmatrix}; \quad r_2 = -\frac{3 + \sqrt{3}}{4}, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ \frac{-1 + \sqrt{3}}{8} \end{pmatrix}.$$

The eigenvalues are negative, hence the critical point $(2, 1/2)$ is a *stable node*, which is *asymptotically stable*.

(d, e).



(f). Except for initial conditions along the coordinate axes, almost all solutions converge to the stable node $(2, 1/2)$.

7. It follows immediately that

$$\begin{aligned}(\sigma_1 X + \sigma_2 Y)^2 - 4\sigma_1\sigma_2 XY &= \sigma_1^2 X^2 + 2\sigma_1\sigma_2 XY + \sigma_2^2 Y^2 - 4\sigma_1\sigma_2 XY \\ &= (\sigma_1 X - \sigma_2 Y)^2.\end{aligned}$$

Since all parameters and variables are *positive*, it follows that

$$(\sigma_1 X + \sigma_2 Y)^2 - 4(\sigma_1\sigma_2 - \alpha_1\alpha_2)XY \geq 0.$$

Hence the radicand in Eq.(39) is *nonnegative*.

10(a). The critical points consist of the solution set of the equations

$$\begin{aligned}x(\epsilon_1 - \sigma_1 x - \alpha_1 y) &= 0 \\ y(\epsilon_2 - \sigma_2 y - \alpha_2 x) &= 0.\end{aligned}$$

If $x = 0$, then either $y = 0$ or $y = \epsilon_2/\sigma_2$. If $\epsilon_1 - \sigma_1 x - \alpha_1 y = 0$, then solving for x results in $x = (\epsilon_1 - \alpha_1 y)/\sigma_1$. Substitution into the *second* equation yields

$$(\sigma_1\sigma_2 - \alpha_1\alpha_2)y^2 - (\sigma_1\epsilon_2 - \epsilon_1\alpha_2)y = 0.$$

Based on the hypothesis, it follows that $(\sigma_1\epsilon_2 - \epsilon_1\alpha_2)y = 0$. So if $\sigma_1\epsilon_2 - \epsilon_1\alpha_2 \neq 0$, then $y = 0$, and the critical points are located at $(0, 0)$, $(0, \epsilon_2/\sigma_2)$ and $(\epsilon_1/\sigma_1, 0)$.

For the case $\sigma_1\epsilon_2 - \epsilon_1\alpha_2 = 0$, y can be arbitrary. From the relation $x = (\epsilon_1 - \alpha_1 y)/\sigma_1$, we conclude that all points on the line $\sigma_1 x + \alpha_1 y = \epsilon_1$ are critical points, in addition to the point $(0, 0)$.

(b). The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} \epsilon_1 - 2\sigma_1 x - \alpha_1 y & -\alpha_1 x \\ -\alpha_2 y & \epsilon_2 - 2\sigma_2 y - \alpha_2 x \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0, 0) = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix},$$

with eigenvalues $r_1 = \epsilon_1$ and $r_2 = \epsilon_2$. Since both eigenvalues are *positive*, the origin is an *unstable node*.

At the point $(0, \epsilon_2/\sigma_2)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0, \epsilon_2/\sigma_2) = \begin{pmatrix} (\epsilon_1\alpha_2 - \sigma_1\epsilon_2)/\alpha_2 & 0 \\ \epsilon_2\alpha_2/\sigma_2 & -\epsilon_2 \end{pmatrix},$$

with eigenvalues $r_1 = (\epsilon_1\alpha_2 - \sigma_1\epsilon_2)/\alpha_2$ and $r_2 = -\epsilon_2$. If $\sigma_1\epsilon_2 - \epsilon_1\alpha_2 > 0$, then both eigenvalues are *negative*. Hence the point $(0, \epsilon_2/\sigma_2)$ is a *stable node*, which is *asymptotically stable*. If $\sigma_1\epsilon_2 - \epsilon_1\alpha_2 < 0$, then the eigenvalues are of opposite sign. Hence the point $(0, \epsilon_2/\sigma_2)$ is a *saddle*, which is *unstable*.

At the point $(\epsilon_1/\sigma_1, 0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(\epsilon_1/\sigma_1, 0) = \begin{pmatrix} -\epsilon_1 & -\epsilon_1\alpha_1/\sigma_1 \\ 0 & (\sigma_1\epsilon_2 - \epsilon_1\alpha_2)/\sigma_1 \end{pmatrix},$$

with eigenvalues $r_1 = (\sigma_1\epsilon_2 - \epsilon_1\alpha_2)/\sigma_1$ and $r_2 = -\epsilon_1$. If $\sigma_1\epsilon_2 - \epsilon_1\alpha_2 > 0$, then the eigenvalues are of *opposite* sign. Hence the point $(\epsilon_1/\sigma_1, 0)$ is a *saddle*, which is *unstable*. If $\sigma_1\epsilon_2 - \epsilon_1\alpha_2 < 0$, then both eigenvalues are *negative*. In that case the point $(\epsilon_1/\sigma_1, 0)$ is a *stable node*, which is *asymptotically stable*.

(c). As shown in Part (a), when $\sigma_1\epsilon_2 - \epsilon_1\alpha_2 = 0$, the set of critical points consists of $(0, 0)$ and all of the points on the straight line $\sigma_1x + \alpha_1y = \epsilon_1$. Based on Part (b), the origin is still an *unstable node*. Setting $y = (\epsilon_1 - \sigma_1x)/\alpha_1$, the Jacobian matrix of the vector field, *along the given straight line*, is

$$\mathbf{J} = \begin{pmatrix} -\sigma_1x & -\alpha_1x \\ -\alpha_2(\epsilon_1 - \sigma_1x)/\alpha_1 & \alpha_2x - \epsilon_1\alpha_2/\sigma_1 \end{pmatrix}.$$

The characteristic equation of the matrix is

$$r^2 + \left[\frac{\epsilon_1\alpha_2 - \alpha_2\sigma_1x + \sigma_1^2x}{\sigma_1} \right] r = 0.$$

Using the given hypothesis, $(\epsilon_1\alpha_2 - \alpha_2\sigma_1x + \sigma_1^2x)/\sigma_1 = \epsilon_2 - \alpha_2x + \sigma_1x$. Hence the characteristic equation can be written as

$$r^2 + [\epsilon_2 - \alpha_2x + \sigma_1x]r = 0.$$

First note that $0 \leq x \leq \epsilon_1/\sigma_1$. Since the coefficient in the quadratic equation is *linear*, and

$$\epsilon_2 - \alpha_2x + \sigma_1x = \begin{cases} \epsilon_2 & \text{at } x = 0 \\ \epsilon_1 & \text{at } x = \epsilon_1/\sigma_1, \end{cases}$$

it follows that the coefficient is *positive* for $0 \leq x \leq \epsilon_1/\sigma_1$. Therefore, along the straight line $\sigma_1x + \alpha_1y = \epsilon_1$, one eigenvalue is *zero* and the other one is *negative*. Hence the continuum of critical points consists of *stable nodes*, which are *asymptotically stable*.

11(a). The critical points are solutions of the system of equations

$$\begin{aligned} x(1 - x - y) + \delta a &= 0 \\ y(0.75 - y - 0.5x) + \delta b &= 0. \end{aligned}$$

Assume solutions of the form

$$\begin{aligned} x &= x_0 + x_1\delta + x_2\delta^2 + \cdots \\ y &= y_0 + y_1\delta + y_2\delta^2 + \cdots \end{aligned}$$

Substitution of the series expansions results in

$$\begin{aligned} x_0(1 - x_0 - y_0) + (x_1 - 2x_1x_0 - x_0y_1 - x_1y_0 + a)\delta + \dots &= 0 \\ y_0(0.75 - y_0 - 0.5x_0) + (0.75y_1 - 2y_0y_1 - x_1y_0/2 - x_0y_1/2 + b)\delta + \dots &= 0. \end{aligned}$$

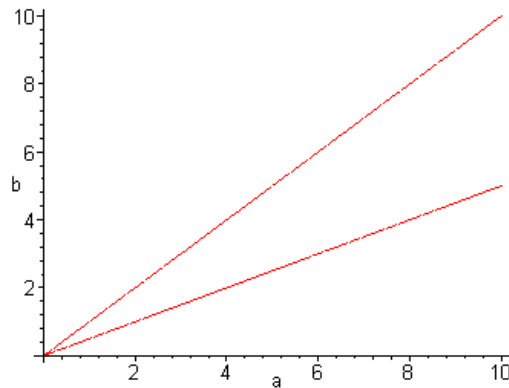
(b). Taking a limit as $\delta \rightarrow 0$, the equations reduce to the original system of equations. It follows that $x_0 = y_0 = 0.5$.

(c). Setting the coefficients of the *linear* terms equal to zero, we find that

$$\begin{aligned} -y_1/2 - x_1/2 + a &= 0 \\ -x_1/4 - y_1/2 + b &= 0, \end{aligned}$$

with solution $x_1 = 4a - 4b$ and $y_1 = -2a + 4b$.

(d). Consider the ab -parameter space. The collection of points for which $b < a$ represents an *increase* in the level of species 1. At points where $b > a$, $x_1\delta < 0$. Likewise, the collection of points for which $2b > a$ represents an *increase* in the level of species 2. At points where $2b < a$, $y_1\delta < 0$.



It follows that if $b < a < 2b$, the level of *both* species will *increase*. This condition is represented by the wedge-shaped region on the graph. Otherwise, the level of one species

will increase, whereas the level of the other species will simultaneously decrease. Only for $a = b = 0$ will both populations remain the same.

13(a). The critical points consist of the solution set of the equations

$$\begin{aligned} -y &= 0 \\ -\gamma y - x(x - 0.15)(x - 2) &= 0. \end{aligned}$$

Setting $y = 0$, the second equation becomes $x(x - 0.15)(x - 2) = 0$, with roots $x = 0$, 0.15 and 2. Hence the critical points are located at $(0, 0)$, $(0.15, 0)$ and $(2, 0)$. The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 0 & -1 \\ -3x^2 + 4.3x - 0.3 & -\gamma \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0, 0) = \begin{pmatrix} 0 & -1 \\ -0.3 & -\gamma \end{pmatrix},$$

with eigenvalues

$$r_{1,2} = -\frac{\gamma}{2} \pm \frac{1}{10} \sqrt{25\gamma^2 + 30}.$$

Regardless of the value of γ , the eigenvalues are real and of opposite sign. Hence $(0, 0)$ is a *saddle*, which is *unstable*.

At the critical point $(0.15, 0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0.15, 0) = \begin{pmatrix} 0 & -1 \\ 0.2775 & -\gamma \end{pmatrix},$$

with eigenvalues

$$r_{1,2} = -\frac{\gamma}{2} \pm \frac{1}{20} \sqrt{100\gamma^2 - 111}.$$

If $100\gamma^2 - 111 \geq 0$, then the eigenvalues are real. Furthermore, since $r_1 r_2 = 0.2775$, both eigenvalues will have the same sign. Therefore the critical point is a *node*, with its stability dependent on the *sign* of γ . If $100\gamma^2 - 111 < 0$, the eigenvalues are complex conjugates. In that case the critical point $(0.15, 0)$ is a *spiral*, with its stability dependent on the *sign* of γ .

At the critical point $(2, 0)$, the coefficient matrix of the linearized system is

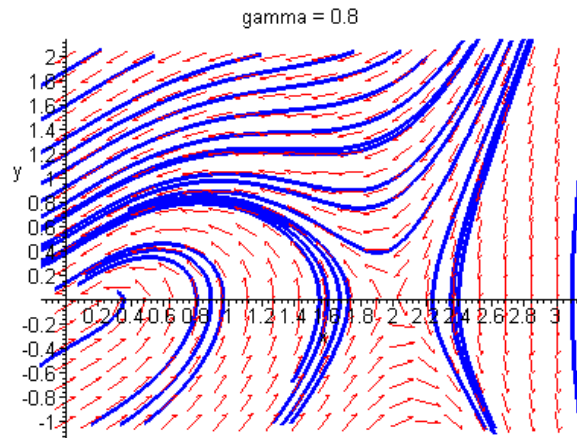
$$\mathbf{J}(2, 0) = \begin{pmatrix} 0 & -1 \\ -3.7 & -\gamma \end{pmatrix},$$

with eigenvalues

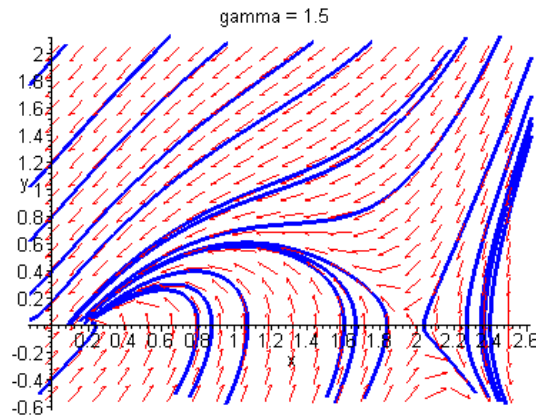
$$r_{1,2} = -\frac{\gamma}{2} \pm \frac{1}{10} \sqrt{25\gamma^2 + 370}.$$

Regardless of the value of γ , the eigenvalues are real and of opposite sign. Hence $(2, 0)$ is a *saddle*, which is *unstable*.

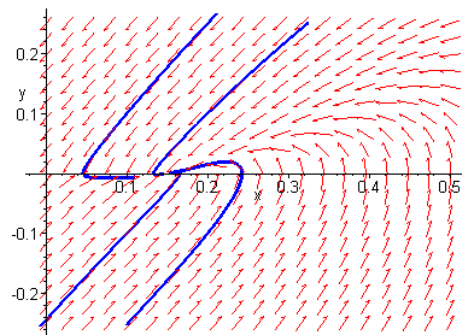
(b).



It is evident that for $\gamma = 0.8$, the critical point $(0.15, 0)$ is a *stable spiral*.

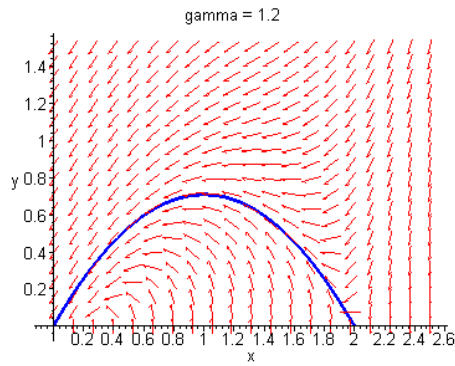
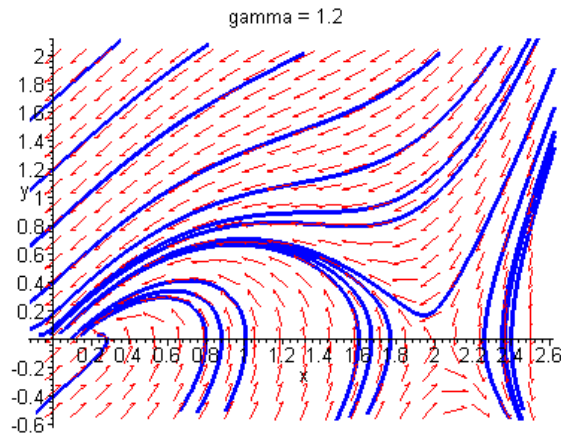


Closer examination shows that for $\gamma = 1.5$, the critical point $(0.15, 0)$ is a *stable node*.



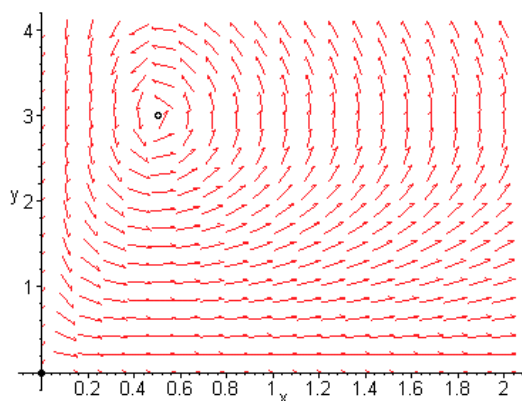
(c). Based on the phase portraits in Part (b), it is apparent that the required value of γ satisfies $0.8 < \gamma < 1.5$. Using the initial condition $x(0) = 2$ and $y(0) = 0.01$, it is possible to solve the initial value problem for various values of γ . A reasonable first guess is $\gamma = \sqrt{1.11}$. This value marks the change in qualitative behavior of the critical

point $(0.15, 0)$. Numerical experiments show that the solution remains positive for $\gamma \approx 1.20$.



Section 9.5

1(a).



(b). The critical points are solutions of the system of equations

$$\begin{aligned}x(1.5 - 0.5y) &= 0 \\y(-0.5 + x) &= 0.\end{aligned}$$

The two critical points are $(0, 0)$ and $(0.5, 3)$.

(c). The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 3/2 - y/2 & -x/2 \\ y & -1/2 + x \end{pmatrix}.$$

At the critical point $(0, 0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0, 0) = \begin{pmatrix} 3/2 & 0 \\ 0 & -1/2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 3/2, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad r_2 = -1/2, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the origin is a *saddle*, which is *unstable*.At the critical point $(0.5, 3)$, the coefficient matrix of the linearized system is

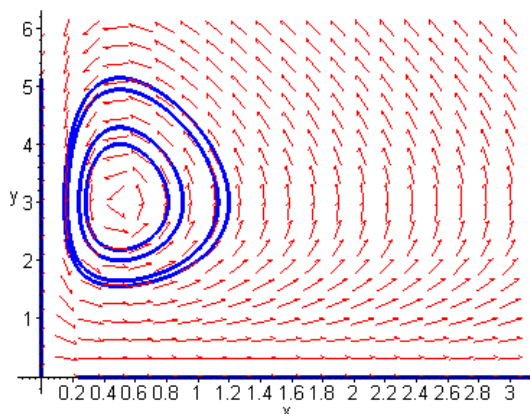
$$\mathbf{J}(0.5, 3) = \begin{pmatrix} 0 & -1/4 \\ 3 & 0 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = i \frac{\sqrt{3}}{2}, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -2i\sqrt{3} \end{pmatrix}; \quad r_2 = -i \frac{\sqrt{3}}{2}, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 2i\sqrt{3} \end{pmatrix}.$$

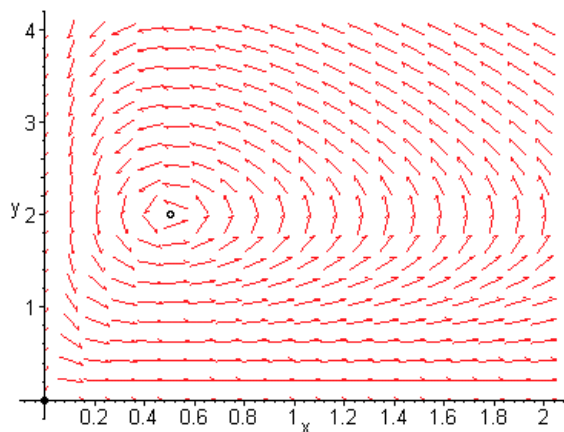
The eigenvalues are purely imaginary. Hence the critical point is a *center*, which is *stable*.

(d, e).



(f). Except for solutions along the coordinate axes, almost all trajectories are closed curves about the critical point (0.5, 3).

2(a).



(b). The critical points are the solution set of the system of equations

$$\begin{aligned} x(1 - 0.5y) &= 0 \\ y(-0.25 + 0.5x) &= 0. \end{aligned}$$

The two critical points are (0, 0) and (0.5, 2).

(c). The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 1 - y/2 & -x/2 \\ y/2 & -1/4 + x/2 \end{pmatrix}.$$

At the critical point (0, 0), the coefficient matrix of the linearized system is

$$\mathbf{J}(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -1/4 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 1, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad r_2 = -1/4, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the origin is a *saddle*, which is *unstable*.

At the critical point $(0.5, 2)$, the coefficient matrix of the linearized system is

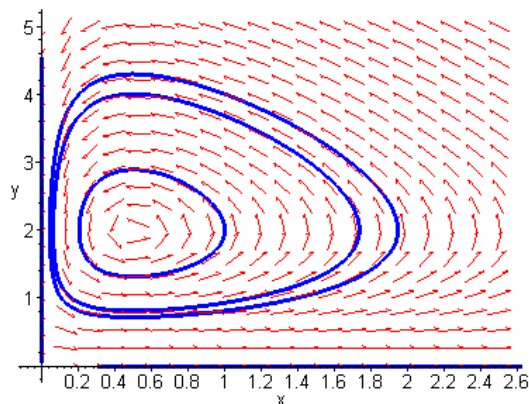
$$\mathbf{J}(0.5, 2) = \begin{pmatrix} 0 & -1/4 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = i/2, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -2i \end{pmatrix}; \quad r_2 = -i/2, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 2i \end{pmatrix}.$$

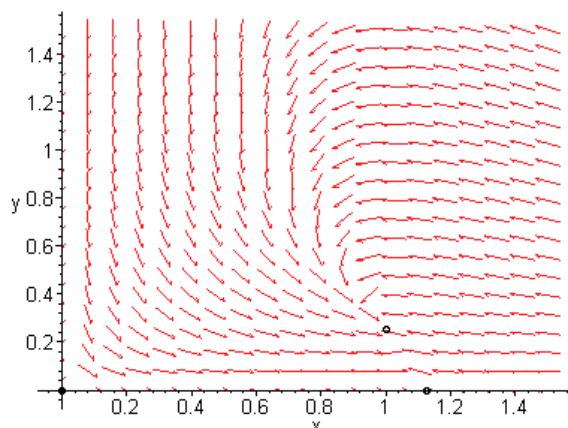
The eigenvalues are purely imaginary. Hence the critical point is a *center*, which is *stable*.

(d, e).



(f). Except for solutions along the coordinate axes, almost all trajectories are closed curves about the critical point $(0.5, 2)$.

4(a).



(b). The critical points are the solution set of the system of equations

$$\begin{aligned}x(9/8 - x - y/2) &= 0 \\ y(-1 + x) &= 0.\end{aligned}$$

The three critical points are $(0, 0)$, $(9/8, 0)$ and $(1, 1/4)$.

(c). The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 9/8 - 2x - y/2 & -x/2 \\ y & -1 + x \end{pmatrix}.$$

At the critical point $(0, 0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0, 0) = \begin{pmatrix} 9/8 & 0 \\ 0 & -1 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 9/8, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad r_2 = -1, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the origin is a *saddle*, which is *unstable*.At the critical point $(9/8, 0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(9/8, 0) = \begin{pmatrix} -9/8 & -9/16 \\ 0 & 1/8 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = -\frac{9}{8}, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad r_2 = \frac{1}{8}, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 9 \\ -20 \end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the critical point $(9/8, 0)$ is a *saddle*, which is *unstable*.

At the critical point $(1, 1/4)$, the coefficient matrix of the linearized system is

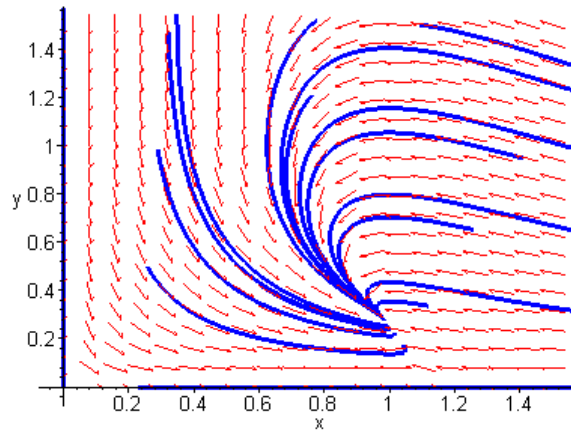
$$\mathbf{J}(1, 1/4) = \begin{pmatrix} -1 & -1/2 \\ 1/4 & 0 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = \frac{-2 + \sqrt{2}}{4}, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} -2 + \sqrt{2} \\ 1 \end{pmatrix}; \quad r_2 = \frac{-2 - \sqrt{2}}{4}, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} -2 - \sqrt{2} \\ 1 \end{pmatrix}.$$

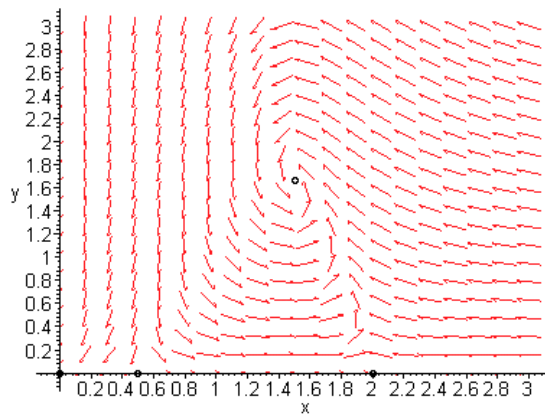
The eigenvalues are both negative. Hence the critical point is a *stable node*, which is *asymptotically stable*.

(d, e).



(f). Except for solutions along the coordinate axes, all solutions converge to the critical point $(1, 1/4)$.

5(a).



(b). The critical points are solutions of the system of equations

$$\begin{aligned}x(-1 + 2.5x - 0.3y - x^2) &= 0 \\ y(-1.5 + x) &= 0.\end{aligned}$$

The four critical points are $(0, 0)$, $(1/2, 0)$, $(2, 0)$ and $(3/2, 5/3)$.

(c). The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} -1 + 5x - 3x^2 - 3y/10 & -3x/10 \\ y & -3/2 + x \end{pmatrix}.$$

At the critical point $(0, 0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0, 0) = \begin{pmatrix} -1 & 0 \\ 0 & -3/2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = -1, \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad r_2 = -3/2, \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are both negative, hence the critical point $(0, 0)$ is a *stable node*, which is *asymptotically stable*.

At the critical point $(1/2, 0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(1/2, 0) = \begin{pmatrix} 3/4 & -3/20 \\ 0 & -1 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = \frac{3}{4}, \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad r_2 = -1, \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 3 \\ 35 \end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the critical point $(1/2, 0)$ is a *saddle*, which is *unstable*.

At the critical point $(2, 0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(2, 0) = \begin{pmatrix} -3 & -3/5 \\ 0 & 1/2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = -3, \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad r_2 = 1/2, \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 6 \\ -35 \end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the critical point $(2, 0)$ is a *saddle*, which is *unstable*.

At the critical point $(3/2, 5/3)$, the coefficient matrix of the linearized system is

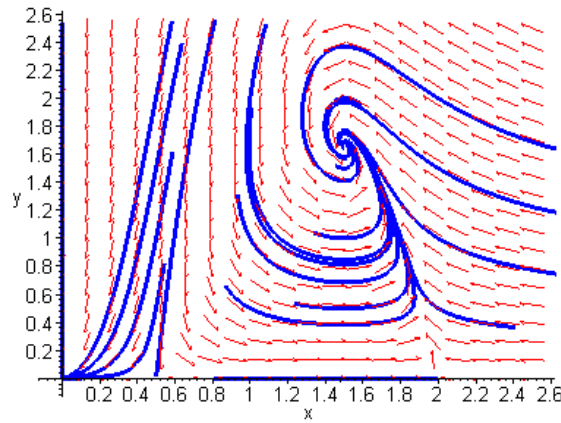
$$\mathbf{J}(3/2, 5/3) = \begin{pmatrix} -3/4 & -9/20 \\ 5/3 & 0 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = \frac{-3 + i\sqrt{39}}{8}, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} \frac{-9+i3\sqrt{39}}{40} \\ 1 \end{pmatrix}; \quad r_2 = \frac{-3 - i\sqrt{39}}{8}, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} \frac{-9-i3\sqrt{39}}{40} \\ 1 \end{pmatrix}.$$

The eigenvalues are complex conjugates. Hence the critical point $(3/2, 5/3)$ is a *stable spiral*, which is *asymptotically stable*.

(d, e).



(f). The single solution curve that converges to the node at $(1/2, 0)$ is a *separatrix*. Except for initial conditions on the coordinate axes, trajectories on either side of the separatrix converge to the node at $(0, 0)$ or the stable spiral at $(3/2, 5/3)$.

6. Given that t is measured from the time that x is a *maximum*, we have

$$x = \frac{c}{\gamma} + \frac{cK}{\gamma} \cos(\sqrt{ac} t)$$

$$y = \frac{a}{\alpha} + K \frac{a}{\alpha} \sqrt{\frac{c}{\alpha}} \sin(\sqrt{ac} t).$$

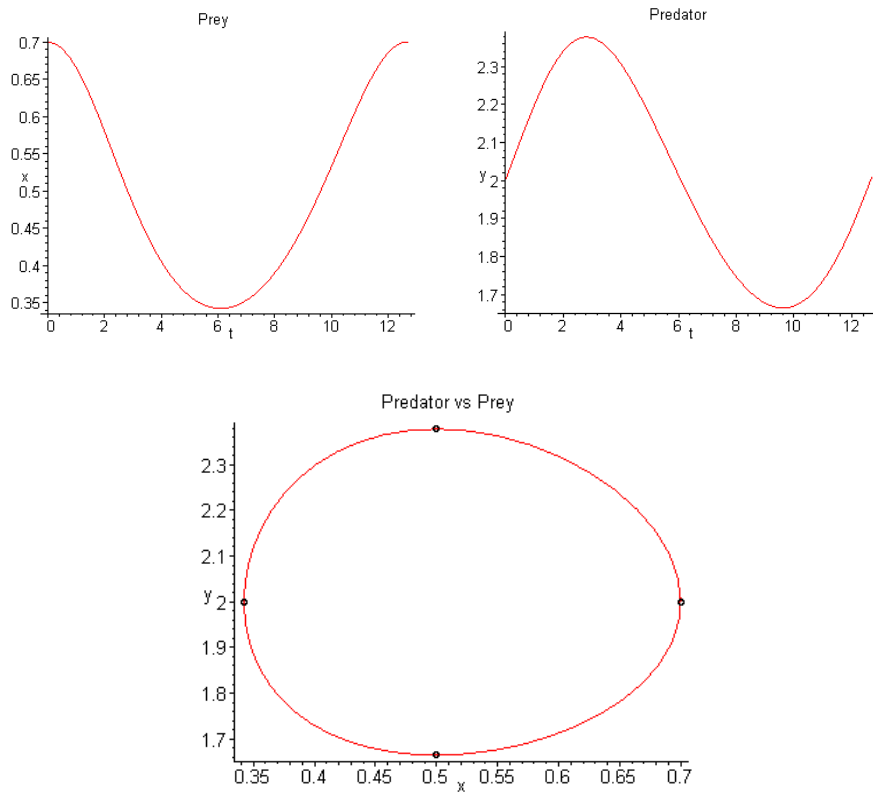
The *period* of oscillation is evidently $T = 2\pi/\sqrt{ac}$. Both populations oscillate about a mean value. The following is based on the properties of the *cos* and *sin* functions

The prey population (x) is *maximum* at $t = 0$ and $t = T$. It is a *minimum* at $t = T/2$. Its rate of increase is greatest at $t = 3T/4$. The rate of *decrease* of the prey population is greatest at $t = T/4$.

The predator population (y) is *maximum* at $t = T/4$. It is a *minimum* at $t = 3T/4$.

The rate of increase of the predator population is greatest at $t = 0$ and $t = T$. The rate of decrease of the predator population is greatest at $t = T/2$.

In the following example, the system in Problem 2 is solved numerically with the initial conditions $x(0) = 0.7$ and $y(0) = 2$. The critical point of interest is at $(0.5, 2)$. Since $a = 1$ and $c = 1/4$, it follows that the period of oscillation is $T = 4\pi$.



8(a). The *period* of oscillation for the linear system is $T = 2\pi/\sqrt{ac}$. In system (2), $a = 1$ and $c = 0.75$. Hence the period is estimated as $T = 2\pi/\sqrt{0.75} \approx 7.2552$.

(b). The estimated period appears to agree with the graphic in Figure 9.5.3.

(c). The critical point of interest is at $(3, 2)$. The system is solved numerically, with $y(0) = 2$ and $x(0) = 3.5, 4.0, 4.5, 5.0$. The resulting periods are shown in the table:

	$x(0) = 3.5$	$x(0) = 4.0$	$x(0) = 4.5$	$x(0) = 5.0$
T	7.26	7.29	7.34	7.42

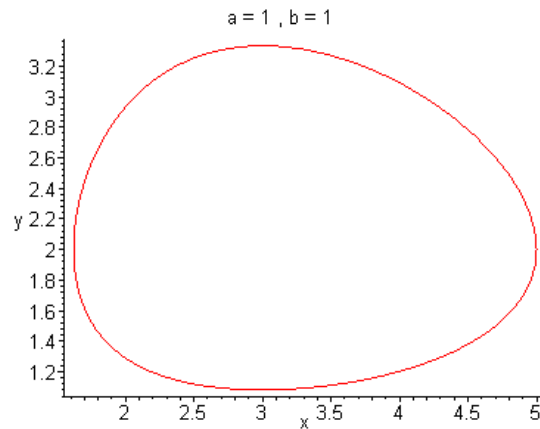
The actual amplitude steadily *increases* as the amplitude increases.

9. The system

$$\begin{aligned} \frac{dx}{dt} &= ax \left(1 - \frac{y}{2}\right) \\ \frac{dy}{dt} &= by \left(-1 + \frac{x}{3}\right) \end{aligned}$$

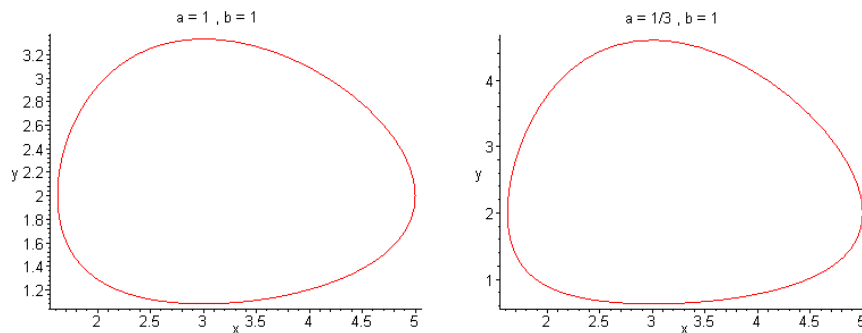
is solved numerically for various values of the parameters. The initial conditions are $x(0) = 5$, $y(0) = 2$.

(a). $a = 1$ and $b = 1$:



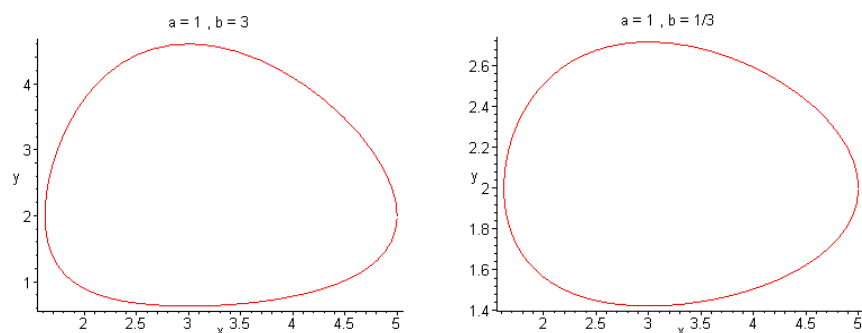
The period is estimated by observing when the trajectory becomes a closed curve. In this case, $T \approx 6.45$.

(b). $a = 3$ and $a = 1/3$, with $b = 1$:



For $a = 3$, $T \approx 3.69$. For $a = 1/3$, $T \approx 11.44$.

(c). $b = 3$ and $b = 1/3$, with $a = 1$:



For $b = 3$, $T \approx 3.82$. For $b = 1/3$, $T \approx 11.06$.

(d). It appears that if one of the parameters is fixed, the period varies *inversely* with the other parameter. Hence one might postulate the relation

$$T = \frac{k}{f(a, b)}.$$

10(a). Since $T = 2\pi/\sqrt{ac}$, we first note that

$$\int_A^{A+T} \cos(\sqrt{ac} t + \phi) dt = \int_A^{A+T} \sin(\sqrt{ac} t + \phi) dt = 0.$$

Hence

$$\bar{x} = \frac{1}{T} \int_A^{A+T} \frac{c}{\gamma} dt = \frac{c}{\gamma} \text{ and } \bar{y} = \frac{1}{T} \int_A^{A+T} \frac{a}{\alpha} dt = \frac{a}{\alpha}.$$

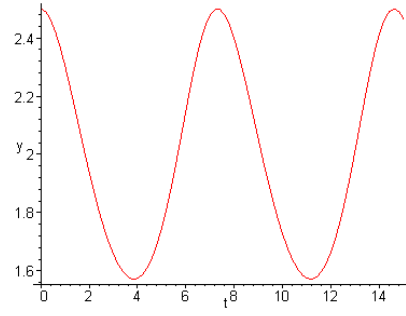
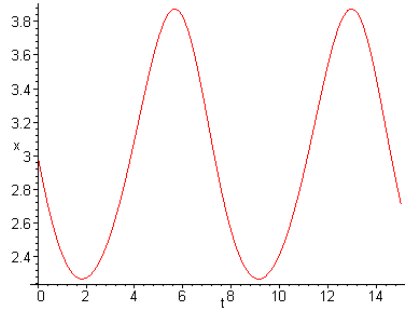
(b). One way to estimate the mean values is to find a horizontal line such that the area above the line is approximately equal to the area under the line. From Figure 9.5.3, it appears that $\bar{x} \approx 3.25$ and $\bar{y} \approx 2.0$. In Example 1, $a = 1$, $c = 0.75$, $\alpha = 0.5$ and $\gamma = 0.25$. Using the result in Part (a), $\bar{x} = 3$ and $\bar{y} = 2$.

(c). The system

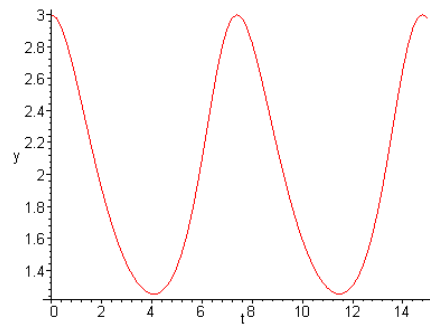
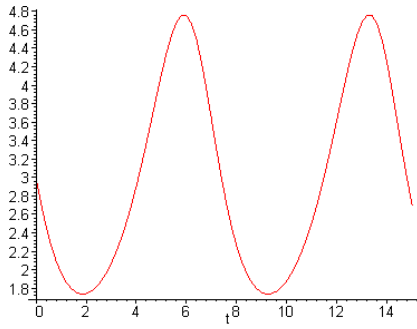
$$\begin{aligned} \frac{dx}{dt} &= x \left(1 - \frac{y}{2} \right) \\ \frac{dy}{dt} &= y \left(-\frac{3}{4} + \frac{x}{4} \right) \end{aligned}$$

is solved numerically for various initial conditions.

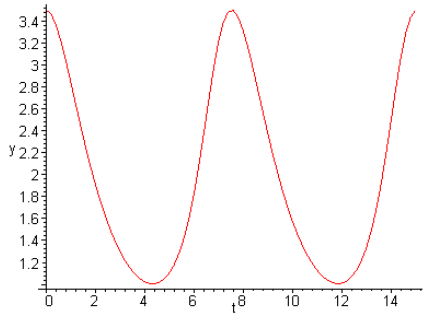
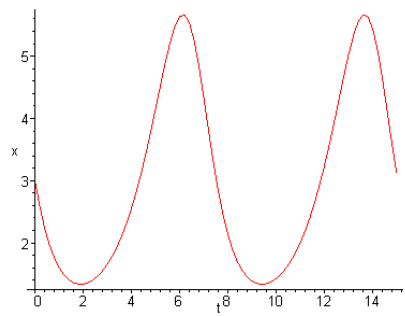
$x(0) = 3$ and $y(0) = 2.5$:



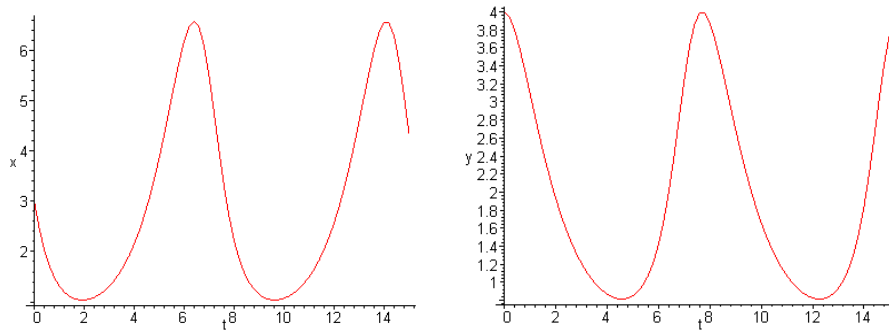
$x(0) = 3$ and $y(0) = 3.0$:



$x(0) = 3$ and $y(0) = 3.5$:



$x(0) = 3$ and $y(0) = 4.0$:



It is evident that the mean values *increase* as the amplitude increases. That is, the mean values increase as the initial conditions move farther from the critical point.

12. The system of equations in model (1) is given by

$$\begin{aligned} \frac{dx}{dt} &= x(a - \alpha y) \\ \frac{dy}{dt} &= y(-c + \gamma x). \end{aligned}$$

Based on the hypothesis, let the *death* rate of the insect population and the predators be $p x$ and $q y$, respectively. The modified system of equations becomes

$$\begin{aligned} \frac{dx}{dt} &= x(a - \alpha y) - p x \\ \frac{dy}{dt} &= y(-c + \gamma x) - q y, \end{aligned}$$

in which $p > 0, q > 0$. The critical points are solutions of the system of equations

$$\begin{aligned} x(a - p - \alpha y) &= 0 \\ y(-c - q + \gamma x) &= 0. \end{aligned}$$

It is easy to see that the critical points are now at $(0, 0)$ and $\left(\frac{c+q}{\gamma}, \frac{a-p}{\alpha}\right)$. Furthermore, since $(c + q)/\gamma > c/\gamma$, the equilibrium level of the insect population has *increased*. On the other hand, since $(a - p)/\alpha < a/\alpha$, equilibrium level of the predators has *decreased*. Indeed, the introduction of insecticide creates a potential to significantly affect the predator population ($a \approx p$).

Section 9.6

2. We consider the function $V(x, y) = ax^2 + cy^2$. The rate of change of V along any trajectory is

$$\begin{aligned}\dot{V} &= V_x \frac{dx}{dt} + V_y \frac{dy}{dt} \\ &= 2ax \left(-\frac{1}{2}x^3 + 2xy^2 \right) + 2cy(-y^3) \\ &= -ax^4 + 4ax^2y^2 - 2cy^4.\end{aligned}$$

Let $u = x^2$, $v = y^2$, $\alpha = -a$, $\beta = 4a$, and $\gamma = -2c$. We then have

$$-ax^4 + 4ax^2y^2 - 2cy^4 = \alpha u^2 + \beta uv + \gamma v^2.$$

If $a > 0$ and $c > 0$, then $V(x, y)$ is *positive definite*. Furthermore, $\alpha < 0$. Recall that Theorem 9.6.4 asserts that if $4\alpha\gamma - \beta^2 = 8ac - 16a^2 > 0$, then the function

$$\alpha u^2 + \beta uv + \gamma v^2$$

is *negative definite*. Hence if $c > 2a$, then $\dot{V}(x, y)$ is *negative definite*. One such example is $V(x, y) = x^2 + 3y^2$. It follows from Theorem 9.6.1 that the origin is an asymptotically stable critical point.

4. Given $V(x, y) = ax^2 + cy^2$, the rate of change of V along any trajectory is

$$\begin{aligned}\dot{V} &= V_x \frac{dx}{dt} + V_y \frac{dy}{dt} \\ &= 2ax(x^3 - y^3) + 2cy(2xy^2 + 4x^2y + 2y^3) \\ &= 2ax^4 + (4c - 2a)xy^3 + 8cx^2y^2 + 4cy^4.\end{aligned}$$

Setting $a = 2c$,

$$\begin{aligned}\dot{V} &= 4cx^4 + 8cx^2y^2 + 4cy^4 \\ &\geq 4cx^4 + 4cy^4.\end{aligned}$$

As long as $a = 2c > 0$, the function $V(x, y)$ is *positive definite* and $\dot{V}(x, y)$ is also *positive definite*. It follows from Theorem 9.6.2 that $(0, 0)$ is an unstable critical point.

5. Given $V(x, y) = c(x^2 + y^2)$, the rate of change of V along any trajectory is

$$\begin{aligned}\dot{V} &= V_x \frac{dx}{dt} + V_y \frac{dy}{dt} \\ &= 2cx[y - xf(x, y)] + 2cy[-x - yf(x, y)] \\ &= -2c(x^2 + y^2)f(x, y).\end{aligned}$$

If $c > 0$, then $V(x, y)$ is *positive definite*. Furthermore, if $f(x, y)$ is *positive* in some neighborhood of the origin, then $\dot{V}(x, y)$ is *negative definite*. Theorem 9.6.1 asserts that

the origin is an asymptotically stable critical point.

On the other hand, if $f(x, y)$ is *negative* in some neighborhood of the origin, then $V(x, y)$ and $\dot{V}(x, y)$ are both *positive definite*. It follows from Theorem 9.6.2 that the origin is an unstable critical point.

9(a). Letting $x = u$ and $y = u'$, we obtain the system of equations

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -g(x) - y.\end{aligned}$$

Since $g(0) = 0$, it is evident that $(0, 0)$ is a critical point of the system. Consider the function

$$V(x, y) = \frac{1}{2}y^2 + \int_0^x g(s)ds.$$

It is clear that $V(0, 0) = 0$. Since $g(u)$ is an *odd* function in a neighborhood of $u = 0$,

$$\int_0^x g(s)ds > 0 \text{ for } x > 0,$$

and

$$\int_0^x g(s)ds = -\int_x^0 g(s)ds > 0 \text{ for } x < 0.$$

Therefore $V(x, y)$ is *positive definite*.

The rate of change of V along any trajectory is

$$\begin{aligned}\dot{V} &= V_x \frac{dx}{dt} + V_y \frac{dy}{dt} \\ &= g(x) \cdot (y) + y[-g(x) - y] \\ &= -y^2.\end{aligned}$$

It follows that $\dot{V}(x, y)$ is only *negative semidefinite*. Hence the origin is a *stable* critical point.

(b). Given

$$V(x, y) = \frac{1}{2}y^2 + \frac{1}{2}y \sin(x) + \int_0^x \sin(s)ds,$$

It is easy to see that $V(0, 0) = 0$. The rate of change of V along any trajectory is

$$\begin{aligned}
 \dot{V} &= V_x \frac{dx}{dt} + V_y \frac{dy}{dt} \\
 &= \left[\sin x + \frac{y}{2} \cos x \right] (y) + \left[y + \frac{1}{2} \sin x \right] [-\sin x - y] \\
 &= \frac{1}{2} y^2 \cos x - \frac{1}{2} \sin^2 x - \frac{y}{2} \sin x - y^2.
 \end{aligned}$$

For $-\pi/2 < x < \pi/2$, we can write $\sin x = x - \alpha x^3/6$ and $\cos x = 1 - \beta x^2/2$, in which $\alpha = \alpha(x)$, $\beta = \beta(x)$. Note that $0 < \alpha, \beta < 1$. Then

$$\dot{V}(x, y) = \frac{y^2}{2} \left(1 - \frac{\beta x^2}{2} \right) - \frac{1}{2} \left(x - \frac{\alpha x^3}{6} \right)^2 - \frac{y}{2} \left(x - \frac{\alpha x^3}{6} \right) - y^2.$$

Using polar coordinates,

$$\begin{aligned}
 \dot{V}(r, \theta) &= -\frac{r^2}{2} [1 + \sin \theta \cos \theta + h(r, \theta)] \\
 &= -\frac{r^2}{2} \left[1 + \frac{1}{2} \sin 2\theta + h(r, \theta) \right].
 \end{aligned}$$

It is easy to show that

$$|h(r, \theta)| \leq \frac{1}{2} r^2 + \frac{1}{72} r^4.$$

So if r is sufficiently small, then $|h(r, \theta)| < 1/2$ and $|\frac{1}{2} \sin 2\theta + h(r, \theta)| < 1$. Hence $\dot{V}(x, y)$ is negative definite.

Now we show that $V(x, y)$ is positive definite. Since $g(u) = \sin u$,

$$V(x, y) = \frac{1}{2} y^2 + \frac{1}{2} y \sin(x) + 1 - \cos x.$$

This time we set

$$\cos x = 1 - \frac{x^2}{2} + \gamma \frac{x^4}{24}.$$

Note that $0 < \gamma < 1$ for $-\pi/2 < x < \pi/2$. Converting to polar coordinates,

$$\begin{aligned}
 V(r, \theta) &= \frac{r^2}{2} \left[1 + \sin \theta \cos \theta - \frac{r^2}{12} \sin \theta \cos^3 \theta - \gamma \frac{r^2}{24} \cos^4 \theta \right] \\
 &= \frac{r^2}{2} \left[1 + \frac{1}{2} \sin 2\theta - \frac{r^2}{12} \sin \theta \cos^3 \theta - \gamma \frac{r^2}{24} \cos^4 \theta \right].
 \end{aligned}$$

Now

$$-\frac{r^2}{12} \sin \theta \cos^3 \theta - \gamma \frac{r^2}{24} \cos^4 \theta > -\frac{1}{8} \text{ for } r < 1.$$

It follows that when $r > 0$,

$$V(r, \theta) > \frac{r^2}{2} \left[\frac{7}{8} + \frac{1}{2} \sin 2\theta \right] \geq \frac{3r^2}{16} > 0.$$

Therefore $V(x, y)$ is indeed *positive definite*, and by Theorem 9.6.1, the origin is an asymptotically stable critical point.

12(a). We consider the linear system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Let $V(x, y) = Ax^2 + Bxy + Cy^2$, in which

$$\begin{aligned} A &= -\frac{a_{21}^2 + a_{22}^2 + (a_{11}a_{22} - a_{12}a_{21})}{2\Delta} \\ B &= \frac{a_{12}a_{22} + a_{11}a_{21}}{\Delta} \\ C &= -\frac{a_{11}^2 + a_{12}^2 + (a_{11}a_{22} - a_{12}a_{21})}{2\Delta}, \end{aligned}$$

and $\Delta = (a_{11} + a_{22})(a_{11}a_{22} - a_{12}a_{21})$. Based on the hypothesis, the coefficients A and B are negative. Therefore, except for the origin, $V(x, y)$ is *negative* on each of the coordinate axes. Along each trajectory,

$$\begin{aligned} \dot{V} &= (2Ax + By)(a_{11}x + a_{12}y) + (2Cy + Bx)(a_{21}x + a_{22}y) \\ &= -x^2 - y^2. \end{aligned}$$

Hence $\dot{V}(x, y)$ is *negative definite*. Theorem 9.6.2 asserts that the origin is an *unstable* critical point.

(b). We now consider the system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} F_1(x, y) \\ G_1(x, y) \end{pmatrix},$$

in which $F_1(x, y)/r \rightarrow 0$ and $G_1(x, y)/r \rightarrow 0$ as $r \rightarrow 0$. Let

$$V(x, y) = Ax^2 + Bxy + Cy^2,$$

in which

$$\begin{aligned}
 A &= \frac{a_{21}^2 + a_{22}^2 + (a_{11}a_{22} - a_{12}a_{21})}{2\Delta} \\
 B &= -\frac{a_{12}a_{22} + a_{11}a_{21}}{\Delta} \\
 C &= \frac{a_{11}^2 + a_{12}^2 + (a_{11}a_{22} - a_{12}a_{21})}{2\Delta},
 \end{aligned}$$

and $\Delta = (a_{11} + a_{22})(a_{11}a_{22} - a_{12}a_{21})$. Based on the hypothesis, $A, B > 0$. Except for the origin, $V(x, y)$ is *positive* on each of the coordinate axes. Along each trajectory,

$$\dot{V} = x^2 + y^2 + (2Ax + By)F_1(x, y) + (2Cy + Bx)G_1(x, y).$$

Converting to polar coordinates, for $r \neq 0$,

$$\begin{aligned}
 \dot{V} &= r^2 + r(2A\cos\theta + B\sin\theta)F_1 + r(2C\sin\theta + B\cos\theta)G_1 \\
 &= r^2 + r^2 \left[(2A\cos\theta + B\sin\theta) \frac{F_1}{r} + (2C\sin\theta + B\cos\theta) \frac{G_1}{r} \right].
 \end{aligned}$$

Since the system is *almost linear*, there is an R such that

$$\left| (2A\cos\theta + B\sin\theta) \frac{F_1}{r} + (2C\sin\theta + B\cos\theta) \frac{G_1}{r} \right| < \frac{1}{2},$$

and hence

$$(2A\cos\theta + B\sin\theta) \frac{F_1}{r} + (2C\sin\theta + B\cos\theta) \frac{G_1}{r} > -\frac{1}{2}$$

for $r < R$. It follows that

$$\dot{V} > \frac{1}{2}r^2$$

as long as $0 < r < R$. Hence \dot{V} is *positive definite* on the domain

$$D = \{(x, y) \mid x^2 + y^2 < R^2\}.$$

By Theorem 9.6.2, the origin is an *unstable* critical point.

Section 9.7

3. The equilibrium solutions of the ODE

$$\frac{dr}{dt} = r(r-1)(r-3)$$

are given by $r_1 = 0$, $r_2 = 1$ and $r_3 = 3$. Note that

$$\frac{dr}{dt} > 0 \text{ for } 0 < r < 1 \text{ and } r > 3; \quad \frac{dr}{dt} < 0 \text{ for } 1 < r < 3.$$

$r = 0$ corresponds to an *unstable* critical point. The equilibrium solution $r_2 = 1$ is *asymptotically stable*, whereas the equilibrium solution $r_3 = 3$ is *unstable*. Since the critical values are *isolated*, a limit cycle is given by

$$r = 1, \theta = t + t_0$$

which is *asymptotically stable*. Another periodic solution is found to be

$$r = 3, \theta = t + t_0$$

which is *unstable*.

5. The equilibrium solutions of the ODE

$$\frac{dr}{dt} = \sin \pi r$$

are given by $r = n$, $n = 0, 1, 2, \dots$. Based on the *sign* of r' in the neighborhood of each critical value, the equilibrium solutions $r = 2k$, $k = 1, 2, \dots$ correspond to *unstable* periodic solutions, with $\theta = t + t_0$. The equilibrium solutions $r = 2k + 1$, $k = 0, 1, 2, \dots$ correspond to *stable* limit cycles, with $\theta = t + t_0$. The solution $r = 0$ represents an *unstable* critical point.

10. Given $F(x, y) = a_{11}x + a_{12}y$ and $G(x, y) = a_{21}x + a_{22}y$, it follows that

$$F_x + G_y = a_{11} + a_{22}.$$

Based on the hypothesis, $F_x + G_y$ is either *positive* or *negative* on the entire plane. By Theorem 9.7.2, the system cannot have a nontrivial periodic solution.

12. Given that $F(x, y) = -2x - 3y - xy^2$ and $G(x, y) = y + x^3 - x^2y$,

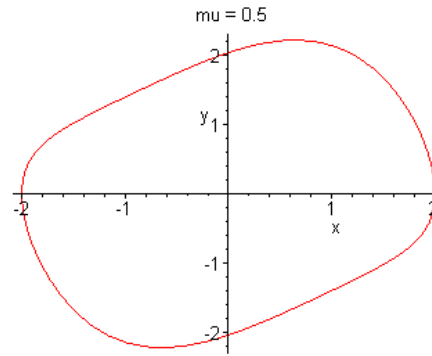
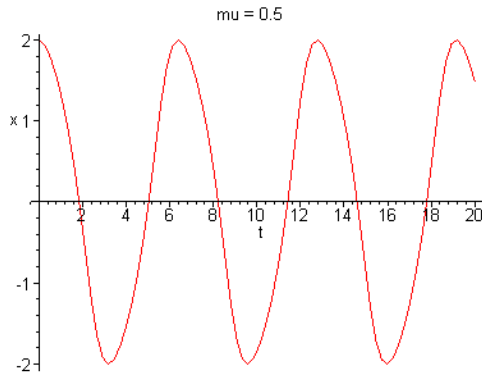
$$F_x + G_y = -1 - x^2 - y^2.$$

Since $F_x + G_y < 0$ on the entire plane, Theorem 9.7.2 asserts that the system cannot have a nontrivial periodic solution.

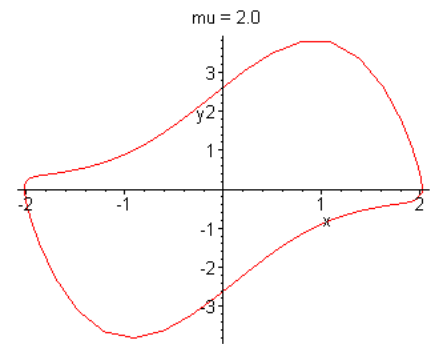
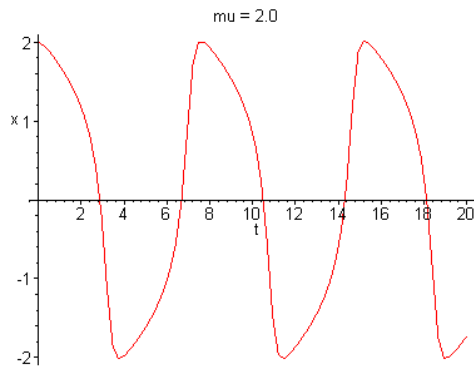
14(a). Based on the given graphs, the following table shows the estimated values:

$\mu = 0.2$	$T \approx 6.29$
$\mu = 1.0$	$T \approx 6.66$
$\mu = 5.0$	$T \approx 11.60$

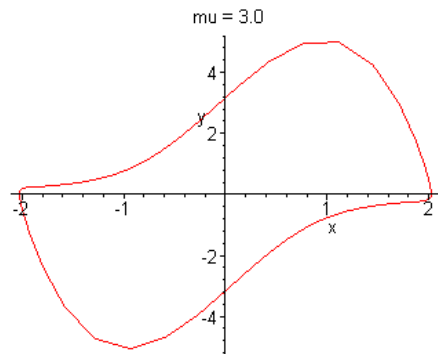
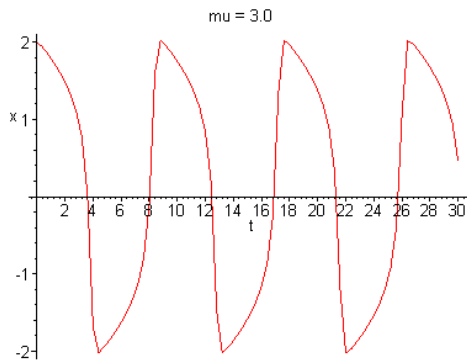
(b). The initial conditions were chosen as $x(0) = 2, y(0) = 0$.



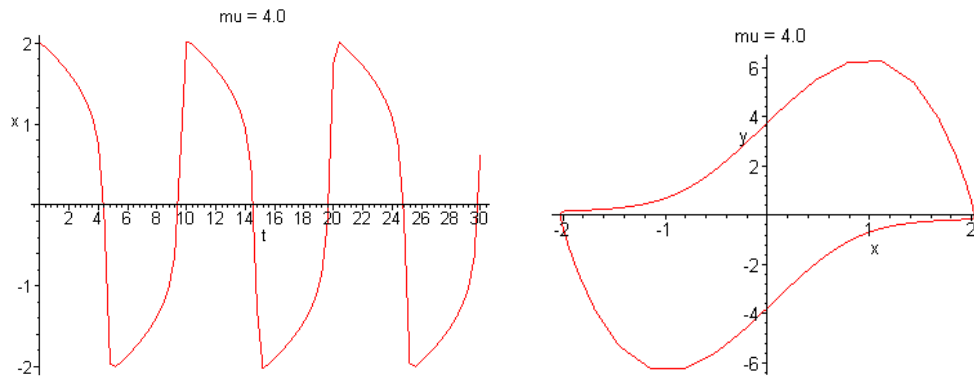
$T \approx 6.38$.



$T \approx 7.65$.

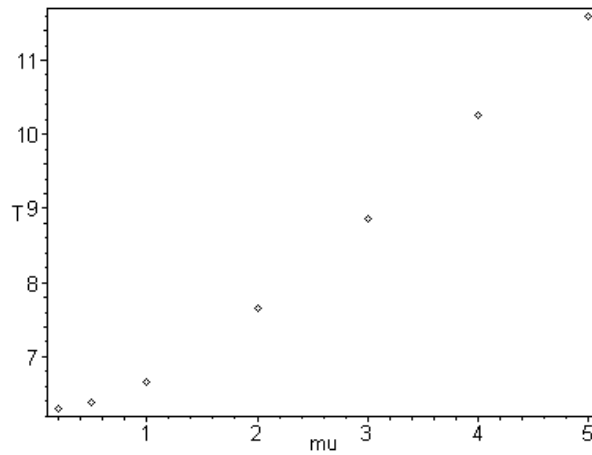


$T \approx 8.86$.



$T \approx 10.25$.

(c). The period, T , appears to be a *quadratic* function of μ .



15(a). Setting $x = u$ and $y = u'$, we obtain the system of equations

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -x + \mu \left(1 - \frac{1}{3} y^2 \right) y. \end{aligned}$$

(b). Evidently, $y = 0$. It follows that $x = 0$. Hence the only critical point of the system is at $(0, 0)$. The components of the vector field are infinitely differentiable everywhere. Therefore the system is *almost linear*.

The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & \mu - \mu y^2 \end{pmatrix}.$$

At the critical point $(0, 0)$, the coefficient matrix of the linearized system is

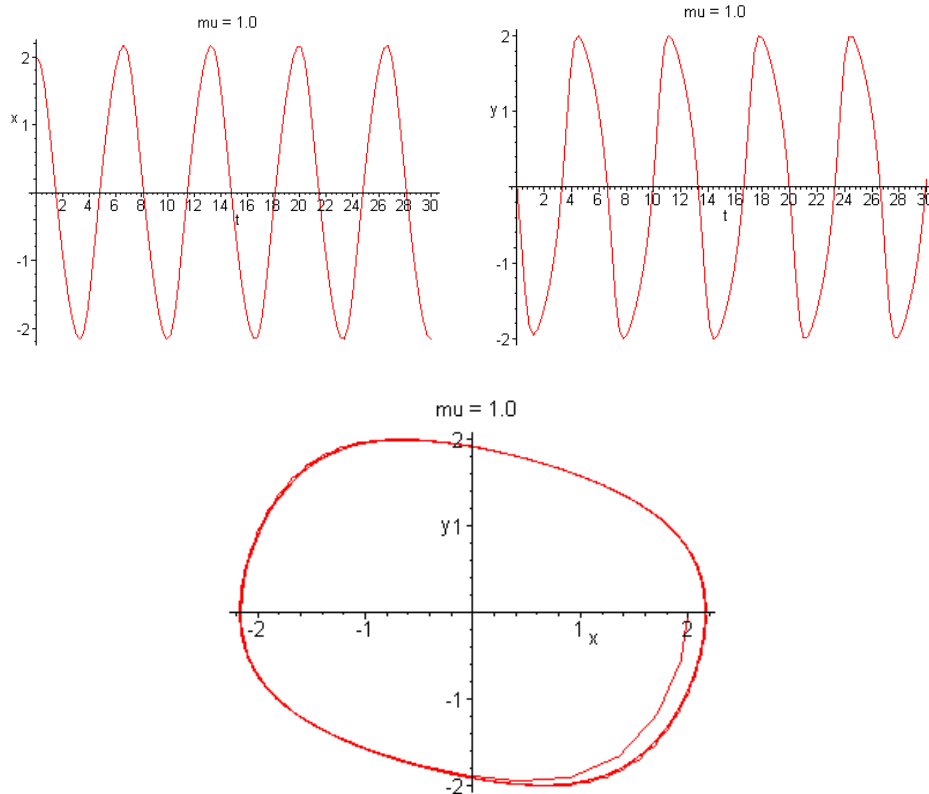
$$\mathbf{J}(0, 0) = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix},$$

with eigenvalues

$$r_{1,2} = \frac{\mu}{2} \pm \frac{1}{2} \sqrt{\mu^2 - 4}.$$

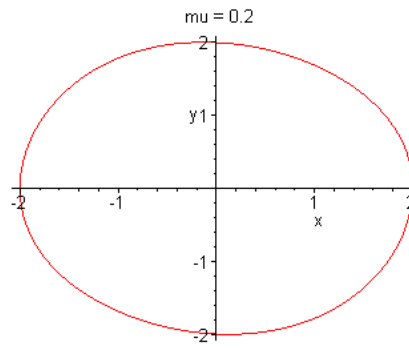
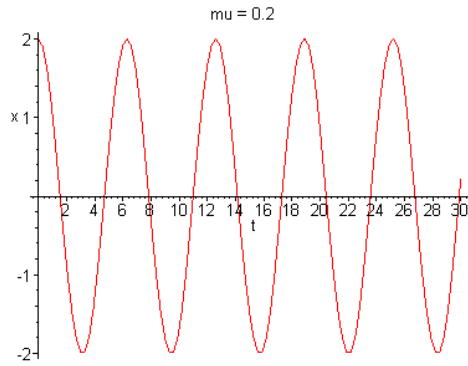
If $\mu = 0$, the equation reduces to the ODE for a simple harmonic oscillator. For the case $0 < \mu < 2$, the eigenvalues are *complex*, and the critical point is an *unstable spiral*. For $\mu \geq 2$, the eigenvalues are *real*, and the origin is an *unstable node*.

(c). The initial conditions were chosen as $x(0) = 2$, $y(0) = 0$.

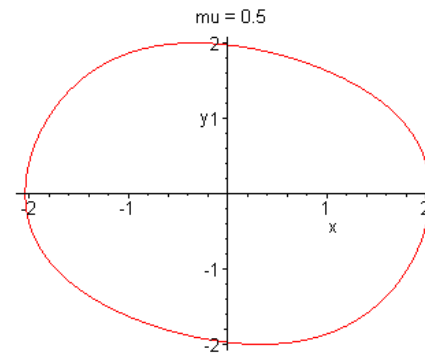
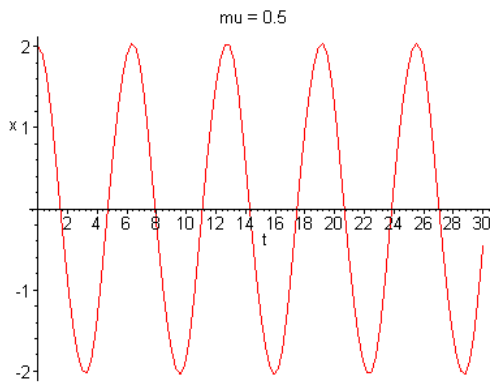


$A \approx 2.16$ and $T \approx 6.65$.

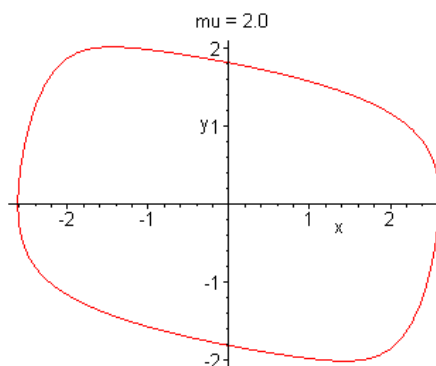
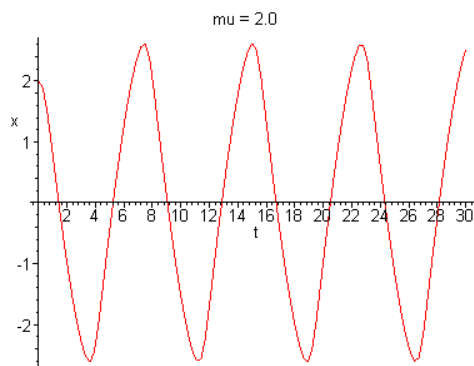
(d).



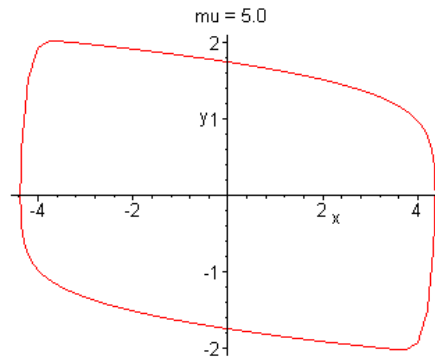
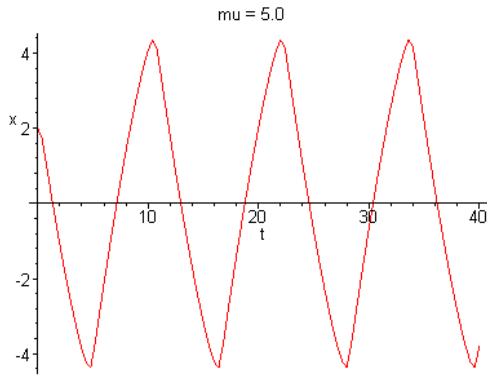
$A \approx 2.00$ and $T \approx 6.30$.



$A \approx 2.04$ and $T \approx 6.38$.



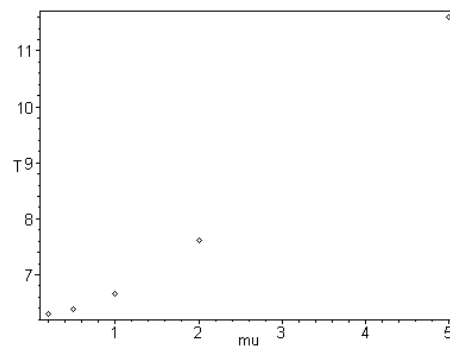
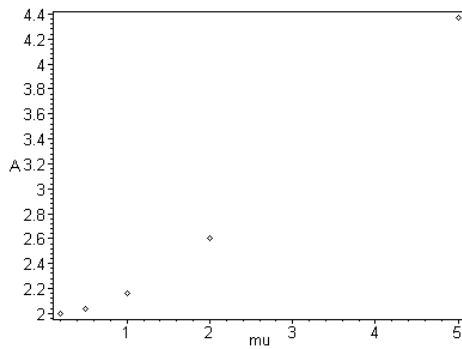
$A \approx 2.6$ and $T \approx 7.62$.



$A \approx 4.37$ and $T \approx 11.61$.

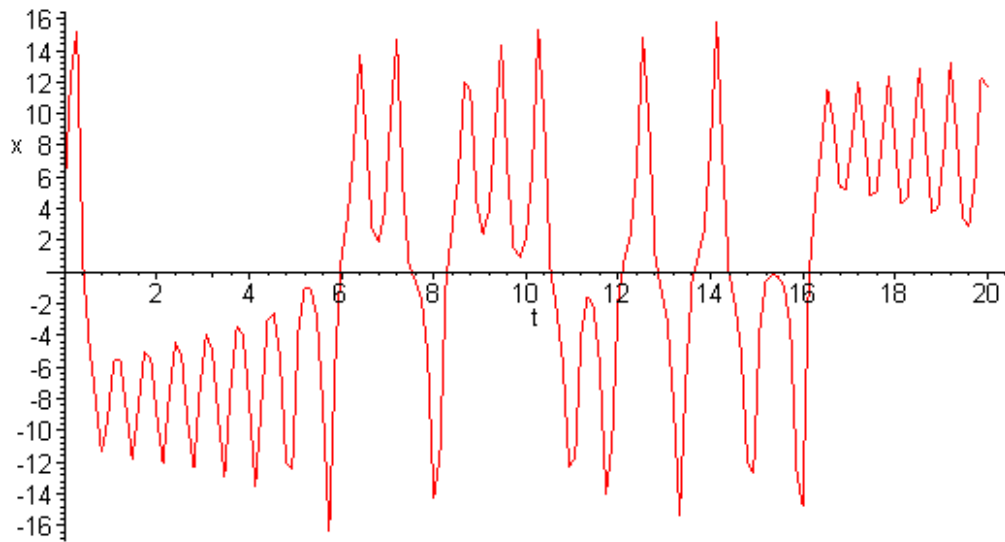
(e).

	A	T
$\mu = 0.2$	2.00	6.30
$\mu = 0.5$	2.04	6.38
$\mu = 1.0$	2.16	6.65
$\mu = 2.0$	2.6	7.62
$\mu = 5.0$	4.37	11.61

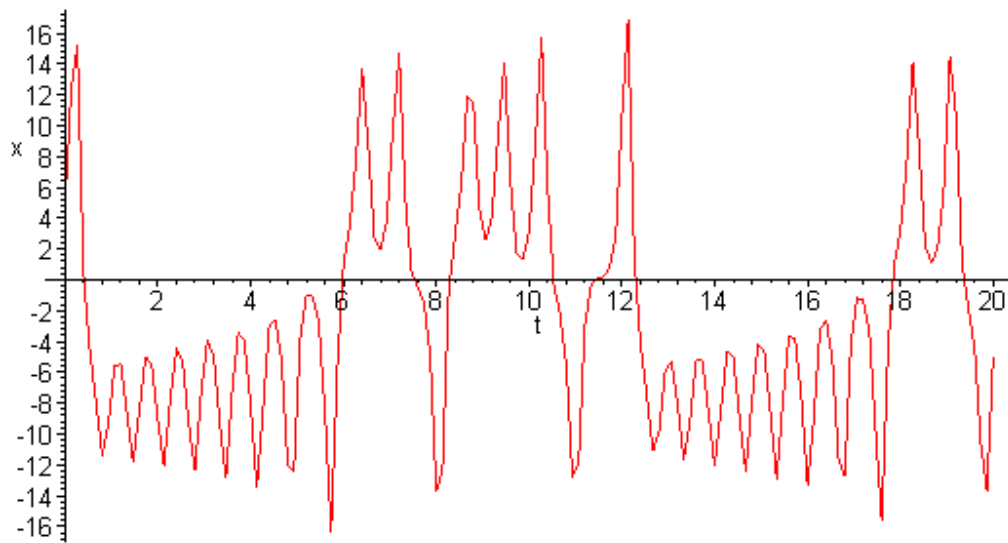


Section 9.8

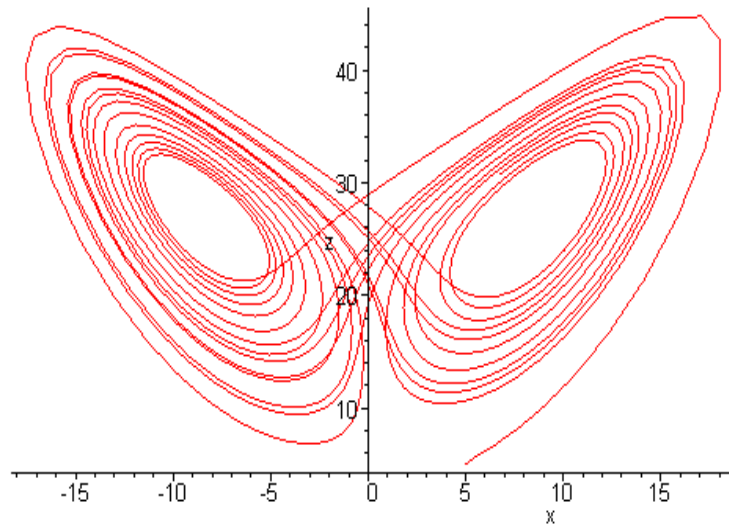
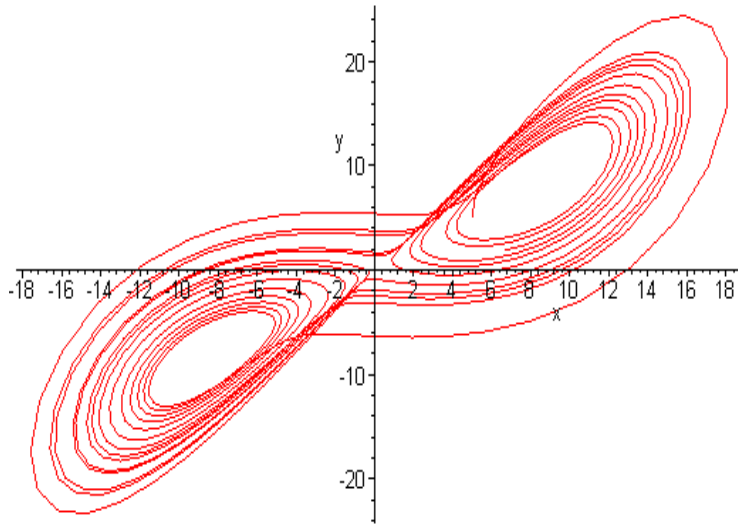
6. $r = 28$, with initial point $(5, 5, 5)$:



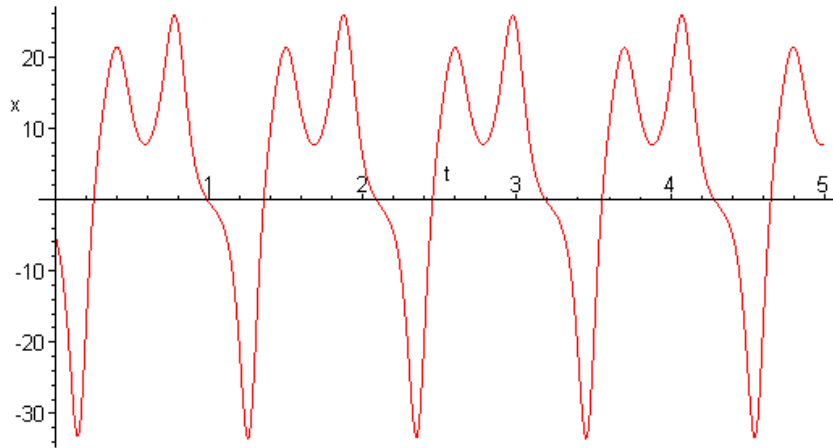
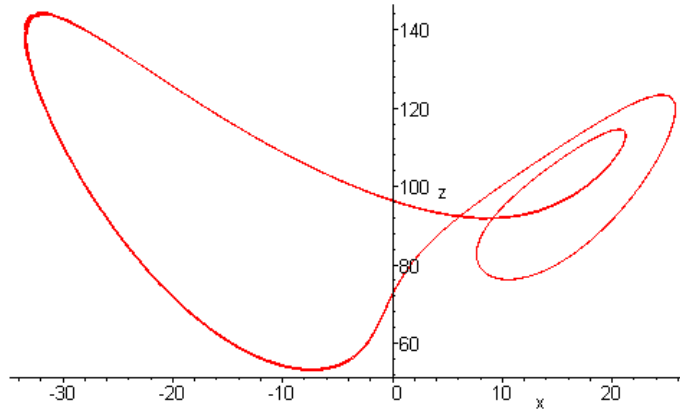
$r = 28$, with initial point $(5.01, 5, 5)$:



7. $r = 28$:

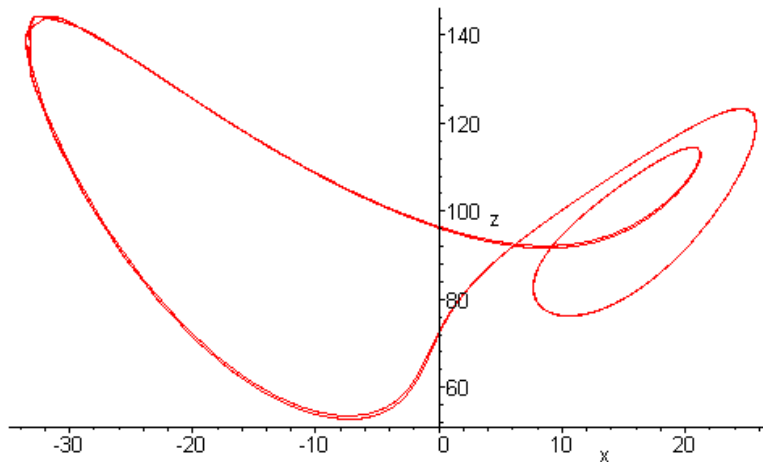


9(a). $r = 100$, initial point $(-5, -13, 55)$:

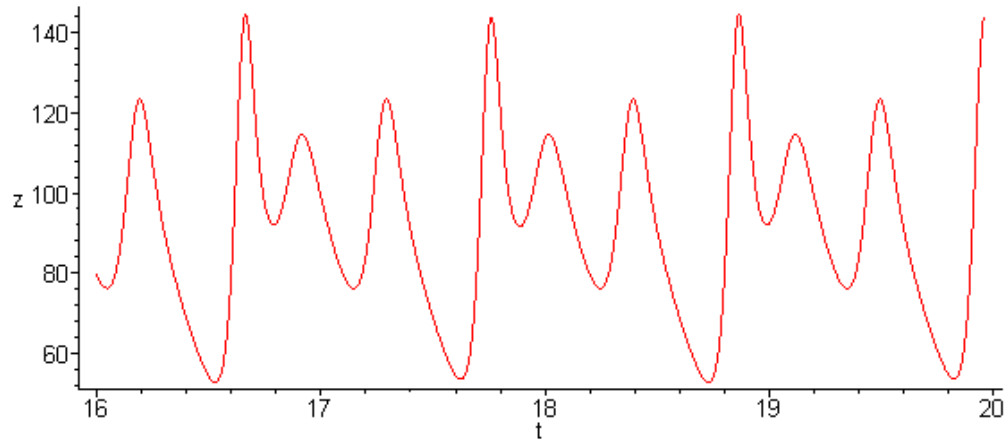


The period appears to be $T \approx 1.12$.

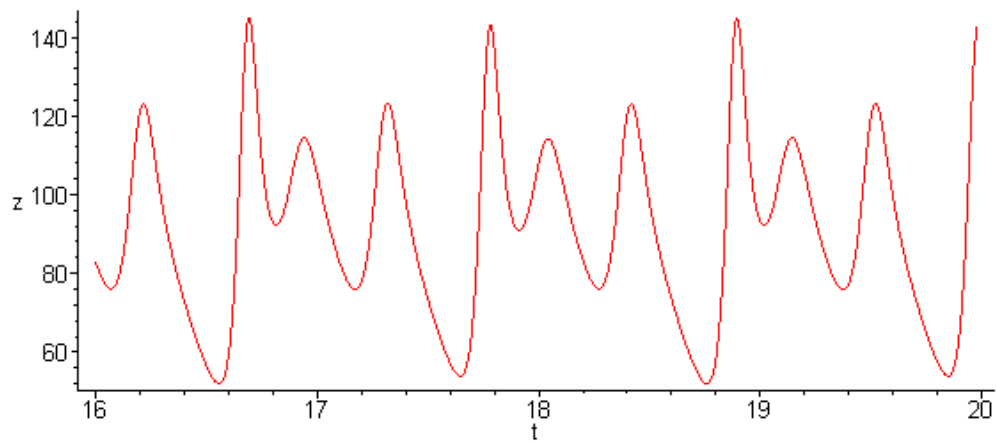
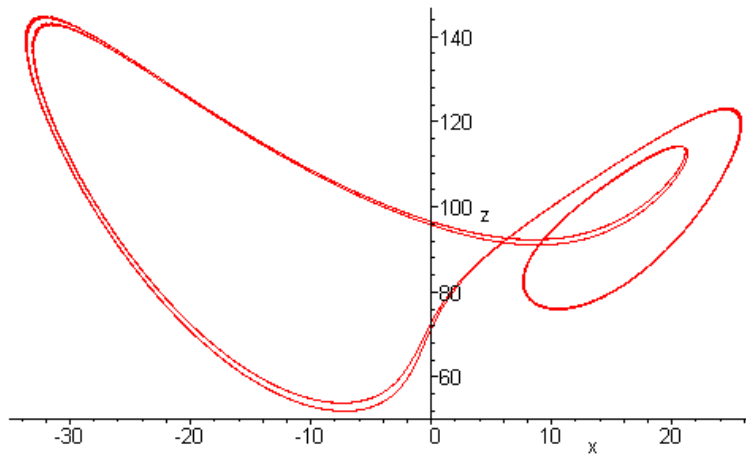
(b). $r = 99.94$, initial point $(-5, -13, 55)$:



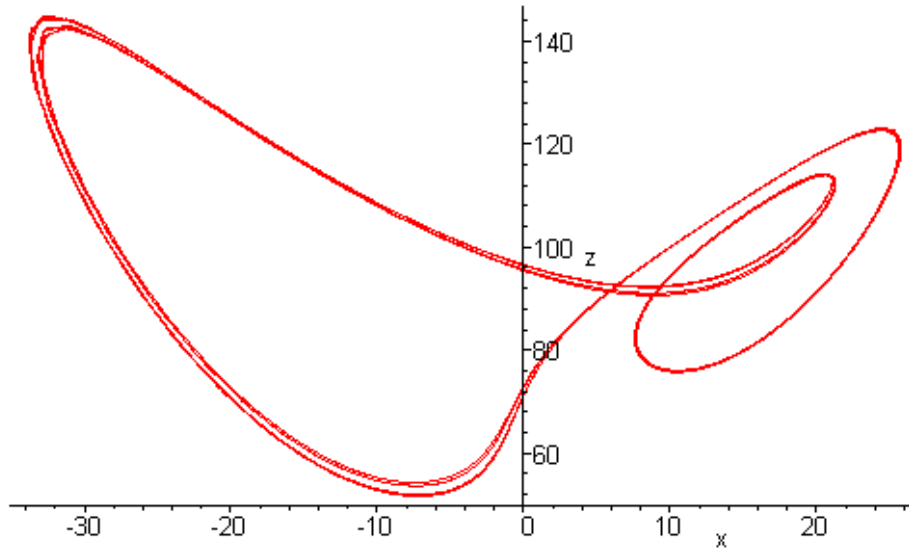
The periodic trajectory appears to have split into two strands, indicative of a period-doubling. Closer examination reveals that the peak values of $z(t)$ are slightly different:



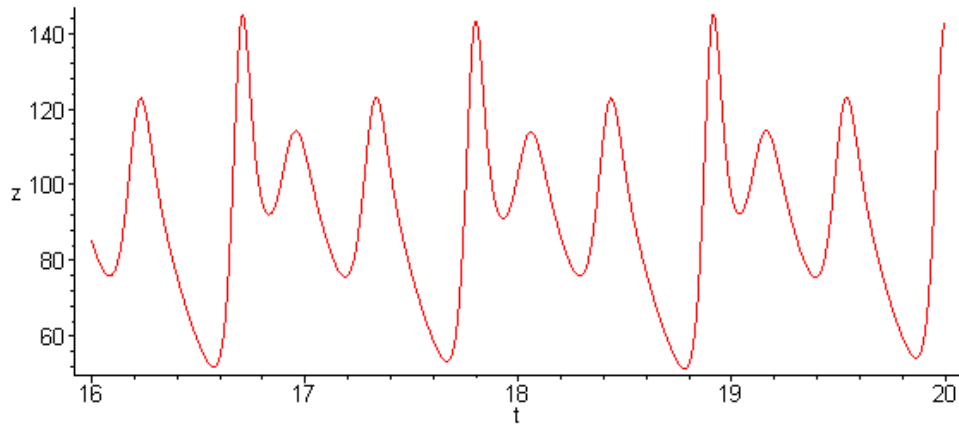
$r = 99.7$, initial point $(-5, -13, 55)$:



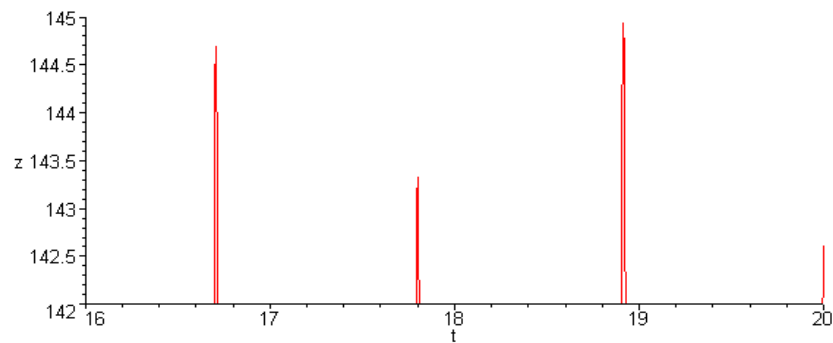
(c). $r = 99.6$, initial point $(-5, -13, 55)$:



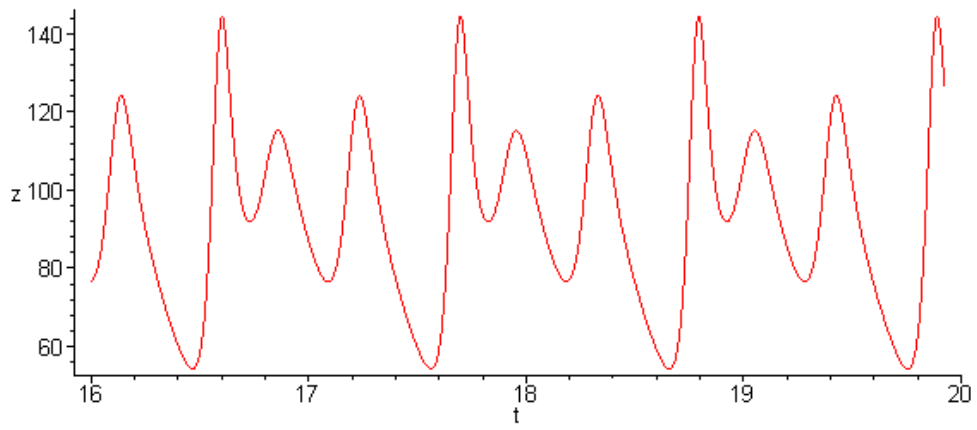
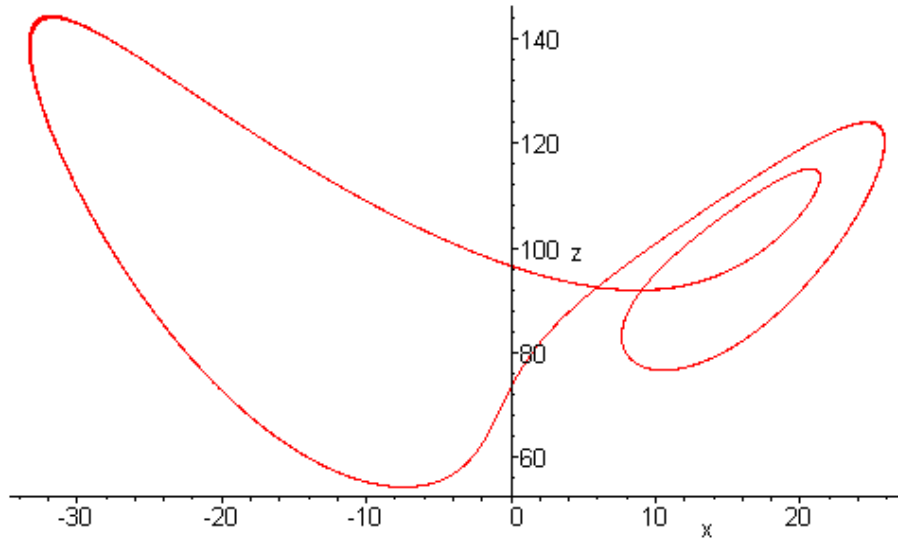
The strands again appear to have split.



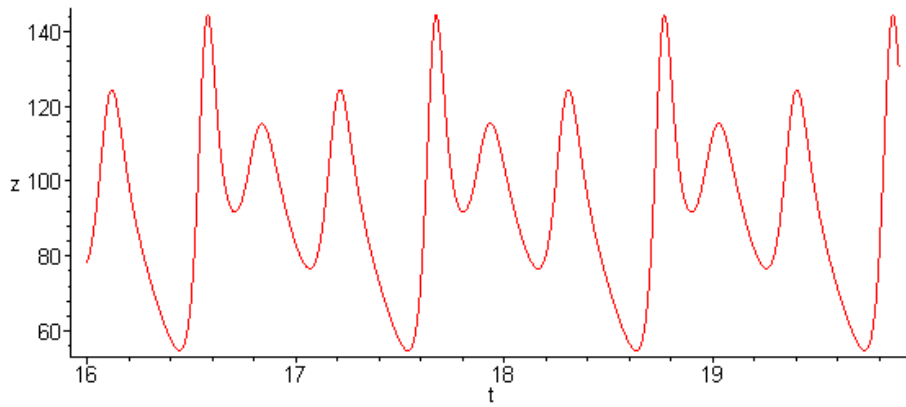
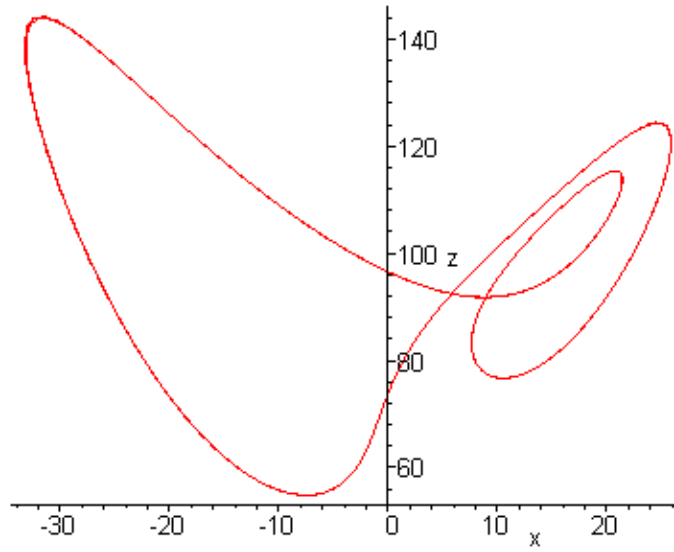
Closer examination reveals that the peak values of $z(t)$ are different:



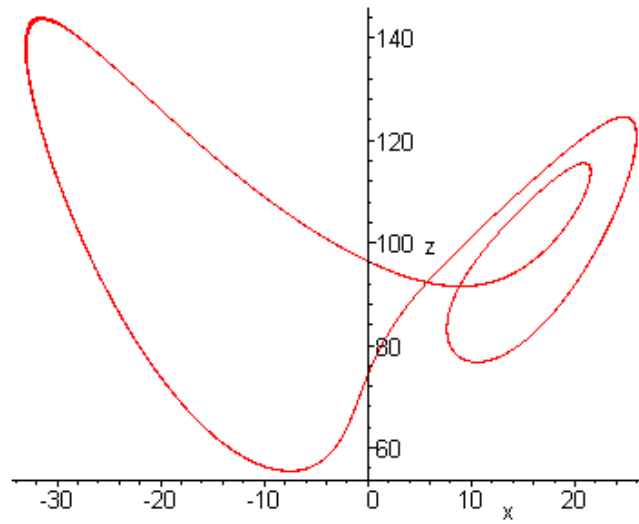
10(a). $r = 100.5$, initial point $(-5, -13, 55)$:



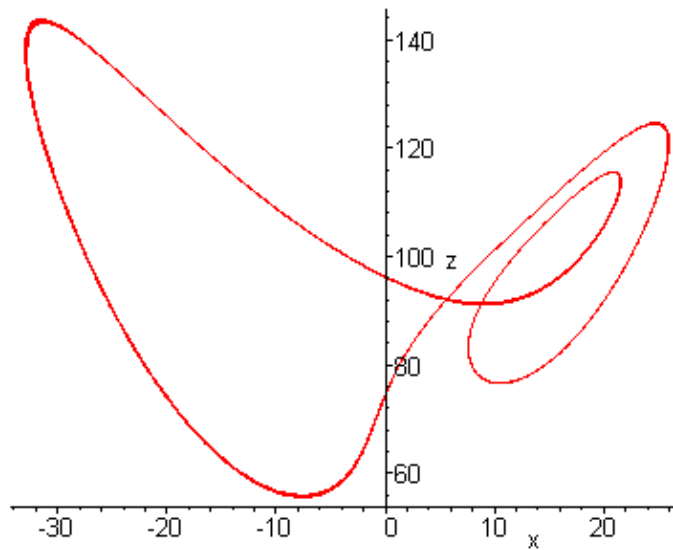
$r = 100.7$, initial point $(-5, -13, 55)$:



(b). $r = 100.8$, initial point $(-5, -13, 55)$:

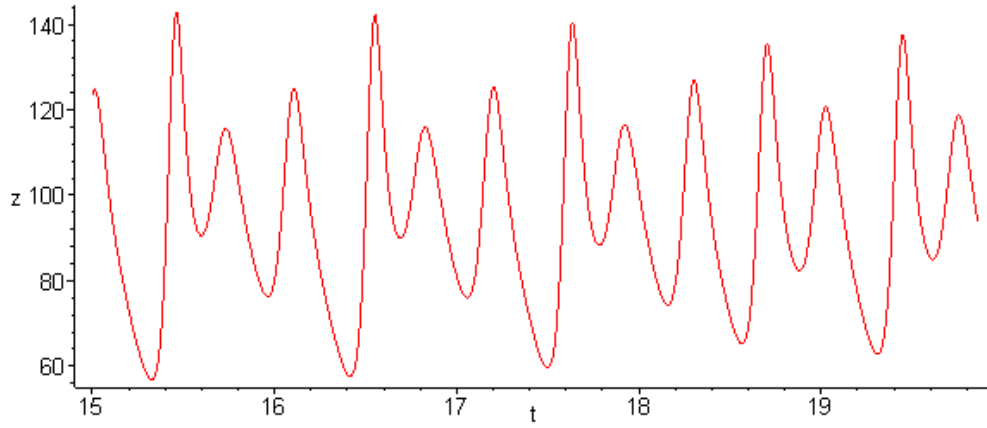
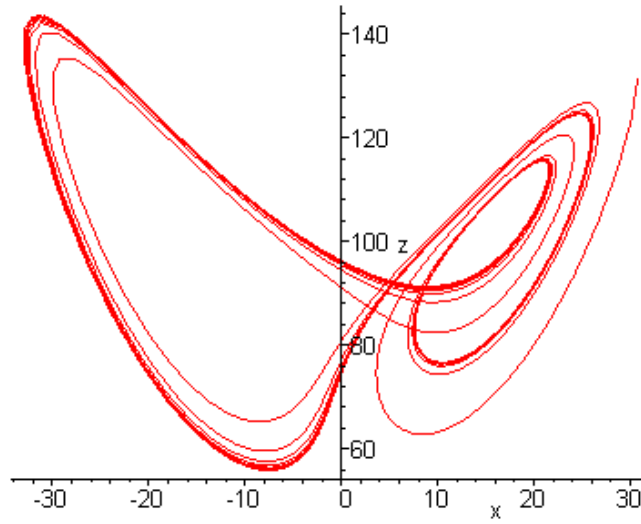


$r = 100.81$, initial point $(-5, -13, 55)$:

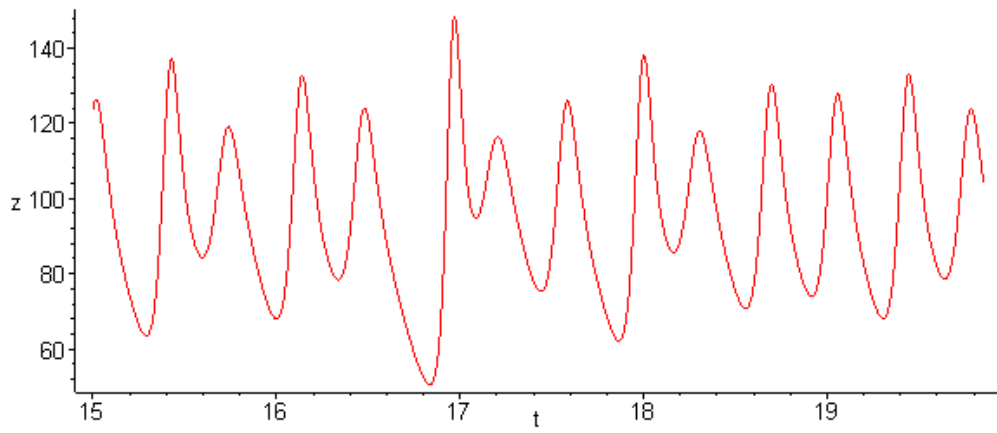
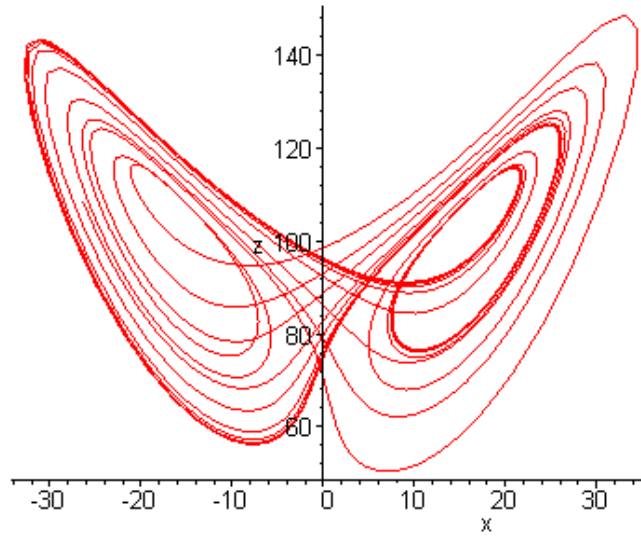


The strands of the periodic trajectory are beginning to split apart.

$r = 100.82$, initial point $(-5, -13, 55)$:



$r = 100.83$, initial point $(-5, -13, 55)$:



$r = 100.84$, initial point $(-5, -13, 55)$:

