INSTRUCTOR'S RESOURCE GUIDE
AND
SOLUTIONS MANUAL

to accompany

FINITE MATHEMATICS

Eighth Edition

Lial • Greenwell • Ritchey

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PREFACE

This book provides several resources for instructors using *Finite Mathematics,* Eighth Edition, by Margaret L. Lial, Raymond N. Greenwell, and Nathan P. Ritchey.

- Hints for teaching *Finite Mathematics* are provided as a resource for faculty.

- One open–response form and one multiple–choice form of a pretest are provided. These tests are an aid to instructors in identifying students who may need assistance.

- One open-response form and one multiple-choice form of a final examination are provided.

- Solutions for nearly all of the even-numbered exercises in the textbook are included. Solutions are usually not provided for exercises with open–response answers.

The following people have made valuable contributions to the production of this *Instructor's Resource Guide and Solutions Manual:* LaurelTech Integrated Publishing Services, editors; Judy Martinez and Sheri Minkner, typists; and Joe Vetere, Senior Author Support/Technology Specialist.

TEACHING HINTS

HINTS FOR TEACHING FINITE MATHEMATICS

Algebra Reference

This chapter is not as important for finite mathematics as it is for calculus. Nevertheless, we have included it in both books for those instructors who wish to cover this material. Some instructors get best results by going through this chapter carefully at the beginning of the semester. Others find it better to refer to it as needed throughout the course. Use whichever method works best for your students. We refer to the chapter as a "Reference" rather than a "Review," and the regular page numbers don't begin until Chapter 1. We hope this will make your students less anxious if you don't cover this material.

Section 1.1

This section and the next may seem fairly basic to students who covered linear functions in high school. Some students have difficulty finding the equation of a line from two points. Emphasize that there is no point-point form.

Perpendicular lines are not used in future chapters and could be skipped if you're in a hurry.

Section 1.2

Linear functions are the only functions students learn about in this section, giving them a gentle introduction to functions. Review graphing lines using intercepts, especially horizontal lines, vertical lines, and lines through the origin.

Supply and demand provides the students' first experience with a mathematical model. Spend time developing both the economics and the mathematics involved.

Stress that for cost, revenue, and profit functions, x represents the number of units. For supply and demand functions, we use the economists' notation of q to represent the number of units.

Emphasize the difference between the profit earned on 100 units sold as opposed to the number of units that must be sold to produce a profit of $100.

Section 1.3

The statistical functions on a calculator can greatly simplify these calculations, allowing more time for discussion and further examples. In this edition, we use "parallel presentation" to allow instructor choice on the extent technology is used. This section may be skipped if you are in a hurry, but your students can benefit from the realistic model and the additional work with equations of lines.

Chapter 2

The echelon method and the Gauss-Jordan method presented in the text are improved variations of the traditional methods. The "leading ones" are postponed until the last step, so as to avoid fractions and decimals. You may want to practice a few examples before presenting this method in class. We also present the traditional Gauss-Jordan method using a graphing calculator, for which keeping track of the fractions presents no difficulty.

Section 2.1

We have found it useful to spend less time on the echelon method and save the larger examples for the Gauss-Jordan method in the next section. Consequently, most of the exercises in this section involve only two equations. Use this section to introduce the concept of solving a system of linear equations and to show how a system can have one solution, no solutions, or an infinite number of solutions. Also use this section to show students how to solve applied exercises. Notice the application exercises in which a dependent system only has a finite number of solutions because the solution is restricted to nonnegative integers.

Emphasize row notation as a way to keep track of and to check the problem solving process.

Stress the guidelines, found before Example 5, for solving an application problem.

Section 2.2

Shown on the next page is a comparison between the improved version and the traditional Gauss-Jordan method. Note the absence of tedious fractions in the improved version.

Solve:

$$\begin{aligned} 4x - 2y - 3z &= -23 \\ -4x + 3y + z &= 11 \\ 8x - 5y + 4z &= 6 \end{aligned}$$

New Method

$$\left[\begin{array}{rrr|r} 4 & -2 & -3 & -23 \\ -4 & 3 & 1 & 11 \\ 8 & -5 & 4 & 6 \end{array}\right]$$

$$\begin{array}{r} \\ R_1 + R_2 \to R_2 \\ -2R_1 + R_3 \to R_3 \end{array} \quad \left[\begin{array}{rrr|r} 4 & -2 & -3 & -23 \\ 0 & 1 & -2 & -12 \\ 0 & -1 & 10 & 52 \end{array}\right]$$

$$\begin{array}{r} R_1 + 2R_2 \to R_1 \\ \\ R_2 + R_3 \to R_3 \end{array} \quad \left[\begin{array}{rrr|r} 4 & 0 & -7 & -47 \\ 0 & 1 & -2 & -12 \\ 0 & 0 & 8 & 40 \end{array}\right]$$

$$\begin{array}{r} 7R_3 + 8R_1 \to R_1 \\ R_3 + 4R_2 \to R_2 \\ \\ \end{array} \quad \left[\begin{array}{rrr|r} 32 & 0 & 0 & -96 \\ 0 & 4 & 0 & -8 \\ 0 & 0 & 8 & 40 \end{array}\right]$$

$$\begin{array}{r} \frac{1}{32}R_1 \to R_1 \\ \frac{1}{4}R_2 \to R_2 \\ \frac{1}{8}R_3 \to R_3 \end{array} \quad \left[\begin{array}{rrr|r} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$(-3, -2, 5)$ is the solution.

Traditional Method

$$\left[\begin{array}{rrr|r} 4 & -2 & -3 & -23 \\ -4 & 3 & 1 & 11 \\ 8 & -5 & 4 & 6 \end{array}\right]$$

$$\begin{array}{r} \frac{1}{4}R_1 \to R_1 \\ \\ \\ \end{array} \quad \left[\begin{array}{rrr|r} 1 & -\frac{1}{2} & -\frac{3}{4} & -\frac{23}{4} \\ -4 & 3 & 1 & 11 \\ 8 & -5 & 4 & 6 \end{array}\right]$$

$$\begin{array}{r} \\ 4R_1 + R_2 \to R_2 \\ -8R_1 + R_3 \to R_3 \end{array} \quad \left[\begin{array}{rrr|r} 1 & -\frac{1}{2} & -\frac{3}{4} & -\frac{23}{4} \\ 0 & 1 & -2 & -12 \\ 0 & -1 & 10 & 52 \end{array}\right]$$

$$\begin{array}{r} \frac{1}{2}R_2 + R_1 \to R_1 \\ \\ R_2 + R_3 \to R_3 \end{array} \quad \left[\begin{array}{rrr|r} 1 & 0 & -\frac{7}{4} & -\frac{47}{4} \\ 0 & 1 & -2 & -12 \\ 0 & 0 & 8 & 40 \end{array}\right]$$

$$\begin{array}{r} \\ \\ \frac{1}{8}R_3 \to R_3 \end{array} \quad \left[\begin{array}{rrr|r} 1 & 0 & -\frac{7}{4} & -\frac{47}{4} \\ 0 & 1 & -2 & -12 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$$\begin{array}{r} \frac{7}{4}R_3 + R_1 \to R_1 \\ 2R_3 + R_2 \to R_2 \\ \\ \end{array} \quad \left[\begin{array}{rrr|r} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$(-3, -2, 5)$ is the solution.

By reworking a problem from Section 2.1 using the Gauss-Jordan method, students will see how closely this method parallels the echelon method given there.

Remind students to operate on the entire row. A common error is to forget the entry to the right of the vertical bar.

Section 2.3

Mention that, as in algebra, only like things can be added or subtracted. In this case, the like things are matrices having the same dimensions.

Section 2.4

Using the visual approach to matrix multiplication given after Example 2, students will have no trouble multiplying matrices. Most will eventually no longer need this tool.

Section 2.5

Explain that the technique used in finding the multiplicative inverse of a matrix is still the Gauss-Jordan method, now with more than one entry per row to the right of the vertical bar.

Students may be resistant to learning another method for solving a system of equations. Stress the advantage of using the inverse method to solve systems having the same matrix of coefficients. See Example 4. Point out that these systems can be found in many different fields of application. See Exercise 60.

Section 2.6

Discuss how the entries of A, the input-output matrix, could be determined. Stress the economic significance of the matrices A, D, X, and AX.

Section 3.1

Emphasize that the test point can be *any* point *not* on the boundary. Choose several points on either side of the boundary and on the boundary itself to illustrate this concept.

Students may fall into the habit of always choosing $(0, 0)$ as the test point. Do a couple of problems where $(0, 0)$ is not available for use as a test point.

Using a different color to shade each half plane for a system of inequalities will make their overlap easier to recognize.

Section 3.2

Use diagrams like Figures 11 and 12 to convince students of the believability of the corner point theorem. Emphasize that a corner point *must* be a point in the feasible region. Also, stress that not all corner points can be found by inspection. Some require solving a system of two linear equations. Have students note the equation of the boundary line next to its graph, so they will know which equations to solve as a system.

Section 3.3

Review the guidelines for setting up an applied problem (Section 2.1) to determine the objective function and all necessary constraints.

Students find those constraints comparing two unknown quantities the most difficult. See Exercise 12 for an example of this type of constraint.

Section 4.1

The simplex method in this chapter is modified from the traditional method along the lines of the Gauss-Jordan method in Chapter 2, eliminating tedious fractions until the last step. The notation of s instead of x for slack and surplus variables will help students remember which variables are the originals and which are slack or surplus variables.

Note the horizontal line in the simplex tableau to separate the constraints from the objective function.

Students may need several examples to be able to pick out the basic variables and to find the basic feasible solution from a matrix.

Section 4.2

Vocabulary is extremely important in this section. An understanding of the terms basic variables, basic feasible solution, indicators, and pivots is a necessity.

Remind students that the simplex method stated before Example 1 works only for problems in standard maximum form.

Example 1 is extremely important because it connects the two methods of solving a linear programming problem. You may want to do a similar example in class. Emphasize the advantages of the simplex method, especially for larger problems.

Section 4.3

If you are in a hurry, either Section 4.3 or 4.4 can be skipped. Section 4.3 is needed if you wish to cover Section 11.3 on game theory and linear programming. If you choose to cover this section and skip Section 4.4, your students will only know how to solve standard minimization and maximization problems, but they at least they will see the profound and amazing theorem of duality. Notice in Exercises 16 and 17 that for maximization problems, shadow costs become shadow profits.

Provide numerous examples for reading the optimal solution from the last row of the final tableau of the dual problem.

Section 4.4

The usual method for solving nonstandard problems (those with mixed constraints) is the two-phase method, which is somewhat complicated for students at this level. We use a modification of this method which students should find simpler.

Stress that slack variables are used for $\leq$ constraints, while surplus variables are used for $\geq$ constraints. Artificial variables only need to be covered if you want to solve constraints with an $=$. Even then, they can be avoided by replacing each $=$ constraint with two inequalities, one with $\leq$ and one with $\geq$.

Emphasize that to use the simplex method to find the optimal feasible solution, one must start with a feasible solution.

In Step 5 of the box "Solving a Nonstandard Problem," our choice of the positive entry that is farthest to the left is arbitrary. If your students choose a different column, they may still come up with the correct answer, and it might even require fewer steps.

Remind students to convert from z to w as the last step in solving a minimization problem.

Chapter 5

The chapter on mathematics of finance does not depend on earlier chapters and may be covered at any time.

Students may feel overwhelmed by the number of formulas presented in Chapter 5. Guidelines for choosing the appropriate formula can be found at the end of the chapter. This summary may be referred to throughout Chapter 5.

Chapter 5 requires numerous financial calculations. Make sure students are familiar with their calculators. The financial features of the TI-83/84 Plus make calculations easy.

Section 5.1

Interest is the key concept in Chapter 5. It is important that students understand that interest is the cost of borrowing money (or the reward for lending money). Both simple and compound interest are covered in the first section.

Point out that as the frequency of compounding increases, so does the amount of interest earned. Also note, however, that this increase in interest gets smaller and smaller as the interest is compounded more frequently. See Exercises 58 and 59.

The effective rate of interest is a topic that students find most useful and interesting. Bring in advertisements for loans that hide the effective rate (the APR) in the fine print.

Chapter 5 is full of symbols and formulas. It is imperative that students become familiar with the notation and know which formula is appropriate for a given problem. A summarization of the formulas in Section 5.1 is found at the end of the section.

Section 5.2

This section starts with an introduction to geometric sequences, which lays the groundwork for developing the future value formula as the sum of a geometric sequence.

Section 5.3

Make sure students understand that the present value formula presented here is for an ordinary annuity only.

Many students have had experience with amortization. Illustrate this topic using examples with present day interest rates. Students may bring in personal examples that may be used in class.

Chapter 6

This chapter has been substantially rewritten and reorganized from the previous edition. It now leads students toward the construction of proofs in Section 5, with quantifiers briefly introduced in Section 6. Proofs are more difficult than truth tables, but they are also far more important. In this edition we use meaningful variables, such as d for "Django is a good dog," rather than generic variables such as p and q. We find (and our students do too) that this makes it easier to keep track of which variable stands for each statement.

This is a nice chapter to cover before sets (Chapter 7), because many of the same ideas appear in both contexts. You should point out the parallels to the students whenever possible.

Section 6.1

The chapter starts with fairly easy material, but the statement "Neither p nor q" can cause trouble. Notice that, in this edition, we have introduced the basic truth tables in this section, and the material on the quantifiers "For all" and "There exists" has been moved to Section 6.

Section 6.2

Students usually find truth tables fun. The alternative method presented in Example 5 helps alleviate any tedium.

Section 6.3 and 6.4

The conditional is probably the most challenging logical operator, perhaps because its usage in mathematical language is just different enough from that of common language to cause confusion. We have found that even after students have studied logic, they still give erroneous answers to tests of reason such as those in Exercises 59 and 60 of Section 6.4. The common translations of $p \to q$ given before Example 3 of Section 6.4 are particularly troublesome. Don't assume your students have mastered this material until you have firm evidence.

Section 6.5

This section is the culmination of the first four sections. The two most important skills are showing an argument is invalid by counterexample, and showing an argument is valid by proof. Look carefully at Examples 5 through 8. This is the most difficult material in the chapter, so students need a lot of practice to master these ideas. The payoff is worth it when students learn to create a proof. The puzzles by Lewis Carroll in Example 6 and Exercises 38-43 are fun.

Section 6.6

Some students won't believe Euler is pronounced "oiler." The material on quantifiers, previously scattered in two sections, is now united here. This is just an introduction; we do not try to teach proofs using quantifiers except by Euler diagrams.

Chapter 7

The material on probability is arranged so that if you are in a hurry to get to Markov chains, you can skip Section 7.6 and all of Chapters 8 and 9.

Section 7.1

Manipulatives are quite useful in this chapter, especially a deck of cards. Further, set brackets may be modeled as a box and the elements of the set as objects inside the box.

Stress the key word for each set operation: "not" for complement; "and" for intersection; "or" for union.

Section 7.2

Mention the order of set operations.

If parentheses are present, simplify within them in the following order:

1) Take all complements.
2) Take the unions or intersections in the order they occur from left to right.

If no parentheses are present, start with 1).

To solve a survey problem, students must first be able to identify what type of object belongs to a certain region before they can determine how many objects belong to that region.

Have students explore the union rule for sets by determining the number of cards that are red or a king in their decks. Compare this problem with the problem of determining how many cards are fives or sevens (disjoint sets).

Section 7.3

Students need to be able to identify the experiment, the number of trials, the sample space, and the event in each probability problem.

Illustrate the basic probability principle using numerous examples with manipulatives.

Section 7.4

Redo the examples used to explore the union rule for sets to explore the related union rule for probability.

The complement rule is most useful for problems that contain statements of the form greater than, less than, etc.

Section 7.5

Sometimes independent events can be thought of as events that have the same sample space. For example, when two cards are drawn one at a time with replacement, both draws have a sample space consisting of all 52 cards. If these cards are drawn, instead, without replacement, the sample space for the second card has been reduced to 51 cards. Emphasize that the notation $P(A|B)$ reminds us how the sample space was reduced.

Section 7.6

Point out that trying to calculate $P(F|E)$ directly is sometimes impossible, too expensive, or too inconvenient. Thus, there is a need for Bayes' theorem which allows for the indirect calculation of $P(F|E)$ using $P(E|F)$. If a tree diagram is employed, then Bayes' theorem can be stated as

$$P(F|E) = \frac{\text{the probability of the branch through } F \text{ and } E}{\text{the sum of the probabilities of all branches ending in } E}$$

Point out that the branch in the numerator will also be one of the branches in the denominator.

Section 8.1

To use the multiplication principle, break down the problem (the task) into parts. Draw a blank for each part. Fill in each blank with the number of ways that part of the task can be completed. Finally, multiply these numbers to obtain the solution.

Permutations are a special case of the multiplication principle that does not allow for repetition.

Section 8.2

An additional way to determine whether to use combinations or permutations is as follows:

1) Give a label to each of the n objects.
2) Pick r objects from the n objects.
3) Rearrange the r objects.
4) If this rearrangement can be considered the same as the original arrangement, use combinations. If it is different, use permutations.

Section 8.3

This section combines the counting techniques of the previous two sections with the basic probability principle.

Section 8.4

Students often have difficulty dealing with the phrases "at least," "at most," "no more than," etc. Have the students work numerous examples that include these phrases.

Section 8.5

In this section, we complete the discussion of binomial probability from the previous section by giving the formula $E(x) = np$ for the expected value in binomial probability. Having students work out expected value in a binomial probability exercise by the definition and by this formula will increase their confidence in both.

Section 9.1

Warn students that the term "average" is ambiguous. Illustrate this concept using the average salary example following Example 6. Have students find the modal salary. Discuss the problems this ambiguity may cause.

Section 9.2

The square of the standard deviation is the variance, while the square root of the variance is the standard deviation. Students often get these confused.

Section 9.3

Note that the standard normal table used in this text is different from the table that is found in many statistics books. Call this to the attention of students. Some may be familiar with the other table.

Students may find it helpful to draw the nonstandard normal curve with x-values first, then convert to z-scores and draw the standard normal curve.

If your students have graphing calculators that give normal probability, they will have no need for the standard normal table in the back of the book. Their answers to exercises and examples, however, may be slightly different from ours, which were found using the table.

Section 9.4

When using the normal approximation to the binomial, students often have difficulty choosing the appropriate x-value(s) on the normal curve. Provide numerous examples to practice this technique.

Chapter 10

Chapter 10 requires a knowledge of the Gauss-Jordan method, matrix multiplication, and matrix inverses, along with probability.

Section 10.2

Remind students that since the equilibrium vector, $V = [v_1 \quad v_2 \quad \ldots \quad v_n]$, is a probability vector, we have the additional equation $v_1 + v_2 + \ldots + v_n = 1$. This equation will always be part of the system of linear equations that is solved to determine V.

Section 10.3

Illustrate the construction of G, the transition matrix for an absorbing Markov chain, very carefully. Make sure the students can identify I_m, Q, R, O, and the number of absorbing states.

Chapter 11

Chapter 11 utilizes matrices (Chapter 2), probability (Chapter 7), and expected value (Chapter 8).

Section 11.1

Begin developing the concept of strategy that is central to the entire chapter. Contrast strategies with states of nature.

There is a lot of specialized vocabulary in this section, which can be quite confusing for students. They need to have an understanding of the following terms: game, two-person game, zero-sum game, strictly determined game, fair game, strategy, dominated strategy, optimum strategy, and value of the game.

Section 11.2

Emphasize that the optimum strategy in a non-strictly determined game occurs when $E_1 = E_2$. Notice the new Exercises 27 and 28, in which we show how the method of this section can be extended to matrices with two rows and more than two columns. Of course, the method can also be extended to matrices with two columns and more than two rows. This forces students to think through the process of Example 3, rather than to just memorize the formulas in the box "Optimum Strategies in a Non-Strictly-Determined Game."

Section 11.3

Section 11.3 requires an understanding of solving linear programming problems by the simplex method (Chapter 4), including Section 4.3 on duality. Point out that the technique presented in this section is used to solve games that don't have a saddle point and in which each player has an arbitrary number of choices.

PRETESTS

AND

ANSWERS

FINITE MATHEMATICS

Name:

Pretest, Form A

Find the value of each of the following expressions.

1. $(.5)^3 \cdot (.2)^2$ **1.** ______

2. $\sqrt{27\left(\frac{1}{3}\right)\left(1+\frac{1}{3}\right)}$ **2.** ______

3. $2000\left(1-\dfrac{1}{2}\right)^4$ **3.** ______

4. $\dfrac{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{4\cdot 3\cdot 2\cdot 1}$ **4.** ______

5. $\dfrac{.5\,(.7)}{.5\,(.7)+2\,(.35)}$ **5.** ______

6. Find the value of $\frac{3(a+2b-1)}{a(b+1)}$ if $a = 6$ and $b = -2$. **6.** ______

Solve each of the following equations.

7. $18 - \dfrac{2}{3}y = \dfrac{5}{6}y$ **7.** ______

8. $3x - (4x + 8) = 25$ **8.** ______

9. $4\,(2z - 3) + 5 = -2\,(z + 6)$ **9.** ______

10. $.03x + .05\,(200 - x) = 9$ **10.** ______

11. Solve the equation $q = \frac{3}{4}p - 8$ for p. **11.** ______

12. Solve the equation $4x - 5y = 8$ for y. **12.** ______

13. Find the coordinates of the point where the graph of $6x - 5y = 10$ crosses the x-axis. **13.** ______

14. Suppose $C = 35x - 250$. Find x when C is 450. **14.** ______

15. A coat that sells for \$125 is put on sale for \$80. Find the percent of markdown. **15.** ______

16. 66 is 120% of what number? **16.** ______

17. Margaret can travel 216 miles on 9 gallons of gas. How many gallons will she need to travel 336 miles?

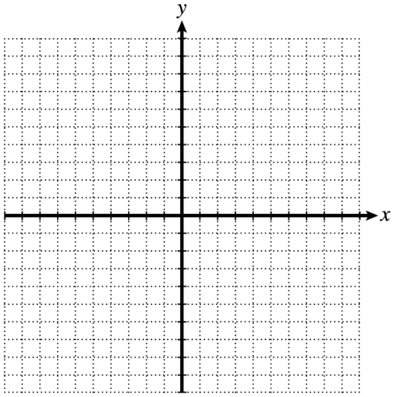
17. ____________________

18. Solve the inequality $3\,(2-x)-4\,(x+6)<-2\,(x-1)\,.$

18. ____________________

19. Graph the equation $5x-2y=10.$

19.



20. Find the slope of the line through the points with coordinates $(4,7)$ and $(-3,5)\,.$

20. ____________________

21. Write an equation in the form $ax+by=c$ for the line through the points with coordinates $(2,-3)$ and $(5,6)\,.$

21. ____________________

22. Solve the following system of equations.

$$\begin{aligned} 3x + 4y &= 14 \\ -2x + 5y &= 29 \end{aligned}$$

22. ____________________

23. Which of the following describes the graph of the equation $x=-4$?

(a) A line with slope -4
(b) A line with slope 4
(c) A horizontal line
(d) A vertical line

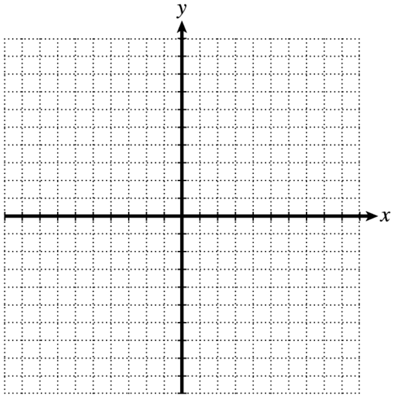
23. ____________________

24. Find the slope of the line with equation $4x-5y=10.$

24. ____________________

25. Graph the inequality $3x+5y\geq 15.$

25.



FINITE MATHEMATICS

Name:

Pretest, Form B

Choose the best answer.

Find the value of each of the following expressions.

1. $(.3)^2 \cdot (.7)^1$ **1.** ________

(a) 2.1 (b) .63 (c) .21 (d) .063

2. $\sqrt{16\left(\frac{1}{2}\right)\left(1-\frac{1}{2}\right)}$ **2.** ________

(a) 8 (b) 4 (c) 2 (d) $4\sqrt{2}$

3. $4000\left(1+\dfrac{1}{2}\right)^3$ **3.** ________

(a) 13,500 (b) 9000 (c) 6000 (d) 3000

4. $\dfrac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$ **4.** ________

(a) 9876 (b) 6789 (c) 3024 (d) 120

5. $\dfrac{.6\,(.3)}{.6\,(.3)+.4\,(.9)}$ **5.** ________

(a) 2 (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) .54

6. Find the value of $\frac{2(a+1-b)}{a(a+1)}$ if $a = 10$ and $b = 5$. **6.** ________

(a) 1 (b) $\frac{6}{55}$ (c) $\frac{1}{11}$ (d) $-\frac{4}{15}$

Solve each of the following equations.

7. $12 - \dfrac{3}{4}r = \dfrac{5}{8}r + 23$ **7.** ________

(a) -8 (b) 8 (c) -1 (d) 4

8. $2x - (3x + 7) = 18$ **8.** ________

(a) 11 (b) -11 (c) 25 (d) -25

9. $-2(k-1)+8=-2(k+3)$ **9.** ________

(a) 4 (b) 10 (c) -6 (d) No solution

10. $.04x+.07(900-x)=51$ **10.** ________

(a) 4000 (b) 400 (c) 317 (d) 2083

11. Solve the equation $q=-\frac{2}{7}p-3$ for p. **11.** ________

(a) $p=\frac{7}{2}q+\frac{21}{2}$ (b) $p=-\frac{7}{2}q-\frac{21}{2}$

(c) $p=\frac{7}{2}q-\frac{21}{2}$ (d) $p=7q+\frac{3}{2}$

12. Solve the equation $3x+6y=7$ for y. **12.** ________

(a) $y=-\frac{1}{2}x+7$ (b) $y=\frac{1}{2}x+\frac{7}{6}$

(c) $y=-2x+\frac{7}{3}$ (d) $y=-\frac{1}{2}x+\frac{7}{6}$

13. Find the coordinates of the point where the graph of $7x-4y=8$ crosses the y-axis. **13.** ________

(a) $(0,7)$ (b) $(0,4)$ (c) $(-2,0)$ (d) $(0,-2)$

14. Suppose $C=20x+500$. Find x when C is 650. **14.** ________

(a) 13,500 (b) 1150 (c) 7.5 (d) 3

15. A boat that sells for \$5000 is marked up to \$8000. What is the percent increase? **15.** ________

(a) 80% (b) 60% (c) 62.5% (d) 37.5%

16. 68 is 85% of what number? **16.** ________

(a) 125 (b) 80 (c) 57.8 (d) 54.4

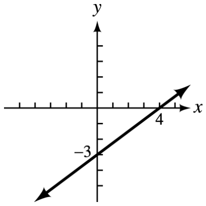
17. Bob can travel 160 miles on 8 gallons of gas. How many gallons will he need to travel 400 miles? **17.** ________

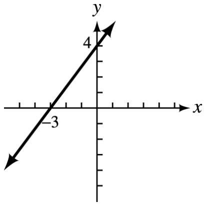
(a) 80 (b) 30 (c) 20 (d) 16

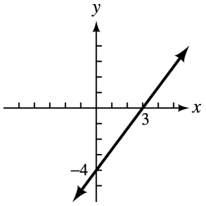
18. Solve the inequality $4(5-2x)-2(x+7)\leq -2-(x-80)$. **18.** ________

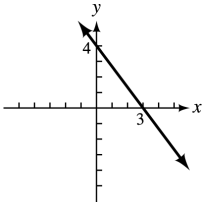
(a) $x\geq -8$ (b) $x\leq -8$ (c) $x\geq 8$ (d) $x\leq 8$

19. Graph the equation $4x - 3y = 12$. **19.** ________

(a)


(b)


(c)


(d)


20. Find the slope of the line through the points with coordinates $(3, 8)$ and $(-2, 4)$ **20.** ________

(a) 6 (b) $\frac{5}{4}$ (c) $\frac{4}{5}$ (d) $-\frac{4}{5}$

21. Write an equation in the form $ax + by = c$ for the line through the points with coordinates $(5, 8)$ and $(6, 4)$. **21.** ________

(a) $4x - y = 12$ (b) $4x + y = 12$

(c) $x + 4y = 37$ (d) $4x + y = 28$

22. Solve the following system of equations. **22.** ________

$$\begin{aligned} x + 2y &= 5 \\ x - 20 &= 3y \end{aligned}$$

(a) $(-50, 25)$ (b) $(11, -3)$ (c) $(-5, 5)$ (d) $(5, -5)$

23. Which of the following describes the graph of the equation $x = 7$? **23.** ________

(a) A vertical line (b) A parabola

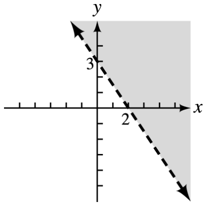
(c) A line with slope 7 (d) A horizontal line

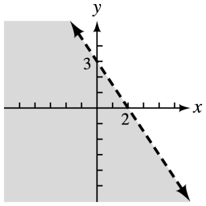
24. Which of the following lines does *not* have slope 4? **24.** ________

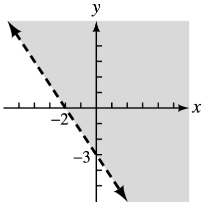
(a) $y = 4x - 8$ (b) $2y - 8x = 7$

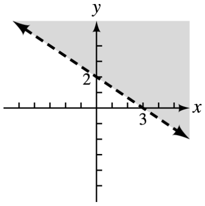
(c) $x = 4$ (d) $4x - y = 19$

25. Graph the inequality $6x + 4y > 12$. **25.** ________

(a)


(b)


(c)


(d)


ANSWERS TO PRETESTS

PRETEST, FORM A

1. $.005$

2. $2\sqrt{3}$

3. 125

4. 1680

5. $\frac{1}{3}$

6. $-\frac{1}{2}$

7. 12

8. -33

9. $-\frac{1}{2}$

10. 50

11. $p = \frac{4}{3}q + \frac{32}{3}$

12. $y = \frac{4}{5}x - \frac{8}{5}$

13. $\left(\frac{5}{3}, 0\right)$

14. 20

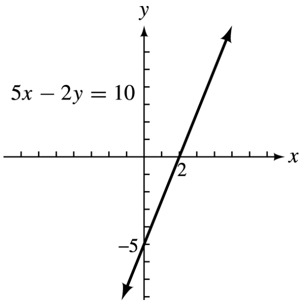
15. 36%

16. 55

17. 14

18. $x > -4$

19.



20. $\frac{2}{7}$

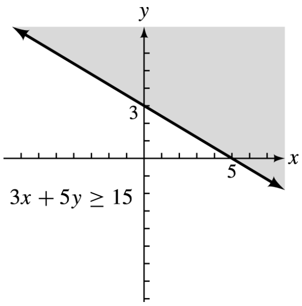
21. $3x - y = 9$

22. $(-2, 5)$

23. (d)

24. $\frac{4}{5}$

25.



PRETEST, FORM B

1. (d)

2. (c)

3. (a)

4. (c)

5. (c)

6. (b)

7. (a)

8. (d)

9. (d)

10. (b)

11. (b)

12. (d)

13. (d)

14. (c)

15. (b)

16. (b)

17. (c)

18. (a)

19. (c)

20. (c)

21. (d)

22. (b)

23. (a)

24. (c)

25. (a)

FINITE MATHEMATICS

Name:

Final Examination, Form A

1. The supply and demand functions for chocolate ice cream are given by

$$p = S(x) = \frac{1}{6}x \text{ and } p = D(x) = 9 - \frac{1}{3}x,$$

where p represents the price in dollars. Find the equilibrium price.

1. ____________________

2. Suppose that a linear cost function for an item is given by $C(x) = 50x + 300$. The items sell for \$75 each. Find the break-even quantity.

2. ____________________

3. If 6 items cost \$900 to produce and 13 items cost \$1600 to produce, find the linear cost function.

3. ____________________

4. The cost in dollars for producing x units of a particular item is given by $C(x) = .37x + 682$. How many units could be produced for a cost of \$978?

4. ____________________

5. Use the echelon method to solve the following system of equations.

$$\begin{aligned} 9x - 8y &= 12 \\ 6x + 4y &= 1 \end{aligned}$$

5. ____________________

6. Use the Gauss-Jordan method to solve the following system of equations.

$$\begin{aligned} 2x + y - z &= -1 \\ x - 2y + 2z &= 7 \\ 3x + y + z &= 4 \end{aligned}$$

6. ____________________

7. Let $A = \begin{bmatrix} 3 & 2 \\ -4 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & -3 \\ 4 & -2 & 5 \end{bmatrix}$.

Find the products AB and BA, if these products exist.

7. ____________________

8. Find the inverse of the following matrix, if the inverse exists.

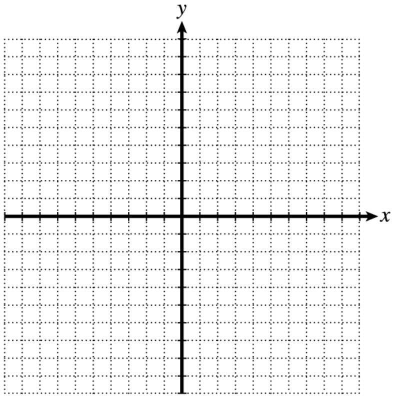
$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

8. ____________________

9. Graph the feasible region for the following system of inequalities.

$$\begin{aligned}&-3 < x < 2\\&0 \le y \le 3\\&x + 2y < 6\end{aligned}$$

9.



10. Give the coordinates of the corner points of the feasible region for the following system.

$$\begin{aligned}&x \le 7\\&y \le x\\&y \ge 3\end{aligned}$$

10. ____________________

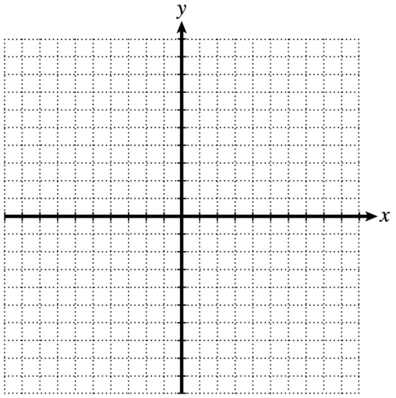
11. For the system given in Problem 10, find the maximum value of the objective function $z = 6x - 3y$.

11. ____________________

12. Use the graphical method to solve the following linear programming problem.

$$\begin{aligned}\text{Maximize}\quad & z = 5x + 2y\\ \text{subject to:}\quad & 3x + 2y \le 6\\ & x + y \le 2\\ & x \ge 0\\ & y \ge 0.\end{aligned}$$

12. ____________________



13. To pour a concrete sidewalk takes 2 hours of preparation and 3 hours of finishing. To pour a concrete patio takes 4 hours of preparation and 3 hours of finishing. There are 8 hours available for preparation and 21 hours available for finishing. ABC Concrete Company makes a profit of $450 on a sidewalk and $700 on a patio. How many sidewalks and patios should the company construct to maximize its profit?

Set up a system of inequalities for this problem, identify all variables used, and give the objective function, but do not solve. **13.** ____________________

14. Write the initial simplex tableau for the linear programming problem given in Problem 13. **14.** ____________________

15. Use the simplex method to solve the linear programming problem given in Problem 13. **15.** ____________________

16. Read the solution from the following simplex tableau.

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{cccccc|c} 0 & 2 & 4 & 1 & 6 & 0 & 200 \\ 1 & 3 & 6 & 0 & 9 & 0 & 350 \\ \hline 0 & -8 & 16 & 0 & 24 & 1 & 0 \end{array}\right] \end{array}$$

16. ____________________

17. Find the maximum value of z in Problem 16. **17.** ____________________

18. State the dual of the following linear programming problem.

$$\begin{array}{ll} \text{Minimize} & w = 6y_1 + 8y_2 \\ \text{subject to:} & 2y_1 + 5y_2 \geq 9 \\ & 2y_1 + 3y_2 \geq 11 \\ & 7y_1 + 2y_2 \geq 5 \\ \text{with} & y_1 \geq 0,\ y_2 \geq 0. \end{array}$$

18. ____________________

19. Find the compound amount if $750 is invested at 8% compounded quarterly for 3 years. **19.** ____________________

20. Jerry Herst wants to have $2400 available for a vacation 2 years from now. How much must he invest today, at 6% compounded monthly, so that he will have the required amount? **20.** ____________________

21. Find the tenth term of the geometric sequence with $a = 5$ and $r = 2$. **21.** ____________________

22. Find the payment necessary to amortize a loan of $10,000 if the interest rate is 8% compounded quarterly and there are 20 quarterly payments. **22.** ____________________

23. Write the negation of

"Some pennies are made of silver."

23. ______________________

24. If p is false and q is true, find the truth value of

$$(q \vee \sim p) \longleftrightarrow (p \rightarrow q)\,.$$

24. ______________________

25. Determine the truth value of the statement

"If it is raining, then if it is not raining it is pouring."

25. ______________________

26. Determine whether the following argument is *valid* or *invalid:*

$$\begin{array}{l} \sim p \longleftrightarrow q \\ \underline{\sim q \qquad\quad} \\ p \end{array}$$

26. ______________________

27. Write "p is a necessary condition for q" symbolically.

27. ______________________

28. Let $U = \{a, b, c, d, e, f, g\}$, $M = \{b, e, f\}$, $N = \{c, f, g\}$, and $P = \{a, b, c, d\}$. List the members of each of the following sets, using set braces.

(a) $(M \cap P) \cup N$

(b) $M' \cap (N \cup P)$

28. (a) ______________________

(b) ______________________

29. Let A, B, and C be three sets. Draw a Venn diagram and use shading to show the set $A \cap (B' \cap C')$.

29.

30. In a survey, 28 people drank coffee with breakfast and 22 drank milk. 8 people drank both, and 5 people drank neither coffee nor milk.

Use a Venn diagram to determine how many people were surveyed.

30. ______________________

31. A die is rolled and then a coin is tossed.

(a) Write the sample space for this experiment.

(b) What is the probability that an even number is rolled and a head is tossed?

31. (a) ______________

(b) ______________

32. A phone number consists of seven digits. How many such numbers have the prefix (first three digits) 487?

32. ______________

33. A bag contains 4 red, 3 white, and 5 blue marbles. How many samples of 4 marbles can be drawn in which 2 marbles are red and 2 marbles are blue?

33. ______________

34. In the experiment described in Problem 33, what is the probability that a sample of 4 marbles contains 2 red and 2 white marbles?

34. ______________

35. Cass is taking a 5-question multiple-choice quiz in which each question has three choices. He guesses on all of the questions.

(a) What is the probability that Cass answers all of the questions correctly?

(b) What is the probability that he answers exactly three questions correctly?

35. (a) ______________

(b) ______________

36. A raffle has a first prize of $400, two second prizes of $75 each, and ten third prizes of $20 each. One thousand tickets are sold at $1 apiece. Find the expected winnings of a person buying 1 ticket.

36. ______________

37. Consider the following list of test scores:

98, 70, 32, 48, 71, 80, 85, 50, 46, 71.

For this data, find each of the following. Round to the nearest tenth when necessary.

(a) The mean

(b) The median

(c) The mode

(d) The range

37. (a) ______________

(b) ______________

(c) ______________

(d) ______________

38. Find the mean for the following data. Round to the nearest hundredth.

Value	Frequency
1	5
3	8
5	10
7	3
9	4

38. ______________

39. Find the standard deviation for the following set of numbers. Round to the nearest hundredth.

14, 5, 9, 3, 11, 12

39. ______________________

40. Suppose 12 coins are tossed. Using the binomial probability formula, find the probability of getting fewer than 5 tails.

40. ______________________

41. The probability that a certain basketball team will win a given game is .62. If the team plays 50 games, find the expected number of wins and the standard deviation. (Round to the nearest hundredth if necessary.

41. ______________________

42. Name three things that must be true of a transition matrix.

42. ______________________

____

____

43. Find the long-range distribution for the following transition matrix.

$$\begin{bmatrix} .8 & .2 \\ .6 & .4 \end{bmatrix}$$

43. ______________________

44. Identify all absorbing states for the following matrix.

$$\begin{bmatrix} .40 & .30 & .30 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

44. ______________________

45. Is the matrix in Problem 44 a transition matrix for an absorbing Markov chain? Why or why not?

45. ______________________

46. Maria Perez has $10,000 to invest. She can invest in a stock fund or a money market fund. There are two states of nature: the market goes up or the market goes down. The following payoff matrix shows the amounts she will have after 2 years under the various circumstances.

$$\begin{array}{r} \\ \text{Buy Stocks} \\ \text{Buy Money Market} \end{array} \begin{array}{cc} \text{Market Up} & \text{Market Down} \\ \left[\begin{array}{c} \$12,050 \\ \$11,025 \end{array}\right. & \left.\begin{array}{c} \$7800 \\ \$11,025 \end{array}\right] \end{array}$$

Which investment should Maria make if she is

(a) an optimist;

(b) a pessimist?

46. (a) ______________________

(b) ______________________

47. In the following game, decide on the payoff when the given strategies are used.

$$\begin{array}{cc|ccc} & & & B & \\ & & 1 & 2 & 3 \\ \hline & 1 & 3 & -4 & 0 \\ A & 2 & 5 & 0 & 2 \\ & 3 & 2 & -5 & -4 \end{array}$$

(a) $(2,3)$

(b) $(3,2)$

47. (a) ______________________

(b) ______________________

48. Remove any dominated strategies in the following game.

$$\begin{bmatrix} -6 & 3 & 1 & -4 \\ 7 & -2 & 1 & -8 \end{bmatrix}$$

48. ______________________

49. For the following game, find any saddle points and the value of the game.

$$\begin{bmatrix} 3 & 4 & -5 \\ 0 & 5 & -1 \\ 1 & 0 & -2 \end{bmatrix}$$

49. ______________________

50. Suppose a game has the payoff matrix

$$M = \begin{bmatrix} 4 & -2 \\ -5 & 6 \end{bmatrix}.$$

Find the expected value of the game for the following strategies for the players A and B.

$$\text{A} = \begin{bmatrix} .6 & .4 \end{bmatrix}; \text{B} = \begin{bmatrix} .2 \\ .8 \end{bmatrix}$$

50. ______________________

FINAL EXAMINATIONS

FINITE MATHEMATICS

Name:

Final Examination, Form B

Choose the best answer.

1. Suppose that the variable cost of producing an item is \$300 and the fixed cost is \$200. Find a linear cost function for production of this item. **1.** ________

(a) $C(x) = 200x + 300$ (b) $C(x) = 300x + 200$

(c) $C(x) = 500x + 200$ (d) $C(x) = 300x$

2. The supply and demand functions for a particular commodity are given by

$$p = S(x) = \frac{2}{7}x \text{ and } p = D(x) = 22 - \frac{1}{2}x,$$

where p represents the price in dollars. Find the equilibrium price. **2.** ________

(a) \$28 (b) \$14 (c) \$8 (d) \$6

3. An item which sells for \$37 has a linear cost function given by $C(x) = 13x + 4800$. Find the break-even quantity. **3.** ________

(a) 366 (b) 200 (c) 7400 (d) 300

4. The cost in dollars for producing x units of a particular item is given by $C(x) = 0.78x + 576$. How many units could be produced for a cost of \$849? **4.** ________

(a) 273 (b) 1088 (c) 350 (d) 1238

5. Give the solution of the system with the following augmented matrix.

$$\left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 6 & 18 \end{array}\right]$$

5. ________

(a) $(-2, 6)$ (b) $(-2, 3)$ (c) $(-2, 18)$ (d) $(-2, 9)$

6. Use the Gauss–Jordan method to solve the following system of equations. Give only the x–value of the solution.

$$\begin{aligned} -4x + y &= -12 \\ 3y + 2z &= -18 \\ 2x - 3z &= 13 \end{aligned}$$

6. ________

(a) 2 (b) -2 (c) 3 (d) No solution

7. If $A = \begin{bmatrix} 6 & 3 \\ 8 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, find AB. **7.** ______

(a) $\begin{bmatrix} 0 & -4 \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ -4 \end{bmatrix}$ (c) $\begin{bmatrix} -6 & 6 \\ -8 & 4 \end{bmatrix}$

(d) Product does not exist.

8. If $A = \begin{bmatrix} 2 & 4 \\ -3 & 5 \end{bmatrix}$, find A^{-1}. **8.** ______

(a) $\begin{bmatrix} -2 & -4 \\ 3 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & -\frac{3}{4} \\ 2 & -\frac{5}{2} \end{bmatrix}$ (c) $\begin{bmatrix} \frac{5}{22} & -\frac{2}{11} \\ \frac{3}{22} & \frac{1}{11} \end{bmatrix}$

(d) A^{-1} does not exist.

9. Describe the graph of the inequality $2y + 4x < 8$.

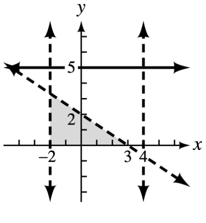
(a) The region to the left of the dashed line $y = -2x + 4$
(b) The region to the right of the dashed line $y = -2x + 4$
(c) The region to the left of and including the solid line $y = -2x + 4$
(d) The region to the right of and including the solid line $y = -2x + 4$ **9.** ______

10. Graph the feasible region for the following system of inequalities.

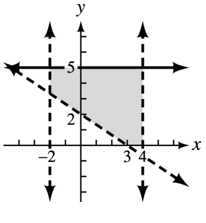
$$-2 < x < 4$$
$$0 \le y \le 5$$
$$2x + 3y > 6$$

10. ______

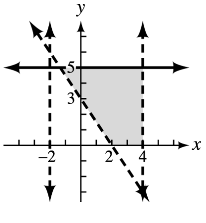
(a)



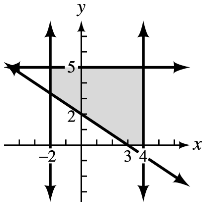
(b)



(c)



(d)



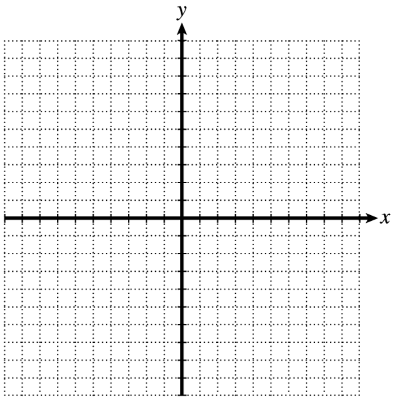
11. If the feasible region for a system has corner points $(0,8)$, $(4,3)$, and $(5,0)$, find the maximum value of the objective function $z = 6x - 3y$. **11.** ______

(a) 36 (b) 32 (c) 30 (d) 24

12. Use the graphical method to solve the following linear programming problem.

Maximize $z = 2x + 3y$

subject to: $2x + 5y \le 10$
$x - y \ge 0$
$x \ge 0$
$y \ge 0.$



12. ________

(a) 10 at $(5, 0)$ (b) $\frac{50}{7}$ at $\left(\frac{10}{7}, \frac{10}{7}\right)$ (c) 15 at $(0, 5)$ (d) No maximum value

13. It takes 3 hours to build a planter box and 2 hours to paint it. It takes 4 hours to build a step-stool and 3 hours to paint it. A man has 12 painting hours and 8 building hours available. If x is the number of planter boxes and y the number of stepstools, which of the following systems of inequalities describes this situation?

13. ________

(a) $3x + 2y \le 8$
$4x + 3y \le 12$
$x \ge 0,\ y \ge 0$

(b) $3x + 4y \le 8$
$2x + 3y \le 12$
$x \ge 0,\ y \ge 0$

(c) $3x + 8y \le 4$
$2x + 12y \le 3$
$x \ge 0,\ y \ge 0$

(d) $3x + 4y \le 12$
$2x + 3y \le 8$
$x \ge 0,\ y \ge 0$

14. Read the solution from the following simplex tableau.

$$\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ 3 & 0 & 3 & 1 & 0 & 17 \\ 4 & 1 & 5 & 0 & 0 & 20 \\ \hline -1 & 0 & 6 & 0 & 1 & 0 \end{array}$$

14. ________

(a) $x_1 = 3,\ x_2 = 0,\ s_1 = 3,\ s_2 = 1,\ z = 17$

(b) $x_1 = -1,\ x_2 = 0,\ s_1 = 6,\ s_2 = 0,\ z = 1$

(c) $x_1 = 0,\ x_2 = 20,\ s_1 = 0,\ s_2 = 17,\ z = 0$

(d) $x_1 = 0,\ x_2 = 20,\ s_1 = 0,\ s_2 = 17,\ z = 1$

15. To solve a linear programming problem with the following initial simplex tableau, which element would be selected as the first pivot?

$$\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ 1 & 5 & 0 & 10 & 0 & 100 \\ 0 & 4 & 1 & 15 & 0 & 200 \\ \hline 0 & -6 & 0 & -10 & 1 & 0 \end{array}$$

15. ________

(a) 10 (b) 15 (c) 5 (d) -6

16. Find the transpose of the following matrix.

$$\begin{bmatrix} 2 & 0 \\ 3 & -1 \\ 1 & 4 \end{bmatrix}$$

16. ________

(a) $\begin{bmatrix} 0 & 2 \\ -1 & 3 \\ 4 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & -1 \\ 1 & 4 \\ 2 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 4 \\ 3 & -1 \\ 2 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 4 \end{bmatrix}$

17. Margaret Murphy opened a savings account with a deposit of $7500. The account pays 8% interest compounded quarterly. If no further deposits and no withdrawals are made, find the balance in Margaret's account at the end of 5 years.

17. ________

(a) $10,500 (b) $34,957.18 (c) $8100 (d) $11,144.61

18. Marc Rossoff wants to have $25,000 available 10 years from now to buy a car. How much must he invest today, at 6% compounded monthly, so that he will have the required amount?

18. ________

(a) $23,584.91 (b) $13,959.87 (c) $13,740.82 (d) $15,000

19. Find the sum of the first five terms of the geometric sequence with $a = 6$ and $r = -\frac{1}{2}$.

19. ________

(a) $\frac{93}{8}$ (b) $-\frac{93}{16}$ (c) $\frac{8}{33}$ (d) $\frac{33}{8}$

20. Find the payment necessary to amortize a loan of $20,000 if the interest rate is 10% compounded quarterly and payments are made quarterly for 10 years.

20. ________

(a) $296.72 (b) $796.72 (c) $500 (d) $800

21. Write the negation of "Some coins are worth one dollar."

21. ________

(a) Some coins are not worth one dollar

(b) No coins are worth one dollar.

(c) If it is a coin, it is worth one dollar.

(d) If it is not worth one dollar, it is a coin.

22. If p is true and q is false, find the true statement.

22. ________

(a) $q \wedge p$ (b) $\sim q \vee \sim p$ (c) $q \wedge \sim p$ (d) $\sim p$

23. Find the valid argument. **23.** ________

(a) $\begin{array}{l} p \vee q \\ p \\ \hline \sim q \end{array}$ (b) $\begin{array}{l} p \rightarrow q \\ q \rightarrow p \\ \hline p \wedge q \end{array}$

(c) $\begin{array}{l} (p \rightarrow q) \wedge (q \rightarrow p) \\ p \\ \hline p \vee q \end{array}$ (d) $\begin{array}{l} (\sim p \vee q) \wedge (\sim p \rightarrow q) \\ p \\ \hline \sim q \end{array}$

24. Write "p is sufficient for q" symbolically. **24.** ________

(a) $p \rightarrow q$ (b) $q \rightarrow p$ (c) $p \longleftrightarrow q$ (d) $\sim p \rightarrow \sim q$

25. How many subsets does the set $\{a, b, c, d, e\}$ have? **25.** ________

(a) 64 (b) 32 (c) 16 (d) 30

26. Which of the following statements is false? **26.** ________

(a) $7 \in \{7, 9, 12\}$ (b) $\{a, b\} \subseteq \{a, b\}$

(c) $6 \notin \{5, 6, 7\}$ (d) $\emptyset \subseteq \{5, 6, 7\}$

27. A survey of members of a health club found that:

24 members swim;
32 members use exercise bikes;
20 members use weight machines;
8 members swim and use weight machines;
13 members use exercise bikes and weight machines;
12 members use exercise bikes only;
5 members swim, use exercise bikes, and use weight machines;
6 members do not swim and do not use either exercise bikes or weight machines

Use a Venn diagram to determine how many members were surveyed.

(a) 120 (b) 82

(c) 48 (d) 54

27. ________

28. Suppose that a single card is drawn from a standard 52-card deck. Find the probability that the card is a black seven. **28.** ________

(a) $\frac{1}{4}$ (b) $\frac{3}{14}$ (c) $\frac{1}{13}$ (d) $\frac{1}{26}$

29. If $P(A) = .3,\ P(B|A) = .6,\ P(B'|A') = .1$, find $P(B'|A)$. **29.** ________

(a) $\frac{2}{5}$ (b) $\frac{7}{19}$ (c) $\frac{4}{25}$ (d) .9

30. Suppose that Marco has 6 shirts, 5 pairs of pants, and 3 pairs of shoes. How many outfits can he create if an outfit consists of 1 shirt, 1 pair of pants, and 1 pair of shoes? **30.** ________

(a) 150 (b) 30 (c) 90 (d) 14

31. Find the number of distinguishable permutations of the letters in the word *moose.* **31.** ________

(a) 120 (b) 60 (c) 30 (d) 5

32. From a group of 6 boys and 3 girls, an after-school reading club of 2 boys and 2 girls is selected. How many such clubs are possible? **32.** ________

(a) 126 (b) 720 (c) 18 (d) 45

33. Georgia is taking a 5-question multiple-choice quiz in which each question has 4 choices. She guesses on all of the questions. What is the probability that she answers exactly 2 of the questions correctly? **33.** ________

(a) $\frac{1}{16}$ (b) $\frac{27}{1024}$ (c) $\frac{135}{512}$ (d) $\frac{45}{512}$

34. If 3 balls are drawn from a bag containing 4 red, 3 blue, and 2 yellow balls, what is the expected number of yellow balls in the sample? **34.** ________

(a) 2 (b) 1 (c) $\frac{2}{9}$ (d) $\frac{2}{3}$

35. Suppose that a student has test scores of 70, 78, 80, and 94. What is the student's mean score? **35.** ________

(a) 322 (b) 79 (c) 80.5 (d) 80

36. Find the median for the following set of numbers.

6, 14, 9, 13, 12, 11 **36.** ________

(a) 10.83 (b) 11.5 (c) 11 (d) No median

37. Find the mode or modes for the following set of numbers.

2, 1, 5, 2, 8, 5, 9 **37.** ________

(a) 2 (b) 5 (c) 2 and 5 (d) No mode

38. Find the standard deviation for the following set of numbers. Round to the nearest hundredth.

15, 13, 20, 8, 22, 12 **38.** ________

(a) 27.20 (b) 5.22 (c) 4.76 (d) 15.88

39. Find the mean for the following grouped data. Round to the nearest hundredth.

39. ________

Interval	Frequency
1-3	8
4-6	12
7-9	20
10-12	32

(a) 8.17 (b) 6.5 (c) 7.17 (d) 9.17

40. The probability that a certain baseball team will win a given game is .46. If the team plays 100 games, find the expected number of wins and the standard deviation. (Round to the nearest hundredth if necessary.) **40.** ________

(a) $\mu = 4.98$; $\sigma = 46$ (b) $\mu = 46$; $\sigma = 6.78$

(c) $\mu = 46$; $\sigma = 4.98$ (d) $\mu = 46$; $\sigma = 7.35$

41. Which one of the following matrices could be a probability vector? **41.** ________

(a) $\begin{bmatrix} .3 & .6 & .1 \\ .4 & .2 & .4 \end{bmatrix}$ (b) $\begin{bmatrix} .7 & .5 & -.2 \end{bmatrix}$

(c) $\begin{bmatrix} .999 & .111 \end{bmatrix}$ (d) $\begin{bmatrix} .36 & .24 & .22 & .18 \end{bmatrix}$

42. Which one of the following matrices could be a transition matrix? **42.** ________

(a) $\begin{bmatrix} .6 & .4 & 0 \\ .1 & 0 & .9 \end{bmatrix}$ (b) $\begin{bmatrix} .75 & .35 \\ .25 & .65 \end{bmatrix}$ (c) $\begin{bmatrix} .32 & .68 \\ .41 & .59 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$

43. Suppose that

$$A = \begin{bmatrix} .2 & .8 \\ .1 & .9 \end{bmatrix}$$

is a transition matrix. What is the probability that state 1 changes to state 2 after two repetitions of the experiment?

43. ________

(a) .89 (b) .88 (c) .80 (d) .12

44. Which one of the following transition matrices is *not* regular?

44. ________

(a) $\begin{bmatrix} .1 & .2 & .7 \\ .3 & .3 & .4 \\ .1 & .1 & .8 \end{bmatrix}$ (b) $\begin{bmatrix} .2 & .4 & .2 \\ 0 & 0 & 1 \\ .1 & .3 & .3 \end{bmatrix}$ (c) $\begin{bmatrix} .9 & .1 \\ .3 & .7 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

45. Find all absorbing states for the following transition matrix.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ .2 & .3 & .5 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

45. ________

(a) State 1 only (b) State 2 only
(c) State 3 only (d) States 1 and 3

46. A farmer must decide whether to irrigate his fields. The payoff matrix for a given week is as follows.

$$\begin{array}{r} \\ \text{Irrigate} \\ \text{Don't Irrigate} \end{array} \begin{array}{c} \begin{array}{cc} \text{Rain} & \text{No Rain} \end{array} \\ \begin{bmatrix} -\$2500 & \$8000 \\ \$6200 & \$2100 \end{bmatrix} \end{array}$$

If there is a 40% chance of rain during the week, what strategy is best?

46. ________

(a) Irrigate (b) Don't irrigate
(c) Both strategies are equally good.
(d) The best strategy cannot be determined from the given information.

47. In the following game, decide on the payoff when the strategy $(3, 1)$ is used.

$$\begin{array}{cc|ccc|} & & & \text{B} & \\ & & 1 & 2 & 3 \\ \hline & 1 & 4 & -2 & 1 \\ \text{A} & 2 & -3 & 0 & 2 \\ & 3 & -6 & 3 & 0 \end{array}$$

47. __________

(a) \$1 from A to B (b) \$1 from B to A

(c) \$6 from A to B (d) \$6 from B to A

48. Remove any dominated strategies in the following game.

$$\begin{bmatrix} -3 & -2 & 5 \\ 2 & 1 & 6 \\ 4 & -2 & -3 \end{bmatrix}$$

48. __________

(a) $\begin{bmatrix} 2 & 1 & 6 \\ 4 & -2 & -3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 6 \\ -2 & -3 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 6 \\ 4 & -3 \end{bmatrix}$

(d) No dominated strategies

49. Find any saddle points for the following game.

$$\begin{bmatrix} 2 & -1 \\ -3 & 4 \\ 6 & 5 \end{bmatrix}$$

49. __________

(a) 6 at $(3, 1)$ (b) -1 at $(2, 1)$

(c) 5 at $(3, 2)$ (d) No saddle point

50. Suppose that a game has payoff matrix

$$M = \begin{bmatrix} -2 & 3 \\ 2 & 0 \end{bmatrix}.$$

Suppose that player A chooses row 1 with probability .3 and player B chooses column 1 with probability .5. Find the expected value of the game.

50. __________

(a) \$.90 (b) \$.85 (c) $-$\$.85 (d) \$1.05

ANSWERS TO FINAL EXAMINATIONS

FINAL EXAMINATION, FORM A

1. \$3

2. 12

3. $C(x) = 100x + 300$

4. 800 units

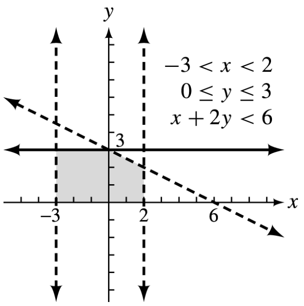
5. $\left(\frac{2}{3}, -\frac{3}{4}\right)$

6. $(1, -1, 2)$

7. $AB = \begin{bmatrix} 11 & -4 & 1 \\ -8 & 2 & 7 \end{bmatrix}$; BA does not exist.

8. $A^{-1} = \begin{bmatrix} \frac{3}{5} & -\frac{1}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} & \frac{2}{5} \\ \frac{3}{5} & -\frac{1}{5} & -\frac{6}{5} \end{bmatrix}$

9.



10. $(3,3)$, $(7,3)$, $(7,7)$

11. 33 at $(7,3)$

12. Maximum of 10 at $(2,0)$

13. Let x = the number of sidewalks;
y = the number of patios.

$$\begin{aligned} 2x + 4y &\le 8 \\ 3x + 3y &\le 21 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$

Maximize $z = 450x + 700y$.

or

Let x_1 = the number of sidewalks;
x_2 = the number of patios.

$$\begin{aligned} 2x_1 + 4x_2 &\le 8 \\ 3x_1 + 3x_2 &\le 21 \\ x_1 &\ge 0 \\ x_2 &\ge 0 \end{aligned}$$

Maximize $z = 450x_1 + 700x_2$.

14. $$\begin{array}{rrrrr|r} x_1 & x_2 & s_1 & s_2 & z & \\ 2 & 4 & 1 & 0 & 0 & 8 \\ 3 & 3 & 0 & 1 & 0 & 21 \\ \hline -450 & -700 & 0 & 0 & 1 & 0 \end{array}$$

15. Construct 4 sidewalks and no patios, for a profit of \$1800.

16. $x_1 = 350,\ x_2 = 0,\ x_3 = 0,\ s_1 = 200,$ $s_2 = 0,\ z = 0$

17. 800

18. Maximize $z = 9x_1 + 11x_2 + 5x_3$
subject to: $2x_1 + 2x_2 + 7x_3 \le 6$
$5x_1 + 3x_2 + 2x_3 \le 8$
with $x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0.$

19. \$951.18

20. \$2129.25

21. 2560

22. \$611.57

23. No pennies are made of silver.

24. True

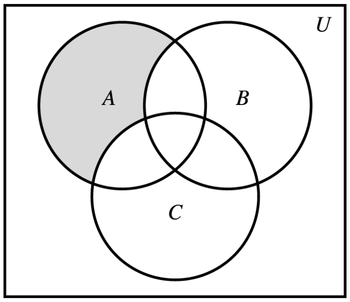
25. True

26. Valid

27. $q \rightarrow p$

28. **(a)** $\{b, c, f, g\}$ **(b)** $\{a, c, d, g\}$

29.



$A \cap (B' \cap C')$

30. 47

31. **(a)** $\{1H, 2H, 3H, 4H, 5H, 6H, 1T, 2T, 3T, 4T, 5T, 6T\}$
(b) $\frac{1}{4}$

32. 10,000

33. 60

34. $\frac{2}{55}$

35. **(a)** $\frac{1}{243}$ **(b)** $\frac{40}{243}$

36. −\$.25

37. **(a)** 65.1 **(b)** 70.5 **(c)** 71 **(d)** 66

38. 4.53

39. 4.24

40. $\approx .1938$

41. $\mu = 31$; $\sigma = 3.43$

42.
1. It must be a square matrix.
2. All entries must be between 0 and 1, inclusive.
3. The sum of the entries in any row must be 1.

43. $\begin{bmatrix} \frac{3}{4} & \frac{1}{4} \end{bmatrix}$

44. States 2 and 3

45. Yes; it is possible to move from state 1 into state 2 or state 3.

46. **(a)** Buy stocks **(b)** Buy money market

47. **(a)** \$2 from B to A **(b)** \$5 from A to B

48. $\begin{bmatrix} -6 & -4 \\ 7 & -8 \end{bmatrix}$

49. -1 at $(2, 3)$; value -1

50. \$1.04

FINAL EXAMINATION, FORM B

1. (b)
2. (c)
3. (b)
4. (c)
5. (b)
6. (a)
7. (b)
8. (c)
9. (a)
10. (b)
11. (c)
12. (a)
13. (b)
14. (c)
15. (a)
16. (d)
17. (d)
18. (c)
19. (d)
20. (b)
21. (b)
22. (b)
23. (c)
24. (a)
25. (b)
26. (c)
27. (d)
28. (d)
29. (a)
30. (c)
31. (b)
32. (d)
33. (c)
34. (d)
35. (c)
36. (b)
37. (c)
38. (b)
39. (a)
40. (c)
41. (d)
42. (c)
43. (b)
44. (d)
45. (d)
46. (a)
47. (c)
48. (c)
49. (c)
50. (b)

SOLUTIONS
TO
EVEN–NUMBERED
EXERCISES

Chapter R

ALGEBRA REFERENCE

R.1 Polynomials

2. $(-4y^2 - 3y + 8) - (2y^2 - 6y - 2)$
$$\begin{aligned} &= (-4y^2 - 3y + 8) + (-2y^2 + 6y + 2) \\ &= -4y^2 - 3y + 8 - 2y^2 + 6y + 2 \\ &= (-4y^2 - 2y^2) + (-3y + 6y) \\ &\quad + (8 + 2) \\ &= -6y^2 + 3y + 10 \end{aligned}$$

4. $2(3r^2 + 4r + 2) - 3(-r^2 + 4r - 5)$
$$\begin{aligned} &= (6r^2 + 8r + 4) + (3r^2 - 12r + 15) \\ &= (6r^2 + 3r^2) + (8r - 12r) \\ &\quad + (4 + 15) \\ &= 9r^2 - 4r + 19 \end{aligned}$$

6. $.83(5r^2 - 2r + 7) - (7.12r^2 + 6.423r - 2)$
$$\begin{aligned} &= (4.15r^2 - 1.66r + 5.81) \\ &\quad + (-7.12r^2 - 6.423r + 2) \\ &= (4.15r^2 - 7.12r^2) \\ &\quad + (-1.66r - 6.423r) + (5.81 + 2) \\ &= -2.97r^2 - 8.083r + 7.81 \end{aligned}$$

8. $(6k - 1)(2k - 3)$
$$\begin{aligned} &= (6k)(2k) + (6k)(-3) + (-1)(2k) \\ &\quad + (-1)(-3) \\ &= 12k^2 - 18k - 2k + 3 \\ &= 12k^2 - 20k + 3 \end{aligned}$$

10. $(9k + q)(2k - q)$
$$\begin{aligned} &= (9k)(2k) + (9k)(-q) + (q)(2k) \\ &\quad + (q)(-q) \\ &= 18k^2 - 9kq + 2kq - q^2 \\ &= 18k^2 - 7kq - q^2 \end{aligned}$$

12. $\left(\frac{3}{4}r - \frac{2}{3}s\right)\left(\frac{5}{4}r + \frac{1}{3}s\right)$
$$\begin{aligned} &= \left(\frac{3}{4}r\right)\left(\frac{5}{4}r\right) + \left(\frac{3}{4}r\right)\left(\frac{1}{3}s\right) + \left(-\frac{2}{3}s\right)\left(\frac{5}{4}r\right) \\ &\quad + \left(-\frac{2}{3}s\right)\left(\frac{1}{3}s\right) \\ &= \frac{15}{16}r^2 + \frac{1}{4}rs - \frac{5}{6}rs - \frac{2}{9}s^2 \\ &= \frac{15}{16}r^2 - \frac{7}{12}rs - \frac{2}{9}s^2 \end{aligned}$$

14. $(6m + 5)(6m - 5)$
$$\begin{aligned} &= (6m)(6m) + (6m)(-5) + (5)(6m) \\ &\quad + (5)(-5) \\ &= 36m^2 - 30m + 30m - 25 \\ &= 36m^2 - 25 \end{aligned}$$

16. $(2p - 1)(3p^2 - 4p + 5)$
$$\begin{aligned} &= (2p)(3p^2) + (2p)(-4p) + (2p)(5) \\ &\quad + (-1)(3p^2) + (-1)(-4p) + (-1)(5) \\ &= 6p^3 - 8p^2 + 10p - 3p^2 + 4p - 5 \\ &= 6p^3 - 11p^2 + 14p - 5 \end{aligned}$$

18. $(k + 2)(12k^3 - 3k^2 + k + 1)$
$$\begin{aligned} &= k(12k^3) + k(-3k^2) + k(k) + k(1) \\ &\quad + 2(12k^3) + 2(-3k^2) + 2(k) + 2(1) \\ &= 12k^4 - 3k^3 + k^2 + k + 24k^3 - 6k^2 \\ &\quad + 2k + 2 \\ &= 12k^4 + 21k^3 - 5k^2 + 3k + 2 \end{aligned}$$

20. $(r - 3s + t)(2r - s + t)$
$$\begin{aligned} &= r(2r) + r(-s) + r(t) - 3s(2r) \\ &\quad - 3s(-s) - 3s(t) + t(2r) + t(-s) \\ &\quad + t(t) \\ &= 2r^2 - rs + rt - 6rs + 3s^2 - 3st \\ &\quad + 2rt - st + t^2 \\ &= 2r^2 - 7rs + 3s^2 + 3rt - 4st + t^2 \end{aligned}$$

22. $(x - 1)(x + 2)(x - 3)$
$$\begin{aligned} &= [x(x + 2) + (-1)(x + 2)](x - 3) \\ &= (x^2 + 2x - x - 2)(x - 3) \\ &= (x^2 + x - 2)(x - 3) \\ &= x^2(x - 3) + x(x - 3) + (-2)(x - 3) \\ &= x^3 - 3x^2 + x^2 - 3x - 2x + 6 \\ &= x^3 - 2x^2 - 5x + 6 \end{aligned}$$

24. $(x - 2y)^3$
$$\begin{aligned} &= [(x - 2y)(x - 2y)](x - 2y) \\ &= (x^2 - 2xy - 2xy + 4y^2)(x - 2y) \\ &= (x^2 - 4xy + 4y^2)(x - 2y) \\ &= (x^2 - 4xy + 4y^2)x + (x^2 - 4xy + 4y^2)(-2y) \\ &= x^3 - 4x^2y + 4xy^2 - 2x^2y + 8xy^2 - 8y^3 \\ &= x^3 - 6x^2y + 12xy^2 - 8y^3 \end{aligned}$$

R.2 Factoring

2. $3y^3 + 24y^2 + 9y$
$= 3y \cdot y^2 + 3y \cdot 8y + 3y \cdot 3$
$= 3y(y^2 + 8y + 3)$

4. $60m^4 - 120m^3n + 50m^2n^2$
$= 10m^2 \cdot 6m^2 - 10m^2 \cdot 12mn$
$+ 10m^2 \cdot 5n^2$
$= 10m^2(6m^2 - 12mn + 5n^2)$

6. $x^2 + 4x - 5 = (x + 5)(x - 1)$
since $5(-1) = -5$ and $-1 + 5 = 4$.

8. $b^2 - 8b + 7 = (b - 7)(b - 1)$
since $(-7)(-1) = 7$ and $-7 + (-1) = -8$.

10. $s^2 + 2st - 35t^2 = (s - 5t)(s + 7t)$
since $(-5t)(7t) = -35t^2$ and $7t + (-5t) = 2t$.

12. $6a^2 - 48a - 120 = 6(a^2 - 8a - 20)$
$= 6(a - 10)(a + 2)$

14. $2x^2 - 5x - 3$

The possible factors of $2x^2$ are $2x$ and x and the possible factors of -3 are -3 and 1, or 3 and -1. Try various combinations until one works.

$$2x^2 - 5x - 3 = (2x + 1)(x - 3)$$

16. $2a^2 - 17a + 30 = (2a - 5)(a - 6)$

18. $21m^2 + 13mn + 2n^2 = (7m + 2n)(3m + n)$

20. $32z^5 - 20z^4a - 12z^3a^2$
$= 4z^3(8z^2 - 5za - 3a^2)$
$= 4z^3(8z + 3a)(z - a)$

22. $9m^2 - 25 = (3m)^2 - (5)^2$
$= (3m + 5)(3m - 5)$

24. $9x^2 + 64$ is the *sum* of two perfect squares. It cannot be factored. It is prime.

26. $m^2 - 6mn + 9n^2$
$= m^2 - 2(3mn) + (3n)^2$
$= (m - 3n)^2$

28. $a^3 - 216$
$= a^3 - 6^3$
$= (a - 6)[(a)^2 + (a)(6) + (6)^2]$
$= (a - 6)(a^2 + 6a + 36)$

30. $64m^3 + 125$
$= (4m)^3 + 5^3$
$= (4m + 5)[(4m)^2 - (4m)(5) + (5)^2]$
$= (4m + 5)(16m^2 - 20m + 25)$

32. $16a^4 - 81b^4$
$= (4a^2)^2 - (9b^2)^2$
$= (4a^2 + 9b^2)(4a^2 - 9b^2)$
$= (4a^2 + 9b^2)[(2a)^2 - (3b)^2]$
$= (4a^2 + 9b^2)(2a + 3b)(2a - 3b)$

R.3 Rational Expressions

2. $\dfrac{25p^3}{10p^2} = \dfrac{5 \cdot 5 \cdot p \cdot p \cdot p}{2 \cdot 5 \cdot p \cdot p} = \dfrac{5p}{2}$

4. $\dfrac{3(t + 5)}{(t + 5)(t - 3)} = \dfrac{3}{t - 3}$

6.
$$\frac{36y^2 + 72y}{9y} = \frac{36y(y + 2)}{9y}$$
$$= \frac{9 \cdot 4 \cdot y(y + 2)}{9 \cdot y}$$
$$= 4(y + 2)$$

8.
$$\frac{r^2 - r - 6}{r^2 + r - 12} = \frac{(r - 3)(r + 2)}{(r + 4)(r - 3)}$$
$$= \frac{r + 2}{r + 4}$$

10.
$$\frac{z^2 - 5z + 6}{z^2 - 4} = \frac{(z - 3)(z - 2)}{(z + 2)(z - 2)}$$
$$= \frac{z - 3}{z + 2}$$

12.
$$\frac{6y^2 + 11y + 4}{3y^2 + 7y + 4} = \frac{(3y + 4)(2y + 1)}{(3y + 4)(y + 1)}$$
$$= \frac{2y + 1}{y + 1}$$

14.
$$\frac{15p^3}{9p^2} \div \frac{6p}{10p^2} = \frac{15p^3}{9p^2} \cdot \frac{10p^2}{6p}$$
$$= \frac{150p^5}{54p^3}$$
$$= \frac{25 \cdot 6p^5}{9 \cdot 6p^3}$$
$$= \frac{25p^2}{9}$$

16. $$\frac{a-3}{16} \div \frac{a-3}{32} = \frac{a-3}{16} \cdot \frac{32}{a-3} = \frac{a-3}{16} \cdot \frac{16 \cdot 2}{a-3} = \frac{2}{1} = 2$$

18. $$\frac{9y-18}{6y+12} \cdot \frac{3y+6}{15y-30} = \frac{9(y-2)}{6(y+2)} \cdot \frac{3(y+2)}{15(y-2)} = \frac{27}{90} = \frac{3 \cdot 3}{10 \cdot 3} = \frac{3}{10}$$

20. $$\frac{6r-18}{9r^2+6r-24} \cdot \frac{12r-16}{4r-12} = \frac{6(r-3)}{3(3r^2+2r-8)} \cdot \frac{4(3r-4)}{4(r-3)} = \frac{6(r-3)}{3(3r-4)(r+2)} \cdot \frac{4(3r-4)}{4(r-3)} = \frac{6}{3(r+2)} = \frac{2}{r+2}$$

22. $$\frac{m^2+3m+2}{m^2+5m+4} \div \frac{m^2+5m+6}{m^2+10m+24} = \frac{m^2+3m+2}{m^2+5m+4} \cdot \frac{m^2+10m+24}{m^2+5m+6} = \frac{(m+1)(m+2)}{(m+4)(m+1)} \cdot \frac{(m+6)(m+4)}{(m+3)(m+2)} = \frac{m+6}{m+3}$$

24. $$\frac{6n^2-5n-6}{6n^2+5n-6} \cdot \frac{12n^2-17n+6}{12n^2-n-6} = \frac{(2n-3)(3n+2)}{(2n+3)(3n-2)} \cdot \frac{(3n-2)(4n-3)}{(3n+2)(4n-3)} = \frac{2n-3}{2n+3}$$

26. $$\frac{3}{p} + \frac{1}{2}$$

Multiply the first term by $\frac{2}{2}$ and the second by $\frac{p}{p}$.

$$\frac{2 \cdot 3}{2 \cdot p} + \frac{p \cdot 1}{p \cdot 2} = \frac{6}{2p} + \frac{p}{2p} = \frac{6+p}{2p}$$

28. $$\frac{1}{6m} + \frac{2}{5m} + \frac{4}{m} = \frac{5 \cdot 1}{5 \cdot 6m} + \frac{6 \cdot 2}{6 \cdot 5m} + \frac{30 \cdot 4}{30 \cdot m} = \frac{5}{30m} + \frac{12}{30m} + \frac{120}{30m} = \frac{5+12+120}{30m} = \frac{137}{30m}$$

30. $$\frac{6}{r} - \frac{5}{r-2} = \frac{6(r-2)}{r(r-2)} - \frac{5r}{r(r-2)} = \frac{6(r-2)-5r}{r(r-2)} = \frac{6r-12-5r}{r(r-2)} = \frac{r-12}{r(r-2)}$$

32. $$\frac{2}{5(k-2)} + \frac{3}{4(k-2)} = \frac{4 \cdot 2}{4 \cdot 5(k-2)} + \frac{5 \cdot 3}{5 \cdot 4(k-2)} = \frac{8}{20(k-2)} + \frac{15}{20(k-2)} = \frac{8+15}{20(k-2)} = \frac{23}{20(k-2)}$$

34. $$\frac{2y}{y^2+7y+12} - \frac{y}{y^2+5y+6} = \frac{2y}{(y+4)(y+3)} - \frac{y}{(y+3)(y+2)} = \frac{2y(y+2)}{(y+4)(y+3)(y+2)} - \frac{y(y+4)}{(y+3)(y+2)(y+4)}$$
$$= \frac{2y(y+2)-y(y+4)}{(y+4)(y+3)(y+2)} = \frac{2y^2+4y-y^2-4y}{(y+4)(y+3)(y+2)} = \frac{y^2}{(y+4)(y+3)(y+2)}$$

36. $$\frac{4m}{3m^2+7m-6}-\frac{m}{3m^2-14m+8}$$
$$=\frac{4m}{(3m-2)(m+3)}-\frac{m}{(3m-2)(m-4)}$$
$$=\frac{4m(m-4)}{(3m-2)(m+3)(m-4)}$$
$$-\frac{m(m+3)}{(3m-2)(m-4)(m+3)}$$
$$=\frac{4m(m-4)-m(m+3)}{(3m-2)(m-4)(m+3)}$$
$$=\frac{4m^2-16m-m^2-3m}{(3m-2)(m+3)(m-4)}$$
$$=\frac{3m^2-19m}{(3m-2)(m+3)(m-4)}$$
$$=\frac{m(3m-19)}{(3m-2)(m+3)(m-4)}$$

38. $$\frac{5x+2}{x^2-1}+\frac{3}{x^2+x}-\frac{1}{x^2-x}$$
$$=\frac{5x+2}{(x+1)(x-1)}+\frac{3}{x(x+1)}-\frac{1}{x(x-1)}$$
$$=\left(\frac{x}{x}\right)\left(\frac{5x+2}{(x+1)(x-1)}\right)+\left(\frac{x-1}{x-1}\right)\left(\frac{3}{x(x+1)}\right)$$
$$-\left(\frac{x+1}{x+1}\right)\left(\frac{1}{x(x-1)}\right)$$
$$=\frac{x(5x+2)+(x-1)(3)-(x+1)(1)}{x(x+1)(x-1)}$$
$$=\frac{5x^2+2x+3x-3-x-1}{x(x+1)(x-1)}$$
$$=\frac{5x^2+4x-4}{x(x+1)(x-1)}$$

R.4 Equations

2. $\frac{5}{6}k-2k+\frac{1}{3}=\frac{2}{3}$

Multiply both sides of the equation by 6.

$$6\left(\frac{5}{6}k\right)-6(2k)+6\left(\frac{1}{3}\right)=6\left(\frac{2}{3}\right)$$
$$\begin{aligned}5k-12k+2&=4\\-7k+2&=4\\-7k&=2\\k&=-\frac{2}{7}\end{aligned}$$

The solution is $-\frac{2}{7}$.

4. $$\begin{aligned}5x+2&=8-3x\\8x+2&=8\\8x&=6\\x&=\frac{3}{4}\end{aligned}$$

The solution is $\frac{3}{4}$.

6. $$\begin{aligned}5(a+3)+4a-5&=-(2a-4)\\5a+15+4a-5&=-2a+4\\9a+10&=-2a+4\\11a+10&=4\\11a&=-6\\a&=-\frac{6}{11}\end{aligned}$$

The solution is $-\frac{6}{11}$.

8. $$\begin{aligned}4[2p-(3-p)+5]&=-7p-2\\4[2p-3+p+5]&=-7p-2\\4[3p+2]&=-7p-2\\12p+8&=-7p-2\\19p+8&=-2\\19p&=-10\\p&=-\frac{10}{19}\end{aligned}$$

The solution is $-\frac{10}{19}$.

10. $$\begin{aligned}x^2&=3+2x\\x^2-2x-3&=0\\(x-3)(x+1)&=0\\x-3=0\quad&\text{or}\quad x+1=0\\x=3\quad&\text{or}\quad x=-1\end{aligned}$$

The solutions are 3 and −1.

12. $$\begin{aligned}2k^2-k&=10\\2k^2-k-10&=0\\(2k-5)(k+2)&=0\\2k-5=0\quad&\text{or}\quad k+2=0\\k=\frac{5}{2}\quad&\text{or}\quad k=-2\end{aligned}$$

The solutions are $\frac{5}{2}$ and −2.

14. $$\begin{aligned}m(m-7)&=-10\\m^2-7m+10&=0\\(m-5)(m-2)&=0\\m-5=0\quad&\text{or}\quad m-2=0\\m=5\quad&\text{or}\quad m=2\end{aligned}$$

The solutions are 5 and 2.

16.
$$\begin{aligned} z(2z+7) &= 4 \\ 2z^2+7z-4 &= 0 \\ (2z-1)(z+4) &= 0 \\ 2z-1=0 \quad &\text{or} \quad z+4=0 \\ z=\frac{1}{2} \quad &\text{or} \quad z=-4 \end{aligned}$$

The solutions are $\frac{1}{2}$ and -4.

18. $3x^2-5x+1=0$

Use the quadratic formula.

$$x=\frac{-(-5)\pm\sqrt{(-5)^2-4(3)(1)}}{2(3)}=\frac{5\pm\sqrt{25-12}}{6}$$

$$x=\frac{5+\sqrt{13}}{6} \quad \text{or} \quad x=\frac{5-\sqrt{13}}{6}$$

$$\approx 1.434 \qquad\qquad \approx .232$$

The solutions are $\frac{5+\sqrt{13}}{6}\approx 1.434$ and $\frac{5-\sqrt{13}}{6}\approx .232$.

20. $p^2+p-1=0$

$$p=\frac{-1\pm\sqrt{1^2-4(1)(-1)}}{2(1)}=\frac{-1\pm\sqrt{5}}{2}$$

The solutions are $\frac{-1+\sqrt{5}}{2}\approx .618$ and $\frac{-1-\sqrt{5}}{2}\approx -1.618$.

22. $2x^2+12x+5=0$

$$\begin{aligned} x &= \frac{-12\pm\sqrt{(12)^2-4(2)(5)}}{2(2)} \\ &= \frac{-12\pm\sqrt{104}}{4}=\frac{-12\pm\sqrt{4\cdot 26}}{4} \\ &= \frac{-12\pm\sqrt{4}\sqrt{26}}{4}=\frac{-12\pm 2\sqrt{26}}{4} \\ &= \frac{2(-6\pm\sqrt{26})}{2\cdot 2}=\frac{-6\pm\sqrt{26}}{2} \end{aligned}$$

The solutions are $\frac{-6+\sqrt{26}}{2}\approx -.450$ and $\frac{-6-\sqrt{26}}{2}\approx -5.550$.

24. $2x^2-7x+30=0$

$$x=\frac{-(-7)\pm\sqrt{(-7)^2-4(2)(30)}}{2(2)}$$

$$x=\frac{7\pm\sqrt{49-240}}{4}$$

$$x=\frac{7\pm\sqrt{-191}}{4}$$

Since there is a negative number under the radical sign, $\sqrt{-191}$ is not a real number. Thus, there are no real number solutions.

26.
$$\begin{aligned} 5m^2+5m &= 0 \\ 5m(m+1) &= 0 \\ 5m=0 \quad \text{or} \quad m+1 &= 0 \\ m=0 \quad \text{or} \quad m &= -1 \end{aligned}$$

The solutions are 0 and -1.

28. $\frac{x}{3}-7=6-\frac{3x}{4}$

Multiply both sides by 12, the least common denominator of 3 and 4.

$$\begin{aligned} 12\left(\frac{x}{3}-7\right) &= 12\left(6-\frac{3x}{4}\right) \\ 12\left(\frac{x}{3}\right)-(12)(7) &= (12)(6)-(12)\left(\frac{3x}{4}\right) \\ 4x-84 &= 72-9x \\ 13x-84 &= 72 \\ 13x &= 156 \\ x &= 12 \end{aligned}$$

The solution is 12.

30. $\frac{5}{2p+3}-\frac{3}{p-2}=\frac{4}{2p+3}$

Multiply both sides by $(2p+3)(p-2)$. Note that $p\neq-\frac{3}{2}$ and $p\neq 2$.

$$(2p+3)(p-2)\left(\frac{5}{2p+3}-\frac{3}{p-2}\right)$$
$$=(2p+3)(p-2)\left(\frac{4}{2p+3}\right)$$

$$(2p+3)(p-2)\left(\frac{5}{2p+3}\right)-(2p+3)(p-2)\left(\frac{3}{p-2}\right)$$
$$=(2p+3)(p-2)\left(\frac{4}{2p+3}\right)$$

$$\begin{aligned} (p-2)(5)-(2p+3)(3) &= (p-2)(4) \\ 5p-10-6p-9 &= 4p-8 \\ -p-19 &= 4p-8 \\ -5p-19 &= -8 \\ -5p &= 11 \\ p &= -\frac{11}{5} \end{aligned}$$

The solutions is $-\frac{11}{5}$.

32. $\dfrac{2y}{y-1} = \dfrac{5}{y} + \dfrac{10-8y}{y^2-y}$

$$\frac{2y}{y-1} = \frac{5}{y} + \frac{10-8y}{y(y-1)}$$

Multiply both sides by $y(y-1)$.
Note that $y \neq 0$ and $y \neq 1$.

$$\begin{aligned}
y(y-1)\left(\frac{2y}{y-1}\right) &= y(y-1)\left[\frac{5}{y} + \frac{10-8y}{y(y-1)}\right] \\
y(y-1)\left(\frac{2y}{y-1}\right) &= y(y-1)\left(\frac{5}{y}\right) \\
&\quad + y(y-1)\left[\frac{10-8y}{y(y-1)}\right] \\
y(2y) &= (y-1)(5) + (10-8y) \\
2y^2 &= 5y - 5 + 10 - 8y \\
2y^2 &= 5 - 3y \\
2y^2 + 3y - 5 &= 0 \\
(2y+5)(y-1) &= 0 \\
2y + 5 = 0 \quad &\text{or} \quad y - 1 = 0 \\
y = -\frac{5}{2} \quad &\text{or} \quad y = 1
\end{aligned}$$

Since $y \neq 1$, 1 is not a solution.
The solution is $-\frac{5}{2}$.

34. $\dfrac{5}{a} + \dfrac{-7}{a+1} = \dfrac{a^2-2a+4}{a^2+a}$

$$a(a+1)\left(\frac{5}{a} + \frac{-7}{a+1}\right) = a(a+1)\left(\frac{a^2-2a+4}{a^2+a}\right)$$

Note that $a \neq 0$ and $a \neq -1$.

$$\begin{aligned}
5(a+1) + (-7)(a) &= a^2 - 2a + 4 \\
5a + 5 - 7a &= a^2 - 2a + 4 \\
5 - 2a &= a^2 - 2a + 4 \\
5 &= a^2 + 4 \\
0 &= a^2 - 1 \\
0 &= (a+1)(a-1) \\
a + 1 = 0 \quad &\text{or} \quad a - 1 = 0 \\
a = -1 \quad &\text{or} \quad a = 1
\end{aligned}$$

Since -1 would make two denominators zero, 1 is the only solution.

36. $\dfrac{2}{x^2-2x-3} + \dfrac{5}{x^2-x-6} = \dfrac{1}{x^2+3x+2}$

$$\frac{2}{(x-3)(x+1)} + \frac{5}{(x-3)(x+2)} = \frac{1}{(x+2)(x+1)}$$

Multiply both sides by $(x-3)(x+1)(x+2)$.
Note that $x \neq 3$, $x \neq -1$, and $x \neq -2$.

$$\begin{aligned}
&(x-3)(x+1)(x+2)\left(\frac{2}{(x-3)(x+1)}\right) \\
&\quad + (x-3)(x+1)(x+2)\left(\frac{5}{(x-3)(x+2)}\right) \\
&\quad = (x-3)(x+1)(x+2)\left(\frac{1}{(x+2)(x+1)}\right)
\end{aligned}$$

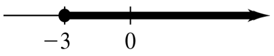
$$\begin{aligned}
2(x+2) + 5(x+1) &= x - 3 \\
2x + 4 + 5x + 5 &= x - 3 \\
7x + 9 &= x - 3 \\
6x + 9 &= -3 \\
6x &= -12 \\
x &= -2
\end{aligned}$$

However, $x \neq -2$. Therefore there is no solution.

R.5 Inequalities

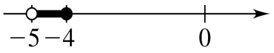
2. $x \geq -3$

Because the inequality sign means "greater than or equal to," the endpoint at -3 is included. This inequality is written in interval notation as $[-3, \infty)$. To graph this interval on a number line, place a solid circle at -3 and draw a heavy arrow pointing to the right.



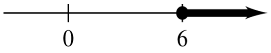
4. $-5 < x \leq -4$

The endpoint at -4 is included, but the endpoint at -5 is not. This inequality is written in interval notation as $(-5, -4]$. To graph this interval, place a an open circle at -5 and a closed circle at -4; then draw a heavy line segment between them.



6. $6 \le x$

This inequality may be written as $x \ge 6$, and is written in interval notation as $[6, \infty)$. Note that the endpoint at 6 is included. To graph this interval, place a closed circle at 6 and draw a heavy arrow pointing to the right.



8. $[2, 7)$

This represents all the numbers between 2 and 7, including 2 but not including 7. This interval can be written as the inequality $2 \le x < 7$.

10. $(3, \infty)$

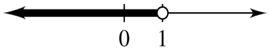
This represents all the numbers to the right of 3, and does not include the endpoint. This interval can be written as the inequality $x > 3$.

12. Notice that neither endpoint is included. The interval shown in the graph can be written as $0 < x < 8$.

14. Notice that the endpoint 0 is not included, but 3 is included. The interval shown in the graph can be written as $x < 0$ or $x \ge 3$.

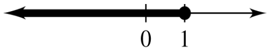
16.
$$\begin{aligned} 6k - 4 &< 3k - 1 \\ 6k &< 3k + 3 \\ 3k &< 3 \\ k &< 1 \end{aligned}$$

The solution in interval notation is $(-\infty, 1)$.



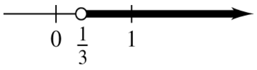
18.
$$\begin{aligned} -2(3y - 8) &\ge 5(4y - 2) \\ -6y + 16 &\ge 20y - 10 \\ -6y + 16 + (-16) &\ge 20y - 10 + (-16) \\ -6y &\ge 20y - 26 \\ -6y + (-20y) &\ge 20y + (-20y) - 26 \\ -26y &\ge -26 \\ -\frac{1}{26}(-26)y &\le -\frac{1}{26}(-26) \\ y &\le 1 \end{aligned}$$

The solution is $(-\infty, 1]$.



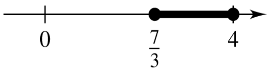
20.
$$\begin{aligned} x + 5(x + 1) &> 4(2 - x) + x \\ x + 5x + 5 &> 8 - 4x + x \\ 6x + 5 &> 8 - 3x \\ 6x &> 3 - 3x \\ 9x &> 3 \\ x &> \frac{1}{3} \end{aligned}$$

The solution is $\left(\frac{1}{3}, \infty\right)$.



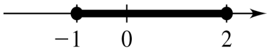
22.
$$\begin{aligned} 8 &\le 3r + 1 \le 13 \\ 8 + (-1) &\le 3r + 1 + (-1) \le 13 + (-1) \\ 7 &\le 3r \le 12 \\ \frac{1}{3}(7) &\le \frac{1}{3}(3r) \le \frac{1}{3}(12) \\ \frac{7}{3} &\le r \le 4 \end{aligned}$$

The solution is $\left[\frac{7}{3}, 4\right]$.



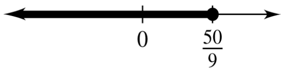
24.
$$\begin{aligned} -1 &\le \frac{5y + 2}{3} \le 4 \\ 3(-1) &\le 3\left(\frac{5y + 2}{3}\right) \le 3(4) \\ -3 &\le 5y + 2 \le 12 \\ -5 &\le 5y \le 10 \\ -1 &\le y \le 2 \end{aligned}$$

The solution is $[-1,\ 2]$.



26.
$$\begin{aligned} \frac{8}{3}(z - 4) &\le \frac{2}{9}(3z + 2) \\ (9)\frac{8}{3}(z - 4) &\le (9)\frac{2}{9}(3z + 2) \\ 24(z - 4) &\le 2(3z + 2) \\ 24z - 96 &\le 6z + 4 \\ 24z &\le 6z + 100 \\ 18z &\le 100 \\ z &\le \frac{100}{18} \\ z &\le \frac{50}{9} \end{aligned}$$

The solution is $(-\infty,\ \frac{50}{9}]$.



28. $(t+6)(t-1) \geq 0$

Solve $(t+6)(t-1) = 0$.

$$\begin{aligned} (t+6)(t-1) &= 0 \\ t = -6 \quad \text{or} \quad t &= 1 \end{aligned}$$

Intervals: $(-\infty, -6),\ (-6, 1),\ (1, \infty)$

For $(-\infty, -6)$, choose -7 to test for t.

$$(-7+6)(-7-1) = (-1)(-8) = 8 \geq 0$$

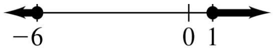
For $(-6,\ 1)$, choose 0.

$$(0+6)(0-1) = (6)(-1) = -6 \ngeq 0$$

For $(1,\ \infty)$, choose 2.

$$(2+6)(2-1) = (8)(1) = 8 \geq 0$$

Because the symbol $\geq$ is used, the endpoints -6 and 1 are included in the solution, $(-\infty, -6] \cup [1, \infty)$.



30. $2k^2 + 7k - 4 > 0$

Solve $2k^2 + 7k - 4 = 0$.

$$\begin{aligned} 2k^2 + 7k - 4 &= 0 \\ (2k-1)(k+4) &= 0 \end{aligned}$$

$$k = \frac{1}{2} \quad \text{or} \quad k = -4$$

Intervals: $(-\infty, -4), \left(-4, \frac{1}{2}\right) \left(\frac{1}{2}, \infty\right)$

For $(-\infty, -4)$, choose -5.

$$2(-5)^2 + 7(-5) - 4 = 11 > 0$$

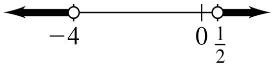
For $\left(-4, \frac{1}{2}\right)$, choose 0.

$$2(0)^2 + 7(0) - 4 = -4 \not> 0$$

For $\left(\frac{1}{2}, \infty\right)$, choose 1.

$$2(1)^2 + 7(1) - 4 = 5 > 0$$

The solution is $(-\infty, -4) \cup \left(\frac{1}{2}, \infty\right)$.



32. $2k^2 - 7k - 15 \leq 0$

Solve $2k^2 - 7k - 15 = 0$.

$$\begin{aligned} 2k^2 - 7k - 15 &= 0 \\ (2k+3)(k-5) &= 0 \end{aligned}$$

$$k = -\frac{3}{2} \quad \text{or} \quad k = 5$$

Intervals: $\left(-\infty, -\frac{3}{2}\right), \left(-\frac{3}{2}, 5\right), (5, \infty)$

For $\left(-\infty, -\frac{3}{2}\right)$, choose -2.

$$2(-2)^2 - 7(-2) - 15 = 7 \nleq 0$$

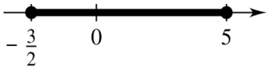
For $\left(-\frac{3}{2}, 5\right)$, choose 0.

$$2(0)^2 - 7(0) - 15 = -15 \leq 0$$

For $(5, \infty)$, choose 6.

$$2(6)^2 - 7(6) - 15 \nleq 0$$

The solution is $\left[-\frac{3}{2}, 5\right]$.



34. $10r^2 + r \leq 2$

Solve $10r^2 + r = 2$.

$$\begin{aligned} 10r^2 + r &= 2 \\ 10r^2 + r - 2 &= 0 \\ (5r-2)(2r+1) &= 0 \end{aligned}$$

$$r = \frac{2}{5} \quad \text{or} \quad r = -\frac{1}{2}$$

Intervals: $\left(-\infty, -\frac{1}{2}\right), \left(-\frac{1}{2}, \frac{2}{5}\right), \left(\frac{2}{5}, \infty\right)$

For $\left(-\infty, -\frac{1}{2}\right)$, choose -1.

$$10(-1)^2 + (-1) = 9 \nleq 2$$

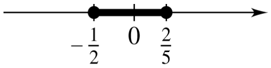
For $\left(-\frac{1}{2}, \frac{2}{5}\right)$, choose 0.

$$10(0)^2 + 0 = 0 \leq 2$$

For $\left(\frac{2}{5}, \infty\right)$, choose 1.

$$10(1)^2 + 1 = 11 \nleq 2$$

The solution is $\left[-\frac{1}{2}, \frac{2}{5}\right]$.



36. $3a^2 + a > 10$

Solve $3a^2 + a = 10$.

$$\begin{aligned} 3a^2 + a &= 10 \\ 3a^2 + a - 10 &= 0 \\ (3a - 5)(a + 2) &= 0 \end{aligned}$$

$$a = \frac{5}{3} \quad \text{or} \quad a = -2$$

Intervals: $(-\infty, -2), \left(-2, \frac{5}{3}\right), \left(\frac{5}{3}, \infty\right)$

For $(-\infty, -2)$, choose -3.

$$3(-3)^2 + (-3) = 24 > 10$$

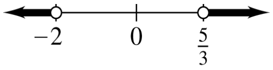
For $\left(-2, \frac{5}{3}\right)$, choose 0.

$$3(0)^2 + 0 = 0 \not> 10$$

For $\left(\frac{5}{3}, \infty\right)$, choose 2.

$$3(2)^2 + 2 = 14 > 10$$

The solution is $(-\infty, -2) \cup \left(\frac{5}{3}, \infty\right)$.



38. $p^2 - 16p > 0$

Solve $\text{p}^2 - 16p = 0$.

$$\begin{aligned} p^2 - 16p &= 0 \\ p(p - 16) &= 0 \\ p = 0 \quad \text{or} \quad p &= 16 \end{aligned}$$

Intervals: $(-\infty, 0),\ (0, 16),\ (16, \infty)$

For $(-\infty, 0)$, choose -1.

$$(-1)^2 - 16(-1) = 17 > 0$$

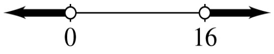
For $(0, 16)$, choose 1.

$$(1)^2 - 16(1) = -15 \not> 0$$

For $(16, \infty)$, choose 17.

$$(17)^2 - 16(17) = 17 > 0$$

The solution is $(-\infty, 0) \cup (16, \infty)$.



40. $\dfrac{r+1}{r-1} > 0$

Solve the equation $\dfrac{r+1}{r-1} = 0$.

$$\begin{aligned} \frac{r+1}{r-1} &= 0 \\ (r-1)\frac{r+1}{r-1} &= (r-1)(0) \\ r + 1 &= 0 \\ r &= -1 \end{aligned}$$

Find the value for which the denominator equals zero.

$$\begin{aligned} r - 1 &= 0 \\ r &= 1 \end{aligned}$$

Intervals: $(-\infty, -1),\ (-1, 1),\ (1, \infty)$

For $(-\infty, -1)$, choose -2.

$$\frac{-2+1}{-2-1} = \frac{-1}{-3} = \frac{1}{3} > 0$$

For $(-1, 1)$, choose 0.

$$\frac{0+1}{0-1} = \frac{1}{-1} = -1 \not> 0$$

For $(1, \infty)$, choose 2.

$$\frac{2+1}{2-1} = \frac{3}{1} = 3 > 0$$

The solution is $(-\infty, -1) \cup (1, \infty)$.

42. $\dfrac{a-5}{a+2} < -1$

Solve the equation $\dfrac{a-5}{a+2} = -1$.

$$\begin{aligned} \frac{a-5}{a+2} &= -1 \\ a - 5 &= -1(a + 2) \\ a - 5 &= -a - 2 \\ 2a &= 3 \\ a &= \frac{3}{2} \end{aligned}$$

Set the denominator equal to zero and solve for a.

$$\begin{aligned} a + 2 &= 0 \\ a &= -2 \end{aligned}$$

Intervals: $(-\infty, -2), \left(-2, \frac{3}{2}\right), \left(\frac{3}{2}, \infty\right)$

For $(-\infty, -2)$, choose -3.

$$\frac{-3-5}{-3+2} = \frac{-8}{-1} = 8 \not< -1$$

For $\left(-2, \frac{3}{2}\right)$, choose 0.

$$\frac{0-5}{0+2} = \frac{-5}{2} = -\frac{5}{2} < -1$$

For $\left(\frac{3}{2}, \infty\right)$, choose 2.

$$\frac{2-5}{2+2} = \frac{-3}{4} = -\frac{3}{4} \not< -1$$

The solution is $\left(-2, \frac{3}{2}\right)$.

44. $\dfrac{a+2}{3+2a} \leq 5$

For the equation $\dfrac{a+2}{3+2a} = 5$.

$$\begin{aligned} \frac{a+2}{3+2a} &= 5 \\ a+2 &= 5(3+2a) \\ a+2 &= 15+10a \\ -9a &= 13 \\ a &= -\frac{13}{9} \end{aligned}$$

Set the denominator equal to zero and solve for a.

$$\begin{aligned} 3+2a &= 0 \\ 2a &= -3 \\ a &= -\frac{3}{2} \end{aligned}$$

Intervals: $\left(-\infty, -\frac{3}{2}\right), \left(-\frac{3}{2}, -\frac{13}{9}\right), \left(-\frac{13}{9}, \infty\right)$

For $\left(-\infty, -\frac{3}{2}\right)$, choose -2.

$$\frac{-2+2}{3+2(-2)} = \frac{0}{-1} = 0 \leq 5$$

For $\left(-\frac{3}{2}, -\frac{13}{9}\right)$, choose -1.46.

$$\frac{-1.46+2}{3+2(-1.46)} = \frac{.54}{.08} = 6.75 \not\leq 5$$

For $\left(-\frac{13}{9}, \infty\right)$, choose 0.

$$\frac{0+2}{3+2(0)} = \frac{2}{3} \leq 5$$

The value $-\frac{3}{2}$ cannot be included in the solution since it would make the denominator zero. The solution is $\left(-\infty, -\frac{3}{2}\right) \cup \left[-\frac{13}{9}, \infty\right)$.

46. $\dfrac{5}{p+1} > \dfrac{12}{p+1}$

Solve the equation $\dfrac{5}{p+1} = \dfrac{12}{p+1}$.

$$\begin{aligned} \frac{5}{p+1} &= \frac{12}{p+1} \\ 5 &= 12 \end{aligned}$$

The equation has no solution.
Set the denominator equal to zero and solve for p.

$$\begin{aligned} p+1 &= 0 \\ p &= -1 \end{aligned}$$

Intervals: $(-\infty, -1)$, $(-1, \infty)$
For $(-\infty, -1)$, choose -2.

$\dfrac{5}{-2+1} = -5$ and $\dfrac{12}{-2+1} = -12$, so

$$\frac{5}{-2+1} > \frac{12}{-2+1}.$$

For $(-1, \infty)$, choose 0.

$\dfrac{5}{0+1} = 5$ and $\dfrac{12}{0+1} = 12$, so

$$\frac{5}{0+1} \not> \frac{12}{0+1}.$$

The solution is $(-\infty, -1)$.

48. $\dfrac{8}{p^2+2p} > 1$

Solve the equation $\dfrac{8}{p^2+2p} = 1$.

$$\begin{aligned} \frac{8}{p^2+2p} &= 1 \\ 8 &= p^2+2p \\ 0 &= p^2+2p-8 \\ 0 &= (p+4)(p-2) \end{aligned}$$

$$\begin{aligned} p+4 &= 0 \quad \text{or} \quad p-2 = 0 \\ p &= -4 \quad \text{or} \quad p = 2 \end{aligned}$$

Set the denominator equal to zero and solve for p.

$$\begin{aligned} p^2+2p &= 0 \\ p(p+2) &= 0 \\ p = 0 \quad \text{or} \quad p+2 &= 0 \\ p &= -2 \end{aligned}$$

Intervals: $(-\infty, -4)$, $(-4, -2)$, $(-2, 0)$, $(0, 2)$, $(2, \infty)$

For $(-\infty,-4)$, choose -5.

$$\frac{8}{(-5)^2+2(-5)}=\frac{8}{15}\not>1$$

For $(-4,-2)$, choose -3.

$$\frac{8}{(-3)^2+2(-3)}=\frac{8}{9-6}=\frac{8}{3}>1$$

For $(-2,0)$, choose -1.

$$\frac{8}{(-1)^2+2(-1)}=\frac{8}{-1}=-8\not>1$$

For $(0,2)$, choose 1.

$$\frac{8}{(1)^2+2(1)}=\frac{8}{3}>1$$

For $(2,\infty)$, choose 3.

$$\frac{8}{(3)^2+(2)(3)}=\frac{8}{15}\not>1$$

The solution is $(-4,-2)\cup(0,2)$.

50. $\dfrac{a^2+2a}{a^2-4}\le 2$

Solve the equation $\dfrac{a^2+2a}{a^2-4}=2$.

$$\begin{aligned}\frac{a^2+2a}{a^2-4}&=2\\ a^2+2a&=2(a^2-4)\\ a^2+2a&=2a^2-8\\ 0&=a^2-2a-8\\ 0&=(a-4)(a+2)\\ a-4=0 \quad\text{or}\quad & a+2=0\\ a=4 \quad\text{or}\quad & a=-2\end{aligned}$$

But -2 is not a possible solution.
Set the denominator equal to zero and solve for a.

$$\begin{aligned}a^2-4&=0\\ (a+2)(a-2)&=0\\ a+2=0 \quad\text{or}\quad & a-2=0\\ a=-2 \quad\text{or}\quad & a=2\end{aligned}$$

Intervals: $(-\infty,-2)$, $(-2,2)$,
$(2,4)$, $(4,\infty)$

For $(-\infty,-2)$, choose -3.

$$\frac{(-3)^2+2(-3)}{(-3)^2-4}=\frac{9-6}{9-4}=\frac{3}{5}\le 2$$

For $(-2,2)$, choose 0.

$$\frac{(0)^2+2(0)}{0-4}=\frac{0}{-4}=0\le 2$$

For $(2,4)$, choose 3.

$$\frac{(3)^2+2(3)}{(3)^2-4}=\frac{9+6}{9-5}=\frac{15}{4}\not\le 2$$

For $(4,\infty)$, choose 5.

$$\frac{(5)^2+2(5)}{(5)^2-4}=\frac{25+10}{25-4}=\frac{35}{21}\le 2$$

The value 4 will satisfy the original inequality, but the values -2 and 2 will not since they make the denominator zero. The solution is $(-\infty,-2)\cup(-2,2)\cup[4,\infty)$.

R.6 Exponents

2. $3^{-4}=\dfrac{1}{3^4}=\dfrac{1}{81}$

4. $(-12)^0=1$, by definition.

6. $-(-3^{-2})=-\left(-\dfrac{1}{3^2}\right)=-\left(-\dfrac{1}{9}\right)=\dfrac{1}{9}$

8. $\left(\dfrac{4}{3}\right)^{-3}=\dfrac{1}{\left(\frac{4}{3}\right)^3}=\dfrac{1}{\frac{64}{27}}=\dfrac{27}{64}$

10. $\dfrac{8^9\cdot 8^{-7}}{8^{-3}}=8^{9+(-7)-(-3)}=8^{9-7+3}=8^5$

12.
$$\begin{aligned}\left(\frac{5^{-6}\cdot 5^3}{5^{-2}}\right)^{-1}&=(5^{-6+3-(-2)})^{-1}\\ &=(5^{-6+3+2})^{-1}=(5^{-1})^{-1}\\ &=5^{(-1)(-1)}=5^1=5\end{aligned}$$

14. $\dfrac{y^9y^7}{y^{13}}=y^{9+7-13}=y^3$

16.
$$\begin{aligned}\frac{(3z^2)^{-1}}{z^5}&=\frac{3^{-1}(z^2)^{-1}}{z^5}=\frac{3^{-1}z^{2(-1)}}{z^5}\\ &=\frac{3^{-1}z^{-2}}{z^5}=3^{-1}z^{-2-5}\\ &=3^{-1}z^{-7}=\frac{1}{3}\cdot\frac{1}{z^7}=\frac{1}{3z^7}\end{aligned}$$

18. $$\begin{aligned}\frac{5^{-2}m^2y^{-2}}{5^2m^{-1}y^{-2}} &= \frac{5^{-2}}{5^2}\cdot\frac{m^2}{m^{-1}}\cdot\frac{y^{-2}}{y^{-2}}\\ &= 5^{-2-2}m^{2-(-1)}y^{-2-(-2)}\\ &= 5^{-2-2}m^{2+1}y^{-2+2}\\ &= 5^{-4}m^3y^0 = \frac{1}{5^4}\cdot m^3\cdot 1\\ &= \frac{m^3}{5^4}\end{aligned}$$

20. $$\begin{aligned}\left(\frac{2c^2}{d^3}\right)^{-2} &= \frac{2^{-2}(c^2)^{-2}}{(d^3)^{-2}}\\ &= \frac{2^{-2}c^{(2)(-2)}}{d^{(3)(-2)}} = \frac{2^{-2}c^{-4}}{d^{-6}}\\ &= \frac{d^6}{2^2c^4}\end{aligned}$$

22. $$\begin{aligned}\left(\frac{a^{-7}b^{-1}}{b^{-4}a^2}\right)^{1/3} &= \left(a^{-7-2}b^{-1-(-4)}\right)^{1/3}\\ &= \left(a^{-9}b^3\right)^{1/3}\\ &= \left(a^{-9}\right)^{1/3}\left(b^3\right)^{1/3}\\ &= a^{-3}b^1\\ &= \frac{b}{a^3}\end{aligned}$$

24. $$\begin{aligned}b^{-2}-a &= \frac{1}{b^2}-a\\ &= \frac{1}{b^2}-a\left(\frac{b^2}{b^2}\right)\\ &= \frac{1}{b^2}-\frac{ab^2}{b^2}\\ &= \frac{1-ab^2}{b^2}\end{aligned}$$

26. $$\begin{aligned}\left(\frac{m}{3}\right)^{-1}+\left(\frac{n}{2}\right)^{-2} &= \left(\frac{3}{m}\right)^1+\left(\frac{2}{n}\right)^2\\ &= \frac{3}{m}+\frac{4}{n^2}\\ &= \left(\frac{3}{m}\right)\left(\frac{n^2}{n^2}\right)+\left(\frac{4}{n^2}\right)\left(\frac{m}{m}\right)\\ &= \frac{3n^2}{mn^2}+\frac{4m}{mn^2}\\ &= \frac{3n^2+4m}{mn^2}\end{aligned}$$

28. $$\begin{aligned}\left(x^{-2}+y^{-2}\right)^{-2} &= \left(\frac{1}{x^2}+\frac{1}{y^2}\right)^{-2}\\ &= \left[\left(\frac{1}{x^2}\right)\left(\frac{y^2}{y^2}\right)+\left(\frac{x^2}{x^2}\right)\left(\frac{1}{y^2}\right)\right]^{-2}\\ &= \left(\frac{y^2}{x^2y^2}+\frac{x^2}{x^2y^2}\right)^{-2}\\ &= \left(\frac{y^2+x^2}{x^2y^2}\right)^{-2} = \left(\frac{x^2y^2}{y^2+x^2}\right)^2\\ &= \frac{(x^2)^2(y^2)^2}{(x^2+y^2)^2} = \frac{x^4y^4}{(x^2+y^2)^2}\end{aligned}$$

30. $27^{1/3} = \sqrt[3]{27} = 3$

32. $-125^{2/3} = -(125^{1/3})^2 = -5^2 = -25$

34. $$\left(\frac{64}{27}\right)^{1/3} = \frac{64^{1/3}}{27^{1/3}} = \frac{4}{3}$$

36. $$625^{-1/4} = \frac{1}{625^{1/4}} = \frac{1}{5}$$

38. $$\begin{aligned}\left(\frac{121}{100}\right)^{-3/2} &= \frac{1}{\left(\frac{121}{100}\right)^{3/2}} = \frac{1}{\left[\left(\frac{121}{100}\right)^{1/2}\right]^3}\\ &= \frac{1}{\left(\frac{11}{10}\right)^3} = \frac{1}{\frac{1331}{1000}} = \frac{1000}{1331}\end{aligned}$$

40. $$\begin{aligned}27^{2/3}\cdot 27^{-1/3} &= 27^{(2/3)+(-1/3)}\\ &= 27^{2/3-1/3}\\ &= 27^{1/3}\end{aligned}$$

42. $$\begin{aligned}&\frac{3^{-5/2}\cdot 3^{3/2}}{3^{7/2}\cdot 3^{-9/2}}\\ &= 3^{(-5/2)+(3/2)-(7/2)-(-9/2)}\\ &= 3^{-5/2+3/2-7/2+9/2}\\ &= 3^0 = 1\end{aligned}$$

44. $$\begin{aligned}\frac{12^{3/4}\cdot 12^{5/4}\cdot y^{-2}}{12^{-1}\cdot (y^{-3})^{-2}} &= \frac{12^{3/4+5/4}\cdot y^{-2}}{12^{-1}\cdot y^{(-3)(-2)}}\\ &= \frac{12^{8/4}\cdot y^{-2}}{12^{-1}\cdot y^6}\\ &= \frac{12^2\cdot y^{-2}}{12^{-1}y^6}\\ &= 12^{2-(-1)}\cdot y^{-2-6}\\ &= 12^3y^{-8}\\ &= \frac{12^3}{y^8}\end{aligned}$$

46. $$\begin{aligned}\frac{8p^{-3}(4p^2)^{-2}}{p^{-5}} &= \frac{8p^{-3}\cdot 4^{-2}p^{(2)(-2)}}{p^{-5}}\\ &= \frac{8p^{-3}4^{-2}p^{-4}}{p^{-5}}\\ &= 8\cdot 4^{-2}p^{(-3)+(-4)-(-5)}\\ &= 8\cdot 4^{-2}p^{-3-4+5}\\ &= 8\cdot 4^{-2}p^{-2}\\ &= 8\cdot\frac{1}{4^2}\cdot\frac{1}{p^2}\\ &= 8\cdot\frac{1}{16}\cdot\frac{1}{p^2}\\ &= \frac{8}{16p^2} = \frac{1}{2p^2}\end{aligned}$$

48. $$\begin{aligned}&\frac{x^{1/3}\cdot y^{2/3}\cdot z^{1/4}}{x^{5/3}\cdot y^{-1/3}\cdot z^{3/4}}\\ &= x^{1/3-(5/3)}y^{(2/3)-(-1/3)}z^{1/4-(3/4)}\\ &= x^{1/3-5/3}y^{2/3+1/3}z^{1/4-3/4}\\ &= x^{-4/3}y^{3/3}z^{-2/4}\\ &= \frac{y}{x^{4/3}z^{2/4}}\\ &= \frac{y}{x^{4/3}z^{1/2}}\end{aligned}$$

50. $$\begin{aligned}&\frac{m^{7/3}\cdot n^{-2/5}\cdot p^{3/8}}{m^{-2/3}\cdot n^{3/5}\cdot p^{-5/8}}\\ &= m^{7/3-(-2/3)}n^{-2/5-(3/5)}p^{3/8-(-5/8)}\\ &= m^{7/3+2/3}n^{-2/5-3/5}p^{3/8+5/8}\\ &= m^{9/3}n^{-5/5}p^{8/8}\\ &= m^3n^{-1}p^1 = \frac{m^3p}{n}\end{aligned}$$

52. $$\begin{aligned}&6x(x^3+7)^2 - 6x^2(3x^2+5)(x^3+7)\\ &= 6x(x^3+7)(x^3+7) - 6x(x)(3x^2+5)(x^3+7)\\ &= 6x(x^3+7)[(x^3+7) - x(3x^2+5)]\\ &= 6x(x^3+7)(x^3+7-3x^3-5x)\\ &= 6x(x^3+7)(-2x^3-5x+7)\end{aligned}$$

54. $$\begin{aligned}&9(6x+2)^{1/2} + 3(9x-1)(6x+2)^{-1/2}\\ &= 3\cdot 3(6x+2)^{-1/2}(6x+2)^1\\ &\qquad + 3(9x-1)(6x+2)^{-1/2}\\ &= 3(6x+2)^{-1/2}[3(6x+2)+(9x-1)]\\ &= 3(6x+2)^{-1/2}(18x+6+9x-1)\\ &= 3(6x+2)^{-1/2}(27x+5)\end{aligned}$$

56. $$\begin{aligned}&(4x^2+1)^2(2x-1)^{-1/2} + 16x(4x^2+1)(2x-1)^{1/2}\\ &= (4x^2+1)(4x^2+1)(2x-1)^{-1/2}\\ &\qquad + 16x(4x^2+1)(2x-1)^{-1/2}(2x-1)\\ &= (4x^2+1)(2x-1)^{-1/2}\\ &\qquad\cdot[(4x^2+1)+16x(2x-1)]\\ &= (4x^2+1)(2x-1)^{-1/2}(4x^2+1+32x^2-16x)\\ &= (4x^2+1)(2x-1)^{-1/2}(36x^2-16x+1)\end{aligned}$$

R.7 Radicals

2. $\sqrt[4]{1296} = \sqrt[4]{6^4} = 6$

4. $\sqrt{50} = \sqrt{25\cdot 2} = \sqrt{25}\sqrt{2} = 5\sqrt{2}$

6. $$\begin{aligned}\sqrt{32y^5} &= \sqrt{(16y^4)(2y)}\\ &= \sqrt{16y^4}\sqrt{2y}\\ &= 4y^2\sqrt{2y}\end{aligned}$$

8. $$\begin{aligned}&4\sqrt{3} - 5\sqrt{12} + 3\sqrt{75}\\ &= 4\sqrt{3} - 5(\sqrt{4}\sqrt{3}) + 3(\sqrt{25}\sqrt{3})\\ &= 4\sqrt{3} - 5(2\sqrt{3}) + 3(5\sqrt{3})\\ &= 4\sqrt{3} - 10\sqrt{3} + 15\sqrt{3}\\ &= (4-10+15)\sqrt{3} = 9\sqrt{3}\end{aligned}$$

10. $$\begin{aligned}&3\sqrt{28} - 4\sqrt{63} + \sqrt{112}\\ &= 3(\sqrt{4}\sqrt{7}) - 4(\sqrt{9}\sqrt{7}) + (\sqrt{16}\sqrt{7})\\ &= 3(2\sqrt{7}) - 4(3\sqrt{7}) + (4\sqrt{7})\\ &= 6\sqrt{7} - 12\sqrt{7} + 4\sqrt{7}\\ &= (6-12+4)\sqrt{7}\\ &= -2\sqrt{7}\end{aligned}$$

12. $$\begin{aligned}&2\sqrt[3]{3} + 4\sqrt[3]{24} - \sqrt[3]{81}\\ &= 2\sqrt[3]{3} + 4\sqrt[3]{8\cdot 3} - \sqrt[3]{27\cdot 3}\\ &= 2\sqrt[3]{3} + 4(2)\sqrt[3]{3} - 3\sqrt[3]{3}\\ &= 2\sqrt[3]{3} + 8\sqrt[3]{3} - 3\sqrt[3]{3}\\ &= 7\sqrt[3]{3}\end{aligned}$$

14. $$\begin{aligned}\sqrt{2x^3y^2z^4} &= \sqrt{x^2y^2z^4\cdot 2x}\\ &= xyz^2\sqrt{2x}\end{aligned}$$

16. $$\begin{aligned}\sqrt[3]{16x^8y^4z^5} &= \sqrt[3]{8x^6y^3z^3\cdot 2x^2yz^2}\\ &= 2x^2yz\sqrt[3]{2x^2yz^2}\end{aligned}$$

18. $$\begin{aligned}&\sqrt{a^3b^5} - 2\sqrt{a^7b^3} + \sqrt{a^3b^9}\\ &= \sqrt{a^2b^4ab} - 2\sqrt{a^6b^2ab} + \sqrt{a^2b^8ab}\\ &= ab^2\sqrt{ab} - 2a^3b\sqrt{ab} + ab^4\sqrt{ab}\\ &= (ab^2 - 2a^3b + ab^4)\sqrt{ab}\\ &= ab\sqrt{ab}(b - 2a^2 + b^3)\end{aligned}$$

20. $\dfrac{5}{\sqrt{7}} = \dfrac{5}{\sqrt{7}} \cdot \dfrac{\sqrt{7}}{\sqrt{7}} = \dfrac{5\sqrt{7}}{7}$

22. $$\begin{aligned}\frac{-3}{\sqrt{12}} &= \frac{-3}{\sqrt{4\cdot 3}}\\ &= \frac{-3}{2\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}\\ &= \frac{-3\sqrt{3}}{6}\\ &= -\frac{\sqrt{3}}{2}\end{aligned}$$

24. $$\begin{aligned}\frac{3}{1-\sqrt{5}} &= \frac{3}{1-\sqrt{5}}\cdot\frac{1+\sqrt{5}}{1+\sqrt{5}}\\ &= \frac{3(1+\sqrt{5})}{1-5}\\ &= \frac{-3(1+\sqrt{5})}{4}\end{aligned}$$

26. $$\begin{aligned}\frac{-2}{\sqrt{3}-\sqrt{2}} &= \frac{-2}{\sqrt{3}-\sqrt{2}}\cdot\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}\\ &= \frac{-2(\sqrt{3}+\sqrt{2})}{3-2}\\ &= \frac{-2(\sqrt{3}+\sqrt{2})}{1}\\ &= -2(\sqrt{3}+\sqrt{2})\end{aligned}$$

28. $$\begin{aligned}\frac{1}{\sqrt{r}-\sqrt{3}} &= \frac{1}{\sqrt{r}-\sqrt{3}}\cdot\frac{\sqrt{r}+\sqrt{3}}{\sqrt{r}+\sqrt{3}}\\ &= \frac{\sqrt{r}+\sqrt{3}}{r-3}\end{aligned}$$

30. $$\begin{aligned}\frac{y-5}{\sqrt{y}-\sqrt{5}} &= \frac{y-5}{\sqrt{y}-\sqrt{5}}\cdot\frac{\sqrt{y}+\sqrt{5}}{\sqrt{y}+\sqrt{5}}\\ &= \frac{(y-5)(\sqrt{y}+\sqrt{5})}{y-5}\\ &= \sqrt{y}+\sqrt{5}\end{aligned}$$

32. $$\begin{aligned}\frac{\sqrt{x}+\sqrt{x+1}}{\sqrt{x}-\sqrt{x+1}} &= \frac{\sqrt{x}+\sqrt{x+1}}{\sqrt{x}-\sqrt{x+1}}\cdot\frac{\sqrt{x}+\sqrt{x+1}}{\sqrt{x}+\sqrt{x+1}}\\ &= \frac{x+2\sqrt{x(x+1)}+(x+1)}{x-(x+1)}\\ &= \frac{2x+2\sqrt{x(x+1)}+1}{-1}\\ &= -2x-2\sqrt{x(x+1)}-1\end{aligned}$$

34. $$\begin{aligned}\frac{1+\sqrt{2}}{2} &= \frac{(1+\sqrt{2})(1-\sqrt{2})}{2(1-\sqrt{2})}\\ &= \frac{1-2}{2(1-\sqrt{2})}\\ &= -\frac{1}{2(1-\sqrt{2})}\end{aligned}$$

36. $$\begin{aligned}&\frac{\sqrt{x}+\sqrt{x+1}}{\sqrt{x}-\sqrt{x+1}}\\ &= \frac{\sqrt{x}+\sqrt{x+1}}{\sqrt{x}-\sqrt{x+1}}\cdot\frac{\sqrt{x}-\sqrt{x+1}}{\sqrt{x}-\sqrt{x+1}}\\ &= \frac{x-(x+1)}{x-2\sqrt{x}\cdot\sqrt{x+1}+(x+1)}\\ &= \frac{-1}{2x-2\sqrt{x(x+1)}+1}\end{aligned}$$

38. $$\begin{aligned}&\sqrt{16-8x+x^2}\\ &= \sqrt{(4-x)^2}\\ &= |4-x|\end{aligned}$$

Since $\sqrt{}$ denotes the nonnegative root, we must have $4-x \geq 0$.

40. $\sqrt{4-25z^2} = \sqrt{(2+5z)(2-5z)}$

This factorization does not produce a perfect square, so the expression $\sqrt{4-25z^2}$ cannot be simplified.

Chapter 1

LINEAR FUNCTIONS

1.1 Slopes and Equations of Lines

2. Find the slope of the line through $(5, -4)$ and $(1, 3)$.

$$\begin{aligned} m &= \frac{3-(-4)}{1-5} \\ &= \frac{3+4}{-4} = -\frac{7}{4} \end{aligned}$$

4. Find the slope of the line through $(1, 5)$ and $(-2, 5)$.

$$m = \frac{5-5}{-2-1} = \frac{0}{-3} = 0$$

6. $y = 3x - 2$

This equation is in slope-intercept form, $y = mx + b$. Thus, the coefficient of the x-term, 3, is the slope.

8. $4x + 7y = 1$

Rewrite the equation in slope-intercept form.

$$\begin{aligned} 7y &= 1 - 4x \\ \frac{1}{7}(7y) &= \frac{1}{7}(1) - \frac{1}{7}(4x) \\ y &= \frac{1}{7} - \frac{4}{7}x \\ y &= -\frac{4}{7}x + \frac{1}{7} \end{aligned}$$

The slope is $-\frac{4}{7}$.

10. The x-axis is the horizontal line $y = 0$. Horizontal lines have a slope of 0.

12. $y = -4$

By rewriting this equation in the slope-intercept form, $y = mx + b$, we get $y = 0x - 4$, with the slope, m, being 0.

14. Find the slope of a line perpendicular to $6x = y - 3$.
First, rewrite the given equation in slope-intercept form.

$$\begin{aligned} 6x &= y - 3 \\ 6x + 3 &= y \\ \text{or} \qquad y &= 6x + 3 \end{aligned}$$

The slope of this line is 6.
Let m be the slope of any line perpendicular to the given line. Then

$$\begin{aligned} 6m &= -1 \\ m &= -\frac{1}{6}. \end{aligned}$$

16. The line goes through $(2, 4)$, with slope $m = -1$.
Use point-slope form.

$$\begin{aligned} y - 4 &= -1(x - 2) \\ y - 4 &= -x + 2 \\ y &= -x + 6 \end{aligned}$$

18. The line goes through $(-8, 1)$, with undefined slope. Since the slope is undefined, the line is vertical. The equation of the vertical line passing through $(-8, 1)$ is $x = -8$.

20. The line goes through $(8, -1)$ and $(4, 3)$.
Find the slope, then use point-slope form with either of the two given points.

$$\begin{aligned} m &= \frac{3-(-1)}{4-8} \\ &= \frac{3+1}{-4} \\ &= \frac{4}{-4} = -1 \end{aligned}$$

$$\begin{aligned} y - (-1) &= -1(x - 8) \\ y + 1 &= -x + 8 \\ y &= -x + 7 \end{aligned}$$

22. The line goes through $\left(-2, \frac{3}{4}\right)$ and $\left(\frac{2}{3}, \frac{5}{2}\right)$.

$$\begin{aligned} m &= \frac{\frac{5}{2} - \frac{3}{4}}{\frac{2}{3} - (-2)} = \frac{\frac{10}{4} - \frac{3}{4}}{\frac{2}{3} + \frac{6}{3}} \\ &= \frac{\frac{7}{4}}{\frac{8}{3}} = \frac{21}{32} \end{aligned}$$

$$\begin{aligned} y - \frac{3}{4} &= \frac{21}{32}[x - (-2)] \\ y - \frac{3}{4} &= \frac{21}{32}x + \frac{42}{32} \\ y &= \frac{21}{32}x + \frac{42}{32} + \frac{3}{4} \\ y &= \frac{21}{32}x + \frac{21}{16} + \frac{12}{16} \\ y &= \frac{21}{32}x + \frac{33}{16} \end{aligned}$$

24. The line goes through $(-1, 3)$ and $(0, 3)$.

$$m = \frac{3 - 3}{-1 - 0} = \frac{0}{-1} = 0$$

This is a horizontal line; the value of y is always 3. The equation of this line is $y = 3$.

26. The line has x-intercept -2 and y-intercept 4. Two points on the line are $(-2, 0)$ and $(0, 4)$. Find the slope; then use slope-intercept form.

$$\begin{aligned} m &= \frac{4 - 0}{0 - (-2)} = \frac{4}{2} = 2 \\ y &= mx + b \\ y &= 2x + 4 \end{aligned}$$

28. The line is horizontal, through $(8, 7)$. The line has an equation of the form $y = k$ where k is the y-coordinate of the point. In this case, $k = 7$, so the equation is $y = 7$.

30. Write the equation of the line through $(2, -5)$, parallel to $y - 4 = 2x$. Rewrite the equation in slope-intercept form.

$$\begin{aligned} y - 4 &= 2x \\ y &= 2x + 4 \end{aligned}$$

The slope of this line is 2.
Use $m = 2$ and the point $(2, -5)$ in the point-slope form.

$$\begin{aligned} y - (-5) &= 2(x - 2) \\ y + 5 &= 2x - 4 \\ y &= 2x - 9 \end{aligned}$$

32. Write the equation of the line through $(-2, 6)$, perpendicular to $2x - 3y = 5$.
Rewrite the equation in slope-intercept form.

$$\begin{aligned} 2x - 3y &= 5 \\ -3y &= -2x + 5 \\ y &= \frac{2}{3}x - \frac{5}{3} \end{aligned}$$

The slope of this line is $\frac{2}{3}$. To find the slope of a perpendicular line, solve

$$\begin{aligned} \frac{2}{3}m &= -1. \\ m &= -\frac{3}{2} \end{aligned}$$

Use $m = -\frac{3}{2}$ and $(-2, 6)$ in the point-slope form.

$$\begin{aligned} y - 6 &= -\frac{3}{2}[x - (-2)] \\ y - 6 &= -\frac{3}{2}(x + 2) \\ y - 6 &= -\frac{3}{2}x - 3 \\ y &= -\frac{3}{2}x + 3 \end{aligned}$$

34. Write the equation of the line with x-intercept $-\frac{2}{3}$, perpendicular to $2x - y = 4$.
Find the slope of the given line.

$$\begin{aligned} 2x - y &= 4 \\ 2x - 4 &= y \end{aligned}$$

The slope of this line is 2. Since the lines are perpendicular, the slope of the needed line is $-\frac{1}{2}$. The line also has an x-intercept of $-\frac{2}{3}$. Thus, it passes through the point $\left(-\frac{2}{3}, 0\right)$.
Using the point-slope form, we have

$$\begin{aligned} y - 0 &= -\frac{1}{2}\left[x - \left(-\frac{2}{3}\right)\right] \\ y &= -\frac{1}{2}\left(x + \frac{2}{3}\right) \\ y &= -\frac{1}{2}x - \frac{1}{3} \end{aligned}$$

36. **(a)** Write the given line in slope-intercept form.

$$\begin{aligned} 2x + 3y &= 6 \\ 3y &= -2x + 6 \\ y &= -\frac{2}{3}x + 2 \end{aligned}$$

This line has a slope of $-\frac{2}{3}$. The desired line has a slope of $-\frac{2}{3}$ since it is parallel to the given line. Use the definition of slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ -\frac{2}{3} &= \frac{2 - (-1)}{k - 4} \\ -\frac{2}{3} &= \frac{3}{k - 4} \\ -2(k - 4) &= (3)(3) \\ -2k + 8 &= 9 \\ -2k &= 1 \\ k &= -\frac{1}{2} \end{aligned}$$

(b) Write the given line in slope-intercept form.

$$\begin{aligned} 5x - 2y &= -1 \\ 2y &= 5x + 1 \\ y &= \frac{5}{2}x + \frac{1}{2} \end{aligned}$$

This line has a slope of $\frac{5}{2}$. The desired line has a slope of $-\frac{2}{5}$ since it is perpendicular to the given line. Use the definition of slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{2 - (-1)}{k - 4} \\ -\frac{2}{5} &= \frac{2 + 1}{k - 4} \\ \frac{-2}{5} &= \frac{3}{k - 4} \\ -2(k - 4) &= (3)(5) \\ -2k + 8 &= 15 \\ -2k &= 7 \\ k &= -\frac{7}{2} \end{aligned}$$

38. Two lines are perpendicular if the product of their slopes is -1.
The slope of the diagonal containing $(4, 5)$ and $(-2, -1)$ is

$$m = \frac{5 - (-1)}{4 - (-2)} = \frac{6}{6} = 1.$$

The slope of the diagonal containing $(-2, 5)$ and $(4, -1)$ is

$$m = \frac{5 - (-1)}{-2 - 4} = \frac{6}{-6} = -1.$$

The product of the slopes is $(1)(-1) = -1$, so the diagonals are perpendicular.

40. The line goes through $(1, 3)$ and $(2, 0)$.

$$m = \frac{3 - 0}{1 - 2} = \frac{3}{-1} = -3$$

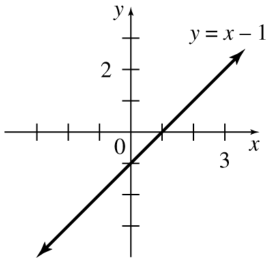
The correct choice is (f).

42. The line goes through $(-2, 0)$ and $(0, 1)$.

$$m = \frac{1 - 0}{0 - (-2)} = \frac{1}{2}$$

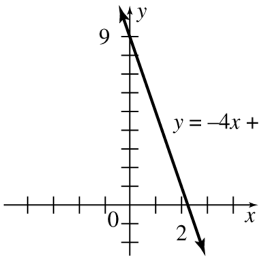
44. $y = x - 1$

Three ordered pairs that satisfy this equation are $(0, -1)$, $(1, 0)$, and $(4, 3)$. Plot these points and draw a line through them.



46. $y = -4x + 9$

Three ordered pairs that satisfy this equation are $(0, 9)$, $(1, 5)$, and $(2, 1)$. Plot these points and draw a line through them.



48. $2x - 3y = 12$

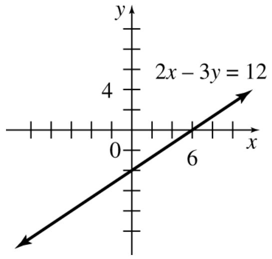
Find the intercepts.
If $y = 0$, then

$$\begin{aligned} 2x - 3(0) &= 12 \\ 2x &= 12 \\ x &= 6 \end{aligned}$$

so the x-intercept is 6.
If $x = 0$, then

$$\begin{aligned} 2(0) - 3y &= 12 \\ -3y &= 12 \\ y &= -4 \end{aligned}$$

so the y-intercept is -4.
Plot the ordered pairs $(6, 0)$ and $(0, -4)$ and draw a line through these points. (A third point may be used as a check.)



50. $3y + 4x = 12$

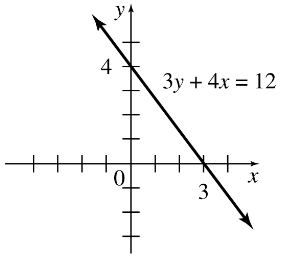
Find the intercepts.
If $y = 0$, then

$$\begin{aligned} 3(0) + 4x &= 12 \\ 4x &= 12 \\ x &= 3, \end{aligned}$$

so the x-intercept is 3.
If $x = 0$, then

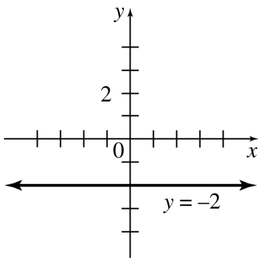
$$\begin{aligned} 3y + 4(0) &= 12 \\ 3y &= 12 \\ y &= 4, \end{aligned}$$

so the y-intercept is 4.
Plot the ordered pairs $(3, 0)$ and $(0, 4)$ and draw a line through these points. (A third point may be used as a check.)



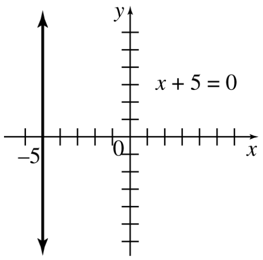
52. $y = -2$

The equation $y = -2$, or, equivalently, $y = 0x - 2$, always gives the same y-value, -2, for any value of x. The graph of this equation is the horizontal line with y-intercept -2.



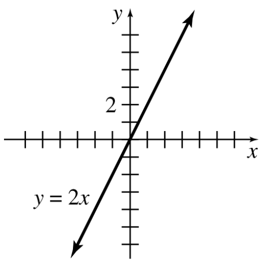
54. $x + 5 = 0$

This equation may be rewritten as $x = -5$. For any value of y, the x-value is -5. Because all ordered pairs that satisfy this equation have the same first number, this equation does not represent a function. The graph is the vertical line with x-intercept -5.



56. $y = 2x$

Three ordered pairs that satisfy this equation are $(0, 0)$, $(-2, -4)$, and $(2, 4)$. Use these points to draw the graph.

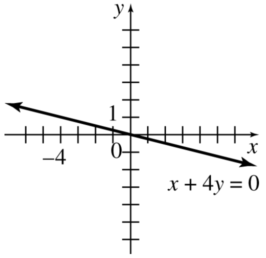


58. $x + 4y = 0$

If $y = 0$, then $x = 0$, so the x-intercept is 0. If $x = 0$, then $y = 0$, so the y-intercept is 0. Both intercepts give the same ordered pair, $(0, 0)$.
To get a second point, choose some other value of x (or y). For example if $x = 4$, then

$$\begin{aligned} x + 4y &= 0 \\ 4 + 4y &= 0 \\ 4y &= -4 \\ y &= -1, \end{aligned}$$

giving the ordered pair $(4, -1)$. Graph the line through $(0, 0)$ and $(4, -1)$.



60. **(a)** The line goes through (2, 27,000) and (5, 63,000).

$$m = \frac{63{,}000 - 27{,}000}{5 - 2} = 12{,}000$$

$$\begin{aligned} y - 27{,}000 &= 12{,}000(x - 2) \\ y - 27{,}000 &= 12{,}000x - 24{,}000 \\ y &= 12{,}000x + 3000 \end{aligned}$$

(b) Let y =100,000; find x.

$$\begin{aligned} 100{,}000 &= 12{,}000x + 3000 \\ 97{,}000 &= 12{,}000x \\ 8.08 &= x \end{aligned}$$

Sales would surpass $100,000 after 8 years, 1 month.

62. **(a)** Using the points $(.7, 1.4)$ and $(5.3, 10.9)$, we obtain

$$\begin{aligned} m &= \frac{10.9 - 1.4}{5.3 - .7} = \frac{9.5}{4.6} \\ &\approx 2.065. \end{aligned}$$

To avoid round-off error, keep all digits for the value of m in your calculator; then round the decimals in the final step.

Use the point-slope form.

$$\begin{aligned} y - 1.4 &= \frac{9.5}{4.6}(x - .7) \\ y - 1.4 &= \frac{9.5}{4.6}x - \frac{9.5}{4.6}(.7) \\ y &= \frac{9.5}{4.6}x - \frac{9.5}{4.6}(.7) + 1.4 \\ y &= 2.065x - .0456 \end{aligned}$$

(b) $y = 2.065(4.9) - .0456 \approx 10.1$ million passengers; this agrees favorably with the FAA predication of 10.3 million.

64. **(a)** Let x = age.

$$\begin{aligned} u &= .85(220 - x) = 187 - .85x \\ l &= .7(220 - x) = 154 - .7x \end{aligned}$$

(b) $u = 187 - .85(20) = 170$
$l = 154 - .7(20) = 140$

The target heart rate zone is 140 to 170 beats per minute.

(c) $u = 187 - .85(40) = 153$
$l = 154 - .7(40) = 126$

The target heart rate zone is 126 to 153 beats per minute.

(d)
$$\begin{aligned} 154 - .7x &= 187 - .85(x + 36) \\ 154 - .7x &= 187 - .85x - 30.6 \\ 154 - .7x &= 156.4 - .85x \\ .15x &= 2.4 \\ x &= 16 \end{aligned}$$

The younger woman is 16; the older woman is $16 + 36 = 52$. $l = .7(220 - 16) \approx 143$ beats per minute.

66. Let $x = 0$ correspond to 1900. Then the "life expectancy from birth" line contains the points $(0, 46)$ and $(100, 76.9)$.

$$m = \frac{76.9 - 46}{100 - 0} = \frac{30.9}{100} = .309$$

Since $(0, 46)$ is one of the points, the line is given by the equation

$$y = .309x + 46.$$

The "life expectancy from age 65" line contains the points $(0, 76)$ and $(100, 82.9)$.

$$m = \frac{82.9 - 76}{100 - 0} = \frac{6.9}{100} = .069$$

Since $(0, 76)$ is one of the points, the line is given by the equation

$$y = .069x + 76.$$

Set the two equations equal to determine where the lines intersect. At this point, life expectancy should increase no further.

$$\begin{aligned} .309x + 46 &= .069x + 76 \\ .24x &= 30 \\ x &= 125 \end{aligned}$$

Determine the y-value when $x = 125$. Use the first equation.

$$\begin{aligned} y &= .309(125) + 46 \\ &= 38.625 + 46 \\ &= 84.625 \end{aligned}$$

Thus, the maximum life expectancy for humans is about 85 years.

68. **(a)** The line goes through the points $(0, 86{,}821)$ and (26, 217,753).

$$\begin{aligned} m &= \frac{217{,}753 - 86{,}821}{26 - 0} \\ &= \frac{130{,}932}{26} \\ &\approx 5035.85 \end{aligned}$$

Since one of the points is $(0, 86{,}821)$, the line is given by the equation

$$y = 5035.85x + 86{,}821.$$

(b) The year 2010 corresponds to $x = 36$.

$$\begin{aligned} y &= 5035.85(36) + 86{,}821 \\ y &\approx 268{,}112 \end{aligned}$$

We predict that the number of immigrants to California in 2010 will be about 268,112.

70. **(a)** Using the points $(0, 9.6)$ and $(31, 19.2)$,

$$\begin{aligned} m &= \frac{19.2 - 9.6}{31 - 0} \\ &= \frac{9.6}{31} \\ &\approx .31 \end{aligned}$$

Since $(0, 9.6)$ is on the line, the equation is given by

$$p = .31t + 9.6.$$

(b) The year 2010 corresponds to $t = 40$.

$$p = .31(40) + 9.6 = 22$$

If the trend continues, about 22% of college students will be 35 and older in 2010.

(c) Let $p = 31$ and solve the equation for t.

$$\begin{aligned} 31 &= .31t + 9.6 \\ 21.4 &= .31t \\ t &\approx 69 \end{aligned}$$

This corresponds to the year 1970 + 69 = 2039.

72. **(a)** If the temperature rises .3C° per decade, it rises .03C° per year.

$$\begin{aligned} m &= .03 \\ b &= 15, \text{ since a point is } (0, 15). \end{aligned}$$

$$T = .03t + 15$$

(b) Let $T = 19$; find t.

$$\begin{aligned} 19 &= .03t + 15 \\ 4 &= .03t \\ 133.3 &= t \\ 133 &\approx t \\ 1970 + 133 &= 2103 \end{aligned}$$

The temperature will rise to 19°C in about the year 2103.

74. **(a)** $m = \dfrac{13{,}150 - 2773}{2000 - 1950} = \dfrac{10{,}377}{50} = 207.54$

This means that each year there is an average increase of about 208 stations.

(b) Use the point-slope form with $(2000, 13{,}150)$.

$$\begin{aligned} y - 13{,}150 &= 207.54(x - 2000) \\ y - 13{,}150 &= 207.54x - 415{,}080 \\ y &= 207.54x - 401{,}930 \end{aligned}$$

(c) Let $y = 15{,}000$ and solve the equation for x.

$$\begin{aligned} 15{,}000 &= 207.54x - 401{,}930 \\ 416{,}930 &= 207.54x \\ x &\approx 2008.9 \end{aligned}$$

The estimated year when it is expected that the number of stations will first exceed 15,000 is 2009.

1.2 Linear Functions and Applications

2. This statement is false.

The graph of $f(x) = -3$ is a horizontal line.

4. This statement is true.

For any value of a,

$$f(0) = a \cdot 0 = 0,$$

so the point $(0, 0)$, which is the origin, lies on the line.

8. \$12 is the fixed cost and \$1 is the cost per hour.

$$\begin{aligned} \text{Let } x &= \text{ number of hours;} \\ C(x) &= \text{ cost of renting a saw for} \\ &\quad\ x \text{ hours.} \end{aligned}$$

Thus,

$$\begin{aligned} C(x) &= \text{ fixed cost } + \text{ (cost per hour)} \\ &\quad \cdot \text{ (number of hours)} \\ C(x) &= 12 + 1x \\ &\quad 12 + x. \end{aligned}$$

10. 50¢ is the fixed cost and 35¢ is the cost per half-hour.

$$\begin{aligned} \text{Let } x &= \text{ the number of half-hours;} \\ C(x) &= \text{ the cost of parking a car for} \\ &\quad\ x \text{ half-hours.} \end{aligned}$$

Thus,

$$\begin{aligned} C(x) &= 50 + 35x \\ &= 35x + 50. \end{aligned}$$

12. Fixed cost, \$100; 50 items cost \$1600 to produce.

$$\begin{aligned} \text{Let } C(x) &= \text{ cost of producing } x \text{ items.} \\ C(x) &= mx + b, \text{ where } b \text{ is the fixed} \\ &\quad\ \text{cost.} \end{aligned}$$

$$C(x) = mx + 100$$

Now,

$C(x) = 1600$ when $x = 50$, so

$$\begin{aligned} 1600 &= m(50) + 100 \\ 1500 &= 50m \\ 30 &= m. \end{aligned}$$

Thus, $C(x) = 30x + 100$.

14. Marginal cost, \$90; 150 items cost \$16,000 to produce.

$$C(x) = 90x + b$$

Now, $C(x) = 16{,}000$ when $x = 150$.

$$\begin{aligned} 16{,}000 &= 90(150) + b \\ 16{,}000 &= 13{,}500 + b \\ 2500 &= b \end{aligned}$$

Thus, $C(x) = 90x + 2500$.

16. For a linear function, the average rate of change will be the same as the slope of the line. If the function is not linear, the average rate of change is the slope of the secant line connecting the beginning and ending points.

18. $D(q) = 16 - \frac{5}{4}q$

(a) $D(0) = 16 - \frac{5}{4}(0) = 16 - 0 = 16$

When 0 can openers are demanded, the price is \$16.

(b) $D(4) = 16 - \frac{5}{4}(4) = 16 - 5 = 11$

When 400 can openers are demanded, the price is \$11.

(c) $D(8) = 16 - \frac{5}{4}(8) = 16 - 10 = 6$

When 800 can openers are demanded, the price is \$6.

(d) Let $D(q) = 8$. Find q.

$$\begin{aligned} 8 &= 16 - \frac{5}{4}q \\ \frac{5}{4}q &= 8 \\ q &= 6.4 \end{aligned}$$

When the price is \$8, 640 can openers are demanded.

(e) Let $D(q) = 10$. Find q.

$$\begin{aligned} 10 &= 16 - \frac{5}{4}q \\ \frac{5}{4}q &= 6 \\ q &= 4.8 \end{aligned}$$

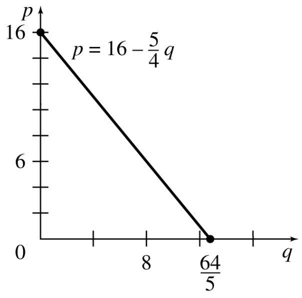
When the price is \$10, 480 can openers are demanded.

(f) Let $D(q) = 12$. Find q.

$$\begin{aligned} 12 &= 16 - \frac{5}{4}q \\ \frac{5}{4}q &= 4 \\ q &= 3.2 \end{aligned}$$

When the price is \$12, 320 can openers are demanded.

(g)



(h) $S(q) = \frac{3}{4}q$

Let $S(q) = 0$. Find q.

$$\begin{aligned} 0 &= \frac{3}{4}q \\ 0 &= q \end{aligned}$$

When the price is \$0, 0 can openers are supplied.

(i) Let $S(q) = 10$. Find q.

$$\begin{aligned} 10 &= \frac{3}{4}q \\ \frac{40}{3} &= q \\ q &= 13.\overline{3} \end{aligned}$$

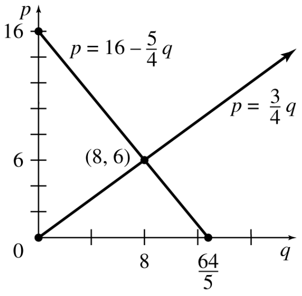
When the price is \$10, about 1333 can openers are supplied.

(j) Let $S(q) = 20$. Find q.

$$\begin{aligned} 20 &= \frac{3}{4}q \\ \frac{80}{3} &= q \\ q &= 26.\overline{6} \end{aligned}$$

When the price is \$20, about 2667 can openers are demanded.

(k)



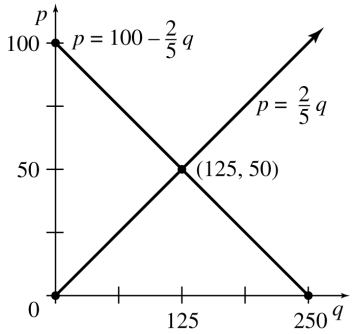
(l)

$$\begin{aligned} D(q) &= S(q) \\ 16 - \frac{5}{4}q &= \frac{3}{4}q \\ 16 &= 2q \\ 8 &= q \\ S(8) &= \frac{3}{4}(8) = 6 \end{aligned}$$

The equilibrium quantity is 800, and the equilibrium price is \$6.

20. $p = S(q) = \frac{2}{5}q$; $p = D(q) = 100 - \frac{2}{5}q$

(a)



(b)

$$\begin{aligned} S(q) &= D(q) \\ \frac{2}{5}q &= 100 - \frac{2}{5}q \\ \frac{4}{5}q &= 100 \\ q &= 125 \\ S(125) &= \frac{2}{5}(125) = 50 \end{aligned}$$

The equilibrium quantity is 125, the equilibrium price is \$50

22. **(a)** $C(x) = mx + b;\ m = 3.50;\ C(60) = 300$

$$C(x) = 3.50x + b$$

Find b.

$$\begin{aligned} 300 &= 3.50(60) + b \\ 300 &= 210 + b \\ 90 &= b \\ C(x) &= 3.50x + 90 \end{aligned}$$

(b) $R(x) = 9x$
$C(x) = R(x)$

$$\begin{aligned} 3.50x + 90 &= 9x \\ 90 &= 5.5x \\ 16.36 &= x \end{aligned}$$

Yoshi must produce and sell 17 shirts.

(c) $P(x) = R(x) - C(x);\ P(x) = 500$

$$\begin{aligned} 500 &= 9x - (3.50x + 90) \\ 500 &= 5.5x - 90 \\ 590 &= 5.5x \\ 107.27 &= x \end{aligned}$$

To make a profit of \$500, Yoshi must produce and sell 108 shirts.

24. **(a)** Using the points $(100, 11.02)$ and $(400, 40.12)$,

$$m = \frac{40.12 - 11.02}{400 - 100} = \frac{29.1}{300} = .097.$$

$$\begin{aligned} y - 11.02 &= .097(x - 100) \\ y - 11.02 &= .097x - 9.7 \\ y &= .097x + 1.32 \\ C(x) &= .097x + 1.32 \end{aligned}$$

(b) The fixed cost is given by the constant in $C(x)$. It is \$1.32.

(c) $$\begin{aligned} C(1000) &= .097(1000) + 1.32 = 97 + 1.32 \\ &= 98.32 \end{aligned}$$

The total cost of producing 1000 cups is \$98.32.

(d) $$\begin{aligned} C(1001) &= .097(1001) + 1.32 = 97.097 + 1.32 \\ &= 98.417 \end{aligned}$$

The total cost of producing 1001 cups is \$98.417.

(e) $$\begin{aligned} \text{Marginal cost} &= 98.417 - 98.32 \\ &= \$.097 \quad \text{or} \quad 9.7¢ \end{aligned}$$

(f) The marginal cost for *any* cup is the slope, \$.097 or 9.7¢. This means the cost of producing one additional cup of coffee would be 9.7¢.

26. **(a)** $(100{,}000)(50) = 5{,}000{,}000$

Sales in 1996 would be 100,000 + 5,000,000 = 5,100,000.

(b) The ordered pairs are (1, 100,000) and (6, 5,100,000).

(c) $$m = \frac{5{,}100{,}000 - 100{,}000}{6 - 1} = \frac{5{,}000{,}000}{5}$$
$$= 1,000,000$$

$$\begin{aligned} y - 100{,}000 &= 1{,}000{,}000(x - 1) \\ y - 100{,}000 &= 1{,}000{,}000x - 1{,}000{,}000 \\ y &= 1{,}000{,}000x - 900{,}000 \\ S(x) &= 1{,}000{,}000x - 900{,}000 \end{aligned}$$

(d) Let $S(x) = 1{,}000{,}000{,}000$. Find x.

$$\begin{aligned} 1{,}000{,}000{,}000 &= 1{,}000{,}000x - 900{,}000 \\ 1{,}000{,}900{,}000 &= 1{,}000{,}000x \\ x &= 1000.9 \approx 1001 \end{aligned}$$

Sales would reach one billion dollars in about 1991 + 1001 = 2992.

(e) According to our linear model, in 2003, $x = 13$ and estimated sales would be

$$S(13) = 1{,}000{,}000(13) - 900{,}000 = 12{,}100{,}000$$

or about \$12,100,000. Sales are growing much faster than linearly if they reached one billion dollars in 2003.

28. $C(x) = 12x + 39;\ R(x) = 25x$

(a) $$\begin{aligned} C(x) &= R(x) \\ 12x + 39 &= 25x \\ 39 &= 13x \\ 3 &= x \end{aligned}$$

The break-even quantity is 3 units.

(b) $$\begin{aligned} P(x) &= R(x) - C(x) \\ P(x) &= 25x - (12x + 39) \\ P(x) &= 13x - 39 \\ P(250) &= 13(250) - 39 \\ &= 3250 - 39 \\ &= 3211 \end{aligned}$$

The profit from 250 units is \$3211.

(c) $P(x) = \$130$; find x.

$$\begin{aligned} 130 &= 13x - 39 \\ 169 &= 13x \\ 13 &= x \end{aligned}$$

For a profit of \$130, 13 units must be produced.

30. $C(x) = 105x + 6000$
$R(x) = 250x$

Set $C(x) = R(x)$ to find the break-even quantity.

$$\begin{aligned} 105x + 6000 &= 250x \\ 6000 &= 145x \\ 41.38 &\approx x \end{aligned}$$

The break-even quantity is about 41 units, so you should decide to produce.

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 250x - (105x + 6000) \\ &= 145x - 6000 \end{aligned}$$

The profit function is $P(x) = 145x - 6000$.

32. $C(x) = 1000x + 5000$
$R(x) = 900x$

$$\begin{aligned} 900x &= 1000x + 5000 \\ -5000 &= 100x \\ -50 &= x \end{aligned}$$

It is impossible to make a profit when the break-even quantity is negative. Cost will always be greater than revenue.

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 900x - (1000x + 5000) \\ &= -100x - 5000 \end{aligned}$$

The profit function is $P(x) = -100x - 500$ (always a loss).

34. Use the formula derived in Example 7 in this section of the textbook.

$$\begin{aligned} F &= \frac{9}{5}C + 32 \\ C &= \frac{5}{9}(F - 32) \end{aligned}$$

(a) $C = 37$; find F.

$$\begin{aligned} F &= \frac{9}{5}(37) + 32 \\ F &= \frac{333}{5} + 32 \\ F &= 98.6 \end{aligned}$$

The Fahrenheit equivalent of 37°C is 98.6°F.

(b) $C = 36.5$; find F.

$$\begin{aligned} F &= \frac{9}{5}(36.5) + 32 \\ F &= 65.7 + 32 \\ F &= 97.7 \end{aligned}$$

$C = 37.5$; find F.

$$\begin{aligned} F &= \frac{9}{5}(37.5) + 32 \\ &= 67.5 + 32 = 99.5 \end{aligned}$$

The range is between 97.7°F and 99.5°F.

1.3 The Least Squares Line

2. For the set of points $(1, 4)$, $(2, 5)$, and $(3, 6)$, $Y = x + 3$. For the set $(4, 1)$, $(5, 2)$, and $(6, 3)$, $Y = x - 3$.

4.

$$\begin{aligned} nb + (\sum x)m &= \sum y \\ (\sum x)b + (\sum x^2)m &= \sum xy \\ nb + (\sum x)m &= \sum y \\ nb &= (\sum y) - (\sum x)m \\ b &= \frac{\sum y - m(\sum x)}{n} \\ (\sum x)\left(\frac{\sum y - m(\sum x)}{n}\right) + (\sum x^2)m &= \sum xy \\ (\sum x)[(\sum y) - m(\sum x)] + nm(\sum x^2) &= n(\sum xy) \\ (\sum x)(\sum y) - m(\sum x)^2 + nm(\sum x^2) &= n(\sum xy) \\ nm(\sum x^2) - m(\sum x)^2 &= n(\sum xy) - (\sum x)(\sum y) \\ m\left[n(\sum x^2) - (\sum x)^2\right] &= n(\sum xy) - (\sum x)(\sum y) \\ m &= \frac{n(\sum xy) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2} \end{aligned}$$

6. (a)

$$\begin{aligned} 10b + 965m &= 95.3 \\ 10b + 965m &= 95.3 \\ 10b &= 95.3 - 965m \\ b &= \frac{95.3 - 965m}{10} \\ b &= 95.3 - 96.5m \end{aligned}$$

$$\begin{aligned} 965b + 93{,}205m &= 9165.1 \\ 965(9.53 - 96.5m) + 93{,}205m &= 9165.1 \\ 9196.45 - 93{,}122.5m + 93,205m &= 9165.1 \\ 82.5m &= -31.35 \\ m &= -.38 \end{aligned}$$

$$\begin{aligned} b &= 9.53 - 96.5(-.38) = 46.2 \\ Y &= -.38x + 46.2 \end{aligned}$$

(b) The year 2004 corresponds to $x = 104$.

$Y = -.38(104) + 46.2 = 6.68$ (in thousands)

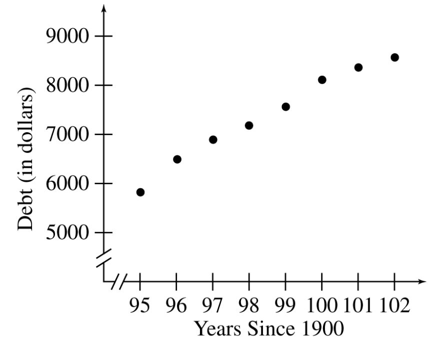
If the trend continues, there will be about 6680 banks in 2004.

(c) $$r = \frac{10(9165.1) - (965)(95.3)}{\sqrt{10(93{,}205) - 965^2} \cdot \sqrt{10(920.47) - 95.3^2}}$$

$$\approx -.986$$

This means that the least squares line fits the data points very well. The negative sign indicates that the number of banks is decreasing as the years increase.

8. (a)



Yes, the pattern is linear.

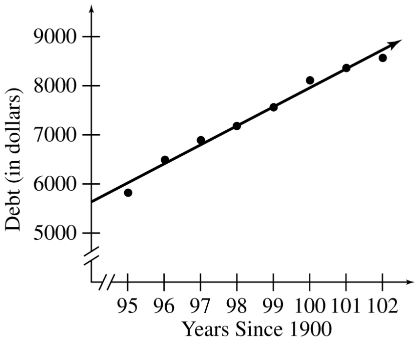
(b)

x	y	xy	x^2	y^2
95	5832	554, 040	9025	34, 012, 224
96	6487	622, 752	9216	42, 081, 169
97	6900	669, 300	9409	47, 610, 000
98	7188	704, 424	9604	51, 667, 344
99	7564	748, 836	9801	57, 214, 096
100	8123	812, 300	10, 000	65, 983, 129
101	8367	845, 067	10, 201	70, 006, 689
102	8562	873, 324	10, 404	73, 307, 844
788	59, 023	5, 830, 043	77, 660	441, 882, 495

$$\begin{aligned} 8b + 788m &= 59{,}023 \\ 8b &= 59{,}023 - 788m \\ b &= \frac{59{,}023 - 788m}{8} \end{aligned}$$

$$\begin{aligned} 788b + 77{,}660m &= 5{,}830{,}043 \\ 788\left(\frac{59{,}023 - 788m}{8}\right) + 77{,}660m &= 5{,}830{,}043 \\ 788(59{,}023 - 788m) + 621{,}280m &= 46{,}640{,}344 \\ 46{,}510{,}124 - 620{,}944m + 621{,}280\text{m} &= 46{,}640{,}344 \\ 336m &= 130{,}220 \\ m &\approx 387.56 \end{aligned}$$

$$\begin{aligned} b &= \frac{59{,}023 - 788\left(\frac{130{,}220}{336}\right)}{8} = -30{,}796.74 \\ Y &= 387.56x - 30{,}796.74 \end{aligned}$$



The line appears to be a good fit.

(c)

$$r = \frac{8(5{,}830{,}043) - (788)(59{,}023)}{\sqrt{8(77{,}660) - 788^2} \cdot \sqrt{8(441{,}882{,}495) - 59{,}023^2}}$$

$$\approx .991$$

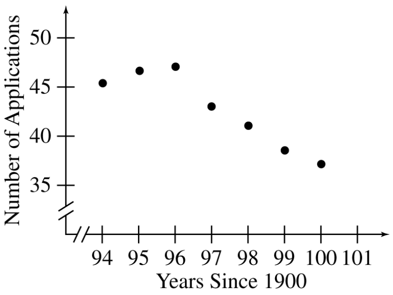
This indicates that the least squares line fits the data points very well.

(d) Let $Y = 10{,}000$ and solve for x.

$$\begin{aligned} 10,000 &= 387.56x - 30{,}796.74 \\ 40{,}796.74 &= 387.56x \\ x &\approx 105 \end{aligned}$$

If the trend continues, household debt will reach \$10,000 in 1900 + 105 = 2005.

10. **(a)**



If all points are included, the pattern is not linear.

(b)

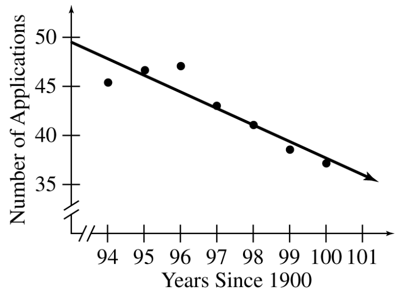
x	y	xy	x^2	y^2
94	45.4	4267.6	8836	2061.16
95	46.6	4427	9025	2171.56
96	47.0	4512	9216	2209
97	43.0	4171	9409	1849
98	41.0	4018	9604	1681
99	38.5	3811.5	9801	1482.25
100	37.1	3710	10,000	1376.41
679	298.6	28,917.1	65,891	12,830.38

$$\begin{aligned} 7b + 679m &= 298.6 \\ 7b &= 298.6 - 679m \\ b &= \frac{298.6 - 679m}{7} \end{aligned}$$

$$\begin{aligned} 679b + 65{,}891m &= 28{,}917.1 \\ 679\left(\frac{298.6 - 679m}{7}\right) + 65{,}891m &= 28{,}917.1 \\ 679(298.6 - 679m) + 461{,}237m &= 202{,}419.7 \\ 202{,}749.4 - 461{,}041m + 461{,}237m &= 202{,}419.7 \\ 196m &= -329.7 \\ m &\approx -1.682 \end{aligned}$$

$$b = \frac{298.6 - 679\left(\frac{-329.7}{196}\right)}{7} = 205.825$$

$$Y = -1.682x + 205.825$$



The line fits the data reasonable well.

(c) $r = \dfrac{7(28{,}917.1) - (679)(298.6)}{\sqrt{7(65{,}891) - (679)^2} \cdot \sqrt{7(12{,}830.38) - 298.6^2}}$

$r \approx -.923$

This indicates a reasonably good fit, as concluded in part (b).

(d) The line lies near most of the points. The most distant two are still relatively close to the line.

12. **(a)**

x	y	xy	x^2	y^2
88.6	20.0	1772	7849.96	400.0
71.6	16.0	1145.6	5126.56	256.0
93.3	19.8	1847.34	8704.89	392.04
84.3	18.4	1551.12	7106.49	338.56
80.6	17.1	1378.26	6496.36	292.41
75.2	15.5	1165.6	5655.04	240.25
69.7	14.7	1024.59	4858.09	216.09
82.0	17.1	1402.2	6724	292.41
69.4	15.4	1068.76	4816.36	237.16
83.3	16.2	1349.46	6938.89	262.44
79.6	15.0	1194	6336.16	225
82.6	17.2	1420.72	6822.76	295.84
80.6	16.0	1289.6	6496.36	256.0
83.5	17.0	1419.5	6972.25	289.0
76.3	14.4	1098.72	5821.69	207.36
1200.6	249.8	20,127.47	96,725.86	4200.56

$$\begin{aligned} 15b + 1200.6m &= 249.8 \\ 1200.6b + 96{,}725.86m &= 20{,}127.47 \end{aligned}$$

$$\begin{aligned} 15b &= 249.8 - 1200.6m \\ b &= \frac{249.8 - 1200.6m}{15} \end{aligned}$$

$$\begin{aligned} 1200.6\left(\frac{249.8 - 1200.6m}{15}\right) + 96{,}725.86m &= 20{,}127.47 \\ 1200.6(249.8 - 1200.6m) + 1{,}450{,}887.9m &= 301{,}912.05 \\ 299{,}909.88 - 1{,}441{,}440.36m &= 301{,}912.05 \\ +1{,}450{,}887.9m & \\ 9447.54m &= 2002.17 \\ m &\approx .212 \end{aligned}$$

$$b = \frac{249.8 - 1200.6(.212)}{15} = -.315$$

$$Y = .212x - .315$$

(b) Let $x = 73$; find Y.

$$\begin{aligned} Y &= .212(73) - .315 \\ &\approx 15.2 \end{aligned}$$

If the temperature were 73°F, you would expect to hear 15.2 chirps per second.

(c) Let $Y = 18$; find x.

$$\begin{aligned} 18 &= .212x - .315 \\ 18.315 &= .212x \\ 86.4 &\approx x \end{aligned}$$

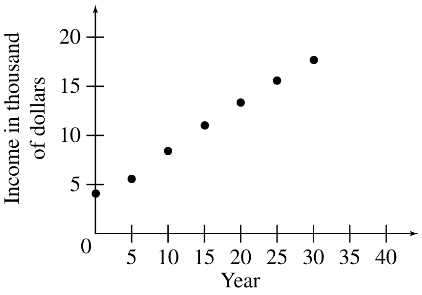
When the crickets are chirping 18 times per second, the temperature is 86.4°F.

(d)

$$r = \frac{15(20{,}127) - (1200.6)(249.8)}{\sqrt{15(96{,}725.86) - (1200.6)^2} \cdot \sqrt{15(4200.56) - (249.8)^2}}$$

$$= .835$$

14. (a)



Yes, the data appear to lie along a straight line.

(b)

$$r = \frac{7(1460.97) - (105)(75.402)}{\sqrt{7(2275) - 105^2} \cdot \sqrt{7(968.270792) - 75.402^2}}$$

$$r \approx .998$$

Yes, there is a strong positive linear correlation between the income and the year.

(c)

$$\begin{aligned} 7b + 105m &= 75.402 \\ 7b &= 75.402 - 105m \\ b &= \frac{75.402 - 105m}{7} \end{aligned}$$

$$\begin{aligned} 105b + 2275m &= 1460.97 \\ 105\left(\frac{75.402 - 105m}{7}\right) + 2275m &= 1460.97 \\ 105(75.402 - 105m) + 15{,}925m &= 10{,}226.79 \\ 7917.21 - 11{,}025m + 15{,}925m &= 10{,}226.79 \\ 4900m &= 2309.58 \\ m &\approx .471 \end{aligned}$$

$$b = \frac{75.402 - 105\left(\frac{2309.58}{4900}\right)}{7} \approx 3.702$$

$$Y = .471x + 3.702$$

(d) The year 2015 corresponds to $x = 45$.

$$\begin{aligned} Y &= .471(45) + 3.702 \\ Y &\approx 24.897 \end{aligned}$$

The predicted poverty level in the year 2015 is $24,897.

16. (a)

x	y	xy	x^2	y^2
150	5000	750,000	22,500	25,000,000
175	5500	962,500	30,625	30,250,000
215	6000	1,290,000	46,225	36,000,000
250	6500	1,625,000	62,500	42,250,000
280	7000	1,960,000	78,400	49,000,000
310	7500	2,325,000	96,100	56,250,000
350	8000	2,800,000	122,500	64,000,000
370	8500	3,145,000	136,900	72,250,000
420	9000	3,780,000	176,400	81,000,000
450	9500	4,275,000	202,500	90,250,000
2970	72,500	22,912,500	974,650	546,250,000

$$\begin{aligned} 10b + 2970m &= 72{,}500 \\ 2970b + 974{,}650m &= 22{,}912{,}500 \end{aligned}$$

$$\begin{aligned} 10b &= 72{,}500 - 2970m \\ b &= 7250 - 297m \end{aligned}$$

$$\begin{aligned} 2970(7250 - 297m) + 974{,}650m &= 22{,}912{,}500 \\ 21{,}532{,}500 - 882{,}090m + 974{,}650m &= 22{,}912{,}500 \\ 92{,}560m &= 1{,}380{,}000 \\ m &= 14.9 \end{aligned}$$

$$\begin{aligned} b &= 7250 - 297(14.9) \approx 2820 \\ Y &= 14.9x + 2820 \end{aligned}$$

(b) Let $x = 150$; find Y.

$$\begin{aligned} Y &= 14.9(150) + 2820 \\ Y &\approx 5060, \text{ compared to actual } 5000 \end{aligned}$$

Let $x = 280$; find Y.

$$\begin{aligned} Y &= 14.9(280) + 2820 \\ &\approx 6990, \text{ compared to actual } 7000 \end{aligned}$$

Let $x = 420$; find Y.

$$\begin{aligned} Y &= 14.9(420) + 2820 \\ &\approx 9080, \text{ compared to actual } 9000 \end{aligned}$$

(c) Let $x = 230$; find Y.

$$\begin{aligned} Y &= 14.9(230) + 2820 \\ &\approx 6250 \end{aligned}$$

Adam would need to buy a 6500 BTU air conditioner.

18. (a)

x	y	xy	x^2	y^2
5	113.4	567	25	12,859.56
15	111.9	1678.5	225	12,521.61
25	111.9	2797.5	625	12,521.61
35	109.7	3839.5	1225	12,034.09
45	106.6	4797	2025	11,363.56
55	105.7	5813.5	3025	11,172.49
65	104.3	6779.5	4225	10,878.49
75	103.7	7777.5	5625	10,753.69
85	101.73	8647.05	7225	10,348.9929
95	101.11	9605.45	9025	10,223.2321
500	1070.04	52,302.5	33,250	114,677.325

$$\begin{aligned}
10b + 500m &= 1070.04\\
500b + 33{,}250\mathrm{m} &= 52{,}302.5\\
10b &= 1070.04 - 500m\\
b &= 107.004 - 50m\\
500(107.004 - 50m) + 33{,}250m &= 52{,}302.5\\
53{,}502 - 25{,}000m + 33{,}250m &= 52{,}302.5\\
8250m &= -1199.5\\
m &= -.1454
\end{aligned}$$

$$\begin{aligned}
b &= 107.004 - 50(-.1454)\\
b &\approx 114.27\\
Y &= -.1454x + 114.27
\end{aligned}$$

(b)

x	y	xy	x^2	y^2
25	144.0	3600.0	625	20, 736
35	135.6	4746	1225	18, 387.36
45	132.0	5940.0	2025	17, 424
55	125.0	6875.0	3025	15, 625
65	118.0	7670.0	4225	13, 924
75	117.48	8811	5625	13,801.5504
85	113.28	9628.8	7225	12, 832.3584
95	113.28	10,761.6	9025	12, 832.3584
480	998.64	58,032.4	33,000	125,562.6272

$$\begin{aligned}
8b + 480m &= 998.64\\
480b + 33{,}000m &= 58{,}032.4\\
8b &= 998.64 - 480m\\
b &= 124.83 - 60m\\
480(124.83 - 60m) + 33{,}000m &= 58{,}032.4\\
59{,}918.4 - 28{,}800m + 33{,}000m &= 58{,}032.4\\
4200m &= -1886\\
m &= -.4490
\end{aligned}$$

$$\begin{aligned}
b &= 124.83 - 60(-.4490)\\
b &= 151.77\\
Y &= -.4490x + 151.77
\end{aligned}$$

(c)
$$\begin{aligned} -.1454x + 114.27 &= -.4490x + 151.77 \\ .3036x &= 37.5 \\ x &\approx 123.52 \approx 124 \\ 1900 + 124 &= 2024 \end{aligned}$$

The women will catch up to the men in the year 2024.

(d)
$$r_{\text{men}} = \frac{10(52{,}302.5) - (500)(1070.04)}{\sqrt{10(33{,}250) - 500^2} \cdot \sqrt{10(114{,}677.325) - (1070.04)^2}} = -.9877$$
$$r_{\text{women}} = \frac{8(58{,}032.4) - (480)(998.64)}{\sqrt{8(33{,}000) - 480^2} \cdot \sqrt{8(125{,}562.6272) - (998.64)^2}} = -9.688$$

Both sets of points closely fit a line with negative slope.

20. (a)

x	y	xy	x^2
1	33	33	1
2	34	68	4
3	36	108	9
4	35	140	16
5	40	200	25
6	44	264	36
7	48	336	49
8	45	360	64
9	46	414	81
10	48	480	100
11	49	539	121
12	49	588	144
13	48	624	169
14	54	756	196
15	57	855	225
120	666	5765	1240

$$\begin{aligned} 15b + 120m &= 666 \\ 120b + 1240m &= 5765 \\ 15b &= 666 - 120m \\ b &= \frac{666 - 120m}{15} \end{aligned}$$

$$\begin{aligned} 120\left(\frac{666 - 120m}{15}\right) + 1240m &= 5765 \\ 8(666 - 120m) + 1240m &= 5765 \\ 5328 - 960m + 1240m &= 5765 \\ 280m &= 437 \\ m &\approx 1.5607 \end{aligned}$$

$$b = \frac{666 - 120(1.5607)}{15} \approx 31.914$$

$$Y = 1.5607x + 31.914$$

(b) Let $Y = 75$ (1 hour and 15 minutes beyond 2 hours); find x.

$$\begin{aligned}75 &= 1.5607x + 31.914\\ 43.086 &= 1.5607x\\ 27.61 &\approx x\end{aligned}$$

If the trend continues, the average completion time will be 3 hours and 15 minutes in the year $1980 + 27.61 \approx 2008$.

Chapter 1 Review Exercises

2. To complete the coefficient of correlation, you need to compute the following quantities: $\sum x$, $\sum y$, $\sum xy$, $\sum x^2$, $\sum y^2$, and n.

4. Through $(4, -1)$ and $(3, -3)$.

$$\begin{aligned}m &= \frac{-3-(-1)}{3-4}\\ &= \frac{-3+1}{-1}\\ &= \frac{-2}{-1} = 2\end{aligned}$$

6. Through the origin and $(0, 7)$

$$m = \frac{7-0}{0-0} = \frac{7}{0}$$

The slope of the line is undefined.

8. $4x - y = 7$

$$\begin{aligned}-y &= -4x + 7\\ y &= 4x - 7\\ m &= 4\end{aligned}$$

10. $3y - 1 = 14$

$$\begin{aligned}3y &= 14 + 1\\ 3y &= 15\\ y &= 5\end{aligned}$$

This is a horizontal line. The slope of a horizontal line is 0.

12. $x = 5y$

$$\begin{aligned}\frac{1}{5}x &= y\\ m &= \frac{1}{5}\end{aligned}$$

14. Through $(8, 0)$, with slope $-\frac{1}{4}$

Use point-slope form.

$$\begin{aligned}y - 0 &= -\frac{1}{4}(x-8)\\ y &= -\frac{1}{4}x + 2\end{aligned}$$

16. Through $(2, -3)$ and $(-3, 4)$

$$m = \frac{4-(-3)}{-3-2} = -\frac{7}{5}$$

Use point-slope form.

$$\begin{aligned}y-(-3) &= -\frac{7}{5}(x-2)\\ y+3 &= -\frac{7}{5}x + \frac{14}{5}\\ y &= -\frac{7}{5}x + \frac{14}{5} - 3\\ y &= -\frac{7}{5}x + \frac{14}{5} - \frac{15}{5}\\ y &= -\frac{7}{5}x - \frac{1}{5}\end{aligned}$$

18. Through $(-2, 5)$, with slope 0

Horizontal lines have 0 slope and an equation of the form $y = k$.
The line passes through $(-2, 5)$ so $k = 5$. An equation of the line is $y = 5$.

20. Through $(0, 5)$, perpendicular to $8x + 5y = 3$
Find the slope of the given line first.

$$\begin{aligned}8x + 5y &= 3\\ 5y &= -8x + 3\\ y &= \frac{-8}{5}x + \frac{3}{5}\\ m &= -\frac{8}{5}\end{aligned}$$

The perpendicular line has $m = \frac{5}{8}$.
Use point-slope form.

$$\begin{aligned}y - 5 &= \frac{5}{8}(x-0)\\ y &= \frac{5}{8}x + 5\end{aligned}$$

22. Through $(3, -5)$, parallel to $y = 4$

Find the slope of the given line.
$y = 0x + 4$, so $m = 0$, and the required line will also have slope 0.
Use the point-slope form.

$$\begin{aligned} y - (-5) &= 0(x - 3) \\ y + 5 &= 0 \\ y &= -5 \end{aligned}$$

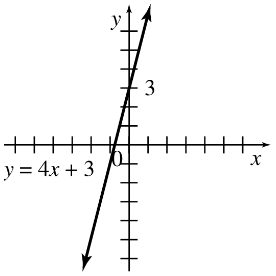
24. $y = 4x + 3$

Let $x = 0$. $\quad y = 4(0) + 3$
$y = 3$

Let $y = 0$. $\quad 0 = 4x + 3$

$$\begin{aligned} -3 &= 4x \\ -\frac{3}{4} &= x \end{aligned}$$

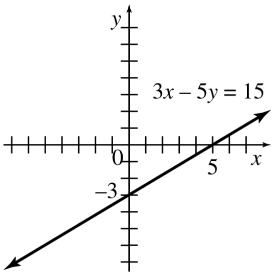
Draw the line through $(0, 3)$ and $\left(-\frac{3}{4}, 0\right)$.



26. $3x - 5y = 15$

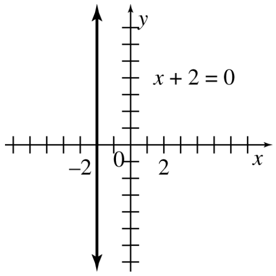
$$\begin{aligned} -5y &= -3x + 15 \\ y &= \frac{3}{5}x - 3 \end{aligned}$$

When $x = 0$, $y = -3$.
When $y = 0$, $x = 5$.
Draw the line through $(0, -3)$ and $(5, 0)$.



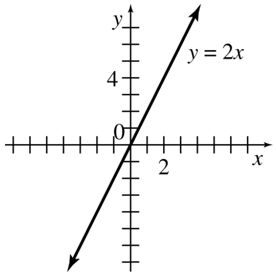
28. $x + 2 = 0$
$x = -2$

This is the vertical line through $(-2, 0)$.



30. $y = 2x$

When $x = 0$, $y = 0$.
When $x = 1$, $y = 2$.
Draw the line through $(0, 0)$ and $(1, 2)$.



32. **(a)** $E = 352 + 42x$ (where x is in thousands)

(b) $R = 130x$ (where x is in thousands)

(c)
$$\begin{aligned} R &> E \\ 130x &> 352 + 42x \\ 88x &> 352 \\ x &> 4 \end{aligned}$$

For a profit to be made, more than 4000 chips must be sold.

34. Using the points $(60, 40)$ and $(100, 60)$,

$$m = \frac{60 - 40}{100 - 60} = \frac{20}{40} = .5.$$

$$\begin{aligned} p - 40 &= .5(q - 60) \\ p - 40 &= .5q - 30 \\ p &= .5q + 10 \\ S(q) &= .5q + 10 \end{aligned}$$

36.
$$\begin{aligned} S(q) &= D(q) \\ .5q + 10 &= -.5q + 72.50 \\ q &= 62.5 \end{aligned}$$

$S(62.5) = .5(62.5) + 10 = 31.25 + 10 = 41.25$

The equilibrium price is \$41.25, and the equilibrium quantity is 62.5 diet pills.

38. Fixed cost is \$2000; 36 units cost \$8480.
Two points on the line are $(0, 2000)$ and $(36, 8480)$, so

$$m = \frac{8480 - 2000}{36 - 0} = \frac{6480}{36} = 180.$$

Use point-slope form.

$$\begin{aligned} y &= 180x + 2000 \\ C(x) &= 180x + 2000 \end{aligned}$$

40. Thirty units cost \$1500; 120 units cost \$5640.
Two points on the line are $(30, 1500)$, $(120, 5640)$, so

$$m = \frac{5640 - 1500}{120 - 30} = \frac{4140}{90} = 46.$$

Use point-slope form.

$$\begin{aligned} y - 1500 &= 46(x - 30) \\ y &= 46x - 1380 + 1500 \\ y &= 46x + 120 \\ C(x) &= 46x + 120 \end{aligned}$$

42. **(a)** $C(x) = 3x + 160$; $R(x) = 7x$

$$\begin{aligned} C(x) &= R(x) \\ 3x + 160 &= 7x \\ 160 &= 4x \\ 40 &= x \end{aligned}$$

The break-even quantity is 40 pounds.

(b) $R(40) = 7 \cdot 40 = \$280$

The revenue for 40 pounds is \$280.

44. Using the points (91, 6) and $(101, 19)$,

$$m = \frac{19 - 6}{101 - 01} = \frac{13}{10} = 1.3$$

$$\begin{aligned} y - 10 &= 1.3(x - 91) \\ y - 10 &= 1.3 - 118.3 \\ y &= 1.3 - 108.3 \end{aligned}$$

46. **(a)**

x	y	xy	x^2	y^2
75	6000	450,000	5625	36,000,000
80	7500	600,000	6400	56,250,000
85	12,000	1,020,000	7225	144,000,000
90	16,000	1,440,000	8100	256,000,000
95	20,400	1,938,000	9025	416,160,000
100	24,900	2,490,000	10,000	620,010,000
525	86,800	7,938,000	46,375	1,528,420,000

$$\begin{aligned} 6b + 525m &= 86{,}800 \\ 6b &= 86{,}800 - 525m \\ b &= \frac{86{,}800 - 525m}{6} \end{aligned}$$

$$\begin{aligned} 525b + 46{,}375m &= 7{,}938{,}000 \\ 525\left(\frac{86{,}800 - 525m}{6}\right) + 46{,}375m &= 7{,}938{,}000 \\ 525(86{,}800 - 525m) + 278{,}250m &= 47{,}628{,}000 \\ 45{,}570{,}000 - 275{,}625m + 278{,}250m &= 47{,}628{,}000 \\ 2625m &= 2{,}058{,}000 \\ m &= 784 \end{aligned}$$

$$\begin{aligned} b &= \frac{86{,}800 - 525(784)}{6} \approx -54{,}133.333 \\ Y &= 784x - 54{,}133.33 \end{aligned}$$

(b) $Y = 784(105) - 54{,}133.33$
$\approx 28{,}186.67$

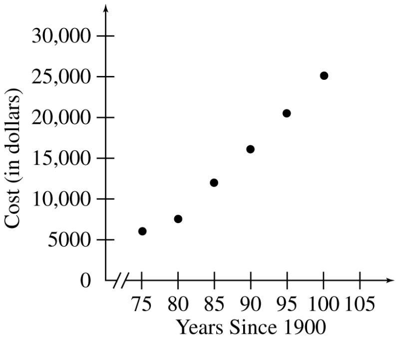
The average cost of a new car in the year 2005 is predicted to be about \$28,187.

(c)

$$r = \frac{6(7{,}938{,}000) - (525)(86{,}800)}{\sqrt{6(46{,}375) - 525^2} \cdot \sqrt{6(1{,}528{,}420{,}000) - 86{,}800^2}}$$
$\approx .993$

Yes, this indicates that the line fits the data points quite well.

(d)



No, the scatterplot suggests that the trend is linear.

48. **(a)**

x	y	xy	x^2	y^2
130	170	22,100	16,900	28,900
138	160	22,080	19,044	25,600
142	173	24,566	20,164	29,929
159	181	28,779	25,281	32,761
165	201	33,165	27,225	40,401
200	192	38,400	40,000	36,864
210	240	50,400	44,100	57,600
250	290	72,500	62,500	84,100
1394	1607	291,990	255,214	336,155

$$\begin{aligned} 8b + 1394m &= 1607 \\ 1394b + 255{,}214m &= 291,990 \end{aligned}$$

$$\begin{aligned} 8b &= 1607 - 1394m \\ b &= \frac{1607-1394m}{8} \end{aligned}$$

$$\begin{aligned} 1394\left(\frac{1607-1394m}{8}\right) + 255{,}214m &= 291{,}990 \\ 1394(1607-1394m) + 2{,}041{,}712m &= 2{,}335{,}920 \\ 2{,}240{,}158 - 1{,}943{,}236m + 2{,}041{,}712m &= 2{,}335{,}920 \\ 98{,}476m &= 95{,}762 \\ m &= .9724400 \\ &\approx .97 \end{aligned}$$

$$\begin{aligned} b &= \frac{1607-1394(.97)}{8} \approx 31.85 \\ Y &= .97x + 31.85 \end{aligned}$$

(b) Let $x = 190$; find Y.

$$\begin{aligned} Y &= .97(190) + 31.85 \\ Y &= 216.15 \approx 216 \end{aligned}$$

The cholesterol level for a person whose blood sugar level is 190 would be about 216.

(c)

$$\begin{aligned} r &= \frac{8(291{,}990) - (1394)(1607)}{\sqrt{8(255{,}214) - 1394^2} \cdot \sqrt{8(336{,}155) - 1607^2}} \\ &= .933814 \approx .93 \end{aligned}$$

50. Using the points $(90, 52.6)$ and $(100, 59.9)$,

$$m = \frac{59.9 - 52.6}{100 - 90} = \frac{7.3}{10} = .73$$

$$\begin{aligned} y - 59.9 &= .73(x - 100) \\ y - 59.9 &= .73x - 73 \\ y &= .73x - 13.1 \end{aligned}$$

Extended Application: Using Extrapolation to Predict Life Expectancy

1.

x	y	xy	x^2	y^2
1950	71.3	139,035	3,802,500	5083.69
1960	73.1	143,276	3,841,600	5343.61
1970	74.7	147,159	3,880,900	5580.09
1980	77.4	153,252	3,920,400	5990.76
1985	78.2	155,227	3,940,225	6115.24
1990	78.8	156,812	3,960,100	6209.44
1995	78.9	157,406	3,980,025	6225.21
13,830	532.4	1,052,167	27,325,750	40,548

$$\begin{aligned} 7b + 13{,}830m &= 532.4 \\ 13{,}830b + 27{,}325{,}750m &= 1{,}052{,}167 \\ 7b &= 532.4 - 13{,}830m \\ b &= \frac{532.4 - 13{,}830m}{7} \end{aligned}$$

$$\begin{aligned} 13{,}830\left(\frac{532.4 - 13{,}830m}{7}\right) + 27{,}325{,}750m &= 1{,}052{,}167 \\ 13{,}830(532.4 - 13{,}830m) + 191{,}280{,}250m &= 7{,}365{,}169 \\ 7{,}363{,}092 - 191{,}268{,}900m + 191{,}280{,}250m &= 7{,}365{,}169 \\ 11{,}350m &= 2077 \\ m &\approx .183 \end{aligned}$$

$$\begin{aligned} b &= \frac{532.4 - 13{,}830(.183)}{7} \approx -285 \\ Y &= -285 + .183x \end{aligned}$$

2. Let $x = 1900$

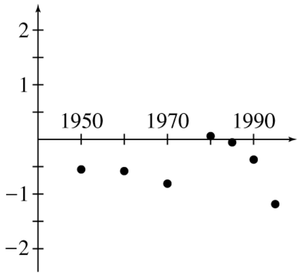
$$\begin{aligned} Y &= -285 + .183(1900) \\ Y &= 62.7 \end{aligned}$$

From the equation, the guess is the life expectancy of females born is 1900 is 62.7 years.

3. The poor prediction isn't surprising, since we were extrapolating far beyond the range of the original data.

4.

x	Predicted value	Residual
1950	71.850	−.550
1960	73.680	−.580
1970	75.510	−.810
1980	77.340	.060
1985	78.255	−.055
1990	79.170	−.370
1995	80.085	−1.185



5. It's not clear that any simple smooth function will fit this data–there seems to be a break in the pattern between 1970 and 1980. This will make it difficult to predict the life expectancy for females born in 2010.

6. You'll get 0 slope and 0 intercept, because you've already subtracted out the linear component of the data.

7. They used a regression equation of some kind to predict this value!

Chapter 2

SYSTEMS OF LINEAR EQUATIONS AND MATRICES

2.1 Solution of Linear Systems by the Echelon Method

In Exercises 2-16 and 20-26, check each solution by substituting it in the original equation of the system.

2. $4x + y = 9 \quad (1)$
$3x - y = 5 \quad (2)$

First use transformation 3 to eliminate the x-term from equation (2). Multiply equation (1) by 3 and add the result to -4 times equation (2).

$$\begin{array}{lrl} & 4x + y = 9 & (1) \\ 3\text{R}_1 + (-4)\text{R}_2 \to \text{R}_2 & 7y = 7 & (3) \end{array}$$

Now use transformation 2 to make the coefficient of the first term in each equation equal to 1.

$$\begin{array}{lrl} \frac{1}{4}\text{R}_1 \to \text{R}_1 & x + \dfrac{1}{4}y = \dfrac{9}{4} & (4) \\ \frac{1}{7}\text{R}_2 \to \text{R}_2 & y = 1 & (5) \end{array}$$

Complete the solution by back-substitution. Substitute 1 for y in equation (4) to get

$$\begin{aligned} x + \frac{1}{4}(1) &= \frac{9}{4} \\ x &= \frac{8}{4} = 2. \end{aligned}$$

The solution is $(2, 1)$.

4. $2x + 7y = -8 \quad (1)$
$-2x + 3y = -12 \quad (2)$

$$\begin{array}{lrl} & 2x + 7y = -8 & (1) \\ \text{R}_1 + \text{R}_2 \to \text{R}_2 & 10y = -20 & (3) \end{array}$$

Make each leading coefficient equal 1.

$$\begin{array}{lrl} \frac{1}{2}\text{R}_1 \to \text{R}_1 & x + \dfrac{7}{2}y = -4 & (4) \\ \frac{1}{10}\text{R}_2 \to \text{R}_2 & y = -2 & (5) \end{array}$$

Substitute -2 for y in equation (4).

$$\begin{aligned} x + \frac{7}{2}(-2) &= -4 \\ x - 7 &= -4 \\ x &= 3 \end{aligned}$$

The solution is $(3, -2)$.

6. $-6x + 2y = 8$
$5x - 2y = -8$

$$\begin{array}{lr} & -6x + 2y = 8 \\ 5\text{R}_1 + 6\text{R}_2 \to \text{R}_2 & -2y = -8 \end{array}$$

Make each leading coefficient equal 1.

$$\begin{array}{lr} -\frac{1}{6}\text{R}_1 \to \text{R}_1 & x - \dfrac{1}{3}y = -\dfrac{4}{3} \\ -\frac{1}{2}\text{R}_2 \to \text{R}_2 & y = 4 \end{array}$$

Back-substitution gives

$$\begin{aligned} x - \frac{1}{3}(4) &= -\frac{4}{3} \\ x - \frac{4}{3} &= -\frac{4}{3} \\ x &= 0. \end{aligned}$$

The solution is $(0, 4)$.

8. $4m + 3n = -1$
$2m + 5n = 3$

$$\begin{array}{lr} & 4m + 3n = -1 \\ \text{R}_1 + (-2)\text{R}_2 \to \text{R}_2 & -7n = -7 \end{array}$$

Make each leading coefficient equal 1.

$$\begin{array}{lr} \frac{1}{4}\text{R}_1 \to \text{R}_1 & m + \dfrac{3}{4}n = -\dfrac{1}{4} \\ -\frac{1}{7}\text{R}_2 \to \text{R}_2 & n = 1 \end{array}$$

Back-substitution gives

$$\begin{aligned} m + \frac{3}{4}(1) &= -\frac{1}{4} \\ m &= -\frac{4}{4} = -1. \end{aligned}$$

The solution is $(-1, 1)$.

10. $12s - 5t = 9$
$3s - 8t = -18$

$$\begin{array}{lr} & 12s - 5t = 9 \\ \text{R}_1 + (-4)\text{R}_2 \to \text{R}_2 & 27t = 81 \\ \frac{1}{12}\text{R}_1 \to \text{R}_1 & s - \dfrac{5}{12}t = \dfrac{3}{4} \\ \frac{1}{27}\text{R}_2 \to \text{R}_2 & t = 3 \end{array}$$

Back substitution gives

$$\begin{aligned} s - \frac{5}{12}(3) &= \frac{3}{4} \\ s - \frac{5}{4} &= \frac{3}{4} \\ s &= \frac{8}{4} = 2. \end{aligned}$$

The solution is $(2, 3)$.

12. $\begin{aligned} 2a + 9b &= 3 \\ 5a + 7b &= -8 \end{aligned}$

$$\begin{aligned} & & 2a + 9b &= 3 \\ 5\text{R}_1 + (-2)\text{R}_2 &\rightarrow \text{R}_2 & 31b &= 31 \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}\text{R}_1 &\rightarrow \text{R}_1 & a + \frac{9}{2}b &= \frac{3}{2} \\ \tfrac{1}{31}\text{R}_2 &\rightarrow \text{R}_2 & b &= 1 \end{aligned}$$

Back-substitution gives

$$\begin{aligned} a + \frac{9}{2}(1) &= \frac{3}{2} \\ a &= -\frac{6}{2} = -3. \end{aligned}$$

The solution is $(-3, 1)$.

14. $\begin{aligned} 9x - 5y &= 1 \\ -18x + 10y &= 1 \end{aligned}$

$$\begin{aligned} & & 9x - 5y &= 1 \\ 2\text{R}_1 + \text{R}_2 &\rightarrow \text{R}_2 & 0 &= 3 \end{aligned}$$

The equation $0 = 3$ is a false statement, which indicates that the system is inconsistent and has no solution.

16. $\begin{aligned} 3x + 5y + 2 &= 0 \\ 9x + 15y + 6 &= 0 \end{aligned}$

Begin by rewriting the equations in standard form.

$$\begin{aligned} 3x + 5y &= -2 \\ 9x + 15y &= -6 \end{aligned}$$

$$\begin{aligned} & & 3x + 5y &= -2 \\ 3\text{R}_1 + (-1)\text{R}_2 &\rightarrow \text{R}_2 & 0 &= 0 \end{aligned}$$

The true statement, $0 = 0$, shows that the two equations have the same graph, which means that there are an infinite number of solutions for the system. All ordered pairs that satisfy the equation $3x + 5y = -2$ are solutions. Solve this equation for x.

$$\begin{aligned} 3x &= -5y - 2 \\ x &= \frac{-5y - 2}{3} \end{aligned}$$

The general solution is the set of all ordered pairs of the form

$$\left(\frac{-5y - 2}{3}, y\right),$$

where y is any real number.

18. The solution of a system with two dependent equations in two variables is *an infinite set of ordered pairs*.

20. $\begin{aligned} \frac{x}{5} + 3y &= 31 \\ 2x - \frac{y}{5} &= 8 \end{aligned}$

Multiply each equation by 5 to eliminate fractions.

$$\begin{aligned} x + 15y &= 155 \\ 10x - y &= 40 \end{aligned}$$

$$\begin{aligned} & & x + 15y &= 155 \\ 10\text{R}_1 + (-1)\text{R}_2 &\rightarrow \text{R}_2 & 151y &= 1510 \end{aligned}$$

$$\begin{aligned} & & x + 15y &= 155 \\ \tfrac{1}{151}\text{R}_2 &\rightarrow \text{R}_2 & y &= 10 \end{aligned}$$

Back-substitution gives

$$\begin{aligned} x + 15(10) &= 155 \\ x &= 5. \end{aligned}$$

The solution is $(5, 10)$.

22. $x + \dfrac{y}{3} = -6$ *(1)*

$\dfrac{x}{5} + \dfrac{y}{4} = -\dfrac{7}{4}$ *(2)*

Multiply equation (1) by 3 and equation (2) by 20 to eliminate fractions. This gives the system

$$\begin{aligned} 3x + y &= -18 \\ 4x + 5y &= -35. \end{aligned}$$

Now proceed as usual.

$$\begin{aligned} & & 3x + y &= -18 \\ 4\text{R}_1 + (-3)\text{R}_2 &\rightarrow \text{R}_2 & -11y &= 33 \end{aligned}$$

$$\begin{aligned} \tfrac{1}{3}\text{R}_1 &\rightarrow \text{R}_1 & x + \frac{1}{3}y &= -6 \\ -\tfrac{1}{11}\text{R}_2 &\rightarrow \text{R}_2 & y &= -3 \end{aligned}$$

Finally, back-substitution gives

$$\begin{aligned} x + \frac{1}{3}(-3) &= -6 \\ x - 1 &= -6 \\ x &= -5. \end{aligned}$$

The solution is $(-5, -3)$.

24. $$\begin{aligned} 2x+y+z&=9 && (\mathit{1})\\ -x-y+z&=1 && (\mathit{2})\\ 3x-y+z&=9 && (\mathit{3}) \end{aligned}$$

First, eliminate the x-terms from equations (2) and (3).

$$\begin{array}{lrl} & 2x+y+z = 9 & (\mathit{1})\\ \mathrm{R}_1+2\mathrm{R}_2\rightarrow\mathrm{R}_2 & -y+3z = 11 & (\mathit{4})\\ 3\mathrm{R}_1+(-2)\mathrm{R}_3\rightarrow\mathrm{R}_3 & 5y+z = 9 & (\mathit{5}) \end{array}$$

Next, eliminate the y-term from equation (5).

$$\begin{array}{lrl} & 2x+y+z = 9 & (\mathit{1})\\ & -y+3z = 11 & (\mathit{4})\\ 5\mathrm{R}_2+\mathrm{R}_3\rightarrow\mathrm{R}_3 & 16z = 64 & (\mathit{6}) \end{array}$$

Now make the coefficient of the first term in each equation equal to 1.

$$\begin{array}{lrl} \frac{1}{2}\mathrm{R}_1\rightarrow\mathrm{R}_1 & x+\dfrac{1}{2}y+\dfrac{1}{2}z = \dfrac{9}{2} & (\mathit{7})\\ -1\mathrm{R}_2\rightarrow\mathrm{R}_2 & y-3z = -11 & (\mathit{8})\\ \frac{1}{16}\mathrm{R}_3\rightarrow\mathrm{R}_3 & z = 4 & (\mathit{9}) \end{array}$$

Complete the solution by back-substitution. Substitute 4 for z in equation (8) to find y.

$$\begin{aligned} y-3(4)&=-11\\ y-12&=-11\\ y&=1 \end{aligned}$$

Finally, substitute 1 for y and 4 for z in equation (7) to find x.

$$\begin{aligned} x+\frac{1}{2}(1)+\frac{1}{2}(4)&=\frac{9}{2}\\ x+\frac{5}{2}&=\frac{9}{2}\\ x&=\frac{4}{2}=2 \end{aligned}$$

The solution is $(2,1,4)$.

26. $$\begin{aligned} 4x-y+3z&=-2 && (\mathit{1})\\ 3x+5y-z&=15 && (\mathit{2})\\ -2x+y+4z&=14 && (\mathit{3}) \end{aligned}$$

First, eliminate the x-terms from equations (2) and (3).

$$\begin{array}{lrl} & 4x-y+3z = -2 & (\mathit{1})\\ -3\mathrm{R}_1+4\mathrm{R}_2\rightarrow\mathrm{R}_2 & 23y-13z = 66 & (\mathit{4})\\ \mathrm{R}_1+2\mathrm{R}_3\rightarrow\mathrm{R}_3 & y+11z = 26 & (\mathit{5}) \end{array}$$

Next, eliminate the y-term from equation (5).

$$\begin{array}{lrl} & 4x-y+3z = -2 & (\mathit{1})\\ & 23y-13z = 66 & (\mathit{4})\\ \mathrm{R}_2+(-23)\mathrm{R}_3\rightarrow\mathrm{R}_3 & -266z = -532 & (\mathit{6}) \end{array}$$

Now make the coefficient of the first term in each equation equal to 1.

$$\begin{array}{lrl} \frac{1}{4}\mathrm{R}_1\rightarrow\mathrm{R}_1 & x-\dfrac{1}{4}y+\dfrac{3}{4}z = -\dfrac{1}{2} & (\mathit{7})\\ \frac{1}{23}\mathrm{R}_2\rightarrow\mathrm{R}_2 & y-\dfrac{13}{23}z = \dfrac{66}{23} & (\mathit{8})\\ -\frac{1}{266}\mathrm{R}_3\rightarrow\mathrm{R}_3 & z = 2 & (\mathit{9}) \end{array}$$

Complete the solution by back-substitution. Substitute 2 for z in equation (8) to find y.

$$\begin{aligned} y-\frac{13}{23}(2)&=\frac{66}{23}\\ y-\frac{26}{23}&=\frac{66}{23}\\ y&=\frac{92}{23}=4 \end{aligned}$$

Finally, substitute 4 for y and 2 for z in equation (7) to find x.

$$\begin{aligned} x-\frac{1}{4}(4)+\frac{3}{4}(2)&=-\frac{1}{2}\\ x+\frac{1}{2}&=-\frac{1}{2}\\ x&=-1 \end{aligned}$$

The solution is $(-1,4,2)$.

28. $$\begin{aligned} 5x+3y+4z&=19 && (\mathit{1})\\ 3x-y+z&=-4 && (\mathit{2}) \end{aligned}$$

$$\begin{array}{lrl} & 5x+3y+4z = 19 & (\mathit{1})\\ 3\mathrm{R}_1+(-5)\mathrm{R}_2\rightarrow\mathrm{R}_2 & 14y+7z = 77 & (\mathit{3}) \end{array}$$

Since there are only two equations, it is not possible to continue with the echelon method as in the previous exercises involving systems with three equations and three variables. To complete the solution, make the coefficient of the first term in each equation equal 1.

$$\begin{array}{lrl} \frac{1}{5}\mathrm{R}_1\rightarrow\mathrm{R}_1 & x+\dfrac{3}{5}y+\dfrac{4}{5}z = \dfrac{19}{5} & (\mathit{4})\\ \frac{1}{14}\mathrm{R}_2\rightarrow\mathrm{R}_2 & y+\dfrac{1}{2}z = \dfrac{11}{2} & (\mathit{5}) \end{array}$$

Solve equation (5) for y in terms of the parameter z.

$$\begin{aligned} y&=\frac{11}{2}-\frac{1}{2}z\\ \text{or}\quad y&=\frac{11-z}{2} \end{aligned}$$

Substitute the first expression for y in equation (4) to solve for x in terms of the parameter z.

$$x+\frac{3}{5}\left(\frac{11}{2}-\frac{1}{2}z\right)+\frac{4}{5}z=\frac{19}{5}$$

$$\begin{aligned} x+\frac{33}{10}-\frac{3}{10}z+\frac{4}{5}z&=\frac{19}{5}\\ x+\frac{33}{10}-\frac{3}{10}z+\frac{8}{10}z&=\frac{38}{10}\\ x+\frac{5}{10}z&=\frac{5}{10}\\ x&=\frac{1}{2}-\frac{1}{2}z\\ \text{or}\quad x&=\frac{1-z}{2} \end{aligned}$$

The solution is $\left(\frac{1-z}{2},\frac{11-z}{2},z\right)$.

30.
$$\begin{aligned} x+2y+3z&=11 \quad \textit{(1)}\\ 2x-\ \ y+\ \ z&=\ \ 2 \quad \textit{(2)} \end{aligned}$$

$$\begin{array}{rl} & x+2y+3z=\ \ 11 \quad \textit{(1)}\\ -2\text{R}_1+\text{R}_2\rightarrow\text{R}_2 & \quad -5y-5z=-20 \quad \textit{(3)} \end{array}$$

$$\begin{array}{rl} & x+2y+3z=11 \quad \textit{(1)}\\ -\frac{1}{5}\text{R}_2\rightarrow\text{R}_2 & \qquad y+\ \ z=\ \ 4 \quad \textit{(4)} \end{array}$$

Since there are only two equations, it is not possible to continue with the echelon method. To complete the solution, solve equation (4) for y in terms of the parameter z.

$$y=4-z$$

Now substitute $4-z$ for y in equation (1) and solve for x in terms of z.

$$\begin{aligned} x+2(4-z)+3z&=11\\ x+8-2z+3z&=11\\ x&=3-z \end{aligned}$$

The solution is $(3-z,4-z,z)$.

32.
$$\begin{aligned} nb+(\textstyle\sum x)m&=\textstyle\sum y \quad \textit{(1)}\\ (\textstyle\sum x)b+(\textstyle\sum x^2)m&=\textstyle\sum xy \quad \textit{(2)} \end{aligned}$$

Multiply equation (1) by $\frac{1}{n}$.

$$b+\frac{\sum x}{n}m=\frac{\sum y}{n} \quad \textit{(3)}$$
$$(\textstyle\sum x)b+(\sum x^2)m=\sum xy \quad \textit{(2)}$$

Eliminate b from equation (2).

$$b+\frac{\sum x}{n}m=\frac{\sum y}{n} \qquad \textit{(3)}$$

$(-\sum x)\text{R}_1+\text{R}_2\rightarrow R_2$

$$\left[-\frac{(\sum x)^2}{n}+\sum x^2\right]m=\frac{-(\sum x)(\sum y)}{n}+\sum xy \ \textit{(4)}$$

Multiply equation (4) by $\dfrac{1}{-\frac{(\sum x)^2}{n}+\sum x^2}$.

$$b+\frac{\sum x}{n}m=\frac{\sum y}{n} \quad \textit{(3)}$$

$$m=\left[\frac{-(\sum x)(\sum y)}{n}+\sum xy\right]\left[\frac{1}{-\frac{(\sum x)^2}{n}+\sum x^2}\right] \qquad \textit{(5)}$$

Simplify the right side of equation (5).

$$m=\left[\frac{-(\sum x)(\sum y)+n\sum xy}{n}\right]\left[\frac{n}{-(\sum x)^2+n(\sum x^2)}\right]$$

$$m=\frac{n\sum xy-(\sum x)(\sum y)}{n(\sum x^2)-(\sum x)^2}$$

From equation (3) we have

$$\begin{aligned} b&=\frac{\sum y}{n}-\frac{\sum x}{n}m\\ b&=\frac{\sum y-m(\sum x)}{n}. \end{aligned}$$

34. Let $x=$ the number of units of ROM chips and $y=$ the number of units of RAM chips.

Use a table to organize the information.

	ROM	RAM	Totals
Hours on Line A	1	3	15
Hours on Line B	2	1	15

The system to be solved is

$$\begin{aligned} x+3y&=15 \quad \textit{(1)}\\ 2x+\ \ y&=15. \quad \textit{(2)} \end{aligned}$$

Eliminate x in equation (2).

$$\begin{array}{rl} & x+3y=\ \ 15 \quad \textit{(1)}\\ (-2)\text{R}_1+\text{R}_2\rightarrow\text{R}_2 & \quad -5y=-15 \quad \textit{(3)} \end{array}$$

Make each leading coefficient equal 1.

$$\begin{array}{rl} & x+3y=15 \quad \textit{(1)}\\ -\frac{1}{5}\text{R}_2\rightarrow\text{R}_2 & \qquad y=\ \ 3 \quad \textit{(4)} \end{array}$$

Substitute 3 for y in equation (1) to get $x=6$.

6 units of ROM chips and 3 units of RAM chips can be produced in a day.

36. Let $x =$ the number of skirts originally in the store
and $y =$ the number of blouses originally in the store.

The system to be solved is

$$\begin{aligned} 45x + 35y &= 51{,}750 \quad (\mathit{1}) \\ 45\left(\frac{1}{2}x\right) + 35\left(\frac{2}{3}y\right) &= 30{,}600. \quad (\mathit{2}) \end{aligned}$$

Simplify each equation. Multiply equation (1) by $\frac{1}{5}$ and equation (2) by $\frac{6}{5}$.

$$\begin{aligned} 9x + 7y &= 10{,}350 \quad (\mathit{3}) \\ 27x + 28y &= 36{,}720 \quad (\mathit{4}) \end{aligned}$$

Eliminate x from equation (4).

$$\begin{array}{ll} & 9x + 7y = 10{,}350 \quad (\mathit{3}) \\ -3R_1 + R_2 \to R_2 & 7y = 5670 \quad (\mathit{5}) \end{array}$$

Make each leading coefficient equal 1.

$$\begin{array}{ll} \frac{1}{9}R_1 \to R_1 & x + \frac{7}{9}y = 1150 \quad (\mathit{6}) \\ \frac{1}{7}R_2 \to R_2 & \phantom{x + \frac{7}{9}}y = 810 \quad (\mathit{7}) \end{array}$$

Substitute 810 for y in equation (6).

$$\begin{aligned} x + \frac{7}{9}(810) &= 1150 \\ x + 630 &= 1150 \\ x &= 520 \end{aligned}$$

Half of the skirts are sold, leaving half in the store, so

$$\frac{1}{2}x = \frac{1}{2}(520) = 260.$$

Two-thirds of the blouses are sold, leaving one-third in the store, so

$$\frac{1}{3}y = \frac{1}{3}(810) = 270.$$

There are 260 skirts and 270 blouses left in the store.

38. Let $x =$ the number of shares of Disney stock
and $y =$ the number of shares of Intel stock.

$$\begin{aligned} 30x + 70y &= 16{,}000 \quad (\mathit{1}) \\ 45x + 105y &= 25{,}500 \quad (\mathit{2}) \end{aligned}$$

Simplify each equation. Multiply equation (1) by $\frac{1}{10}$ and equation (2) by $\frac{1}{15}$.

$$\begin{aligned} 3x + 7y &= 1600 \quad (\mathit{3}) \\ 3x + 7y &= 1700 \quad (\mathit{4}) \end{aligned}$$

Since $3x+7y$ cannot equal both 1600 and 1700 for one point (x, y), we have an inconsistent system. Therefore, this situation is not possible.

40. Let $x =$ the number of fives,
$y =$ the number of tens, and
$z =$ the number of twenties.

Since the number of fives is three times the number of tens, $x = 3y$.

The system to be solved is

$$\begin{aligned} x + y + z &= 70 \quad (\mathit{1}) \\ x - 3y &= 0 \quad (\mathit{2}) \\ 5x + 10y + 20z &= 960. \quad (\mathit{3}) \end{aligned}$$

Eliminate x in equations (2) and (3).

$$\begin{array}{ll} & x + y + z = 70 \quad (\mathit{1}) \\ R_1 + (-1)R_2 \to R_2 & 4y + z = 70 \quad (\mathit{4}) \\ -5R_1 + R_3 \to R_3 & 5y + 15z = 610 \quad (\mathit{5}) \end{array}$$

Eliminate y in equation (5).

$$\begin{array}{ll} & x + y + z = 70 \quad (\mathit{1}) \\ & 4y + z = 70 \quad (\mathit{4}) \\ -5R_2 + 4R_3 \to R_3 & 55z = 2090 \quad (\mathit{6}) \end{array}$$

Make each leading coefficient equal 1.

$$\begin{array}{ll} & x + y + z = 70 \quad (\mathit{1}) \\ \frac{1}{4}R_2 \to R_2 & y + \frac{1}{4}z = \frac{35}{2} \quad (\mathit{7}) \\ \frac{1}{55}R_3 \to R_3 & z = 38 \quad (\mathit{8}) \end{array}$$

Substitute 38 for z in equation (7) to get $y = 8$. Finally, substitute 38 for z and 8 for y in equation (1) to get $x = 24$.

There are 24 fives, 8 tens, and 38 twenties.

42. Let $x =$ the number of long-sleeve blouses,
$y =$ the number of short-sleeve blouses, and
$z =$ the number of sleeveless blouses.

Make a table.

	Long Sleeve	Short Sleeve	Sleeveless	Totals
Cutting	1.5	1	.5	380
Sewing	1.2	.9	.6	330

The system to be solved is

$$\begin{aligned} 1.5x + y + .5z &= 380 \quad (\mathit{1}) \\ 1.2x + .9y + .6z &= 330. \quad (\mathit{2}) \end{aligned}$$

Simplify the equations. Multiply equation (1) by 2 and equation (2) by $\frac{10}{3}$.

$$\begin{aligned} 3x+2y+z &= 760 \quad (3)\\ 4x+3y+2z &= 1100 \quad (4)\end{aligned}$$

Make the leading coefficient of equation (3) equa1 1.

$$\begin{array}{ll} \frac{1}{3}\mathrm{R}_1 \to \mathrm{R}_1 & x+\dfrac{2}{3}y+\dfrac{1}{3}z = \dfrac{760}{3} \quad (5)\\ & 4x+3y+2z=1100 \quad (4)\end{array}$$

Eliminate x from equation (4).

$$\begin{array}{lr} & x+\dfrac{2}{3}y+\dfrac{1}{3}z=\dfrac{760}{3} \quad (5)\\ -4\mathrm{R}_1+\mathrm{R}_2\to \mathrm{R}_2 & \dfrac{1}{3}y+\dfrac{2}{3}z=\dfrac{260}{3}\quad (6)\end{array}$$

Make the leading coefficient of equation (6) equal 1.

$$\begin{array}{lr} & x+\dfrac{2}{3}y+\dfrac{1}{3}z=\dfrac{760}{3} \quad (5)\\ 3\mathrm{R}_2\to \mathrm{R}_2 & y+2z=260 \quad (7)\end{array}$$

From equation (7), $y=260-2z$. Substitute this into equation (5).

$$\begin{aligned} x+\frac{2}{3}(260-2z)+\frac{1}{3}z&=\frac{760}{3}\\ x+\frac{520}{3}-\frac{4}{3}z+\frac{1}{3}z&=\frac{760}{3}\\ x-z&=\frac{240}{3}\\ x&=z+80\end{aligned}$$

The solution is $(z+80, 260-2z, z)$. In this problem $x, y,$ and z must be nonnegative, so

$$\begin{aligned} 260-2z&\geq 0\\ -2z&\geq -260\\ z&\leq 130.\end{aligned}$$

Therefore, the plant should make $z + 80$ long-sleeve blouses, $260 - 2z$ short-sleeve blouses, and z sleeveless blouses with $0\leq z\leq 130$.

44. Let $x =$ the number of EZ models,
$y =$ the number of compact models, and
$z =$ the number of commercial models.

Make a table.

	EZ	Compact	Commercial	Totals
Weight	10	20	60	440
Space	10	8	28	248

$$\begin{aligned} 10x+20y+60z&=440 \quad (1)\\ 10x+8y+28z&=248 \quad (2)\end{aligned}$$

Eliminate x from equation (2).

$$\begin{array}{lr} & 10x+20y+60z=440 \quad (1)\\ \mathrm{R}_1+(-1)\mathrm{R}_2\to \mathrm{R}_2 & 12y+32z=192 \quad (3)\end{array}$$

Make the leading coefficients equal 1.

$$\begin{array}{ll} \frac{1}{10}\mathrm{R}_1\to \mathrm{R}_1 & x+2y+6z=44 \quad (4)\\ \frac{1}{12}\mathrm{R}_2\to \mathrm{R}_2 & \quad\; y+\dfrac{8}{3}z=16 \quad (5)\end{array}$$

Solve equation (5) for y.

$$\begin{aligned} y&=16-\frac{8}{3}z\\ y&=\frac{48-8z}{3}\\ y&=\frac{8(6-z)}{3}\end{aligned}$$

Substitute this expression for y into equation (4) and solve for x.

$$x+2\left[\frac{8(6-z)}{3}\right]+6z=44$$

$$\begin{aligned} x&=44-6z-\frac{16(6-z)}{3}\\ x&=\frac{132-18z-96+16z}{3}\\ x&=\frac{36-2z}{3}\\ x&=\frac{2(18-z)}{3}\end{aligned}$$

The solution of the system is

$$\left(\frac{2(18-z)}{3},\frac{8(6-z)}{3},z\right).$$

The solutions must be nonnegative integers. Therefore, $0 \leq z \leq 6$. (Any larger values of z would cause y to be negative which would make no sense in the problem.)

Values of z	Solutions
0	$(12,16,0)$
1	$\left(\frac{34}{3},\frac{40}{3},1\right)$
2	$\left(\frac{32}{3},\frac{32}{3},2\right)$
3	$(10,8,3)$
4	$\left(\frac{28}{3},\frac{16}{3},4\right)$
5	$\left(\frac{26}{3},\frac{8}{3},5\right)$
6	$(8,0,6)$

Ignore solutions which contain values that are not integers. There are three possible solutions:

1. 12 EZ models, 16 compact models, and 0 commercial models;
2. 10 EZ models, 8 compact models, and 3 commercial models; or
3. 8 EZ models, 0 compact models, and 6 commercial models.

46. **(a)** For the first equation, the first sighting in 1980 was on day $y = 759 - .338(1980) = 89.76$, or during the eighty-ninth day of the year. Since 1980 was a leap year, the eighty-ninth day fell on March 29.
For the second equation, the first sighting in 1980 was on day $y = 1637 - .779(1980) = 94.58$, or during the ninety-fourth day of the year. Since 1980 was a leap year, the ninety-fourth day fell on April 3.

(b) $y = 759 - .338x \quad (1)$
$y = 1637 - .779x \quad (2)$

Rewrite equations so that variables are on the left side and constant term is on the right side.

$$\begin{aligned} .338x + y &= 759 \quad (3) \\ .779x + y &= 1637 \quad (4) \end{aligned}$$

Eliminate y from equation (4).

$$\begin{array}{rr} & .338x + y = 759 \quad (3) \\ -1\text{R}_1 + \text{R}_2 \to \text{R}_2 & .441x = 878 \quad (5) \end{array}$$

Make leading coefficient for equation (5) equal 1.

$$\begin{array}{rr} & .338x + y = 759 \quad (3) \\ \frac{1}{.441}\text{R}_2 \to \text{R}_2 & x = \dfrac{878}{.441} \quad (6) \end{array}$$

The two estimates agree in the year closest to $x = \frac{878}{.441} \approx 1990.93$ so they agree in 1991. The estimated number of days into the year when a robin can be expected is

$$.338\left(\frac{878}{.441}\right) + y = 759$$

$$y \approx 86.$$

48. **(a)** Since 8 and 9 must be two of the four numbers combined using addition, subtraction, multiplication, and/or division to get 24, begin by finding two numbers to use with 8 and 9. One possibility is 8 and 3 since $(9 - 8) \cdot 8 \cdot 3 = 24$. If we can find values of x and y such that either $x + y = 8$ and $3x + 2y = 3$, or $x + y = 3$ and $3x + 2y = 8$, we will have found a solution. Solving the first system gives $x = -13$ and $y = 21$. This, however, does not satisfy the condition that x and y be single-digit positive integers. Solving the second system gives $x = 2$ and $y = 1$. Since both of these values are single-digit positive integers, we have one possible system. Thus, one system is

$$\begin{cases} x + y = 3 \\ 3x + 2y = 8 \end{cases}.$$

Its solution is $(2, 1)$. These values of x and y give the numbers 8, 9, 8, and 3 on the game card. These numbers can be combined as $(9 - 8) \cdot 8 \cdot 3$ to make 24.

2.2 Solution of Linear Systems by the Gauss-Jordan Method

2. $3x + 5y = -13$
$2x + 3y = -9$

The equations are in proper form, so the augmented matrix is

$$\left[\begin{array}{cc|c} 3 & 5 & -13 \\ 2 & 3 & -9 \end{array}\right].$$

4. $4x - 2y + 3z = 4$
$3x + 5y + z = 7$
$5x - y + 4z = 7$

The equations are in proper form, so the augmented matrix is

$$\left[\begin{array}{ccc|c} 4 & -2 & 3 & 4 \\ 3 & 5 & 1 & 7 \\ 5 & -1 & 4 & 7 \end{array}\right].$$

6. $\left[\begin{array}{cc|r}1&0&5\\0&1&-3\end{array}\right]$

is equivalent to the system

$$\begin{aligned}x &= 5\\ y &= -3.\end{aligned}$$

8. $\left[\begin{array}{ccc|c}1&0&0&4\\0&1&0&2\\0&0&1&3\end{array}\right]$

is equivalent to the system

$$\begin{aligned}x &= 4\\ y &= 2\\ z &= 3.\end{aligned}$$

12. Replace R_3 by $-1R_1 + 3R_3$.
The original matrix is

$$\left[\begin{array}{rrr|r}3&2&6&18\\2&-2&5&7\\1&0&5&20\end{array}\right].$$

The resulting matrix is

$$\left[\begin{array}{rrr|r}3&2&6&18\\2&-2&5&7\\0&-2&9&42\end{array}\right].$$

14. Replace R_1 by $R_3 + (-3)R_1$.

The original matrix is

$$\left[\begin{array}{rrr|r}1&0&4&21\\0&6&5&30\\0&0&12&15\end{array}\right].$$

The resulting matrix is

$$\left[\begin{array}{rrr|r}-3&0&0&-48\\0&6&5&30\\0&0&12&15\end{array}\right].$$

16. Replace R_3 by $\frac{1}{4}R_3$.

The original matrix is

$$\left[\begin{array}{ccc|r}1&0&0&6\\0&1&0&5\\0&0&4&12\end{array}\right].$$

The resulting matrix is

$$\left[\begin{array}{ccc|c}1&0&0&6\\0&1&0&5\\0&0&1&3\end{array}\right].$$

18. $$\begin{aligned}x + 2y &= 5\\ 2x + y &= -2\end{aligned}$$

To begin, write the augmented matrix for the given system.

$$\left[\begin{array}{cc|r}1&2&5\\2&1&-2\end{array}\right]$$

The third row operation is used to change the 2 in row 2 to 0.

$$-2R_1 + R_2 \to R_2 \qquad \left[\begin{array}{rr|r}1&2&5\\0&-3&-12\end{array}\right]$$

Next, change the 2 in row 1 to 0.

$$2R_2 + 3R_1 \to R_1 \qquad \left[\begin{array}{rr|r}3&0&-9\\0&-3&-12\end{array}\right]$$

Finally, change the first nonzero number in each row to 1.

$$\begin{aligned}\tfrac{1}{3}R_1 &\to R_1\\ -\tfrac{1}{3}R_2 &\to R_2\end{aligned} \qquad \left[\begin{array}{cc|r}1&0&-3\\0&1&4\end{array}\right]$$

The final matrix is equivalent to the system

$$\begin{aligned}x &= -3\\ y &= 4,\end{aligned}$$

so the solution of the original system is $(-3, 4)$.

20. $$\begin{aligned}4x - 2y &= 3\\ -2x + 3y &= 1\end{aligned}$$

The augmented matrix for the system is

$$\left[\begin{array}{rr|c}4&-2&3\\-2&3&1\end{array}\right].$$

$$R_1 + 2R_2 \to R_2 \qquad \left[\begin{array}{rr|c}4&-2&3\\0&4&5\end{array}\right]$$

$$R_2 + 2R_1 \to R_1 \qquad \left[\begin{array}{cc|r}8&0&11\\0&4&5\end{array}\right]$$

$$\begin{aligned}\tfrac{1}{8}R_1 &\to R_1\\ \tfrac{1}{4}R_2 &\to R_2\end{aligned} \qquad \left[\begin{array}{cc|c}1&0&\frac{11}{8}\\0&1&\frac{5}{4}\end{array}\right]$$

The solution is $\left(\frac{11}{8}, \frac{5}{4}\right)$.

22. $$\begin{aligned} x + 2y &= 1 \\ 2x + 4y &= 3 \end{aligned}$$

The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 4 & 3 \end{array}\right].$$

$$-2\text{R}_1 + \text{R}_2 \rightarrow \text{R}_2 \qquad \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & 1 \end{array}\right]$$

The second equation says $0 = 1$ (which is impossible), so the system is inconsistent and has no solution.

24. $$\begin{aligned} x - y &= 1 \\ -x + y &= -1 \end{aligned}$$

The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ -1 & 1 & -1 \end{array}\right].$$

$$\text{R}_1 + \text{R}_2 \rightarrow \text{R}_2 \qquad \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

The row of zeros indicates dependent equations. (Both equations have the same line as their graph.) The remaining equation is $x - y = 1$. Solving for x gives $x = y + 1$. There are an infinite number of solutions, each of the form $(y + 1, y)$, for any real number y.

26. $$\begin{aligned} x &= 1 - y \\ 2x &= z \\ 2z &= -2 - y \end{aligned}$$

Put the equations in proper form to obtain the system

$$\begin{aligned} x + y &= 1 \\ 2x - z &= 0 \\ y + 2z &= -2. \end{aligned}$$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 2 & 0 & -1 & 0 \\ 0 & 1 & 2 & -2 \end{array}\right].$$

$$-2\text{R}_1 + \text{R}_2 \rightarrow \text{R}_2 \qquad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -2 & -1 & -2 \\ 0 & 1 & 2 & -2 \end{array}\right]$$

$$\begin{array}{r} \text{R}_2 + 2\text{R}_1 \rightarrow \text{R}_1 \\ \\ \text{R}_2 + 2\text{R}_3 \rightarrow \text{R}_3 \end{array} \qquad \left[\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 0 & -2 & -1 & -2 \\ 0 & 0 & 3 & -6 \end{array}\right]$$

$$\begin{array}{r} \text{R}_3 + 3\text{R}_1 \rightarrow \text{R}_1 \\ \text{R}_3 + 3\text{R}_2 \rightarrow \text{R}_2 \\ \\ \end{array} \qquad \left[\begin{array}{ccc|c} 6 & 0 & 0 & -6 \\ 0 & -6 & 0 & -12 \\ 0 & 0 & 3 & -6 \end{array}\right]$$

$$\begin{array}{r} \frac{1}{6}\text{R}_1 \rightarrow \text{R}_1 \\ -\frac{1}{6}\text{R}_2 \rightarrow \text{R}_2 \\ \frac{1}{3}\text{R}_3 \rightarrow \text{R}_3 \end{array} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

The solution is $(-1, 2, -2)$.

28. $$\begin{aligned} x \qquad - z &= -3 \\ y + z &= 9 \\ -2x + 3y + 5z &= 33 \end{aligned}$$

Write the augmented matrix and use row operations.

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 9 \\ -2 & 3 & 5 & 33 \end{array}\right]$$

$$2\text{R}_1 + \text{R}_3 \rightarrow \text{R}_3 \qquad \left[\begin{array}{ccc|c} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 9 \\ 0 & 3 & 3 & 27 \end{array}\right]$$

$$-3\text{R}_2 + \text{R}_3 \rightarrow \text{R}_3 \qquad \left[\begin{array}{ccc|c} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The last row indicates an infinite number of solutions. The remaining equations are

$$x - z = -3 \quad \text{and} \quad y + z = 9.$$

Solve these for x and y, the solutions are

$$(z - 3, -z + 9, z)$$

for any real number z.

30. $$\begin{aligned} x+3y-6z&=7\\ 2x-y+2z&=0\\ x+y+2z&=-1\end{aligned}$$

The augmented matrix is

$$\left[\begin{array}{rrr|r} 1 & 3 & -6 & 7\\ 2 & -1 & 2 & 0\\ 1 & 1 & 2 & -1\end{array}\right].$$

$$\begin{array}{r} \\ -2R_1+R_2\to R_2\\ -1R_1+R_3\to R_3\end{array}\left[\begin{array}{rrr|r} 1 & 3 & -6 & 7\\ 0 & -7 & 14 & -14\\ 0 & -2 & 8 & -8\end{array}\right]$$

$$\begin{array}{r} 3R_2+7R_1\to R_1\\ \\ 2R_2+(-7)R_3\to R_3\end{array}\left[\begin{array}{rrr|r} 7 & 0 & 0 & 7\\ 0 & -7 & 14 & -14\\ 0 & 0 & -28 & 28\end{array}\right]$$

$$\begin{array}{r} \\ \\ R_3+2R_2\to R_2\end{array}\left[\begin{array}{rrr|r} 7 & 0 & 0 & 7\\ 0 & -14 & 0 & 0\\ 0 & 0 & -28 & 28\end{array}\right]$$

$$\begin{array}{r} \frac{1}{7}R_1\to R_1\\ -\frac{1}{14}R_2\to R_2\\ -\frac{1}{28}R_3\to R_3\end{array}\left[\begin{array}{rrr|r} 1 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & -1\end{array}\right]$$

The solution is $(1, 0, -1)$.

32. $$\begin{aligned} 3x-6y+3z&=11\\ 2x+y-z&=2\\ 5x-5y+2z&=6\end{aligned}$$

Write the augmented matrix and use row operations.

$$\left[\begin{array}{rrr|r} 3 & -6 & 3 & 11\\ 2 & 1 & -1 & 2\\ 5 & -5 & 2 & 6\end{array}\right].$$

$$\begin{array}{r} \\ -2R_1+3R_2\to R_2\\ -5R_1+3R_3\to R_3\end{array}\left[\begin{array}{rrr|r} 3 & -6 & 3 & 11\\ 0 & 15 & -9 & -16\\ 0 & 15 & -9 & -37\end{array}\right]$$

$$\begin{array}{r} 2R_2+5R_1\to R_1\\ \\ -R_2+R_3\to R_3\end{array}\left[\begin{array}{rrr|r} 15 & 0 & -3 & 23\\ 0 & 15 & -9 & -16\\ 0 & 0 & 0 & -21\end{array}\right]$$

The last row indicates inconsistent equations. There is no solution to the system.

34. $$\begin{aligned} 3x+2y-z&=-16\\ 6x-4y+3z&=12\\ 5x-2y+2z&=4\end{aligned}$$

Write the augmented matrix and use row operations.

$$\left[\begin{array}{rrr|r} 3 & 2 & -1 & -16\\ 6 & -4 & 3 & 12\\ 5 & -2 & 2 & 4\end{array}\right]$$

$$\begin{array}{r} \\ -2R_1+R_2\to R_2\\ -5R_1+3R_3\to R_3\end{array}\left[\begin{array}{rrr|r} 3 & 2 & -1 & -16\\ 0 & -8 & 5 & 44\\ 0 & -16 & 11 & 92\end{array}\right]$$

$$\begin{array}{r} R_2+4R_1\to R_1\\ \\ -2R_2+R_3\to R_3\end{array}\left[\begin{array}{rrr|r} 12 & 0 & 1 & -20\\ 0 & -8 & 5 & 44\\ 0 & 0 & 1 & 4\end{array}\right]$$

$$\begin{array}{r} -1R_3+R_1\to R_1\\ -5R_3+R_2\to R_2\\ \end{array}\left[\begin{array}{rrr|r} 12 & 0 & 0 & -24\\ 0 & -8 & 0 & 24\\ 0 & 0 & 1 & 4\end{array}\right]$$

$$\begin{array}{r} \frac{1}{12}R_1\to R_1\\ -\frac{1}{8}R_2\to R_2\\ \end{array}\left[\begin{array}{rrr|r} 1 & 0 & 0 & -2\\ 0 & 1 & 0 & -3\\ 0 & 0 & 1 & 4\end{array}\right]$$

Read the solution from the last column of the matrix. The solution is $(-2, -3, 4)$.

36. $$\begin{aligned} 3x-5y-2z&=-9\\ -4x+3y+z&=11\\ 8x-5y+4z&=6\end{aligned}$$

Write the augmented matrix and use row operations.

$$\left[\begin{array}{rrr|r} 3 & -5 & -2 & -9\\ -4 & 3 & 1 & 11\\ 8 & -5 & 4 & 6\end{array}\right]$$

$$\begin{array}{r} \\ 4R_1+3R_2\to R_2\\ -8R_1+3R_3\to R_3\end{array}\left[\begin{array}{rrr|r} 3 & -5 & -2 & -9\\ 0 & -11 & -5 & -3\\ 0 & 25 & 28 & 90\end{array}\right]$$

$$\begin{array}{r} -5R_2+11R_1\to R_1\\ \\ 25R_2+11R_3\to R_3\end{array}\left[\begin{array}{rrr|r} 33 & 0 & 3 & -84\\ 0 & -11 & -5 & -3\\ 0 & 0 & 183 & 915\end{array}\right]$$

$$\begin{array}{r} -R_3+61R_1\to R_1\\ 5R_3+183R_2\to R_2\\ \end{array}\left[\begin{array}{rrr|r} 2013 & 0 & 0 & -6039\\ 0 & -2013 & 0 & 4026\\ 0 & 0 & 183 & 915\end{array}\right]$$

$$\begin{array}{r} \frac{1}{2013}R_1\to R_1\\ -\frac{1}{2013}R_2\to R_2\\ \frac{1}{183}R_3\to R_3\end{array}\left[\begin{array}{rrr|r} 1 & 0 & 0 & -3\\ 0 & 1 & 0 & -2\\ 0 & 0 & 1 & 5\end{array}\right]$$

Read the solutlion from the last column of the matrix. The solution is $(-3, -2, 5)$.

38.
$$\begin{aligned} x+3y-2z-\ w &= 9\\ 2x+4y\qquad +2w &= 10\\ -3x-5y+2z-\ w &= -15\\ x-\ y-3z+2w &= 6 \end{aligned}$$

$$\left[\begin{array}{rrrr|r} 1 & 3 & -2 & -1 & 9\\ 2 & 4 & 0 & 2 & 10\\ -3 & -5 & 2 & -1 & -15\\ 1 & -1 & -3 & 2 & 6 \end{array}\right]$$

$$\begin{array}{r} \\ -2R_1+R_2\to R_2\\ 3R_1+R_3\to R_3\\ -1R_1+R_4\to R_4 \end{array} \left[\begin{array}{rrrr|r} 1 & 3 & -2 & -1 & 9\\ 0 & -2 & 4 & 4 & -8\\ 0 & 4 & -4 & -4 & 12\\ 0 & -4 & -1 & 3 & -3 \end{array}\right]$$

$$\begin{array}{r} 3R_2+2R_1\to R_1\\ \\ 2R_2+R_3\to R_3\\ -2R_2+R_4\to R_4 \end{array} \left[\begin{array}{rrrr|r} 2 & 0 & 8 & 10 & -6\\ 0 & -2 & 4 & 4 & -8\\ 0 & 0 & 4 & 4 & -4\\ 0 & 0 & -9 & -5 & 13 \end{array}\right]$$

$$\begin{array}{r} -2R_3+R_1\to R_1\\ -1R_3+R_2\to R_2\\ \\ 9R_3+4R_4\to R_4 \end{array} \left[\begin{array}{rrrr|r} 2 & 0 & 0 & 2 & 2\\ 0 & -2 & 0 & 0 & -4\\ 0 & 0 & 4 & 4 & -4\\ 0 & 0 & 0 & 16 & 16 \end{array}\right]$$

$$\begin{array}{r} R_4+(-8)R_1\to R_1\\ \\ R_4+(-4)R_3\to R_3\\ \\ \end{array} \left[\begin{array}{rrrr|r} -16 & 0 & 0 & 0 & 0\\ 0 & -2 & 0 & 0 & -4\\ 0 & 0 & -16 & 0 & 32\\ 0 & 0 & 0 & 16 & 16 \end{array}\right]$$

$$\begin{array}{r} -\frac{1}{16}R_1\to R_1\\ -\frac{1}{2}R_2\to R_2\\ -\frac{1}{16}R_3\to R_3\\ \frac{1}{16}R_4\to R_4 \end{array} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 2\\ 0 & 0 & 1 & 0 & -2\\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

The solution is $(0, 2, -2, 1)$.

40.
$$\begin{aligned} 4x-3y+\ z+\ \ w &= 21\\ -2x-\ y+2z+\ 7w &= 2\\ 10x\qquad -5z-20w &= 15 \end{aligned}$$

$$\left[\begin{array}{rrrr|r} 4 & -3 & 1 & 1 & 21\\ -2 & -1 & 2 & 7 & 2\\ 10 & 0 & -5 & -20 & 15 \end{array}\right]$$

Interchange rows 1 and 2.

$$\left[\begin{array}{rrrr|r} -2 & -1 & 2 & 7 & 2\\ 4 & -3 & 1 & 1 & 21\\ 10 & 0 & -5 & -20 & 15 \end{array}\right]$$

$$\begin{array}{r} \\ 2R_1+R_2\to R_2\\ 5R_1+R_3\to R_3 \end{array} \left[\begin{array}{rrrr|r} -2 & -1 & 2 & 7 & 2\\ 0 & -5 & 5 & 15 & 25\\ 0 & -5 & 5 & 15 & 25 \end{array}\right]$$

$$\begin{array}{r} \\ -\frac{1}{5}R_2\to R_2\\ \frac{1}{5}R_3\to R_3 \end{array} \left[\begin{array}{rrrr|r} -2 & -1 & 2 & 7 & 2\\ 0 & 1 & -1 & -3 & -5\\ 0 & -1 & 1 & 3 & 5 \end{array}\right]$$

$$\begin{array}{r} R_2+R_1\to R_1\\ \\ R_2+R_3\to R_3 \end{array} \left[\begin{array}{rrrr|r} -2 & 0 & 1 & 4 & -3\\ 0 & 1 & -1 & -3 & -5\\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The last row indicates that there are an infinite number of solutions. The remaining equations are

$$-2x+z+4w=-3 \quad \text{and} \quad y-z-3w=-5$$

Solve these for x and y to obtain

$$x=\frac{z+4w+3}{2} \quad \text{and} \quad y=z+3w-5.$$

There are an infinite number of solutions, each of the form

$$\left(\frac{z+4w+3}{2}, z+3w-5, z, w\right),$$

or

$$(1.5+.5z+2w, -5+z+3w, z, w),$$

for any real numbers z and w.

42.
$$\begin{aligned} 28.6x+94.5y+16.0z-2.94w &= 198.3\\ 16.7x+44.3y-27.3z+\ 8.9w &= 254.7\\ 12.5x-38.7y+92.5z+22.4w &= 562.7\\ 40.1x-28.3y+17.5z-10.2w &= 375.4 \end{aligned}$$

This exercise should be solved by graphing calculator or computer methods. The solution, which may vary slightly, is

$$(11.844, -1.153, .609, 14.004).$$

44. Let $x =$ the number of hours to hire the Garcia firm

and $y =$ the number of hours to hire the Wong firm.

The system to be solved is

$$\begin{aligned} 10x+20y &= 500 \quad (1)\\ 30x+10y &= 750 \quad (2)\\ 5x+10y &= 250. \quad (3) \end{aligned}$$

Write the augmented matrix of the system.

$$\left[\begin{array}{cc|c} 10 & 20 & 500 \\ 30 & 10 & 750 \\ 5 & 10 & 250 \end{array}\right]$$

$$\begin{array}{r} \frac{1}{10}\text{R}_1 \rightarrow \text{R}_1 \\ \frac{1}{10}\text{R}_2 \rightarrow \text{R}_2 \\ \frac{1}{5}\text{R}_3 \rightarrow \text{R}_3 \end{array} \quad \left[\begin{array}{cc|c} 1 & 2 & 50 \\ 3 & 1 & 75 \\ 1 & 2 & 50 \end{array}\right]$$

$$\begin{array}{r} -3\text{R}_1 + \text{R}_2 \rightarrow \text{R}_2 \\ -1\text{R}_1 + \text{R}_3 \rightarrow \text{R}_3 \end{array} \quad \left[\begin{array}{cc|c} 1 & 2 & 50 \\ 0 & -5 & -75 \\ 0 & 0 & 0 \end{array}\right]$$

$$-\frac{1}{5}\text{R}_2 \rightarrow \text{R}_2 \quad \left[\begin{array}{cc|c} 1 & 2 & 50 \\ 0 & 1 & 15 \\ 0 & 0 & 0 \end{array}\right]$$

$$-2\text{R}_2 + \text{R}_1 \rightarrow \text{R}_1 \quad \left[\begin{array}{cc|c} 1 & 0 & 20 \\ 0 & 1 & 15 \\ 0 & 0 & 0 \end{array}\right]$$

The solution is $(20, 15)$. Hire the Garcia firm for 20 hr and the Wong firm for 15 hr.

46. Let $x =$ the number of chairs produced each week,
$y =$ the number of cabinets produced each week, and
$z =$ the number of buffets produced each week.

Make a table to organize the information.

	Chair	Cabinet	Buffet	Totals
Cutting	.2	.5	.3	1950
Assembly	.3	.4	.1	1490
Finishing	.1	.6	.4	2160

The system to be solved is

$$\begin{aligned} .2x + .5y + .3z &= 1950 \\ .3x + .4y + .1z &= 1490 \\ .1x + .6y + .4z &= 2160. \end{aligned}$$

Write the augmented matrix of the system.

$$\left[\begin{array}{ccc|c} .2 & .5 & .3 & 1950 \\ .3 & .4 & .1 & 1490 \\ .1 & .6 & .4 & 2160 \end{array}\right]$$

$$\begin{array}{r} 10\text{R}_1 \rightarrow \text{R}_1 \\ 10\text{R}_2 \rightarrow \text{R}_2 \\ 10\text{R}_3 \rightarrow \text{R}_3 \end{array} \quad \left[\begin{array}{ccc|c} 2 & 5 & 3 & 19{,}500 \\ 3 & 4 & 1 & 14{,}900 \\ 1 & 6 & 4 & 21{,}600 \end{array}\right]$$

Interchange rows 1 and 3.

$$\left[\begin{array}{ccc|c} 1 & 6 & 4 & 21{,}600 \\ 3 & 4 & 1 & 14{,}900 \\ 2 & 5 & 3 & 19{,}500 \end{array}\right]$$

$$\begin{array}{r} -3\text{R}_1 + \text{R}_2 \rightarrow \text{R}_2 \\ -2\text{R}_1 + \text{R}_3 \rightarrow \text{R}_3 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 6 & 4 & 21{,}600 \\ 0 & -14 & -11 & -49{,}900 \\ 0 & -7 & -5 & -23{,}700 \end{array}\right]$$

$$-\frac{1}{14}\text{R}_2 \rightarrow \text{R}_2 \quad \left[\begin{array}{ccc|c} 1 & 6 & 4 & 21{,}600 \\ 0 & 1 & \frac{11}{14} & \frac{24{,}950}{7} \\ 0 & -7 & -5 & -23{,}700 \end{array}\right]$$

$$\begin{array}{r} -6\text{R}_2 + \text{R}_1 \rightarrow \text{R}_1 \\ \\ 7\text{R}_2 + \text{R}_3 \rightarrow \text{R}_3 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & -\frac{5}{7} & \frac{1500}{7} \\ 0 & 1 & \frac{11}{14} & \frac{24{,}950}{7} \\ 0 & 0 & \frac{1}{2} & 1250 \end{array}\right]$$

$$2\text{R}_3 \rightarrow \text{R}_3 \quad \left[\begin{array}{ccc|c} 1 & 0 & -\frac{5}{7} & \frac{1500}{7} \\ 0 & 1 & \frac{11}{14} & \frac{24{,}950}{7} \\ 0 & 0 & 1 & 2500 \end{array}\right]$$

$$\begin{array}{r} \frac{5}{7}\text{R}_3 + \text{R}_1 \rightarrow \text{R}_1 \\ -\frac{11}{14}\text{R}_3 + \text{R}_2 \rightarrow \text{R}_2 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2000 \\ 0 & 1 & 0 & 1600 \\ 0 & 0 & 1 & 2500 \end{array}\right]$$

The solution is $(2000, 1600, 2500)$. Therefore, 2000 chairs, 1600 cabinets, and 2500 buffets should be produced.

48. **(a)** Let x be the number of trucks used, y be the number of vans, and z be the number of station wagons. We first obtain the equations given here.

$$\begin{aligned} 2x + 3y + 3z &= 25 \\ 2x + 4y + 5z &= 33 \\ 3x + 2y + z &= 22 \end{aligned}$$

Write the augmented matrix and use row operations.

$$\left[\begin{array}{ccc|c} 2 & 3 & 3 & 25 \\ 2 & 4 & 5 & 33 \\ 3 & 2 & 1 & 22 \end{array}\right]$$

$$\begin{array}{r} -1R_1 + R_2 \to R_2 \\ -3R_1 + 2R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 2 & 3 & 3 & 25 \\ 0 & 1 & 2 & 8 \\ 0 & -5 & -7 & -31 \end{array}\right]$$

$$\begin{array}{r} -3R_2 + R_1 \to R_1 \\ \\ 5R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 2 & 0 & -3 & 1 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 3 & 9 \end{array}\right]$$

$$\begin{array}{r} R_3 + R_1 \to R_1 \\ -2R_3 + 3R_2 \to R_2 \end{array} \left[\begin{array}{ccc|c} 2 & 0 & 0 & 10 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 3 & 9 \end{array}\right]$$

$$\begin{array}{r} \frac{1}{2}R_1 \to R_1 \\ \frac{1}{3}R_2 \to R_2 \\ \frac{1}{3}R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Read the solution from the last column of the matrix. The solution is 5 trucks, 2 vans, and 3 station wagons.

(b) The system of equations is now

$$\begin{aligned} 2x + 3y + 3z &= 25 \\ 2x + 4y + 5z &= 33. \end{aligned}$$

Write the augmented matrix and use row operations.

$$\left[\begin{array}{ccc|c} 2 & 3 & 3 & 25 \\ 2 & 4 & 5 & 33 \end{array}\right]$$

$$-R_1 + R_2 \to R_2 \quad \left[\begin{array}{ccc|c} 2 & 3 & 3 & 25 \\ 0 & 1 & 2 & 8 \end{array}\right]$$

$$-3R_2 + R_1 \to R_1 \quad \left[\begin{array}{ccc|c} 2 & 0 & -3 & 1 \\ 0 & 1 & 2 & 8 \end{array}\right]$$

Obtain a one in row 1, column 1.

$$\tfrac{1}{2}R_1 \to R_1 \quad \left[\begin{array}{ccc|c} 1 & 0 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 1 & 2 & 8 \end{array}\right]$$

The last row indicates multiple solutions are possible. The remaining equations are

$$x - \frac{3}{2}z = \frac{1}{2} \quad \text{and} \quad y + 2z = 8.$$

Solving these for x and y, we have

$$x = \frac{3}{2}z + \frac{1}{2} \quad \text{and} \quad y = -2z + 8.$$

The form of the solution is $(\frac{3}{2}z + \frac{1}{2}, -2z + 8, z)$. Since the solutions must be whole numbers,

$$\begin{array}{rcl} \frac{3}{2}z + \frac{1}{2} \geq 0 & \quad\text{and}\quad & -2z + 8 \geq 0 \\ \frac{3}{2}z \geq -\frac{1}{2} & & -2z \geq -8 \\ z \geq -\frac{1}{3} & & z \leq 4 \end{array}$$

Thus, there are 4 possible solutions but each must be checked to determine if they produce whole numbers for x and y.

When $z = 0$, $\left(\frac{1}{2}, 8, 0\right)$ which is not realistic.

When $z = 1$, $(2, 6, 1)$.

When $z = 2$, $\left(\frac{7}{2}, 4, 2\right)$ which is not realistic.

When $z = 3$, $(5, 2, 3)$.

When $z = 4$, $\left(\frac{13}{2}, 0, 4\right)$ which is not realistic.

The company has 2 options. Either use 2 trucks, 6 vans, and 1 station wagon or use 5 trucks, 2 vans, and 3 station wagons.

50. Let $x =$ the amount borrowed at 13%,
$y =$ the amount borrowed at 14%,
and $z =$ the amount borrowed at 12%.

(a) The system to be solved is

$$\begin{aligned} x + \quad y + \quad z &= 25{,}000 && (1) \\ .13x + .14y + .12z &= 3240 && (2) \\ y &= \frac{1}{2}x + 2000. && (3) \end{aligned}$$

Multiply equation (2) by 100 and equation (3) by 2. Then rewrite the system.

$$\begin{aligned} x + \quad y + \quad z &= \quad 25{,}000 && (1) \\ 13x + 14y + 12z &= 324{,}000 && (4) \\ -x + \quad 2y \qquad &= \qquad 4000 && (5) \end{aligned}$$

Write the augmented matrix of the system.

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 25{,}000 \\ 13 & 14 & 12 & 324{,}000 \\ -1 & 2 & 0 & 4000 \end{array}\right]$$

$$\begin{array}{r} \\ -13R_1 + R_2 \to R_2 \\ R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{rrr|r} 1 & 1 & 1 & 25{,}000 \\ 0 & 1 & -1 & -1000 \\ 0 & 3 & 1 & 29{,}000 \end{array}\right]$$

$$\begin{array}{r} -1R_2 + R_1 \to R_1 \\ \\ -3R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{rrr|r} 1 & 0 & 2 & 26{,}000 \\ 0 & 1 & -1 & -1000 \\ 0 & 0 & 4 & 32{,}000 \end{array}\right]$$

$$\begin{array}{r} \\ \\ \frac{1}{4}R_3 \to R_3 \end{array} \left[\begin{array}{rrr|r} 1 & 0 & 2 & 26{,}000 \\ 0 & 1 & -1 & -1000 \\ 0 & 0 & 1 & 8000 \end{array}\right]$$

$$\begin{array}{r} -2R_3 + R_1 \to R_1 \\ R_3 + R_2 \to R_2 \\ \end{array} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 10{,}000 \\ 0 & 1 & 0 & 7000 \\ 0 & 0 & 1 & 8000 \end{array}\right]$$

The solution is (10,000, 7000, 8000). Borrow \$10,000 at 13%, \$7000 at 14%, and \$8000 at 12%.

(b) If the condition is dropped, refer to the first two rows of the fourth augmented matrix of part (a).

$$\left[\begin{array}{rrr|r} 1 & 0 & 2 & 26{,}000 \\ 0 & 1 & -1 & -1000 \end{array}\right]$$

This gives

$$\begin{aligned} x &= 26{,}000 - 2z \\ y &= z - 1000. \end{aligned}$$

Since all values must be nonnegative,

$$\begin{array}{rcr} 26{,}000 - 2z \geq 0 & \text{and} & z - 1000 \geq 0 \\ z \leq 13{,}000 & \text{and} & z \geq 1000. \end{array}$$

Therefore, the amount borrowed at 12% must be between \$1000 and \$13,000. If $z = 5000$, then

$$\begin{aligned} x &= 26{,}000 - 2(5000) = 16{,}000 \text{ and} \\ y &= 5000 - 1000 = 4000. \end{aligned}$$

Therefore, \$16,000 is borrowed at 13% and \$4000 at 14%.

(c) Substitute $z = 6000$ into equations (1), (4), and (5) from part (a) to obtain the system

$$\begin{array}{rcrl} x + y + 6000 & = & 25,000 & (6) \\ 13x + 14y + 12(6000) & = & 324{,}000 & (7) \\ -x + 2y & = & 4000. & (5) \end{array}$$

This gives the system

$$\begin{array}{rcrl} x + y & = & 19{,}000 & (8) \\ 13x + 14y & = & 252{,}000 & (9) \\ -x + 2y & = & 4000. & (5) \end{array}$$

The augmented matrix for this system is

$$\left[\begin{array}{rr|r} 1 & 1 & 19{,}000 \\ 13 & 14 & 252{,}000 \\ -1 & 2 & 4000 \end{array}\right].$$

$$\begin{array}{r} \\ -13R_1 + R_2 \to R_2 \\ R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{rr|r} 1 & 1 & 19{,}000 \\ 0 & 1 & 5000 \\ 0 & 3 & 23{,}000 \end{array}\right].$$

$$\begin{array}{r} \\ \\ -3R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{rr|r} 1 & 1 & 19{,}000 \\ 0 & 1 & 5000 \\ 0 & 0 & 8000 \end{array}\right].$$

The equation that corresponds to the last row of this matrix is $0x + 0y = 8000$, so there is no solution if \$6000 is borrowed at 12%.

52. Let $x_1 =$ the number of units from first supplier for Roseville,

$x_2 =$ the number of units from first supplier for Akron,

$x_3 =$ the number of units from second supplier for Roseville, and

$x_4 =$ the number of units from second supplier for Akron.

Roseville needs 40 units so

$$x_1 + x_3 = 40.$$

Akron needs 75 units so

$$x_2 + x_4 = 75.$$

The manufacturer orders 75 units from the first supplier so

$$x_1 + x_2 = 75.$$

The total cost is \$10,750 so

$$70x_1 + 90x_2 + 80x_3 + 120x_4 = 10{,}750.$$

The system to be solve is

$$\begin{aligned}
x_1 \qquad\quad + x_3 \qquad\quad &= 40\\
x_2 \qquad\quad + \quad x_4 &= 75\\
x_1 + \quad x_2 \qquad\qquad\qquad &= 75\\
70x_1 + 90x_2 + 80x_3 + 120x_4 &= 10{,}750.
\end{aligned}$$

Write augmented matrix and use row operations.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 40\\ 0 & 1 & 0 & 1 & 75\\ 1 & 1 & 0 & 0 & 75\\ 70 & 90 & 80 & 120 & 10{,}750 \end{array}\right]$$

$$\begin{array}{r} \\ \\ -1R_1+R_3\to R_3\\ -70R_1+R_4\to R_4 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 40\\ 0 & 1 & 0 & 1 & 75\\ 0 & 1 & -1 & 0 & 35\\ 0 & 90 & 10 & 120 & 7950 \end{array}\right]$$

$$\begin{array}{r} \\ \\ -1R_2+R_3\to R_3\\ -90R_2+R_4\to R_4 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 40\\ 0 & 1 & 0 & 1 & 75\\ 0 & 0 & -1 & -1 & -40\\ 0 & 0 & 10 & 30 & 1200 \end{array}\right]$$

$$\begin{array}{r} \\ \\ -1R_3\to R_3\\ 10R_3+R_4\to R_4 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 40\\ 0 & 1 & 0 & 1 & 75\\ 0 & 0 & 1 & 1 & 40\\ 0 & 0 & 0 & 20 & 800 \end{array}\right]$$

$$\begin{array}{r} \\ \\ \\ \frac{1}{20}R_4\to R_4 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 40\\ 0 & 1 & 0 & 1 & 75\\ 0 & 0 & 1 & 1 & 40\\ 0 & 0 & 0 & 1 & 40 \end{array}\right]$$

$$\begin{array}{r} \\ -1R_4+R_2\to R_2\\ -1R_4+R_3\to R_3\\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 40\\ 0 & 1 & 0 & 0 & 35\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 40 \end{array}\right]$$

$$\begin{array}{r} -1R_3+R_1\to R_1\\ \\ \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 40\\ 0 & 1 & 0 & 0 & 35\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 40 \end{array}\right]$$

The solution of the system is $x_1 = 40, x_2 = 35, x_3 = 0, x_4 = 40$, or $(40, 35, 0, 40)$. The first supplier should send 40 units to Roseville and 35 units to Akron. The second supplier should send 0 units to Roseville and 40 units to Akron.

54. **(a)** The other two equations are

$$\begin{aligned} x_2 + x_3 &= 700\\ x_3 + x_4 &= 600. \end{aligned}$$

(b) The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1000\\ 1 & 1 & 0 & 0 & 1100\\ 0 & 1 & 1 & 0 & 700\\ 0 & 0 & 1 & 1 & 600 \end{array}\right].$$

$$\begin{array}{r} \\ \\ -1R_1+R_2\to R_2\\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1000\\ 0 & 1 & 0 & -1 & 100\\ 0 & 1 & 1 & 0 & 700\\ 0 & 0 & 1 & 1 & 600 \end{array}\right]$$

$$\begin{array}{r} \\ \\ -1R_2+R_3\to R_3\\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1000\\ 0 & 1 & 0 & -1 & 100\\ 0 & 0 & 1 & 1 & 600\\ 0 & 0 & 1 & 1 & 600 \end{array}\right]$$

$$\begin{array}{r} \\ \\ \\ -1R_3+R_4\to R_4 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1000\\ 0 & 1 & 0 & -1 & 100\\ 0 & 0 & 1 & 1 & 600\\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Let x_4 be arbitrary. Solve the first three equations for x_1, x_2, and x_3.

$$\begin{aligned} x_1 &= 1000 - x_4\\ x_2 &= 100 + x_4\\ x_3 &= 600 - x_4 \end{aligned}$$

The solution is $(1000 - x_4, 100 + x_4, 600 - x_4, x_4)$.

(c) For x_4, we see that $x_4 \geq 0$ and $x_4 \leq 600$ since $600 - x_4$ must be nonnegative. Therefore, $0 \leq x_4 \leq 600$.

(d) x_1: If $x_4 = 0$, then $x_1 = 1000$.
If $x_4 = 600$, then $x_1 = 1000 - 600 = 400$.

Therefore, $400 \leq x_1 \leq 1000$.

x_2: If $x_4 = 0$, then $x_2 = 100$.
If $x_4 = 600$, then $x_2 = 100 + 600 = 700$.

Therefore, $100 \leq x_2 \leq 700$.

x_3: If $x_4 = 0$, then $x_3 = 600$.
If $x_4 = 600$, then $x_3 = 600 - 600 = 0$.

Therefore, $0 \leq x_3 \leq 600$.

(e) If you know the number of cars entering or leaving three of the intersections, then the number entering or leaving the fourth is automatically determined because the number leaving must equal the number entering.

56. Let $x =$ the number of grams of group A,
$y =$ the number of grams of group B, and
$z =$ the number of grams of group C.

(a) The system to be solved is

$$\begin{aligned} x+y+z &= 400 \quad (\mathit{1})\\ x \qquad &= \frac{1}{3}y \quad (\mathit{2})\\ x \quad +z &= \ 2y. \quad (\mathit{3}) \end{aligned}$$

Rewrite equations (2) and (3) in proper form and multiply both sides of equation (2) by 3.

$$\begin{aligned} x+\ y+z &= 400\\ 3x-\ y \quad &= \ 0\\ x-2y+z &= \ 0 \end{aligned}$$

Write the augmented matrix.

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 400\\ 3 & -1 & 0 & 0\\ 1 & -2 & 1 & 0 \end{array}\right]$$

$$\begin{array}{r} \\ -3\text{R}_1+\text{R}_2\to\text{R}_2\\ -1\text{R}_1+\text{R}_3\to\text{R}_3 \end{array} \left[\begin{array}{rrr|r} 1 & 1 & 1 & 400\\ 0 & -4 & -3 & -1200\\ 0 & -3 & 0 & -400 \end{array}\right]$$

$$\begin{array}{r} \\ \\ -\frac{1}{3}\text{R}_3\to\text{R}_3 \end{array} \left[\begin{array}{rrr|r} 1 & 1 & 1 & 400\\ 0 & -4 & -3 & -1200\\ 0 & 1 & 0 & \frac{400}{3} \end{array}\right]$$

Interchange rows 2 and 3.

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 400\\ 0 & 1 & 0 & \frac{400}{3}\\ 0 & -4 & -3 & -1200 \end{array}\right]$$

$$\begin{array}{r} -1\text{R}_2+\text{R}_1\to\text{R}_1\\ \\ 4\text{R}_2+\text{R}_3\to\text{R}_3 \end{array} \left[\begin{array}{rrr|r} 1 & 0 & 1 & \frac{800}{3}\\ 0 & 1 & 0 & \frac{400}{3}\\ 0 & 0 & -3 & -\frac{2000}{3} \end{array}\right]$$

$$\begin{array}{r} \\ \\ -\frac{1}{3}\text{R}_3\to\text{R}_3 \end{array} \left[\begin{array}{rrr|r} 1 & 0 & 1 & \frac{800}{3}\\ 0 & 1 & 0 & \frac{400}{3}\\ 0 & 0 & 1 & \frac{2000}{9} \end{array}\right]$$

$$\begin{array}{r} -1\text{R}_3+\text{R}_1\to\text{R}_1\\ \\ \\ \end{array} \left[\begin{array}{rrr|r} 1 & 0 & 0 & \frac{400}{9}\\ 0 & 1 & 0 & \frac{400}{3}\\ 0 & 0 & 1 & \frac{2000}{9} \end{array}\right]$$

The solution is $\left(\frac{400}{9}, \frac{400}{3}, \frac{2000}{9}\right)$. Include $\frac{400}{9}$ g of group A, $\frac{400}{3}$ g of group B, and $\frac{2000}{9}$ g of group C.

(b) If the requirement that the diet include one-third as much of A as of B is dropped, refer to the first two rows of the fifth augmented matrix in part (a).

$$\left[\begin{array}{rrr|r} 1 & 0 & 1 & \frac{800}{3}\\ 0 & 1 & 0 & \frac{400}{3} \end{array}\right]$$

This gives

$$\begin{aligned} x &= \frac{800}{3}-z\\ y &= \frac{400}{3}. \end{aligned}$$

Therefore, for any positive number of grams of group C, there should be C grams less than $\frac{800}{3}$ g of group A and $\frac{400}{3}$ g of group B.

(c) Since there was a unique solution for the original problem, by adding an additional condition, the only possible solution would be the one from part (a). However, by substituting those values of A, B, and C for $x, y,$ and z in the equation for the additional condition, $.02x+.02y+.03z = 8.00$, the values do not work. Therefore, a solution is not possible.

58. Let x be the number of two-person tents, y be the number of three-person tents, and z be the number of four-person tents that were ordered. The problem then is to solve the following system of equations.

$$\begin{aligned} 2x+\quad 3y+\quad 4z &= \quad 166\\ 150x+200y+250z &= 11,150\\ 3x+\quad 5y+\quad 8z &= \quad 289 \end{aligned}$$

To simplify matters, divide each term in the second equation by 50 to obtain $3x+4y+5z = 223$. Write the augmented matrix and use row operations.

$$\left[\begin{array}{rrr|r} 2 & 3 & 4 & 166 \\ 3 & 4 & 5 & 223 \\ 3 & 5 & 8 & 289 \end{array}\right]$$

$$\begin{array}{r} \\ -3\text{R}_1 + \text{R}_2 \to \text{R}_2 \\ -3\text{R}_1 + 2\text{R}_3 \to \text{R}_3 \end{array} \left[\begin{array}{rrr|r} 2 & 3 & 4 & 166 \\ 0 & -1 & -2 & -52 \\ 0 & 1 & 4 & 80 \end{array}\right]$$

$$\begin{array}{r} 3\text{R}_2 + \text{R}_1 \to \text{R}_1 \\ \\ \text{R}_2 + \text{R}_3 \to \text{R}_3 \end{array} \left[\begin{array}{rrr|r} 2 & 0 & -2 & 10 \\ 0 & -1 & -2 & -52 \\ 0 & 0 & 2 & 28 \end{array}\right]$$

$$\begin{array}{r} \text{R}_3 + \text{R}_1 \to \text{R}_1 \\ \text{R}_3 + \text{R}_2 \to \text{R}_2 \\ \\ \end{array} \left[\begin{array}{rrr|r} 2 & 0 & 0 & 38 \\ 0 & -1 & 0 & -24 \\ 0 & 0 & 2 & 28 \end{array}\right]$$

$$\begin{array}{r} \frac{1}{2}\text{R}_1 \to \text{R}_1 \\ -1\text{R}_2 \to \text{R}_2 \\ \frac{1}{2}\text{R}_3 \to \text{R}_3 \end{array} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 19 \\ 0 & 1 & 0 & 24 \\ 0 & 0 & 1 & 14 \end{array}\right]$$

Read the solution from the last column of the matrix. The solution is $x = 19, y = 24$, and $z = 14$ so 19 two-person, 24 three-person, and 14 four-person tents were ordered.

60. Let $x =$ the number of species A,
$y =$ the number of species B, and
$z =$ the number of species C.

Use a chart to organize the information.

		Species			
		A	B	C	Totals
	I	1.32	2.1	.86	490
Food	II	2.9	.95	1.52	897
	III	1.75	.6	2.01	653

The system to be solved is

$$\begin{aligned} 1.32x + 2.1y + .86z &= 490 \\ 2.9x + .95y + 1.52z &= 897 \\ 1.75x + .6y + 2.01z &= 653. \end{aligned}$$

Use graphing calculator or computer methods to solve this system. The solution, which may vary slightly, is to stock about 244 fish of species A, 39 fish of species B, and 101 fish of species C.

62. **(a)** Bulls:

The number of white ones was one half plus one third the number of black greater than the brown.

$$\begin{aligned} X &= \left(\frac{1}{2} + \frac{1}{3}\right) Y + T \\ X &= \frac{5}{6} Y + T \\ 6X &= 5Y + 6T \\ 6X - 5Y &= 6T \end{aligned}$$

The number of the black, one quarter plus one fifth the number of the spotted greater than the brown.

$$\begin{aligned} Y &= \left(\frac{1}{4} + \frac{1}{5}\right) Z + T \\ Y &= \frac{9}{20} Z + T \\ 20Y &= 9Z + 20T \\ 20Y - 9Z &= 20T \end{aligned}$$

The number of the spotted, one sixth and one seventh the number of the white greater than the brown.

$$\begin{aligned} Z &= \left(\frac{1}{6} + \frac{1}{7}\right) X + T \\ Z &= \frac{13}{42} X + T \\ 42Z &= 13X + 42T \\ 42Z - 13X &= 42T \end{aligned}$$

So the system of equations for the bulls is

$$\begin{aligned} 6X - 5Y &= 6T \\ 20Y - 9Z &= 20T \\ 42Z - 13X &= 42T. \end{aligned}$$

Cows:

The number of white ones was one third plus one quarter of the total black cattle.

$$\begin{aligned} x &= \left(\frac{1}{3} + \frac{1}{4}\right) (Y + y) \\ x &= \frac{7}{12}(Y + y) \\ 12x &= 7Y + 7y \\ 12x - 7y &= 7Y \end{aligned}$$

The number of the black, one quarter plus one fifth the total of the spotted cattle.

$$\begin{aligned} y &= \left(\frac{1}{4} + \frac{1}{5}\right) (Z + z) \\ y &= \frac{9}{20}(Z + z) \\ 20y &= 9Z + 9z \\ 20y - 9z &= 9Z \end{aligned}$$

The number of the spotted, one fifth plus one sixth the total of the brown cattle.

$$\begin{aligned} z &= \left(\frac{1}{5}+\frac{1}{6}\right)(T+t) \\ z &= \frac{11}{30}(T+t) \\ 30z &= 11T+11t \\ 30z-11t &= 11T \end{aligned}$$

The number of the brown, one sixth plus one seventh the total of the white cattle.

$$\begin{aligned} t &= \left(\frac{1}{6}+\frac{1}{7}\right)(X+x) \\ t &= \frac{13}{42}(X+x) \\ 42t &= 13X+13x \\ 42t-13x &= 13X \end{aligned}$$

So the system of equations for the cows is

$$\begin{aligned} 12x-7y &= 7Y \\ 20y-9z &= 9Z \\ 30z-11t &= 11T \\ -13x+42t &= 13X \end{aligned}$$

(b) For $T = 4{,}149{,}387$, the 3×3 system to be solved is

$$\begin{aligned} 6X - 5Y \phantom{{}-9Z} &= 24{,}896{,}322 \\ 20Y - 9Z &= 82{,}987{,}740 \\ -13X \phantom{{}-5Y} + 42Z &= 174{,}274{,}254 \end{aligned}$$

Write the augmented matrix of the system.

$$\left[\begin{array}{rrr|r} 6 & -5 & 0 & 24{,}896{,}322 \\ 0 & 20 & -9 & 82{,}987{,}740 \\ -13 & 0 & 42 & 174{,}274{,}254 \end{array}\right]$$

This exercise should be solved by graphing calculator or computer methods. The solution is $X = 10{,}366{,}482$ white bulls, $Y = 7{,}460{,}514$ black bulls, and $Z = 7{,}358{,}060$ spotted bulls.
For $X = 10{,}366{,}482$, $Y = 7{,}460{,}514$, and $Z = 7{,}358{,}060$, the 4×4 system to be solved is

$$\begin{aligned} 12x-7y &= 52{,}223{,}598 \\ 20y-9z &= 66{,}222{,}540 \\ 30z-11t &= 45{,}643{,}257 \\ -13x+42t &= 134{,}764{,}266 \end{aligned}$$

Write the augmented matrix of the system.

$$\left[\begin{array}{rrrr|r} 12 & -7 & 0 & 0 & 52{,}223{,}598 \\ 0 & 20 & -9 & 0 & 66{,}222{,}540 \\ 0 & 0 & 30 & -11 & 45{,}643{,}257 \\ -13 & 0 & 0 & 42 & 134{,}764{,}266 \end{array}\right]$$

This exercise should be solved by graphing calculator or computer methods. The solution is $x = 7{,}206{,}360$ white cows, $y = 4{,}893{,}246$ black cows, $z = 3{,}515{,}820$ spotted cows, and $t = 5{,}439{,}213$ brown cows.

64. (a) $5.4 = a(8)^2 + b(8) + c$

$$\begin{aligned} 5.4 &= 64a+8b+c \\ 6.3 &= a(13)^2+b(13)+c \\ 6.3 &= 169a+13b+c \\ 5.6 &= a(18)^2+b(18)+c \\ 5.6 &= 324a+18b+c \end{aligned}$$

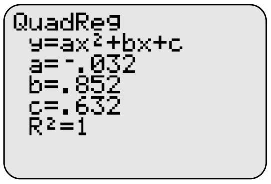
The linear system to be solved is

$$\begin{aligned} 64a + 8b + c &= 5.4 \\ 169a + 13b + c &= 6.3 \\ 324a + 18b + c &= 5.6. \end{aligned}$$

Use a graphing calculator or computer methods to solve this system. The solution is $a = -.032, b = .852$, and $c = .632$. Thus, the equation is

$$y = -.032x^2 + .852x + .632.$$

(b)



```
QuadReg
 y=ax²+bx+c
 a=⁻.032
 b=.852
 c=.632
 R²=1
```

The answer obtained using Gauss–Jordan elimination is the same as the answer obtained using the quadratic regression feature on a graphing calculator.

66. (a) $x_{11} + x_{12} + x_{21} = 1$

$$\begin{aligned} x_{11}+x_{12}+x_{22} &= 1 \\ x_{11}+x_{21}+x_{22} &= 1 \\ x_{12}+x_{21}+x_{22} &= 1 \end{aligned}$$

Write the augmented matrix of the system.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

$$\begin{array}{l} -1\mathrm{R}_1 + \mathrm{R}_2 \rightarrow \mathrm{R}_2 \\ -1\mathrm{R}_1 + \mathrm{R}_3 \rightarrow \mathrm{R}_3 \end{array} \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

Since $-1 = 1$ modulo 2, replace -1 with 1.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

Interchange rows 2 and 3.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

$$\begin{array}{r} -1\mathrm{R}_2 + \mathrm{R}_1 \to \mathrm{R}_1 \\ \\ \\ -\mathrm{R}_2 + \mathrm{R}_4 \to \mathrm{R}_4 \end{array} \left[\begin{array}{cccr|c} 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

Again, replace -1 with 1.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} -1\mathrm{R}_3 + \mathrm{R}_1 \to \mathrm{R}_1 \\ \\ \\ -1\mathrm{R}_3 + \mathrm{R}_4 \to \mathrm{R}_4 \end{array} \left[\begin{array}{cccr|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{array}\right]$$

Replace -1 with 1.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ -1\mathrm{R}_4 + \mathrm{R}_2 \to \mathrm{R}_2 \\ -1\mathrm{R}_4 + \mathrm{R}_3 \to \mathrm{R}_3 \\ \\ \end{array} \left[\begin{array}{cccc|r} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

Finally, replace -1 with 1.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

The solution $(1, 1, 1, 1)$ corresponds to $x_{11} = 1$, $x_{12} = 1, x_{21} = 1$, and $x_{22} = 1$. Since 1 indicates that a button is pushed, the strategy required to turn all the lights out is to push every button one time.

(b) $x_{11} + x_{12} + x_{21} = 0$
$x_{11} + x_{12} + x_{22} = 1$
$x_{11} + x_{21} + x_{22} = 1$
$x_{12} + x_{21} + x_{22} = 0$

Write the augmented matrix of the system.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$\begin{array}{r} \\ -1\mathrm{R}_1 + \mathrm{R}_2 \to \mathrm{R}_2 \\ -1\mathrm{R}_1 + \mathrm{R}_3 \to \mathrm{R}_3 \\ \\ \end{array} \left[\begin{array}{crrc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

Replace -1 with 1.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

Interchange rows 2 and 3.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$\begin{array}{r} -1\mathrm{R}_2 + \mathrm{R}_1 \to \mathrm{R}_1 \\ \\ \\ -1\mathrm{R}_2 + \mathrm{R}_4 \to \mathrm{R}_4 \end{array} \left[\begin{array}{cccr|r} 1 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{array}\right]$$

Replace -1 with 1.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} -1\mathrm{R}_3 + \mathrm{R}_1 \to \mathrm{R}_1 \\ \\ \\ -1\mathrm{R}_3 + \mathrm{R}_4 \to \mathrm{R}_4 \end{array} \left[\begin{array}{cccr|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

Replace -1 with 1.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{array}{r} \\ -1\mathrm{R}_4 + \mathrm{R}_2 \to \mathrm{R}_2 \\ -1\mathrm{R}_4 + \mathrm{R}_3 \to \mathrm{R}_3 \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

The solution $(0, 1, 1, 0)$ corresponds to $x_{11} = 0, x_{12} = 1, x_{21} = 1$, and $x_{22} = 0$. Since 1 indicates that a button is pushed and 0 indicates that it is not, the strategy required to turn all the lights out is to push the button in the first row, second column, and push the button in the second row first column.

2.3 Addition and Subtraction of Matrices

2. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = [1 \quad 2 \quad 3]$

This statement is false. For two matrices to be equal, they must be the same size, and each pair of corresponding elements must be equal. These two matrices are different sizes.

4. $\begin{bmatrix} 3 & 5 & 2 & 8 \\ 1 & -1 & 4 & 0 \end{bmatrix}$ is a 4×2 matrix.

This statement is false. Since the matrix has 2 rows and 4 columns, it is a 2×4 matrix.

6. $\begin{bmatrix} 2 & 4 & -1 \\ 3 & 7 & 5 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -1 \\ 3 & 7 & 5 \end{bmatrix}$

This statement is false since the matrices are different sizes.

8. $\begin{bmatrix} -9 & 6 & 2 \\ 4 & 1 & 8 \end{bmatrix}$

This matrix has 2 rows and 3 columns, so it is a 2×3 matrix. The additive inverse is

$$\begin{bmatrix} 9 & -6 & -2 \\ -4 & -1 & -8 \end{bmatrix}.$$

10. $\begin{bmatrix} 8 & -2 & 4 & 6 & 3 \end{bmatrix}$

The matrix has 1 row and 5 columns, so it is a 1×5 matrix. It is a row matrix since it has only 1 row. The additive inverse is

$$\begin{bmatrix} -8 & 2 & -4 & -6 & -3 \end{bmatrix}.$$

12. $[-9]$

This matrix has 1 row and 1 column, so it is a 1×1 square matrix. It is also a row matrix since it has only 1 row, and a column matrix because it has only 1 column. The additive inverse is $[9]$.

14. Since A is a 5×2 matrix, and since A and K can be added, we know that K is also a 5×2 matrix. Also, since $A + K = A$, all entries of K must be 0.

16. $\begin{bmatrix} -5 \\ y \end{bmatrix} = \begin{bmatrix} -5 \\ 8 \end{bmatrix}$

Two matrices can be equal only if they are the same size and corresponding elements are equal. These matrices will be equal if $y = 8$.

18. $\begin{bmatrix} 9 & 7 \\ r & 0 \end{bmatrix} = \begin{bmatrix} m-3 & n+5 \\ 8 & 0 \end{bmatrix}$

The matrices are the same size, so they will be equal if corresponding elements are equal.

$$\begin{aligned} 9 &= m - 3 & 7 &= n + 5 & r &= 8 \\ 12 &= m & 2 &= n \end{aligned}$$

Thus, $m = 12$, $n = 2$, and $r = 8$.

20. $\begin{bmatrix} a+2 & 3z+1 & 5m \\ 4k & 0 & 3 \end{bmatrix} + \begin{bmatrix} 3a & 2z & 5m \\ 2k & 5 & 6 \end{bmatrix}$

$$= \begin{bmatrix} 10 & -14 & 80 \\ 10 & 5 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 4a+2 & 5z+1 & 10m \\ 6k & 5 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & -14 & 80 \\ 10 & 5 & 9 \end{bmatrix}$$

$$\begin{aligned} 4a + 2 &= 10 & 5z + 1 &= -14 \\ a &= 2 & z &= -3 \end{aligned}$$

$$\begin{aligned} 10m &= 80 & 6k &= 10 \\ m &= 8 & k &= \frac{5}{3} \end{aligned}$$

Thus, $a = 2$, $z = -3$, $m = 8$, and $k = \frac{5}{3}$.

22. $\begin{bmatrix} 1 & 5 \\ 2 & -3 \\ 3 & 7 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 8 & 5 \\ -1 & 9 \end{bmatrix} = \begin{bmatrix} 1+2 & 5+3 \\ 2+8 & -3+5 \\ 3+(-1) & 7+9 \end{bmatrix}$

$$= \begin{bmatrix} 3 & 8 \\ 10 & 2 \\ 2 & 16 \end{bmatrix}$$

24. $\begin{bmatrix} 1 & 3 & -2 \\ 4 & 7 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 6 & -5 \\ 0 & 4 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 3 & -2 \\ 4 & 7 & 1 \end{bmatrix} + \begin{bmatrix} -3 & -6 & 5 \\ 0 & -4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -3 & 3 \\ 4 & 3 & -1 \end{bmatrix}$$

26. $$\begin{bmatrix}2&1\\5&-3\\-7&2\\9&0\end{bmatrix}+\begin{bmatrix}1&-8&0\\5&3&2\\-6&7&-5\\2&-1&0\end{bmatrix}$$

This operation is not possible because the matrices are different sizes. Only matrices of the same size can be added.

28. $$\begin{bmatrix}4&3\\1&2\end{bmatrix}-\begin{bmatrix}1&1\\1&0\end{bmatrix}+\begin{bmatrix}1&1\\1&4\end{bmatrix}$$

$$=\begin{bmatrix}4&3\\1&2\end{bmatrix}+\begin{bmatrix}-1&-1\\-1&0\end{bmatrix}+\begin{bmatrix}1&1\\1&4\end{bmatrix}$$

$$=\begin{bmatrix}4+(-1)+1&3+(-1)+1\\1+(-1)+1&2+0+4\end{bmatrix}$$

$$=\begin{bmatrix}4&3\\1&6\end{bmatrix}$$

30. $$\begin{bmatrix}-1&-1\\-1&0\end{bmatrix}+\begin{bmatrix}4&3\\1&2\end{bmatrix}+\begin{bmatrix}1&1\\1&4\end{bmatrix}$$

$$=\begin{bmatrix}-1+4+1&-1+3+1\\-1+1+1&0+2+4\end{bmatrix}=\begin{bmatrix}4&3\\1&6\end{bmatrix}$$

32. $$\begin{bmatrix}4k-8y\\6z-3x\\2k+5a\\-4m+2n\end{bmatrix}-\begin{bmatrix}5k+6y\\2z+5x\\4k+6a\\4m-2n\end{bmatrix}$$

$$=\begin{bmatrix}4k-8y\\6z-3x\\2k+5a\\-4m+2n\end{bmatrix}+\begin{bmatrix}-5k-6y\\-2z-5x\\-4k-6a\\-4m+2n\end{bmatrix}$$

$$=\begin{bmatrix}-k-14y\\4z-8x\\-2k-a\\-8m+4n\end{bmatrix}$$

34. Verify that $X+T=T+X$.

$$X+T=\begin{bmatrix}x+r&y+s\\z+t&w+u\end{bmatrix}$$

$$T+X=\begin{bmatrix}r+x&s+y\\t+z&u+w\end{bmatrix}$$

Because of the commutative property for addition of real numbers, $x+r=r+x$. This also applies to the other corresponding elements, so we conclude that $T+X=X+T$.

36. Verify that $X+(-X)=O$.

$$X+(-X)=\begin{bmatrix}x&y\\z&w\end{bmatrix}+\begin{bmatrix}-x&-y\\-z&-w\end{bmatrix}$$

$$=\begin{bmatrix}x+(-x)&y+(-y)\\z+(-z)&w+(-w)\end{bmatrix}$$

$$=\begin{bmatrix}0&0\\0&0\end{bmatrix}=O$$

38. All of these properties are valid for matrices that are not square, as long as all necessary sums exist. The sizes of all matrices in each equation must be the same.

40. **(a)**

$$\begin{array}{l|rrr|} & \text{I} & \text{II} & \text{III} \\ \text{Bread} & 88 & 105 & 60 \\ \text{Milk} & 48 & 72 & 40 \\ \text{PB} & 16 & 21 & 0 \\ \text{Cold cuts} & 112 & 147 & 50 \end{array}$$

(b) $$\begin{bmatrix}88+.25(88)&105+\frac{1}{3}(105)&60+.1(60)\\48+.25(48)&72+\frac{1}{3}(72)&40+.1(40)\\16+.25(16)&21+\frac{1}{3}(21)&0+.1(0)\\112+.25(112)&147+\frac{1}{3}(147)&50+.1(50)\end{bmatrix}$$

$$=\begin{bmatrix}110&140&66\\60&96&44\\20&28&0\\140&196&55\end{bmatrix}$$

(c) Add the final matrices from parts (a) and (b).

$$\begin{bmatrix}88&105&60\\48&72&40\\16&21&0\\112&147&50\end{bmatrix}+\begin{bmatrix}110&140&66\\60&96&44\\20&28&0\\140&196&55\end{bmatrix}$$

$$=\begin{bmatrix}198&245&126\\108&168&84\\36&49&0\\252&343&105\end{bmatrix}$$

42. **(a)**
$$\begin{array}{l} \text{Length} \\ \text{Weight} \end{array} \begin{bmatrix} 5.6 & 6.4 & 6.9 & 7.6 & 6.1 \\ 144 & 138 & 149 & 152 & 146 \end{bmatrix}$$

(b)
$$\begin{array}{l} \text{Length} \\ \text{Weight} \end{array} \begin{bmatrix} 10.2 & 11.4 & 11.4 & 12.7 & 10.8 \\ 196 & 196 & 225 & 250 & 230 \end{bmatrix}$$

(c) Subtract the matrix in part (a) from the one in part (b).

$$\begin{bmatrix} 10.2 & 11.4 & 11.4 & 12.7 & 10.8 \\ 196 & 196 & 225 & 250 & 230 \end{bmatrix} - \begin{bmatrix} 5.6 & 6.4 & 6.9 & 7.6 & 6.1 \\ 144 & 138 & 149 & 152 & 146 \end{bmatrix}$$

$$= \begin{bmatrix} 4.6 & 5.0 & 4.5 & 5.1 & 4.7 \\ 52 & 58 & 76 & 98 & 84 \end{bmatrix} \begin{array}{l} \text{Change in length} \\ \text{Change in weight} \end{array}$$

(d) Add the new matrix to the one from part (b).

$$\begin{bmatrix} 10.2 & 11.4 & 11.4 & 12.7 & 10.8 \\ 196 & 196 & 225 & 250 & 230 \end{bmatrix} + \begin{bmatrix} 1.8 & 1.5 & 2.3 & 1.8 & 2.0 \\ 25 & 22 & 29 & 33 & 20 \end{bmatrix}$$

$$= \begin{bmatrix} 12.0 & 12.9 & 13.7 & 14.5 & 12.8 \\ 221 & 218 & 254 & 283 & 250 \end{bmatrix} \begin{array}{l} \text{Final length} \\ \text{Final weight} \end{array}$$

44. **(a)** The matrix for the death rate of male drivers is

$$\begin{array}{lc} & \text{Number of Passengers} \\ & \begin{array}{cccc} 0 & 1 & 2 & \geq 3 \end{array} \\ \begin{array}{l} \text{Age 16} \\ \text{Age 17} \\ \text{Ages 30–59} \end{array} & \begin{bmatrix} 2.61 & 4.39 & 6.29 & 9.08 \\ 1.63 & 2.77 & 4.61 & 6.92 \\ .92 & .75 & .62 & .54 \end{bmatrix} \end{array}$$

(b) The matrix for the death rate of female drivers is

$$\begin{array}{lc} & \text{Number of Passengers} \\ & \begin{array}{cccc} 0 & 1 & 2 & \geq 3 \end{array} \\ \begin{array}{l} \text{Age 16} \\ \text{Age 17} \\ \text{Ages 30–59} \end{array} & \begin{bmatrix} 1.38 & 1.72 & 1.94 & 3.31 \\ 1.26 & 1.48 & 2.82 & 2.28 \\ .41 & .33 & .27 & .40 \end{bmatrix} \end{array}$$

(c) The matrix showing the difference between the death rates of males and females is

$$\begin{bmatrix} 2.61 & 4.39 & 6.29 & 9.08 \\ 1.63 & 2.77 & 4.61 & 6.92 \\ .92 & .75 & .62 & .54 \end{bmatrix} - \begin{bmatrix} 1.38 & 1.72 & 1.94 & 3.31 \\ 1.26 & 1.48 & 2.82 & 2.28 \\ .41 & .33 & .27 & .40 \end{bmatrix}$$

$$= \begin{bmatrix} 2.61 & 4.39 & 6.29 & 9.08 \\ 1.63 & 2.77 & 4.61 & 6.92 \\ .92 & .75 & .62 & .54 \end{bmatrix} + \begin{bmatrix} -1.38 & -1.72 & -1.94 & -3.31 \\ -1.26 & -1.48 & -2.82 & -2.28 \\ -.41 & -.33 & -.27 & -.40 \end{bmatrix}$$

$$= \begin{bmatrix} 1.23 & 2.67 & 4.35 & 5.77 \\ .37 & 1.29 & 1.79 & 4.64 \\ .51 & .42 & .35 & .14 \end{bmatrix}$$

46. **(a)** The matrix for the educational attainment of males is

$$\begin{array}{c|cc|}
 & \begin{array}{c}\text{4 Years of}\\ \text{High School}\\ \text{or More}\end{array} & \begin{array}{c}\text{4 Years of}\\ \text{College}\\ \text{or More}\end{array} \\
1960 & 39.5 & 9.7 \\
1970 & 51.9 & 13.5 \\
1980 & 67.3 & 20.1 \\
1990 & 77.7 & 24.4 \\
1995 & 81.7 & 26.0 \\
2000 & 84.2 & 27.8
\end{array}$$

(b) The matrix for the educational attainment of females is

$$\begin{array}{c|cc|}
 & \begin{array}{c}\text{4 Years of}\\ \text{High School}\\ \text{or More}\end{array} & \begin{array}{c}\text{4 Years of}\\ \text{College}\\ \text{or More}\end{array} \\
1960 & 42.5 & 5.8 \\
1970 & 52.8 & 8.1 \\
1980 & 65.8 & 12.8 \\
1990 & 77.5 & 18.4 \\
1995 & 81.6 & 20.2 \\
2000 & 84.0 & 23.6
\end{array}$$

(c) The matrix showing how much more (or less) education males have attained than females is

$$\begin{bmatrix} 39.5 & 9.7 \\ 51.9 & 13.5 \\ 67.3 & 20.1 \\ 77.7 & 24.4 \\ 81.7 & 26.0 \\ 84.2 & 27.8 \end{bmatrix} - \begin{bmatrix} 42.5 & 5.8 \\ 52.8 & 8.1 \\ 65.8 & 12.8 \\ 77.5 & 18.4 \\ 81.6 & 20.2 \\ 84.0 & 23.6 \end{bmatrix} = \begin{bmatrix} 39.5 & 9.7 \\ 51.9 & 13.5 \\ 67.3 & 20.1 \\ 77.7 & 24.4 \\ 81.7 & 26.0 \\ 84.2 & 27.8 \end{bmatrix} + \begin{bmatrix} -42.5 & -\ 5.8 \\ -52.8 & -\ 8.1 \\ -65.8 & -12.8 \\ -77.5 & -18.4 \\ -81.6 & -20.2 \\ -84.0 & -23.6 \end{bmatrix} = \begin{bmatrix} -3.0 & 3.9 \\ -\ .9 & 5.4 \\ 1.5 & 7.3 \\ .2 & 6.0 \\ .1 & 5.8 \\ .2 & 4.2 \end{bmatrix}$$

48. **(a)**

$$\begin{array}{c|cccc|}
 & \text{M} & \text{J} & \text{Ca} & \text{Cl} \\
\text{M} & 1 & 1 & 1 & 1 \\
\text{J} & 0 & 1 & 0 & 0 \\
\text{Ca} & 0 & 1 & 1 & 1 \\
\text{Cl} & 0 & 1 & 1 & 1
\end{array}$$

(b) Rows 1 and 2 will stay the same. Since the cats now like Musk, the zeros in rows 3 and 4 change to ones.

$$\begin{array}{c|cccc|}
 & \text{M} & \text{J} & \text{Ca} & \text{Cl} \\
\text{M} & 1 & 1 & 1 & 1 \\
\text{J} & 0 & 1 & 0 & 0 \\
\text{Ca} & 1 & 1 & 1 & 1 \\
\text{Cl} & 1 & 1 & 1 & 1
\end{array}$$

2.4 Multiplication of Matrices

In Exercises 2-6, let

$$A = \begin{bmatrix} -2 & 4 \\ 0 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} -6 & 2 \\ 4 & 0 \end{bmatrix}.$$

2. $-3B = -3\begin{bmatrix} -6 & 2 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 18 & -6 \\ -12 & 0 \end{bmatrix}$

4. $5A = 5\begin{bmatrix} -2 & 4 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -10 & 20 \\ 0 & 15 \end{bmatrix}$

6.
$$\begin{aligned} 3A - 10B &= 3\begin{bmatrix} -2 & 4 \\ 0 & 3 \end{bmatrix} - 10\begin{bmatrix} -6 & 2 \\ 4 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -6 & 12 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} -60 & 20 \\ 40 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -6 & 12 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 60 & -20 \\ -40 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 54 & -8 \\ -40 & 9 \end{bmatrix} \end{aligned}$$

8. A is 3×3, and B is 3×3.

The number of columns of A is the same as the number of rows of B, so the product AB exists; its size is 3×3. The number of columns of B is the same as the number of rows of A, so the product BA also exists; its size is 3×3.

10. Matrix A size Matrix B size

$4 \times \underline{\mathbf{3}}$ $\underline{\mathbf{3}} \times 6$

Since the number of columns of A, 3, is the same as the number of rows of B, 3, the product AB exists. Its size is 4×6.

Matrix B size Matrix A size

$3 \times \underline{\mathbf{6}}$ $\underline{\mathbf{4}} \times 3$

The product BA does not exist since the number of columns of B, 6, is not the same as the number of rows of A, 4.

12. A is 7×3, and B is 2×7.

The product AB does not exist, since the number of columns of A is not the same as the number of rows of B.

The product BA exists; its size is 2×3.

14. The product matrix AB has the same number of *rows* as A and the same number of *columns* as B.

16. $\begin{bmatrix} -1 & 5 \\ 7 & 0 \end{bmatrix}\begin{bmatrix} 6 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \cdot 6 + 5 \cdot 2 \\ 7 \cdot 6 + 0 \cdot 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 42 \end{bmatrix}$

18. $\begin{bmatrix} 5 & 2 \\ 7 & 6 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 4 & 0 \\ 2 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 5 \cdot 1 + 2 \cdot 2 & 5 \cdot 4 + 2(-1) & 5 \cdot 0 + 2 \cdot 2 \\ 7 \cdot 1 + 6 \cdot 2 & 7 \cdot 4 + 6(-1) & 7 \cdot 0 + 6 \cdot 2 \\ 1 \cdot 1 + 0 \cdot 2 & 1 \cdot 4 + 0(-1) & 1 \cdot 0 + 0 \cdot 2 \end{bmatrix} = \begin{bmatrix} 9 & 18 & 4 \\ 19 & 22 & 12 \\ 1 & 4 & 0 \end{bmatrix}$

20. $\begin{bmatrix} 6 & 0 & -4 \\ 1 & 2 & 5 \\ 10 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \cdot 1 + 0 \cdot 2 + (-4)0 \\ 1 \cdot 1 + 2 \cdot 2 + 5 \cdot 0 \\ 10 \cdot 1 + (-1)2 + 3 \cdot 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 8 \end{bmatrix}$

22. $\begin{bmatrix} -9 & 2 & 1 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} -9 \cdot 2 + 2(-1) + 1 \cdot 4 \\ 3 \cdot 2 + 0(-1) + 0 \cdot 4 \end{bmatrix} = \begin{bmatrix} -16 \\ 6 \end{bmatrix}$

24. $\begin{bmatrix} -1 & 5 \\ 7 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -1 \cdot 1 + 5 \cdot 3 & -1 \cdot 2 + 5 \cdot 4 \\ 7 \cdot 1 + 0 \cdot 3 & 7 \cdot 2 + 0 \cdot 4 \end{bmatrix} = \begin{bmatrix} 14 & 18 \\ 7 & 14 \end{bmatrix}$

26. $\begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 6(-1) & 6 \cdot 1 & 6 \cdot 1 \\ 5(-1) & 5 \cdot 1 & 5 \cdot 1 \\ 4(-1) & 4 \cdot 1 & 4 \cdot 1 \end{bmatrix} = \begin{bmatrix} -6 & 6 & 6 \\ -5 & 5 & 5 \\ -4 & 4 & 4 \end{bmatrix}$

28. $\begin{bmatrix} 4 & 3 \\ 1 & 2 \\ 0 & -5 \end{bmatrix} \left(\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 10 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 4 & 3 \\ 1 & 2 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} 20 \\ 10 \end{bmatrix} = \begin{bmatrix} 110 \\ 40 \\ -50 \end{bmatrix}$

30. $\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 7 & 0 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 16 & -10 \\ 8 & -5 \end{bmatrix} = \begin{bmatrix} 22 & -8 \\ 11 & -4 \end{bmatrix}$

32. Verify that $(PX)T = P(XT)$.

$$\begin{aligned}
(PX)T &= \left(\begin{bmatrix} m & n \\ p & q \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} \right) \begin{bmatrix} r & s \\ t & u \end{bmatrix} \\
&= \begin{bmatrix} mx + nz & my + nw \\ px + qz & py + qw \end{bmatrix} \begin{bmatrix} r & s \\ t & u \end{bmatrix} \\
&= \begin{bmatrix} (mx + nz)r + (my + nw)t & (mx+nz)s + (my+nw)u \\ (px + qz)r + (py + qw)t & (px+qz)s + (py+qw)u \end{bmatrix} \\
&= \begin{bmatrix} mxr + nzr + myt + nwt & mxs + nzs + myu + nwu \\ pxr + qzr + pyt + qwt & pxs + qzs + pyu + qwu \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
P(XT) &= \begin{bmatrix} m & n \\ p & q \end{bmatrix} \left(\begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} r & s \\ t & u \end{bmatrix} \right) \\
&= \begin{bmatrix} m & n \\ p & q \end{bmatrix} \begin{bmatrix} xr + yt & xs + yu \\ zr + wt & zs + wu \end{bmatrix} \\
&= \begin{bmatrix} m(xr + yt) + n(zr + wt) & m(xs + yu) + n(zs + wu) \\ p(xr + yt) + q(zr + wt) & p(xs + yu) + q(zs + wu) \end{bmatrix} \\
&= \begin{bmatrix} mxr+myt+nzr+nwt & mxs+myu+nzs+nwu \\ pxr+pyt+qzr+qwt & pxs+pyu+qzs+qwu \end{bmatrix} \\
&= \begin{bmatrix} mxr+nzr+myt+nwt & mxs+nzs+myu+nwu \\ pxr+qzr+pyt+qwt & pxs+qzs+pyu+qwu \end{bmatrix}
\end{aligned}$$

Thus, $(PX)T = P(XT)$.

34. Prove that $k(X+T) = kX + kT$ for any real number k.

$$\begin{aligned} k(X+T) &= k\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix} + \begin{bmatrix} r & s \\ t & u \end{bmatrix}\right) \\ &= k\left(\begin{bmatrix} x+r & y+s \\ z+t & w+u \end{bmatrix}\right) \\ &= \begin{bmatrix} k(x+r) & k(y+s) \\ k(z+t) & k(w+u) \end{bmatrix} \\ &= \begin{bmatrix} kx+kr & ky+ks \\ kz+kt & kw+ku \end{bmatrix} \quad \begin{matrix} \textit{Distributive property} \\ \textit{for real numbers} \end{matrix} \\ &= \begin{bmatrix} kx & ky \\ kz & kw \end{bmatrix} + \begin{bmatrix} kr & ks \\ kt & ku \end{bmatrix} \\ &= k\begin{bmatrix} x & y \\ z & w \end{bmatrix} + k\begin{bmatrix} r & s \\ t & u \end{bmatrix} \\ &= kX + kT \end{aligned}$$

36. (a)

$$IP = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} m & n \\ p & q \end{bmatrix} = \begin{bmatrix} m & n \\ p & q \end{bmatrix} = P$$

Thus, $IP = P$.

$$PI = \begin{bmatrix} m & n \\ p & q \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} m & n \\ p & q \end{bmatrix} = P$$

Thus, $PI = P$.

$$IX = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & y \\ z & w \end{bmatrix} = X$$

Thus, $IX = X$.

(b) $IT = T$

The matrix I is called an identity matrix because it acts like the multiplicative identity for real numbers, which is 1.

If x is a real number,

$$1 \cdot x = x \cdot 1 = x.$$

If X is a 2×2 matrix,

$$IX = XI = X.$$

(c) I is called an identity matrix, because if a matrix is multiplied by I (assuming that these matrices are 2×2), the resulting matrix is the matrix you originally started with. Hence, I maintains the identity of any 2×2 matrix under multiplication.

38. $A = \begin{bmatrix} 1 & 2 \\ -3 & 5 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $B = \begin{bmatrix} -4 \\ 12 \end{bmatrix}$

If $AX = B$, then

$$\begin{bmatrix} 1 & 2 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 \\ 12 \end{bmatrix}.$$

Using the definition of matrix multiplication,

$$\begin{bmatrix} 1x_1 + 2x_2 \\ -3x_1 + 5x_2 \end{bmatrix} = \begin{bmatrix} -4 \\ 12 \end{bmatrix}.$$

By the definition of equality of matrices, corresponding elements must be equal, so

$$\begin{aligned} x_1 + 2x_2 &= -4 \quad \textit{(1)} \\ -3x_1 + 5x_2 &= 12. \quad \textit{(2)} \end{aligned}$$

This is a linear system of two equations in two variables, x_1 and x_2.
Solve this system by the elimination (addition) method. Multiply equation (1) by 3 and add the result to equation (2).

$$\begin{array}{r} 3x_1 + 6x_2 = -12 \\ \underline{-3x_1 + 5x_2 = 12} \\ 11x_2 = 0 \\ x_2 = 0 \end{array}$$

Substitute 0 for x_2 in equation (1) to find x_1.

$$\begin{aligned} x_1 + 2(0) &= -4 \\ x_1 &= -4 \end{aligned}$$

The solution of the system is $(-4, 0)$.

Substitute -4 for x_1 and 0 for x_2 in the matrix equation to check this result.

$$\begin{bmatrix} 1 & 2 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} -4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1(-4) + 2(0) \\ -3(-4) + 5(0) \end{bmatrix} = \begin{bmatrix} -4 \\ 12 \end{bmatrix}.$$

40. This exercise should be solved by graphing calculator or computer methods. The answers are as follows:

(a) $CD = \begin{bmatrix} 44 & 75 & -60 & -33 & 11 \\ 20 & 169 & -164 & 18 & 105 \\ 113 & -82 & 239 & 218 & -55 \\ 119 & 83 & 7 & 82 & 106 \\ 162 & 20 & 175 & 143 & 74 \end{bmatrix}$;

(b) $DC = \begin{bmatrix} 110 & 96 & 30 & 226 & 37 \\ -94 & 127 & 134 & -87 & -33 \\ -52 & 126 & 193 & 153 & 22 \\ 117 & 56 & -55 & 147 & 57 \\ 54 & 69 & 58 & 37 & 31 \end{bmatrix}$;

(c) No, $CD \neq DC$.

42. Exercise 41 illustrates the distributive property.

44. (a)
$$\begin{array}{c} \\ \text{S} \\ \text{C} \end{array} \begin{array}{c} \begin{array}{ccc} \text{CC} & \text{MM} & \text{AD} \end{array} \\ \begin{bmatrix} .5 & .4 & .3 \\ .2 & .3 & .3 \end{bmatrix} \end{array}$$

(b)
$$\begin{array}{c} \\ \text{SD} \\ \text{MC} \\ \text{M} \end{array} \begin{array}{c} \begin{array}{cc} \text{S} & \text{C} \end{array} \\ \begin{bmatrix} 4 & 3 \\ 2 & 5 \\ 1 & 7 \end{bmatrix} \end{array}$$

(c) $\begin{bmatrix} 4 & 3 \\ 2 & 5 \\ 1 & 7 \end{bmatrix} \begin{bmatrix} .5 & .4 & .3 \\ .2 & .3 & .3 \end{bmatrix}$

$$\begin{array}{c} \\ \text{SD} \\ = \text{MC} \\ \text{M} \end{array} \begin{array}{c} \begin{array}{ccc} \text{CC} & \text{MM} & \text{AD} \end{array} \\ \begin{bmatrix} 2.6 & 2.5 & 2.1 \\ 2 & 2.3 & 2.1 \\ 1.9 & 2.5 & 2.4 \end{bmatrix} \end{array}$$

(d) Look at the entry in row 3, column 2 of the last matrix. The cost is \$2.50.

(e) $\begin{bmatrix} 2.6 & 2.5 & 2.1 \\ 2 & 2.3 & 2.1 \\ 1.9 & 2.5 & 2.4 \end{bmatrix} \begin{bmatrix} 100 \\ 200 \\ 500 \end{bmatrix} = \begin{bmatrix} 1810 \\ 1710 \\ 1890 \end{bmatrix}$

The total sugar and chocolate cost is \$1810 in San Diego, \$1710 in Mexico City, and \$1890 in Managua, so the order can be produced for the lowest cost in Mexico City.

46. $P = \begin{bmatrix} 0 & 30 \\ 10 & 20 \\ 20 & 20 \end{bmatrix}$, $Q = \begin{bmatrix} 10 & 2 & 0 & 2 \\ 50 & 1 & 20 & 2 \end{bmatrix}$,

$R = \begin{bmatrix} 20 \\ 180 \\ 60 \\ 25 \end{bmatrix}$

(a) $QR = \begin{bmatrix} 10 & 2 & 0 & 2 \\ 50 & 1 & 20 & 2 \end{bmatrix} \begin{bmatrix} 20 \\ 180 \\ 60 \\ 25 \end{bmatrix}$

$$= \begin{bmatrix} 10(20) + 2(180) + 0(60) + 2(25) \\ 50(20) + 1(180) + 20(60) + 2(25) \end{bmatrix}$$

$$= \begin{bmatrix} 610 \\ 2430 \end{bmatrix}$$

The rows represent the cost of materials for each type of house.

(b) From part (a), $QR = \begin{bmatrix} 610 \\ 2430 \end{bmatrix}$.

$$P(QR) = \begin{bmatrix} 0 & 30 \\ 10 & 20 \\ 20 & 20 \end{bmatrix} \begin{bmatrix} 610 \\ 2430 \end{bmatrix} = \begin{bmatrix} 72{,}900 \\ 54{,}700 \\ 60{,}800 \end{bmatrix}$$

$$(PQ)R = \left(\begin{bmatrix} 0 & 30 \\ 10 & 20 \\ 20 & 20 \end{bmatrix} \begin{bmatrix} 10 & 2 & 0 & 2 \\ 50 & 1 & 20 & 2 \end{bmatrix} \right) \begin{bmatrix} 20 \\ 180 \\ 60 \\ 25 \end{bmatrix}$$

$$= \begin{bmatrix} 1500 & 30 & 600 & 60 \\ 1100 & 40 & 400 & 60 \\ 1200 & 60 & 400 & 80 \end{bmatrix} \begin{bmatrix} 20 \\ 180 \\ 60 \\ 25 \end{bmatrix}$$

$$= \begin{bmatrix} 72{,}900 \\ 54{,}700 \\ 60{,}800 \end{bmatrix}$$

Therefore, $P(QR) = (PQ)R$.

48. **(a)** To increase by 25%, multiply by 1.25 or $\frac{5}{4}$. To increase by $\frac{1}{3}$, multiply by $1\frac{1}{3}$ or $\frac{4}{3}$. To increase by 10%, multiply by 1.10 or $\frac{11}{10}$. The matrix is

$$\begin{bmatrix} \frac{5}{4} \\ \frac{4}{3} \\ \frac{11}{10} \end{bmatrix}.$$

(b) $$\begin{bmatrix} 88 & 105 & 60 \\ 48 & 72 & 40 \\ 16 & 21 & 0 \\ 112 & 147 & 50 \end{bmatrix} \begin{bmatrix} \frac{5}{4} \\ \frac{4}{3} \\ \frac{11}{10} \end{bmatrix} = \begin{bmatrix} 316 \\ 200 \\ 48 \\ 391 \end{bmatrix}$$

50. **(a)** $$XY = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 3 & 2 & 2 & 1 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 7 \\ 0 & 10 & 1 \\ 0 & 15 & 2 \\ 10 & 12 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 52 & 27 \\ 25 & 62 & 35 \\ 30 & 72 & 43 \end{bmatrix}$$

The rows give the amounts of fat, carbohydrates, and protein, respectively, in each of the daily meals.

(b) $$YZ = \begin{bmatrix} 5 & 0 & 7 \\ 0 & 10 & 1 \\ 0 & 15 & 2 \\ 10 & 12 & 8 \end{bmatrix} \begin{bmatrix} 8 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 75 \\ 45 \\ 70 \\ 168 \end{bmatrix}$$

The rows give the number of calories in one exchange of each of the food groups.

(c) Use the matrices found for XY and YZ from parts (a) and (b).

$$(XY)Z = \begin{bmatrix} 20 & 52 & 27 \\ 25 & 62 & 35 \\ 30 & 72 & 43 \end{bmatrix} \begin{bmatrix} 8 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 503 \\ 623 \\ 743 \end{bmatrix}$$

$$X(YZ) = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 3 & 2 & 2 & 1 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 75 \\ 45 \\ 70 \\ 168 \end{bmatrix} = \begin{bmatrix} 503 \\ 623 \\ 743 \end{bmatrix}$$

The rows give the number of calories in each meal.

52. $$\frac{1}{6}\left(\begin{bmatrix} 60.0 & 68.3 \\ 63.8 & 72.5 \\ 64.5 & 73.6 \\ 67.2 & 74.7 \end{bmatrix} + \frac{5}{6} \begin{bmatrix} 68.0 & 75.6 \\ 70.7 & 78.1 \\ 72.7 & 79.4 \\ 74.3 & 79.9 \end{bmatrix} \right)$$

$$= \frac{1}{6}\left(\begin{bmatrix} 60.0 & 68.3 \\ 63.8 & 72.5 \\ 64.5 & 73.6 \\ 67.2 & 74.7 \end{bmatrix} + 5 \begin{bmatrix} 68.0 & 75.6 \\ 70.7 & 78.1 \\ 72.7 & 79.4 \\ 74.3 & 79.9 \end{bmatrix} \right)$$

$$= \frac{1}{6} \begin{bmatrix} 400 & 446.3 \\ 417.3 & 446 \\ 428 & 470.6 \\ 438.7 & 474.2 \end{bmatrix}$$

$$\approx \begin{bmatrix} 66.7 & 74.4 \\ 69.6 & 77.2 \\ 71.3 & 78.4 \\ 73.1 & 79.0 \end{bmatrix}$$

2.5 Matrix Inverses

2. $$\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 5(3) + 7(-2) & 5(-7) + 7(5) \\ 2(3) + 3(-2) & 2(-7) + 3(5) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3(5) + (-7)(2) & 3(7) + (-7)(3) \\ (-2)(5) + 5(2) & (-2)(7) + 5(3) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Since the products obtained by multiplying the matrices in either order are both the 2×2 identity matrix, the given matrices are inverses of each other.

4. $\begin{bmatrix}-1 & 2\\ 3 & -5\end{bmatrix}\begin{bmatrix}-5 & -2\\ -3 & -1\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix} \neq I$

Since this product is not the 2×2 identity matrix, the given matrices are not inverses of each other.

6. $\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & -2\\ 1 & -1 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 1\\ 1 & 0 & 0\\ 0 & -1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix} \neq I$

Since this product is not the 3×3 identity matrix, the given matrices are not inverses of each other.

8. $\begin{bmatrix}-1 & 0 & 2\\ 3 & 1 & 0\\ 0 & 2 & -3\end{bmatrix}\begin{bmatrix}-\frac{1}{5} & \frac{4}{15} & -\frac{2}{15}\\ \frac{3}{5} & \frac{1}{5} & \frac{2}{5}\\ \frac{2}{5} & \frac{2}{15} & -\frac{1}{15}\end{bmatrix}$

$$= \begin{bmatrix}\frac{1}{5}+0+\frac{4}{5} & -\frac{4}{15}+0+\frac{4}{15} & \frac{2}{15}+0-\frac{2}{15}\\ -\frac{3}{5}+\frac{3}{5}+0 & \frac{12}{15}+\frac{3}{15}+0 & -\frac{2}{5}+\frac{2}{5}+0\\ 0+\frac{6}{5}-\frac{6}{5} & 0+\frac{2}{5}-\frac{2}{5} & 0+\frac{4}{5}+\frac{1}{5}\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = I$$

$$\begin{bmatrix}-\frac{1}{5} & \frac{4}{15} & -\frac{2}{15}\\ \frac{3}{5} & \frac{1}{5} & \frac{2}{5}\\ \frac{2}{5} & \frac{2}{15} & -\frac{1}{15}\end{bmatrix}\begin{bmatrix}-1 & 0 & 2\\ 3 & 1 & 0\\ 0 & 2 & -3\end{bmatrix}$$

$$= \begin{bmatrix}\frac{1}{5}+\frac{4}{5}+0 & 0+\frac{4}{15}-\frac{4}{15} & -\frac{2}{5}+0+\frac{2}{5}\\ -\frac{3}{5}+\frac{3}{5}+0 & 0+\frac{1}{5}+\frac{4}{5} & \frac{6}{5}+0-\frac{6}{5}\\ -\frac{2}{5}+\frac{2}{5}+0 & 0+\frac{2}{15}-\frac{2}{15} & \frac{4}{5}+0+\frac{1}{5}\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = I$$

Since the products obtained by multiplying the matrices in either order are both the 3×3 identity matrix, the given matrices are inverses of each other.

10. Since the inverse of A is A^{-1}, $(A^{-1})^{-1}$ would be the inverse of the inverse of A, which is A.

12. Find the inverse of $A = \begin{bmatrix}-1 & 2\\ -2 & -1\end{bmatrix}$, if it exists.

Write the augmented matrix $[A|I]$.

$$[A|I] = \left[\begin{array}{cc|cc}-1 & 2 & 1 & 0\\ -2 & -1 & 0 & 1\end{array}\right]$$

Perform row operations on $[A|I]$ to get a matrix of the form $[I|B]$.

$$-2\mathrm{R}_1 + \mathrm{R}_2 \rightarrow \mathrm{R}_2 \quad \left[\begin{array}{cc|cc}-1 & 2 & 1 & 0\\ 0 & -5 & -2 & 1\end{array}\right]$$

$$2\mathrm{R}_2 + 5\mathrm{R}_1 \rightarrow \mathrm{R}_1 \quad \left[\begin{array}{cc|cc}-5 & 0 & 1 & 2\\ 0 & -5 & -2 & 1\end{array}\right]$$

$$\begin{array}{r}-\frac{1}{5}\mathrm{R}_1 \rightarrow \mathrm{R}_1\\ -\frac{1}{5}\mathrm{R}_2 \rightarrow \mathrm{R}_2\end{array} \quad \left[\begin{array}{cc|cc}1 & 0 & -\frac{1}{5} & -\frac{2}{5}\\ 0 & 1 & \frac{2}{5} & -\frac{1}{5}\end{array}\right]$$

The last augmented matrix is of the form $[I|B]$, so the desired inverse is

$$A^{-1} = \begin{bmatrix}-\frac{1}{5} & -\frac{2}{5}\\ \frac{2}{5} & -\frac{1}{5}\end{bmatrix}.$$

14. Find the inverse of $A = \begin{bmatrix}-1 & -2\\ 3 & 4\end{bmatrix}$, if it exists.

$$[A|I] = \left[\begin{array}{cc|cc}-1 & -2 & 1 & 0\\ 3 & 4 & 0 & 1\end{array}\right]$$

$$3R_1 + \mathrm{R}_2 \rightarrow \mathrm{R}_2 \quad \left[\begin{array}{cc|cc}-1 & -2 & 1 & 0\\ 0 & -2 & 3 & 1\end{array}\right]$$

$$-1\mathrm{R}_2 + \mathrm{R}_1 \rightarrow \mathrm{R}_1 \quad \left[\begin{array}{cc|cc}-1 & 0 & -2 & -1\\ 0 & -2 & 3 & 1\end{array}\right]$$

$$\begin{array}{r}-1\mathrm{R}_1 \rightarrow \mathrm{R}_1\\ -\frac{1}{2}\mathrm{R}_2 \rightarrow \mathrm{R}_2\end{array} \quad \left[\begin{array}{cc|cc}1 & 0 & 2 & 1\\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2}\end{array}\right]$$

$$A^{-1} = \begin{bmatrix}2 & 1\\ -\frac{3}{2} & -\frac{1}{2}\end{bmatrix}$$

16. Find the inverse of $A = \begin{bmatrix}5 & 10\\ -3 & -6\end{bmatrix}$, if it exists.

$$[A|I] = \left[\begin{array}{cc|cc}5 & 10 & 1 & 0\\ -3 & -6 & 0 & 1\end{array}\right]$$

$$3\mathrm{R}_1 + 5\mathrm{R}_2 \rightarrow \mathrm{R}_2 \quad \left[\begin{array}{cc|cc}5 & 10 & 1 & 0\\ 0 & 0 & 3 & 5\end{array}\right]$$

At this point, the matrix should be changed so that the first row, second column element will be 0. Since this cannot be done using row operations, the inverse of the given matrix does not exist.

18. Find the inverse of $A = \begin{bmatrix}1 & 0 & 1\\ 0 & -1 & 0\\ 2 & 1 & 1\end{bmatrix}$, if it exists.

$$[A|I]=\left[\begin{array}{rrr|rrr}1&0&1&1&0&0\\0&-1&0&0&1&0\\2&1&1&0&0&1\end{array}\right]$$

$$\begin{array}{r}\\ \\ -2\mathrm{R}_1+\mathrm{R}_3\to\mathrm{R}_3\end{array}\left[\begin{array}{rrr|rrr}1&0&1&1&0&0\\0&-1&0&0&1&0\\0&1&-1&-2&0&1\end{array}\right]$$

$$\begin{array}{r}\\ \\ \mathrm{R}_2+\mathrm{R}_3\to\mathrm{R}_3\end{array}\left[\begin{array}{rrr|rrr}1&0&1&1&0&0\\0&-1&0&0&1&0\\0&0&-1&-2&1&1\end{array}\right]$$

$$\begin{array}{r}\\ \mathrm{R}_3+\mathrm{R}_1\to\mathrm{R}_1\\ \\ \end{array}\left[\begin{array}{rrr|rrr}1&0&0&-1&1&1\\0&-1&0&0&1&0\\0&0&-1&-2&1&1\end{array}\right]$$

$$\begin{array}{r}\\ -1\mathrm{R}_2\to\mathrm{R}_2\\ -1\mathrm{R}_3\to\mathrm{R}_3\end{array}\left[\begin{array}{rrr|rrr}1&0&0&-1&1&1\\0&1&0&0&-1&0\\0&0&1&2&-1&-1\end{array}\right]$$

$$A^{-1}=\begin{bmatrix}-1&1&1\\0&-1&0\\2&-1&-1\end{bmatrix}$$

20. Find the inverse of $A=\begin{bmatrix}2&0&4\\3&1&5\\-1&1&-2\end{bmatrix}$, if it exists.

$$[A|I]=\left[\begin{array}{rrr|rrr}2&0&4&1&0&0\\3&1&5&0&1&0\\-1&1&-2&0&0&1\end{array}\right]$$

$$\begin{array}{r}\\ 3\mathrm{R}_1+(-2)\mathrm{R}_2\to\mathrm{R}_2\\ \mathrm{R}_1+2\mathrm{R}_3\to\mathrm{R}_3\end{array}\left[\begin{array}{rrr|rrr}2&0&4&1&0&0\\0&-2&2&3&-2&0\\0&2&0&1&0&2\end{array}\right]$$

$$\begin{array}{r}\\ \\ \mathrm{R}_2+\mathrm{R}_3\to\mathrm{R}_3\end{array}\left[\begin{array}{rrr|rrr}2&0&4&1&0&0\\0&-2&2&3&-2&0\\0&0&2&4&-2&2\end{array}\right]$$

$$\begin{array}{r}-2\mathrm{R}_3+\mathrm{R}_1\to\mathrm{R}_1\\ -1\mathrm{R}_3+\mathrm{R}_2\to\mathrm{R}_2\\ \\ \end{array}\left[\begin{array}{rrr|rrr}2&0&0&-7&4&-4\\0&-2&0&-1&0&-2\\0&0&2&4&-2&2\end{array}\right]$$

$$\begin{array}{r}\frac{1}{2}\mathrm{R}_1\to\mathrm{R}_1\\ -\frac{1}{2}\mathrm{R}_2\to\mathrm{R}_2\\ \frac{1}{2}\mathrm{R}_3\to\mathrm{R}_3\end{array}\left[\begin{array}{rrr|rrr}1&0&0&-\frac{7}{2}&2&-2\\0&1&0&\frac{1}{2}&0&1\\0&0&1&2&-1&1\end{array}\right]$$

$$A^{-1}=\begin{bmatrix}-\frac{7}{2}&2&-2\\\frac{1}{2}&0&1\\2&-1&1\end{bmatrix}$$

22. Find the inverse of $A=\begin{bmatrix}2&0&4\\1&0&-1\\3&0&-2\end{bmatrix}$, if it exists.

$$[A|I]=\left[\begin{array}{rrr|rrr}2&0&4&1&0&0\\1&0&-1&0&1&0\\3&0&-2&0&0&1\end{array}\right]$$

$$\begin{array}{r}\\ \mathrm{R}_1+(-2)\mathrm{R}_2\to\mathrm{R}_2\\ 3\mathrm{R}_1+(-2)\mathrm{R}_3\to\mathrm{R}_3\end{array}\left[\begin{array}{rrr|rrr}2&0&4&1&0&0\\0&0&6&1&-2&0\\0&0&16&3&0&-2\end{array}\right]$$

Since the second column is all zeros, it will not be possible too get a 1 in the second row, second column position. Therefore, the inverse of the given matrix does not exist.

24. Find the inverse of $A=\begin{bmatrix}2&2&-4\\2&6&0\\-3&-3&5\end{bmatrix}$, if it exists.

$$[A|I]=\left[\begin{array}{rrr|rrr}2&2&-4&1&0&0\\2&6&0&0&1&0\\-3&-3&5&0&0&1\end{array}\right]$$

$$\begin{array}{r}\\ -\mathrm{R}_1+\mathrm{R}_2\to\mathrm{R}_2\\ 3\mathrm{R}_1+2\mathrm{R}_3\to\mathrm{R}_3\end{array}\left[\begin{array}{rrr|rrr}2&2&-4&1&0&0\\0&4&4&-1&1&0\\0&0&-2&3&0&2\end{array}\right]$$

$$\begin{array}{r}\\ \mathrm{R}_2+(-2)\mathrm{R}_1\to\mathrm{R}_1\\ \\ \end{array}\left[\begin{array}{rrr|rrr}-4&0&12&-3&1&0\\0&4&4&-1&1&0\\0&0&-2&3&0&2\end{array}\right]$$

$$\begin{array}{r}6\mathrm{R}_3+\mathrm{R}_1\to\mathrm{R}_1\\ 2\mathrm{R}_3+\mathrm{R}_2\to\mathrm{R}_2\\ \\ \end{array}\left[\begin{array}{rrr|rrr}-4&0&0&15&1&12\\0&4&0&5&1&4\\0&0&-2&3&0&2\end{array}\right]$$

$$\begin{array}{r}-\frac{1}{4}\mathrm{R}_1\to\mathrm{R}_1\\ \frac{1}{4}\mathrm{R}_2\to\mathrm{R}_2\\ -\frac{1}{2}\mathrm{R}_3\to\mathrm{R}_3\end{array}\left[\begin{array}{rrr|rrr}1&0&0&-\frac{15}{4}&-\frac{1}{4}&-3\\0&1&0&\frac{5}{4}&\frac{1}{4}&1\\0&0&1&-\frac{3}{2}&0&-1\end{array}\right]$$

$$A^{-1}=\begin{bmatrix}-\frac{15}{4}&-\frac{1}{4}&-3\\\frac{5}{4}&\frac{1}{4}&1\\-\frac{3}{2}&0&-1\end{bmatrix}$$

26. Find the inverse of $A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 2 & -1 & 1 & -1 \\ 3 & 3 & 2 & -2 \\ 1 & 2 & 1 & 0 \end{bmatrix}$, if it exists.

$$[A|I] = \left[\begin{array}{rrrr|rrrr} 1 & 1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 2 & -1 & 1 & -1 & 0 & 1 & 0 & 0 \\ 3 & 3 & 2 & -2 & 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Carry out these row operations to get the next matrix: $-2\mathrm{R}_1 + \mathrm{R}_2 \rightarrow \mathrm{R}_2$, $-3\mathrm{R}_1 + \mathrm{R}_3 \rightarrow \mathrm{R}_3$, $-1\mathrm{R}_1 + \mathrm{R}_4 \rightarrow \mathrm{R}_4$.

$$\left[\begin{array}{rrrr|rrrr} 1 & 1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & -3 & 1 & -5 & -2 & 1 & 0 & 0 \\ 0 & 0 & 2 & -8 & -3 & 0 & 1 & 0 \\ 0 & 1 & 1 & -2 & -1 & 0 & 0 & 1 \end{array}\right]$$

Carry out the row operations $\mathrm{R}_2 + 3\mathrm{R}_1 \rightarrow \mathrm{R}_1$, $\mathrm{R}_2 + 3\mathrm{R}_4 \rightarrow \mathrm{R}_4$

$$\left[\begin{array}{rrrr|rrrr} 3 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -3 & 1 & -5 & -2 & 1 & 0 & 0 \\ 0 & 0 & 2 & -8 & -3 & 0 & 1 & 0 \\ 0 & 0 & 4 & -11 & -5 & 1 & 0 & 3 \end{array}\right]$$

$\mathrm{R}_3 + (-2)\mathrm{R}_1 \rightarrow \mathrm{R}_1$, $\mathrm{R}_3 + (-2)\mathrm{R}_2 \rightarrow \mathrm{R}_2$, $-2\mathrm{R}_3 + \mathrm{R}_4 \rightarrow \mathrm{R}_4$

$$\left[\begin{array}{rrrr|rrrr} -6 & 0 & 0 & -10 & -5 & -2 & 1 & 0 \\ 0 & 6 & 0 & 2 & 1 & -2 & 1 & 0 \\ 0 & 0 & 2 & -8 & -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 & 1 & 1 & -2 & 3 \end{array}\right]$$

$2\mathrm{R}_4 + \mathrm{R}_1 \rightarrow \mathrm{R}_1$, $-2\mathrm{R}_4 + 5\mathrm{R}_2 \rightarrow \mathrm{R}_2$, $8\mathrm{R}_4 + 5\mathrm{R}_3 \rightarrow \mathrm{R}_3$

$$\left[\begin{array}{rrrr|rrrr} -6 & 0 & 0 & 0 & -3 & 0 & -3 & 6 \\ 0 & 30 & 0 & 0 & 3 & -12 & 9 & -6 \\ 0 & 0 & 10 & 0 & -7 & 8 & -11 & 24 \\ 0 & 0 & 0 & 5 & 1 & 1 & -2 & 3 \end{array}\right]$$

$-\frac{1}{6}\mathrm{R}_1 \rightarrow \mathrm{R}_1$, $\frac{1}{30}\mathrm{R}_2 \rightarrow \mathrm{R}_2$, $\frac{1}{10}\mathrm{R}_3 \rightarrow \mathrm{R}_3$, $\frac{1}{5}\mathrm{R}_4 \rightarrow \mathrm{R}_4$

$$\left[\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & -1 \\ 0 & 1 & 0 & 0 & \frac{1}{10} & -\frac{2}{5} & \frac{3}{10} & -\frac{1}{5} \\ 0 & 0 & 1 & 0 & -\frac{7}{10} & \frac{4}{5} & -\frac{11}{10} & \frac{12}{5} \\ 0 & 0 & 0 & 1 & \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} & \frac{3}{5} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & -1 \\ \frac{1}{10} & -\frac{2}{5} & \frac{3}{10} & -\frac{1}{5} \\ -\frac{7}{10} & \frac{4}{5} & -\frac{11}{10} & \frac{12}{5} \\ \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} & \frac{3}{5} \end{bmatrix}$$

28. $\begin{aligned} -x + 2y &= 15 \\ -2x - y &= 20 \end{aligned}$

This system may be written as the matrix equation

$$\begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15 \\ 20 \end{bmatrix}.$$

In Exercise 12, it was calculated that

$$\begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{1}{5} & -\frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix},$$

and we now know that $X = A^{-1}B$ is the solution to $AX = B$.

Thus,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} & -\frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} 15 \\ 20 \end{bmatrix} = \begin{bmatrix} -11 \\ 2 \end{bmatrix},$$

and $(-11, 2)$ is the solution of the system.

30. $\begin{aligned} -x - 2y &= 8 \\ 3x + 4y &= 24 \end{aligned}$

This system may be written as the matrix equation

$$\begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 24 \end{bmatrix}.$$

In Exercise 14, it was calculated that

$$\begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 1 \\ -\frac{3}{2} & -\frac{1}{2} \end{bmatrix},$$

and since $X = A^{-1}B$, we get

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -\frac{3}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 8 \\ 24 \end{bmatrix} = \begin{bmatrix} 40 \\ -24 \end{bmatrix}.$$

Therefore, the solution is $(40, -24)$.

32. $\begin{aligned} 3x - 6y &= 1 \\ -5x + 9y &= -1 \end{aligned}$

This system may be written as the matrix equation

$$\begin{bmatrix} 3 & -6 \\ -5 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Calculate the inverse of

$$A = \begin{bmatrix} 3 & -6 \\ -5 & 9 \end{bmatrix}.$$

$$A^{-1} = \begin{bmatrix} -3 & -2 \\ -\frac{5}{3} & -1 \end{bmatrix},$$

so $$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ -\frac{5}{3} & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -\frac{2}{3} \end{bmatrix}.$$

The solution is $\left(-1, -\frac{2}{3}\right)$.

34. $$\begin{aligned} x + 3y &= -14 \\ 2x - y &= 7 \end{aligned}$$

This system may be written as the matrix equation

$$\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -14 \\ 7 \end{bmatrix}.$$

Calculate the inverse of

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}.$$

The inverse is

$$A^{-1} = \begin{bmatrix} \frac{1}{7} & \frac{3}{7} \\ \frac{2}{7} & -\frac{1}{7} \end{bmatrix},$$

so $X = A^{-1}B$ leads to

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{7} & \frac{3}{7} \\ \frac{2}{7} & -\frac{1}{7} \end{bmatrix} \begin{bmatrix} -14 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix},$$

which indicates that $(1, -5)$ is the solution.

36. $$\begin{aligned} 2x \qquad + 4z &= -8 \\ 3x + y + 5z &= 2 \\ -x + y - 2z &= 4 \end{aligned}$$

This system may be written as the matrix equation

$$\begin{bmatrix} 2 & 0 & 4 \\ 3 & 1 & 5 \\ -1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -8 \\ 2 \\ 4 \end{bmatrix}.$$

In Exercise 20, it was calculated that

$$\begin{bmatrix} 2 & 0 & 4 \\ 3 & 1 & 5 \\ -1 & 1 & -2 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{7}{2} & 2 & -2 \\ \frac{1}{2} & 0 & 1 \\ 2 & -1 & 1 \end{bmatrix},$$

so $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{7}{2} & 2 & -2 \\ \frac{1}{2} & 0 & 1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} -8 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 24 \\ 0 \\ -14 \end{bmatrix}.$$

The solution is $(24, 0, -14)$.

38. $$\begin{aligned} 2x + 2y - 4z &= 12 \\ 2x + 6y \qquad &= 16 \\ -3x - 3y + 5z &= -20 \end{aligned}$$

This system may be written as the matrix equation

$$\begin{bmatrix} 2 & 2 & -4 \\ 2 & 6 & 0 \\ -3 & -3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 16 \\ -20 \end{bmatrix}.$$

In Exercise 24, it was calculated that

$$\begin{bmatrix} 2 & 2 & -4 \\ 2 & 6 & 0 \\ -3 & -3 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{15}{4} & -\frac{1}{4} & -3 \\ \frac{5}{4} & \frac{1}{4} & 1 \\ -\frac{3}{2} & 0 & -1 \end{bmatrix},$$

so $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{15}{4} & -\frac{1}{4} & -3 \\ \frac{5}{4} & \frac{1}{4} & 1 \\ -\frac{3}{2} & 0 & -1 \end{bmatrix} \begin{bmatrix} 12 \\ 16 \\ -20 \end{bmatrix}$$

$$= \begin{bmatrix} 11 \\ -1 \\ 2 \end{bmatrix}.$$

The solution is $(11, -1, 2)$.

40. $$\begin{aligned} x \qquad + z &= 3 \\ y + 2z &= 8 \\ -x + y \qquad &= 4 \end{aligned}$$

The system may be written as the matrix equation

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 4 \end{bmatrix}.$$

Calculate that

$$A^{-1} = \begin{bmatrix} 2 & -1 & 1 \\ 2 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix}.$$

$$X = A^{-1}B = \begin{bmatrix} 2 & -1 & 1 \\ 2 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix}$$

The solution is $(2, 6, 1)$.

42.
$$\begin{aligned} x+\ \ y\qquad\ +2w&=3\\ 2x-\ \ y+\ \ z-\ \ w&=3\\ 3x+3y+2z-2w&=5\\ x+2y+\ \ z\qquad&=3 \end{aligned}$$

This system may be written as the matrix equation

$$\begin{bmatrix} 1 & 1 & 0 & 2\\ 2 & -1 & 1 & -1\\ 3 & 3 & 2 & -2\\ 1 & 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} x\\ y\\ z\\ w \end{bmatrix}=\begin{bmatrix} 3\\ 3\\ 5\\ 3 \end{bmatrix}.$$

The inverse of this coefficient matrix was calculated in Exercise 26. Use that result to obtain

$$\begin{bmatrix} x\\ y\\ z\\ w \end{bmatrix}=\begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & -1\\ \frac{1}{10} & -\frac{2}{5} & \frac{3}{10} & -\frac{1}{5}\\ -\frac{7}{10} & \frac{4}{5} & -\frac{11}{10} & \frac{12}{5}\\ \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} & \frac{3}{5} \end{bmatrix}\begin{bmatrix} 3\\ 3\\ 5\\ 3 \end{bmatrix}$$

$$=\begin{bmatrix} 1\\ 0\\ 2\\ 1 \end{bmatrix}.$$

The solution is $(1, 0, 2, 1)$.

In Exercises 44-48, let $A=\begin{bmatrix} a & b\\ c & d \end{bmatrix}$.

44.
$$\begin{aligned} AI&=\begin{bmatrix} a & b\\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0\\ 0 & 1 \end{bmatrix}\\ &=\begin{bmatrix} a\cdot 1+b\cdot 0 & a\cdot 0+b\cdot 1\\ c\cdot 1+d\cdot 0 & c\cdot 0+d\cdot 1 \end{bmatrix}\\ &=\begin{bmatrix} a & b\\ c & d \end{bmatrix}=A \end{aligned}$$

46.
$$\left[\begin{array}{cc|cc} a & b & 1 & 0\\ c & d & 0 & 1 \end{array}\right]$$

$$-c\mathrm{R}_1+a\mathrm{R}_2\rightarrow\mathrm{R}_2\quad\left[\begin{array}{cc|cc} a & b & 1 & 0\\ 0 & ad-bc & -c & a \end{array}\right]$$

$$-\frac{b}{ad-bc}\mathrm{R}_2+\mathrm{R}_1\rightarrow\mathrm{R}_1\quad\left[\begin{array}{cc|cc} a & 0 & \frac{ad}{ad-bc} & \frac{-ab}{ad-bc}\\ 0 & ad-bc & -c & a \end{array}\right]$$

$$\begin{array}{r} \frac{1}{a}\mathrm{R}_1\rightarrow\mathrm{R}_1\\ \frac{1}{ad-bc}\mathrm{R}_2\rightarrow\mathrm{R}_2 \end{array}\quad\left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc}\\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

Thus, $A^{-1}=\begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc}\\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$, if $ad-bc\neq 0$.

48. Show that $AA^{-1}=I$.

From Exercise 46,

$$A^{-1}=\begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc}\\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}.$$

$$\begin{aligned} AA^{-1}&=\begin{bmatrix} a & b\\ c & d \end{bmatrix}\begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc}\\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}\\ &=\begin{bmatrix} \frac{ad-bc}{ad-bc} & \frac{-ab+ab}{ad-bc}\\ \frac{cd-cd}{ad-bc} & \frac{-bc+ad}{ad-bc} \end{bmatrix}=\begin{bmatrix} 1 & 0\\ 0 & 1 \end{bmatrix}=I. \end{aligned}$$

50. Assume that matrix A has two inverses B and C. Then

$$\begin{aligned} AB=BA&=I\quad (1)\\ \text{and}\quad AC=CA&=I.\quad (2) \end{aligned}$$

Multiply equation (1) by C.

$$C(AB)=C(BA)=CI$$

Since matrix multiplication is associative,

$$C(AB)=(CA)B.$$

Since I is an identity matrix,

$$CI=C.$$

Combining these results, we have

$$\begin{aligned} C(AB)&=C\\ (CA)B&=C. \end{aligned}$$

From equation (2), $CA=I$, giving

$$\begin{aligned} IB&=C\\ B&=C. \end{aligned}$$

Thus, if it exists, the inverse of a matrix is unique.

52. Use a graphing calculator or a computer to perform this calculation. With entries rounded to 6 places, the answer is

$$(CD)^{-1}$$
$$=\begin{bmatrix} .010146 & -.011883 & .002772 & .020724 & -.012273\\ .006353 & .014233 & -.001861 & -.029146 & .019225\\ -.000638 & .006782 & -.004823 & -.022658 & .019344\\ -.005261 & .003781 & .006192 & .004837 & -.006910\\ -.012252 & -.001177 & -.006126 & .006744 & .002792 \end{bmatrix}.$$

54. Use a graphing calculator or a computer to determine that, no, $C^{-1}D^{-1}$ and $(CD)^{-1}$ are not equal.

56. Use a graphing calculator or a computer to obtain

$$X = \begin{bmatrix} .62963 \\ .148148 \\ .259259 \end{bmatrix}.$$

58. Use a graphing calculator or a computer to obtain

$$X = \begin{bmatrix} .489558 \\ 1.00104 \\ 2.11853 \\ -1.20793 \\ -.961346 \end{bmatrix}.$$

60. Let $x =$ the number of transistors,
$y =$ the number of resistors, and
$z =$ the number of computer chips.

Solve the following system:

$$\begin{aligned} 3x + 3y + 2z &= \text{amount of copper available;} \\ x + 2y + z &= \text{amount of zinc available;} \\ 2x + y + 2z &= \text{amount of glass available.} \end{aligned}$$

First, find the inverse of the coefficient matrix $\begin{bmatrix} 3 & 3 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 3 & 3 & 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ R_1 + (-3)R_2 \to R_2 \\ 2R_1 + (-3)R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 3 & 3 & 2 & 1 & 0 & 0 \\ 0 & -3 & -1 & 1 & -3 & 0 \\ 0 & 3 & -2 & 2 & 0 & -3 \end{array}\right]$$

$$\begin{array}{r} R_2 + R_1 \to R_1 \\ \\ R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 3 & 0 & 1 & 2 & -3 & 0 \\ 0 & -3 & -1 & 1 & -3 & 0 \\ 0 & 0 & -3 & 3 & -3 & -3 \end{array}\right]$$

$$\begin{array}{r} R_3 + 3R_1 \to R_1 \\ R_3 + (-3)R_2 \to R_2 \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 9 & 0 & 0 & 9 & -12 & -3 \\ 0 & 9 & 0 & 0 & 6 & -3 \\ 0 & 0 & -3 & 3 & -3 & -3 \end{array}\right]$$

$$\begin{array}{r} \frac{1}{9}R_1 \to R_1 \\ \frac{1}{9}R_2 \to R_2 \\ -\frac{1}{3}R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -\frac{4}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & -\frac{4}{3} & -\frac{1}{3} \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ -1 & 1 & 1 \end{bmatrix}$$

(a) 810 units of copper, 410 units of zinc, and 490 units of glass

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -\frac{4}{3} & -\frac{1}{3} \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 810 \\ 410 \\ 490 \end{bmatrix} = \begin{bmatrix} 100 \\ 110 \\ 90 \end{bmatrix}$$

100 transistors, 110 resistors, and 90 computer chips can be made.

(b) 765 units of copper, 385 units of zinc, and 470 units of glass

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -\frac{4}{3} & -\frac{1}{3} \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 765 \\ 385 \\ 470 \end{bmatrix} = \begin{bmatrix} 95 \\ 100 \\ 90 \end{bmatrix}$$

95 transistors, 100 resistors, and 90 computer chips can be made.

(c) 1010 units of copper, 500 units of zinc, and 610 units of glass

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -\frac{4}{3} & -\frac{1}{3} \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1010 \\ 500 \\ 610 \end{bmatrix} = \begin{bmatrix} 140 \\ 130 \\ 100 \end{bmatrix}$$

140 transistors, 130 resistors, and 100 computer chips can be made.

62. Let $x =$ the number of pounds of pretzels,
$y =$ the number of pounds of dried fruit, and
$z =$ the number of pounds of nuts.

The total amount of trail mix is

$$x + y + z.$$

The cost of the trail mix is

$$3x + 4y + 8z.$$

Twice as many pretzels as dried fruit means

$$x = 2y.$$

(a) The system to be solved is

$$\begin{aligned} x + \ y + \ z &= 140 \\ 3x + 4y + 8z &= 140(6) \\ x \qquad\quad &= 2y, \end{aligned}$$

which can be rewritten as

$$\begin{aligned} x + \ y + \ z &= 140 \\ 3x + 4y + 8z &= 840 \\ x - 2y \qquad &= \ 0. \end{aligned}$$

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & 8 \\ 1 & -2 & 0 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$,

$$B = \begin{bmatrix} 140 \\ 840 \\ 0 \end{bmatrix}.$$

Thus, $AX = B$, and

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & 8 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 140 \\ 840 \\ 0 \end{bmatrix}.$$

Use row operations to obtain the inverse of the coefficient matrix.

$$A^{-1} = \begin{bmatrix} \frac{8}{7} & -\frac{1}{7} & \frac{2}{7} \\ \frac{4}{7} & -\frac{1}{14} & -\frac{5}{14} \\ -\frac{5}{7} & \frac{3}{14} & \frac{1}{14} \end{bmatrix}$$

Since $X = A^{-1}B$,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{8}{7} & -\frac{1}{7} & \frac{2}{7} \\ \frac{4}{7} & -\frac{1}{14} & -\frac{5}{14} \\ -\frac{5}{7} & \frac{3}{14} & \frac{1}{14} \end{bmatrix} \begin{bmatrix} 140 \\ 840 \\ 0 \end{bmatrix} = \begin{bmatrix} 40 \\ 20 \\ 80 \end{bmatrix}.$$

Use 40 lb of pretzels, 20 lb of dried fruit, and 80 lb of nuts.

(b) For 112 lb at \$6.50/lb, the matrix of constants is changed to

$$B = \begin{bmatrix} 112 \\ 112(6.50) \\ 0 \end{bmatrix} = \begin{bmatrix} 112 \\ 728 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{8}{7} & -\frac{1}{7} & \frac{2}{7} \\ \frac{4}{7} & -\frac{1}{14} & -\frac{5}{14} \\ -\frac{5}{7} & \frac{3}{14} & \frac{1}{14} \end{bmatrix} \begin{bmatrix} 112 \\ 728 \\ 0 \end{bmatrix} = \begin{bmatrix} 24 \\ 12 \\ 76 \end{bmatrix}$$

Use 24 lb of pretzels, 12 lb of dried fruit, and 76 lb of nuts.

(c) For 126 lb at \$5/lb, the matrix of constants is changed to

$$B = \begin{bmatrix} 126 \\ 126(5) \\ 0 \end{bmatrix} = \begin{bmatrix} 126 \\ 630 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{8}{7} & -\frac{1}{7} & \frac{2}{7} \\ \frac{4}{7} & -\frac{1}{14} & -\frac{5}{14} \\ -\frac{5}{7} & \frac{3}{14} & \frac{1}{14} \end{bmatrix} \begin{bmatrix} 126 \\ 630 \\ 0 \end{bmatrix} = \begin{bmatrix} 54 \\ 27 \\ 45 \end{bmatrix}$$

Use 54 lb of pretzels, 27 lb of dried fruit, and 45 lb of nuts.

64. **(a)** First, divide the letters and spaces of the sentence into groups of 3, writing each group as a column vector.

$$\begin{bmatrix} A \\ l \\ l \end{bmatrix}, \begin{bmatrix} \text{(space)} \\ i \\ s \end{bmatrix}, \begin{bmatrix} \text{(space)} \\ f \\ a \end{bmatrix}, \begin{bmatrix} i \\ r \\ \text{(space)} \end{bmatrix},$$

$$\begin{bmatrix} i \\ n \\ \text{(space)} \end{bmatrix}, \begin{bmatrix} l \\ o \\ v \end{bmatrix}, \begin{bmatrix} e \\ \text{(space)} \\ a \end{bmatrix}, \begin{bmatrix} n \\ d \\ \text{(space)} \end{bmatrix}, \begin{bmatrix} w \\ a \\ r \end{bmatrix}$$

Next, convert each letter into a number, assigning 1 to A, 2 to B, and so on, with the number 27 used to represent each space between words.

$$\begin{bmatrix} 1 \\ 12 \\ 12 \end{bmatrix}, \begin{bmatrix} 27 \\ 9 \\ 19 \end{bmatrix}, \begin{bmatrix} 27 \\ 6 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 18 \\ 27 \end{bmatrix}, \begin{bmatrix} 9 \\ 14 \\ 27 \end{bmatrix},$$

$$\begin{bmatrix} 12 \\ 15 \\ 22 \end{bmatrix}, \begin{bmatrix} 5 \\ 27 \\ 1 \end{bmatrix}, \begin{bmatrix} 14 \\ 4 \\ 27 \end{bmatrix}, \begin{bmatrix} 23 \\ 1 \\ 18 \end{bmatrix}$$

Now find the product of the coding matrix presented in Example 6, $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 3 \\ 4 & 2 & 1 \end{bmatrix}$, and each column vector above. This produces a new set of vectors which represents the coded message.

$$\begin{bmatrix} 85 \\ 50 \\ 40 \end{bmatrix}, \begin{bmatrix} 130 \\ 120 \\ 145 \end{bmatrix}, \begin{bmatrix} 49 \\ 63 \\ 121 \end{bmatrix}, \begin{bmatrix} 171 \\ 117 \\ 99 \end{bmatrix}, \begin{bmatrix} 159 \\ 113 \\ 91 \end{bmatrix},$$

$$\begin{bmatrix} 145 \\ 105 \\ 100 \end{bmatrix}, \begin{bmatrix} 90 \\ 40 \\ 75 \end{bmatrix}, \begin{bmatrix} 134 \\ 113 \\ 91 \end{bmatrix}, \begin{bmatrix} 98 \\ 101 \\ 112 \end{bmatrix}$$

The message will be transmitted as 85, 50, 40, 130, 120, 145, 49, 63, 121, 171, 117, 99, 159, 113, 91, 145, 105, 100, 90, 40, 75, 134, 113, 91, 98, 101, 112.

(b) First, divide the coded message into groups of three numbers and form each group into a column vector.

$$\begin{bmatrix} 138 \\ 81 \\ 102 \end{bmatrix}, \begin{bmatrix} 101 \\ 67 \\ 109 \end{bmatrix}, \begin{bmatrix} 162 \\ 124 \\ 173 \end{bmatrix}, \begin{bmatrix} 210 \\ 150 \\ 165 \end{bmatrix}$$

Next, find the product of the decoding matrix presented in Example 6, the inverse of matrix A in part (a) above, $A^{-1} = \begin{bmatrix} -.2 & .2 & .2 \\ .4 & -.6 & .2 \\ 0 & .4 & -.2 \end{bmatrix}$, and each of the column vectors above. This produces a new set of vectors which represents the decoded message.

$$\begin{bmatrix} 9 \\ 27 \\ 12 \end{bmatrix}, \begin{bmatrix} 15 \\ 22 \\ 5 \end{bmatrix}, \begin{bmatrix} 27 \\ 25 \\ 15 \end{bmatrix}, \begin{bmatrix} 21 \\ 27 \\ 27 \end{bmatrix}$$

Lastly, convert each number into a letter, assigning A to 1, B to 2, and so on, with the number 27 used to represent each space between words.
Decoded message: I love you

2.6 Input-Output Models

2. $A = \begin{bmatrix} .2 & .04 \\ .6 & .05 \end{bmatrix}, D = \begin{bmatrix} 3 \\ 10 \end{bmatrix}$

To find the production matrix X, first calculate $I - A$.

$$I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} .2 & .04 \\ .6 & .05 \end{bmatrix}$$
$$= \begin{bmatrix} .8 & -.04 \\ -.6 & .95 \end{bmatrix}$$

Next, use row operations to find the inverse of $I - A$.

$$(I - A)^{-1} = \begin{bmatrix} 1.29 & .054 \\ .815 & 1.087 \end{bmatrix}$$

Since $X = (I - A)^{-1}D$,

$$X = \begin{bmatrix} 1.29 & .054 \\ .815 & 1.087 \end{bmatrix} \begin{bmatrix} 3 \\ 10 \end{bmatrix} = \begin{bmatrix} 4.4 \\ 13.3 \end{bmatrix}.$$

4. $A = \begin{bmatrix} .01 & .03 \\ .05 & .05 \end{bmatrix}, D = \begin{bmatrix} 100 \\ 200 \end{bmatrix}$

To find $X = (I - A)^{-1}D$, first calculate $I - A$.

$$I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} .01 & .03 \\ .05 & .05 \end{bmatrix}$$
$$= \begin{bmatrix} .99 & -.03 \\ -.05 & .95 \end{bmatrix}$$

Then use row operations to find the inverse of $I - A$.

$$(I - A)^{-1} = \begin{bmatrix} 1.011715 & .03195 \\ .0532 & 1.0543 \end{bmatrix}$$

Since $X = (I - A)^{-1}D$,

$$X = \begin{bmatrix} 1.011715 & .03195 \\ .0532 & 1.0543 \end{bmatrix} \begin{bmatrix} 100 \\ 200 \end{bmatrix}$$
$$= \begin{bmatrix} 107.6 \\ 216.2 \end{bmatrix}.$$

6. $A = \begin{bmatrix} .1 & .5 & 0 \\ 0 & .3 & .4 \\ .1 & .2 & .1 \end{bmatrix}, D = \begin{bmatrix} 10 \\ 4 \\ 2 \end{bmatrix}$

$$I - A = \begin{bmatrix} .9 & -.5 & 0 \\ 0 & .7 & -.4 \\ -.1 & -.2 & .9 \end{bmatrix}$$

$$(I - A)^{-1} = \begin{bmatrix} 1.158 & .947 & .421 \\ .084 & 1.705 & .758 \\ .147 & .484 & 1.33 \end{bmatrix}$$

$$X = (I - A)^{-1}D = \begin{bmatrix} 1.158 & .947 & .421 \\ .084 & 1.705 & .758 \\ .147 & .484 & 1.33 \end{bmatrix} \begin{bmatrix} 10 \\ 4 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} 16.21 \\ 9.18 \\ 6.06 \end{bmatrix}$$

8. $$\begin{array}{c|ccc} & \text{A} & \text{B} & \text{C} \\ \hline \text{A} & .2 & .1 & .5 \\ \text{B} & .4 & .3 & .4 \\ \text{C} & .4 & .6 & .1 \end{array} = A$$

Calculate $I - A$, and then set $(I - A)X = O$ to find X.

$$I - A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} .2 & .1 & .5 \\ .4 & .3 & .4 \\ .4 & .6 & .1 \end{bmatrix}$$

$$= \begin{bmatrix} .8 & -.1 & -.5 \\ -.4 & .7 & -.4 \\ -.4 & -.6 & .9 \end{bmatrix}$$

$$(I - A)X = \begin{bmatrix} .8 & -.1 & -.5 \\ -.4 & .7 & -.4 \\ -.4 & -.6 & .9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Multiply to get

$$\begin{bmatrix} .8x_1 - .1x_2 - .5x_3 \\ -.4x_1 + .7x_2 - .4x_3 \\ -.4x_1 - .6x_2 + .9x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The last matrix equation corresponds to the following system.

$$\begin{aligned} .8x_1 - .1x_2 - .5x_3 &= 0 \\ -.4x_1 + .7x_2 - .4x_3 &= 0 \\ -.4x_1 - .6x_2 + .9x_3 &= 0 \end{aligned}$$

Solving this system with x_3 as the parameter will give the solution

$$\left(\frac{3}{4}x_3, x_3, x_3\right).$$

If $x_3 = 4$, then $x_1 = 3$ and $x_2 = 4$, so the production of A, B, and C should be in the ratio 3:4:4.

10. This exercise should be solved using a graphing calculator or a computer. The answer is

$$\begin{bmatrix} 9972 \\ 4956 \\ 9364 \\ 3045 \end{bmatrix}.$$

Values have been rounded.

12. $A = \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{5} & 0 \end{bmatrix}$, $D = \begin{bmatrix} 500 \\ 1000 \end{bmatrix}$

$$X = (I - A)^{-1}D$$

$$= \begin{bmatrix} 1 & -\frac{1}{3} \\ -\frac{1}{5} & 1 \end{bmatrix}^{-1} \begin{bmatrix} 500 \\ 1000 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{15}{14} & \frac{5}{14} \\ \frac{3}{14} & \frac{15}{14} \end{bmatrix} \begin{bmatrix} 500 \\ 1000 \end{bmatrix}$$

$$= \begin{bmatrix} 892.9 \\ 1178.6 \end{bmatrix}$$

Produce about 892.9 metric tons of wheat and about 1178.6 metric tons of oil.

14. $$A = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \end{bmatrix}, \ D = \begin{bmatrix} 1000 \\ 1000 \\ 1000 \end{bmatrix}$$

$$I - A = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ -\frac{1}{3} & 1 & 0 \\ 0 & -\frac{1}{4} & 1 \end{bmatrix}$$

$$(I - A)^{-1} = \begin{bmatrix} 1.043 & .130 & .5217 \\ .3478 & 1.043 & .1739 \\ .0870 & .2609 & 1.043 \end{bmatrix}$$

$$X = (I - A)^{-1}D = \begin{bmatrix} 1694.7 \\ 1564.7 \\ 1390.9 \end{bmatrix}$$

Produce about 1694.7 units of agriculture, about 1564.7 units of manufacturing, and about 1390.9 units of transportation.

16. $$A = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & 0 \end{bmatrix}, \ D = \begin{bmatrix} 500 \\ 500 \\ 500 \end{bmatrix}$$

$$I - A = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & 1 & -\frac{1}{4} \\ -\frac{1}{2} & -\frac{1}{4} & 1 \end{bmatrix}$$

$$(I - A)^{-1} = \begin{bmatrix} 1.538 & .9231 & .6154 \\ .6154 & 1.436 & .5128 \\ .9231 & .8205 & 1.436 \end{bmatrix}$$

$$X = (I - A)^{-1}D = \begin{bmatrix} 1538.3 \\ 1282.1 \\ 1589.8 \end{bmatrix}$$

Produce about 1538.3 units of agriculture, about

1282.1 units of manufacturing, and about 1589.8 units of transportation.

18. For this economy,

$$A = \begin{bmatrix} .1 & .2 & .1 \\ .2 & .1 & .05 \\ 0 & .05 & .1 \end{bmatrix}.$$

Use a graphing calculator or a computer to find $(I - A)^{-1}D$ where

$$D = \begin{bmatrix} 1000 \\ 1000 \\ 1000 \end{bmatrix}.$$

The solution, which may vary slightly, is

$$\begin{bmatrix} 1583.91 \\ 1529.54 \\ 1196.09 \end{bmatrix}.$$

Produce about 1584 units of oil, 1530 units of corn, and 1196 units of coffee.

20. Use a graphing calculator or a computer to find $(I-A)^{-1}D$. The solution, which may vary slightly, is

$$\begin{bmatrix} 18.2 \\ 73.2 \\ 66.7 \end{bmatrix}.$$

Values have been rounded.

Produce \$18.2 billion of agriculture, \$73.2 billion of manufacturing, and \$66.7 billion of households.

22. **(a)** Use a graphing calculator or a computer to find $(I - A)^{-1}D$. The solution, which may vary slightly, is

$$\begin{bmatrix} 183{,}464 \\ 304{,}005 \\ 42{,}037 \end{bmatrix}.$$

Values have been rounded.

In 100,000 RMB, produce about 183,000 for agriculture, 304,000 for industry/construction, and 42,000 for transportation/commerce.

(b) The entries in matrix $(I-A)^{-1}$ are called multipliers, and they give the desired economic values.

$$(I - A)^{-1} \approx \begin{bmatrix} 1.24 & .34 & .04 \\ .30 & 1.85 & .14 \\ .03 & .08 & 1.02 \end{bmatrix}$$

(Each entry has been rounded to two decimal places.)

Since we are interested in the result when there is a 1 RMB increase in demand for agricultural exports, we are interested in the first column of this matrix. Interpreting the multipliers shown, we find that an increase of 1 RMB in demand for agricultural exports will result in a 1.24 RMB increase in production of agricultural commodities, a .30 RMB increase in production of industrial/construction commodities, and a .03 RMB increase in production of transportation/commercial commodities.

24. **(a)** Use a graphing calculator or a computer to find $(I - B)^{-1}C$. The solution, which may vary slightly, is

$$\begin{bmatrix} 3 \\ 60 \\ 27 \\ 42 \\ 1002 \end{bmatrix}.$$

Values have been rounded.

A \$50 million increase in manufacturing demand will result in a \$3 million production increase in natural resources, a \$60 million production increase in manufacturing, a \$27 million production increase in trade and services, a \$42 million production increase in personal consumption, and 1002 new jobs.

(b) The matrix $(I - B)^{-1}$ is

$$\begin{bmatrix} 1.1 & .1 & .0 & .0 & 0 \\ .2 & 1.2 & .2 & .1 & 0 \\ .7 & .5 & 1.9 & .7 & 0 \\ 1.3 & .8 & 1.3 & 1.6 & 0 \\ 40.9 & 20.0 & 39.2 & 16.0 & 1 \end{bmatrix}$$

The bottom row of this matrix indicates the total employment requirement, per million dollars, of a sector. For example, the total employment requirement, per million dollars, of natural resource output is 40.9 employees.

26. Find the value of $I - A$, then set $(I - A)X = O$.

$$\begin{aligned}(I-A)X &= \left(\begin{bmatrix}1&0\\0&1\end{bmatrix} - \begin{bmatrix}\frac14&\frac12\\\frac34&\frac12\end{bmatrix}\right)\begin{bmatrix}x_1\\x_2\end{bmatrix}\\ &= \begin{bmatrix}\frac34&-\frac12\\-\frac34&\frac12\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}\\ &= \begin{bmatrix}\frac34x_1-\frac12x_2\\-\frac34x_1+\frac12x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}\end{aligned}$$

Thus,

$$\begin{aligned}\tfrac34x_1-\tfrac12x_2&=0\\ \tfrac34x_1&=\tfrac12x_2\\ x_1&=\tfrac23x_2.\end{aligned}$$

If $x_2 = 3$, then $x_1 = 2$. Therefore, produce 2 units of yams for every 3 units of pigs.

28. Find the value of $I - A$, then set $(I - A)X = O$.

$$\begin{aligned}(I-A)X &= \left(\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \begin{bmatrix}\frac13&\frac12&0\\\frac13&\frac14&\frac14\\\frac13&\frac14&\frac34\end{bmatrix}\right)\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\\ &= \begin{bmatrix}\frac23&-\frac12&0\\-\frac13&\frac34&-\frac14\\-\frac13&-\frac14&\frac14\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\\ &= \begin{bmatrix}\frac23x_1-\frac12x_2\\-\frac13x_1+\frac34x_2-\frac14x_3\\-\frac13x_1-\frac14x_2+\frac14x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}\end{aligned}$$

The system to be solved is

$$\begin{aligned}\tfrac23x_1-\tfrac12x_2\qquad\quad&=0\\ -\tfrac13x_1+\tfrac34x_2-\tfrac14x_3&=0\\ -\tfrac13x_1-\tfrac14x_2+\tfrac14x_3&=0.\end{aligned}$$

Write the augmented matrix of the system.

$$\left[\begin{array}{rrr|r}\frac23&-\frac12&0&0\\-\frac13&\frac34&-\frac14&0\\-\frac13&-\frac14&\frac14&0\end{array}\right]$$

$$\begin{array}{r}\frac32R_1\to R_1\\12R_2\to R_2\\12R_3\to R_3\end{array}\quad\left[\begin{array}{rrr|r}1&-\frac34&0&0\\-4&9&-3&0\\-4&-3&3&0\end{array}\right]$$

$$\begin{array}{r}\\4R_1+R_2\to R_2\\4R_1+R_3\to R_3\end{array}\quad\left[\begin{array}{rrr|r}1&-\frac34&0&0\\0&6&-3&0\\0&-6&3&0\end{array}\right]$$

$$\begin{array}{r}\\\frac16R_2\to R_2\\\\\end{array}\quad\left[\begin{array}{rrr|r}1&-\frac34&0&0\\0&1&-\frac12&0\\0&-6&3&0\end{array}\right]$$

$$\begin{array}{r}\frac34R_2+R_1\to R_1\\\\6R_2+R_3\to R_3\end{array}\quad\left[\begin{array}{rrr|r}1&0&-\frac38&0\\0&1&-\frac12&0\\0&0&0&0\end{array}\right]$$

Use x_3 as the parameter. Therefore, $x_1 = \frac38x_3$ and $x_2 = \frac12x_3$, and the solution is $\left(\frac38x_3, \frac12x_3, x_3\right)$. If $x_3 = 8$, then $x_1 = 3$ and $x_2 = 4$.

Produce 3 units of agriculture to every 4 units of manufacturing and 8 units of transportation.

Chapter 2 Review Exercises

4. $\dfrac{x}{2}+\dfrac{y}{4}=3$

$\dfrac{x}{4}-\dfrac{y}{2}=4$

First, multiply both equations by 4 to clear fractions.

$$\begin{aligned}2x+y&=12\\x-2y&=16\end{aligned}$$

Now proceed by the echelon method.

$$\begin{array}{rrl}&2x+y=&12\\R_1+(-2)R_2\to R_2&5y=&-20\\[1ex]\frac12R_1\to R_1&x+\dfrac12y=&6\\[1ex]\frac15R_2\to R_2&y=&-4\end{array}$$

Back-substitution gives

$$\begin{aligned}x+\frac12(-4)&=6\\x&=8.\end{aligned}$$

The solution is $(8, -4)$.

6.
$$\begin{aligned} x - y &= 3 \\ 2x + 3y + z &= 13 \\ 3x - 2z &= 21 \end{aligned}$$

Use the echelon method, applying the appropriate row operations.

$$\begin{array}{rrl} & x - y & = 3 \\ 2R_1 + (-1)R_2 \to R_2 & -5y - z & = -7 \\ 3R_1 + (-1)R_3 \to R_3 & -3y + 2z & = -12 \end{array}$$

$$\begin{array}{rrl} & x - y & = 3 \\ & -5y - z & = -7 \\ 3R_2 + (-5)R_3 \to R_3 & -13z & = 39 \end{array}$$

$$\begin{array}{rrl} & x - y & = 3 \\ -\frac{1}{5}R_2 \to R_2 & y + \dfrac{1}{5}z & = \dfrac{7}{5} \\ -\frac{1}{13}R_3 \to R_3 & z & = -3 \end{array}$$

Back-substitution gives

$$y + \frac{1}{5}(-3) = \frac{7}{5}$$
$$y = \frac{10}{5} = 2,$$

and

$$x - 2 = 3$$
$$x = 5.$$

The solution is $(5, 2, -3)$.

8.
$$\begin{aligned} x + 2y &= -9 \\ 4x + 9y &= 41 \end{aligned}$$

Write the system in augmented matrix form and apply row operations.

$$\left[\begin{array}{cc|c} 1 & 2 & -9 \\ 4 & 9 & 41 \end{array}\right]$$

$$-4R_1 + R_2 \to R_2 \quad \left[\begin{array}{cc|c} 1 & 2 & -9 \\ 0 & 1 & 77 \end{array}\right]$$

$$-2R_2 + R_1 \to R_1 \quad \left[\begin{array}{cc|r} 1 & 0 & -163 \\ 0 & 1 & 77 \end{array}\right]$$

The solution is $(-163, 77)$.

10.
$$\begin{aligned} x - 2z &= 5 \\ 3x + 2y &= 8 \\ -x + 2z &= 10 \end{aligned}$$

Write the system in augmented matrix form and apply row operations.

$$\left[\begin{array}{rrr|r} 1 & 0 & -2 & 5 \\ 3 & 2 & 0 & 8 \\ -1 & 0 & 2 & 10 \end{array}\right]$$

$$\begin{array}{r} \\ -3R_1 + R_2 \to R_2 \\ R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{rrr|r} 1 & 0 & -2 & 5 \\ 0 & 2 & 6 & -7 \\ 0 & 0 & 0 & 15 \end{array}\right]$$

The last row says that $0 = 15$, which is false, so the system is inconsistent and there is no solution.

In Exercises 12 and 14, corresponding elements must be equal.

12. $\begin{bmatrix} 2 & 3 \\ 5 & q \end{bmatrix} = \begin{bmatrix} a & b \\ c & 9 \end{bmatrix}$

Size: 2×2; $a = 2$, $b = 3$, $c = 5$, $q = 9$; square matrix

14. $\begin{bmatrix} m & 4 & z & -1 \end{bmatrix} = \begin{bmatrix} 12 & k & -8 & r \end{bmatrix}$

Size: 1×4; $m = 12$, $k = 4$, $z = -8$, $r = -1$; row matrix

16.
$$A + C = \begin{bmatrix} 4 & 10 \\ -2 & -3 \\ 6 & 9 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ -1 & 3 \\ 4 & 7 \end{bmatrix}$$
$$= \begin{bmatrix} 9 & 10 \\ -3 & 0 \\ 10 & 16 \end{bmatrix}$$

18.
$$3C + 2A = 3\begin{bmatrix} 5 & 0 \\ -1 & 3 \\ 4 & 7 \end{bmatrix} + 2\begin{bmatrix} 4 & 10 \\ -2 & -3 \\ 6 & 9 \end{bmatrix}$$
$$= \begin{bmatrix} 15 & 0 \\ -3 & 9 \\ 12 & 21 \end{bmatrix} + \begin{bmatrix} 8 & 20 \\ -4 & -6 \\ 12 & 18 \end{bmatrix}$$
$$= \begin{bmatrix} 23 & 20 \\ -7 & 3 \\ 24 & 39 \end{bmatrix}$$

20. $2A - 5C = 2\begin{bmatrix} 4 & 10 \\ -2 & -3 \\ 6 & 9 \end{bmatrix} - 5\begin{bmatrix} 5 & 0 \\ -1 & 3 \\ 4 & 7 \end{bmatrix}$

$$= \begin{bmatrix} 8 & 20 \\ -4 & -6 \\ 12 & 18 \end{bmatrix} - \begin{bmatrix} 25 & 0 \\ -5 & 15 \\ 20 & 35 \end{bmatrix}$$

$$= \begin{bmatrix} -17 & 20 \\ 1 & -21 \\ -8 & -17 \end{bmatrix}$$

22. A is 3×2 and C is 3×2, so finding the product AC is not possible.

$$\begin{array}{cc} A & C \\ 3 \times \mathbf{2} & \mathbf{3} \times 2 \end{array}$$

(The inner 2 numbers must match.)

24. $ED = \begin{bmatrix} 1 & 3 & -4 \end{bmatrix}\begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix} = [1\cdot 6 + 3\cdot 1 + (-4)0] = [9]$

26. $EA = \begin{bmatrix} 1 & 3 & -4 \end{bmatrix}\begin{bmatrix} 4 & 10 \\ -2 & -3 \\ 6 & 9 \end{bmatrix}$

$$= \begin{bmatrix} 1\cdot 4 + 3(-2) + (-4)6 & 1\cdot 10 + 3(-3) + (-4)9 \end{bmatrix}$$

$$= \begin{bmatrix} -26 & -35 \end{bmatrix}$$

28. Find the inverse of $B = \begin{bmatrix} 2 & 3 & -2 \\ 2 & 4 & 0 \\ 0 & 1 & 2 \end{bmatrix}$, if it exists.

Write the augmented matrix to obtain

$$[B|I] = \left[\begin{array}{ccc|ccc} 2 & 3 & -2 & 1 & 0 & 0 \\ 2 & 4 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right].$$

$$-1\text{R}_1 + \text{R}_2 \to \text{R}_2 \quad \left[\begin{array}{ccc|ccc} 2 & 3 & -2 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} -3\text{R}_2 + \text{R}_1 \to \text{R}_1 \\ \\ -1\text{R}_2 + \text{R}_3 \to \text{R}_3 \end{array} \quad \left[\begin{array}{ccc|ccc} 2 & 0 & -8 & 4 & -3 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right]$$

No inverse exists, since the third row is all zeros.

30. Find the inverse of $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$, if it exists.

The augmented matrix is

$$[A|I] = \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 5 & 3 & 0 & 1 \end{array}\right].$$

$$5\text{R}_1 + (-2)\text{R}_2 \to \text{R}_2 \quad \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & -1 & 5 & -2 \end{array}\right]$$

$$\text{R}_2 + \text{R}_1 \to \text{R}_1 \quad \left[\begin{array}{cc|cc} 2 & 0 & 6 & -2 \\ 0 & -1 & 5 & -2 \end{array}\right]$$

$$\begin{array}{r} \frac{1}{2}\text{R}_1 \to \text{R}_1 \\ -1\text{R}_2 \to \text{R}_2 \end{array} \quad \left[\begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & 1 & -5 & 2 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

32. Find the inverse of $A = \begin{bmatrix} 2 & 0 \\ -1 & 5 \end{bmatrix}$, if it exists.

The augmented matrix is

$$[A|I] = \left[\begin{array}{cc|cc} 2 & 0 & 1 & 0 \\ -1 & 5 & 0 & 1 \end{array}\right].$$

$$\text{R}_1 + 2\text{R}_2 \to \text{R}_2 \quad \left[\begin{array}{cc|cc} 2 & 0 & 1 & 0 \\ 0 & 10 & 1 & 2 \end{array}\right]$$

$$\begin{array}{r} \frac{1}{2}\text{R}_1 \to \text{R}_1 \\ \frac{1}{10}\text{R}_2 \to \text{R}_2 \end{array} \quad \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{10} & \frac{1}{5} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{10} & \frac{1}{5} \end{bmatrix}$$

34. Find the inverse of $A=\begin{bmatrix}2&-1&0\\1&0&1\\1&-2&0\end{bmatrix}$, if it exists.

The augmented matrix is

$$[A|I]=\left[\begin{array}{rrr|rrr}2&-1&0&1&0&0\\1&0&1&0&1&0\\1&-2&0&0&0&1\end{array}\right].$$

$$\begin{array}{r}\\ \mathrm{R}_1+(-2)\mathrm{R}_2\to\mathrm{R}_2\\ \mathrm{R}_1+(-2)\mathrm{R}_3\to\mathrm{R}_3\end{array}\left[\begin{array}{rrr|rrr}2&-1&0&1&0&0\\0&-1&-2&1&-2&0\\0&3&0&1&0&-2\end{array}\right]$$

$$\begin{array}{r}-1\mathrm{R}_2+\mathrm{R}_1\to\mathrm{R}_1\\ \\ 3\mathrm{R}_2+\mathrm{R}_3\to\mathrm{R}_3\end{array}\left[\begin{array}{rrr|rrr}2&0&2&0&2&0\\0&-1&-2&1&-2&0\\0&0&-6&4&-6&-2\end{array}\right]$$

$$\begin{array}{r}\mathrm{R}_3+3\mathrm{R}_1\to\mathrm{R}_1\\ \mathrm{R}_3+(-3)\mathrm{R}_2\to\mathrm{R}_2\\ \\ \end{array}\left[\begin{array}{rrr|rrr}6&0&0&4&0&-2\\0&3&0&1&0&-2\\0&0&-6&4&-6&-2\end{array}\right]$$

$$\begin{array}{r}\frac{1}{6}\mathrm{R}_1\to\mathrm{R}_1\\ \frac{1}{3}\mathrm{R}_2\to\mathrm{R}_2\\ -\frac{1}{6}\mathrm{R}_3\to\mathrm{R}_3\end{array}\left[\begin{array}{rrr|rrr}1&0&0&\frac{2}{3}&0&-\frac{1}{3}\\0&1&0&\frac{1}{3}&0&-\frac{2}{3}\\0&0&1&-\frac{2}{3}&1&\frac{1}{3}\end{array}\right]$$

$$A^{-1}=\begin{bmatrix}\frac{2}{3}&0&-\frac{1}{3}\\\frac{1}{3}&0&-\frac{2}{3}\\-\frac{2}{3}&1&\frac{1}{3}\end{bmatrix}$$

36. Find the inverse of $A=\begin{bmatrix}1&3&6\\4&0&9\\5&15&30\end{bmatrix}$, if it exists.

$$[A|I]=\left[\begin{array}{rrr|rrr}1&3&6&1&0&0\\4&0&9&0&1&0\\5&15&30&0&0&1\end{array}\right]$$

$$\begin{array}{r}\\ -4\mathrm{R}_1+\mathrm{R}_2\to\mathrm{R}_2\\ -5\mathrm{R}_1+\mathrm{R}_3\to\mathrm{R}_3\end{array}\left[\begin{array}{rrr|rrr}1&3&6&1&0&0\\0&-12&-15&-4&1&0\\0&0&0&-5&0&1\end{array}\right]$$

The last row is all zeros, so no inverse exists.

38. $A=\begin{bmatrix}2&4\\-1&-3\end{bmatrix}$, $B=\begin{bmatrix}8\\3\end{bmatrix}$

The matrix equation to be solved is $AX=B$, or

$$\begin{bmatrix}2&4\\-1&-3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}8\\3\end{bmatrix}.$$

Calculate the inverse of the coefficient matrix A to obtain

$$\begin{bmatrix}2&4\\-1&-3\end{bmatrix}^{-1}=\begin{bmatrix}\frac{3}{2}&2\\-\frac{1}{2}&-1\end{bmatrix}.$$

Now $X=A^{-1}B$, so

$$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\frac{3}{2}&2\\-\frac{1}{2}&-1\end{bmatrix}\begin{bmatrix}8\\3\end{bmatrix}=\begin{bmatrix}18\\-7\end{bmatrix}.$$

40. $A=\begin{bmatrix}1&0&2\\-1&1&0\\3&0&4\end{bmatrix}$, $B=\begin{bmatrix}8\\4\\-6\end{bmatrix}$

By the usual method, we find that the inverse of the coefficient matrix is

$$A^{-1}=\begin{bmatrix}-2&0&1\\-2&1&1\\\frac{3}{2}&0&-\frac{1}{2}\end{bmatrix}.$$

Since $X=A^{-1}B$,

$$X=\begin{bmatrix}-2&0&1\\-2&1&1\\\frac{3}{2}&0&-\frac{1}{2}\end{bmatrix}\begin{bmatrix}8\\4\\-6\end{bmatrix}=\begin{bmatrix}-22\\-18\\15\end{bmatrix}.$$

42. $2x+\ \ y=5$
$3x-2y=4$

The coefficient matrix is

$$A=\begin{bmatrix}2&1\\3&-2\end{bmatrix}.$$

Find that the inverse of A is

$$A^{-1}=\begin{bmatrix}\frac{2}{7}&\frac{1}{7}\\\frac{3}{7}&-\frac{2}{7}\end{bmatrix},$$

and use $X=A^{-1}B$ to find that

$$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\frac{2}{7}&\frac{1}{7}\\\frac{3}{7}&-\frac{2}{7}\end{bmatrix}\begin{bmatrix}5\\4\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}.$$

The solution is $(2,1)$.

44.
$$\begin{aligned} x + y + z &= 1 \\ 2x + y \phantom{{}+z} &= -2 \\ 3y + z &= 2 \end{aligned}$$

The coefficient matrix is

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}.$$

Find that the inverse of A is

$$A^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & -\frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{6}{5} & -\frac{3}{5} & -\frac{1}{5} \end{bmatrix}.$$

Now $X = A^{-1}B$, so

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & -\frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{6}{5} & -\frac{3}{5} & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$$

The solution is $(-1, 0, 2)$.

46. $A = \begin{bmatrix} .01 & .05 \\ .04 & .03 \end{bmatrix}$, $D = \begin{bmatrix} 200 \\ 300 \end{bmatrix}$

$X = (I - A)^{-1}D$

$$\begin{aligned} I - A &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} .01 & .05 \\ .04 & .03 \end{bmatrix} \\ &= \begin{bmatrix} .99 & -.05 \\ -.04 & .97 \end{bmatrix} \end{aligned}$$

Use row operations to find the inverse of $I - A$, which is

$$(I - A)^{-1} = \begin{bmatrix} 1.0122 & .0522 \\ .0417 & 1.0331 \end{bmatrix}.$$

Since $X = (I - A)^{-1}D$, the production matrix is

$$X = \begin{bmatrix} 1.0122 & .0522 \\ .0417 & 1.0331 \end{bmatrix} \begin{bmatrix} 200 \\ 300 \end{bmatrix} = \begin{bmatrix} 218.1 \\ 318.3 \end{bmatrix}.$$

48.
$$\begin{aligned} x + 2y + z &= 7 \quad (1) \\ 2x - y - z &= 2 \quad (2) \\ 3x - 3y + 2z &= -5 \quad (3) \end{aligned}$$

(a) To solve the system by the echelon method, begin by eliminating x in equations (2) and (3).

$$\begin{array}{lrl} & x + 2y + z = 7 & (1) \\ -2\mathrm{R}_1 + \mathrm{R}_2 \rightarrow \mathrm{R}_2 & -5y - 3z = -12 & (4) \\ -3\mathrm{R}_1 + \mathrm{R}_3 \rightarrow \mathrm{R}_3 & -9y - z = -26 & (5) \end{array}$$

Eliminate y in equation (5).

$$\begin{array}{lrl} & x + 2y + z = 7 & (1) \\ & -5y - 3z = -12 & (4) \\ -9\mathrm{R}_2 + 5\mathrm{R}_3 \rightarrow \mathrm{R}_3 & 22z = -22 & (6) \end{array}$$

Make each leading coefficient equal 1.

$$\begin{array}{lrl} & x + 2y + z = 7 & (1) \\ -\frac{1}{5}\mathrm{R}_2 \rightarrow \mathrm{R}_2 & y + \dfrac{3}{5}z = \dfrac{12}{5} & (7) \\ \frac{1}{22}\mathrm{R}_3 \rightarrow \mathrm{R}_3 & z = -1 & (8) \end{array}$$

Substitute -1 for z in equation (7) to get $y = 3$. Substitute -1 for z and 3 for y in equation (1) to get $x = 2$.

The solution is $(2, 3, -1)$.

(b) The same system is to be solved using the Gauss-Jordan method. Write the augmented matrix and use row operations.

$$\left[\begin{array}{rrr|r} 1 & 2 & 1 & 7 \\ 2 & -1 & -1 & 2 \\ 3 & -3 & 2 & -5 \end{array}\right]$$

$$\begin{array}{r} -2\mathrm{R}_1 + \mathrm{R}_2 \rightarrow \mathrm{R}_2 \\ -3\mathrm{R}_1 + \mathrm{R}_3 \rightarrow \mathrm{R}_3 \end{array} \quad \left[\begin{array}{rrr|r} 1 & 2 & 1 & 7 \\ 0 & -5 & -3 & -12 \\ 0 & -9 & -1 & -26 \end{array}\right]$$

$$\begin{array}{r} 2\mathrm{R}_2 + 5\mathrm{R}_1 \rightarrow \mathrm{R}_1 \\ \\ -9\mathrm{R}_2 + 5\mathrm{R}_3 \rightarrow \mathrm{R}_3 \end{array} \quad \left[\begin{array}{rrr|r} 5 & 0 & -1 & 11 \\ 0 & -5 & -3 & -12 \\ 0 & 0 & 22 & -22 \end{array}\right]$$

$$\begin{array}{r} \mathrm{R}_3 + 22\mathrm{R}_1 \rightarrow \mathrm{R}_1 \\ 3\mathrm{R}_3 + 22\mathrm{R}_2 \rightarrow \mathrm{R}_2 \\ \end{array} \quad \left[\begin{array}{rrr|r} 110 & 0 & 0 & 220 \\ 0 & -110 & 0 & -330 \\ 0 & 0 & 22 & -22 \end{array}\right]$$

$$\begin{array}{r} \frac{1}{110}\mathrm{R}_1 + \mathrm{R}_1 \\ -\frac{1}{110}\mathrm{R}_2 \rightarrow \mathrm{R}_2 \\ \frac{1}{22}\mathrm{R}_3 \rightarrow \mathrm{R}_3 \end{array} \quad \left[\begin{array}{rrr|r} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

The corresponding system is

$$\begin{aligned} x &= 2 \\ y &= 3 \\ z &= -1. \end{aligned}$$

The solution is $(2, 3, -1)$

(c) The system can be written as a matrix equation $AX = B$ by writing

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & -1 \\ 3 & -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ -5 \end{bmatrix}.$$

(d) The inverse of the coefficient matrix A can be found by using row operations.

$$[A|I] = \left[\begin{array}{rrr|rrr} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & -1 & -1 & 0 & 1 & 0 \\ 3 & -3 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ -2R_1+R_2\to R_2 \\ -3R_1+R_3\to R_3 \end{array} \left[\begin{array}{rrr|rrr} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -5 & -3 & -2 & 1 & 0 \\ 0 & -9 & -1 & -3 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} 2R_2+5R_1\to R_1 \\ \\ -9R_2+5R_3\to R_3 \end{array} \left[\begin{array}{rrr|rrr} 5 & 0 & -1 & 1 & 2 & 0 \\ 0 & -5 & -3 & -2 & 1 & 0 \\ 0 & 0 & 22 & 3 & -9 & 5 \end{array}\right]$$

$$\begin{array}{r} R_3+22R_1\to R_1 \\ 3R_3+22R_2\to R_2 \\ \\ \end{array} \left[\begin{array}{rrr|rrr} 110 & 0 & 0 & 25 & 35 & 5 \\ 0 & -110 & 0 & -35 & -5 & 15 \\ 0 & 0 & 22 & 3 & -9 & 5 \end{array}\right]$$

The inverse of matrix A is

$$A^{-1} = \begin{bmatrix} \frac{5}{22} & \frac{7}{22} & \frac{1}{22} \\ \frac{7}{22} & \frac{1}{22} & -\frac{3}{22} \\ \frac{3}{22} & -\frac{9}{22} & \frac{5}{22} \end{bmatrix} \approx \begin{bmatrix} .23 & .32 & .05 \\ .32 & .05 & -.14 \\ .14 & -.41 & .23 \end{bmatrix}.$$

(e) Since $X = A^{-1}B$,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{5}{22} & \frac{7}{22} & \frac{1}{22} \\ \frac{7}{22} & \frac{1}{22} & -\frac{3}{22} \\ \frac{3}{22} & -\frac{9}{22} & \frac{5}{22} \end{bmatrix} \begin{bmatrix} 7 \\ 2 \\ -5 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}.$$

Once again, the solution is $(2, 3, -1)$.

50. Let $x =$ number of shares of \$32 stock,
and $y =$ number of shares of \$23 stock.

Then

$$32x + 23y = 10{,}100,$$

and

$$1.2x + 1.4y = 540.$$

The augmented matrix is

$$\left[\begin{array}{rr|r} 32 & 23 & 10{,}100 \\ 1.2 & 1.4 & 540 \end{array}\right].$$

$$\begin{array}{r} \\ -1.2R_1+32R_2\to R_2 \end{array} \left[\begin{array}{rr|r} 32 & 23 & 10{,}100 \\ 0 & 17.2 & 5160 \end{array}\right]$$

$$\begin{array}{r} -23R_2+17.2R_1\to R_1 \\ \\ \end{array} \left[\begin{array}{rr|r} 550.4 & 0 & 55{,}040 \\ 0 & 17.2 & 5160 \end{array}\right]$$

$$\begin{array}{r} \frac{1}{550.4}R_1\to R_1 \\ \frac{1}{17.2}R_2\to R_2 \end{array} \left[\begin{array}{rr|r} 1 & 0 & 100 \\ 0 & 1 & 300 \end{array}\right]$$

She should buy 100 shares of the first stock and 300 shares of the second stock.

52. Let $x =$ Tulsa's number of gallons,
$y =$ New Orleans' number of gallons, and
$z =$ Ardmore's number of gallons.

The system that may be written is

$$\begin{aligned} .5x + .4y + .3z &= 219{,}000 && \textit{Chicago} \\ .2x + .4y + .4z &= 192{,}000 && \textit{Dallas} \\ .3x + .2y + .3z &= 144{,}000. && \textit{Atlanta} \end{aligned}$$

The augmented matrix is

$$\left[\begin{array}{rrr|r} .5 & .4 & .3 & 219{,}000 \\ .2 & .4 & .4 & 192{,}000 \\ .3 & .2 & .3 & 144{,}000 \end{array}\right].$$

$$\begin{array}{r} \\ 2R_1+(-5)R_2\to R_2 \\ 3R_1+(-5)R_3\to R_3 \end{array} \left[\begin{array}{rrr|r} .5 & .4 & .3 & 219{,}000 \\ 0 & -1.2 & -1.4 & -522{,}000 \\ 0 & .2 & -.6 & -63{,}000 \end{array}\right]$$

$$\begin{array}{r} -2R_3+\ R_1\to R_1 \\ \\ R_2+6R_3\to R_3 \end{array} \left[\begin{array}{rrr|r} .5 & 0 & 1.5 & 345{,}000 \\ 0 & -1.2 & -1.4 & -522{,}000 \\ 0 & 0 & -5 & -900{,}000 \end{array}\right]$$

$$\begin{array}{r} .3R_3+\ R_1\to R_1 \\ -14R_3+50R_2\to R_2 \\ \\ \end{array} \left[\begin{array}{rrr|r} .5 & 0 & 0 & 75{,}000 \\ 0 & -60 & 0 & -13{,}500{,}000 \\ 0 & 0 & -5 & -900{,}000 \end{array}\right]$$

$$\begin{array}{r} 2R_1\to R_1 \\ -\frac{1}{60}R_2\to R_2 \\ -\frac{1}{5}R_3\to R_3 \end{array} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 150{,}000 \\ 0 & 1 & 0 & 225{,}000 \\ 0 & 0 & 1 & 180{,}000 \end{array}\right]$$

Thus, 150,000 gal were produced at Tulsa, 225,000 gal at New Orleans, and 180,000 gal at Ardmore.

54. (a)
$$\begin{array}{l} \text{High} \\ \text{Medium} \\ \text{Coated} \end{array} \begin{bmatrix} 3170 \\ 2360 \\ 1800 \end{bmatrix}$$

(b)
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

(c)
$$\begin{bmatrix} 10 & 5 & 8 \\ 12 & 0 & 4 \\ 0 & 10 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3170 \\ 2360 \\ 1800 \end{bmatrix}$$

(d)
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 & 5 & 8 \\ 12 & 0 & 4 \\ 0 & 10 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 3170 \\ 2360 \\ 1800 \end{bmatrix}$$
$$= \begin{bmatrix} -.154 & .212 & .0769 \\ -.231 & .192 & .2154 \\ .462 & -.385 & -.231 \end{bmatrix} \begin{bmatrix} 3170 \\ 2360 \\ 1800 \end{bmatrix}$$
$$= \begin{bmatrix} 150 \\ 110 \\ 140 \end{bmatrix}$$

56. **(a)** Use a graphing calculator or a computer to find $(I - A)^{-1}$. The solution, which may vary slightly, is

$$\begin{bmatrix} 1.30 & .045 & .567 & .012 & .068 & .020 \\ .204 & 1.03 & .183 & .004 & .022 & .006 \\ .155 & .038 & 1.12 & .020 & .114 & .034 \\ .018 & .021 & .028 & 1.08 & .016 & .033 \\ .537 & .525 & .483 & .279 & 1.73 & .419 \\ .573 & .346 & .497 & .536 & .087 & 1.94 \end{bmatrix}$$

Values have been rounded.

The value in row 2, column 1 of this matrix, .204, indicates that every \$1 of increased demand for livestock will result in an increase of production demand of \$.204 in crops.

(b) Use a graphing calculator or computer to find $(I-A)^{-1}D$. The solution, which may vary slightly, is

$$\begin{bmatrix} 3855 \\ 1476 \\ 2726 \\ 1338 \\ 8439 \\ 10{,}256 \end{bmatrix}.$$

Values have been rounded.

In millions of dollars, produce \$3855 in livestock, \$1476 in crops, \$2726 in food products, \$1338 in mining and manufacturing, \$8439 in households, and \$10,256 in other business sectors.

58. **(a)** The X-ray passes through cells B and C, so the attenuation value for beam 3 is $b + c$.

(b) Beam 1: $a + b = .8$
Beam 2: $a + c = .55$
Beam 3: $b + c = .65$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} .8 \\ .55 \\ .65 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} .8 \\ .55 \\ .65 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} .8 \\ .55 \\ .65 \end{bmatrix}$$
$$= \begin{bmatrix} .35 \\ .45 \\ .2 \end{bmatrix}$$

The solution is $(.35, .45, .2)$, so A is tumorous, B is bone, and C is healthy.

(c) For patient X,

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} .54 \\ .40 \\ .52 \end{bmatrix} = \begin{bmatrix} .21 \\ .33 \\ .19 \end{bmatrix}.$$

A and C are healthy; B is tumorous.

For patient Y,

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} .65 \\ .80 \\ .75 \end{bmatrix} = \begin{bmatrix} .35 \\ .3 \\ .45 \end{bmatrix}.$$

A and B are tumorous; C is bone.

For patient Z,

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} .51 \\ .49 \\ .44 \end{bmatrix} = \begin{bmatrix} .28 \\ .23 \\ .21 \end{bmatrix}.$$

A could be healthy or tumorous; B and C are healthy.

60. The matrix representing the rates per 1000 athlete-exposures for specific injuries that caused a player wearing either shield to miss one or more events is

$$\begin{bmatrix} 3.54 & 1.41 \\ 1.53 & 1.57 \\ .34 & .29 \\ 7.53 & 6.21 \end{bmatrix}.$$

Since an equal number of players wear each type of shield and the total number of athlete-exposures for the league in a season is 8000, each type of shield is worn by 4000 players. Since the rates are given per 1000 athletic-exposures, the matrix representing the number of 1000 athlete-exposures for each type of shield is

$$\begin{bmatrix} 4 \\ 4 \end{bmatrix}.$$

The product of these matrices is

$$\begin{bmatrix} 20 \\ 12 \\ 3 \\ 55 \end{bmatrix}.$$

Values have been rounded.

There would be about 20 head and face injuries, 12 concussions, 3 neck injuries, and 55 other injuries.

62. $\frac{1}{2}W_1 + \frac{\sqrt{2}}{2}W_2 = 150 \quad (1)$

$\frac{\sqrt{3}}{2}W_1 - \frac{\sqrt{2}}{2}W_2 = \quad 0 \quad (2)$

Adding equations (1) and (2) gives

$$\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) W_1 = 150.$$

Multiply by 2.

$$(1+\sqrt{3})W_1 = 300$$
$$W_1 = \frac{300}{1+\sqrt{3}} \approx 110$$

From equation (2),

$$\frac{\sqrt{3}}{2}W_1 = \frac{\sqrt{2}}{2}W_2$$
$$W_2 = \frac{\sqrt{3}}{\sqrt{2}}W_1.$$

Substitute $\frac{300}{1+\sqrt{3}}$ from above for W_1.

$$W_2 = \frac{\sqrt{3}}{\sqrt{2}} \cdot \frac{300}{1+\sqrt{3}} = \frac{300\sqrt{3}}{(1+\sqrt{3})\sqrt{2}} \approx 134$$

Therefore, $W_1 \approx 110$ lb and $W_2 \approx 134$ lb.

64. (a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \end{bmatrix} y = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} x \\ x \end{bmatrix} + \begin{bmatrix} 0 \\ 2y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ x+2y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Since corresponding elements must be equal, $x = 1$ and $x+2y = 2$. Substituting $x = 1$ in the second equation gives $y = \frac{1}{2}$. Note that $x = 1$ and $y = \frac{1}{2}$ are the values that balance the equation.

(b) $x\text{CO}_2 + y\text{H}_2 + z\text{CO} = \text{H}_2\text{O}$

$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} y + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} z = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ 0 \\ 2x \end{bmatrix} + \begin{bmatrix} 0 \\ 2y \\ 0 \end{bmatrix} + \begin{bmatrix} z \\ 0 \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x+z \\ 2y \\ 2x+z \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Since corresponding elements must be equal, $x + z = 0, 2y = 2$, and $2x+z = 1$. Solving $2y = 2$ gives $y = 1$. Solving the system $\begin{cases} x+z=0 \\ 2x+z=1 \end{cases}$ gives $x = 1$ and $z = -1$. Thus, the values that balance the equation are $x = 1$, $y = 1$, and $z = -1$.

Extended Application: Contagion

1. $PQ = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 2 & 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 2 & 1 \end{bmatrix}$$

2. In the product PQ, $a_{23} = 0$, so there were no contacts.

3. In the product PQ, column 3 has all zeros. The third person had no contacts with the first group.

4. In the product PQ, the entries in columns 2 and 4 both have a sum of 4 while the entries in column 5 have a sum of 3. In Q, columns 2, 4, and 5 have a sum of 2, 2, and 3, respectively. Therefore, the second, fourth, and fifth persons in the third group each had a total of 6 first-and second-order contacts.

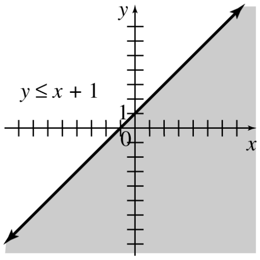
Chapter 3

LINEAR PROGRAMMING: THE GRAPHICAL METHOD

3.1 Graphing Linear Inequalities

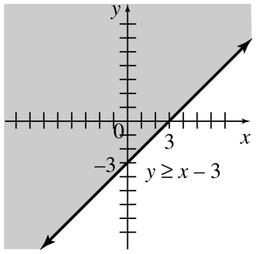
2. $y \leq x + 1$

The boundary is the line $y = x + 1$. Graph a solid line using the points $(0, 1)$ and $(-1, 0)$. Use $(0, 0)$ as a test point to get $0 \leq 1$, which is true. Shade all points below the line, or the half-plane containing $(0, 0)$.



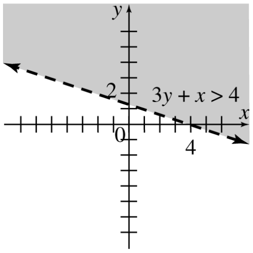
4. $y \geq x - 3$

The boundary is the line $y = x - 3$. Graph a solid line using the points $(0, -3)$ and $(3, 0)$. Use $(0, 0)$ as a test point to get $0 \geq 0 - 3$, which is true. Shade all points above the line, or the half-plane containing $(0, 0)$.



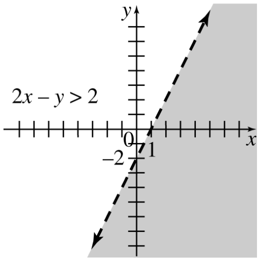
6. $3y + x > 4$

The boundary is the line $3y + x = 4$. Graph a dashed line using the points $(4, 0)$ and $\left(0, \frac{4}{3}\right)$. Use $(0, 0)$ as a test point to get $0 + 0 > 4$, which is false. Shade the half-plane that does not contain $(0, 0)$, or all points above the line.



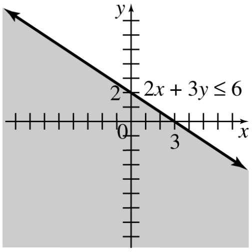
8. $2x - y > 2$

The boundary is the line $2x - y = 2$. Graph a dashed line using the points $(0, -2)$ and $(1, 0)$. Use $(0, 0)$ as a test point to get $0 - 0 > 2$, which is false. Shade the half-plane that does not contain $(0, 0)$, or all points below the line.



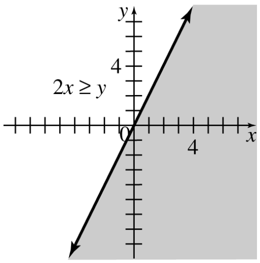
10. $2x + 3y \leq 6$

The boundary is the line $2x + 3y = 6$. Graph a solid line using $(0, 2)$ and $(3, 0)$. Use $(0, 0)$ as a test point to get $0 + 0 \leq 6$, which is true. Shade all points below the line, or the half-plane containing $(0, 0)$.



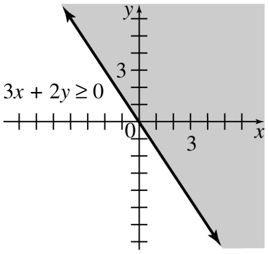
12. $2x \geq y$

The boundary is $2x = y$. Graph a solid line using the points $(0, 0)$ and $(2, 4)$. Since $(0, 0)$ is on the line, use $(3, 1)$ as a test point to get $2 \cdot 3 \geq 1$, which is true. Shade the half-plane containing $(3, 1)$, or all points below the line.



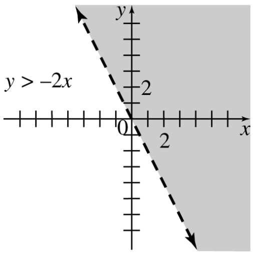
14. $3x + 2y \geq 0$

The boundary is $3x + 2y = 0$. Graph a solid line using $(0, 0)$ and $(2, -3)$. Use $(4, 1)$ as a test point to get $12 + 2 \geq 0$, which is true. Shade the half-plane containing $(4, 1)$, or all points above the line.



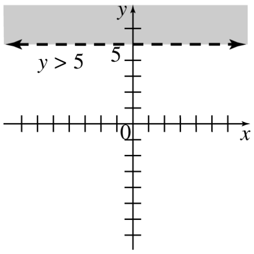
16. $y > -2x$

The boundary is $y = -2x$. Graph a dashed line using the points $(0, 0)$ and $(2, -4)$. Use $(4, 1)$ as a test point to get $1 > -8$, which is true. Shade the half-plane containing $(4, 1)$, or all points above the line.



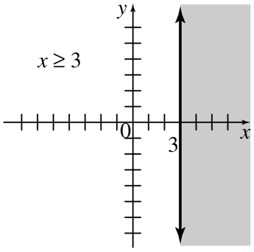
18. $y > 5$

The boundary is $y = 5$, a horizontal dashed line. $y > 5$ is the set of points in the half-plane above this line.



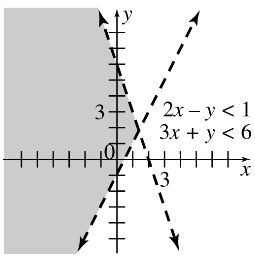
20. $x \geq 3$

The boundary is $x = 3$, a solid vertical line. The graph of $x \geq 3$ is the set of points which are on or to the right of this line.



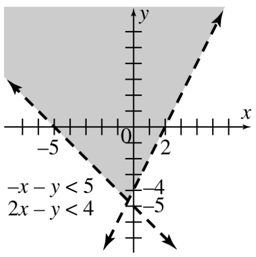
22. $2x - y < 1$
$3x + y < 6$

Graph $2x - y < 1$ as the half-plane above the dashed line $2x - y = 1$. Graph $3x + y < 6$ as the half-plane below the dashed line $3x + y = 6$. Shade the overlapping part of these two half-planes to show the feasible region for this system.



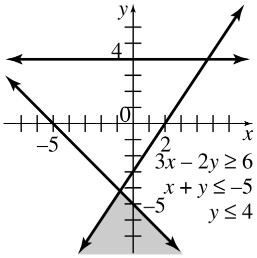
24. $-x - y < 5$
$2x - y < 4$

Graph $-x - y < 5$ as the half-plane above the dashed line $-x - y = 5$. Graph $2x - y < 4$ as the half-plane above the dashed line $2x - y = 4$. Shade the overlapping part of these two half-planes to show the feasible region.



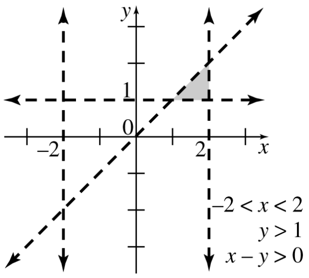
26. $3x - 2y \geq 6$
$x + y \leq -5$
$y \leq 4$

Graph the solid lines $3x - 2y = 6$, $x + y = -5$, and $y = 4$. Use $(0, 0)$ as a test point. $0 - 0 \geq 6$ is false, $0 + 0 \leq -5$ is false, and $0 \leq 4$ is true. Shade all points below each line. The feasible region is the overlap of the three half-planes.



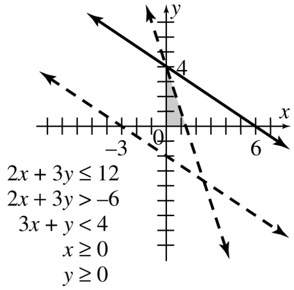
28. $-2 < x < 2$
$y > 1$
$x - y > 0$

The inequality $-2 < x < 2$ defines a vertical region from $x = -2$ to $x = 2$ with dashed boundary lines $x = 2$ and $x = -2$. The inequality $y > 1$ defines the half-plane above the dashed line $y = 1$. For $x - y > 0$, use $(4, 1)$ as a test point. $4 - 1 > 0$ is true. Shade all points below $x - y = 0$. The feasible region is a small triangle bordered above by $x - y = 0$, below by $y = 1$, and on the right by $x = 2$. $x = -2$ does not affect the feasible region.



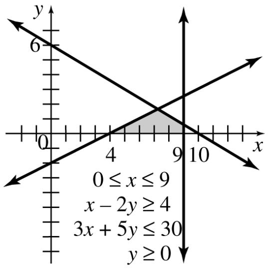
30. $2x + 3y \leq 12$
$2x + 3y > -6$
$3x + y < 4$
$x \geq 0$
$y \geq 0$

The inequalities $x \geq 0$ and $y \geq 0$ restrict the feasible region to the first quadrant. Graph $2x + 3y = 12$ as a solid line, $2x + 3y = -6$ as a dashed line, and $3x + y = 4$ as a dashed line. $2x + 3y = -6$ does not cross quadrant I and has no effect on the answer. For $2x + 3y = 12$ and $3x + y = 4$, use test point $(0, 0)$. $0 + 0 \leq 12$ is true, and $0 + 0 < 4$ is true. Shade all points below each line and in quadrant I. The feasible region is the overlap of these regions.



32. $$\begin{aligned} 0 \le x &\le 9 \\ x - 2y &\ge 4 \\ 3x + 5y &\le 30 \\ y &\ge 0 \end{aligned}$$

$0 \le x \le 9$ and $y \ge 0$ restrict the region to the first quadrant. Graph $x = 9$, $x - 2y = 4$, and $3x + 5y = 30$ as solid lines. Use test point $(0, 0)$. $0 - 0 \ge 4$ is false, and $0 + 0 \le 30$ is true. Shade all points from $x = 0$ to $x = 9$, below $x - 2y = 4$, and below $3x + 5y = 30$. The feasible region is the region in the first quadrant which is bounded above by $x - 2y = 4$ and $3x + 5y = 30$, below by $y = 0$, and on the right by $x = 9$.

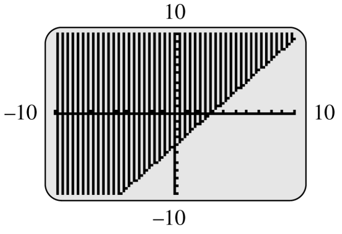


34. $4x - 3y < 12$

Use a graphing calculator. The boundary line is the graph of $4x - 3y = 12$. Solve this equation for y.

$$\begin{aligned} -3y &= -4x + 12 \\ y &= \frac{-4}{-3}x + \frac{12}{-3} \\ y &= \frac{4}{3}x - 4 \end{aligned}$$

Enter $y_1 = \frac{4}{3}x - 4$ and graph it. Using the origin as a test point, we obtain $0 < 12$ which is true. Shade the region that contains the origin.



36. $6x - 4y > 8$
$3x + 2y > 4$

Use a graphing calculator. One boundary line is the graph of $6x - 4y = 8$. Solve this equation for y.

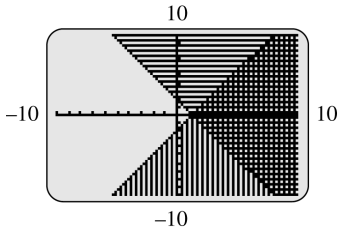
$$\begin{aligned} -4y &= -6x + 8 \\ y &= \frac{-6}{-4}x + \frac{8}{-4} \\ y &= \frac{3}{2}x - 2 \end{aligned}$$

Enter $y_1 = \frac{3}{2}x - 2$ and graph it. Using the origin as a test point, we obtain $0 > 8$ which is false. Shade the region that does not contain the origin.

The other boundary line is the graph of $3x + 2y = 4$. Solve this equation for y.

$$\begin{aligned} 2y &= -3x + 4 \\ y &= -\frac{3}{2}x + 2 \end{aligned}$$

Enter $y_2 = -\frac{3}{2}x + 2$ and graph it. Using the origin as a test point, we obtain $0 > 4$ which is false. Shade the region that does not contain the origin. The overlap of the two graphs is the feasible region.



38. (a)

	Glazed	Unglazed	Maximum
Number Made	x	y	
Time on Wheel	$\frac{1}{2}$	1	8
Time in Kiln	1	6	20

(b) On the wheel, x glazed planters require $\frac{1}{2} \cdot x = \frac{1}{2}x$ hr and y unglazed planters require $1 \cdot y = y$ hr. Since the wheel is available for at most 8 hr per day,

$$\frac{1}{2}x + y \le 8.$$

In the kiln, x glazed planters require $1 \cdot x = x$ hr and y unglazed planters require $6 \cdot y = 6y$ hr.

Since the kiln is available for at most 20 hr per day,

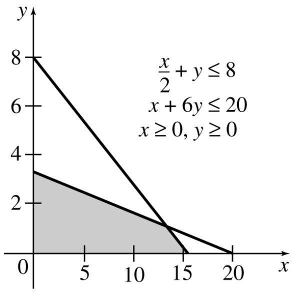
$$x + 6y \le 20.$$

Since it is not possible to produce a negative number of pots,

$$x \ge 0 \text{ and y} \ge 0.$$

Thus, we have the system

$$\begin{aligned}\frac{1}{2}x + y &\le 8\\ x + 6y &\le 20\\ x &\ge 0\\ y &\ge 0.\end{aligned}$$



(c) Yes, 5 glazed and 2 unglazed planters can be made, since the point $(5, 2)$ lies within the feasible region.

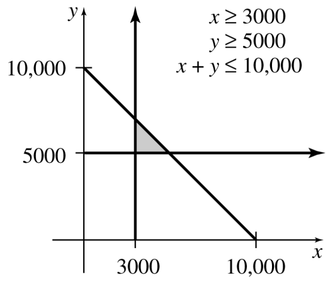
From the graph, it looks like the point $(10, 2)$ might lie right on a boundary of the feasible region. However, $(10, 2)$ does not satisfy the inequality $x + 6y \le 20$, so the point is definitely outside the feasible region. Therefore, 10 glazed and 2 unglazed planters cannot be made.

40. (a)

$$\begin{aligned}x &\ge 3000\\ y &\ge 5000\\ x + y &\le 10,000\end{aligned}$$

(b) The first inequality gives the half-plane to the right of the vertical line $x = 3000$, including the points on the line. The second inequality gives the half-plane above the horizontal line $y = 5000$, including the points on the line. The third inequality gives the half-plane below the line $x + y \le 10,000$, including the points on the line.

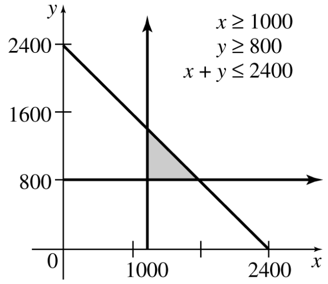
Shade the region where the three half-planes overlap to show the feasible region.



42. (a)

$$\begin{aligned}x &\ge 1000\\ y &\ge 800\\ x + y &\le 2400\end{aligned}$$

(b) The first inequality gives the set of points on and to the right of the vertical line $x = 1000$. The second inequality gives the set of points on and above the horizontal line $y = 800$. The third inequality gives the set of points on and below the line $x + y = 2400$. Shade the region where the three graphs overlap to show the feasible region.



44. (a)

	Number	Emissions	Cost
Type 1	x	.5	.16
Type 2	y	.3	.20
Maximum		1.8	.8

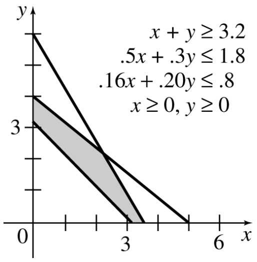
The manufacturer produces at least 3.2 million barrels annually, so

$$x + y \ge 3.2.$$

Operating costs are not to exceed \$.8 million, so $.16x + .20y \le .8$. Total emissions must not exceed 1.8 million lb, so $.5x + .3y \le 1.8$. We obtain the system

$$\begin{aligned}x + y &\ge 3.2\\ .16x + .20y &\le \ .8\\ .5x + .3y &\le 1.8\\ x &\ge 0\\ y &\ge 0.\end{aligned}$$

(b) Using the above system, graph solid lines and shade appropriate half-planes to get the feasible region.



3.2 Solving Linear Programming Problems Graphically

2. Make a table indicating the value of the objective function at each corner point.

Corner Point	Value of $z = 6x - y$
$(1, 5)$	$6(1) - 5 = 1$ Minimum
$(6, 8)$	$6(6) - 8 = 28$
$(9, 1)$	$6(9) - 1 = 53$ Maximum
$(1, 2)$	$6(1) - 2 = 4$

The maximum value of 53 occurs at $(9, 1)$. The minimum value of 1 occurs at $(1, 5)$.

4.

Corner Point	Value of $z = .35x + 1.25y$
$(0, 15)$	$.35(0) + 1.25(15) = 18.75$
$(6, 18)$	$.35(6) + 1.25(18) = 24.60$ Maximum
$(10, 9)$	$.35(10) + 1.25(9) = 14.75$
$(12, 0)$	$.35(12) + 1.25(0) = 4.20$
$(0, 0)$	$.35(0) + 1.25(0) = 0$ Minimum

The maximum is 24.6 at $(6, 18)$; the minimum is 0 at $(0, 0)$.

6. (a) $z = 4x + y$

Corner Point	Value of $z = 4x + y$
$(0, 10)$	$4(0) + 10 = 10$ Minimum
$(2, 4)$	$4(2) + 4 = 12$
$(5, 2)$	$4(5) + 2 = 22$
$(15, 0)$	$4(15) + 0 = 60$

The minimum value is 10 at $(0, 10)$. There is no maximum because the feasible region is unbounded.

(b) $z = 5x + 6y$

Corner Point	Value of $z = 5x + 6y$
$(0, 10)$	$5(0) + 6(10) = 60$
$(2, 4)$	$5(2) + 6(4) = 34$ Minimum
$(5, 2)$	$5(5) + 6(2) = 37$
$(15, 0)$	$5(15) + 6(0) = 75$

The minimum value is 34 at $(2, 4)$. There is no maximum because the feasible region is unbounded.

(c) $z = x + 2y$

Corner Point	Value of $z = x + 2y$
$(0, 10)$	$0 + 2(10) = 20$
$(2, 4)$	$2 + 2(4) = 10$
$(5, 2)$	$5 + 2(2) = 9$ Minimum
$(15, 0)$	$15 + 2(0) = 15$

The minimum value is 9 at $(5, 2)$. There is no maximum because the feasible region is unbounded.

(d) $z = x + 6y$

Corner Point	Value of $z = x + 6y$
$(0, 10)$	$0 + 6(10) = 60$
$(2, 4)$	$2 + 6(4) = 26$
$(5, 2)$	$5 + 6(2) = 17$
$(15, 0)$	$15 + 6(0) = 15$ Minimum

The minimum value is 15 at $(15, 0)$. There is no maximum because the feasible region is unbounded.

8. Minimize $z = x + 3y$

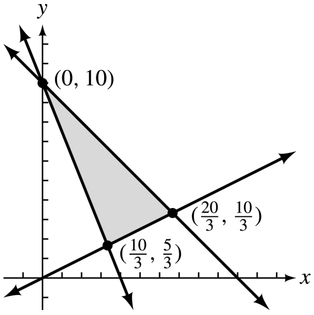
$$\text{subject to:}\quad \begin{aligned} x + y &\le 10 \\ 5x + 2y &\ge 20 \\ -x + 2y &\ge 0 \\ x &\ge 0 \\ y &\ge 0. \end{aligned}$$

Graph the feasible region, and identify the corner points. The corner point $\left(\frac{20}{3}, \frac{10}{3}\right)$ can be found by solving the system

$$\begin{aligned} x + y &= 10 \\ -x + 2y &= 0, \end{aligned}$$

and the corner point $\left(\frac{10}{3}, \frac{5}{3}\right)$ can be found by solving the system

$$\begin{aligned} 5x + 2y &= 20 \\ -x + 2y &= 0. \end{aligned}$$



Corner Point	Value of $z = x + 3y$
$(0, 10)$	$0 + 3(10) = 30$
$\left(\frac{20}{3}, \frac{10}{3}\right)$	$\frac{20}{3} + 3\left(\frac{10}{3}\right) = \frac{50}{3}$
$\left(\frac{10}{3}, \frac{5}{3}\right)$	$\frac{10}{3} + 3\left(\frac{5}{3}\right) = \frac{25}{3}$ Minimum

The minimum value is $\frac{25}{3}$ when $x = \frac{10}{3}$ and $y = \frac{5}{3}$.

10. Maximize $z = 10x + 8y$

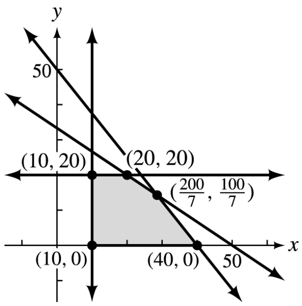
subject to:
$$\begin{aligned} 2x + 3y &\le 100 \\ 5x + 4y &\le 200 \\ x &\ge 10 \\ 0 \le y &\le 20. \end{aligned}$$

Graph the feasible region, and identify the corner points. The corner point $(20, 20)$ can be found by solving the system

$$\begin{aligned} 2x + 3y &= 100 \\ y &= 20, \end{aligned}$$

and the corner point $\left(\frac{200}{7}, \frac{100}{7}\right)$ can be found by solving the system

$$\begin{aligned} 2x + 3y &= 100 \\ 5x + 4y &= 200. \end{aligned}$$



Corner Point	Value of $z = 10x + 8y$
$(10, 0)$	$10(10) + 8(0) = 100$
$(10, 20)$	$10(10) + 8(20) = 260$
$(20, 20)$	$10(20) + 8(20) = 360$
$\left(\frac{200}{7}, \frac{100}{7}\right)$	$10\left(\frac{200}{7}\right) + 8\left(\frac{100}{7}\right) = 400$ Maximum
$(40, 0)$	$10(40) + 8(0) = 400$ Maximum

The maximum value is 400 when $x = \frac{200}{7}$ and $y = \frac{100}{7}$, as well as when $x = 40$ and $y = 0$ and at all points in between.

12. Maximize $z = 4x + 5y$

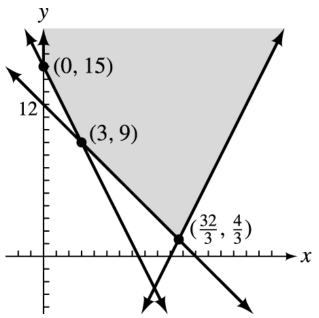
subject to:
$$\begin{aligned} 10x - 5y &\le 100 \\ 20x + 10y &\ge 150 \\ x + y &\ge 12 \\ x &\ge 0 \\ y &\ge 0. \end{aligned}$$

Graph the feasible region and identify the corner points. The corner point $\left(\frac{32}{3}, \frac{4}{3}\right)$ can be found by solving the system

$$\begin{aligned} 10x - 5y &= 100 \\ x + y &= 12, \end{aligned}$$

and the corner point $(3, 9)$ can be found by solving the system

$$\begin{aligned} 20x + 10y &= 150 \\ x + y &= 12. \end{aligned}$$



Since the region is unbounded, there is no maximum value, hence no solution.

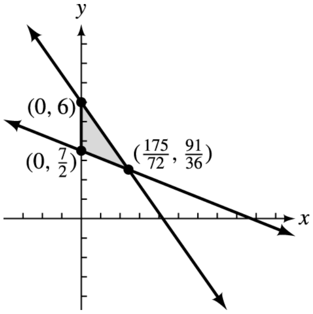
14. Minimize $z = 3x + 2y$ subject to the following constraints, with $x \geq 0$ and $y \geq 0$.

(a) $10x + 7y \leq 42$
$4x + 10y \geq 35$

Graph the feasible region and identify the corner points. The corner point $\left(\frac{175}{72}, \frac{91}{36}\right)$ can be found by solving the system

$$\begin{aligned} 10x + 7y &= 42 \\ 4x + 10y &= 35 \end{aligned}$$

by the elimination method.



Corner Point	Value of $z = 3x + 2y$
$\left(0, \frac{7}{2}\right)$	$3(0) + 2\left(\frac{7}{2}\right) = 7$ Minimum
$(0, 6)$	$3(0) + 2(6) = 12$
$\left(\frac{175}{72}, \frac{91}{36}\right)$	$3\left(\frac{175}{72}\right) + 2\left(\frac{91}{36}\right) = \frac{889}{72} \approx 12.35$

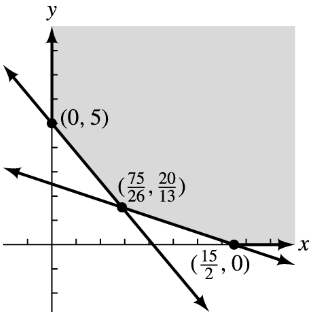
The minimum value is 7 when $x = 0$ and $y = \frac{7}{2}$.

(b) $6x + 5y \geq 25$
$2x + 6y \geq 15$

Graph the feasible region, and identify the corner points. The corner point $\left(\frac{75}{26}, \frac{20}{13}\right)$ can be found by solving the system

$$\begin{aligned} 6x + 5y &= 25 \\ 2x + 6y &= 15 \end{aligned}$$

by the elimination method.



Corner Point	Value of $z = 3x + 2y$
$(0, 5)$	$3(0) + 2(5) = 10$ Minimum
$\left(\frac{15}{2}, 0\right)$	$3\left(\frac{15}{2}\right) + 2(0) = \frac{45}{2} = 22.5$
$\left(\frac{75}{26}, \frac{20}{13}\right)$	$3\left(\frac{75}{26}\right) + 2\left(\frac{20}{13}\right) = \frac{305}{26} \approx 11.7$

The minimum value is 10 when $x = 0$ and $y = 5$.

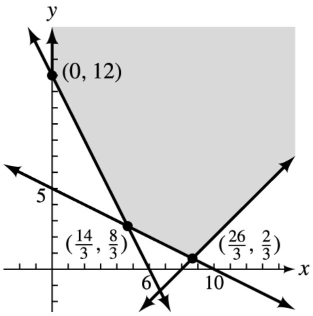
(c) $x + 2y \geq 10$
$2x + y \geq 12$
$x - y \leq 8$

Graph the region and identify the corner points. The corner point $\left(\frac{14}{3}, \frac{8}{3}\right)$ can be found by solving the system

$$\begin{aligned} x + 2y &= 10 \\ 2x + y &= 12, \end{aligned}$$

and the corner point $\left(\frac{26}{3}, \frac{2}{3}\right)$ can be found by solving the system

$$\begin{aligned} x + 2y &= 10 \\ x - y &= 8. \end{aligned}$$



Corner Point	Value of $z = 3x + 2y$
$\left(\frac{14}{3}, \frac{8}{3}\right)$	$3\left(\frac{14}{3}\right) + 2\left(\frac{8}{3}\right) = \frac{58}{3} = 19\frac{1}{3}$ Minimum
$\left(\frac{26}{3}, \frac{2}{3}\right)$	$3\left(\frac{26}{3}\right) + 2\left(\frac{2}{3}\right) = \frac{82}{3} = 27\frac{1}{3}$
$(0, 12)$	$3(0) + 2(12) = 24$

The minimum value is $\frac{58}{3}$ when $x = \frac{14}{3}$ and $y = \frac{8}{3}$.

3.3 Applications of Linear Programming

2. Let x represent the number of cows. Then $\frac{1}{3}x$ is the number of acres used by x cows. Let y represent the number of sheep. Then $\frac{1}{4}y$ is the number of acres used by y sheep. There are at least 120 acres of pasture, so the inequality is

$$\frac{1}{3}x + \frac{1}{4}y \geq 120.$$

4. If x represents the number of small computers sold, then $3x$ is the number of hours spent. If y represents the number of large computers sold, then $5y$ is the number of hours spent. Since Pauline works no more than 45 hr/wk, the inequality is

$$3x + 5y \leq 45.$$

6. Let x represent the number of gallons of light oil and y represent the number of gallons of heavy oil. The total must not exceed 120 gal, so the inequality is

$$x + y \leq 120.$$

(Note that the price per gallon is not used in setting up this inequality.)

8. Let $x =$ the number of refrigerators to ship to warehouse A
and $y =$ the number of refrigerators to ship to warehouse B.

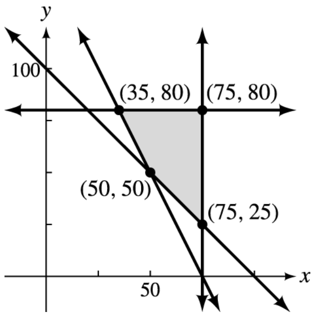
Minimize $z = 12x + 10y$

subject to:

$$\begin{aligned} x + y &\geq 100 \\ x &\leq 75 \\ y &\leq 80 \\ 4x + 2y &\geq 300 \\ x &\geq 0 \\ y &\geq 0. \end{aligned}$$

Graph the feasible region in quadrant I, and identify the corner points.

$$\begin{aligned} x + \ \ y &= 100 \\ 4x + 2y &= 300. \end{aligned}$$



The corner points are: $(35, 80)$, which is the intersection of $y = 80$ and $4x + 2y = 300$; $(75, 80)$; $(75, 25)$, which is the intersection of $x = 75$ and $x + y = 100$; and $(50, 50)$, which is the interection of $x + y = 100$ and $4x + 2y = 300$.

Corner Point	Value of $z = 12x + 10y$
$(35, 80)$	1220
$(75, 80)$	1700
$(75, 25)$	1150
$(50, 50)$	1100 Minimum

Ship 50 refrigerators to warehouse A and 50 to warehouse B for a minimum cost of \$1100.

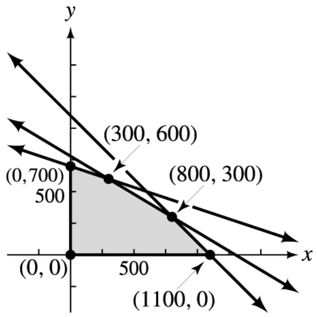
10. (a) Let $x =$ the number of bargain sets
and $y =$ the number of deluxe sets.

Maximize $z = 100x + 150y$

subject to:

$$\begin{aligned} 3x + 5y &\leq 3900 \\ x + 3y &\leq 2100 \\ 2x + 2y &\leq 2200 \\ x &\geq 0 \\ y &\geq 0. \end{aligned}$$

Graph the feasible region in quadrant I.



The corner points $(0, 0)$, $(1100, 0)$, and $(0, 700)$ can be identified from the graph. The coordinates of the corner point $(800, 300)$ can be found by solving the system

$$\begin{aligned} 3x + 5y &= 3900 \\ 2x + 2y &= 2200. \end{aligned}$$

The coordinates of the final corner point, (300, 600), can be found by solving the system

$$x + 3y = 2100$$
$$3x + 5y = 3900.$$

Corner Point	Value of $z = 100x + 150y$	
(300, 600)	120,000	
(800, 300)	125,000	Maximum
(1100, 0)	110,000	
(0, 700)	105,000	
(0, 0)	0	

A maximum profit of \$125,000 is obtained by producing 800 bargain sets and 300 deluxe sets.

(b) The objective function changes to

$$z = 100x + 170y.$$

The corner points stay the same.

Corner Point	Value of $z = 100x + 170y$	
(300, 600)	132,000	Maximum
(800, 300)	131,000	
(1100, 0)	110,000	
(0, 700)	119,000	
(0, 0)	0	

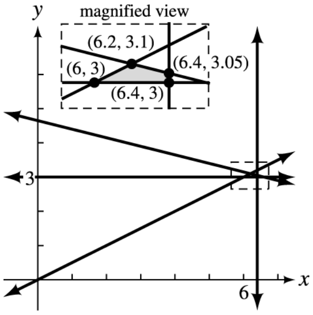
A maximum profit of \$132,000 is obtained by producing 300 bargain sets and 600 deluxe sets.

12. Let $x =$ the number of gallons of gasoline in millions to produce each day
and $y =$ the number of gallons of fuel oil in millions to produce each day.

Maximize $z = 1.25x + 1.00y$

subject to:
$$x \geq 2y$$
$$y \geq 3$$
$$x \leq 6.4$$
$$.25x + 1y \leq 4.65$$
$$x \geq 0$$
$$y \geq 0.$$

Graph the feasible region in quadrant I, and find the corner points.



Corner Point	Value of $z = 1.25x + 1.00y$	
(6, 3)	10.5	
(6.2, 3.1)	10.85	
(6.4, 3.05)	11.05	Maximum
(6.4, 3)	11	

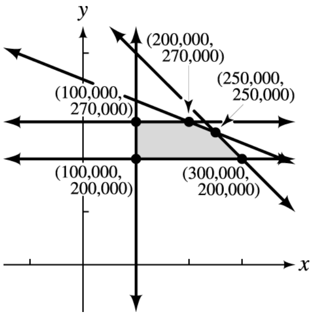
Produce 6.4 million gal of gasoline and 3.05 million gal of fuel oil for a maximum revenue of \$11.05 million (or \$11,050,000).

14. Let $x =$ the number of hectares of coffee
and $y =$ the number of hectares of cocoa.

Maximize $z = 220x + 550y$

subject to:
$$x + y \leq 500,000$$
$$x \geq 100,000$$
$$200,000 \leq y \leq 270,000$$
$$2x + 5y \leq 1,750,000$$
$$x \geq 0$$
$$y \geq 0.$$

Graph the feasible region in quadrant I, and identify the corner points.



Corner Point	Value of $z = 220x + 550y$	
(100,000, 200,000)	132,000,000	
(100,000, 270,000)	170,500,000	
(200,000, 270,000)	192,500,000	Maximum
(250,000, 250,000)	192,500,000	Maximum
(300,000, 200,000)	176,000,000	

A maximum profit of \$192,500,000 is obtained by growing 250,000 hectares of each crop or by growing 200,000 hectares of coffee and 270,000 hectares of cocoa (or any point on the line between these two points).

16. Let $x =$ the number of boxes shipped from warehouse I to San Jose
and $y =$ the number of boxes shipped from warehouse I to Memphis.

Then, $350 - x$ is the number of boxes shipped from warehouse II to San Jose. $300 - y$ is the number of boxes shipped from warehouse II to Memphis. The constraints are represented by the following inequalities:

$$\begin{aligned} x + y &\le 370 \\ (350 - x) + (300 - y) \le 290 \text{ or } x + y &\ge 360 \\ x &\le 350 \\ y &\le 300 \\ x &\ge 0 \\ y &\ge 0. \end{aligned}$$

Minimize the shipping cost

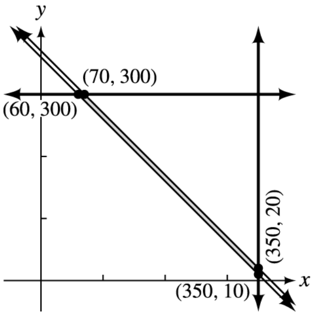
$$z = .25x + .22y + .23(350 - x) + .21(300 - y)$$

or

$$z = .02x + .01y + 143.5$$

subject to the above constraints.

Graph the feasible region in quadrant I, and identify the corner points.



Corner Point	Value of $z = .02x + .01y + 143.5$
$(60, 300)$	147.7 Minimum
$(70, 300)$	147.9
$(350, 20)$	150.7
$(350, 10)$	150.6

Thus, a minimum cost of \$147.70 is obtained when 60 boxes are shipped from warehouse I to San Jose, 300 boxes are shipped from warehouse I to Memphis, and 290 boxes are shipped from warehouse II to San Jose. Notice that no boxes are sent from warehouse II to Memphis.

18. 1 Zeta + 2 Beta must not exceed 1000; thus (b) is the correct choice.

20. \$4.00 Zeta + \$5.25 Beta equals the total contribution margin; (c) is the correct choice.

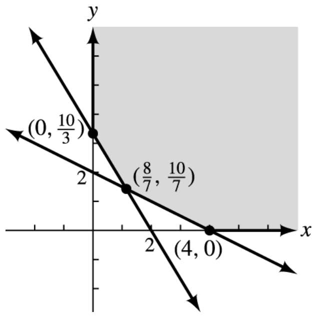
22. Let $x =$ the number of species I prey
and $y =$ the number of species II prey.

		Protein	Fat
Species	I	5	2
	II	3	4

Minimize $z = 2x + 3y$

$$\begin{aligned} \text{subject to: } 5x + 3y &\ge 10 \\ 2x + 4y &\ge 8 \\ x &\ge 0 \\ y &\ge 0. \end{aligned}$$

Graph the feasible region in quadrant I, and identify the corner points $\left(0, \frac{10}{3}\right)$, $(4, 0)$, and the intersection of $5x + 3y = 10$ and $2x + 4y = 8$, which is $\left(\frac{8}{7}, \frac{10}{7}\right)$.



Corner Point	Value of $z = 2x + 3y$
$\left(0, \frac{10}{3}\right)$	10
$(4, 0)$	8
$\left(\frac{8}{7}, \frac{10}{7}\right)$	$\frac{46}{7}$ Minimum

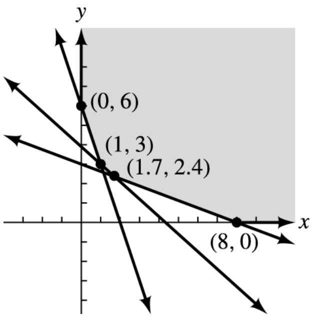
The minimum value is $\frac{46}{7} \approx 6.57$ units of energy. $\frac{8}{7}$ units of species I and $\frac{10}{7}$ units of species II will meet the daily food requirements with the least expenditure of energy. However, a predator probably can catch and digest only whole numbers of prey. This problem shows that it is important to consider whether a model produces a realistic answer to a problem.

24. Let $x =$ the number of Brand X pills and $y =$ the number of Brand Y pills.

Minimize $z = .05x + .04y$

$$\begin{aligned} \text{subject to: } 3000x + 1000y &\geq 6000 \\ 45x + 50y &\geq 195 \\ 75x + 200y &\geq 600 \\ x &\geq 0 \\ y &\geq 0. \end{aligned}$$

Graph the feasible region in quadrant I.



The corner points $(0, 6)$ and $(8, 0)$ can be identified from the graph. The coordinates of the corner point $(1, 3)$ can be found by solving the system

$$\begin{aligned} 45x + \quad 50y &= \quad 195 \\ 3000x + 1000y &= 6000. \end{aligned}$$

The approximate coordinates of the corner point $(1.7, 2.4)$ can be found by solving the system

$$\begin{aligned} 45x + \quad 50y &= 195 \\ 75x + 200y &= 600. \end{aligned}$$

Corner Point	Value of $z = .05x + .04y$
$(1, 3)$	.17 Minimum
$(1.7, 2.4)$	.18
$(8, 0)$	.40
$(0, 6)$	.24

A minimum daily cost of 17¢ is incurred by taking 1 Brand X pill and 3 Brand Y pills.

26. Let $x =$ the number of square feet of window space and $y =$ the number of square feet of wall space.

Maximize $z = x + y$

$$\begin{aligned} \text{subject to: } \quad x &\geq \frac{1}{6}y \\ 10x + 20y &\leq 12{,}000 \\ .32x + .20y &\leq 160 \\ x &\geq 0 \\ y &\geq 0. \end{aligned}$$

Graph the feasible region on a graphing calculator and identify the corner points.

Corner Point	Value of $z = x + y$
$(0, 0)$	0
$(92.31, 553.85)$	646.16
$(181.82, 509.09)$	690.91 Maximum
$(500, 0)$	500

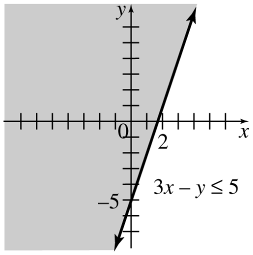
The maximum total area is 690.91 sq ft occurring when 181.82 sq ft is used for windows and 509.09 sq ft is used for walls.

Chapter 3 Review Exercises

2. There is no limit to the number of constraints in the graphical method. We are, however, limited to two variables.

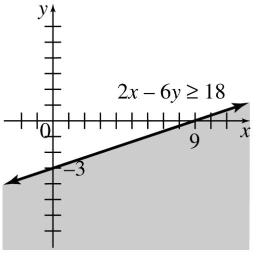
4. $3x - y \leq 5$

Graph $3x - y = 5$ as a solid line using the points $\left(\frac{5}{3}, 0\right)$ and $(0, -5)$. The test point $(0, 0)$ gives $0 \leq 5$, which is true. Shade the half-plane containing $(0, 0)$, that is, all points above the line.



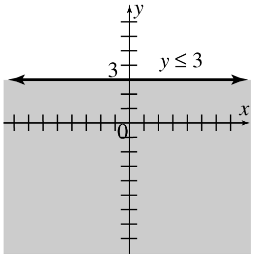
6. $2x - 6y \geq 18$

Graph $2x - 6y = 18$ as a solid line using the points $(9, 0)$ and $(0, -3)$. The test point $(0, 0)$ gives $0 \geq 18$, which is false. Shade the half-plane that does not contain $(0, 0)$, that is, all points below the line.



8. $y \le 3$

Graph the horizontal solid line $y = 3$, and shade the half-plane below the line.



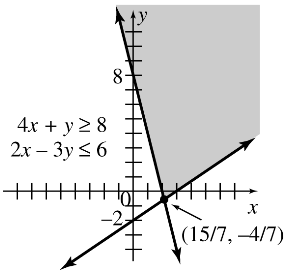
10. $$\begin{aligned} 4x + y &\ge 8 \\ 2x - 3y &\le 6 \end{aligned}$$

Graph $4x + y = 8$ as a solid line using $(2, 0)$ and $(0, 8)$, and graph $2x - 3y = 6$ as a solid line using $(3, 0)$ and $(0, -2)$. The test point $(0, 0)$ gives $0 + 0 \ge 8$, which is false and $0 - 0 \le 6$, which is true. Shade the half-planes above $4x + y = 8$ and above $2x - 3y = 6$. The final shaded region is the overlapping part of these two half-planes, which is the region above both lines.

The only corner point is the intersection of the two boundary lines. To find the coordinates of this point, solve the system

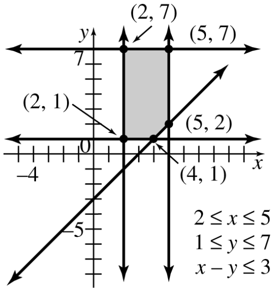
$$\begin{aligned} 4x + y &= 8 \\ 2x - 3y &= 6. \end{aligned}$$

The corner point is $\left(\frac{15}{7}, -\frac{4}{7}\right)$.



12. $$\begin{aligned} 2 \le x &\le 5 \\ 1 \le y &\le 7 \\ x - y &\le 3 \end{aligned}$$

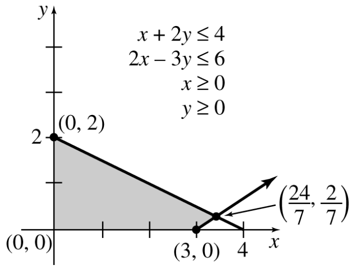
Graph $x = 2$, $x = 5$, $y = 1$, $y = 7$, and $x - y = 3$ as solid lines. $(0, 0)$ can be used as a test point for each inequality to help graph the feasible region. The corner points are $(2, 1)$, $(2, 7)$, $(5, 7)$, $(5, 2)$, and $(4, 1)$.



14. $$\begin{aligned} x + 2y &\le 4 \\ 2x - 3y &\le 6 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$

Graph $x + 2y = 4$ and $2x - 3y = 6$ as solid lines. The inequalities $x \ge 0$ and $y \ge 0$ restrict the region to quadrant I. Use $(0, 0)$ as a test point to get $0 + 0 \le 4$ and $0 - 0 \le 6$, which are true. The feasible region contains all points on or below $x + 2y = 4$ and on or above $2x - 3y = 6$ in quadrant I.

The corner points $(0, 0)$, $(0, 2)$, and $(3, 0)$ can be identified from the graph. The fourth corner point, $\left(\frac{24}{7}, \frac{2}{7}\right)$, is the intersection of the lines $x + 2y = 4$ and $2x - 3y = 6$.



16. Evaluate the objective function $z = 2x + 4y$ at all the corner points.

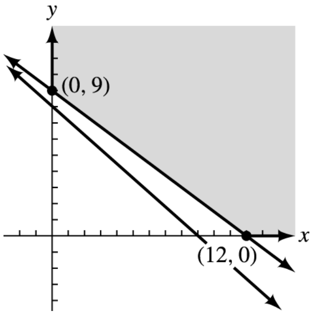
Corner Point	Value of $z = 2x + 4y$	
$(0, 8)$	$2(0) + 4(8) = 32$	
$(8, 8)$	$2(8) + 4(8) = 48$	Maximum
$(5, 2)$	$2(5) + 4(2) = 18$	
$(2, 0)$	$2(2) + 4(0) = 4$	Minimum

The maximum is 48 at $(8, 8)$; the minimum is 4 at $(2, 0)$.

18. Minimize $z = 3x + 2y$

$$\begin{aligned} \text{subject to: } 8x + 9y &\geq 72 \\ 6x + 8y &\geq 72 \\ x &\geq 0 \\ y &\geq 0. \end{aligned}$$

Graph the feasible region, which contains all points in quadrant I on or above $6x + 8y = 72$, and identify the corner points.



Evaluate the objective function at each corner point.

Corner Point	Value of $z = 3x + 2y$	
$(0, 9)$	$3(0) + 2(9) = 18$	Minimum
$(12, 0)$	$3(12) + 2(0) = 36$	

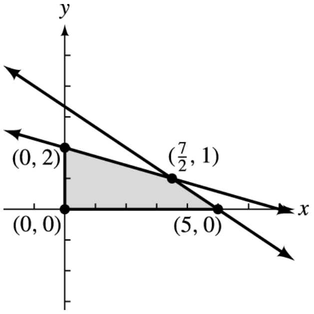
The minimum value is 18 at $(0, 9)$.

20. Maximize $z = 4x + 3y$

$$\begin{aligned} \text{subject to: } 2x + 7y &\leq 14 \\ 2x + 3y &\leq 10 \\ x &\geq 0 \\ y &\geq 0. \end{aligned}$$

Graph the feasible region, which is the overlap of all points on or below $2x + 3y = 10$ and $2x + 7y = 14$, and identify the corner points. The corner point $\left(\frac{7}{2}, 1\right)$ can be found by solving the system

$$\begin{aligned} 2x + 7y &= 14 \\ 2x + 3y &= 10. \end{aligned}$$



Evaluate the objective function at each corner point.

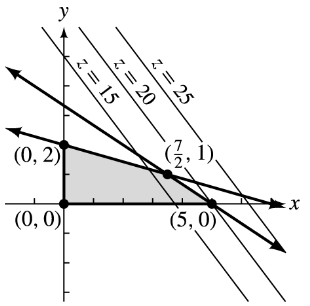
Corner Point	Value of $z = 4x + 3y$	
$(0, 2)$	$4(0) + 3(2) = 6$	
$(0, 0)$	$4(0) + 3(0) = 0$	
$(5, 0)$	$4(5) + 3(0) = 20$	Maximum
$\left(\frac{7}{2}, 1\right)$	$4\left(\frac{7}{2}\right) + 3(1) = 17$	

The maximum value is 20 at $(5, 0)$.

24. Maximize $z = 4x + 3y$

$$\begin{aligned} \text{subject to: } 2x + 7y &\leq 14 \\ 2x + 3y &\leq 10 \\ x &\geq 0 \\ y &\geq 0. \end{aligned}$$

Using the method of Exercise 23, $z = 20$ or $4x + 3y = 20$ is the line that is as far from the origin as possible but still touches the feasible region. It touches the feasible region at $(5, 0)$. Note that a value of z greater than 20, such as $z = 25$, will produce a line that does not intersect the feasible region, while a value of z less than 20, such as $z = 15$, will produce a line that intersects the feasible region in more than one point.

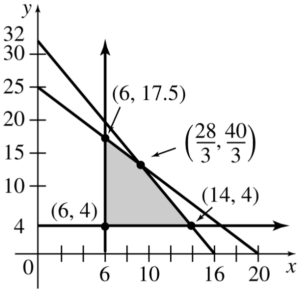


The maximum value of z is 20 when $x = 5$ and $y = 0$.

26. Let $x =$ the number of units of special pizza and $y =$ the number of units of basic pizza.

The system of inequalities is

$$\begin{aligned} x &\geq 6 \\ y &\geq 4 \\ 5x + 4y &\leq 100 \\ 2x + y &\leq 32 \\ x &\geq 0 \\ y &\geq 0. \end{aligned}$$



28. Maximize $z = 20x + 15y$.

Identify the corner points from the graph of the feasible region in Exercise 26.

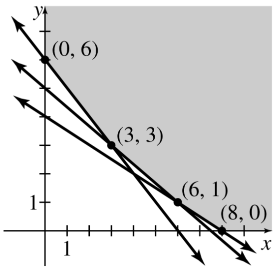
Corner Point	Value of $z = 20x + 15y$	
$(6, 4)$	180	
$(6, 17.5)$	382.5	
$\left(\frac{28}{3}, \frac{40}{3}\right)$	$386.\overline{6}$	Maximum
$(14, 4)$	340	

Produce $\frac{28}{3}$ units of special pizza and $\frac{40}{3}$ units of basic pizza for a maximum revenue of \$386.67.

30. Let $x =$ the number of Atlantic boathouses. and $y =$ the number of Pacific boathouses.

Minimize $z = 30{,}000x + 40{,}000y$

subject to:
$$\begin{aligned} 1000x + 2000y &\geq 8000 \\ 3000x + 3000y &\geq 18{,}000 \\ 2000x + 3000y &\geq 15{,}000 \\ x &\geq 0 \\ y &\geq 0. \end{aligned}$$



The corner points $(0, 6)$ and $(8, 0)$ can be identified from the graphs. The coordinates of the corner point $(3, 3)$ can be found by solving the system

$$\begin{aligned} 3000x + 3000y &= 18{,}000 \\ 2000x + 3000y &= 15{,}000. \end{aligned}$$

The coordinates of the point $(6, 1)$ can be found by solving the system

$$\begin{aligned} 1000x + 2000y &= 8000 \\ 2000x + 3000y &= 15{,}000. \end{aligned}$$

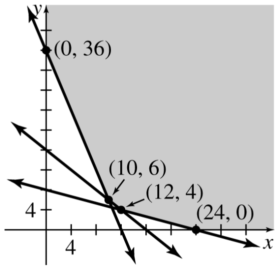
Corner Point	Value of $z = 30{,}000x + 40{,}000y$	
$(0, 6)$	240,000	
$(3, 3)$	210,000	Minimum
$(6, 1)$	220,000	
$(8, 0)$	240,000	

A minimum cost of \$210,000 is incurred by building 3 of each model.

32. Let $x =$ ounces of Health Trough and $y =$ ounces of Power Gunk.

Minimize $z = 8x + 4y$

subject to:
$$\begin{aligned} 30x + 10y &\geq 360 \\ 10x + 10y &\geq 160 \\ 10x + 30y &\geq 240 \\ x &\geq 0 \\ y &\geq 0. \end{aligned}$$



The corner points $(0, 36)$ and $(24, 0)$ can be identified from the graph. The coordinates of the corner point $(10, 6)$ can be found by solving the following system.

$$\begin{aligned} 30x + 10y &= 360 \\ 10x + 10y &= 160. \end{aligned}$$

The coordinates of the corner point $(12, 4)$ can be found by solving the system

$$\begin{aligned} 10x + 10y &= 160 \\ 10x + 30y &= 240. \end{aligned}$$

Corner Point	Value of $z = 8x + 4y$	
$(0, 36)$	144	
$(10, 6)$	104	Minimum
$(12, 4)$	112	
$(24, 0)$	192	

Ten ounces of Health Trough and six ounces of Power Gunk should be used for a minimum cholesterol intake of 104 mg.

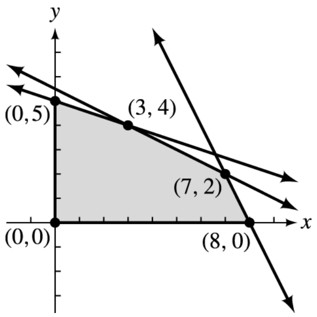
34. Let $x =$ the number of hours Ron should spend with his math tutor
and $y =$ the number of hours he should spend with his accounting tutor.

The number of points he expects to gain on the two tests combined is $z = 3x + 5y$. The given information translates into the following problem.

Maximize $z = 3x + 5y$

subject to:
$$\begin{aligned} 20x + 40y &\le 220 && \text{(finances)} \\ x + \frac{1}{2}y &\le 8 && \text{(aspirin)} \\ x + 3y &\le 15 && \text{(sleep)} \\ x &\ge 0 \\ y &\ge 0. \end{aligned}$$

Sketch the feasible region.



The corner points of the feasible region are $(0, 0)$, $(0, 5)$, $(3, 4)$, $(7, 2)$, and $(8, 0)$. Evaluate the objective function at each corner point.

Corner Point	Value of $z = 3x + 5y$	
$(0, 0)$	0	
$(0, 5)$	25	
$(3, 4)$	29	
$(7, 2)$	31	Maximum
$(8, 0)$	24	

Therefore, Ron should spend 7 hours with the math tutor and 2 hours with the accounting tutor in order to gain a maximum of 31 points.

Chapter 4

LINEAR PROGRAMMING: THE SIMPLEX METHOD

4.1 Slack Variables and the Pivot

2. $3x_1 + 5x_2 \le 100$

Add s_1 to the given inequality to obtain

$$3x_1 + 5x_2 + s_1 = 100.$$

4. $8x_1 + 6x_2 + 5x_3 \le 250$

Add s_1 to the given inequality to obtain

$$8x_1 + 6x_2 + 5x_3 + s_1 = 250.$$

6.

$$\begin{array}{ll}
\text{Maximize} & z = 1.2x_1 + 3.5x_2 \\
\text{subject to:} & 2.4x_1 + 1.5x_2 \le 10 \\
 & 1.7x_1 + 1.9x_2 \le 15 \\
\text{with} & x_1 \ge 0, x_2 \ge 0.
\end{array}$$

(a) We need one slack variable for each inequality. Thus, 2 are needed.

(b) We will use s_1 and s_2 for the slack variables.

(c) $2.4x_1 + 1.5x_2 \le 10$ becomes

$$2.4x_1 + 1.5x_2 + s_1 = 10.$$

$1.7x_1 + 1.9x_2 \le 15$ becomes

$$1.7x_1 + 1.9x_2 + s_2 = 15.$$

8.

$$\begin{array}{ll}
\text{Maximize} & z = 12x_1 + 15x_2 + 10x_3 \\
\text{subject to:} & 2x_1 + 2x_2 + x_3 \le 8 \\
 & x_1 + 4x_2 + 3x_3 \le 12 \\
\text{with} & x_1 \ge 0, x_2 \ge 0, x_3 \ge 0.
\end{array}$$

(a) There are two inequalities, so 2 slack variables are needed.

(b) Use s_1 and s_2 for the slack variables.

(c)

$$\begin{array}{rl}
2x_1 + 2x_2 + x_3 + s_1 \phantom{{}+s_2} &= 8 \\
x_1 + 4x_2 + 3x_3 \phantom{{}+s_1} + s_2 &= 12
\end{array}$$

10.

$$\begin{array}{cccccc|c}
x_1 & x_2 & x_3 & s_1 & s_2 & z & \\
0 & 2 & 1 & 1 & 3 & 0 & 5 \\
1 & 5 & 0 & 1 & 2 & 0 & 8 \\
\hline
0 & -2 & 0 & 1 & 1 & 1 & 10
\end{array}$$

x_1 and x_3 are the basic variables. The solution is $x_1 = 8, x_2 = 0, x_3 = 5, s_1 = 0, s_2 = 0$, and $z = 10$.

12.

$$\begin{array}{ccccccc|c}
x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\
0 & 2 & 0 & 5 & 2 & 2 & 0 & 15 \\
0 & 3 & 1 & 0 & 1 & 2 & 0 & 2 \\
7 & 4 & 0 & 0 & 3 & 5 & 0 & 35 \\
\hline
0 & -4 & 0 & 0 & 4 & 3 & 2 & 40
\end{array}$$

x_1, x_3, and s_1 are the basic variables. The solution is $x_1 = 5, x_2 = 0, x_3 = 2, s_1 = 3, s_2 = 0, s_3 = 0$, and $z = 20$.

14.

$$\begin{array}{cccccc|c}
x_1 & x_2 & x_3 & s_1 & s_2 & z & \\
5 & 4 & 1 & 1 & 0 & 0 & 50 \\
3 & 3 & \boxed{2} & 0 & 1 & 0 & 40 \\
\hline
-1 & -2 & -4 & 0 & 0 & 1 & 0
\end{array}$$

Clear the x_3 column.

$$\begin{array}{r|cccccc|c}
 & x_1 & x_2 & x_3 & s_1 & s_2 & z & \\
-R_2+2R_1 \to R_1 & 7 & 5 & 0 & 2 & -1 & 0 & 60 \\
 & 3 & 3 & \boxed{2} & 0 & 1 & 0 & 40 \\
\hline
2R_2+\ R_3 \to R_3 & 5 & 4 & 0 & 0 & 2 & 1 & 80
\end{array}$$

x_3 and s_1 are now basic. Thus, the solution is $x_1 = 0, x_2 = 0, x_3 = 20, s_1 = 30, s_2 = 0$, and $z = 80$.

16.

$$\begin{array}{ccccccc|c}
x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\
4 & 2 & 3 & 1 & 0 & 0 & 0 & 22 \\
2 & 2 & \boxed{5} & 0 & 1 & 0 & 0 & 28 \\
1 & 3 & 2 & 0 & 0 & 1 & 0 & 45 \\
\hline
-3 & -2 & -4 & 0 & 0 & 0 & 1 & 0
\end{array}$$

Clear the x_3 column.

$$\begin{array}{r|ccccccc|c}
 & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\
-3R_2+5R_1 \to R_1 & 14 & 4 & 0 & 5 & -3 & 0 & 0 & 26 \\
 & 2 & 2 & \boxed{5} & 0 & 1 & 0 & 0 & 28 \\
-2R_2+5R_3 \to R_3 & 1 & 11 & 0 & 0 & -2 & 5 & 0 & 169 \\
\hline
4R_2+5R_4 \to R_4 & -7 & -2 & 0 & 0 & 4 & 0 & 5 & 112
\end{array}$$

x_3, s_1, and s_3 are now basic. Thus, the solution is $x_1 = 0, x_2 = 0, x_3 = \frac{28}{5}, s_1 = \frac{26}{5}, s_2 = 0, s_3 = \frac{169}{5}$, and $z = \frac{112}{5}$.

18.
$$\begin{array}{c} \begin{array}{cccccccc|c} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & z & \end{array} \\ \left[\begin{array}{cccccccc|c} 1 & 2 & 3 & 1 & 1 & 0 & 0 & 0 & 115 \\ 2 & 1 & 8 & 5 & 0 & 1 & 0 & 0 & 200 \\ \boxed{1} & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 50 \\ \hline -2 & -1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Clear the x_1 column.

$$\begin{array}{r} \\ -R_3 + R_1 \to R_1 \\ -2R_3 + R_2 \to R_2 \\ \\ 2R_3 + R_4 \to R_4 \end{array} \begin{array}{c} \begin{array}{cccccccc|c} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & z & \end{array} \\ \left[\begin{array}{cccccccc|c} 0 & 2 & 2 & 1 & 1 & 0 & -1 & 0 & 65 \\ 0 & 1 & 6 & 5 & 0 & 1 & -2 & 0 & 100 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 50 \\ \hline 0 & -1 & 1 & -1 & 0 & 0 & 2 & 1 & 100 \end{array}\right] \end{array}$$

x_1, s_1, and s_2 are now basic. Thus, the solution is $x_1 = 50, x_2 = 0, x_3 = 0, x_4 = 0, s_1 = 65, s_2 = 100, s_3 = 0$, and $z = 100$.

20. Find $x_1 \geq 0$ and $x_2 \geq 0$ such that

$$\begin{aligned} 2x_1 + 3x_2 &\leq 100 \\ 5x_1 + 4x_2 &\leq 200 \end{aligned}$$

and $z = x_1 + 3x_2$ is maximized. We need two slack variables. Add s_1 and s_2 to get the system

$$\begin{aligned} 2x_1 + 3x_2 + s_1 \quad\quad &= 100 \\ 5x_1 + 4x_2 \quad + s_2 \quad &= 200 \\ -x_1 - 3x_2 \quad\quad + z &= \quad 0. \end{aligned}$$

The initial simplex tableau is

$$\begin{array}{c} \begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc|c} 2 & 3 & 1 & 0 & 0 & 100 \\ 5 & 4 & 0 & 1 & 0 & 200 \\ \hline -1 & -3 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}.$$

22. Find $x_1 \geq 0$ and $x_2 \geq 0$ such that

$$\begin{aligned} x_1 + \quad x_2 &\leq 10 \\ 5x_1 + 3x_2 &\leq 75 \end{aligned}$$

and $z = 4x_1 + 2x_2$ is maximized.
Add s_1 and s_2 to get the system

$$\begin{aligned} x_1 + \quad x_2 + s_1 \quad\quad &= 10 \\ 5x_1 + 3x_2 \quad + s_2 \quad &= 75 \\ -4x_1 - 2x_2 \quad\quad + z &= \quad 0. \end{aligned}$$

The initial simplex tableau is

$$\begin{array}{c} \begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 10 \\ 5 & 3 & 0 & 1 & 0 & 75 \\ \hline -4 & -2 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}.$$

24. Find $x_1 \geq 0$ and $x_2 \geq 0$ such that

$$\begin{aligned} 10x_1 + \quad 4x_2 &\leq 100 \\ 20x_1 + 10x_2 &\leq 150 \end{aligned}$$

and $z = 4x_1 + 5x_2$ is maximized.
Add s_1 and s_2 to get

$$\begin{aligned} 10x_1 + \quad 4x_2 + s_1 \quad\quad &= 100 \\ 20x_1 + 10x_2 \quad + s_2 \quad &= 150 \\ -4x_1 - \quad 5x_2 \quad\quad + z &= \quad 0. \end{aligned}$$

The initial simplex tableau is

$$\begin{array}{c} \begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc|c} 10 & 4 & 1 & 0 & 0 & 100 \\ 20 & 10 & 0 & 1 & 0 & 150 \\ \hline -4 & -5 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}.$$

26. Let x_1 represent the number of racing bicycles, x_2 the number of touring bicycles, and x_3 the number of mountain bicycles. Organize the information in a table.

	Racing	Touring	Mountain	Amount Available
Steel	17	27	34	91,800
Aluminum	12	21	15	42,000
Profit	\$8	\$12	\$22	

Using this information, the problem may be stated as:

Find $x_1 \geq 0, x_2 \geq 0$, and $x_3 \geq 0$ such that

$$\begin{aligned} 17x_1 + 27x_2 + 34x_3 &\leq 91{,}800 \\ 12x_1 + 21x_2 + 15x_3 &\leq 42{,}000 \end{aligned}$$

and $z = 8x_1 + 12x_2 + 22x_3$ is maximized.

Introduce slack variables s_1 and s_2, and the problem can be restated as:

Find $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, s_1 \geq 0$, and $s_2 \geq 0$ such that

$$\begin{aligned} 17x_1 + 27x_2 + 34x_3 + s_1 \quad &= 91{,}800 \\ 12x_1 + 21x_2 + 15x_3 \quad + s_2 &= 42{,}000 \end{aligned}$$

and $z = 8x_1 + 12x_2 + 22x_3$ is maximized.

Rewrite the objective function as

$$-8x_1 - 12x_2 - 22x_3 + z = 0.$$

The initial simplex tableau is

$$\begin{array}{c} \begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{cccccc|c} 17 & 27 & 34 & 1 & 0 & 0 & 91{,}800 \\ 12 & 21 & 15 & 0 & 1 & 0 & 42{,}000 \\ \hline -8 & -12 & -22 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}.$$

28. Let x_1 represent the number of Basic sets, x_2 the number of Regular sets, and x_3 the number of Deluxe sets. Organize the information in a table.

	Basic Set	Regular Set	Deluxe Set	Number Available
Utility Knife	2	2	3	800
Chef's Knife	1	1	1	400
Slicer	0	1	1	200
Profit	\$30	\$40	\$60	

Using this information, the problem may be stated as:

Find $x_1 \geq 0, x_2 \geq 0$, and $x_3 \geq 0$ such that

$$\begin{aligned} 2x_1 + 2x_2 + 3x_3 &\leq 800 \\ x_1 + x_2 + x_3 &\leq 400 \\ x_2 + x_3 &\leq 200 \end{aligned}$$

and $z = 30x_1 + 40x_2 + 60x_3$ is maximized.

Introduce slack variables s_1, s_2, and s_3, and the problem can be restated as:

Find $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, s_1 \geq 0, s_2 \geq 0$, and $s_3 \geq 0$ such that

$$\begin{aligned} 2x_1 + 2x_2 + 3x_3 + s_1 &= 800 \\ x_1 + x_2 + x_3 + s_2 &= 400 \\ x_2 + x_3 + s_3 &= 200 \end{aligned}$$

and $z = 30x_1 + 40x_2 + 60x_3$ is maximized.

Rewrite the objective function as

$$-30x_1 - 40x_2 - 60x_3 + z = 0.$$

The initial simplex tableau is

$$\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ 2 & 2 & 3 & 1 & 0 & 0 & 0 & 800 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 400 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 200 \\ \hline -30 & -40 & -60 & 0 & 0 & 0 & 1 & 0 \end{array}.$$

4.2 Maximization Problems

2.

$$\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ 2 & 2 & 1 & 1 & 0 & 0 & 10 \\ 1 & 2 & 3 & 0 & 1 & 0 & 15 \\ \hline -3 & -2 & -1 & 0 & 0 & 1 & 0 \end{array}$$

The solution which can be read from this initial simplex tableau is not optimal. We will try to improve the value of z. To choose the pivot, first identify the most negative indicator, which is -3, in the first column. This means that the variable x_1 will be made basic. Now form quotients by dividing each number on the right side of the tableau by the corresponding number from the column with the most negative indicator, which is column 1.

$$\begin{array}{r|cccccc|c} \text{Quotients} & x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ 10/2 = 5 & \boxed{2} & 2 & 1 & 1 & 0 & 0 & 10 \\ 15/1 = 15 & 1 & 2 & 3 & 0 & 1 & 0 & 15 \\ \hline & -3 & -2 & -1 & 0 & 0 & 1 & 0 \end{array}$$

Of the two quotients, the smallest is 5 (from row 1), so the 2 in the first row, first column is the pivot.

$$\begin{array}{r|cccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ & 2 & 2 & 1 & 1 & 0 & 0 & 10 \\ -R_1+2R_2 \rightarrow R_2 & 0 & 2 & 5 & -1 & 2 & 0 & 20 \\ \hline 3R_1+2R_3 \rightarrow R_3 & 0 & 2 & 1 & 3 & 0 & 2 & 30 \end{array}$$

None of the indicators in this tableau are negative, so we have found the optimal solution. To simplify the reading of the final tableau, create a 1 in the columns corresponding to the basic variables and z.

$$\begin{array}{r|cccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ \frac{1}{2}R_1 \rightarrow R_1 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 5 \\ \frac{1}{2}R_2 \rightarrow R_2 & 0 & 1 & \frac{5}{2} & -\frac{1}{2} & 1 & 0 & 10 \\ \hline \frac{1}{2}R_3 \rightarrow R_3 & 0 & 1 & \frac{1}{2} & \frac{3}{2} & 0 & 1 & 15 \end{array}$$

All indicators are positive, so the maximum is 15 when $x_1 = 5, x_2 = 0, x_3 = 0, s_1 = 0$, and $s_2 = 10$.

4.

$$\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ 2 & 2 & 1 & 1 & 0 & 0 & 0 & 50 \\ 1 & 1 & \boxed{3} & 0 & 1 & 0 & 0 & 40 \\ 4 & 2 & 5 & 0 & 0 & 1 & 0 & 80 \\ \hline -2 & -3 & -5 & 0 & 0 & 0 & 1 & 0 \end{array}$$

The most negative indicator is -5. Of the quotients $\frac{50}{1} = 50$, $\frac{40}{3} = 13\frac{1}{3}$, and $\frac{80}{5} = 16$, the smallest is $\frac{40}{3}$, so pivot on the 3 in row 2, column 3.

$$\begin{array}{r|ccccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ -R_2+3R_1 \rightarrow R_1 & 5 & \boxed{5} & 0 & 3 & -1 & 0 & 0 & 110 \\ & 1 & 1 & 3 & 0 & 1 & 0 & 0 & 40 \\ -5R_2+3R_3 \rightarrow R_3 & 7 & 1 & 0 & 0 & -5 & 3 & 0 & 40 \\ \hline 5R_2+3R_4 \rightarrow R_4 & -1 & -4 & 0 & 0 & 5 & 0 & 3 & 200 \end{array}$$

The solution is still not optimal. The most negative indicator is −4. Of the quotients $\frac{110}{5} = 22$, $\frac{40}{1} = 40$, and $\frac{40}{1} = 40$, the smallest is 22, so pivot on the 5 in row 1, column 2.

$$\begin{array}{r|ccccccc|c|}
 & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\
 & 5 & 5 & 0 & 3 & -1 & 0 & 0 & 110 \\
-R_1 + 5R_2 \to R_2 & 0 & 0 & 15 & -3 & 6 & 0 & 0 & 90 \\
-R_1 + 5R_3 \to R_3 & 30 & 0 & 0 & -3 & -24 & 15 & 0 & 90 \\ \hline
4R_1 + 5R_4 \to R_4 & 15 & 0 & 0 & 12 & 21 & 0 & 15 & 1440
\end{array}$$

None of the indicators in this tableau are negative, so we have found the optimal solution. To simplify the reading of the final tableau, create a 1 in the columns corresponding to the basic variables and z.

$$\begin{array}{r|ccccccc|c|}
 & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\
\frac{1}{5}R_1 \to R_1 & 1 & 1 & 0 & \frac{3}{5} & -\frac{1}{5} & 0 & 0 & 22 \\
\frac{1}{15}R_2 \to R_2 & 0 & 0 & 1 & -\frac{1}{5} & \frac{2}{5} & 0 & 0 & 6 \\
\frac{1}{15}R_3 \to R_3 & 2 & 0 & 0 & -\frac{1}{5} & -\frac{8}{5} & 1 & 0 & 6 \\ \hline
\frac{1}{15}R_4 \to R_4 & 1 & 0 & 0 & \frac{4}{5} & \frac{7}{5} & 0 & 1 & 96
\end{array}$$

The maximum is 96 when $x_1 = 0, x_2 = 22, x_3 = 6, s_1 = 0, s_2 = 0$, and $s_3 = 6$.

6.

$$\begin{array}{cccccc|c}
x_1 & x_2 & x_3 & s_1 & s_2 & z & \\
3 & 2 & 4 & 1 & 0 & 0 & 18 \\
2 & \boxed{1} & 5 & 0 & 1 & 0 & 8 \\ \hline
-1 & -4 & -2 & 0 & 0 & 1 & 0
\end{array}$$

The most negative indicator is −4. Of the quotients $\frac{18}{2} = 9$ and $\frac{8}{1} = 8$, the smallest is 8, so pivot on the 1 in row 2, column 2.

$$\begin{array}{r|cccccc|c|}
 & x_1 & x_2 & x_3 & s_1 & s_2 & z & \\
-2R_2 + R_1 \to R_1 & -1 & 0 & -6 & 1 & -2 & 0 & 2 \\
 & 2 & 1 & 5 & 0 & 1 & 0 & 8 \\ \hline
4R_2 + R_3 \to R_3 & 7 & 0 & 18 & 0 & 4 & 1 & 32
\end{array}$$

This solution is optimal. The basic variables are x_2 and s_1. The maximum is 32 when $x_1 = 0, x_2 = 8, x_3 = 0, s_1 = 2$, and $s_2 = 0$.

8. Maximize $z = 2x_1 + 3x_2$

subject to: $3x_1 + 5x_2 \le 29$

$2x_1 + x_2 \le 10$

with $x_1 \ge 0, x_2 \ge 0$.

Introduce s_1 and s_2 as slack variables. Rewrite the objective function as

$$-2x_1 - 3x_2 + z = 0.$$

The initial tableau as follows.

$$\begin{array}{ccccc|c}
x_1 & x_2 & s_1 & s_2 & z & \\
3 & \boxed{5} & 1 & 0 & 0 & 29 \\
2 & 1 & 0 & 1 & 0 & 10 \\ \hline
-2 & -3 & 0 & 0 & 1 & 0
\end{array}$$

Pivot on the 5 in row 1, column 2.

$$\begin{array}{r|ccccc|c|}
 & x_1 & x_2 & s_1 & s_2 & z & \\
 & 3 & 5 & 1 & 0 & 0 & 29 \\
-R_1 + 5R_2 \to R_2 & \boxed{7} & 0 & -1 & 5 & 0 & 21 \\ \hline
3R_1 + 5R_3 \to R_3 & -1 & 0 & 3 & 0 & 5 & 87
\end{array}$$

Pivot on the 7 in row 2, column 1.

$$\begin{array}{r|ccccc|c|}
 & x_1 & x_2 & s_1 & s_2 & z & \\
-3R_2 + 7R_1 \to R_1 & 0 & 35 & 10 & -15 & 0 & 140 \\
 & 7 & 0 & -1 & 5 & 0 & 21 \\ \hline
R_2 + 7R_3 \to R_3 & 0 & 0 & 20 & 5 & 35 & 630
\end{array}$$

Create a 1 in the columns corresponding to x_1, x_2, and z.

$$\begin{array}{r|ccccc|c|}
 & x_1 & x_2 & s_1 & s_2 & z & \\
\frac{1}{35}R_1 \to R_1 & 0 & 1 & \frac{2}{7} & -\frac{3}{7} & 0 & 4 \\
\frac{1}{7}R_2 \to R_2 & 1 & 0 & -\frac{1}{7} & \frac{5}{7} & 0 & 3 \\ \hline
\frac{1}{35}R_3 \to R_3 & 0 & 0 & \frac{4}{7} & \frac{1}{7} & 1 & 18
\end{array}$$

This is optimal. The maximum is 18 when $x_1 = 3, x_2 = 4, s_1 = 0$, and $s_2 = 0$.

10. Maximize $z = 1.2x_1 + 3.5x_2$

subject to: $2.4x_1 + 1.5x_2 \le 10$

$1.7x_1 + 1.9x_2 \le 15$

with $x_1 \ge 0, x_2 \ge 0$.

The initial tableau follows.

$$\begin{array}{ccccc|c}
x_1 & x_2 & s_1 & s_2 & z & \\
2.4 & \boxed{1.5} & 1 & 0 & 0 & 10 \\
1.7 & 1.9 & 0 & 1 & 0 & 15 \\ \hline
-1.2 & -3.5 & 0 & 0 & 1 & 0
\end{array}$$

Pivot on the 1.5 in row 1, column 2.

$$\begin{array}{r|ccccc|c|}
 & x_1 & x_2 & s_1 & s_2 & z & \\
 & 2.4 & 1.5 & 1 & 0 & 0 & 10 \\
-1.9R_1 + 1.5R_2 \to R_2 & -2.01 & 0 & -1.9 & 1.5 & 0 & 3.5 \\ \hline
3.5R_1 + 1.5R_3 \to R_3 & 6.6 & 0 & 3.5 & 0 & 1.5 & 35
\end{array}$$

Create a 1 in the columns corresponding to x_2, s_2, and z.

$$\begin{array}{r|rrrrr|r|}
 & x_1 & x_2 & s_1 & s_2 & z & \\
\frac{1}{1.5}R_1 \to R_1 & 1.6 & 1 & .67 & 0 & 0 & 6.67 \\
\frac{1}{1.5}R_2 \to R_2 & -1.34 & 0 & -1.267 & 1 & 0 & 2.33 \\
\cline{2-7}
\frac{1}{1.5}R_3 \to R_3 & 4.4 & 0 & 2.33 & 0 & 1 & 23.33
\end{array}$$

The maximum is 23.33 (or $\frac{70}{3}$) when $x_1 = 0, x_2 = 6.67$ (or $\frac{20}{3}$), $s_1 = 0$, and $s_2 = 2.33$ (or $\frac{7}{3}$).

12. Maximize $z = 12x_1 + 15x_2 + 5x_3$

subject to: $2x_1 + 2x_2 + x_3 \le 8$

$x_1 + 4x_2 + 3x_3 \le 12$

with $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0$.

The initial tableau is as follows.

$$\begin{array}{|rrrrrr|r|}
x_1 & x_2 & x_3 & s_1 & s_2 & z & \\
2 & 2 & 1 & 1 & 0 & 0 & 8 \\
1 & \boxed{4} & 3 & 0 & 1 & 0 & 12 \\
\hline
-12 & -15 & -5 & 0 & 0 & 1 & 0
\end{array}$$

Pivot on the 4 in row 2, column 2.

$$\begin{array}{r|rrrrrr|r|}
 & x_1 & x_2 & x_3 & s_1 & s_2 & z & \\
-R_2 + 2R_1 \to R_1 & 3 & 0 & -1 & 2 & -1 & 0 & 4 \\
 & 1 & \boxed{4} & 3 & 0 & 1 & 0 & 12 \\
\cline{2-8}
15R_2 + 4R_3 \to R_3 & -33 & 0 & 25 & 0 & 15 & 4 & 180
\end{array}$$

Pivot on the 3 in row 1, column 1.

$$\begin{array}{r|rrrrrr|r|}
 & x_1 & x_2 & x_3 & s_1 & s_2 & z & \\
 & \boxed{3} & 0 & -1 & 2 & -1 & 0 & 4 \\
-R_1 + 3R_2 \to R_2 & 0 & 12 & 10 & -2 & 4 & 0 & 32 \\
\cline{2-8}
11R_1 + R_3 \to R_3 & 0 & 0 & 14 & 22 & 4 & 4 & 224
\end{array}$$

Create a 1 in the columns corresponding to x_1, x_2, and z.

$$\begin{array}{r|rrrrrr|r|}
 & x_1 & x_2 & x_3 & s_1 & s_2 & z & \\
\frac{1}{3}R_1 \to R_1 & 1 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & \frac{4}{3} \\
\frac{1}{12}R_2 \to R_2 & 0 & 1 & \frac{5}{6} & -\frac{1}{6} & \frac{1}{3} & 0 & \frac{8}{3} \\
\cline{2-8}
\frac{1}{4}R_3 \to R_3 & 0 & 0 & \frac{7}{2} & \frac{11}{2} & 1 & 1 & 56
\end{array}$$

This solution is optimal. The maximum is 56 when $x_1 = \frac{4}{3}, x_2 = \frac{8}{3}, x_3 = 0, s_1 = 0$, and $s_2 = 0$.

14. Maximize $z = x_1 + x_2 + 4x_3 + 5x_4$

subject to: $x_1 + 2x_2 + 3x_3 + x_4 \le 115$

$2x_1 + x_2 + 8x_3 + 5x_4 \le 200$

$x_1 + x_3 \le 50$

with $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0, x_4 \ge 0$.

$$\begin{array}{|rrrrrrrr|r|}
x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & z & \\
1 & 2 & 3 & 1 & 1 & 0 & 0 & 0 & 115 \\
2 & 1 & 8 & \boxed{5} & 0 & 1 & 0 & 0 & 200 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 50 \\
\hline
-1 & -1 & -4 & -5 & 0 & 0 & 0 & 1 & 0
\end{array}$$

Pivot on the 5 in row 2, column 4

$$\begin{array}{r|rrrrrrrr|r|}
 & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & z & \\
-R_2 + 5R_1 \to R_1 & 3 & 9 & 7 & 0 & 5 & -1 & 0 & 0 & 375 \\
 & 2 & 1 & 8 & 5 & 0 & 1 & 0 & 0 & 200 \\
 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 50 \\
\cline{2-10}
R_2 + R_4 \to R_4 & 1 & 0 & 4 & 0 & 0 & 1 & 0 & 1 & 200
\end{array}$$

Create a 1 in the columns corresponding to x_4 and s_1.

$$\begin{array}{r|rrrrrrrr|r|}
 & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & z & \\
 & \frac{3}{5} & \frac{9}{5} & \frac{7}{5} & 0 & 1 & -\frac{1}{5} & 0 & 0 & 75 \\
\frac{1}{5}R_1 \to R_1 & \frac{2}{5} & \frac{1}{5} & \frac{8}{5} & 1 & 0 & \frac{1}{5} & 0 & 0 & 40 \\
\frac{1}{5}R_2 \to R_2 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 50 \\
\cline{2-10}
 & 1 & 0 & 4 & 0 & 0 & 1 & 0 & 1 & 200
\end{array}$$

This solution is optimal. The maximum is 200 when $x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 40, s_1 = 75, s_2 = 0$, and $s_3 = 50$.

18. Let x_1 be the number of bargain sets and x_2 be the number of deluxe sets.

Then $3x_1 + 5x_2 \le 3900$

$x_1 + 3x_2 \le 2100$

and $2x_1 + 2x_2 \le 2200$.

To maximize $z = 100x_1 + 150x_2$, introduce s_1, s_2, and s_3 as slack variables. The initial tableau will be as follows.

$$\begin{array}{|rrrrrr|r|}
x_1 & x_2 & s_1 & s_2 & s_3 & z & \\
3 & 5 & 1 & 0 & 0 & 0 & 3900 \\
1 & \boxed{3} & 0 & 1 & 0 & 0 & 2100 \\
2 & 2 & 0 & 0 & 1 & 0 & 2200 \\
\hline
-100 & -150 & 0 & 0 & 0 & 1 & 0
\end{array}$$

Pivot on the 3 in row 2, column 2.

$$\begin{array}{r|rrrrrr|r|}
 & x_1 & x_2 & s_1 & s_2 & s_3 & z & \\
-5R_2 + 3R_1 \to R_1 & \boxed{4} & 0 & 3 & -5 & 0 & 0 & 1200 \\
 & 1 & 3 & 0 & 1 & 0 & 0 & 2100 \\
-2R_2 + 3R_3 \to R_3 & 4 & 0 & 0 & -2 & 3 & 0 & 2400 \\
\cline{2-8}
50R_2 + R_4 \to R_4 & -50 & 0 & 0 & 50 & 0 & 1 & 105{,}000
\end{array}$$

Pivot on the 4 in row 1, column 1.

$$\begin{array}{r|cccccc|r|}
 & x_1 & x_2 & s_1 & s_2 & s_3 & z & \\
 & 4 & 0 & 3 & -5 & 0 & 0 & 1200\\
-\mathrm{R}_1+4\mathrm{R}_2\to\mathrm{R}_2 & 0 & 12 & -3 & 9 & 0 & 0 & 7200\\
-\mathrm{R}_1+\mathrm{R}_3\to\mathrm{R}_3 & 0 & 0 & -3 & \boxed{3} & 3 & 0 & 1200\\
\hline
25\mathrm{R}_1+2\mathrm{R}_4\to\mathrm{R}_4 & 0 & 0 & 75 & -25 & 0 & 2 & 240{,}000
\end{array}$$

Pivot on the 3 in row 3, column 4.

$$\begin{array}{r|cccccc|r|}
 & x_1 & x_2 & s_1 & s_2 & s_3 & z & \\
5\mathrm{R}_3+3\mathrm{R}_1\to\mathrm{R}_1 & 12 & 0 & -6 & 0 & 15 & 0 & 9600\\
-3\mathrm{R}_3+\mathrm{R}_2\to\mathrm{R}_2 & 0 & 12 & 6 & 0 & -9 & 0 & 3600\\
 & 0 & 0 & -3 & 3 & 3 & 0 & 1200\\
\hline
25\mathrm{R}_3+3\mathrm{R}_4\to\mathrm{R}_4 & 0 & 0 & 150 & 0 & 75 & 6 & 750{,}000
\end{array}$$

Create a 1 in the columns corresponding to x_1, x_2, s_2, and z.

$$\begin{array}{r|cccccc|r|}
 & x_1 & x_2 & s_1 & s_2 & s_3 & z & \\
\frac{1}{12}\mathrm{R}_1\to\mathrm{R}_1 & 1 & 0 & -\frac{1}{2} & 0 & \frac{5}{4} & 0 & 800\\
\frac{1}{12}\mathrm{R}_2\to\mathrm{R}_2 & 0 & 1 & \frac{1}{2} & 0 & -\frac{3}{4} & 0 & 300\\
\frac{1}{3}\mathrm{R}_3\to\mathrm{R}_3 & 0 & 0 & -1 & 1 & 1 & 0 & 400\\
\hline
\frac{1}{6}\mathrm{R}_4\to\mathrm{R}_4 & 0 & 0 & 25 & 0 & \frac{25}{2} & 1 & 125{,}000
\end{array}$$

The maximum profit is \$125,000 when $x_1 = 800$ and $x_2 = 300$, that is, when 800 bargain sets and 300 deluxe sets are produced.

20. **(a)** Let x_1 be the number of loaves of raisin bread and x_2 be the number of raisin cakes.

Then $x_1 + 5x_2 \le 150$

$x_1 + 2x_2 \le 90$

and $2x_1 + x_2 \le 150$.

To maximize $z = 1.75x_1 + 4x_2$, add s_1, s_2, and s_3 as slack variables. The initial tableau will be as follows.

$$\begin{array}{|cccccc|r|}
x_1 & x_2 & s_1 & s_2 & s_3 & z & \\
1 & \boxed{5} & 1 & 0 & 0 & 0 & 150\\
1 & 2 & 0 & 1 & 0 & 0 & 90\\
2 & 1 & 0 & 0 & 1 & 0 & 150\\
\hline
-\frac{7}{4} & -4 & 0 & 0 & 0 & 1 & 0
\end{array}$$

Pivot on the 5 in row 1, column 2.

$$\begin{array}{r|cccccc|r|}
 & x_1 & x_2 & s_1 & s_2 & s_3 & z & \\
 & 1 & 5 & 1 & 0 & 0 & 0 & 150\\
-2\mathrm{R}_1+5\mathrm{R}_2\to\mathrm{R}_2 & \boxed{3} & 0 & -2 & 5 & 0 & 0 & 150\\
-\mathrm{R}_1+5\mathrm{R}_3\to\mathrm{R}_3 & 9 & 0 & -1 & 0 & 5 & 0 & 600\\
\hline
4\mathrm{R}_1+5\mathrm{R}_4\to\mathrm{R}_4 & -\frac{19}{4} & 0 & 4 & 0 & 0 & 5 & 600
\end{array}$$

Pivot on the 3 in row 2, column 1.

$$\begin{array}{r|cccccc|r|}
 & x_1 & x_2 & s_1 & s_2 & s_3 & z & \\
-\mathrm{R}_2+3\mathrm{R}_1\to\mathrm{R}_1 & 0 & 15 & 5 & -5 & 0 & 0 & 300\\
 & 3 & 0 & -2 & 5 & 0 & 0 & 150\\
-3\mathrm{R}_2+\mathrm{R}_3\to\mathrm{R}_3 & 0 & 0 & 5 & -15 & 5 & 0 & 150\\
\hline
\frac{19}{4}\mathrm{R}_2+3\mathrm{R}_4\to\mathrm{R}_4 & 0 & 0 & \frac{5}{2} & \frac{95}{4} & 0 & 15 & \frac{5025}{2}
\end{array}$$

Create a 1 in the columns corresponding to x_1, x_2, s_3, and z.

$$\begin{array}{r|cccccc|c|} & x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ \frac{1}{15}R_1 \to R_1 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & 0 & 0 & 20 \\ \frac{1}{3}R_2 \to R_2 & 1 & 0 & -\frac{2}{3} & \frac{5}{3} & 0 & 0 & 50 \\ \frac{1}{5}R_3 \to R_3 & 0 & 0 & 1 & -3 & 1 & 0 & 30 \\ \hline \frac{1}{15}R_4 \to R_4 & 0 & 0 & \frac{1}{6} & \frac{19}{12} & 0 & 1 & \frac{335}{2} \end{array}$$

The optimal solution occurs when $x_1 = 50$ and $x_2 = 20$; that is, when 50 loaves of raisin bread and 20 raisin cakes are baked.

(b) $\frac{335}{2} = 167.5$; the maximum gross income is \$167.50.

(c) When $x_1 = 50$ and $x_2 = 20$, the number of units used are as follows.

Flour: $50 + 5(20) = 150$

This is the total amount of available flour.

Sugar: $50 + 2(20) = 90$

This is the total amount of available sugar.

Raisins: $2(50) + 20 = 120$

This leaves $150 - 120$, or 30 units, of raisins.

22. **(a)** The tableau and set up were explained in Exercise 28 of Section 4.1.

$$\begin{array}{|ccccccc|c|} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ 2 & 2 & 3 & 1 & 0 & 0 & 0 & 800 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 400 \\ 0 & 1 & \boxed{1} & 0 & 0 & 1 & 0 & 200 \\ \hline -30 & -40 & -60 & 0 & 0 & 0 & 1 & 0 \end{array}$$

Pivot on the 1 in row 3, column 3.

$$\begin{array}{r|ccccccc|c|} & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ -3R_3 + R_1 \to R_1 & \boxed{2} & -1 & 0 & 1 & 0 & -3 & 0 & 200 \\ -R_3 + R_2 \to R_2 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 200 \\ & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 200 \\ \hline 60R_3 + R_4 \to R_4 & -30 & 20 & 0 & 0 & 0 & 60 & 1 & 12{,}000 \end{array}$$

Pivot on the 2 in row 1, column 1.

$$\begin{array}{r|ccccccc|c|} & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ & 2 & -1 & 0 & 1 & 0 & -3 & 0 & 200 \\ -R_1 + 2R_2 \to R_2 & 0 & 1 & 0 & -1 & 2 & 1 & 0 & 200 \\ & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 200 \\ \hline 15R_1 + R_4 \to R_4 & 0 & 5 & 0 & 15 & 0 & 15 & 1 & 15{,}000 \end{array}$$

Create a 1 in the column corresponding to x_1.

$$\begin{array}{r|ccccccc|c|} & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{3}{2} & 0 & 100 \\ \frac{1}{2}R_1 \to R_1 & 0 & 1 & 0 & -1 & 2 & 1 & 0 & 200 \\ & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 200 \\ \hline & 0 & 5 & 0 & 15 & 0 & 15 & 1 & 15{,}000 \end{array}$$

The maximum profit is \$15,000 when $x_1 = 100, x_2 = 0$, and $x_3 = 200$, that is, when 100 basic sets, no regular sets, and 200 deluxe sets are made.

(b) Even though regular sets make a larger profit, there are only 200 slicers available. Since slicers are used in regular and deluxe sets, and deluxe sets account for \$20 more profit, slicers should be used in deluxe sets (as many as possible) with any leftovers used in regular sets.

24. **(a)** Let x_1 represent the number of toy trucks and x_2 the number of toy fire engines.

$$\begin{array}{ll} \text{Maximize} & z = 8.50x_1 + 12.10x_2 \\ \text{subject to:} & -2x_1 + 3x_2 \leq \quad 0 \\ & x_1 \qquad \leq \ 6700 \\ & \qquad x_2 \leq \ 5500 \\ & x_1 + \ x_2 \leq 12,000 \\ \text{with} & x_1 \geq 0, x_2 \geq 0. \end{array}$$

This exercise should be solved by graphing calculator or computer methods. The answer is to produce 6700 trucks and 4467 fire engines for a maximum profit of \$110,997.

(b) Many solutions are possible.

(c) Many solutions are possible.

26. **(a)** Look at the first table, which has to do with the profits. The profit-maximization formula is

$$\$2\text{A} + \$5\text{B} + \$4\text{C} = X,$$

so the answer is choice (1).

(b) Look at the "Painting" row of the second chart. The "Painting" constraint is

$$1\text{A} + 2\text{B} + 2\text{C} \leq 38{,}000,$$

so the answer is choice (3).

28. Maximize $z = 100x + 150y$

$$\begin{array}{ll} \text{subject to:} & 3x + 5y \leq 3900 \\ & x + 3y \leq 2100 \\ & 2x + 2y \leq 2200 \\ \text{with} & x \geq 0, y \geq 0. \end{array}$$

Using Excel, we enter the variables x and y in cells A1 and B1, respectively. Enter the x- and y-coordinates of the initial corner point of the feasible region, $(0, 0)$, in cells A2 and B2, respectively, and NAME these cells x and y, respectively. In cells C2, C4, C5, C6, C7, and C8, enter the formula for the function to maximize and each of the constraints: $100x + 150y, 3x + 5y, x + 3y, 2x + 2y, x$, and y. Since x and y have been set to 0, all the cells containing formulas should also show the value 0, as below.

	A	B	C
1	x	y	
2	0	0	0
3			
4			0
5			0
6			0
7			0
8			0

Using the SOLVER, ask Excel to maximize the value in cell C2 subject to the constraints C4 $\leq$ 3900, C5 $\leq$ 2100,C6 $\leq$ 2200,C7 $\geq$ 0, C8 $\geq$ 0. Make sure you have checked off the box *Assume Linear Model* in SOLVER OPTIONS.

Excel returns the following values and allows you to choose a report.

	A	B	C
1	x	y	
2	800	300	125000
3			
4			3900
5			1700
6			2200
7			800
8			300

Select the sensitivity report. The report will appear on a new sheet of the spread sheet.

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
\$A\$2	x	800	0	100	50	10
\$B\$2	y	300	0	150	16.66666667	50

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
\$C\$8		300	0	0	300	1E+30
\$C\$4		3900	25	3900	400	600
\$C\$7		800	0	0	800	1E+30
\$C\$5		1700	0	2100	1E+30	400
\$C\$6		2200	12.5	2200	400	400

For the bargain sets, the allowable increase is \$50 and the allowable decrease is \$10. So the profit from the bargain sets can be as high as \$100 + \$50 = \$150 or as low as \$100 − \$10 = \$90 and the original solution is still optimal. For the deluxe sets, the allowable increase is \$16.67 and the allowable decrease is \$50. So the profit from the deluxe sets can be as high as \$150 + \$16.67 = \$166.67 or as low as \$150 − \$50 = \$100 and the original solution is still optimal.

30. **(a)** Let x_1 represent the number of species A, x_2 represent the number of species B, and x_3 represent the number of species C.

$$\begin{array}{ll}
\text{Maximize} & z = 1.62x_1 + 2.14x_2 + 3.01x_3 \\
\text{subject to:} & 1.32x_1 + 2.1x_2 + \ .86x_3 \leq 490 \\
 & \ \ 2.9x_1 + .95x_2 + 1.52x_3 \leq 897 \\
 & 1.75x_1 + \ .6x_2 + 2.01x_3 \leq 653 \\
\text{with} & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.
\end{array}$$

Use a graphing calculator or computer to solve this problem and find that the answer is to stock none of species A, 114 of species B, and 291 of species C for a maximum combined weight of 1119.72 kg.

(b) When $x_1 = 0$, $x_2 = 114$, and $x_2 = 291$, the number of units used are as follows.

Food I: $1.32(0) + 2.1 + (114) + .86(291) = 489.66$

or 490 units, which is the total amount available of Food I.

Food II: $2.9(0) + .95(114) + 1.52(291) = 550.62$

or 551 units, which leaves $897 - 551$, or 346 units of Food II available.

Food III: $1.75(0) + .6(114) + 2.01(291) = 653.31$

or 653 units, which is the total amount available of Food III.

(c) Many answers are possible. The idea is to choose average weights for species B and C that are considerably smaller than the average weight chosen for species A, so that species A dominates the objective function.

(d) Many answers are possible. The idea is to choose average weights for species A and B that are considerably smaller than the average weight chosen for species C.

32. **(a)** Let x_1 = number of large fund-raising parties, x_2 = number of letters requesting funds, and x_3 = number of dinner parties.

$$\begin{array}{llcl} \text{Maximize} & z = 200{,}000x_1 + 100{,}000x_2 + 600{,}000x_3 & & \\ \text{subject to:} & x_1 + \quad x_2 + \quad x_3 & \le & 25 \\ & 3000x + 1000x_2 + 12{,}000x_3 & \le & 102,000 \\ \text{with} & x_1 \ge 0, x_2 \ge 0, x_3 \ge 0. & & \end{array}$$

We need two slack variables. The initial simplex tableau is as follows.

$$\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ 1 & 1 & 1 & 1 & 0 & 0 & 25 \\ 3000 & 1000 & \boxed{12,000} & 0 & 1 & 0 & 102{,}000 \\ \hline -200,000 & -100,000 & -600,000 & 0 & 0 & 1 & 0 \end{array}$$

Pivot on the 12,000 in row 2, column 3.

$$\begin{array}{r|cccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ \text{R}_2 + (-12,000\text{R}_1) \to \text{R}_1 & \boxed{-9000} & -11,000 & 0 & -12,000 & 1 & 0 & -198,000 \\ & 3000 & 1000 & 12,000 & 0 & 1 & 0 & 102{,}000 \\ \hline 50\text{R}_2 + \text{R}_3 \to \text{R}_3 & -50,000 & -50,000 & 0 & 0 & 50 & 1 & 5{,}100{,}000 \end{array}$$

Pivot on -9000 in row 1, column 1.

$$\begin{array}{r|cccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ & -9000 & -11,000 & 0 & -12,000 & 1 & 0 & -198,000 \\ \text{R}_1 + 3\text{R}_2 \to \text{R}_2 & 0 & -8000 & 36,000 & -12,000 & 4 & 0 & 108{,}000 \\ \hline 50\text{R}_1 + (-9)\text{R}_3 \to \text{R}_3 & 0 & -100,000 & 0 & -600,000 & -400 & -9 & -55{,}800{,}000 \end{array}$$

Create a 1 in the columns corresponding to x_1, x_3, and z.

$$\begin{array}{r|cccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ -\frac{1}{9000}\text{R}_1 \to \text{R}_1 & 1 & \frac{11}{9} & 0 & \frac{4}{3} & -\frac{1}{9000} & 0 & 22 \\ \frac{1}{36{,}000}\text{R}_2 \to \text{R}_2 & 0 & -\frac{2}{9} & 1 & -\frac{1}{3} & \frac{1}{9000} & 0 & 3 \\ \hline -\frac{1}{9}\text{R}_3 \to \text{R}_3 & 0 & \frac{100{,}000}{9} & 0 & \frac{200{,}000}{3} & \frac{400}{9} & 1 & 6{,}200{,}000 \end{array}$$

The maximum amount of money is \$6,200,000 when $x_1 = 22, x_2 = 0$, and $x_3 = 3$, that is, when 22 fund-raising parties, no mailings, and 3 dinner parties are planned.

4.3 Minimization Problems; Duality

2. The transpose of a matrix is found by exchanging the rows and columns. The transpose of

$$\begin{bmatrix} 2 & 5 & 8 & 6 & 0 \\ 1 & -1 & 0 & 12 & 14 \end{bmatrix}$$

is

$$\begin{bmatrix} 2 & 1 \\ 5 & -1 \\ 8 & 0 \\ 6 & 12 \\ 0 & 14 \end{bmatrix}.$$

4. The transpose of

$$\begin{bmatrix} 1 & 11 & 15 \\ 0 & 10 & -6 \\ 4 & 12 & -2 \\ 1 & -1 & 13 \\ 2 & 25 & -1 \end{bmatrix}$$

is

$$\begin{bmatrix} 1 & 0 & 4 & 1 & 2 \\ 11 & 10 & 12 & -1 & 25 \\ 15 & -6 & -2 & 13 & -1 \end{bmatrix}.$$

6.

$$\begin{array}{ll} \text{Maximize} & z = 8x_1 + 3x_2 + x_3 \\ \text{subject to:} & 7x_1 + 6x_2 + 8x_3 \le 18 \\ & 4x_1 + 5x_2 + 10x_3 \le 20 \\ \text{with} & x_1 \ge 0, x_2 \ge 0, x_3 \ge 0. \end{array}$$

To find the dual, first write the augmented matrix for the problem.

$$\left[\begin{array}{ccc|c} 7 & 6 & 8 & 18 \\ 4 & 5 & 10 & 20 \\ \hline 8 & 3 & 1 & 0 \end{array}\right]$$

Then form the transpose of this matrix.

$$\left[\begin{array}{cc|c} 7 & 4 & 8 \\ 6 & 5 & 3 \\ 8 & 10 & 1 \\ \hline 18 & 20 & 0 \end{array}\right]$$

The dual problem is:

$$\begin{array}{ll} \text{Minimize} & 18y_1 + 20y_2 \\ \text{subject to:} & 7y_1 + 4y_2 \ge 8 \\ & 6y_1 + 5y_2 \ge 3 \\ & 8y_1 + 10y_2 \ge 1 \\ \text{with} & y_1 \ge 0, y_2 \ge 0. \end{array}$$

8.

$$\begin{array}{ll} \text{Minimize} & w = y_1 + y_2 + 4y_3 \\ \text{subject to:} & y_1 + 2y_2 + 3y_3 \ge 115 \\ & 2y_1 + y_2 + 8y_3 \ge 200 \\ & y_1 + y_3 \ge 50 \\ \text{with} & y_1 \ge 0, y_2 \ge 0, y_3 \ge 0. \end{array}$$

Write the augmented matrix for the problem.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 115 \\ 2 & 1 & 8 & 200 \\ 1 & 0 & 1 & 50 \\ \hline 1 & 1 & 4 & 0 \end{array}\right]$$

Form the transpose of this matrix.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 8 & 1 & 4 \\ \hline 115 & 200 & 50 & 0 \end{array}\right]$$

The dual problem is:

$$\begin{array}{ll} \text{Maximize} & 115x_1 + 200x_2 + 50x_3 \\ \text{subject to:} & x_1 + 2x_2 + x_3 \le 1 \\ & 2x_1 + x_2 \le 1 \\ & 3x_1 + 8x_2 + x_3 \le 4 \\ \text{with} & x_1 \ge 0, x_2 \ge 0, x_3 \ge 0. \end{array}$$

10. Find $y_1 \ge 0$ and $y_2 \ge 0$ such that

$$\begin{aligned} 3y_1 + y_2 &\ge 12 \\ y_1 + 4y_2 &\ge 16 \end{aligned}$$

and $w = 2y_1 + y_2$ is minimized.

Write the augmented matrix for this problem.

$$\left[\begin{array}{cc|c} 3 & 1 & 12 \\ 1 & 4 & 16 \\ \hline 2 & 1 & 0 \end{array}\right]$$

Form the transpose to get the matrix for the dual problem.

$$\left[\begin{array}{cc|c} 3 & 1 & 2 \\ 1 & 4 & 1 \\ \hline 12 & 16 & 0 \end{array}\right]$$

Use this matrix to write the dual problem:

Find $x_1 \ge 0$ and $x_2 \ge 0$ such that

$$\begin{aligned} 3x_1 + x_2 &\le 2 \\ x_1 + 4x_2 &\le 1 \end{aligned}$$

and $z = 12x_1 + 16x_2$ is maximized.

Introduce slack variables and write the initial tableau.

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} 3 & 1 & 1 & 0 & 0 & 2 \\ 1 & \boxed{4} & 0 & 1 & 0 & 1 \\ \hline -12 & -16 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Pivot on the 4 in row 2, column 2.

$$\begin{array}{r} \\ -\mathrm{R}_2+4\mathrm{R}_1 \to \mathrm{R}_1 \\ \\ 4\mathrm{R}_2+\ \mathrm{R}_3 \to \mathrm{R}_3 \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} \boxed{11} & 0 & 4 & -1 & 0 & 7 \\ 1 & 4 & 0 & 1 & 0 & 1 \\ \hline -8 & 0 & 0 & 4 & 1 & 4 \end{array}\right] \end{array}$$

Pivot on the 11 in row 1, column 1.

$$\begin{array}{r} \\ \\ -\mathrm{R}_1+11\mathrm{R}_2 \to \mathrm{R}_2 \\ 8\mathrm{R}_1+11\mathrm{R}_3 \to \mathrm{R}_3 \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} 11 & 0 & 4 & -1 & 0 & 7 \\ 0 & 44 & -4 & 12 & 0 & 4 \\ \hline 0 & 0 & 32 & 36 & 11 & 100 \end{array}\right] \end{array}$$

Create a 1 in the columns corresponding to x_1, x_2, and z.

$$\begin{array}{r} \\ \frac{1}{11}\mathrm{R}_1 \to \mathrm{R}_1 \\ \frac{1}{44}\mathrm{R}_2 \to \mathrm{R}_2 \\ \frac{1}{11}\mathrm{R}_3 \to \mathrm{R}_3 \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & \frac{4}{11} & -\frac{1}{11} & 0 & \frac{7}{11} \\ 0 & 1 & -\frac{1}{11} & \frac{3}{11} & 0 & \frac{1}{11} \\ \hline 0 & 0 & \frac{32}{11} & \frac{36}{11} & 1 & \frac{100}{11} \end{array}\right] \end{array}$$

The minimum is $\frac{100}{11}$ when $y_1 = \frac{32}{11}$ and $y_2 = \frac{36}{11}$.

12.

$$\begin{array}{ll} \text{Minimize} & w = 3y_1 + 2y_2 \\ \text{subject to:} & 2y_1 + 3y_2 \geq 60 \\ & y_1 + 4y_2 \geq 40 \\ \text{with} & y_1 \geq 0, y_2 \geq 0. \end{array}$$

Write the augmented matrix for this problem.

$$\left[\begin{array}{cc|c} 2 & 3 & 60 \\ 1 & 4 & 40 \\ \hline 3 & 2 & 0 \end{array}\right]$$

Transpose to get the matrix for the dual problem.

$$\left[\begin{array}{cc|c} 2 & 1 & 3 \\ 3 & 4 & 2 \\ \hline 60 & 40 & 0 \end{array}\right]$$

Write the dual problem from this matrix:

$$\begin{array}{ll} \text{Maximize} & z = 60x_1 + 40x_2 \\ \text{subject to:} & 2x_1 + x_2 \leq 3 \\ & 3x_1 + 4x_2 \leq 2 \\ \text{with} & x_1 \geq 0, x_2 \geq 0. \end{array}$$

Write the initial tableau.

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} 2 & 1 & 1 & 0 & 0 & 3 \\ \boxed{3} & 4 & 0 & 1 & 0 & 2 \\ \hline -60 & -40 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Pivot on the 3 in row 2, column 1.

$$\begin{array}{r} \\ -2\mathrm{R}_2+3\mathrm{R}_1 \to \mathrm{R}_1 \\ \\ 20\mathrm{R}_2+\ \mathrm{R}_3 \to \mathrm{R}_3 \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} 0 & -5 & 3 & -2 & 0 & 5 \\ 3 & 4 & 0 & 1 & 0 & 2 \\ \hline 0 & 40 & 0 & 20 & 1 & 40 \end{array}\right] \end{array}$$

Create a 1 in the columns corresponding to x_1 and s_1.

$$\begin{array}{r} \\ \frac{1}{3}\mathrm{R}_1 \to \mathrm{R}_1 \\ \\ \frac{1}{3}\mathrm{R}_3 \to \mathrm{R}_3 \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} 0 & -\frac{5}{3} & 1 & -\frac{2}{3} & 0 & \frac{5}{3} \\ 1 & \frac{4}{3} & 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ \hline 0 & 40 & 0 & 20 & 1 & 40 \end{array}\right] \end{array}$$

This solution is optimal. The minimum is 40 when $y_1 = 0$ and $y_2 = 20$.

14.

$$\begin{array}{ll} \text{Minimize} & w = 3y_1 + 2y_2 \\ \text{subject to:} & y_1 + 2y_2 \geq 10 \\ & y_1 + \ y_2 \geq \ 8 \\ & 2y_1 + \ y_2 \geq 12 \\ \text{with} & y_1 \geq 0, y_2 \geq 0. \end{array}$$

Write the augmented matrix.

$$\left[\begin{array}{cc|c} 1 & 2 & 10 \\ 1 & 1 & 8 \\ 2 & 1 & 12 \\ \hline 3 & 2 & 0 \end{array}\right]$$

Transpose to get the matrix for the dual problem.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 2 & 1 & 1 & 2 \\ \hline 10 & 8 & 12 & 0 \end{array}\right]$$

Write the dual problem:

$$\begin{array}{ll} \text{Maximize} & z = 10x_1 + 8x_2 + 12x_3 \\ \text{subject to:} & x_1 + x_2 + 2x_3 \leq 3 \\ & 2x_1 + x_2 + \ x_3 \leq 2 \\ \text{with} & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0. \end{array}$$

Write the initial tableau.

$$\begin{array}{c} \begin{array}{rrrrrrr} x_1 & x_2 & x_3 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{rrrrrr|r} 1 & 1 & \boxed{2} & 1 & 0 & 0 & 3 \\ 2 & 1 & 1 & 0 & 1 & 0 & 2 \\ \hline -10 & -8 & -12 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Pivot on the 2 in row 1, column 3.

$$\begin{array}{rc} & \begin{array}{rrrrrrr} x_1 & x_2 & x_3 & s_1 & s_2 & z & \end{array} \\ \begin{array}{r} \\ 2\mathrm{R}_2 - \mathrm{R}_1 \to \mathrm{R}_2 \\ \mathrm{R}_3 + 6\mathrm{R}_1 \to \mathrm{R}_3 \end{array} & \left[\begin{array}{rrrrrr|r} 1 & 1 & 2 & 1 & 0 & 0 & 3 \\ \boxed{3} & 1 & 0 & -1 & 2 & 0 & 1 \\ \hline -4 & -2 & 0 & 6 & 0 & 1 & 18 \end{array}\right] \end{array}$$

Pivot on the 3 in row 2, column 1.

$$\begin{array}{rc} & \begin{array}{rrrrrrr} x_1 & x_2 & x_3 & s_1 & s_2 & z & \end{array} \\ \begin{array}{r} 3\mathrm{R}_1 - \mathrm{R}_2 \to \mathrm{R}_1 \\ \\ 4\mathrm{R}_2 + 3\mathrm{R}_3 \to \mathrm{R}_3 \end{array} & \left[\begin{array}{rrrrrr|r} 0 & 2 & 6 & 4 & -2 & 0 & 8 \\ 3 & 1 & 0 & -1 & 2 & 0 & 1 \\ \hline 0 & -2 & 0 & 14 & 8 & 3 & 58 \end{array}\right] \end{array}$$

Pivot on the 1 in row 2, column 2.

$$\begin{array}{rc} & \begin{array}{rrrrrrr} x_1 & x_2 & x_3 & s_1 & s_2 & z & \end{array} \\ \begin{array}{r} \mathrm{R}_1 - 2\mathrm{R}_2 \to \mathrm{R}_1 \\ \\ \mathrm{R}_3 + 2\mathrm{R}_2 \to \mathrm{R}_3 \end{array} & \left[\begin{array}{rrrrrr|r} -6 & 0 & 6 & 6 & -6 & 0 & 6 \\ 3 & 1 & 0 & -1 & 2 & 0 & 1 \\ \hline 6 & 0 & 0 & 12 & 12 & 3 & 60 \end{array}\right] \end{array}$$

Create a 1 in the columns corresponding to x_2, x_3, and z.

$$\begin{array}{rc} & \begin{array}{rrrrrrr} x_1 & x_2 & x_3 & s_1 & s_2 & z & \end{array} \\ \begin{array}{r} \frac{1}{6}\mathrm{R}_1 \to \mathrm{R}_1 \\ \\ \frac{1}{3}\mathrm{R}_3 \to \mathrm{R}_3 \end{array} & \left[\begin{array}{rrrrrr|r} -1 & 0 & 1 & 1 & -1 & 0 & 1 \\ 3 & 1 & 0 & -1 & 2 & 0 & 1 \\ \hline 2 & 0 & 0 & 4 & 4 & 1 & 20 \end{array}\right] \end{array}$$

This solution is optimal. The minimum is 20 when $y_1 = 4$ and $y_2 = 4$.

16. **(a)** The initial matrix for the original problem is

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 100 \\ 400 & 160 & 280 & 20{,}000 \\ \hline 120 & 40 & 60 & 0 \end{array}\right].$$

The transposed matrix, for the dual problem, is

$$\left[\begin{array}{rr|r} 1 & 400 & 120 \\ 1 & 160 & 40 \\ 1 & 280 & 60 \\ \hline 100 & 20{,}000 & 0 \end{array}\right].$$

Minimize $w = 100y_1 + 20{,}000y_2$

subject to: $y_1 + 400y_2 \geq 120$

$y_1 + 160y_2 \geq 40$

$y_1 + 280y_2 \geq 60$

with $y_1 \geq 0, y_2 \geq 0$.

(b) We apply the simplex algorithm to the original maximization problem. The initial tableau is

$$\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ 1 & 1 & 1 & 1 & 0 & 0 & 100 \\ \boxed{400} & 160 & 280 & 0 & 1 & 0 & 20{,}000 \\ \hline -120 & -40 & -60 & 0 & 0 & 1 & 0 \end{array}$$

Pivot on the 400 in row 2, column 1.

$$\begin{array}{r|cccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ 400\text{R}_1 - \text{R}_2 \to \text{R}_1 & 0 & 240 & 120 & 400 & -1 & 0 & 20{,}000 \\ & 400 & 160 & 280 & 0 & 1 & 0 & 20{,}000 \\ \hline \frac{3}{10}\text{R}_2 + \text{R}_3 \to \text{R}_3 & 0 & 8 & 24 & 0 & .3 & 1 & 6000 \end{array}$$

Create a 1 in the columns corresponding to x_1 and s_1.

$$\begin{array}{r|cccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ \frac{1}{400}\text{R}_1 \to \text{R}_1 & 0 & .6 & .3 & 1 & -\frac{1}{400} & 0 & 50 \\ \frac{1}{400}\text{R}_2 \to \text{R}_2 & 1 & .4 & .7 & 0 & \frac{1}{400} & 0 & 50 \\ \hline & 0 & 8 & 24 & 0 & .3 & 1 & 6000 \end{array}$$

This solution is optimal. A maximum profit of \$6000 is achieved by planting 50 acres of potatoes, 0 acres of corn, and 0 acres of cabbage.

From the dual solution, the shadow cost of acreage is 0 and of capital is $\frac{3}{10}$.

$$\text{New profit} = 6000 + 0(-10) + \left(\frac{3}{10}\right)1000 = \$6300$$

Now calculate the number of acres of each:

$$\begin{aligned} \text{Profit} &= 120\text{P} + 40\text{C} + 60\text{B} \\ 6300 &= 120\text{P} + 40(0) + 60(0) \\ \text{P} &= 52.5. \end{aligned}$$

The farmer will make a profit of \$6300 by planting 52.5 acres of potatoes and no corn or cabbage.

(c) $\text{New profit} = 6000 + 0(10) + \frac{3}{10}(-1000)$

$= \$5700$

Calculate the number of acres of each:

$$\begin{aligned} \text{Profit} &= 120\text{P} + 40\text{C} + 60\text{B} \\ 5700 &= 120\text{P} + 40(0) + 60(0) \\ \text{P} &= 47.5. \end{aligned}$$

The farmer will make a profit of \$5700 by planting 47.5 acres of potatoes and no corn or cabbage.

18. **(a)** Let $y_1 =$ number of units of regular beer
and $y_2 =$ number of units of light beer.

Minimize $w = 36{,}000y_1 + 48{,}000y_2$

subject to:

$$\begin{aligned} y_1 \qquad\qquad &\geq 12 \\ y_2 &\geq 10 \\ 100,000y_1 + 300,000y_2 &\geq 7,000,000 \\ y_1 + \qquad y_2 &\geq 42 \end{aligned}$$

with $y_1 \geq 0, y_2 \geq 0.$

Write the augmented matrix for this problem.

$$\left[\begin{array}{rr|r} 1 & 0 & 12 \\ 0 & 1 & 10 \\ 100{,}000 & 300{,}000 & 7{,}000{,}000 \\ 1 & 1 & 42 \\ \hline 36{,}000 & 48{,}000 & 0 \end{array}\right]$$

Form the transpose of this matrix for the dual problem.

$$\left[\begin{array}{rrrr|r} 1 & 0 & 100{,}000 & 1 & 36{,}000 \\ 0 & 1 & 300{,}000 & 1 & 48{,}000 \\ \hline 12 & 10 & 7{,}000{,}000 & 42 & 0 \end{array}\right]$$

The dual problem is:

$$\begin{array}{ll} \text{Maximize} & 12x_1 + 10x_2 + 7,000,000x_3 + 42x_4 \\ \text{subject to:} & x_1 \qquad + 100,000x_3 + x_4 \le 36,000 \\ & \qquad x_2 + 300,000x_3 + x_4 \le 48,000 \\ \text{with} & x_1 \ge 0, x_2 \ge 0, x_3 \ge 0, x_4 \ge 0. \end{array}$$

Write the initial tableau.

$$\begin{array}{rrrrrrr|r} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & z & \\ 1 & 0 & 100{,}000 & 1 & 1 & 0 & 0 & 36{,}000 \\ 0 & 1 & \boxed{300,000} & 1 & 0 & 1 & 0 & 48{,}000 \\ \hline -12 & -10 & -7,000,000 & -42 & 0 & 0 & 1 & 0 \end{array}$$

Pivot on the 300,000 in row 2, column 3.

$$\begin{array}{l|rrrrrrr|r} & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & z & \\ -R_2 + 3R_1 \to R_1 & 3 & -1 & 0 & \boxed{2} & 3 & -1 & 0 & 60{,}000 \\ & 0 & 1 & 300,000 & 1 & 0 & 1 & 0 & 48{,}000 \\ \hline 70R_2 + 3R_3 \to R_3 & -36 & 40 & 0 & -56 & 0 & 70 & 3 & 3{,}360{,}000 \end{array}$$

Pivot on the 2 in row 1, column 4.

$$\begin{array}{l|rrrrrrr|r} & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & z & \\ & 3 & -1 & 0 & 2 & 3 & -1 & 0 & 60{,}000 \\ -R_1 + 2R_2 \to R_2 & -3 & 3 & 600,000 & 0 & -3 & 3 & 0 & 36{,}000 \\ \hline 28R_1 + R_3 \to R_3 & 48 & 12 & 0 & 0 & 84 & 42 & 3 & 5{,}040{,}000 \end{array}$$

Create a 1 in the columns corresponding to x_3, x_4, and z.

$$\begin{array}{l|rrrrrrr|r} & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & z & \\ \frac{1}{2}R_1 \to R_1 & \frac{3}{2} & -\frac{1}{2} & 0 & 1 & \frac{3}{2} & -\frac{1}{2} & 0 & 30{,}000 \\ \frac{1}{600,000}R_2 \to R_2 & -\frac{1}{200,000} & \frac{1}{200,000} & 1 & 0 & -\frac{1}{200,000} & \frac{1}{200,000} & 0 & \frac{3}{50} \\ \hline \frac{1}{3}R_3 \to R_3 & 16 & 4 & 0 & 0 & 28 & 14 & 1 & 1{,}680{,}000 \end{array}$$

The minimum value of w is 1,680,000 when $y_1 = 28$ and $y_2 = 14$. Therefore, 28 units of regular beer and 14 units of light beer should be made for a minimum cost of $1,680,000.

(b) The shadow cost for revenue is $.06. An increase in $500,000 in revenue will increase costs to

$$\$1,680,000 + .06(500,000) = \$1,710,000.$$

20. Let $y_1 =$ the number of political interviews conducted
and $y_2 =$ the number of market interviews conducted.

The problem is:

$$\begin{array}{ll} \text{Minimize} & w = 45y_1 + 55y_2 \\ \text{subject to:} & y_1 + y_2 \geq 8 \\ & 8y_1 + 10y_2 \geq 60 \\ & 6y_1 + 5y_2 \geq 40 \\ \text{with} & y_1 \geq 0, y_2 \geq 0. \end{array}$$

Write the augmented matrix.

$$\left[\begin{array}{cc|c} 1 & 1 & 8 \\ 8 & 10 & 60 \\ 6 & 5 & 40 \\ \hline 45 & 55 & 0 \end{array}\right]$$

Transpose to get the matrix for the dual problem.

$$\left[\begin{array}{ccc|c} 1 & 8 & 6 & 45 \\ 1 & 10 & 5 & 55 \\ \hline 8 & 60 & 40 & 0 \end{array}\right]$$

Write the dual problem:

$$\begin{array}{ll} \text{Maximize} & z = 8x_1 + 60x_2 + 40x_3 \\ \text{subject to:} & x_1 + 8x_2 + 6x_3 \leq 45 \\ & x_1 + 10x_2 + 5x_3 \leq 55 \\ \text{with} & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0. \end{array}$$

Write the initial tableau.

$$\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ 1 & 8 & 6 & 1 & 0 & 0 & 45 \\ 1 & \boxed{10} & 5 & 0 & 1 & 0 & 55 \\ \hline -8 & -60 & -40 & 0 & 0 & 1 & 0 \end{array}$$

Pivot on the 10 in row 2, column 2.

$$\begin{array}{r|cccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ -4\text{R}_2+5\text{R}_1 \to \text{R}_1 & 1 & 0 & \boxed{10} & 5 & -4 & 0 & 5 \\ & 1 & 10 & 5 & 0 & 1 & 0 & 55 \\ \hline 6\text{R}_2+\text{R}_3 \to \text{R}_3 & -2 & 0 & -10 & 0 & 6 & 1 & 330 \end{array}$$

Pivot on the 10 in row 1, column 3.

$$\begin{array}{r|cccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ & \boxed{1} & 0 & 10 & 5 & -4 & 0 & 5 \\ -\text{R}_1+2\text{R}_2 \to \text{R}_2 & 1 & 20 & 0 & -5 & 6 & 0 & 105 \\ \hline \text{R}_1+\text{R}_3 \to \text{R}_3 & -1 & 0 & 0 & 5 & 2 & 1 & 335 \end{array}$$

Pivot on the 1 in row 1, column 1.

$$\begin{array}{r|cccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ & 1 & 0 & 10 & 5 & -4 & 0 & 5 \\ -\text{R}_1+\text{R}_2 \to \text{R}_2 & 0 & 20 & -10 & -10 & \boxed{10} & 0 & 100 \\ \hline \text{R}_1+\text{R}_3 \to \text{R}_3 & 0 & 0 & 10 & 10 & -2 & 1 & 340 \end{array}$$

Pivot on the 10 in row 2, column 5.

$$\begin{array}{r|cccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ 2\text{R}_2 + 5\text{R}_1 \to \text{R}_1 & 5 & 40 & 30 & 5 & 0 & 0 & 225 \\ & 0 & 20 & -10 & -10 & 10 & 0 & 100 \\ \hline \text{R}_2 + 5\text{R}_3 \to \text{R}_3 & 0 & 20 & 40 & 40 & 0 & 5 & 1800 \end{array}$$

Create a 1 in the columns corresponding to $x_1, s_2,$ and z.

$$\begin{array}{r|cccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ \frac{1}{5}\text{R}_1 \to \text{R}_1 & 1 & 8 & 6 & 1 & 0 & 0 & 45 \\ \frac{1}{10}\text{R}_2 \to \text{R}_2 & 0 & 2 & -1 & -1 & 1 & 0 & 10 \\ \hline \frac{1}{5}\text{R}_3 \to \text{R}_3 & 0 & 4 & 8 & 8 & 0 & 1 & 360 \end{array}$$

The minimum time spent is 360 min when $y_1 = 8$ and $y_2 = 0$, that is, when 8 political interviews and no market interviews are done.

22. Organize the information in a table.

	Units of Nutrient A (per bag)	Units of Nutrient B (per bag)	Cost (per bag)
Feed 1	1	2	\$3
Feed 2	3	1	\$2
Minimum	7	4	

Let $y_1 =$ the number of bags of feed 1
and $y_2 =$ the number of bags of feed 2.

(a) We want the cost to equal \$7 for 7 units of A and 4 units of B exactly. Therefore, use a system of equations rather than a system of inequalities.

$$\begin{aligned} 3y_1 + 2y_2 &= 7 \\ y_1 + 3y_2 &= 7 \\ 2y_1 + y_2 &= 4 \end{aligned}$$

Use Gauss-Jordan elimination to solve this system of equations.

$$\left[\begin{array}{cc|c} 3 & 2 & 7 \\ 1 & 3 & 7 \\ 2 & 1 & 4 \end{array}\right]$$

$$\begin{array}{r} \\ -\mathrm{R}_1+3\mathrm{R}_2\to\mathrm{R}_2 \\ -2\mathrm{R}_1+3\mathrm{R}_3\to\mathrm{R}_3 \end{array} \left[\begin{array}{cc|c} 3 & 2 & 7 \\ 0 & 7 & 14 \\ 0 & -1 & -2 \end{array}\right]$$

$$\begin{array}{r} -2\mathrm{R}_2+7\mathrm{R}_1\to\mathrm{R}_1 \\ \\ \mathrm{R}_2+7\mathrm{R}_3\to\mathrm{R}_3 \end{array} \left[\begin{array}{cc|c} 21 & 0 & 21 \\ 0 & 7 & 14 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{r} \frac{1}{21}\mathrm{R}_1\to\mathrm{R}_1 \\ \frac{1}{7}\mathrm{R}_2\to\mathrm{R}_2 \\ \\ \end{array} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

Thus, $y_1 = 1$ and $y_2 = 2$, so use 1 bag of feed 1 and 2 bags of feed 2. The cost will be $3(1) + 2(2) = \$7$ as desired. The number of units of A is $1(1) + 3(2) = 7$, and the number of units of B is $2(1) + 1(2) = 4$.

(b)

	Units of Nutrient A (per bag)	Units of Nutrient B (per bag)	Cost (per bag)
Feed 1	1	2	\$3
Feed 2	3	1	\$2
Minimum	5	4	

The problem is:

$$\begin{array}{ll} \text{Minimize} & w = 3y_1 + 2y_2 \\ \text{subject to:} & y_1 + 3y_2 \geq 5 \\ & 2y_1 + y_2 \geq 4 \\ \text{with} & y_1 \geq 0, y_2 \geq 0. \end{array}$$

The dual problem is as follows.

$$\begin{array}{ll} \text{Maximize} & z = 5x_1 + 4x_2 \\ \text{subject to:} & x_1 + 2x_2 \leq 3 \\ & 3x_1 + x_2 \leq 2 \\ \text{with} & x_1 \geq 0, x_2 \geq 0. \end{array}$$

The initial tableau is as follows.

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 0 & 3 \\ \boxed{3} & 1 & 0 & 1 & 0 & 2 \\ \hline -5 & -4 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Pivot as indicated.

$$\begin{array}{rc} & \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \begin{array}{r} -\mathrm{R}_2+3\mathrm{R}_1\to\mathrm{R}_1 \\ \\ 5\mathrm{R}_2+3\mathrm{R}_3\to\mathrm{R}_3 \end{array} & \left[\begin{array}{ccccc|c} 0 & \boxed{5} & 3 & -1 & 0 & 7 \\ 3 & 1 & 0 & 1 & 0 & 2 \\ \hline 0 & -7 & 0 & 5 & 3 & 10 \end{array}\right] \end{array}$$

$$\begin{array}{rc} & \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \begin{array}{r} \\ -\mathrm{R}_1+5\mathrm{R}_2\to\mathrm{R}_2 \\ 7\mathrm{R}_1+5\mathrm{R}_3\to\mathrm{R}_3 \end{array} & \left[\begin{array}{ccccc|c} 0 & 5 & 3 & -1 & 0 & 7 \\ 15 & 0 & -3 & 6 & 0 & 3 \\ \hline 0 & 0 & 21 & 18 & 15 & 99 \end{array}\right] \end{array}$$

Create a 1 in the columns corresponding to x_1, x_2, and z.

$$\begin{array}{rc} & \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \begin{array}{r} \frac{1}{5}\mathrm{R}_1\to\mathrm{R}_1 \\ \frac{1}{15}\mathrm{R}_2\to\mathrm{R}_2 \\ \frac{1}{15}\mathrm{R}_3\to\mathrm{R}_3 \end{array} & \left[\begin{array}{ccccc|c} 0 & 1 & \frac{3}{5} & -\frac{1}{5} & 0 & \frac{7}{5} \\ 1 & 0 & -\frac{1}{5} & \frac{2}{5} & 0 & \frac{1}{5} \\ \hline 0 & 0 & \frac{7}{5} & \frac{6}{5} & 1 & \frac{33}{5} \end{array}\right] \end{array}$$

Reading from the final column of the final tableau, $x_2 = \$1.40$ is the cost of nutrient B and $x_1 = \$.20$ is the cost of nutrient A. With 5 units of A and 4 units of B, this gives a minimum cost of

$$5(\$.20) + 4(\$1.40) = \$6.60$$

as given in the lower right corner. 1.4 (or $\frac{7}{5}$) bags of feed 1 and 1.2 (or $\frac{6}{5}$) bags of feed 2 should be used.

24. Let $y_1 =$ the number of #1 pills
and $y_2 =$ the number of #2 pills.

Organize the given information in a table.

	Vitamin A	Vitamin B_1	Vitamin C	Cost
#1	8	1	2	\$.10
#2	2	1	7	\$.20
Total Needed	16	5	20	

The problem is:

$$\begin{array}{ll} \text{Minimize} & w = .1y_1 + .2y_2 \\ \text{subject to:} & 8y_1 + 2y_2 \geq 16 \\ & y_1 + y_2 \geq 5 \\ & 2y_1 + 7y_2 \geq 20 \\ \text{with} & y_1 \geq 0, y_2 \geq 0\ . \end{array}$$

The dual problem is as follows.

$$\begin{array}{ll} \text{Maximize} & z = 16x_1 + 5x_2 + 20x_3 \\ \text{subject to:} & 8x_1 + x_2 + 2x_3 \leq .1 \\ & 2x_1 + x_2 + 7x_3 \leq .2 \\ \text{with} & x_1 \geq 0, x_2 \geq 0. \end{array}$$

The initial tableau is as follows.

$$\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ 8 & 1 & 2 & 1 & 0 & 0 & .1 \\ 2 & 1 & \boxed{7} & 0 & 1 & 0 & .2 \\ \hline -16 & -5 & -20 & 0 & 0 & 1 & 0 \end{array}$$

Pivot as indicated.

$$\begin{array}{c|cccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ -2R_2 + \ 7R_1 \rightarrow R_1 & \boxed{52} & 5 & 0 & 7 & -2 & 0 & .3 \\ & 2 & 1 & 7 & 0 & 1 & 0 & .2 \\ \hline 20R_2 + 7R_3 \rightarrow R_3 & -72 & -15 & 0 & 0 & 20 & 7 & 4 \end{array}$$

$$\begin{array}{c|cccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ & 52 & \boxed{5} & 0 & 7 & -2 & 0 & .3 \\ -R_1 + 26R_2 \rightarrow R_2 & 0 & 21 & 182 & -7 & 28 & 0 & 4.9 \\ \hline 18R_1 + 13R_3 \rightarrow R_3 & 0 & -105 & 0 & 126 & 224 & 91 & 57.4 \end{array}$$

$$\begin{array}{c|cccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ & 52 & 5 & 0 & 7 & -2 & 0 & .3 \\ -21R_1 + 5R_2 \rightarrow R_2 & -1092 & 0 & 910 & -182 & 182 & 0 & 18.2 \\ \hline 21R_1 + \ R_3 \rightarrow R_3 & 1092 & 0 & 0 & 273 & 182 & 91 & 63.7 \end{array}$$

Create a 1 in the columns corresponding to x_2, x_3, and z.

$$\begin{array}{c|cccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ \frac{1}{5}R_1 \rightarrow R_1 & \frac{52}{5} & 1 & 0 & \frac{7}{5} & -\frac{2}{5} & 0 & .06 \\ \frac{1}{910}R_2 \rightarrow R_2 & -\frac{6}{5} & 0 & 1 & -\frac{1}{5} & \frac{1}{5} & 0 & .02 \\ \hline \frac{1}{91}R_3 \rightarrow R_3 & 12 & 0 & 0 & 3 & 2 & 1 & .7 \end{array}$$

From the last row, the minimum value is .7 when $y_1 = 3$ and $y_2 = 2$. Mark should buy 3 of pill #1 and 2 of pill #2 for a minimum cost of 70¢.

26. Let $y_1 =$ the number of units of ingredient I;
$y_2 =$ the number of units of ingredient II;
and $y_3 =$ the number of units of ingredient III.

The problem is:

$$\begin{array}{ll} \text{Minimize} & w = 4y_1 + 7y_2 + 5y_3 \\ \text{subject to:} & 4y_1 + \ \ y_2 + 10y_3 \geq 10 \\ & 3y_1 + 2y_2 + \ \ \ y_3 \geq 12 \\ & \qquad 4y_2 + \ 5y_3 \geq 20 \\ \text{with} & y_1 \geq 0, y_2 \geq 0, y_3 \geq 0\ . \end{array}$$

The dual problem is as follows.

$$\begin{array}{ll} \text{Maximize} & z = 10x_1 + 12x_2 + 20x_3 \\ \text{subject to:} & \quad 4x_1 + 3x_2 \qquad\quad \leq 4 \\ & \quad\ x_1 + 2x_2 + 4x_3 \leq 7 \\ & \ 10x_1 + \ \ x_2 + 5x_3 \leq 5 \\ \text{with} & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0. \end{array}$$

The initial tableau is as follows.

$$\left[\begin{array}{ccccccc|c}
x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\
4 & 3 & 0 & 1 & 0 & 0 & 0 & 4 \\
1 & 2 & 4 & 0 & 1 & 0 & 0 & 7 \\
10 & 1 & \boxed{5} & 0 & 0 & 1 & 0 & 5 \\
\hline
-10 & -12 & -20 & 0 & 0 & 0 & 1 & 0
\end{array}\right]$$

Pivot as indicated.

$$\begin{array}{r|ccccccc|c|}
 & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\
 & 4 & \boxed{3} & 0 & 1 & 0 & 0 & 0 & 4 \\
-4\mathrm{R}_3+5\mathrm{R}_2 \to \mathrm{R}_2 & -35 & 6 & 0 & 0 & 5 & -4 & 0 & 15 \\
 & 10 & 1 & 5 & 0 & 0 & 1 & 0 & 5 \\
\hline
4\mathrm{R}_3+\mathrm{R}_4 \to \mathrm{R}_4 & 30 & -8 & 0 & 0 & 0 & 4 & 1 & 20
\end{array}$$

$$\begin{array}{r|ccccccc|c|}
 & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\
 & 4 & 3 & 0 & 1 & 0 & 0 & 0 & 4 \\
-2\mathrm{R}_1+\ \mathrm{R}_2 \to \mathrm{R}_2 & -43 & 0 & 0 & -2 & 5 & -4 & 0 & 7 \\
-\mathrm{R}_1+3\mathrm{R}_3 \to \mathrm{R}_3 & 26 & 0 & 15 & -1 & 0 & 3 & 0 & 11 \\
\hline
8\mathrm{R}_1+3\mathrm{R}_4 \to \mathrm{R}_4 & 122 & 0 & 0 & 8 & 0 & 12 & 3 & 92
\end{array}$$

Create a 1 in the columns corresponding to x_2, x_3, and z.

$$\begin{array}{r|ccccccc|c|}
 & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\
\frac{1}{3}\mathrm{R}_1 \to \mathrm{R}_1 & \frac{4}{3} & 1 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{4}{3} \\
 & -43 & 0 & 0 & -2 & 5 & -4 & 0 & 7 \\
\frac{1}{15}\mathrm{R}_3 \to \mathrm{R}_3 & \frac{26}{15} & 0 & 1 & -\frac{1}{15} & 0 & \frac{1}{5} & 0 & \frac{11}{5} \\
\hline
\frac{1}{3}\mathrm{R}_4 \to \mathrm{R}_4 & \frac{122}{3} & 0 & 0 & \frac{8}{3} & 0 & 4 & 1 & \frac{92}{3}
\end{array}$$

From the last row, the minimum value is $\frac{92}{3}$ when $y_1 = \frac{8}{3}, y_2 = 0$, and $y_3 = 4$. The biologist can meet his needs at a minimum cost of \$30.67 by using $\frac{8}{3}$ units of ingredient I and 4 units of ingredient III. (Ingredient II should not be used at all.)

4.4 Nonstandard Problems

2. $5x_1 + 8x_2 \leq 10$
$6x_1 + 2x_2 \geq \ \ 7$

We need one slack variable, s_1, and one surplus variable, s_2. The system becomes

$$\begin{aligned}
5x_1 + 8x_2 + s_1 \qquad &= 10 \\
6x_1 + 2x_2 \qquad - s_2 &= \ \ 7.
\end{aligned}$$

4. $$\begin{aligned} 2x_1 \quad &+ x_3 \le 40\\ x_1 + x_2 \quad &\ge 18\\ x_1 \quad &+ x_3 \ge 20 \end{aligned}$$

We need one slack variable, s_1, and two surplus variables, s_2 and s_3.
The system becomes

$$\begin{aligned} 2x_1 \quad + x_3 + s_1 \quad &= 40\\ x_1 + x_2 \quad - s_2 \quad &= 18\\ x_1 \quad + x_3 \quad - s_3 &= 20. \end{aligned}$$

6. Minimize $w = 8y_1 + 3y_2 + y_3$
subject to: $7y_1 + 6y_2 + 8y_3 \ge 18$
$4y_1 + 5y_2 + 10y_3 \ge 20$
with $y_1 \ge 0, y_2 \ge 0, y_3 \ge 0.$

To minimize $w = 8y_1 + 3y_2 + y_3$,
we maximize $z = -w = -8y_1 - 3y_2 - y_3$.
The constraints are not changed.

8. Minimize $w = y_1 + y_2 + 4y_3$
subject to: $y_1 + 2y_2 + 3y_3 \ge 115$
$2y_1 + y_2 + y_3 \le 200$
$y_1 + y_3 \ge 50$
with $y_1 \ge 0, y_2 \ge 0, y_3 \ge 0.$

To minimize $w = y_1 + y_2 + 4y_3$,
we maximize $z = -w = -y_1 - y_2 - 4y_3$.
The constraints are not changed.

10. Find $x_1 \ge 0$ and $x_2 \ge 0$ such that

$$\begin{aligned} 3x_1 + 4x_2 &\ge 48\\ 2x_1 + 4x_2 &\le 60 \end{aligned}$$

and $z = 6x_1 + 8x_2$ is maximized.

Introducing one surplus variable and one slack variable, the system becomes

$$\begin{aligned} 3x_1 + 4x_2 - s_1 \quad &= 48\\ 2x_1 + 4x_2 \quad + s_2 &= 60. \end{aligned}$$

The initial simplex tableau is

$$\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ \boxed{3} & 4 & -1 & 0 & 0 & 48\\ 2 & 4 & 0 & 1 & 0 & 60\\ \hline -6 & -8 & 0 & 0 & 1 & 0 \end{array}.$$

The initial basic solution is not feasible since $s_1 = -48$. Pivot on the 3 in row 1, column 1.

$$\begin{array}{r|ccccc|c} & x_1 & x_2 & s_1 & s_2 & z & \\ & 3 & 4 & -1 & 0 & 0 & 48\\ -2R_1 + 3R_2 \to R_2 & 0 & 4 & \boxed{2} & 3 & 0 & 84\\ \hline 2R_1 + R_3 \to R_3 & 0 & 0 & -2 & 0 & 1 & 96 \end{array}$$

Pivot on the 2 in row 2, column 3.

$$\begin{array}{r|ccccc|c} & x_1 & x_2 & s_1 & s_2 & z & \\ R_2 + 2R_1 \to R_1 & 6 & 12 & 0 & 3 & 0 & 180\\ & 0 & 4 & 2 & 3 & 0 & 84\\ \hline R_2 + R_3 \to R_3 & 0 & 4 & 0 & 3 & 1 & 180 \end{array}$$

$$\begin{array}{r|ccccc|c} & x_1 & x_2 & s_1 & s_2 & z & \\ \frac{1}{6}R_1 \to R_1 & 1 & 2 & 0 & \frac{1}{2} & 0 & 30\\ \frac{1}{2}R_2 \to R_2 & 0 & 2 & 1 & \frac{3}{2} & 0 & 42\\ \hline & 0 & 4 & 0 & 3 & 1 & 180 \end{array}$$

The maximum is 180 when $x_1 = 30$ and $x_2 = 0$.

12. Find $x_1 \ge 0, x_2 \ge 0$, and $x_3 \ge 0$ such that

$$\begin{aligned} x_1 + x_2 + 2x_3 &\le 38\\ 2x_1 + x_2 + x_3 &\ge 24 \end{aligned}$$

and $z = 3x_1 + 2x_2 + 2x_3$ is maximized.

The initial simplex tableau is

$$\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ 1 & 1 & 2 & 1 & 0 & 0 & 38\\ \boxed{2} & 1 & 1 & 0 & -1 & 0 & 24\\ \hline -3 & -2 & -2 & 0 & 0 & 1 & 0 \end{array}.$$

The initial basic solution is not feasible since $s_2 = -24$. Pivot on the 2 in row 2, column 1.

$$\begin{array}{r|cccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ -R_2 + 2R_1 \to R_1 & 0 & 1 & 3 & 2 & \boxed{1} & 0 & 52\\ & 2 & 1 & 1 & 0 & -1 & 0 & 24\\ \hline 3R_2 + 2R_3 \to R_3 & 0 & -1 & -1 & 0 & -3 & 2 & 72 \end{array}$$

Pivot on the 1 in row 1, column 5.

$$\begin{array}{r|cccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ & 0 & 1 & 3 & 2 & 1 & 0 & 52\\ R_1 + R_2 \to R_2 & 2 & 2 & 4 & 2 & 0 & 0 & 76\\ \hline 3R_1 + R_3 \to R_3 & 0 & 2 & 8 & 6 & 0 & 2 & 228 \end{array}$$

$$\begin{array}{r|cccccc|c}
 & x_1 & x_2 & x_3 & s_1 & s_2 & z & \\
 & 0 & 1 & 3 & 2 & 1 & 0 & 52 \\
\frac{1}{2}R_2 \to R_2 & 1 & 1 & 2 & 1 & 0 & 0 & 38 \\
\hline
\frac{1}{2}R_3 \to R_3 & 0 & 1 & 4 & 3 & 0 & 1 & 114
\end{array}$$

The maximum is 114 when $x_1 = 38, x_2 = 0$, and $x_3 = 0$.

14. Find $x_1 \geq 0$ and $x_2 \geq 0$ such that

$$\begin{aligned} x_1 + 2x_2 &\leq 18 \\ x_1 + 3x_2 &\geq 12 \\ 2x_1 + 2x_2 &\leq 24 \end{aligned}$$

and $z = 5x_1 - 10x_2$ is maximized.
Introduce slack and surplus variables to get the system

$$\begin{aligned} x_1 + 2x_2 + s_1 \qquad\qquad &= 18 \\ x_1 + 3x_2 \qquad - s_2 \qquad &= 12 \\ 2x_1 + 2x_2 \qquad\qquad + s_3 &= 24. \end{aligned}$$

The initial tableau is

$$\begin{array}{cccccc|c}
x_1 & x_2 & s_1 & s_2 & s_3 & z & \\
1 & 2 & 1 & 0 & 0 & 0 & 18 \\
\boxed{1} & 3 & 0 & -1 & 0 & 0 & 12 \\
2 & 2 & 0 & 0 & 1 & 0 & 24 \\
\hline
-5 & 10 & 0 & 0 & 0 & 1 & 0
\end{array}.$$

$s_2 = -12$ is not a feasible solution. Pivot on the 1 in row 1, column 1.

$$\begin{array}{r|cccccc|c}
 & x_1 & x_2 & s_1 & s_2 & s_3 & z & \\
-R_2 + R_1 \to R_1 & 0 & -1 & 1 & 1 & 0 & 0 & 6 \\
 & 1 & 3 & 0 & -1 & 0 & 0 & 12 \\
-2R_2 + R_3 \to R_3 & 0 & -4 & 0 & \boxed{2} & 1 & 0 & 0 \\
\hline
5R_2 + R_4 \to R_4 & 0 & 25 & 0 & -5 & 0 & 1 & 60
\end{array}$$

Pivot on the 2 in row 3, column 4.

$$\begin{array}{r|cccccc|c}
 & x_1 & x_2 & s_1 & s_2 & s_3 & z & \\
-R_3 + 2R_1 \to R_1 & 0 & 6 & 2 & 0 & -1 & 0 & 12 \\
R_3 + 2R_2 \to R_2 & 2 & 2 & 0 & 0 & 1 & 0 & 24 \\
 & 0 & -4 & 0 & 2 & 1 & 0 & 0 \\
\hline
5R_3 + 2R_4 \to R_4 & 0 & 30 & 0 & 0 & 5 & 2 & 120
\end{array}$$

$$\begin{array}{r|cccccc|c}
 & x_1 & x_2 & s_1 & s_2 & s_3 & z & \\
\frac{1}{2}R_1 \to R_1 & 0 & 3 & 1 & 0 & -\frac{1}{2} & 0 & 6 \\
\frac{1}{2}R_2 \to R_2 & 1 & 1 & 0 & 0 & \frac{1}{2} & 0 & 12 \\
\frac{1}{2}R_3 \to R_3 & 0 & -2 & 0 & 1 & \frac{1}{2} & 0 & 0 \\
\hline
\frac{1}{2}R_4 \to R_4 & 0 & 15 & 0 & 0 & \frac{5}{2} & 1 & 60
\end{array}$$

The maximum is 60 when $x_1 = 12$ and $x_2 = 0$.

16. Minimize $\quad w = 3y_1 + 2y_2 + 3y_3$
subject to: $\quad 2y_1 + 3y_2 + 6y_3 \leq 60$
$\qquad y_1 + 4y_2 + 5y_3 \geq 40$
with $\quad y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$

Let $z = -w = -3y_1 - 2y_2 - 3y_3$. Maximize z.

The initial simplex tableau is

$$\begin{array}{cccccc|c}
y_1 & y_2 & y_3 & s_1 & s_2 & z & \\
\boxed{2} & 3 & 6 & 1 & 0 & 0 & 60 \\
1 & 4 & 5 & 0 & -1 & 0 & 40 \\
\hline
3 & 2 & 3 & 0 & 0 & 1 & 0
\end{array}$$

The initial basic solution is not feasible since $s_2 = -40$. Pivot on the 2 in row 1, column 1.

$$\begin{array}{r|cccccc|c}
 & y_1 & y_2 & y_3 & s_1 & s_2 & z & \\
 & 2 & 3 & 6 & 1 & 0 & 0 & 60 \\
-R_1 + 2R_2 \to R_2 & 0 & 5 & \boxed{4} & -1 & -2 & 0 & 20 \\
\hline
-3R_1 + 2R_3 \to R_3 & 0 & -5 & -12 & -3 & 0 & 2 & -180
\end{array}$$

There are negative indicators, so now pivot on the 4 in row 2, column 3.

$$\begin{array}{r|cccccc|c}
 & y_1 & y_2 & y_3 & s_1 & s_2 & z & \\
-3R_2 + 2R_1 \to R_1 & 4 & -9 & 0 & \boxed{5} & 6 & 0 & 60 \\
 & 0 & 5 & 4 & -1 & -2 & 0 & 20 \\
\hline
3R_2 + \ R_3 \to R_3 & 0 & 10 & 0 & -6 & -6 & 2 & -120
\end{array}$$

Pivot on the 5 in row 1, column 4.

$$\begin{array}{r|cccccc|c}
 & y_1 & y_2 & y_3 & s_1 & s_2 & z & \\
 & 4 & -9 & 0 & 5 & 6 & 0 & 60 \\
R_1 + 5R_2 \to R_2 & 4 & \boxed{16} & 20 & 0 & -4 & 0 & 160 \\
\hline
6R_1 + 5R_3 \to R_3 & 24 & -4 & 0 & 0 & 6 & 10 & -240
\end{array}$$

Pivot on the 16 in row 2, column 2.

$$\begin{array}{r|cccccc|c}
 & y_1 & y_2 & y_3 & s_1 & s_2 & z & \\
9R_2 + 16R_1 \to R_1 & \boxed{100} & 0 & 180 & 80 & 60 & 0 & 2400 \\
 & 4 & 16 & 20 & 0 & -4 & 0 & 160 \\
\hline
R_2 + \ 4R_3 \to R_3 & 100 & 0 & 20 & 0 & 20 & 40 & -800
\end{array}$$

$$\begin{array}{r|cccccc|c}
 & y_1 & y_2 & y_3 & s_1 & s_2 & z & \\
\frac{1}{80}R_1 \to R_1 & \frac{5}{4} & 0 & \frac{9}{4} & 1 & \frac{3}{4} & 0 & 30 \\
\frac{1}{16}R_2 \to R_2 & \frac{1}{4} & 1 & \frac{5}{4} & 0 & -\frac{1}{4} & 0 & 10 \\
\hline
\frac{1}{40}R_3 \to R_3 & \frac{5}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 1 & -20
\end{array}$$

The minimum is $w = 20$ when $y_1 = 0$, $y_2 = 10$, and $y_3 = 0$.

18. Maximize $z = 10x_1 + 9x_2$
subject to: $x_1 + x_2 = 30$
$x_1 + x_2 \geq 25$
$2x_1 + x_2 \leq 40$
with $x_1 \geq 0, x_2 \geq 0.$

With artificial, slack, and surplus variables, we have

$$\begin{aligned} x_1 + x_2 + a \qquad\qquad &= 30 \\ x_1 + x_2 \qquad - s_1 \qquad &= 25 \\ 2x_1 + x_2 \qquad\qquad + s_2 &= 40. \end{aligned}$$

($Note$: Since $x_1+x_2 = 30$, it is clear that $s_1 = 5$.)

The initial tableau is

$$\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & a & z & \\ 1 & 1 & 0 & 0 & 1 & 0 & 30 \\ 1 & 1 & -1 & 0 & 0 & 0 & 25 \\ \boxed{2} & 1 & 0 & 1 & 0 & 0 & 40 \\ \hline -10 & -9 & 0 & 0 & 0 & 1 & 0 \end{array}.$$

Pivot on the 2 in row 3, column 1.

$$\begin{array}{r|cccccc|c} & x_1 & x_2 & s_1 & s_2 & a & z & \\ -R_3 + 2R_1 \to R_1 & 0 & 1 & 0 & -1 & 2 & 0 & 20 \\ -R_3 + 2R_2 \to R_2 & 0 & \boxed{1} & -2 & -1 & 0 & 0 & 10 \\ & 2 & 1 & 0 & 1 & 0 & 0 & 40 \\ \hline 5R_3 + R_4 \to R_4 & 0 & -4 & 0 & 5 & 0 & 1 & 200 \end{array}$$

Pivot on the 1 in row 2, column 2.

$$\begin{array}{r|cccccc|c} & x_1 & x_2 & s_1 & s_2 & a & z & \\ -R_2 + R_1 \to R_1 & 0 & 0 & \boxed{2} & 0 & 2 & 0 & 10 \\ & 0 & 1 & -2 & -1 & 0 & 0 & 10 \\ -R_2 + R_3 \to R_3 & 2 & 0 & 2 & 2 & 0 & 0 & 30 \\ \hline 4R_2 + R_4 \to R_4 & 0 & 0 & -8 & 1 & 0 & 1 & 240 \end{array}$$

Pivot on the 2 in row 1, column 3.

$$\begin{array}{r|cccccc|c} & x_1 & x_2 & s_1 & s_2 & a & z & \\ & 0 & 0 & 2 & 0 & 2 & 0 & 10 \\ R_1 + R_2 \to R_2 & 0 & 1 & 0 & -1 & 2 & 0 & 20 \\ -R_1 + R_3 \to R_3 & 2 & 0 & 0 & 2 & -2 & 0 & 20 \\ \hline 4R_1 + R_4 \to R_4 & 0 & 0 & 0 & 1 & 8 & 1 & 280 \end{array}$$

$$\begin{array}{r|cccccc|c} & x_1 & x_2 & s_1 & s_2 & a & z & \\ \frac{1}{2}R_1 \to R_1 & 0 & 0 & 1 & 0 & 1 & 0 & 5 \\ & 0 & 1 & 0 & -1 & 2 & 0 & 20 \\ \frac{1}{2}R_3 \to R_3 & 1 & 0 & 0 & 1 & -1 & 0 & 10 \\ \hline & 0 & 0 & 0 & 1 & 8 & 1 & 280 \end{array}$$

The maximum is 280 when $x_1 = 10$ and $x_2 = 20$.

20. Minimize $w = 15y_1 + 12y_2 + 18y_3$
subject to: $y_1 + 2y_2 + 3y_3 \leq 12$
$3y_1 + y_2 + 3y_3 \geq 18$
$y_1 + y_2 + y_3 = 10$
with $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$

Let $z = -w = -15y_1 - 12y_2 - 18y_3$ and maximize z. Introduce the slack variable s_1, the surplus variable s_2, and the artificial variable a_1. The initial tableau is as follows.

$$\begin{array}{ccccccc|c} y_1 & y_2 & y_3 & s_1 & s_2 & a_1 & z & \\ 1 & 2 & 3 & 1 & 0 & 0 & 0 & 12 \\ 3 & 1 & 3 & 0 & -1 & 0 & 0 & 18 \\ \boxed{1} & 1 & 1 & 0 & 0 & 1 & 0 & 10 \\ \hline 15 & 12 & 18 & 0 & 0 & 0 & 1 & 0 \end{array}$$

First, eliminate the artifical variable a_1. Pivot on the 1 in row 3, column 1.

$$\begin{array}{r|ccccccc|c} & y_1 & y_2 & y_3 & s_1 & s_2 & a_1 & z & \\ -R_3+R_1 \to R_1 & 0 & 1 & 2 & 1 & 0 & -1 & 0 & 2 \\ -3R_3+R_2 \to R_2 & 0 & -2 & 0 & 0 & -1 & -3 & 0 & -12 \\ & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 10 \\ \hline -15R_3+R_4 \to R_4 & 0 & -3 & 3 & 0 & 0 & -15 & 1 & -150 \end{array}$$

Now $a_1 = 0$, so we can drop the a_1 column.

$$\begin{array}{cccccc|c} y_1 & y_2 & y_3 & s_1 & s_2 & z & \\ 0 & \boxed{1} & 2 & 1 & 0 & 0 & 2 \\ 0 & -2 & 0 & 0 & -1 & 0 & -12 \\ 1 & 1 & 1 & 0 & 0 & 0 & 10 \\ \hline 0 & -3 & 3 & 0 & 0 & 1 & -150 \end{array}$$

Pivot on the 1 in row 1, column 2.

$$\begin{array}{r|cccccc|c} & y_1 & y_2 & y_3 & s_1 & s_2 & z & \\ & 0 & 1 & 2 & 1 & 0 & 0 & 2 \\ 2R_1 + R_2 \to R_2 & 0 & 0 & 4 & 2 & -1 & 0 & -8 \\ -R_1 + R_3 \to R_3 & 1 & 0 & -1 & -1 & 0 & 0 & 8 \\ \hline 3R_1 + R_4 \to R_4 & 0 & 0 & 9 & 3 & 0 & 1 & -144 \end{array}$$

The maximum value of $z = -w$ is -144. Therefore, the minimum value of w is 144 when $y_1 = 8, y_2 = 2$, and $y_3 = 0$.

24. Let $y_1 =$ amount shipped from S_1 to D_1,
$y_2 =$ amount shipped from S_1 to D_2,
$y_3 =$ amount shipped from S_2 to D_1,
and $y_4 =$ amount shipped from S_2 to D_2.

$$\begin{array}{ll}
\text{Minimize} & w = 30y_1 + 20y_2 + 25y_3 + 22y_4 \\
\text{subject to:} & y_1 \phantom{{}+y_2} + y_3 \phantom{{}+y_4} \geq 3000 \\
 & \phantom{y_1+{}} y_2 \phantom{{}+y_3} + y_4 \geq 5000 \\
 & y_1 + y_2 \phantom{{}+y_3+y_4} = 5000 \\
 & \phantom{y_1+y_2+{}} y_3 + y_4 = 5000 \\
 & 2y_1 + 6y_2 + 5y_3 + 4y_4 \leq 40,000 \\
\text{with} & y_1 \geq 0, y_2 \geq 0, y_3 \geq 0, y_4 \geq 0.
\end{array}$$

Maximize $z = -w = -30y_1 - 20y_2 - 25y_3 - 22y_4$.

$$\begin{array}{cccccccccc|c}
y_1 & y_2 & y_3 & y_4 & s_1 & s_2 & s_3 & a_1 & a_2 & z & \\
1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 3000 \\
0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 5000 \\
\boxed{1} & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 5000 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 5000 \\
2 & 6 & 5 & 4 & 0 & 0 & 1 & 0 & 0 & 0 & 40{,}000 \\
\hline
30 & 20 & 25 & 22 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}$$

Pivot on the 1 in row 3, column 1 to remove the a columns.

$$\begin{array}{r|cccccccccc|c}
 & y_1 & y_2 & y_3 & y_4 & s_1 & s_2 & s_3 & a_1 & a_2 & z & \\
-R_1 + R_3 \rightarrow R_1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 2000 \\
 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 5000 \\
 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 5000 \\
 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 5000 \\
-2R_3 + R_5 \rightarrow R_5 & 0 & 4 & 5 & 4 & 0 & 0 & 1 & -2 & 0 & 0 & 30{,}000 \\
\hline
-30R_3 + R_6 \rightarrow R_6 & 0 & -10 & 25 & 22 & 0 & 0 & 0 & -30 & 0 & 1 & -150,000
\end{array}$$

Column a_1 can be removed since $a_1 = 0$.

$$\begin{array}{ccccccccc|c}
y_1 & y_2 & y_3 & y_4 & s_1 & s_2 & s_3 & a_2 & z & \\
0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 2000 \\
0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 5000 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5000 \\
0 & 0 & \boxed{1} & 1 & 0 & 0 & 0 & 1 & 0 & 5000 \\
0 & 4 & 5 & 4 & 0 & 0 & 1 & 0 & 0 & 30{,}000 \\
\hline
0 & -10 & 25 & 22 & 0 & 0 & 0 & 0 & 1 & -150,000
\end{array}$$

Pivot on the 1 in row 4, column 3.

$$\begin{array}{r|ccccccccc|c}
 & y_1 & y_2 & y_3 & y_4 & s_1 & s_2 & s_3 & a_2 & z & \\
R_4 + R_1 \rightarrow R_1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 7000 \\
 & 0 & \boxed{1} & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 5000 \\
 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5000 \\
 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 5000 \\
-5R_4 + R_5 \rightarrow R_5 & 0 & 4 & 0 & -1 & 0 & 0 & 1 & -5 & 0 & 5000 \\
\hline
-25R_4 + R_6 \rightarrow R_6 & 0 & -10 & 0 & -3 & 0 & 0 & 0 & -25 & 1 & -275,000
\end{array}$$

Column a_2 can be removed since $a_2 = 0$. Pivot on the 1 in row 2, column 2 since the basic solution is not yet feasible ($s_2 = -5000$).

$$\begin{array}{r|cccccccc|c|}
 & y_1 & y_2 & y_3 & y_4 & s_1 & s_2 & s_3 & z & \\
-R_2 + R_1 \to R_1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 2000 \\
 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 5000 \\
-R_2 + R_3 \to R_3 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 5000 \\
-4R_2 + R_5 \to R_5 & 0 & 0 & 0 & \boxed{-5} & 0 & 4 & 1 & 0 & -15{,}000 \\
\hline
10R_2 + R_6 \to R_6 & 0 & 0 & 0 & 7 & 0 & -10 & 0 & 1 & -225,000
\end{array}$$

Pivot on the -5 in row 5, column 4.

$$\begin{array}{r|cccccccc|c|}
 & y_1 & y_2 & y_3 & y_4 & s_1 & s_2 & s_3 & z & \\
 & 0 & 0 & 0 & 0 & 1 & \boxed{1} & 0 & 0 & 2000 \\
R_5 + 5R_2 \to R_2 & 0 & 5 & 0 & 0 & 0 & -1 & 1 & 0 & 10{,}000 \\
R_5 - 5R_3 \to R_3 & -5 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -15,000 \\
R_5 + 5R_4 \to R_4 & 0 & 0 & 5 & 0 & 0 & 4 & 1 & 0 & 10{,}000 \\
 & 0 & 0 & 0 & -5 & 0 & 4 & 1 & 0 & -15{,}000 \\
\hline
7R_5 + 5R_6 \to R_6 & 0 & 0 & 0 & 0 & 0 & -22 & 7 & 5 & -1,230,000
\end{array}$$

Pivot on the 1 in row 1, column 6.

$$\begin{array}{r|cccccccc|c|}
 & y_1 & y_2 & y_3 & y_4 & s_1 & s_2 & s_3 & z & \\
 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 2000 \\
R_1 + R_2 \to R_2 & 0 & 5 & 0 & 0 & 1 & 0 & 1 & 0 & 12{,}000 \\
R_1 + R_3 \to R_3 & -5 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -13,000 \\
-4R_1 + R_4 \to R_4 & 0 & 0 & 5 & 0 & -4 & 0 & 1 & 0 & 2000 \\
-4R_1 + R_5 \to R_5 & 0 & 0 & 0 & -5 & -4 & 0 & 1 & 0 & -23{,}000 \\
\hline
22R_1 + R_6 \to R_6 & 0 & 0 & 0 & 0 & 22 & 0 & 7 & 5 & -1,186,000
\end{array}$$

$$\begin{array}{r|cccccccc|c|}
 & y_1 & y_2 & y_3 & y_4 & s_1 & s_2 & s_3 & z & \\
 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 2000 \\
\frac{1}{5}R_2 \to R_2 & 0 & 1 & 0 & 0 & \frac{1}{5} & 0 & \frac{1}{5} & 0 & 2400 \\
-\frac{1}{5}R_3 \to R_3 & 1 & 0 & 0 & 0 & -\frac{1}{5} & 0 & -\frac{1}{5} & 0 & 2600 \\
\frac{1}{5}R_4 \to R_4 & 0 & 0 & 1 & 0 & -\frac{4}{5} & 0 & \frac{1}{5} & 0 & 400 \\
-\frac{1}{5}R_5 \to R_5 & 0 & 0 & 0 & 1 & \frac{4}{5} & 0 & -\frac{1}{5} & 0 & 4600 \\
\hline
\frac{1}{5}R_6 \to R_6 & 0 & 0 & 0 & 0 & \frac{22}{5} & 0 & \frac{7}{5} & 1 & -237,200
\end{array}$$

Here, $y_1 = 2600, y_2 = 2400, y_3 = 400, y_4 = 4600$, and $z = -w = 237,200$. Therefore, ship 2600 barrels from S_1 to D_1, 2400 barrels from S_1 to D_2, 400 barrels from S_2 to D_1, and 4600 barrels from S_2 to D_2 for a minimum cost of \$237,200.

26. Let x_1 = the number of pounds of bluegrass seed,
x_2 = the number of pounds of rye seed,
and x_3 = the number of pounds of Bermuda seed.

Minimize $w = 12x_1 + 15x_2 + 5x_3$

subject to: $x_1 \geq .2\,(x_1 + x_2 + x_3)$ or $.8x_1 - .2x_2 - .2x_3 \geq 0$
$x_3 \leq \frac{2}{3}x_2$ or $-2x_2 + 3x_3 \leq 0$
$x_1 + x_2 + x_3 \geq 5000$

with $x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.$

Maximize $z = -w = -12x_1 - 15x_2 - 5x_3.$

$$\left[\begin{array}{ccccccc|c}
x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\
.8 & -.2 & -.2 & -1 & 0 & 0 & 0 & 0 \\
0 & -2 & 3 & 0 & 1 & 0 & 0 & 0 \\
\boxed{1} & 1 & 1 & 0 & 0 & -1 & 0 & 5000 \\
\hline
12 & 15 & 5 & 0 & 0 & 0 & 1 & 0
\end{array}\right]$$

Since the basic solution is not feasible, pivot on the 1 in row 3, column 1.

$$\begin{array}{r}
\\
.8R_3 + (-R_1) \to R_1 \\
\\
\\
-12R_3 + R_4 \to R_4
\end{array}
\left[\begin{array}{ccccccc|c}
x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\
0 & 1 & \boxed{1} & 1 & 0 & .8 & 0 & 4000 \\
0 & -2 & 3 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & -1 & 0 & 5000 \\
\hline
0 & 3 & -7 & 0 & 0 & 12 & 1 & -60{,}000
\end{array}\right]$$

Pivot on the 1 in row 1, column 3.

$$\begin{array}{r}
\\
\\
-3R_1 + R_2 \to R_2 \\
-R_1 + R_3 \to R_3 \\
7R_1 + R_4 \to R_4
\end{array}
\left[\begin{array}{ccccccc|c}
x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\
0 & 1 & 1 & 1 & 0 & .8 & 0 & 40{,}000 \\
0 & \boxed{-5} & 0 & -3 & 1 & -2.4 & 0 & -12{,}000 \\
1 & 0 & 0 & -1 & 0 & -1.8 & 0 & 1000 \\
\hline
0 & 10 & 0 & 7 & 0 & 17.6 & 1 & -32{,}000
\end{array}\right]$$

Pivot on the -5 in row 2, column 2.

$$\begin{array}{r}
\\
R_2 + 5R_1 \to R_1 \\
\\
\\
2R_2 + R_4 \to R_4
\end{array}
\left[\begin{array}{ccccccc|c}
x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\
0 & 0 & 5 & 2 & 1 & 1.6 & 0 & 8000 \\
0 & -5 & 0 & -3 & 1 & -2.4 & 0 & -12{,}000 \\
1 & 0 & 0 & -1 & 0 & -1.8 & 0 & 1000 \\
\hline
0 & 0 & 0 & 1 & 2 & 12.8 & 1 & -56{,}000
\end{array}\right]$$

Create a 1 in the columns corresponding to x_2 and x_3.

$$\begin{array}{r}
\\
\frac{1}{5}R_1 \to R_1 \\
-\frac{1}{5}R_2 \to R_2 \\
\\
\\
\end{array}
\left[\begin{array}{ccccccc|c}
x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\
0 & 0 & 1 & \frac{2}{5} & \frac{1}{5} & .32 & 0 & 1600 \\
0 & 1 & 0 & \frac{3}{5} & -\frac{1}{5} & .48 & 0 & 2400 \\
1 & 0 & 0 & -1 & 0 & -1.8 & 0 & 1000 \\
\hline
0 & 0 & 0 & 1 & 2 & 12.8 & 1 & -56{,}000
\end{array}\right]$$

Here, $x_1 = 1000$, $x_2 = 2400$, $x_3 = 1600$, and $z = -w = 56{,}000$. Therefore, use 1000 lb of bluegrass, 2400 lb of rye, and 1600 lb of Bermuda for a minimum cost of 56,000¢, that is, $560.

28. Let $x_1 =$ the amount invested in government securities,
$x_2 =$ the amount invested in municipal bonds,
and $x_3 =$ the amount invested in mutual funds.

$$\begin{array}{lccccccl}
\text{Maximize} & \multicolumn{7}{l}{z = .07x_1 + .06x_2 + .10x_3} \\
\text{subject to:} & x_1 & + & x_2 & + & x_3 & = & 100{,}000 \\
 & x_1 & & & & & \geq & 40{,}000 \\
 & & & x_2 & + & x_3 & \geq & 50{,}000 \\
 & .02x_1 & + & .01x_2 & + & .03x_3 & \leq & 2400 \text{ or } 2x_1 + x_2 + 3x_3 \leq 240,000 \\
\text{with} & \multicolumn{7}{l}{x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.}
\end{array}$$

$$\begin{array}{cccccccc|c}
x_1 & x_2 & x_3 & a_1 & s_1 & s_2 & s_3 & z & \\
\boxed{1} & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 100{,}000 \\
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 40{,}000 \\
0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 50{,}000 \\
2 & 1 & 3 & 0 & 0 & 0 & 1 & 0 & 240{,}000 \\
\hline
-.07 & -.06 & -.10 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}$$

Pivot on the 1 in row 1, column 1.

$$\begin{array}{r cccccccc|c}
 & x_1 & x_2 & x_3 & a_1 & s_1 & s_2 & s_3 & z & \\
 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 100{,}000 \\
\mathrm{R}_1 + (-\mathrm{R}_2) \to \mathrm{R}_2 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 60{,}000 \\
 & 0 & 1 & \boxed{1} & 0 & 0 & -1 & 0 & 0 & 50{,}000 \\
-2\mathrm{R}_1 + \mathrm{R}_4 \to \mathrm{R}_4 & 0 & -1 & 1 & -2 & 0 & 0 & 1 & 0 & 40{,}000 \\
\hline
7\mathrm{R}_1 + 100\mathrm{R}_5 \to \mathrm{R}_5 & 0 & 1 & -3 & 7 & 0 & 0 & 0 & 100 & 700{,}000
\end{array}$$

Since $a_1 = 0$, we can now eliminate column a_1. Pivot on the 1 in row 3, column 3 since the basic solution is not feasible $(s_2 = -50{,}000)$.

$$\begin{array}{r ccccccc|c}
 & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\
-\mathrm{R}_3 + \mathrm{R}_1 \to \mathrm{R}_1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 50{,}000 \\
-\mathrm{R}_3 + \mathrm{R}_2 \to \mathrm{R}_2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 10{,}000 \\
 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 50{,}000 \\
\mathrm{R}_3 - \mathrm{R}_4 \to \mathrm{R}_4 & 0 & \boxed{2} & 0 & 0 & -1 & -1 & 0 & 10{,}000 \\
\hline
3\mathrm{R}_3 + \mathrm{R}_5 \to \mathrm{R}_5 & 0 & 4 & 0 & 0 & -3 & 0 & 100 & 850{,}000
\end{array}$$

Pivot on the 2 in row 4, column 2 since the basic solution is not feasible $(s_3 = -10{,}000)$.

$$\begin{array}{r ccccccc|c}
 & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\
 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 50{,}000 \\
 & 0 & 0 & 0 & 1 & \boxed{1} & 0 & 0 & 10{,}000 \\
-\mathrm{R}_4 + 2\mathrm{R}_3 \to \mathrm{R}_3 & 0 & 0 & 2 & 0 & -1 & 1 & 0 & 90{,}000 \\
 & 0 & 2 & 0 & 0 & -1 & -1 & 0 & 10{,}000 \\
\hline
-2\mathrm{R}_4 + \mathrm{R}_5 \to \mathrm{R}_5 & 0 & 0 & 0 & 0 & -1 & 2 & 100 & 830{,}000
\end{array}$$

Since all basic solutions are feasible, continue with the simplex method by pivoting on the 1 in row 2, column 5.

$$\begin{array}{r|rrrrrrr|r} & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ \hline -\text{R}_2 + \text{R}_1 \to \text{R}_1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 40{,}000 \\ & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 10{,}000 \\ \text{R}_2 + \text{R}_3 \to \text{R}_3 & 0 & 0 & 2 & 1 & 0 & 1 & 0 & 100{,}000 \\ \text{R}_2 + \text{R}_4 \to \text{R}_4 & 0 & 2 & 0 & 1 & 0 & -1 & 0 & 20{,}000 \\ \hline \text{R}_2 + \text{R}_5 \to \text{R}_5 & 0 & 0 & 0 & 1 & 0 & 2 & 100 & 840{,}000 \end{array}$$

Create a 1 in the columns corresponding to x_2, x_3, and z.

$$\begin{array}{r|rrrrrrr|r} & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ \hline & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 40{,}000 \\ & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 10{,}000 \\ \frac{1}{2}\text{R}_3 \to \text{R}_3 & 0 & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 50{,}000 \\ \frac{1}{2}\text{R}_4 \to \text{R}_4 & 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 10{,}000 \\ \hline \frac{1}{100}\text{R}_5 \to \text{R}_5 & 0 & 0 & 0 & .01 & 0 & .02 & 1 & 8400 \end{array}$$

Here, $x_1 = 40,000$, $x_2 = 10,000$, $x_3 = 50,000$, and $z = 8400$. Therefore, invest \$40,000 in government securities, \$10,000 in municipal bonds, and \$50,000 in mutual funds for maximum interest of \$8400.

30. Let $x_1 =$ the amount of chemical I,
$x_2 =$ the amount of chemical II,
and $x_3 =$ the amount of chemical III.

$$\begin{array}{ll} \text{Minimize} & w = 1.09x_1 + .87x_2 + .65x_3 \\ \text{subject to:} & x_1 + x_2 + x_3 \geq 750 \\ & .09x_1 + .04x_2 + .03x_3 \geq 30 \\ & 3x_2 = 4x_3 \\ \text{with} & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0. \end{array}$$

Use a graphing calculator or computer to find that we should use 59.21 kg of chemical I, 394.74 kg of chemical II, and 296.05 kg of chemical III for a minimum cost of \$600.39.

32. Let y_1 = the number of gallons of ingredient 1, y_2 = the number of gallons of ingredient 2, y_3 = the number of gallons of ingredient 3, y_4 = the number of gallons of ingredient 4, y_5 = the number of gallons of ingredient 5, and y_6 = the number of gallons of water.

Note that 10%(15,000) = 1500, and .01(15,000) = 150.

The problem becomes:

Minimize

$w = .48y_1 + .32y_2 + .53y_3 + .28y_4 + .43y_5 + .04y_6$

subject to:

$$\begin{array}{rrrrrrl} .28y_1 & +.19y_2 & +.43y_3 & +.57y_4 & +.22y_5 & & \leq 1500 \\ & & y_3 + & y_4 & & & \geq 150 \\ & y_2 & & + & y_5 & & \geq 150 \\ y_1 & & + & y_4 & & & \geq 150 \\ y_1 + & y_2 + & y_3 + & y_4 + & y_5 + & y_6 & = 15{,}000 \end{array}$$

with $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0, y_4 \geq 0, y_5 \geq 0, y_6 \geq 0.$

This exercise should be solved by graphing calculator or computer methods. The answer is to use 0 gal of ingredient 1, 150 gal of 2, 0 gal of 3, 150 gal of 4, 0 gal of 5, and 14,700 gal of water for a minimum cost of \$678.

Chapter 4 Review Exercises

2. If a surplus variable cannot be made nonnegative, then the inequality which represents one of the constraints can never exist. This means that no solution is possible.

4. Maximize $z = 25x_1 + 30x_2$

subject to:
$$\begin{array}{rcrcl} 3x_1 & + & 5x_2 & \le & 47 \\ x_1 & + & x_2 & \le & 25 \\ 5x_1 & + & 2x_2 & \le & 35 \\ 2x_1 & + & x_2 & \le & 30 \end{array}$$

with $x_1 \ge 0,\ x_2 \ge 0.$

(a) Add s_1, s_2, s_3, and s_4 as slack variables to obtain

$$\begin{array}{rcrcrcrcrcrcl} 3x_1 & + & 5x_2 & + & s_1 & & & & & & & = & 47 \\ x_1 & + & x_2 & & & + & s_2 & & & & & = & 25 \\ 5x_1 & + & 2x_2 & & & & & + & s_3 & & & = & 35 \\ 2x_1 & + & x_2 & & & & & & & + & s_4 & = & 30. \end{array}$$

(b) The initial tableau is

$$\left[\begin{array}{ccccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & s_4 & z & \\ 3 & 5 & 1 & 0 & 0 & 0 & 0 & 47 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 25 \\ 5 & 2 & 0 & 0 & 1 & 0 & 0 & 35 \\ 2 & 1 & 0 & 0 & 0 & 1 & 0 & 30 \\ \hline -25 & -30 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

6. Maximize $z = 2x_1 + 3x_2 + 4x_3$

subject to:
$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & \ge & 100 \\ 2x_1 & + & 3x_2 & & & \le & 500 \\ x_1 & & & + & 2x_3 & \le & 350 \end{array}$$

with $x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0.$

(a) Introduce s_1 as a surplus variable and s_2 and s_3 as slack variables to obtain

$$\begin{array}{rcrcrcrcrcrcl} x_1 & + & x_2 & + & x_3 & - & s_1 & & & & & = & 100 \\ 2x_1 & + & 3x_2 & & & & & + & s_2 & & & = & 500 \\ x_1 & & & + & 2x_3 & & & & & + & s_3 & = & 350. \end{array}$$

(b) The initial tableau is

$$\left[\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ 1 & 1 & 1 & -1 & 0 & 0 & 0 & 100 \\ 2 & 3 & 0 & 0 & 1 & 0 & 0 & 500 \\ 1 & 0 & 2 & 0 & 0 & 1 & 0 & 350 \\ \hline -2 & -3 & -4 & 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

8.
$$\left[\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ 2 & 1 & 1 & 0 & 0 & 10 \\ 1 & \boxed{3} & 0 & 1 & 0 & 16 \\ \hline -2 & -3 & 0 & 0 & 1 & 0 \end{array}\right]$$

The most negative indicator is in the x_2 column. The smallest quotient is $\frac{16}{3}$, since $\frac{10}{1} = 10$ and $\frac{16}{3} = 5.3$. Pivot on the 3 in row 2, column 2.

$$\begin{array}{r|ccccc|c} & x_1 & x_2 & s_1 & s_2 & z & \\ -R_2 + 3R_1 \to R_1 & \boxed{5} & 0 & 3 & -1 & 0 & 14 \\ & 1 & 3 & 0 & 1 & 0 & 16 \\ \hline R_2 + R_3 \to R_3 & -1 & 0 & 0 & 1 & 1 & 16 \end{array}$$

Pivot on the 5 in row 1, column 1.

$$\begin{array}{r|ccccc|c} & x_1 & x_2 & s_1 & s_2 & z & \\ & 5 & 0 & 3 & -1 & 0 & 14 \\ -R_1 + 5R_2 \to R_2 & 0 & 15 & -3 & 6 & 0 & 66 \\ \hline R_1 + 5R_3 \to R_3 & 0 & 0 & 3 & 4 & 5 & 94 \end{array}$$

None of the indicators in this tableau are negative, so we have found the optimal solution. To simplify the reading of the final tableau, create a 1 in the columns corresponding to the basic variables (x_1 and x_2) and z.

$$\begin{array}{r|ccccc|c} & x_1 & x_2 & s_1 & s_2 & z & \\ \frac{1}{5}R_1 \to R_1 & 1 & 0 & \frac{3}{5} & -\frac{1}{5} & 0 & \frac{14}{5} \\ \frac{1}{15}R_2 \to R_2 & 0 & 1 & -\frac{1}{5} & \frac{2}{5} & 0 & \frac{22}{5} \\ \hline \frac{1}{5}R_3 \to R_3 & 0 & 0 & \frac{3}{5} & \frac{4}{5} & 1 & \frac{94}{5} \end{array}$$

The maximum is 18.8 $\left(\text{or } \frac{94}{5}\right)$ when $x_1 = 2.8$ $\left(\text{or } \frac{14}{5}\right)$, $x_2 = 4.4$ $\left(\text{or } \frac{22}{5}\right)$, $s_1 = 0$, and $s_2 = 0$.

10.

$$\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ 3 & \boxed{6} & -1 & 0 & 0 & 0 & 28 \\ 1 & 1 & 0 & 1 & 0 & 0 & 12 \\ 2 & 1 & 0 & 0 & 1 & 0 & 16 \\ \hline -1 & -2 & 0 & 0 & 0 & 1 & 0 \end{array}$$

Pivot on the 6 in row 1, column 2.

$$\begin{array}{r|cccccc|c} & x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ & 3 & 6 & -1 & 0 & 0 & 0 & 28 \\ -R_1+6R_2 \to R_2 & 3 & 0 & \boxed{1} & 6 & 0 & 0 & 44 \\ -R_1+6R_3 \to R_3 & 9 & 0 & 1 & 0 & 6 & 0 & 68 \\ \hline R_1+3R_4 \to R_4 & 0 & 0 & -1 & 0 & 0 & 3 & 28 \end{array}$$

Pivot on the 1 in row 2, column 3.

$$\begin{array}{r|cccccc|c} & x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ R_2+R_1 \to R_1 & 6 & 6 & 0 & 6 & 0 & 0 & 72 \\ & 3 & 0 & 1 & 6 & 0 & 0 & 44 \\ -R_2+R_3 \to R_3 & 6 & 0 & 0 & -6 & 6 & 0 & 24 \\ \hline R_2+R_4 \to R_4 & 3 & 0 & 0 & 6 & 0 & 3 & 72 \end{array}$$

$$\begin{array}{r|cccccc|c} & x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ \frac{1}{6}R_1 \to R_1 & 1 & 1 & 0 & 1 & 0 & 0 & 12 \\ & 3 & 0 & 1 & 6 & 0 & 0 & 44 \\ \frac{1}{6}R_3 \to R_3 & 1 & 0 & 0 & -1 & 1 & 0 & 4 \\ \hline \frac{1}{3}R_4 \to R_4 & 1 & 0 & 0 & 2 & 0 & 1 & 24 \end{array}$$

The maximum is 24 when $x_1 = 0,\ x_2 = 12,$ $s_1 = 44,\ s_2 = 0,$ and $s_3 = 4$.

12. Minimize $w = 20y_1 + 15y_2 + 18y_3$

subject to:

$$\begin{aligned} 2y_1 + y_2 + y_3 &\geq 112 \\ y_1 + y_2 + y_3 &\geq 80 \\ y_1 + y_2 &\geq 45 \end{aligned}$$

with $y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0.$

To minimize $w = 20y_1 + 15y_2 + 18y_3$, we maximize $z = -w = -20y_1 - 15y_2 - 18y_3$ subject to the same constraints.

Using the dual:

Maximize $z = 112x_1 + 80x_2 + 45x_3$

subject to:

$$\begin{aligned} 2x_1 + x_2 + x_3 &\leq 20 \\ x_1 + x_2 + x_3 &\leq 15 \\ x_1 + x_2 &\leq 18 \end{aligned}$$

with $x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.$

14.

$$\begin{array}{ccccccc|c} y_1 & y_2 & y_3 & s_1 & s_2 & s_3 & z & \\ 1 & 0 & 0 & 3 & 1 & 2 & 0 & 12 \\ 0 & 0 & 1 & 4 & 5 & 3 & 0 & 5 \\ 0 & 1 & 0 & -2 & 7 & -6 & 0 & 8 \\ \hline 0 & 0 & 0 & 5 & 7 & 3 & 1 & -172 \end{array}$$

Here y_1, y_2, and y_3 are basic. A maximum of -172 gives a minimum of 172 when $y_1 = 12,$ $y_2 = 8,\ y_3 = 5,\ s_1 = 0,\ s_2 = 0,$ and $s_3 = 0$.

16.

$$\begin{array}{cccccc|c} y_1 & y_2 & y_3 & s_1 & s_2 & z & \\ 5 & 1 & 0 & 7 & -1 & 0 & 100 \\ -2 & 0 & 1 & 1 & 3 & 0 & 27 \\ \hline 12 & 0 & 0 & 7 & 2 & 1 & -640 \end{array}$$

Here y_2 and y_3 are basic. A maximum of -640 gives a minimum of 640 when $y_1 = 0,\ y_2 = 100,$ $y_3 = 27,\ s_1 = 0,$ and $s_2 = 0$.

18. Minimize $w = 3y_1 + y_2$

subject to:

$$\begin{aligned} y_1 + y_2 &\geq 5 \\ 3y_1 + 6y_2 &\leq 18 \end{aligned}$$

$$y_1 \geq 0,\ y_2 \geq 0$$

Let $z = -w = -3y_1 - y_2$ and maximize z. Introduce the surplus variable s_1 and the slack variable s_2. The initial tableau is as follows.

$$\begin{array}{ccccc|c} y_1 & y_2 & s_1 & s_2 & z & \\ \boxed{1} & 1 & -1 & 0 & 0 & 5 \\ 3 & 6 & 0 & 1 & 0 & 18 \\ \hline 3 & 1 & 0 & 0 & 1 & 0 \end{array}$$

The initial basic solution is not feasible since $s_1 = -5$. Pivot on the 1 in row 1, column 1.

$$\begin{array}{r|ccccc|c} & y_1 & y_2 & s_1 & s_2 & z & \\ & 1 & 1 & -1 & 0 & 0 & 5 \\ -3R_1+R_2 \to R_2 & 0 & \boxed{3} & 3 & 1 & 0 & 3 \\ \hline -3R_1+R_3 \to R_3 & 0 & -2 & 3 & 0 & 1 & -15 \end{array}$$

Pivot on the 3 in row 2, column 2.

$$\begin{array}{r|ccccc|c} & y_1 & y_2 & s_1 & s_2 & z & \\ -R_2+3R_1 \to R_1 & 3 & 0 & -6 & -1 & 0 & 12 \\ & 0 & 3 & 3 & 1 & 0 & 3 \\ \hline 2R_2+3R_3 \to R_3 & 0 & 0 & 15 & 2 & 3 & -39 \end{array}$$

$$\begin{array}{r|ccccc|c} & y_1 & y_2 & s_1 & s_2 & z & \\ \frac{1}{3}R_1 \to R_1 & 1 & 0 & -2 & -\frac{1}{3} & 0 & 4 \\ \frac{1}{3}R_2 \to R_2 & 0 & 1 & 1 & \frac{1}{3} & 0 & 1 \\ \hline \frac{1}{3}R_3 \to R_3 & 0 & 0 & 5 & \frac{2}{3} & 1 & -13 \end{array}$$

The maximum value of $z = -w$ is -13. Therefore, the minimum value of w is 13 when $y_1 = 4$ and $y_2 = 1$.

20. Minimize $w = 24y_1 + 30y_2 + 36y_3$
subject to:

$$\begin{aligned} 5y_1 + 10y_2 + 15y_3 &\geq 1200 \\ y_1 + y_2 + y_3 &\leq 50 \\ y_1 \geq 0,\ y_2 \geq 0, y_3 &\geq 0 \end{aligned}$$

Let $z = -w = -24y_1 - 30y_2 - 36y_3$ and maximize z.
Introduce surplus variable s_1 and slack variable s_2.
The initial tableau is as follows.

$$\left[\begin{array}{cccccc|c} y_1 & y_2 & y_3 & s_1 & s_2 & z & \\ 5 & 10 & 15 & -1 & 0 & 0 & 1200 \\ \boxed{1} & 1 & 1 & 0 & 1 & 0 & 50 \\ \hline 24 & 30 & 36 & 0 & 0 & 1 & 0 \end{array}\right]$$

The initial basic solution is not feasible since $s_1 = -1200$.
Pivot on the 1 in row 2, column 1.

$$\begin{array}{r} \\ -5R_2+R_1 \to R_1 \\ \\ -24R_2+R_3 \to R_3 \end{array} \left[\begin{array}{cccccc|c} y_1 & y_2 & y_3 & s_1 & s_2 & z & \\ 0 & 5 & 10 & -1 & -5 & 0 & 950 \\ 1 & \boxed{1} & 1 & 0 & 1 & 0 & 50 \\ \hline 0 & 6 & 12 & 0 & -24 & 1 & -1200 \end{array}\right]$$

Pivot on the 1 in row 2, column 2.

$$\begin{array}{r} \\ -5R_2+R_1 \to R_1 \\ \\ -6R_2+R_3 \to R_3 \end{array} \left[\begin{array}{cccccc|c} y_1 & y_2 & y_3 & s_1 & s_2 & z & \\ -5 & 0 & 5 & -1 & -10 & 0 & 700 \\ 1 & 1 & \boxed{1} & 0 & 1 & 0 & 50 \\ \hline -6 & 0 & 6 & 0 & -30 & 1 & -1500 \end{array}\right]$$

Pivot on the 1 in row 2, column 3.

$$\begin{array}{r} \\ -5R_2+R_1 \to R_1 \\ \\ -6R_2+R_3 \to R_3 \end{array} \left[\begin{array}{cccccc|c} y_1 & y_2 & y_3 & s_1 & s_2 & z & \\ -10 & -5 & 0 & -1 & -15 & 0 & 450 \\ 1 & 1 & 1 & 0 & 1 & 0 & 50 \\ \hline -12 & -6 & 0 & 0 & -36 & 1 & -1800 \end{array}\right]$$

Now $s_1 = -450$ is not a feasible solution, but it is not possible to choose a pivot point. Therefore there is no solution.

22. A dual can be used to solve any standard minimization problem.

24. **(a)** Find matrices A, B, C, and X such that the problem

$$\begin{array}{ll} \text{Maximize} & z = 3x_1 + 2x_2 + x_3 \\ \text{subject to:} & 2x_1 + x_2 + x_3 \leq 150 \\ & 2x_1 + 2x_2 + 8x_3 \leq 200 \\ & 2x_1 + 3x_2 + x_3 \leq 320 \\ \text{with} & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0 \end{array}$$

can be written as

$$\begin{array}{ll} \text{Maximize} & CX \\ \text{subject to:} & AX \leq B \\ \text{with} & X \geq O. \end{array}$$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & 8 \\ 2 & 3 & 1 \end{bmatrix}, B = \begin{bmatrix} 150 \\ 200 \\ 320 \end{bmatrix},$$

$$C = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(b) To write the dual, write the augmented matrix for the given problem.

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 150 \\ 2 & 2 & 8 & 200 \\ 2 & 3 & 1 & 320 \\ \hline 3 & 2 & 1 & 0 \end{array}\right]$$

Now form the transpose.

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 3 \\ 1 & 2 & 3 & 2 \\ 1 & 8 & 1 & 1 \\ \hline 150 & 200 & 320 & 0 \end{array}\right]$$

The dual is now stated as:

$$\begin{array}{ll} \text{Minimize} & w = 150y_1 + 200y_2 + 320y_3 \\ \text{subject to:} & 2y_1 + 2y_2 + 2y_3 \geq 3 \\ & y_1 + 2y_2 + 3y_3 \geq 2 \\ & y_1 + 8y_2 + y_3 \geq 1 \\ \text{with} & y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0. \end{array}$$

This can be stated as:

$$\begin{array}{ll} \text{Minimize} & B^TY \\ \text{subject to:} & A^TY \geq C \\ \text{with} & Y \geq O. \end{array}$$

26. **(a)** Let x_1 = the amount invested in oil leases;
x_2 = the amount invested in bonds;
and x_3 = the amount invested in stock.

(b) We want to maximize

$$z = .15x_1 + .09x_2 + .05x_3.$$

(c) The constraints are

$$\begin{aligned} x_1 + x_2 + x_3 &\leq 50{,}000 \\ x_1 + x_2 \phantom{{}+ x_3} &\leq 15{,}000 \\ x_1 \phantom{{}+ x_2} + x_3 &\leq 25{,}000. \end{aligned}$$

28. **(a)** Let y_1 = the number of kilograms of canned whole tomatoes produced
and y_2 = the number of kilograms of tomato sauce produced.

(b) The minimum cost function is

$$w = 4y_1 + 3.25y_2.$$

(c) The constraints are

$$\begin{aligned} y_1 + y_2 &\leq 3{,}000{,}000 \\ y_1 &\geq 800{,}000 \\ y_2 &\geq 80{,}000 \\ 6y_1 + 3y_2 &\geq 6{,}600{,}000. \end{aligned}$$

30. Based on the information given in Exercise 26, the initial tableau is as follows.

$$\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 50{,}000 \\ \boxed{1} & 1 & 0 & 0 & 1 & 0 & 0 & 15{,}000 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 25{,}000 \\ \hline -.15 & -.09 & -.05 & 0 & 0 & 0 & 1 & 0 \end{array}$$

Continue by pivoting on each indicated entry.

$$\begin{array}{r|ccccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ -R_2+R_1 \to R_1 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 35{,}000 \\ & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 15{,}000 \\ -R_2+R_3 \to R_3 & 0 & -1 & \boxed{1} & 0 & -1 & 1 & 0 & 10{,}000 \\ \hline .15R_2+R_4 \to R_4 & 0 & .06 & -.05 & 0 & .15 & 0 & 1 & 2250 \end{array}$$

$$\begin{array}{r|ccccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ -R_3+R_1 \to R_1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 25{,}000 \\ & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 15{,}000 \\ & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 10{,}000 \\ \hline .05R_3+R_4 \to R_4 & 0 & .01 & 0 & 0 & .1 & .05 & 1 & 2750 \end{array}$$

The maximum value is $z = 2750$ when $x_1 = 15{,}000$, $x_2 = 0$, and $x_3 = 10{,}000$. He should invest \$15,000 in oil leases and \$10,000 in stock for a maximum return of \$2750.

32. Based on Exercise 28, the initial tableau is as follows.

$$\begin{array}{rrrrrrr|r}
y_1 & y_2 & s_1 & s_2 & s_3 & s_4 & z & \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 3{,}000{,}000 \\
\boxed{1} & 0 & 0 & -1 & 0 & 0 & 0 & 800{,}000 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 80{,}000 \\
6 & 3 & 0 & 0 & 0 & -1 & 0 & 6{,}600{,}000 \\
\hline
4 & 3.25 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}$$

Pivot on the 1 in row 2, column 1 since the basic solution is not feasible.

$$\begin{array}{r|rrrrrrr|r}
 & y_1 & y_2 & s_1 & s_2 & s_3 & s_4 & z & \\
-R_2+R_1 \to R_1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 2{,}200{,}000 \\
 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 800{,}000 \\
 & 0 & \boxed{1} & 0 & 0 & -1 & 0 & 0 & 80{,}000 \\
-6R_2+R_4 \to R_4 & 0 & 3 & 0 & 6 & 0 & -1 & 0 & 1{,}800{,}000 \\
\hline
-4R_2+R_5 \to R_5 & 0 & 3.25 & 0 & 4 & 0 & 0 & 1 & -3{,}200{,}000
\end{array}$$

Pivot on the 1 in row 3, column 2 since the basic solution is not feasible.

$$\begin{array}{r|rrrrrrr|r}
 & y_1 & y_2 & s_1 & s_2 & s_3 & s_4 & z & \\
-R_3+R_1 \to R_1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1{,}400{,}000 \\
 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 800{,}000 \\
 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 80{,}000 \\
-3R_3+R_4 \to R_4 & 0 & 0 & 0 & \boxed{6} & 3 & -1 & 0 & 1{,}560{,}000 \\
\hline
-3.25R_3+R_5 \to R_5 & 0 & 0 & 0 & 4 & 3.25 & 0 & 1 & -3{,}460{,}000
\end{array}$$

Pivot on the 6 in row 4, column 4 since basic solution is not feasible.

$$\begin{array}{r|rrrrrrr|r}
 & y_1 & y_2 & s_1 & s_2 & s_3 & s_4 & z & \\
R_4+(-6R_1) \to R_1 & 0 & 0 & -6 & 0 & -3 & -1 & 0 & -6{,}840{,}000 \\
R_4+6R_2 \to R_2 & 6 & 0 & 0 & 0 & 3 & -1 & 0 & 6{,}360{,}000 \\
 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 80{,}000 \\
 & 0 & 0 & 0 & 6 & 3 & -1 & 0 & 1{,}560{,}000 \\
\hline
2R_4+3R_5 \to R_5 & 0 & 0 & 0 & 0 & -3.75 & -2 & -3 & 13{,}500{,}000
\end{array}$$

Create a 1 in the columns corresponding to y_1, y_2, s_1, s_2, and z.

$$\begin{array}{r|rrrrrrr|r}
 & y_1 & y_2 & s_1 & s_2 & s_3 & s_4 & z & \\
-\frac{1}{6}R_1 \to R_1 & 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{6} & 0 & 1{,}140{,}000 \\
\frac{1}{6}R_2 \to R_2 & 1 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{6} & 0 & 1{,}060{,}000 \\
 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 80{,}000 \\
\frac{1}{6}R_4 \to R_4 & 0 & 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{6} & 0 & 260{,}000 \\
\hline
-\frac{1}{3}R_5 \to R_5 & 0 & 0 & 0 & 0 & 1.25 & \frac{2}{3} & 1 & -4{,}500{,}000
\end{array}$$

The final tableau gives the solution $y_2 = 1{,}060{,}000$, $y_2 = 80{,}000$, and $z = 4{,}500{,}000$. Use 1,060,000 kg of whole tomatoes and 80,000 kg for sauce for a minimum cost of \$4,500,000.

34. Let $y_1 =$ the number of packages of Sun Hill
and $y_2 =$ the number of packages of Bear Valley.

The problem is:

$$\begin{array}{ll} \text{Minimize} & w = 3y_1 + 2y_2 \\ \text{subject to:} & 10y_1 + 2y_2 \geq 20 \\ & 4y_1 + 4y_2 \geq 24 \\ & 2y_1 + 8y_2 \geq 24 \\ \text{with} & y_1 \geq 0, y_2 \geq 0. \end{array}$$

(a) Change this to a maximization problem by letting $z = -w = -3y_1 - 2y_2$. Now maximize $z = -3y_1 - 2y_2$ subject to the constraints above. Begin by inserting surplus variables to set up the first tableau.

The initial simplex tableau is as follows.

$$\begin{array}{cccccc|c}
y_1 & y_2 & s_1 & s_2 & s_3 & z & \\
\boxed{10} & 2 & -1 & 0 & 0 & 0 & 20 \\
4 & 4 & 0 & -1 & 0 & 0 & 24 \\
2 & 8 & 0 & 0 & -1 & 0 & 24 \\
\hline
3 & 2 & 0 & 0 & 0 & 1 & 0
\end{array}$$

Pivot on the 10 in row 1, column 1.

$$\begin{array}{r|cccccc|c}
 & y_1 & y_2 & s_1 & s_2 & s_3 & z & \\
 & 10 & 2 & -1 & 0 & 0 & 0 & 20 \\
-4R_1 + 10R_2 \rightarrow R_2 & 0 & 32 & 4 & -10 & 0 & 0 & 160 \\
-R_1 + \ \ 5R_3 \rightarrow R_3 & 0 & \boxed{38} & 1 & 0 & -5 & 0 & 100 \\
\hline
-3R_1 + 10R_4 \rightarrow R_4 & 0 & 14 & 3 & 0 & 0 & 10 & -60
\end{array}$$

Pivot on the 38 in row 3, column 2.

$$\begin{array}{r|cccccc|c}
 & y_1 & y_2 & s_1 & s_2 & s_3 & z & \\
-R_3 + 19R_1 \rightarrow R_1 & 190 & 0 & -20 & 0 & 5 & 0 & 280 \\
-16R_3 + 19R_2 \rightarrow R_2 & 0 & 0 & \boxed{60} & -190 & 80 & 0 & 1440 \\
 & 0 & 38 & 1 & 0 & -5 & 0 & 100 \\
\hline
-7R_3 + 19R_4 \rightarrow R_4 & 0 & 0 & 50 & 0 & 35 & 190 & -1840
\end{array}$$

Pivot on the 60 in row 2, column 3.

$$\begin{array}{r|cccccc|c}
 & y_1 & y_2 & s_1 & s_2 & s_3 & z & \\
R_2 + 3R_1 \rightarrow R_1 & 570 & 0 & 0 & -190 & 95 & 0 & 2280 \\
 & 0 & 0 & 60 & -190 & \boxed{80} & 0 & 1440 \\
-R_2 + 60R_3 \rightarrow R_3 & 0 & 2280 & 0 & 190 & -380 & 0 & 4560 \\
\hline
-5R_2 + \ \ 6R_4 \rightarrow R_4 & 0 & 0 & 0 & 950 & -190 & 1140 & -18,240
\end{array}$$

Pivot on the 80 in row 2, column 5.

$$\begin{array}{r|cccccc|c}
 & y_1 & y_2 & s_1 & s_2 & s_3 & z & \\
-19R_2 + 16R_1 \rightarrow R_1 & 9120 & 0 & -1140 & 570 & 0 & 0 & 9120 \\
 & 0 & 0 & 60 & -190 & 80 & 0 & 1440 \\
19R_2 + \ \ 4R_3 \rightarrow R_3 & 0 & 9120 & 1140 & -2850 & 0 & 0 & 45{,}600 \\
\hline
19R_2 + \ \ 8R_4 \rightarrow R_4 & 0 & 0 & 1140 & 3990 & 0 & 9120 & -118{,}560
\end{array}$$

Create a 1 in the columns corresponding to y_1, y_2, and z.

$$\begin{array}{r|cccccc|c}
 & y_1 & y_2 & s_1 & s_2 & s_3 & z & \\
\frac{1}{9120}\mathrm{R}_1 \to \mathrm{R}_1 & 1 & 0 & -\frac{1}{8} & \frac{1}{16} & 0 & 0 & 1 \\
 & 0 & 0 & 60 & -190 & 80 & 0 & 1440 \\
\frac{1}{9120}\mathrm{R}_3 \to \mathrm{R}_3 & 0 & 1 & \frac{1}{8} & -\frac{5}{16} & 0 & 0 & 5 \\
\hline
\frac{1}{9120}\mathrm{R}_4 \to \mathrm{R}_4 & 0 & 0 & \frac{1}{8} & \frac{7}{16} & 0 & 1 & -13
\end{array}$$

The maximum value of z is -13 when $y_1 = 1$ and $y_2 = 5$. Hence the minimum value of w is 13 when $y_1 = 1$ and $y_2 = 5$. Thus the minimum cost is \$13 for 1 package of Sun Hill and 5 packages of Bear Valley.

(b) The dual problem is as follows.

$$\begin{array}{ll}
\text{Maximize} & z = 20x_1 + 24x_2 + 24x_3 \\
\text{subject to:} & 10x_1 + 4x_2 + 2x_3 \le 3 \\
 & 2x_1 + 4x_2 + 8x_3 \le 2 \\
\text{with} & x_1 \ge 0, x_2 \ge 0, x_3 \ge 0.
\end{array}$$

The initial simplex tableau is as follows.

$$\begin{array}{cccccc|c}
x_1 & x_2 & x_3 & s_1 & s_2 & z & \\
10 & 4 & 2 & 1 & 0 & 0 & 3 \\
2 & \boxed{4} & 8 & 0 & 1 & 0 & 2 \\
\hline
-20 & -24 & -24 & 0 & 0 & 1 & 0
\end{array}$$

Pivot about the 4 in row 2, column 2.

$$\begin{array}{r|cccccc|c}
 & x_1 & x_2 & x_3 & s_1 & s_2 & z & \\
-\mathrm{R}_2 + \mathrm{R}_1 \to \mathrm{R}_1 & \boxed{8} & 0 & -6 & 1 & -1 & 0 & 1 \\
 & 2 & 4 & 8 & 0 & 1 & 0 & 2 \\
\hline
6\mathrm{R}_2 + \mathrm{R}_3 \to \mathrm{R}_3 & -8 & 0 & 24 & 0 & 6 & 1 & 12
\end{array}$$

Pivot about the 8 in row 1, column 1.

$$\begin{array}{r|cccccc|c}
 & x_1 & x_2 & x_3 & s_1 & s_2 & z & \\
 & 8 & 0 & -6 & 1 & -1 & 0 & 1 \\
-\mathrm{R}_1 + 4\mathrm{R}_2 \to \mathrm{R}_2 & 0 & 16 & 38 & -1 & 5 & 0 & 7 \\
\hline
\mathrm{R}_1 + \mathrm{R}_3 \to \mathrm{R}_3 & 0 & 0 & 18 & 1 & 5 & 1 & 13
\end{array}$$

$$\begin{array}{r|cccccc|c}
 & x_1 & x_2 & x_3 & s_1 & s_2 & z & \\
\frac{1}{8}\mathrm{R}_1 \to \mathrm{R}_1 & 1 & 0 & -\frac{3}{4} & \frac{1}{8} & -\frac{1}{8} & 0 & \frac{1}{8} \\
\frac{1}{16}\mathrm{R}_2 \to \mathrm{R}_2 & 0 & 1 & \frac{19}{8} & -\frac{1}{16} & \frac{5}{16} & 0 & \frac{7}{16} \\
\hline
 & 0 & 0 & 18 & 1 & 5 & 1 & 13
\end{array}$$

The minimum value of w is 13 when $y_1 = 1$ and $y_2 = 5$, that is, you should buy 1 package of Sun Hill and 5 packages of Bear Valley for a minimum cost of \$13.

(c) The shadow cost for peanuts is based on the variable x_1 and can be read from the final simplex tableau: $\frac{1}{8}$. To get 8 more ounces of peanuts will cost an additional $8\left(\frac{1}{8}\right) = \$1$, so the total cost will then be \$14.

Extended Application: Using Integer Programming in the Stock Cutting Problem

1. **(a)** With Plan A you will need to buy 8 timbers: one cut will give you the two 4-ft lengths, two more cuts will give you two each of the 3-ft and 5-ft lengths, the two 3-ft pieces will come out of another full length, leaving 2 ft over, and all four 6-ft lengths will come out of 8-ft pieces. Your total waste amounts to 10 ft.

(b) There's no advantage to Plan B; you still need 8 pieces of lumber: two cuts will give you four 4-ft lengths, two more cuts will give you two each of the 3-ft and 5-ft lengths, the two 5-ft lengths will come out of 8-ft pieces as will each of the two 6-ft lengths, leaving a total waste of 10 ft.

(c) If the original timbers were 9 ft in length, you could but 6 timbers and cut the lengths for either Plan A or Plan B with no waste.

2. Four patterns not in the minimizer's list are, for example,

$14|14|33|33|, 31|33|33|, 33|33|33|$, and $14|17|31|33|$.

3. The patterns are 31|33|36, 17|17|33|33, 14|17|33|36, and 14|14|36|36.

4. $\frac{808}{35{,}600} \approx 2.3\%$

5. A leftover piece less than 14 inches wide can't be used for any standard width, but a leftover piece of 22 inches, for example, could be cut to make either the 14-inch or the 17-inch standard width. So it might be better to reserve the choice about how to cut this leftover until more orders come in.

6. The highest value is 34: choose weights 2, 2.5 and 4.5.

Chapter 5

MATHEMATICS OF FINANCE

5.1 Simple and Compound Interest

2. If we use 365 days,

$$A = 11,280\left[1 + .11\left(\frac{85}{365}\right)\right] \approx 11,568.95.$$

The maturity value is $11,568.95. Using 365 days in a year is more advantageous to the borrower.

6. $25,000 at 7% for 9 mo

Use the formula for simple interest.

$$\begin{aligned} I &= Prt \\ &= 25,000(.07)\left(\frac{9}{12}\right) \\ &= 1312.50 \end{aligned}$$

The simple interest is $1312.50.

8. $1974 at 6.3% for 7 mo

$$\begin{aligned} I &= Prt \\ &= 1974(.063)\left(\frac{7}{12}\right) \\ &\approx 72.54 \end{aligned}$$

The simple interest is $72.54.

10. $5147.18 at 10.1% for 58 days

$$\begin{aligned} I &= Prt \\ &= 5147.18(.101)\left(\frac{58}{360}\right) \\ &\approx 83.76 \end{aligned}$$

The simple interest is $83.76.

14. The formula for the compound amount is

$$A = P(1+i)^n.$$

To find A, substitute $P = 1000$, $i = .06$, and $n = 8$.

$$\begin{aligned} A &= 1000(1+.06)^8 \\ &= 1000(1.06)^8 \\ &\approx 1593.85 \end{aligned}$$

The compound amount is $1593.85.

16. To find the compound amount A, substitute $P = 470$, $i = \frac{.10}{2} = .05$, and $n = 12(2) = 24$ in the formula for the compound amount.

$$\begin{aligned} A &= P(1+i)^n \\ &= 470(1.05)^{24} \\ &\approx 1515.80 \end{aligned}$$

The compound amount is $1515.80.

18. Use the formula for compound amount with $P = 6500$, $i = \frac{.12}{4}$, and $n = 6(4) = 24$.

$$\begin{aligned} A &= P(1+i)^n \\ &= 6500\left(1 + \frac{.12}{4}\right)^{24} \\ &= 6500(1.03)^{24} \\ &\approx 13,213.16 \end{aligned}$$

The compound amount is $13,213.16.

20. $15,902.74 at 9.8% compounded annually for 7 yr

Use the formula for present value for compound interest with $A = 15{,}902.74$, $i = .098$, and $n = 7$.

$$P = \frac{A}{(1+i)^n} = \frac{15{,}902.74}{(1+.098)^7} \approx 8265.24$$

The present value is $8265.24.

22. $2000 at 9% compounded semiannually for 8 yr

Here $A = 2000$, $i = \frac{.09}{2} = .045$, and $n = 8(2) = 16$.

$$P = \frac{A}{(1+i)^n} = \frac{2000}{(1.045)^{16}} \approx 988.94$$

The present value is $988.94.

24. $8800 at 10% compounded quarterly for 5 yr

Here $A = 8800$, $i = \frac{.10}{4} = .025$, and $n = 5(4) = 20$.

$$P = \frac{A}{(1+i)^n} = \frac{8800}{(1.025)^{20}} \approx 5370.38$$

The present value is $5370.38.

28. 3% compounded quarterly

Use the formula for effective rate with $r = .03$ and $m = 4$.

$$\begin{aligned} r_e &= \left(1+\frac{r}{m}\right)^m - 1 \\ &= \left(1+\frac{.03}{4}\right)^4 - 1 \\ &\approx .03034 \end{aligned}$$

The effective rate is 3.034%.

30. 8.25% compounded semiannually

Here $r = .0825$ and $m = 2$.

$$\begin{aligned} r_e &= \left(1+\frac{r}{m}\right)^m - 1 \\ &= \left(1+\frac{.0825}{2}\right)^2 - 1 \\ &\approx .08420 \end{aligned}$$

The effective rate is 8.420%.

32. Use the formula for future value for simple interest.

$$\begin{aligned} A &= P(1+rt) \\ &= 25,900\left[1+.084\left(\frac{11}{12}\right)\right] \\ &= 27,894.30. \end{aligned}$$

She repaid $27,894.30.

34. The interest is

$$\$101{,}133.33 \ - \ 100{,}000 \ = \$1133.33.$$

Use the formula for simple interest.

$$\begin{aligned} I &= Prt \\ 1133.33 &= 100{,}000r\left(\frac{60}{360}\right) \\ .068 &\approx r \end{aligned}$$

The interest rate was about 6.8%.

36. The interest per share is

$$(24-22)+.50 = 2.50.$$

Use the simple interest formula with $I = 2.50$, $P = 22$, and $t = 1$, and solve for r.

$$\begin{aligned} I &= Prt \\ 2.50 &= 22r(1) \\ .114 &\approx r \end{aligned}$$

The interest rate was about 11.4%.

38. $P = 50,000,\ i = \frac{.12}{12},\ n = 12(4) = 48$

First find the compound amount.

$$\begin{aligned} A &= P(1+i)^n \\ &= 50,000\left(1+\frac{.12}{12}\right)^{48} \\ &= 50,000(1.01)^{48} \\ &\approx \$80,611.30 \end{aligned}$$

$$\begin{aligned} \text{Amount of interest} &= A - P \\ &= \$80,611.30 - \$50,000 \\ &= \$30,611.30 \end{aligned}$$

The business will pay $30,611.30 in interest.

40. First find the compound amount. This is the total amount required to pay off the loan, including both interest and repayment of principal. Here $P = 80,000,\ i = \frac{.10}{4} = .025$, and $n = 5(4) = 20$.

$$\begin{aligned} A &= P(1+i)^n \\ &= 80,000(1.025)^{20} \\ &\approx 131,089.32 \end{aligned}$$

Now find the amount of interest.

$$\begin{aligned} I &= A - P \\ &= 131,089.32 - 80,000 \\ &= 51,089.32 \end{aligned}$$

The interest on the loan will amount to $51,089.32.

42. Use the formula for present value for compound interest with $A = 20,000,\ i = \frac{.08}{4} = .02$, and $n = 5(4) = 20$.

$$P = \frac{A}{(1+i)^n} = \frac{20,000}{(1.02)^{20}} \approx 13,459.43$$

George should invest $13,459.43.

44. Use the formula for compound amount to find the value of $1000 in 5 yr.

$$\begin{aligned} A &= P(1+i)^n \\ &= 1000(1.06)^5 \\ &\approx 1338.23 \end{aligned}$$

In 5 yr, $1000 will be worth $1338.23. Since this is larger than the $1210 one would receive in 5 yr, it would be more profitable to take the $1000 now.

46. Use the formula for effective rate with $r_e = .0201$ and $m = 12$.

$$\begin{aligned}
r_e &= \left(1+\frac{r}{m}\right)^m - 1\\
.0201 &= \left(1+\frac{r}{12}\right)^{12} - 1\\
1.0201 &= \left(1+\frac{r}{12}\right)^{12}\\
(1.0201)^{1/12} &= 1+\frac{r}{12}\\
1.001660 &\approx 1+\frac{r}{12}\\
.001660 &\approx \frac{r}{12}\\
.0199 &\approx r
\end{aligned}$$

Thus, the actual rate is 1.99%.

48. Use 8% compounded quarterly for 20 yr. Then $i = \frac{.08}{4} = .02$ and $n = 20(4) = 80$.

$$A = P\,(1+i)^n = P(1.02)^{80}$$

For $10,000,

$$A = \ 10{,}000(1.02)^{80} \approx \ 48{,}754.39,$$

that is, $48,754.39.

For $149,000,

$$A = 149,000(1.02)^{80} \approx 726,440.43,$$

that is, $726,440.43.

For $1,000,000,

$$A = 1,000,000(1.02)^{80} \approx 4,875,439.16,$$

that is, $4,875,439.16.

50. $2 = (1+.05)^n$
$2 = (1.05)^n$

Try various values for n.

$$(1.05)^{14} \approx 1.979932 \approx 2$$

Thus, $n \approx 14$. It would take about 14 yr for the general level of prices in the economy to double at the annual inflation rate of 5%.

52. Find n such that

$$2 = (1.02)^n$$

By trying various values of n, we see that $n \approx 35$ is approximately correct because

$$(1.02)^{35} \approx 1.999890 \approx 2.$$

It will take about 35 yr before the utilities will need to double their generating capacity.

54. Let $P = 150,000$, $i = -.024$, and $n = 8$.

$$\begin{aligned}
A &= P(1+i)^n\\
&= 150,000(1-.024)^8\\
&= 150,000(.976)^8\\
&= 123,506.50
\end{aligned}$$

After 8 yr, the amount on deposit will be $123,506.50.

56. Use the formula

$$A = P(1+i)^n$$

with $P = \frac{2}{8}$ cent $= \$.0025$ and $r = .04$ compounded quarterly for 2000 yr.

$$\begin{aligned}
A &= .0025\left(1+\frac{.04}{4}\right)^{4(2000)}\\
&= .0025(1.01)^{8000}\\
&\approx 9.31 \times 10^{31}
\end{aligned}$$

The money would be worth $\$9.31 \times 10^{31}$ 2000 yr later.

58. Use the formula

$$A = P(1+i)^n$$

with $P = 10,000$ and $r = .05$ for 10 yr.

(a) If interest is compounding annually,

$$\begin{aligned}
A &= \ 10{,}000(1+.05)^{10}\\
&\approx \ 16{,}288.95.
\end{aligned}$$

The future value is $16,288.95.

(b) If interest is compounding quarterly,

$$\begin{aligned}
A &= \ 10{,}000\left(1+\frac{.05}{4}\right)^{40}\\
&\approx \ 16{,}436.19.
\end{aligned}$$

The future value is $16,436.19.

(c) If interest is compounding monthly,

$$\begin{aligned} A = &\ 10{,}000\left(1+\frac{.05}{12}\right)^{120} \\ \approx &\ 16{,}470.09. \end{aligned}$$

The future value is $16,470.09.

(d) If interest is compounding daily,

$$\begin{aligned} A = &\ 10{,}000\left(1+\frac{.05}{365}\right)^{3650} \\ \approx &\ 16{,}486.65. \end{aligned}$$

The future value is $16,486.65.

60. First consider the case of earning interest at a rate of k per annum compounded quarterly for all 8 yr and earning $2203.76 on the $1000 investment.

$$2203.76 = 1000\left(1+\frac{k}{4}\right)^{8(4)}$$

$$2.20376 = \left(1+\frac{k}{4}\right)^{32}$$

Use a calculator to raise both sides to the power $\frac{1}{32}$.

$$1.025 = 1+\frac{k}{4}$$

$$.025 = \frac{k}{4}$$

$$.1 = k$$

Next consider the actual investments. The $1000 was invested for the first 5 yr at a rate of j per annum compounded semiannually.

$$A = 1000\left(1+\frac{j}{2}\right)^{5(2)}$$

$$A = 1000\left(1+\frac{j}{2}\right)^{10}$$

This amount was then invested for the remaining 3 yr at $k = .1$ per annum compounded quarterly for a final compound amount of $1990.76.

$$1990.76 = A\left(1+\frac{.1}{4}\right)^{3(4)}$$

$$1990.76 = A(1.025)^{12}$$

$$1480.24 \approx A$$

Recall that $A = 1000\left(1+\frac{j}{2}\right)^{10}$ and substitute this value into the above equation.

$$1480.24 = 1000\left(1+\frac{j}{2}\right)^{10}$$

$$1.48024 = \left(1+\frac{j}{2}\right)^{10}$$

Use a calculator to raise both sides to the power $\frac{1}{10}$.

$$1.04 \approx 1+\frac{j}{2}$$

$$.04 = \frac{j}{2}$$

$$.08 = j$$

The ratio of k to j is

$$\frac{k}{j} = \frac{.1}{.08} = \frac{10}{8} = \frac{5}{4}.$$

5.2 Future Value of an Annuity

2. $a = 5,\ r = 3$

$$a_n = ar^{n-1}$$
$$a_5 = 5(3)^4 = 405$$

4. $a = -6;\ r = 2$

$a_5 = ar^4 = -6(2)^4 = -96$

6. $a = 12;\ r = -2$

$$a_5 = ar^4 = 12(-2)^4 = 192$$

8. $a = 729;\ r = \frac{1}{3}$

$$a_5 = ar^4 = 729\left(\frac{1}{3}\right)^4 = 9$$

10. $a = 3;\ r = 3;\ n = 4$

$$S_n = \frac{a(r^n-1)}{r-1}$$

$$S_4 = \frac{3(3^4-1)}{3-1} = 120$$

12. $a = 6;\ r = \frac{1}{2};\ n = 4$

$$S_n = \frac{a(r^n-1)}{r-1}$$

$$S_4 = \frac{6\left[\left(\frac{1}{2}\right)^4-1\right]}{\frac{1}{2}-1} = \frac{6\left(\frac{15}{16}\right)}{\frac{1}{2}} = \frac{45}{4}$$

14. $a = 81;\ r = -\frac{2}{3};\ n = 4$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_4 = \frac{81\left[\left(-\frac{2}{3}\right)^4 - 1\right]}{-\frac{2}{3} - 1} = \frac{81\left(-\frac{65}{81}\right)}{-\frac{5}{3}} = 39$$

16. $$s_{\overline{n}|i} = \frac{(1 + i)^n - 1}{i}$$

$$s_{\overline{20}|.06} = \frac{(1.06)^{20} - 1}{.06} \approx 36.78559$$

18. $s_{\overline{18}|.015} = \dfrac{(1.015)^{18} - 1}{.015} \approx 20.48938$

22. $R = 1000;\ i = .06;\ n = 5$

Use the formula for the future value of an ordinary annuity.

$$\begin{aligned} S &= Rs_{\overline{n}|i} = 1000s_{\overline{5}|.06} \\ &= 1000\left[\frac{(1.06)^5 - 1}{.06}\right] \\ &\approx 5637.09 \end{aligned}$$

The future value is \$5637.09.

24. $R = 29,500;\ i = .058;\ n = 15$

$$\begin{aligned} S &= R\left[\frac{(1 + i)^n - 1}{i}\right] \\ &= 29,500\left[\frac{(1 + .058)^{15} - 1}{.058}\right] \\ &\approx 676,272.05 \end{aligned}$$

The future value is \$676,272.05.

26. $R = 3700;\ i = \frac{.08}{2} = .04;\ n = 11(2) = 22$

$$\begin{aligned} S &= R\left[\frac{(1 + i)^n - 1}{i}\right] \\ &= 3700\left[\frac{(1 + .04)^{22} - 1}{.04}\right] \\ &\approx 126,717.49 \end{aligned}$$

The future value is \$126,717.49.

28. $R = 4600;\ i = \frac{.0873}{4} = .021825;\ n = 9(4) = 36$

$$\begin{aligned} S &= R\left[\frac{(1 + i)^n - 1}{i}\right] \\ &= 4600\left[\frac{(1 + .021825)^{36} - 1}{.021825}\right] \\ &\approx 247,752.70 \end{aligned}$$

The future value is \$247,752.70.

30. $R = 42,000;\ i = \frac{.1005}{2} = .05025;\ n = 12(2) = 24$

$$\begin{aligned} S &= R\left[\frac{(1 + i)^n - 1}{i}\right] \\ &= 42,000\left[\frac{(1 + .05025)^{24} - 1}{.05025}\right] \\ &\approx 1,875,230.74 \end{aligned}$$

The future value is \$1,875,230.74.

32. $R = 1400;\ i = .08;\ n = 10$

To find the future value of an annuity due, use the formula for the future value of an ordinary annuity, but include one additional time period and subtract the amount of one payment.

$$\begin{aligned} S &= R\left[\frac{(1 + i)^{n+1} - 1}{i}\right] - R \\ &= 1400\left[\frac{(1 + .08)^{11} - 1}{.08}\right] - 1400 \\ &\approx 21,903.68 \end{aligned}$$

The future value is \$21,903.68.

34. $R = 4000;\ i = .06;\ n = 11$

$$\begin{aligned} S &= R\left[\frac{(1 + i)^{n+1} - 1}{i}\right] - R \\ &= 4000\left[\frac{(1 + .06)^{12} - 1}{.06}\right] - 4000 \\ &\approx 63,479.76 \end{aligned}$$

The future value is \$63,479.76.

36. $R = 750;\ i = \frac{.059}{12} = .00491\overline{6};\ n = 15(12) = 180$

$$\begin{aligned} S &= R\left[\frac{(1 + i)^{n+1} - 1}{i}\right] - R \\ &= 750\left[\frac{(1 + .00491\overline{6})^{181} - 1}{.00491\overline{6}}\right] - 750 \\ &\approx 217,328.08 \end{aligned}$$

The future value is \$217,328.08.

38. $R = 1500;\ i = \frac{.056}{2} = .028;\ n = 11(2) = 22$

$$\begin{aligned} S &= R\left[\frac{(1 + i)^{n+1} - 1}{i}\right] - R \\ &= 1500\left[\frac{(1 + .028)^{23} - 1}{.028}\right] - 1500 \\ &\approx 46,034.09 \end{aligned}$$

The future value is $\$46,034.09$.

40. $S = 100,000;\ i = \frac{.08}{2} = .04;\ n = 9(2) = 18$

Let R represent the amount to be deposited into the sinking fund each year. Solve the formula

$$S = Rs_{\overline{n}|i}$$

for R and proceed.

$$\begin{aligned} R &= \frac{S}{s_{\overline{n}|i}} \\ &= \frac{100,000}{s_{\overline{18}|.04}} \\ &= \frac{100,000(.04)}{(1+.04)^{18}-1} \\ &\approx 3899.33 \end{aligned}$$

The periodic payment is \$3899.33.

42. $S = 8500;\ i = .08;\ n = 7$

$$\begin{aligned} R &= \frac{S}{s_{\overline{n}|i}} \\ &= \frac{8500}{s_{\overline{7}|.08}} \\ &= \frac{8500(.08)}{(1+.08)^{7}-1} \\ &\approx 952.62 \end{aligned}$$

The payment is \$952.62.

44. $S = 75,000;\ i = \frac{.06}{2} = .03;\ n = 4\frac{1}{2}(2) = 9$

$$\begin{aligned} R &= \frac{S}{s_{\overline{n}|i}} \\ &= \frac{75,000}{s_{\overline{9}|.03}} \\ &= \frac{75,000(.03)}{(1+.03)^{9}-1} \\ &\approx 7382.54 \end{aligned}$$

The payment is \$7382.54.

46. $S = 50,000;\ i = \frac{.079}{4} = .01975;\ n = 2\frac{1}{2}(4) = 10$

$$\begin{aligned} R &= \frac{S}{s_{\overline{n}|i}} \\ &= \frac{50,000}{s_{\overline{10}|.01975}} \\ &= \frac{50,000(.01975)}{(1+.01975)^{10}-1} \\ &\approx 4571.55 \end{aligned}$$

The payment is \$4571.55.

48. **(a)** $R = 12,000;\ i = .08;\ n = 9$

$$\begin{aligned} S &= R\left[\frac{(1+i)^n-1}{i}\right] \\ &= 12,000\left[\frac{(1+.08)^9-1}{.08}\right] \\ &\approx 149,850.69 \end{aligned}$$

The final amount is \$149,850.69.

(b) $R = 12,000;\ i = .06;\ n = 9$

$$\begin{aligned} S &= R\left[\frac{(1+i)^n-1}{i}\right] \\ &= 12{,}000\left[\frac{(1+.06)^9-1}{.06}\right] \\ &\approx 137{,}895.79 \end{aligned}$$

She will have $\$137,895.79$.

(c) The amount that would be lost is the difference between the two amounts in parts (a) and (b), which is

$$\$149{,}850.69 - 137{,}895.79 = \$11{,}954.90.$$

50. $R = 80;\ i = \frac{.075}{12} = .00625;\ n = 3(12) + 9 = 45$

Because the deposits are made at the beginning of each month, this is an annuity due.

$$\begin{aligned} S &= R\left[\frac{(1+i)^{n+1}-1}{i}\right] - R \\ &= 80\left[\frac{(1+.00625)^{46}-1}{.00625}\right] - 80 \\ &\approx 4168.30 \end{aligned}$$

The account will have \$4168.30 in it.

52. This may be considered an annuity due, since payments are made at the beginning of each year, starting with the day the daughter is born. However, a payment should not be subtracted at the end, since a twenty-second payment is made on her twenty-first birthday. Thus, the future value is given by

$$S = R\left[\frac{(1+i)^{n+1}-1}{i}\right],$$

where $R = 1000;\ i = .095;$ and $n = 21$. Therefore,

$$\begin{aligned} S &= 1000\left[\frac{(1.095)^{22}-1}{.095}\right] \\ &\approx 66,988.91 \end{aligned}$$

There will be \$66,988.91 in the account at the end of the day on the daughter's twenty-first birthday.

54. From ages 50 to 60, we have an ordinary annuity with $R = 1200$, $i = \frac{.07}{4} = .0175$, and $n = 10(4) = 40$. Use the formula for the future value of an ordinary annuity.

$$\begin{aligned} S &= R\left[\frac{(1+i)^n - 1}{i}\right] \\ &= 1200\left[\frac{(1.0175)^{40} - 1}{.0175}\right] \\ &\approx 68,680.96 \end{aligned}$$

At age 60, the value of the retirement account is \$68,680.96. This amount now earns 9% interest compounded monthly for 5 yr. Use the formula for compound amount with $P = 68,680.96$, $i = \frac{.09}{12} = .0075$, and $n = 5(12) = 60$ to find the value of this amount after 5 yr.

$$\begin{aligned} A &= P(1+i)^n \\ &= 68,680.96(1.0075)^{60} \\ &\approx 107,532.48 \end{aligned}$$

The value of the amount she withdraws from the retirement account will be \$107,532.48 when she reaches 65.

The deposits of \$300 at the end of each month into the mutual fund form another ordinary annuity. Use the formula for the future value of an ordinary annuity with $R = 300$, $i = \frac{.09}{12} = .0075$, and $n = 5(12) = 60$.

$$\begin{aligned} S &= R\left[\frac{(1+i)^n - 1}{i}\right] \\ &= 300\left[\frac{(1.0075)^{60} - 1}{.0075}\right] \\ &\approx 22,627.24 \end{aligned}$$

The value of this annuity after 5 yr is \$22,627.24.

The total amount in the mutual fund account when the woman reaches age 65 will be

$$\$107,532.48 + 22,627.24 = \$130,159.72.$$

56. For the first 12 yr, we have an annuity due. To find the amount in this account after 12 yr, use the formula for the future value of an annuity due with $R = 10,000$, $i = .05$, and $n = 12$.

$$\begin{aligned} S &= R\left[\frac{(1+i)^{n+1} - 1}{i}\right] - R \\ &= 10{,}000\left[\frac{(1.05)^{13} - 1}{.05}\right] - 10{,}000 \\ &\approx 167{,}129.83 \end{aligned}$$

This amount, \$167,129.83, now earns 6% interest compounded semiannually for another 9 yr, but no new deposits are made. Use the formula for compound amount with $P = 167{,}129.83$, $i = \frac{.06}{2} = .03$, and $n = 9(2) = 18$.

$$\begin{aligned} A &= P(1+i)^n \\ &= 167{,}129.83(1.03)^{18} \\ &\approx 284{,}527.35 \end{aligned}$$

The final amount on deposit after 21 yr is \$284,527.35.

58. This is a sinking fund with $S = 12,000$, $i = \frac{.06}{2} = .03$, and $n = 4(2) = 8$.

$$\begin{aligned} R &= \frac{S}{s_{\overline{n}|i}} \\ &= \frac{12,000}{s_{\overline{8}|.03}} \\ &= \frac{12,000(.03)}{(1+.03)^8 - 1} \\ &\approx 1349.48 \end{aligned}$$

Each payment should be \$1349.48.

60. Use the formula for future value of an ordinary annuity with $R = \frac{2000}{2} = 1000$, $i = \frac{.06}{2} = .03$, and $n = 25(2) = 50$.

$$\begin{aligned} S &= R\left[\frac{(1+i)^n - 1}{i}\right] \\ &= 1000\left[\frac{(1.03)^{50} - 1}{.03}\right] \\ &\approx 112,796.87 \end{aligned}$$

The total amount in the account will be \$112,796.87.

62. Use the formula for future value of an ordinary annuity with $R = \frac{2000}{2} = 1000$, $i = \frac{.04}{2} = .02$, and $n = 25(2) = 50$.

$$\begin{aligned} S &= R\left[\frac{(1+i)^n - 1}{i}\right] \\ &= 1000\left[\frac{(1.02)^{50} - 1}{.02}\right] \\ &\approx 84,579.40 \end{aligned}$$

The total amount in the account will be \$84,579.40.

64. Let $x =$ the annual interest rate.

$n = 20(12) = 240$

Graph $y_1 = 147{,}126$ and

$$y_2 = 300\left[\frac{\left(1+\frac{x}{12}\right)^{240}-1}{\frac{x}{12}}\right].$$

The x-coordinate of the point of intersection is .06499984. Thus, the annual interest rate was about 6.5%.

66. **(a)** Compare the future amounts for an ordinary annuity with $R = 1{,}350{,}000$ and $i = .08$ to compound amounts with $P = 7{,}000{,}000$ and $i = .08$ for different values of n, starting with $n = 1$.

n	$S = R\left[\frac{(1+i)^n-1}{i}\right]$	$A = P(1+i)^n$
1	$1{,}350{,}000\left[\frac{1.08-1}{.08}\right]$	
	$= \$1{,}350{,}000.00$	$\$7{,}560{,}000.00$
2	$1{,}350{,}000\left[\frac{(1.08)^2-1}{.08}\right]$	
	$= \$2{,}808{,}000.00$	$\$8{,}164{,}800.00$
3	$1{,}350{,}000\left[\frac{(1.08)^3-1}{.08}\right]$	
	$= \$4{,}382{,}640.00$	$\$8{,}817{,}984.00$
4	$1{,}350{,}000\left[\frac{(1.08)^4-1}{.08}\right]$	
	$= \$6{,}083{,}251.20$	$\$9{,}523{,}422.72$
5	$1{,}350{,}000\left[\frac{(1.08)^5-1}{.08}\right]$	
	$= \$7{,}919{,}911.30$	$\$10{,}285{,}296.54$
6	$1{,}350{,}000\left[\frac{(1.08)^6-1}{.08}\right]$	
	$= \$9{,}903{,}504.20$	$\$11{,}108{,}120.26$
7	$1{,}350{,}000\left[\frac{(1.08)^7-1}{.08}\right]$	
	$= \$12{,}045{,}784.54$	$\$11{,}996{,}769.88$

After 7 yr, the investors would do better by winning the lottery.

(b) Repeat the calculations from part (a), but change the interest rate to $i = .12$.

n	$S = R\left[\frac{(1+i)^n-1}{i}\right]$	$A = P(1+i)^n$
1	$\$1{,}350{,}000.00$	$\$7{,}840{,}000.00$
2	$\$2{,}862{,}000.00$	$\$8{,}780{,}800.00$
3	$\$4{,}555{,}440.00$	$\$9{,}834{,}496.00$
4	$\$6{,}452{,}092.80$	$\$11{,}014{,}635.52$
5	$\$8{,}576{,}343.93$	$\$12{,}336{,}391.78$
6	$\$10{,}955{,}505.21$	$\$13{,}816{,}758.80$
7	$\$13{,}620{,}165.83$	$\$15{,}474{,}769.85$
8	$\$16{,}604{,}585.73$	$\$17{,}331{,}742.23$
9	$\$19{,}947{,}136.02$	$\$19{,}411{,}551.30$

After 9 yr, the investors would do better by winning the lottery.

68. This exercise should be solved by graphing calculator or computer methods. The answers, which may vary slightly, are as follows.

(a) The amount of each interest payment is \$120.

(b) The amount of each payment is \$681.83, except the last payment, which is \$681.80. A table showing the amount in the sinking fund after each deposit is as follows.

Payment Number	Amount of Deposit	Interest Earned	Total
1	\$681.83	\$0	\$681.83
2	\$681.83	\$54.55	\$1418.12
3	\$681.83	\$113.46	\$2213.50
4	\$681.83	\$177.08	\$3072.41
5	\$681.80	\$245.79	\$4000.00

5.3 Present Value of an Annuity; Amortization

2. The symbol $s_{\overline{n}|i}$ represents the expression

$$\frac{(1+i)^n-1}{i},$$

which is choice (b).

4.
$$\begin{aligned} a_{\overline{n}|i} &= \frac{1-(1+i)^{-n}}{i} \\ a_{\overline{10}|.03} &= \frac{1-(1+.03)^{-10}}{.03} \\ &= \frac{1-(1.03)^{-10}}{.03} \\ &\approx 8.53020 \end{aligned}$$

6.
$$\begin{aligned} a_{\overline{n}|i} &= \frac{1-(1+i)^{-n}}{i} \\ a_{\overline{32}|.029} &= \frac{1-(1+.029)^{-32}}{.029} \approx 20.66906 \end{aligned}$$

8. Payments of \$890 each year for 16 yr at 8% compounded annually

Use the formula for present value of an annuity with $R = 890$, $i = .08$, and $n = 16$.

$$\begin{aligned} P &= R\left[\frac{1-(1+i)^{-n}}{i}\right] \\ &= 890\left[\frac{1-(1.08)^{-16}}{.08}\right] \\ &\approx 7877.72 \end{aligned}$$

The present value is \$7877.72.

10. Payments of \$10,000 semiannually for 15 yr at 10% compounded semiannually

Use the formula for present value of an annuity with $R = 10{,}000$, $i = \frac{.10}{2} = .05$, and $n = 15(2) = 30$.

$$\begin{aligned} P &= R\left[\frac{1-(1+i)^{-n}}{i}\right] \\ &= 10{,}000\left[\frac{1-(1.05)^{-30}}{.05}\right] \\ &\approx 153{,}724.51 \end{aligned}$$

The present value is \$153,724.51.

12. Payments of \$15,806 quarterly for 3 yr at 10.8% compounded quarterly

Here $R = 15{,}806$, $i = \frac{.108}{4} = .027$, and $n = 3(4) = 12$.

$$\begin{aligned} P &= R\left[\frac{1-(1+i)^{-n}}{i}\right] \\ &= 15{,}806\left[\frac{1-(1.027)^{-12}}{.027}\right] \\ &\approx 160{,}188.18 \end{aligned}$$

The present value is \$160,188.18.

14. 4% compounded annually

We want the present value, P, of an annuity with $R = 10{,}000$, $i = .04$, and $n = 15$.

$$\begin{aligned} P &= R\left[\frac{1-(1+i)^{-n}}{i}\right] \\ &= 10{,}000\left[\frac{1-(1.04)^{-15}}{.04}\right] \\ &\approx 111{,}183.87 \end{aligned}$$

The required lump sum is \$111,183.87.

18. $P = 41{,}000$; $i = \frac{.10}{2} = .05$; $n = 10$

Use the formula for amortization payments.

$$\begin{aligned} R &= \frac{Pi}{1-(1+i)^{-n}} \\ R &= \frac{41{,}000(.05)}{1-(1+.05)^{-10}} \\ &\approx 5309.69 \end{aligned}$$

Payments of \$5309.69 are necessary to amortize this loan.

20. $P = 140{,}000$; $i = \frac{.12}{4} = .03$; $n = 15$

$$\begin{aligned} R &= \frac{Pi}{1-(1+i)^{-n}} \\ R &= \frac{140{,}000(.03)}{1-(1+.03)^{-15}} \\ &\approx 11{,}727.32 \end{aligned}$$

Each payment is \$11,727.32.

22. $P = 5500$; $i = \frac{.125}{12} = .01041\overline{6}$; $n = 24$

$$\begin{aligned} R &= \frac{Pi}{1-(1+i)^{-n}} \\ &= \frac{5500(.01041\overline{6})}{1-(1+.01041\overline{6})^{-24}} \\ &\approx 260.19 \end{aligned}$$

Each payment is \$260.19.

24. The "Portion to Principal" column of the table indicates that \$87.10 of the 11th payment of \$88.85 is used to reduce the debt.

26. The amount of interest paid in the last 4 months of the loan is

$$\$3.47 + 2.61 + 1.75 + .88 = \$8.71.$$

28. \$4000 deposited every 6 mo for 10 yr at 6% compounded semiannually will be worth

$$4000 \cdot s_{\overline{20}|.03} \approx \$107,481.50.$$

(Note that $i = \frac{.06}{2} = .03$ and $n = 10(2) = 20$.)

For the lump sum investment of x dollars, use $i = \frac{.08}{2} = .02$ and $n = 10(4) = 40$ in the formula $A = P(1+i)^n$. Our unknown amount x will be worth $x(1.02)^{40}$, so

$$\begin{aligned} x(1.02)^{40} &= 107,481.50 \\ x &\approx 48,677.34. \end{aligned}$$

About \$48,677.34 should be invested today.

30. $P = 170,892$, $i = \frac{.0811}{12} = .0067583$, $n = 30(12) = 360$

$$\begin{aligned} R &= \frac{Pi}{1-(1+i)^{-n}} \\ R &= \frac{170,892(.0067583)}{1-(1+.0067583)^{-360}} \\ &\approx 1267.07 \end{aligned}$$

The monthly payment is \$1267.07.

32. $P = 96,511$, $i = \frac{.0957}{12} = .007975$, $n = 25(12) = 300$

$$\begin{aligned} R &= \frac{Pi}{1-(1+i)^{-n}} \\ R &= \frac{96,511(.007975)}{1-(1+.007975)^{-300}} \\ &\approx 847.91 \end{aligned}$$

The monthly payment is \$847.91.

34. (a) $R = 30$, $i = .0125$, $n = 12(3) = 36$

Use the formula for the present value of an annuity

$$P = 30\left[\frac{1-(1+.0125)^{-36}}{.0125}\right] \approx 865.42$$

Since there was a down payment of \$600, the cost of the stereo system will be

$$\$865.42 + 600 = \$1465.42.$$

(b) Since a payment of \$30 was paid each month for 36 months, the total of the payments will be 36(\$30) = \$1080. From part (a), the cost of the stereo system will be \$865.42. Therefore, the total amount of interest paid will be

$$\$1080 - 865.42 = \$214.58.$$

36. Use the formula for amortization payments with $R = 15,000$, $i = \frac{.10}{2} = .05$, and $n = 4(2) = 8$.

$$\begin{aligned} R &= \frac{Pi}{1-(1+i)^{-n}} \\ &= \frac{15,000(.05)}{1-(1.05)^{-8}} \\ &\approx 2320.83 \end{aligned}$$

The amount of each payment is \$2320.83.

38. (a) There is no interest with 0% financing, so the monthly payment is 25,500 ÷ 60 = \$425.

(b) $P = 21{,}500$, $i = \dfrac{.045}{12} = 0.00375$, $n = 48$

$$\begin{aligned} P &= R\left[\frac{1-(1+i)^{-n}}{i}\right] \\ 21{,}500 &= R\left[\frac{1-(1+.00375)^{-48}}{.00375}\right] \\ R &\approx 490.27 \end{aligned}$$

At 4.5% for 48 months, the monthly payment is \$490.27 and the total amount paid back is 490.27 × 48 = \$23,532.96.

$P = 21{,}500, i = \dfrac{.069}{12} = .00575, n = 60$

$$\begin{aligned} P &= R\left[\frac{1-(1+i)^{-n}}{i}\right] \\ 21{,}500 &= R\left[\frac{1-(1+.00575)^{-60}}{.00575}\right] \\ R &\approx 424.71 \end{aligned}$$

At 6.9% for 60 months, the monthly payment is \$424.71 and the total amount paid back is 424.71 × 60 = \$25,482.60.

(c) If the person can afford the higher monthly payments, the best deal is to choose the \$4000 cash back option and finance \$21,500 at 4.5% for 48 months because the total amount repaid is the least. However, if the person can only afford about \$425 per month, then the other two deals are about the same.

40. $P = 35,000$ at 7.43% compounded monthly for 10 yr. Thus, $i = \frac{.0743}{12} = .006191\overline{6}$ and $n = 10(12) = 120$.

$$\begin{aligned} R &= \frac{Pi}{1-(1+i)^{-n}} \\ &= \frac{35,000(.006191\overline{6})}{1-(1+.006191\overline{6})^{-120}} \\ &\approx 414.18 \end{aligned}$$

The monthly payment is \$414.18. The total interest is given by

$$120(414.18) - 35,000 = 14,701.60.$$

The total interest is \$14,701.60.

42. The amount of each annual payment is

$$R = \frac{4000}{a_{\overline{4}|.08}} \approx \$1207.68.$$

On the first payment, the firm owes interest of

$$I = Prt = 4000(.08)(1) = \$320.$$

Therefore, from the first payment, \$320 goes to interest and the balance,

$$\$1207.68 - 320 = \$887.68,$$

goes to principal. The principal at the end of one year is

$$\$4000 - 887.68 = \$3112.32.$$

The interest for the second year is

$$I = Prt = 3112.32(.08)(1) = \$248.99.$$

Of the second payment, \$248.99 goes to interest and

$$\$1207.68 - 248.99 = \$958.69$$

goes to principal. Continuing in this way, we obtain the following amortization schedule.

Payment Number	Amount of Payment	Interest for Period	Portion to Principal	Principal at End of Period
0	———	———	———	\$4000.00
1	\$1207.68	\$320.00	\$887.68	\$3112.32
2	\$1207.68	\$248.99	\$958.69	\$2153.63
3	\$1207.68	\$172.29	\$1035.39	\$1118.24
4	\$1207.70	\$89.46	\$1118.24	\$0.00

44. The total price is 8(\$1048) = \$8384. Since there is a \$1200 down payment, the amount financed is

$$\$8384 - 1200 = \$7184.$$

$i = \frac{.12}{12} = .01,\ n = 4(12) = 48$

The amount of each monthly payment is

$$R = \frac{7184}{a_{\overline{48}|.01}} \approx \$189.18.$$

For the first payment, the interest is

$$7184(.12)\left(\frac{1}{12}\right) = \$71.84.$$

The portion applied to principal is

$$\$189.18 - 71.84 = \$117.34.$$

The principal at the end of this period is

$$\$7184 - 117.34 = \$7066.66.$$

Continue in this way to complete the amortization schedule for the first four payments.

Payment Number	Amount of Payment	Interest for Period	Portion to Principal	Principal at End of Period
0	———	———	———	\$7184.00
1	\$189.18	\$71.84	\$117.34	\$7066.66
2	\$189.18	\$70.67	\$118.51	\$6948.15
3	\$189.18	\$69.48	\$119.70	\$6828.45
4	\$189.18	\$68.28	\$120.90	\$6707.55

46. The total amount of the loan is

$$14,000 + 7200 - 1200 = 20,000.$$

We have \$20,000 at 12% compounded semiannually for 5 yr.

(a) $i = \frac{.12}{2} = .06,\ n = 5(2) = 10$

$$\begin{aligned} R &= \frac{Pi}{1-(1+i)^{-n}} \\ &= \frac{20,000(.06)}{1-(1+.06)^{-10}} \approx 2717.36 \end{aligned}$$

The amount of each payment is \$2717.36.

(b) Graph

$$y_1 = 2717.36\left[\frac{1-1.06^{-(10-x)}}{.06}\right] \text{ and } y_2 = 5000.$$

The x-coordinate of the point of intersection is 7.9923292 or approximately 8. Therefore, 2 payments are left.

48. This is an amortization problem with $P = 25,000$. R represents the amount of each annual withdrawal.

(a) $i = .06,\ n = 8$

$$\begin{aligned} R &= \frac{Pi}{1-(1+i)^{-n}} \\ &= \frac{25,000(.06)}{1-(1+.06)^{-8}} \\ &\approx 4025.90 \end{aligned}$$

She will be able to withdraw about \$4025.90/yr for the 8 yr.

(b) $i = .06,\ n = 12$

$$\begin{aligned} R &= \frac{Pi}{1-(1+i)^{-n}} \\ &= \frac{25,000(.06)}{1-(1+.06)^{-12}} \\ &\approx 2981.93 \end{aligned}$$

She will be able to withdraw about \$2981.93/yr for the 12 yr.

50. This exercise should be solved by graphing calculator or computer methods. The amortization schedule, which may vary slightly, is as follows.

Payment Number	Amount of Payment	Interest for Period	Portion to Principal	Principal at End of Period
0	–	–	–	\$37,947.50
1	\$5783.49	\$3225.54	\$2557.95	\$35,389.55
2	\$5783.49	\$3008.11	\$2775.38	\$32,614.17
3	\$5783.49	\$2772.20	\$3011.29	\$29,602.88
4	\$5783.49	\$2516.24	\$3267.25	\$26,335.63
5	\$5783.49	\$2238.53	\$3544.96	\$22,790.67
6	\$5783.49	\$1937.21	\$3846.28	\$18,944.39
7	\$5783.49	\$1610.27	\$4173.22	\$14,771.17
8	\$5783.49	\$1255.55	\$4527.94	\$10,243.23
9	\$5783.49	\$870.67	\$ 4912.82	\$5330.41
10	\$5783.49	\$453.08	\$ 5330.41	\$0.00

Chapter 5 Review Exercises

2. $I = Prt$

$$\begin{aligned} &= 4902(.095)\left(\frac{11}{12}\right) \\ &\approx 426.88 \end{aligned}$$

The simple interest is \$426.88.

4. $I = Prt$

$$\begin{aligned} &= 3478(.074)\left(\frac{88}{360}\right) \\ &\approx 62.91 \end{aligned}$$

The simple interest is \$62.91.

6. \$2800 at 6% compounded annually for 10 yr

Use the formula for compound amount with $P = 2800,\ i = .06$, and $n = 10(1) = 10$.

$$\begin{aligned} A &= P(1+i)^n \\ &= 2800(1.06)^{10} \\ &\approx 5014.37 \end{aligned}$$

The compound amount is \$5014.37.

8. \$312.45 at 6% compounded semiannually for 16 yr

Use the formula for compound amount with $P = 312.45,\ i = \frac{.06}{2} = .03$, and $n = 16(2) = 32$.

$$\begin{aligned} A &= P(1+i)^n \\ &= 312.45(1.03)^{32} \\ &\approx 804.58 \end{aligned}$$

The compound amount is \$804.58.

10. \$3954 at 8% compounded annually for 12 yr

Here $P = 3954,\ i = .08$, and $n = 12(1) = 12$. First, find the compound amount.

$$\begin{aligned} A &= P(1+i)^n \\ &= 3954(1.08)^{12} \\ &\approx 9956.84 \end{aligned}$$

The compound amount is \$9956.84. To find the amount of interest earned, subtract the initial deposit from the compound amount.

$$\begin{aligned} \text{Amount of interest} &= A - P \\ &= \$9956.84 - 3954 \\ &= \$6002.84 \end{aligned}$$

12. $12,903.45 at 10.37% compounded quarterly for 29 quarters

Here $P = 12,903.45$, $i = \frac{.1037}{4}$, and $n = 29$.

$$\begin{aligned} A &= P(1+i)^n \\ &= 12,903.45\left(1+\frac{.1037}{4}\right)^{29} \\ &\approx 27,105.57 \end{aligned}$$

The compound amount is $27,105.57.

$$\begin{aligned} \text{Amount of interest} &= A - P \\ &= \$27,105.57 - 12,903.45 \\ &= \$14,202.12 \end{aligned}$$

16. $17,650 in 4 yr, 8% compounded quarterly

Use the formula for present value for compound interest with $A = 17,650$, $i = \frac{.08}{4} = .02$, and $n = 4(4) = 16$.

$$\begin{aligned} P &= \frac{A}{(1+i)^n} \\ &= \frac{17,650}{(1.02)^{16}} \\ &\approx 12,857.07 \end{aligned}$$

The present value is $\$12,857.07$.

18. $2388.90 in 44 mo, 5.93% compounded monthly

$A = 2388.90$, $i = \frac{5.93\%}{12} = \frac{.0593}{12}$, $n = 44$

$$\begin{aligned} P &= \frac{A}{(1+i)^n} \\ &= \frac{2388.90}{\left(1+\frac{.0593}{12}\right)^{44}} \\ &\approx 1923.09 \end{aligned}$$

The present value is $1923.09.

20. $a = 4$, $r = \frac{1}{2}$

The first four terms are

$$4, 4\left(\frac{1}{2}\right), 4\left(\frac{1}{2}\right)^2, 4\left(\frac{1}{2}\right)^3$$

or

$$4, 2, 1, \frac{1}{2}.$$

22. $a = -2$, $r = -2$

For the fifth term, $n = 5$, so

$$a_5 = ar^{n-1} = -2(-2)^4 = -2(16) = -32.$$

The fifth term is -32.

24. $a = 8000$, $r = -\frac{1}{2}$, $n = 5$

$$\begin{aligned} S_n &= \frac{a(r^n - 1)}{r-1} \\ S_5 &= \frac{8000\left[\left(-\frac{1}{2}\right)^5 - 1\right]}{-\frac{1}{2} - 1} \\ &= \frac{8000\left(-\frac{33}{32}\right)}{-\frac{3}{2}} = \frac{-8250}{-\frac{3}{2}} \\ &= (-8250)\left(-\frac{2}{3}\right) = 5500 \end{aligned}$$

26.

$$\begin{aligned} s_{\overline{n}|i} &= \frac{(1+i)^n - 1}{i} \\ s_{\overline{20}|.05} &= \frac{(1.05)^{20} - 1}{.05} \approx 33.06595 \end{aligned}$$

28. $R = 500$, $i = \frac{.06}{2} = .03$, $n = 8(2) = 16$

This is an ordinary annuity.

$$\begin{aligned} S &= Rs_{\overline{n}|i} \\ S &= 500s_{\overline{16}|.03} \\ &= 500\left[\frac{(1.03)^{16} - 1}{.03}\right] \\ &\approx 10,078.44 \end{aligned}$$

The future value is $10,078.44.

30. $R = 4000$, $i = \frac{.06}{4} = .015$, $n = 7(4) = 28$

This is an ordinary annuity.

$$\begin{aligned} S &= R\left[\frac{(1+i)^n - 1}{i}\right] \\ S &= 4000\left[\frac{(1+.015)^{28} - 1}{.015}\right] \\ &\approx 137,925.91 \end{aligned}$$

The future value is $137,925.91.

32. $R = 672$, $i = \frac{.08}{4} = .02$, $n = 7(4) = 28$

This is an annuity due, so we use the formula for future value of an ordinary annuity, but include one additional time period and subtract the amount of one payment.

$$\begin{aligned} S &= R\left[\frac{(1+i)^{n+1} - 1}{i}\right] - R \\ &= 672\left[\frac{(1+.02)^{29} - 1}{.02}\right] - 672 \\ &\approx 25,396.38 \end{aligned}$$

The future value is $25,396.38.

36. $S = 57,000,\ i = \frac{.06}{2} = .03,\ n = \left(8\frac{1}{2}\right)2 = 17$

Let R represent the amount of each payment.

$$\begin{aligned} S &= Rs_{\overline{n}|i} \\ 57,000 &= Rs_{\overline{17}|.03} \\ R &= \frac{57,000}{s_{\overline{17}|.03}} \\ &= \frac{57,000(.03)}{(1+.03)^{17}-1} \\ &\approx 2619.29 \end{aligned}$$

The amount of each payment is \$2619.29.

38. $S = 1,056,788,\ i = \frac{.0812}{12} \approx .0067667,$
$n = \left(4\frac{1}{2}\right)12 = 54$

Let R represent the amount of each payment.

$$\begin{aligned} S &= Rs_{\overline{n}|i} \\ 1,056,788 &= Rs_{\overline{54}|.0067667} \\ R &= \frac{1,056,788}{s_{\overline{54}|.0067667}} \\ &= \frac{1,056,788(.0067667)}{(1+.0067667)^{54}-1} \\ &\approx 16,277.35 \end{aligned}$$

The amount of each payment is \$16,277.35.

40. $R = 1500,\ i = \frac{.08}{4} = .02,\ n = 7(4) = 28$

$$\begin{aligned} P &= R\left[\frac{1-(1+i)^{-n}}{i}\right] \\ P &= 1500\left[\frac{1-(1+.02)^{-28}}{.02}\right] \\ &\approx 31,921.91 \end{aligned}$$

The present value is \$31,921.91.

42. $R = 877.34,\ i = \frac{.094}{12} \approx .0078333,\ n = 17$

$$\begin{aligned} P &= R\left[\frac{1-(1+i)^{-n}}{i}\right] \\ P &= 877.34\left[\frac{1-(1+.0078333)^{-17}}{.0078333}\right] \\ &\approx 13,913.48 \end{aligned}$$

The present value is \$13,913.48.

44. $P = 80,000,\ i = .08,\ n = 9$

$$\begin{aligned} R &= \frac{Pi}{1-(1+i)^{-n}} \\ &= \frac{80,000(.08)}{1-(1+.08)^{-9}} \\ &\approx 12,806.38 \end{aligned}$$

The amount of each payment is \$12,806.38.

46. $P = 32,000,\ i = \frac{.094}{4} = .0235,\ n = 17$

$$\begin{aligned} R &= \frac{Pi}{1-(1+i)^{-n}} \\ &= \frac{32,000(.0235)}{1-(1+.0235)^{-17}} \\ &\approx 2305.07 \end{aligned}$$

The amount of each payment is \$2305.07.

48. $P = 56,890,\ i = \frac{.1074}{12} = .00895,\ n = 25(12) = 300$

$$\begin{aligned} R &= \frac{Pi}{1-(1+i)^{-n}} \\ &= \frac{56,890(.00895)}{1-(1+.00895)^{-300}} \\ &\approx 546.93 \end{aligned}$$

The monthly payment is \$546.93.

50. Look in the column titled "Interest for Period." \$896.06 of the fifth payment of \$1022.64 is interest.

52. In the first 3 months of the loan, the total amount of interest paid is

$$\$899.58 + 898.71 + 897.83 = \$2696.12.$$

54. Here $P = 5800,\ r = 10.3\% = .103$, and $t = \frac{10}{12}$.

$$\begin{aligned} I &= Prt \\ &= 5800(.103)\left(\frac{10}{12}\right) \\ &\approx 497.83 \end{aligned}$$

The interest he will pay is \$497.83. The total amount he will owe is

$$\$5800 + 497.83 = \$6297.83.$$

56. Here $P = 28,000,\ r = 11.5\% = .115$, and $I = 3255$.

Use the formula for simple interest.

$$\begin{aligned} I &= Prt \\ t &= \frac{I}{Pr} \\ &= \frac{3255}{28,000(.115)} \\ &\approx 1.011 \end{aligned}$$

The loan is for about 1.011 yr; convert this to months.

$$1.011 \text{ yr} \left(\frac{12 \text{ mo}}{1 \text{ yr}}\right) \approx 12.13 \text{ mo}$$

The loan is for about 12.13 mo.

58. $A = 7500,\ i = \frac{.10}{2} = .05,\ n = 3(2) = 6$

Let P represent the lump sum.

$$\begin{aligned} A &= P(1+i)^n \\ P &= \frac{A}{(1+i)^n} \\ &= \frac{7500}{(1.05)^6} \\ &\approx 5596.62 \end{aligned}$$

She should deposit about \$5596.62 today.

60. Suppose you receive \$6000/yr at age 55 until age 75. Then $R = 6000,\ i = .08$, and $n = 20$.

$$S = 6000\left[\frac{(1+.08)^{20}-1}{.08}\right] \approx 274,571.79$$

You would receive a total of \$274,571.79.

Suppose you receive \$12,000/yr at age 65 until age 75. Then $R = 12,000,\ i = .08$, and $n = 10$.

$$S = 12,000\left[\frac{(1+.08)^{10}-1}{.08}\right] \approx 173,838.75$$

You would receive a total of \$173,838.75.

Receiving half the pension at 55 would produce the larger amount.

62. Use the formula for amortization payments with $P = 28,000,\ i = \frac{.12}{4} = .03$, and $n = 6\frac{1}{2}(4) = 26$.

$$R = \frac{Pi}{1-(1+i)^{-n}} = \frac{28,000(.03)}{1-(1.03)^{-26}} \approx 1566.27$$

The amount of each payment is \$1566.27.

64. Use the formula for compound amount with $P = 3250,\ i = .09$, and $n = 4$.

$$A = P(1+i)^n = 3250(1.09)^4 \approx 4587.64$$

Mark must pay back \$4587.64.

66. Use the formula for amortization payments with $P = 115,700,\ i = \frac{.105}{12} = .00875$, and $n = 300$.

$$\begin{aligned} R &= \frac{Pi}{1-(1+i)^{-n}} \\ &= \frac{115,700(.00875)}{1-(1+.00875)^{-300}} \\ &\approx 1092.42 \end{aligned}$$

Each monthly payment will be about \$1092.42. The total amount of interest will be

$$300(\$1092.42) - 115,700 = \$212,026.$$

68. **(a)** There is no interest with 0% financing, so the monthly payment is 31,500 ÷ 60 = \$525.

(b) $P = 28{,}000, i = \dfrac{.049}{12}, n = 48$

$$\begin{aligned} P &= R\left[\frac{1-(1+i)^{-n}}{i}\right] \\ 28{,}000 &= R\left[\frac{1-(1+\frac{.049}{12})^{-48}}{\frac{.049}{12}}\right] \\ R &\approx 643.55 \end{aligned}$$

At 4.9% for 48 months, the monthly payment is \$643.55 and the total amount paid back is 643.55 × 48 = \$30,890.40.

$P = 28{,}000,\ i = \dfrac{.055}{12},\ n = 60$

$$\begin{aligned} P &= R\left[\frac{1-(1+i)^{-n}}{i}\right] \\ 28{,}000 &= R\left[\frac{1-(1+\frac{.055}{12})^{-60}}{\frac{.055}{12}}\right] \\ R &\approx 534.83 \end{aligned}$$

At 5.5% for 60 months, the monthly payment is \$534.83 and the total amount paid back is $534.83 \times 60 = \$32{,}089.80$.

(c) If the person can afford the higher monthly payments, the best deal is to choose the \$3500 cash back option and finance \$28,000 at 4.9% for 48 months because the total amount repaid is the least. However, if the person cannot afford \$643.55 per month, then the best deal is 0% financing because the monthly payments are the lowest and the total amount repaid is less than the third financing option.

70. The death benefit grows to

$$10,000(1.05)^7 \approx 14,071.$$

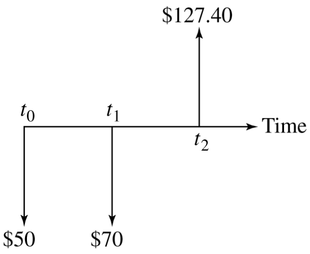
This 14,071 is the present value of an annuity due with $P = 14,071$, $i = \frac{.03}{12} = .0025$, and $n = 120$. Let X represent the amount of each monthly payment.

$$\begin{aligned} P &= R \cdot a_{\overline{n+1}|i} - R \\ 14,071 &= X \cdot a_{\overline{121}|.0025} - X \\ 14,071 &= (a_{\overline{121}|.0025} - 1)X \\ 14,071 &\approx (104.301 - 1)X \\ 14,071 &\approx 103.301X \\ 135 &\approx X \end{aligned}$$

Each payment is about \$135, which corresponds to choice (d).

Extended Application: Time, Money, and Polynomials

1.



The polynomial equation is

$$50(1+i)^2 + 70(1+i) - 127.40 = 0.$$

Let $x = 1 + i$. The equation becomes

$$50x^2 + 70x - 127.40 = 0.$$

Use the quadratic formula to solve the equation for x.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-70 \pm \sqrt{70^2 - 4(50)(-127.40)}}{2(50)}$$

Reject $x = \frac{-70-\sqrt{30,380}}{100}$ because it is negative. Thus,

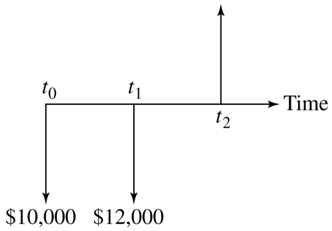
$$x = \frac{-70 + \sqrt{30,380}}{100} \approx 1.04298.$$

Since $x = 1 + i$,

$$\begin{aligned} 1 + i &= 1.04298 \\ i &\approx .043. \end{aligned}$$

Thus, the YTM is 4.3%.

2. (a)



(b) $\begin{aligned} A &= 1.05(10,000) + .045(1.05)(10,000) \\ &\quad + 1.045(12,000) \\ &= 23,512.5 \end{aligned}$

At the end of the second year, \$23,512.50 was in the account.

(c) The polynomial equation is

$$10,000(1+i)^2 + 12,000(1+i) = 23,512.50 = 0.$$

Let $x = 1 + i$. The equation becomes

$$10,000x^2 + 12,000x - 23,512.50 = 0.$$

Use the quadratic formula to solve for x.

$$x = \frac{-12,000 \pm \sqrt{12,000^2 - 4(10,000)(-23,512.50)}}{2(10,000)}$$

Reject

$$x = \frac{-12,000 - \sqrt{12,000^2 - 4(10,000)(-23,512.50)}}{20,000}$$

because it is negative. Thus,

$$x = \frac{-12,000 + \sqrt{12,000^2 - 4(10,000)(-23,512.50)}}{20,000}$$

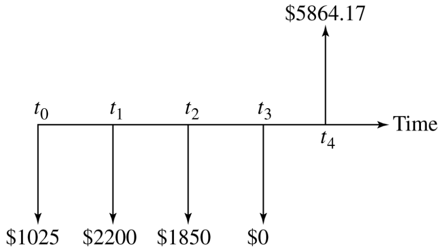
$$\approx 1.04658.$$

Since $x = 1 + i$,

$$\begin{aligned} 1 + i &= 1.04658 \\ i &= .04658. \end{aligned}$$

Thus, the YTM is 4.7%. As might be expected, the YTM is between 4.5% and 5%.

3. **(a)**



(b) The polynomial equation is

$$\begin{aligned} &1025(1 + i)^4 + 2200(1 + i)^3 \\ &\quad + 1850(1 + i)^2 - 5864.17 = 0. \end{aligned}$$

Let $x = 1 + i$. The equation becomes

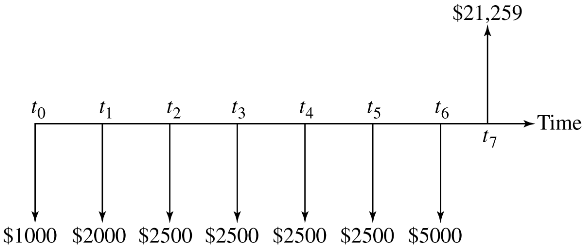
$$1025x^4 + 2200x^3 + 1850x^2 - 5864.17 = 0.$$

Let $f(x) = 1025x^4 + 2200x^3 + 1850x^2 - 5864.17$. Since $0 < i < 1$, then $1 < x < 2$ and

$$\begin{aligned} f(1) &= -789.17; \\ f(1.1) &= 803.2325; \\ f(1.05) &= -31.8761; \\ f(1.052) &= .00172. \end{aligned}$$

The YTM is approximately 5.2%.

4. **(a)**



(b) The polynomial equation is

$$\begin{aligned} &1000(1 + i)^7 + 2000(1 + i)^6 + 2500(1 + i)^5 \\ &+2500(1 + i)^4 + 2500(1 + i)^3 + 2500(1 + i)^2 \\ &+5000(1 + i) - 21,259 = 0. \end{aligned}$$

(c) Let $x = 1 + i$ and

$$\begin{aligned} f(x) = 1000(&x^7 + 2x^6 + 2.5x^5 + 2.5x^4 + 2.5x^3 \\ &+ 2.5x^2 + 5x - 21.259). \end{aligned}$$

Then

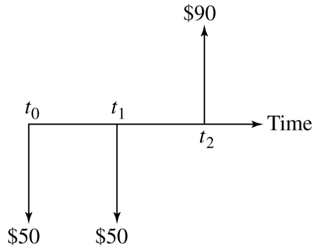
$$\begin{aligned} f(1.0507) &= 7.1216; \\ f(1.0505) &= -6.9040. \end{aligned}$$

Since $f(1.0507)$ is positive and $f(1.0505)$ is negative, the value of x that makes $f(x)$ zero is between 1.0507 and 1.0505.

$$\begin{aligned} 1.0505 < 1 &+ i < 1.0507 \\ .0505 &< i < .0507 \\ 5.05\% &< i < 5.07\% \end{aligned}$$

(d) Graph $y_1 = f(x)$. The graph intersects the x-axis at $x = 1.0505985$. Therefore, $i = .0505985$ or 5.06%.

5. **(a)**



The polynomial equation is

$$50(1 + i)^2 + 50(1 + i) - 90 = 0.$$

Let $x = 1 + i$. The equation becomes

$$\begin{aligned} 50x^2 + 50x - 90 &= 0 \\ 5x^2 + 5x - 9 &= 0. \end{aligned}$$

Solve for x using the quadratic formula.

$$x = \frac{-5 \pm \sqrt{5^2 - 4(5)(-9)}}{2(5)}$$

$$x = \frac{-5 \pm \sqrt{205}}{10}$$

$$x = \frac{-5 + \sqrt{205}}{10} \quad \text{or} \quad x = \frac{-5 - \sqrt{205}}{10}$$

$$\approx .93178 \qquad\qquad \approx -1.93178$$

Then

$$\begin{aligned} 1 + i &= .93178 &\text{or}\quad 1 + i &= -1.93178 \\ i &= -.06822 & i &= -2.93178 \\ i &\approx -6.8\% &\text{or}\quad i &\approx -293.2\%. \end{aligned}$$

(b) For $i = -6.8\%$ use the formula for compound amount for each \$50 investment. For the first investment $P = 50$ and $n = 2$.

$$\begin{aligned} A &= P(1 + i)^n \\ &= 50(1 - .068)^2 \\ &= 43.4312 \end{aligned}$$

For the second investment $P = 50$ and $n = 1$.

$$\begin{aligned} A &= P(1+i)^n \\ &= 50(1-.068)^1 \\ &= 46.6 \end{aligned}$$

Each value of A seems to be a reasonable future value for the investment considered. Also, note that

$$43.4312 + 46.6 = 90.0312.$$

For $i = -293.2\%$ use the formula for compound amount for each \$50 investment. For the first investment $P = 50$ and $n = 2$.

$$\begin{aligned} A &= P(1+i)^n \\ &= 50(1-2.932)^2 \\ &= 186.6312 \end{aligned}$$

For the second investment $P = 50$ and $n = 1$.

$$\begin{aligned} A &= P(1+i)^n \\ &= 50(1-2.932)^1 \\ &= -96.6 \end{aligned}$$

Although $186.6312 - 96.6 = 90.0312$, neither value seems like a reasonable future value for the investment considered. Only $i = -6.8\%$ seems a reasonable interpretation in the context to the problem.

Chapter 6

LOGIC

6.1 Statements and Quantifiers

2. Because the declarative sentence "The ZIP code for Manistee, MI, is 49660" has the property of being true or false, it is considered a statement.

4. "Yield to oncoming traffic" is a command, not a declarative sentence and therefore, it is not a statement.

6. "$5 + 8 = 12$ and $4 - 3 = 1$" is a declarative sentence that is false and therefore, is considered a statement.

8. "Behave yourself and sit down" is a command, not a declarative sentence, and therefore, is not considered a statement.

10. "Tomorrow is not Sunday" is a compound statement because it contains the logical connective "not."

12. "The sign on the back of the car read 'California or bust!"' is not compound because only one assertion is being made.

14. "If Mike is a politician, then Jerry is a crook" is a compound statement because it consists of two simple statements combined by the connective "if ... then."

16. The negation of "The flowers are to be watered" is "The flowers are not to be watered."

18. A negation for "$x < -6$" (without using a slash sign) would be "$x \geq -6$."

20. A negation for "$r \leq 19$" would be "$r > 19$."

24. A translation for "$\sim o$" is "He is not 48 years old."

26. A translation for "$g \vee o$" is "She has green eyes or he is 48 years old."

28. A translation for "$g \wedge \sim o$" is "She has green eyes and he is not 48 years old."

30. A translation for "$\sim g \wedge \sim o$" is "She does not have green eyes and he is not 48 years old."

32. A translation for "$\sim (g \vee \sim o)$" is "It is not the case that she has green eyes or he is not 48 years old."

34. If q is true, then $q \vee (q \wedge \sim p)$ is true, since only one part of the disjunction statement must be true for the compound statement to be true.

36. If $p \vee q$ is false, and p is false, then q must also be false. Observe that both parts of the disjunction must be false for a disjunction to be false.

38. If $\sim (p \wedge q)$ is false, then both p and q must be true. This will assure that the conjunction itself is true making its negation false.

40. Since $q = \text{T}$,

$$\begin{aligned} \sim q &= \sim \text{T} \\ &= \text{F}. \end{aligned}$$

Thus, $\sim q$ is false.

42. Since p is false and q is true, we may consider the "and" statement as

$$\begin{gathered} \text{F} \wedge \text{T} \\ \text{F}, \end{gathered}$$

by the logical definition of an "and" statement. That is, $p \wedge q$ is false.

44. With the given truth values for p and q we may consider $\sim p \wedge q$ as

$$\begin{gathered} \sim \text{F} \wedge \text{T} \\ \text{T} \wedge \text{T} \\ \text{T}, \end{gathered}$$

by the logical definition of "$\wedge$". Thus, the compound statement is true.

46. Replacing p and q with the given truth values, we have

$$\begin{gathered} \text{F} \wedge \sim \text{T} \\ \text{F} \wedge \text{F} \\ \text{F}. \end{gathered}$$

Thus the compound statement $p \wedge \sim q$ is false.

48. Replacing p and q with the given truth values, we have

$$\begin{gathered}\sim(\sim\mathrm{F} \vee \sim\mathrm{T})\\ \sim(\mathrm{T} \vee \mathrm{F})\\ \sim \mathrm{T}\\ \mathrm{F}.\end{gathered}$$

Thus, the statement $\sim(\sim p \vee \sim q)$ is false.

50. Replacing p and q with the given truth values, we have

$$\begin{gathered}\sim[(\sim\mathrm{F} \wedge \sim\mathrm{T}) \vee \sim\mathrm{T}]\\ \sim[(\mathrm{T} \wedge \mathrm{F}) \vee \mathrm{F}]\\ \sim[\mathrm{F} \vee \mathrm{F}]\\ \sim\mathrm{F}\\ \mathrm{T}.\end{gathered}$$

Thus, the statement $\sim[(\sim p \wedge \sim q) \vee \sim q]$ is true.

52. The statement "$6 \geq 2$" is true because $6 > 2$.
The statement "$6 \geq 6$" is true because $6 = 6$.

54. Replacing p, q, and r with the given truth values, we have

$$\begin{gathered}(\mathrm{F} \vee \sim\mathrm{F}) \wedge \mathrm{T}\\ (\mathrm{F} \vee \mathrm{T}) \wedge \mathrm{T}\\ \mathrm{T} \wedge \mathrm{T}\\ \mathrm{T}.\end{gathered}$$

Thus, the statement $(q \vee \sim r) \wedge p$ is true.

56. Replacing p, q and r with the given truth values, we have

$$\begin{gathered}(\sim\mathrm{T} \wedge \mathrm{F}) \vee \sim\mathrm{F}\\ (\mathrm{F} \wedge \mathrm{F}) \vee \mathrm{T}\\ \mathrm{F} \vee \mathrm{T}\\ \mathrm{T}.\end{gathered}$$

Thus, the statement $(\sim p \wedge q) \vee \sim r$ is true.

58. Replacing p, q and r with the given truth values, we have

$$\begin{gathered}(\sim\mathrm{F} \wedge \sim\mathrm{F}) \vee (\sim\mathrm{F} \wedge \mathrm{F})\\ (\mathrm{T} \wedge \mathrm{T}) \vee (\mathrm{T} \wedge \mathrm{F})\\ \mathrm{T} \vee \mathrm{F}\\ \mathrm{T}.\end{gathered}$$

Thus, the statement $(\sim r \wedge \sim q) \vee (\sim r \wedge q)$ is true.

60. Replacing p, q and r with the given truth values, we have

$$\begin{gathered}\sim[\mathrm{F} \vee (\sim\mathrm{F} \wedge \sim\mathrm{T})]\\ \sim[\mathrm{F} \vee (\mathrm{T} \wedge \mathrm{F})]\\ \sim[\mathrm{F} \vee \mathrm{F}]\\ \sim\mathrm{F}\\ \mathrm{T}.\end{gathered}$$

Thus, the statement $\sim[r \vee (\sim q \wedge \sim p)]$ is true.

62. Since p and q are false, we have

$$\begin{gathered}\mathrm{F} \vee \sim\mathrm{F}\\ \mathrm{F} \vee \mathrm{T}\\ \mathrm{T}.\end{gathered}$$

The statement $p \vee \sim q$ is true.

64. Since p is false and r is true, we have

$$\begin{gathered}\sim\mathrm{F} \wedge \sim\mathrm{T}\\ \mathrm{T} \wedge \mathrm{F}\\ \mathrm{F}.\end{gathered}$$

The statement $\sim p \wedge \sim r$ is false.

66. Since p and q are false and r is true, we have

$$\begin{gathered}\sim\mathrm{F} \vee (\sim\mathrm{T} \vee \sim\mathrm{F})\\ \mathrm{T} \vee (\mathrm{F} \vee \mathrm{T})\\ \mathrm{T} \vee \mathrm{T}\\ \mathrm{T}.\end{gathered}$$

The statement $\sim p \vee (\sim r \vee \sim q)$ is true.

68. Since p and q are false and r is true, we have

$$\begin{gathered}\sim(\mathrm{F} \vee \sim\mathrm{F}) \vee \sim\mathrm{T}\\ \sim(\mathrm{F} \vee \mathrm{T}) \vee \mathrm{F}\\ \sim\mathrm{T} \vee \mathrm{F}\\ \mathrm{F} \vee \mathrm{F}\\ \mathrm{F}.\end{gathered}$$

The statement $\sim(p \vee \sim q) \vee \sim r$ is false.

70. **(a)** This is a compound statement, formed using "The Ohio laws against discrimination require that all creditors make credit equally available to all creditworthy customers" and "The Ohio laws against discrimination require that credit reporting agencies maintain separate credit histories on each individual upon request.

(b), (c), (d) These are simple statements, not compound ones.

72. **(a), (b), (d)** These are declarative sentences that are true or false and therefore statements.

(c) This is a command, not a declarative sentence, and therefore not a statement.

74. The negation is formed by adding "not": "Life insurance payments are not not subject to income tax," which simplifies to "Life insurance payments are subject to income tax.

76. **(a)** This is a question, not a declarative sentence, and therefore not a statement.

(b), (c), (d), (e) There are declarative sentences that are true or false and therefore statements.

78. The negation of "A number of viral illnesses can cause swollen glands," which means "Some viral illnesses can cause swollen glands," is "No viral illnesses cause swollen glands" or "Viral illnesses do not cause swollen glands."

80. **(a), (b)** These are commands, not declarative sentences, and therefore not statements.

(c) This is compound statement, formed using the statements "The court won't do it for you" and "Hiring an attorney is usually not cost-effective given the small amount of money involved."

(d) This is a compound statement, formed using the statements "You can't marry," "You're at least 18 years old, and "You have the permission of your parents or guardian."

(e) This is a simple statement, not a compound one.

84. "Chris collects videotapes and Jack does not play the tuba" may be symbolized as $c \wedge \sim j$.

86. "Chris does not collect videotapes or Jack plays the tuba" may be symbolized as $\sim c \vee j$.

88. "Neither Chris collects videotapes nor Jack plays the tuba" may be symbolized as $\sim (c \vee j)$ or equivalently, $\sim c \wedge \sim j$.

90. Assume that c is true and that j is true. Under these conditions, the statements in Exercises 84-89 have the following truth values.

84. $c \wedge \sim j$: False, because c is true and $\sim j$ is false.
85. $\sim c \vee \sim j$: False, because $\sim c$ is false and $\sim j$ is false.
86. $\sim c \vee j$: True, because $\sim c$ is false but j is true.
87. $j \wedge \sim c$: False, because j is true and $\sim c$ is false.
88. $\sim c \wedge \sim j$: False, because $\sim c$ and $\sim j$ are both false, so their conjunction is false.
89. $(j \vee c) \wedge [\sim (j \wedge c)]$: False, because $j \vee c$ is true but $\sim (j \wedge c)$ is false.

The only statement that is true is the statement in Exercises 86.

6.2 Truth Tables and Equivalent Statements

2. Since there are three simple statements (p, q, and r), we have $2^3 = 8$ rows in the truth table.

4. Since there are five simple statements (p, q, r, s, and t), we have $2^5 = 32$ rows in the truth table.

6. Since there are eight simple statements (p, q, r, s, m, n, u, and v), we have $2^8 = 256$ rows in the truth table.

8. It is not possible for a truth table of a compound statement to have exactly 48 rows, because 48 is not a whole number power of 2.

10. $\sim p \vee \sim q$

p	q	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

12. $p \vee \sim q$

p	q	$\sim q$	$p \vee \sim q$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

14. $(p \wedge \sim q) \wedge p$

p	q	$\sim q$	$p \wedge \sim q$	$(p \wedge \sim q) \wedge p$
T	T	F	F	F
T	F	T	T	T
F	T	F	F	F
F	F	T	F	F

In Exercises 16–24 to save space we are using the alternative method, filling in columns in the order indicated by the numbers. Observe that columns with the same number are combined (by the logical definition of the connective) to get the next numbered column. Note that this is different from the way the numbered columns are used in the textbook. Remember that the last column (highest numbered column) completed yields the truth values for the complete compound statement. Be sure to align truth values under the appropriate logical connective or simple statement.

16. $\sim p \vee (\sim q \wedge \sim p)$

p	q	$\sim p$	$\vee$	$(\sim q$	$\wedge$	$\sim p)$
T	T	F	F	F	F	F
T	F	F	F	T	F	F
F	T	T	T	F	F	T
F	F	T	T	T	T	T
		1	4	2	3	2

18. $(\sim p \wedge \sim q) \vee (\sim p \vee q)$

p	q	$(\sim p$	$\wedge$	$\sim q)$	$\vee$	$(\sim p$	$\vee$	$q)$
T	T	F	F	F	T	F	T	T
T	F	F	F	T	F	F	F	F
F	T	T	F	F	T	T	T	T
F	F	T	T	T	T	T	T	F
		1	2	1	5	3	4	3

20. $r \vee (p \wedge \sim q)$

p	q	r	r	$\vee$	$(p$	$\wedge$	$\sim q)$
T	T	T	T	T	T	F	F
T	T	F	F	F	T	F	F
T	F	T	T	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	F	F	F
F	T	F	F	F	F	F	F
F	F	T	T	T	F	F	T
F	F	F	F	F	F	F	T
			1	4	2	3	2

22. $(\sim r \vee \sim p) \wedge (\sim p \vee \sim q)$

p	q	r	$(\sim r$	$\vee$	$\sim p)$	$\wedge$	$(\sim p$	$\vee$	$\sim q)$
T	T	T	F	F	F	F	F	F	F
T	T	F	T	T	F	F	F	F	F
T	F	T	F	F	F	F	F	T	T
T	F	F	T	T	F	T	F	T	T
F	T	T	F	T	T	T	T	T	F
F	T	F	T	T	T	T	T	T	F
F	F	T	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T	T
			1	2	1	5	3	4	3

24. $(\sim r \vee s) \wedge (\sim p \wedge q)$

p	q	r	s	$(\sim r$	$\vee$	$s)$	$\wedge$	$(\sim p$	$\wedge$	$q)$
T	T	T	T	F	T	T	F	F	F	T
T	T	T	F	F	F	F	F	F	F	T
T	T	F	T	T	T	T	F	F	F	T
T	T	F	F	T	T	F	F	F	F	T
T	F	T	T	F	T	T	F	F	F	F
T	F	T	F	F	F	F	F	F	F	F
T	F	F	T	T	T	T	F	F	F	F
T	F	F	F	T	T	F	F	F	F	F
F	T	T	T	F	T	T	T	T	T	T
F	T	T	F	F	F	F	F	T	T	T
F	T	F	T	T	T	T	T	T	T	T
F	T	F	F	T	T	F	T	T	T	T
F	F	T	T	F	T	T	F	T	F	F
F	F	T	F	F	F	F	F	T	F	F
F	F	F	T	T	T	T	F	T	F	F
F	F	F	F	T	T	F	F	T	F	F
				1	2	1	5	3	4	3

26. "I am not going or she is going" has the symbolic form $\sim p \vee q$. Its negation, $\sim(\sim p \vee q)$, is equivalent, by De Morgan's law, to $p \wedge \sim q$. The word translation for the negation is "I am going and she is not going."

28. "1/2 is a positive number and −12 is less than zero" is of the form $p \wedge q$. The negation, $\sim(p \wedge q)$, is equivalent, by De Morgan's law, to $\sim p \vee \sim q$. The word translation for the negation is "1/2 is not a positive number or $-12 \geq$ zero." (Note that the inequality "$\geq$ " is equivalent to "not less than.")

30. “Linda Nelson tried to sell the book, but she was unable to do so” is of the form $p \wedge q$. The negation, $\sim(p \wedge q)$, equivalent, by De Morgan’s law, to $\sim p \vee \sim q$. The word translation for the negation is “Linda Nelson did not try to sell the book, or she was able to do so.”

32. “$3 < 10$ or $7 \neq 2$” is of form $p \vee \sim q$. The negation, $\sim(p \vee \sim q)$, is equivalent, by De Morgan’s law, to $\sim p \wedge q$. A translation for the negation is “$3 \geq 10$ and $7 = 2$. (Note that the inequality “$\geq$ ” is equivalent to “$\not<$ ”).

34. “The lawyer and the client appeared in court” is of the form $p \wedge q$. The negation, $\sim(p \wedge q)$, is equivalent, by De Morgan’s law, to $\sim p \vee \sim q$. The word translation for the negation is “The lawyer did not appear in court or the client did not appear in court.”

36. “$3 + 1 = 4 \underline{\vee} 2 + 5 = 7$” is false since both component statements are true.

38. “$3 + 1 = 7 \underline{\vee} 2 + 5 = 7$” is true since the first component statement is false and the second is true.

40. The negation of $p \wedge q$ is $\sim p \vee \sim q$. The Ohio laws against discrimination either do not require that all creditors make credit equally available to all credit-worthy customers or they do not require that credit reporting agencies maintain separate credit histories on each individual upon request.

44. **(a)** p: Tissue samples may be taken from almost anywhere in the body.
q: The procedure used depends on the site.
The statement is $p \wedge q$. The negation is $\sim p \vee \sim q$.
Negation: Either tissue samples may not be taken from almost anywhere in the body or the procedure need does not depend on the site.

(b) p: The doctor or nurse holds your tongue down with a depressor.
q: The doctor or nurse uses a plastic stick with a sterile cotton end (swab) to collect a fluid sample from your throat.
The statement is $p \wedge q$. The negation is $\sim p \vee \sim q$.
Negation: Either the doctor or nurse does not hold your tongue down with a depressor or the doctor or nurse does not use a plastic stick with a sterile cotton end (swab) to collect a fluid sample from your throat.

(c) p: The doctor holds your tongue down with a depressor.
q: The doctor wipes a swab over your tonsils and the back of your throat.
The statement is $p \wedge q$. The negation is $\sim p \vee \sim q$.
Negation: Either the doctor does not hold your tongue down with a depressor or the doctor does not wipe a swab over your tonsils and the back of your throat.

46. Inclusive, since the “and” case is allowed.

48. p: You could reroll the die again for your Large Straight.
q: You could set aside the 2 Twos and roll for your Twos or for 3 of a Kind.
The statement is $p \vee q$.
Its negation is $\sim p \wedge \sim q$.
Negation: You cannot reroll the die again for your Large Straight and you cannot set aside the 2 Twos and roll for your Twos or for 3 of a Kind.

6.3 The Conditional and Circuits

2. The statement “If the consequent of a conditional statement is true, the conditional statement is true” is true, since a true consequent is always associated with a true conditional statement (i.e., it doesn’t matter what the truth value of the antecedent is, if the consequent itself is true).

4. The statement “If p is true, then $\sim p \rightarrow (q \vee r)$ is true” is true since the antecedent, $\sim p$, is false.

6. “Given that $\sim p$ is false and q is false, the conditional $p \rightarrow q$ is true” is a false statement since the antecedent, p, must be true, so that $p \rightarrow q$ is false.

10. “$(6 \geq 6) \rightarrow F$” is a false statement, since the antecedent is true and the consequent is false.

12. “$(4^2 \neq 16) \rightarrow (4 - 4 = 8)$” is a true statement, since a false antecedent always yields a true conditional statement.

14. “$(s \wedge p) \rightarrow m$,” expressed in words, becomes “If she has a snake for a pet and he trains ponies, then they raise monkeys.”

16. “$(\sim s \vee \sim m) \rightarrow \sim p$,” expressed in words, becomes “If she does not have a snake for a pet or they do not raise monkeys, then he does not train ponies.”

18. The statement "If the play is cancelled, then it does not rain" can be symbolized as "$p \to \sim r$."

20. The statement "The play is cancelled, and if it rains then I do not ride my bike" can be symbolized as "$p \wedge (r \to \sim b)$."

22. The statement "It rains if the play is cancelled" can be symbolized as "$p \to r$."

24. Replacing p and r with the given truth values, we have

$$\begin{gathered}\sim \mathrm{F} \to \sim \mathrm{F} \\ \mathrm{T} \to \mathrm{T} \\ \mathrm{T}.\end{gathered}$$

Thus, the statement $\sim p \to \sim r$ is true.

26. Replacing q and r with the given truth values, we have

$$\begin{gathered}\sim \mathrm{T} \to \mathrm{F} \\ \mathrm{F} \to \mathrm{F} \\ \mathrm{T}.\end{gathered}$$

Thus, the statement $\sim q \to r$ is true.

28. Replacing r and p with the given truth values, we have

$$\begin{gathered}(\sim \mathrm{F} \vee \mathrm{F}) \to \mathrm{F} \\ (\mathrm{T} \vee \mathrm{F}) \to \mathrm{F} \\ \mathrm{T} \to \mathrm{F} \\ \mathrm{F}.\end{gathered}$$

Thus, the statement $(\sim r \vee p) \to p$ is false.

30. Replacing $p, q,$ and r with the given truth values, we have

$$\begin{gathered}(\sim \mathrm{F} \wedge \sim \mathrm{T}) \to (\mathrm{F} \wedge \sim \mathrm{F}) \\ (\mathrm{T} \wedge \mathrm{F}) \to (\mathrm{F} \wedge \mathrm{T}) \\ \mathrm{F} \to \mathrm{F} \\ \mathrm{T}.\end{gathered}$$

Thus, the statement $(\sim p \wedge \sim q) \to (p \wedge \sim r)$ is true.

32. Replacing $p, q,$ and r with the given truth values, we have

$$\begin{gathered}(\mathrm{F} \to \sim \mathrm{T}) \wedge (\mathrm{F} \to \mathrm{F}) \\ (\mathrm{F} \to \mathrm{F}) \wedge \mathrm{T} \\ \mathrm{T} \wedge \mathrm{T} \\ \mathrm{T}.\end{gathered}$$

Thus, the statement $(p \to \sim q) \wedge (p \to r)$ is true.

36. $p \to \sim q$

p	q	p	$\to$	$\sim q$
T	T	T	F	F
T	F	T	T	T
F	T	F	T	F
F	F	F	T	T
		1	2	1

38. $(\sim q \to \sim p) \to \sim q$

p	q	$(\sim q$	$\to$	$\sim p)$	$\to$	$\sim q$
T	T	F	T	F	F	F
T	F	T	F	F	T	T
F	T	F	T	T	F	F
F	F	T	T	T	T	T
		1	2	1	4	3

40. $(p \wedge q) \to (p \vee q)$

p	q	$(p$	$\wedge$	$q)$	$\to$	$(p$	$\vee$	$q)$
T	T	T	T	T	T	T	T	T
T	F	T	F	F	T	T	T	F
F	T	F	T	T	T	F	T	T
F	F	F	F	F	T	F	F	F
		1	2	1	5	3	4	3

Since this statement is always true (column 5), it is a tautology.

42. $r \to (p \wedge \sim q)$

p	q	r	r	$\to$	$(p$	$\wedge$	$\sim q)$
T	T	T	T	F	T	F	F
T	T	F	F	T	T	F	F
T	F	T	T	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	F	F	F	F
F	T	F	F	T	F	F	F
F	F	T	T	F	F	F	T
F	F	F	F	T	F	F	T
			1	4	2	3	2

44. $(\sim r \rightarrow s) \wedge (p \rightarrow \sim q)$

p	q	r	s	$(\sim$	$r \rightarrow$	$s)$	$\vee$	$(p$	$\rightarrow$	$\sim q)$
T	T	T	T	F	T	T	T	T	F	F
T	T	T	F	F	T	F	T	T	F	F
T	T	F	T	T	T	T	T	T	F	F
T	T	F	F	T	F	F	F	T	F	F
T	F	T	T	F	T	T	T	T	T	T
T	F	T	F	F	T	F	T	T	T	T
T	F	F	T	T	T	T	T	T	T	T
T	F	F	F	T	F	F	T	T	T	T
F	T	T	T	F	T	T	T	F	T	F
F	T	T	F	F	T	F	T	F	T	F
F	T	F	T	T	T	T	T	F	T	F
F	T	F	F	T	F	F	T	F	T	F
F	F	T	T	F	T	T	T	F	T	T
F	F	T	F	F	T	F	T	F	T	T
F	F	F	T	T	T	T	T	F	T	T
F	F	F	F	T	F	F	T	F	T	T
				1	2	1	5	3	4	3

46. The statement is not a tautology if at least one F appears in the final column of a truth table, since a tautology requires all T's in the final column.

48. The negation of "If the English measures are not converted to metric measures, then the spacecraft will crash on the surface of Mars" is "The English measures are not converted to metric measures and the spacecraft does not crash on the surface of Mars."

50. The negation of "If loving you is wrong, I don't want to be right" is "Loving you is wrong and I want to be right."

52. An equivalent statement to "If the check is in the mail, I'll be surprised" is "The check is not in the mail or I'll be surprised."

54. An equivalent statement to "If I say yes, she says no" is "I do not say yes or she says no."

56.

p	q	$\sim$	$(p$	$\rightarrow$	$q)$	p	$\wedge$	$\sim q$
T	T	F	T	T	T	T	F	F
T	F	T	T	F	F	T	T	T
F	T	F	F	T	T	F	F	F
F	F	F	F	T	F	F	F	T
		3	1	2	1	1	2	1

Since the truth values in the final columns for each statement are the same, the statements are equivalent.

58.

p	q	q	$\rightarrow$	p	$\sim p$	$\rightarrow$	$\sim q$
T	T	T	T	T	F	T	F
T	F	F	T	T	F	T	T
F	T	T	F	F	T	F	F
F	F	F	T	F	T	T	T
		1	2	1	1	2	1

Since the truth values in the final columns for each statement are the same, the statements are equivalent.

60.

p	q	p	$\rightarrow$	q	$\sim q$	$\rightarrow$	$\sim p$
T	T	T	T	T	F	T	F
T	F	T	F	F	T	F	F
F	T	F	T	T	F	T	T
F	F	F	T	F	T	T	T
		1	2	1	1	2	1

Since the truth values in the final columns for each statement are the same, the statements are equivalent.

62.

p	q	$\sim p$	$\wedge$	q	$\sim p$	$\rightarrow$	q
T	T	F	F	T	F	T	T
T	F	F	F	F	F	T	F
F	T	T	T	T	T	T	T
F	F	T	F	F	T	F	F
		1	2	1	1	2	1

Since the truth values in the final columns for each statement are not the same, the statements are not equivalent.

64.

p	q	p	$\wedge$	q	q	$\wedge$	p
T	T	T	T	T	T	T	T
T	F	T	F	F	F	F	T
F	T	F	F	T	T	F	F
F	F	F	F	F	F	F	F
		1	2	1	3	4	3

The column labeled 2 and 4 are identical.

66.

p	q	r	$(p$	$\wedge$	$q)$	$\wedge$	r	p	$\wedge$	$(q$	$\wedge$	$r)$
T	T	T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	F	T	F	T	F	F
T	F	T	T	F	F	F	T	T	F	F	F	T
T	F	F	T	F	F	F	F	T	F	F	F	F
F	T	T	F	F	T	F	T	F	F	T	T	T
F	T	F	F	F	T	F	F	F	F	T	F	F
F	F	T	F	F	F	F	T	F	F	F	F	T
F	F	F	F	F	F	F	F	F	F	F	F	F
			1	2	1	4	3	5	8	6	7	6

The columns labeled 4 and 8 are identical.

68.

p	q	r	p	$\wedge$	$(q$	$\vee$	$r)$	$(p$	$\wedge$	$q)$	$\vee$	$(p$	$\wedge$	$r)$
T	T	T	T	T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	F	T	T	T	T	T	F	F
T	F	T	T	T	F	T	T	T	F	F	T	T	T	T
T	F	F	T	F	F	F	F	T	F	F	F	T	F	F
F	T	T	F	F	T	T	T	F	F	T	F	F	F	T
F	T	F	F	F	T	T	F	F	F	T	F	F	F	F
F	F	T	F	F	F	T	T	F	F	F	F	F	F	T
F	F	F	F	F	F	F	F	F	F	F	F	F	F	F
			1	4	2	3	2	5	6	5	9	7	8	7

The columns labeled 4 and 9 are identical.

70.

p	q	$(p$	$\vee$	$q)$	$\wedge$	p
T	T	T	T	T	T	T
T	F	T	T	F	T	T
F	T	F	T	T	F	F
F	F	F	F	F	F	F
		1	2	1	4	3

The p column and the column labeled 4 are identical.

72. In the diagram, a parallel circuit is shown, which corresponds to $r \vee q$. This circuit, in turn, is in series with p. Thus, the logical statement is

$$p \wedge (r \vee q).$$

74. The diagram shows q in parallel with a series circuit consisting of p and the parallel circuit involving q and $\sim p$. Thus, the logical statement is

$$q \vee [p \wedge (q \vee \sim p)].$$

One pair of equivalent statements listed in the text includes

$$p \wedge (q \vee \sim p) \equiv (p \wedge q) \vee (p \wedge \sim p).$$

Since $(p \wedge \sim p)$ is never true, $p \wedge (q \vee \sim p)$ simplifies to

$$(p \wedge q) \vee F \equiv (p \wedge q).$$

Thus,

$$\begin{aligned} q \vee [p \wedge (q \vee \sim p)] &\equiv q \vee (p \wedge q) \\ &\equiv (q \vee p) \wedge (q \wedge q) \\ &\equiv (q \vee p) \wedge q \\ &\equiv q. \end{aligned}$$

76. The diagram shows two parallel circuits, $\sim p \vee q$ and $\sim p \vee \sim q$ which are parallel to each other. Thus, the total circuit can be represented as

$$(\sim p \vee q) \vee (\sim p \vee \sim q).$$

This circuit can be simplified using the following equivalencies.

$$\begin{aligned}(\sim p \vee q) \vee (\sim p \vee \sim q) &\equiv \sim p \vee q \vee \sim p \vee \sim q\\ &\equiv \sim p \vee q \vee \sim q\\ &\equiv \sim p \vee (q \vee \sim q)\\ &\equiv \sim p \vee \mathrm{T}\\ &\equiv \mathrm{T}\end{aligned}$$

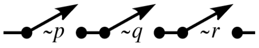
78. In the diagram, series circuit $\sim p \wedge \sim q$ is parallel to the parallel circuit $p \vee q$. This entire circuit is in series with the parallel circuit $p \vee q$ and p. The logical statement is

$$\{[(\sim p \wedge \sim q) \vee (p \vee q)] \wedge (p \vee q)\} \wedge p.$$

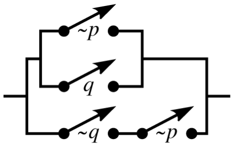
This statement simplifies to p as follows:

$$\begin{aligned}&\{[(\sim p \wedge \sim q) \vee (p \vee q)] \wedge (p \vee q)\} \wedge p\\ &\quad\equiv \{[(\sim (p \vee q) \vee (p \vee q)] \wedge (p \vee q)\} \wedge p\\ &\quad\equiv [T \wedge (p \vee q)] \wedge p\\ &\quad\equiv (p \vee q) \wedge p\\ &\quad\equiv p.\end{aligned}$$

80. The logical statement $(\sim p \wedge \sim q) \wedge \sim r$ can be represented by the following circuit.



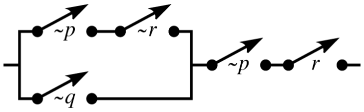
82. The logical statement $(\sim q \wedge \sim p) \vee (\sim p \vee q)$ can be represented by the following circuit.



The statement $(\sim q \wedge \sim p) \vee (\sim p \vee q)$ simplifies to $\sim p \vee q$ as follows:

$$\begin{aligned}&(\sim q \wedge \sim p) \vee (\sim p \vee q)\\ &\quad\equiv [\sim q \vee (\sim p \vee q)] \wedge [\sim p \vee (\sim p \vee q)]\\ &\quad\equiv [\sim q \vee \sim p \vee q] \wedge [\sim p \vee \sim p \vee q]\\ &\quad\equiv [(\sim q \vee q) \vee \sim p] \wedge [(\sim p \vee \sim p) \vee q]\\ &\quad\equiv (\mathrm{T} \vee \sim p) \wedge (\sim p \vee q)\\ &\quad\equiv \mathrm{T} \wedge (\sim p \vee q)\\ &\quad\equiv \sim p \vee q.\end{aligned}$$

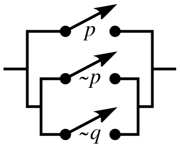
84. The statement $[(\sim p \wedge \sim r) \vee \sim q] \wedge (\sim p \wedge r)$ can be represented by the following circuit.



The statement $[(\sim p \wedge \sim r) \vee \sim q] \wedge (\sim p \wedge r)$ simplifies to $(\sim p \wedge r) \wedge \sim q$ in the following manner. Both $[(\sim p \wedge \sim r) \vee \sim q]$ and $(\sim p \wedge r)$ must be true. But if $(\sim p \wedge r)$ is true, then $(\sim p \wedge \sim r)$ is false. If $(\sim p \wedge \sim r)$ is false, then $\sim q$ must be true for the original disjunction to be true. Thus,

$$\begin{aligned}&[(\sim p \wedge \sim r) \vee \sim q] \wedge (\sim p \wedge r)\\ &\quad\equiv (\mathrm{F} \vee \sim q) \wedge (\sim p \wedge r)\\ &\quad\equiv \sim q \wedge (\sim p \wedge r)\\ &\quad\equiv (\sim p \wedge r) \wedge \sim q.\end{aligned}$$

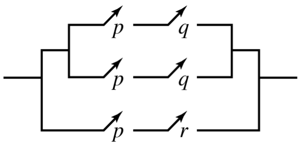
86. The logical statement $\sim p \rightarrow (\sim p \vee \sim q)$ can be represented by the following circuit.



The statement $\sim p \rightarrow (\sim p \vee \sim q)$ simplifies to T as follows:

$$\begin{aligned}\sim p \rightarrow (\sim p \vee \sim q) &\equiv p \vee (\sim p \vee \sim q)\\ &\equiv p \vee \sim p \vee \sim q\\ &\equiv (p \vee \sim p) \vee \sim q\\ &\equiv \mathrm{T} \vee \sim q\\ &\equiv \mathrm{T}.\end{aligned}$$

88. The logical statement $[(p \wedge q) \vee (p \wedge q)] \vee (p \wedge r)$ can be represented by the following circuit.



The statement simplifies to $p \wedge (q \vee r)$ as follows:

$$\begin{aligned}[(p \wedge q) \vee (p \wedge q)] \vee (p \wedge r) &\equiv (p \wedge q) \vee (p \wedge r)\\ &\equiv p \wedge (q \vee r).\end{aligned}$$

90. Referring to Figures 10 and 11 of Example 6 in the text:

Cost per year of the circuit in Figure 10
= number of switches × \$.03 × 24 hrs × 365 days
$= 4 \times .03 \times 24 \times 365$
= \$1051.20.

Cost per year of the circuit in Figure 11
= number of switches × \$.03 × 24 hrs × 365 days
$= 3 \times .03 \times 24 \times 365$
= \$788.40

The savings is \$1051.20 − \$788.40 = \$262.80.

92. The form of each statement is $p \rightarrow q$, which is equivalent to $\sim p \vee q$. The negation has the form $p \wedge \sim q$.

(a) Statement: Either you do not create an artistic work or invention for which you get a government patent or copyright or you may depreciate your costs over the life of the patent or copyright.
Negation: You create an artistic work or invention for which you get a government patent or copyright and you may not depreciate your costs over the life of the patent or copyright.

(b) Statement: Either you are not married at the end of the year or you may file a joint return with your spouse.
Negation: You are married at the end of the year and you may not file a joint return with your spouse.

(c) Statement: Either you do not receive your company's stock as payment for your services or you include the value of the stock as pay in the year you receive it.
Negation: You receive your company's stock as payment for your services and you do not include the value of the stock as pay in the year you receive it.

94. The form of each statement is $p \rightarrow q$, which is equivalent to $\sim p \vee q$. The negation has the form $p \wedge \sim q$.

(a) Statement: Either you are not married or you can't get married again.
Negation: You are married and you can get married again.

(b) Statement: Your job will cost \$500 or less or your contractor is legally required to put it in writing.
Negation: Your job will cost more than \$500 and your contractor is not legally required to put it in writing.

(c) Statement: Your application for citizenship is not denied or you can appeal in federal court.
Negation: Your application for citizenship is denied and you cannot appeal in federal court.

6.4 More on the Conditional

Wording may vary in the answers to Exercises 2–10.

2. The direct statement: If you lead, then I will follow.

(a) *Converse:* If I follow, then you lead.

(b) *Inverse:* If you do not lead, then I will not follow.

(c) *Contrapositive:* If I do not follow, then you do not lead.

4. The direct statement: If I had a nickel for each time that happened, I would be rich.

(a) *Converse:* If I were rich, then I would have a nickel for each time that happened.

(b) *Inverse:* If I did not have a nickel for each time that happened, then I would not be rich.

(c) *Contrapositive:* If I were not rich, then I would not have a nickel for each time that happened.

It is helpful to restate the direct statement in an if ... then form for Exercises 6 and 8.

6. The direct statement: If it's milk, then it contains calcium.

(a) *Converse:* If it contains calcium, then it's milk.

(b) *Inverse:* If it's not milk, then it does not contain calcium.

(c) *Contrapositive:* If it does not contain calcium, then it's not milk.

8. The direct statement: If it is a rolling stone, then it gathers no moss

(a) *Converse:* If it gathers no moss, then it is a rolling stone.

(b) *Inverse:* If it is not a rolling stone, then it gathers moss.

(c) *Contrapositive:* If it gathers moss, then it is not a rolling stone.

10. The direct statement: If there's smoke, then there's fire.

(a) *Converse:* If there's fire, then there's smoke.

(b) *Inverse:* If there's no smoke, then there's no fire.

(c) *Contrapositive:* If there's no fire, then there's no smoke.

12. The direct statement: $\sim p \rightarrow q$.

(a) *Converse:* $q \rightarrow \sim p$.

(b) *Inverse:* $p \rightarrow \sim q$.

(c) *Contrapositive:* $\sim q \rightarrow p$.

14. The direct statement: $\sim q \rightarrow \sim p$.

(a) *Converse:* $\sim p \rightarrow \sim q$.

(b) *Inverse:* $q \rightarrow p$.

(c) *Contrapositive:* $p \rightarrow q$.

16. The direct statement: $(r \vee \sim q) \rightarrow p$.

(a) *Converse:* $p \rightarrow (r \vee \sim q)$.

(b) *Inverse:* $\sim (r \vee \sim q) \rightarrow \sim p$ or $(\sim r \wedge q) \rightarrow \sim p$.

(c) *Contrapositive:* $\sim p \rightarrow \sim (r \vee \sim q)$ or $\sim p \rightarrow (\sim r \wedge q)$.

20. The statement "If I finish studying, I'll go to the party" becomes "If I finish studying, then I'll go to the party."

22. The statement "'Today is Wednesday' implies that yesterday was Tuesday" becomes "If Today is Wednesday, then yesterday was Tuesday."

24. The statement "Being in Fort Lauderdale is sufficient for being in Florida" becomes "If you are in Fort Lauderdale, then you are in Florida."

26. The statement "Being an environmentalist is necessary for being elected" becomes "If one is elected, then one is an environmentalist."

28. The statement "The principal will hire more teachers only if the school board approves" becomes "If the principal hires more teachers, then the school board approves."

30. The statement "Jesse will be a liberal when pigs fly" becomes "If pigs fly, then Jesse will be a liberal."

32. The statement "A parallelogram is a four-sided figure with opposite sides parallel" becomes "If the figure is a parallelogram, then it is a four-sided figure with opposite sides parallel."

34. The statement "A square is rectangle with two adjacent sides equal" becomes "If the figure is a square, then it is a rectangle with two adjacent sides equal."

36. The statement "An integer whose units digit is 0 or 5 is divisible by 5" becomes "If an integer has a units digit of 0 or 5, then it is divisible by 5."

42. The statement "$3 + 1 \neq 6$ if and only if $8 \neq 8$" is <u>false</u> since this is a biconditional consisting of a true and a false statement.

44. The statement "$6 \times 2 = 14$ if and only iff $9 + 7 \neq 16$" is <u>true</u>, since this is a biconditional consisting of two false statements.

46. "The moon is made of green cheese if and only if Hawaii is one of the United States" is <u>false</u>, since this is a biconditional consisting of a false and a true statement.

48. For $p \to q$, the converse is $q \to p$, the inverse is $\sim p \to \sim q$, and the contrapositive is $\sim q \to \sim p$.

Converse: If you may avoid paying the annual fee billed on this statement, then you close your account within 30 days from the date this statement was mailed.
Inverse: If you do not close your account within 30 days from the date this statement was mailed, you may not avoid paying the annual fee billed on this statement.
Contrapositive: If you may not avoid paying the annual fee billed on this statement, then you do not close your account within 30 days from the date this statement was mailed.
The converse and the inverse are equivalent, and the contrapositive and the original statement are equivalent.

50. **(a)** Let p represent "there are triplets," let q represent "the most persistent stands to gain an extra meal," and let r represent "it may eat at the expense of another." Then the statement can be written as $p \to (q \wedge r)$.

(b) The contrapositive is $\sim (q \wedge r) \to p$, which is equivalent to $(\sim q \vee \ \sim r) \to p$: If the most persistent does not stand to gain an extra meal or it does not eat at the expense of another, then there are not triplets.

52. **(a)** if . . . then form: If you make a stitch in time, then it will save you nine (stitches).
contrapositive: If you will not save nine (stitches), then you do not make a stitch in time.
or form: You do not make a stitch in time or it will save you nine (stitches).

(b) if . . . then form: If a stone rolls, then it gathers no moss.
contrapositive: If a stone gathers moss, then it is not rolling.
or form: A stone does not roll or it gathers no moss.

(c) if . . . then form: If they are birds of a feather, then they flock together.
contrapositive: If they do not flock together, then they are not birds of a feather.
or form: They are not birds of a feather or they flock together.

54. If liberty and equality are not best attained when all persons share alike in the government to the utmost, then they are not, as is thought by some, chiefly to be found in democracy.

56. The statement can be written as

$$(d \to l) \wedge \sim (l \to d)$$

58. If there is an R.P.F. alliance, there there is a Modéré incumbent. *Converse*: If there is a Modéré incumbent, then there is an R.P.F. alliance. *Inverse*: If there is not an R.P.F. alliance, then there is not a Modéré incumbent. *Contrapositive*: If there is not a Modéré incumbent, then there is not an R.P.F. alliance.
The contrapositive is equivalent to the original.

60. "worked on the weekend": Must be turned over to see whether the employee got a day off.
"did not work on the weekend": Need not be turned over, since it does not describe an employee who worked on the weekend.
"did get a day off": Need not be turned over, since it cannot describe an employee who worked on the weekend without getting a day off.
"did not get a day off": Must be turned over to see whether the other side says "worked on the weekend."

6.5 Analyzing Arguments and Proofs

2. Let p represent "Billy Joel comes to town," q represent "I will go to the concert," and r represent "I'll call in sick for work." The argument is then represented symbolically by:

$$\begin{array}{c} p \to q \\ \underline{q \to r} \\ p \to r. \end{array}$$

This is the <u>valid</u> argument form Reasoning by Transitivity.

4. Let p represent "Teri Lovelace works hard enough" and q represent "she will get a promotion." The argument is then represented symbolically by:

$$\begin{array}{l} p \rightarrow q \\ \underline{p \qquad} \\ q. \end{array}$$

This is the <u>valid</u> argument form Modus Ponens.

6. Let p represent "He doesn't have to get up at 5:30 A.M." and q represent "he is ecstatic." The argument is then represented symbolically by:

$$\begin{array}{l} p \rightarrow q \\ \underline{q \qquad} \\ p. \end{array}$$

Since this is the form Fallacy of the Converse, it is invalid and considered a <u>fallacy</u>.

8. Let p represent "Pedro Martinez pitches" and q represent "the Red Sox win." The argument is then represented symbolically by:

$$\begin{array}{l} \quad p \rightarrow q \\ \underline{\sim q \qquad} \\ \sim p. \end{array}$$

This is the <u>valid</u> argument form Modus Tollens.

10. Let p represent "I have seen farther than others" and q represent " I have stood on the shoulders of giants." The argument is then represented symbolically by:

$$\begin{array}{l} \quad p \rightarrow q \\ \underline{\sim p \qquad} \\ \sim q. \end{array}$$

Since this is the form Fallacy of the Inverse, it is invalid and considered a <u>fallacy</u>.

12. Let p represent "She uses e-commerce" and q represent "she pays by credit card." The argument is then represented symbolically by:

$$\begin{array}{l} \quad p \vee q \text{ (or } q \vee p) \\ \underline{\sim q \qquad} \\ \quad p. \end{array}$$

Since this is the form Disjunctive Syllogism, it is a <u>valid</u> argument.

To show validity for the arguments in the following exercises we must show that the conjunction of the premises implies the conclusion. That is, the conditional statement $[P_1 \wedge P_2 \wedge \ldots \wedge P_n] \rightarrow C$ *must be a tautology.*

14.
1. $p \vee \sim q$ T
2. $\underline{p \qquad}$ T

$\sim q$ F

The argument is <u>invalid</u>. When $p = \text{T}$ and $q = \text{T}$, the premises are true but the conclusion is false.

16.
1. $\sim p \rightarrow q$ T
2. $\underline{\quad p \qquad}$ T

$\sim q$ F

The argument is <u>invalid</u>. When $p = \text{T}$ and $q = \text{T}$, the premises are true but the conclusion is false.

18. The argument is <u>valid</u>.

1.	$p \rightarrow \sim q$	Premise
2.	q	Premise
3.	$\sim p$	1, 2, Modus Tollens

20. The argument is <u>valid</u>.

1.	$p \vee q$	Premise
2.	$\sim p$	Premise
3.	$r \rightarrow \sim q$	Premise
4.	q	1, 2, Disjunctive Syllogism
5.	$\sim r$	3, 4, Modus Tollens

22. The argument is <u>valid</u>.

1.	$p \rightarrow q$	Premise
2.	$r \rightarrow \sim q$	Premise
3.	$q \rightarrow \sim r$	2, Contrapositive
4.	$p \rightarrow \sim r$	1, 3, Transitivity

24. The argument is <u>valid</u>.

1.	$p \rightarrow q$	Premise
2.	$\sim p \rightarrow r$	Premise
3.	$s \rightarrow \sim q$	Premise
4.	$\sim r \rightarrow p$	2, Contrapositive
5.	$\sim r \rightarrow q$	1, 4, Transitivity
6.	$q \rightarrow \sim s$	3, Contrapositive
7.	$\sim r \rightarrow \sim s$	5, 6, Transitivity

26. Make a truth table for the statement $p \rightarrow (p \vee q)$.

p	q	p	$\rightarrow$	$(p$	$\vee$	$q)$
T	T	T	T	T	T	T
T	F	T	T	T	T	F
F	T	F	T	F	T	T
F	F	F	T	F	F	F
		2	3	1	2	1

30. Let i represent "that tree is infested with pine bark beetles," d represent "it will die," and p represent "people plant trees on Arbor Day." The argument is then represented symbolically by:

$$\begin{array}{l} i \to d \\ p \to \sim d \\ \hline p \to \sim i. \end{array}$$

The argument is <u>valid</u>.

1.	$i \to d$	Premise
2.	$p \to \sim d$	Premise
3.	$\sim d \to \sim i$	1, Contrapositive
4.	$p \to \sim i$	2, 3, Transitivity

32. Let p represent "I've got you under my skin," q represent "you are deep in the heart of me," and r represent "you are really a part of me" The argument is then represented symbolically by:

$$\begin{array}{ll} p \to q & \text{T} \\ q \to \sim r & \text{T} \\ q \vee r & \text{T} \\ \hline p \to r. & \text{F} \end{array}$$

The argument is <u>invalid</u>. When $p = \text{T}$, $q = \text{T}$, and $r = \text{F}$, the premises are true but the conclusion is false.

34. Let p represent "Peyton leads the league in passing," c represent "the Colts will be in the playoffs," and m represent "Marv loves the Colts." The argument is then represented symbolically by

$$\begin{array}{l} p \to c \\ m \vee p \\ \sim m \\ \hline c \end{array}$$

The argument is <u>valid</u>.

1.	$p \to c$	Premise
2.	$m \vee p$	Premise
3.	$\sim m$	Premise
4.	p	2, 3, Disjunctive Syllogism
5.	c	1, 4, Modus Ponens

36. The argument is <u>valid</u>.

Let p represent "my car starts," q represent "the battery is strong," r represent "I got a jump," and s represent "I'm taking the train."
The argument is then represented symbolically by

$$\begin{array}{l} p \to (q \vee r) \\ p \vee s \\ \sim q \wedge \sim r \\ \hline s \end{array}$$

1.	$p \to (q \vee r)$	Premise
2.	$p \vee s$	Premise
3.	$\sim q \wedge \sim r$	Premise
4.	$\sim (q \vee r)$	3, De Morgan's Law
5.	$\sim p$	1, 4, Modus Tollens
6.	s	2, 5, Disjunctive Syllogism

38. **(a)** $d \to \sim w$ **(b)** $o \to w$ or $w \to \sim o$ **(c)** $p \to d$ **(d)** $p \to \sim o$, *Conclusion*: If it is my poultry, then it is not an officer. In Lewis Carroll's words, "My poultry are not officers."

40. **(a)** $b \to \sim t$ or $t \to \sim b$ **(b)** $w \to c$ **(c)** $\sim b \to h$ **(d)** $\sim w \to \sim p$ or $p \to w$ **(e)** $c \to t$ **(f)** $p \to h$, *Conclusion*: If one is a pawnbroker, then one is honest. In Lewis Carroll's words, "No pawnbroker is dishonest."

42. **(a)** $d \to p$ **(b)** $\sim t \to \sim i$ **(c)** $r \to \sim f$ or $f \to \sim r$ **(d)** $o \to d$ or $\sim d \to \sim o$ **(e)** $\sim c \to i$ **(f)** $b \to s$ **(g)** $p \to f$ **(h)** $\sim o \to \sim c$ or $c \to o$ **(i)** $s \to \sim t$ or $t \to \sim s$ **(j)** $b \to \sim r$, *Conclusion*: If it is written by Brown, then I can't read it. In Lewis Carroll's words, "I cannot read any of Brown's letters."

6.6 Analyzing Arguments with Quantifiers

2. Let $p(x)$ represent "x is a student that is present" and $c(x)$ represent "x will get another chance."

(a) $\forall x[p(x) \to c(x)]$

(b) $\exists x[p(x) \wedge \sim c(x)]$

(c) There is a student present who will not get another chance.

4. Let $b(x)$ represent "x is a business" and $s(x)$ represent "x is like show business."

(a) $\forall x[b(x) \rightarrow \sim s(x)]$

(b) $\exists x[b(x) \wedge s(x)]$

(c) There is a business like show business.

6. Let $d(x)$ represent "x is a day" and $b(x)$ represent "x is better than others."

(a) $\exists x[d(x) \wedge b(x)]$

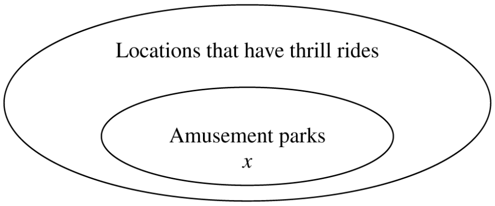
(b) $\forall x[d(x) \rightarrow \sim b(x)]$

(c) No days are better than others.

8. **(a)** Let $a(x)$ represent "x is an amusement park" and $t(x)$ represent "x has thrill rides." Let g represent Great America. We can represent the argument symbolically as follows.

$$\begin{array}{l} \forall x[a(x) \rightarrow t(x)] \\ a(g) \\ \hline t(g) \end{array}$$

(b) Draw an Euler diagram where the region representing "amusement parks" must be inside the region representing "locations that have thrill rides" so that the first premise is true.



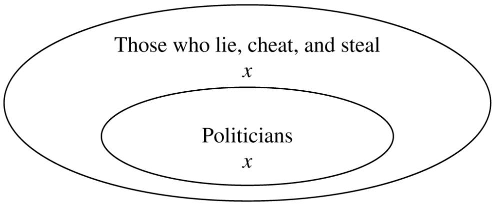
x represents *Great America*

Let x represent the amusement park "Great America". By the second premise, x must lie in the "amusement parks" region. Since this forces the conclusion to be true, the argument is valid.

10. **(a)** Let $p(x)$ represent "x is a politician" and $l(x)$ represent "x lies, cheats, and steals." Let t represent that man. We can represent the argument symbolically as follows.

$$\begin{array}{l} \forall x[p(x) \rightarrow l(x)] \\ l(t) \\ \hline p(t) \end{array}$$

(b) Draw an Euler diagram where the region representing "politicians" must be inside the region representing "those who lie, cheat, and steal" so that the first premise is true.



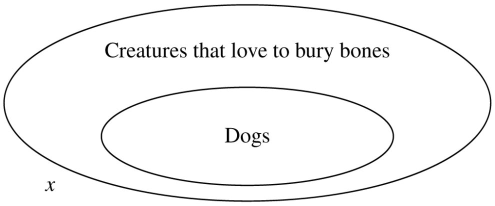
x represents that man

Let x represent "that man." By the second premise, x must lie in the "those who lie, cheat, and steal" region. Thus, he could be inside or outside the inner region. Since this allows for a false conclusion (he doesn't have to be in the "politicians" region for both premises to be true), the argument is invalid.

12. **(a)** Let $d(x)$ represent "x is a dog" and $b(x)$ represent "x loves to bury bones." Let a represent Archie. We can represent the argument symbolically as follows.

$$\begin{array}{l} \forall x[d(x) \rightarrow b(x)] \\ \sim b(a) \\ \hline \sim d(a) \end{array}$$

(b) Draw an Euler diagram where the region representing "dogs" must be inside the region representing "creatures that love to bury bones" so that the first premise is true.



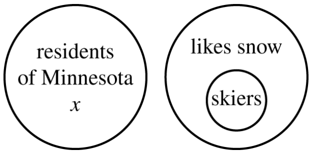
x represents Archie

Let x represent "Archie." By the second premise, x must lie outside the region representing "creatures that love to bury bones." Since this forces the conclusion to be true, the argument is valid.

14. **(a)** Let $r(x)$ represent "x is a resident of Minnesota" let $l(x)$ represent "x likes snow," and let $s(x)$ represent "x is a skier." We can represent the argument symbolically as follows.

$$\begin{array}{l} \exists x[r(x) \wedge \sim l(x)] \\ \underline{\forall x[s(x) \rightarrow l(x)]} \\ \exists x[r(x) \wedge \sim s(x)] \end{array}$$

Draw an Euler diagram where the region representing "residents of Minnesota" lies completely outside the region representing "likes snow." This keeps the first premise true.

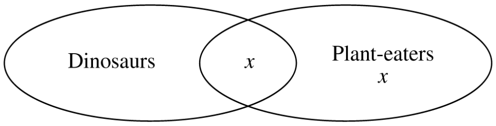


By the second premise the region representing "skiers" must be inside the region "likes snow." Then since x lies outside "likes snow" it also lies outside "skiers," hence the conclusion is true and so the argument is valid.

16. **(a)** Let $d(x)$ represent "x is a dinosaur" and $p(x)$ represent "x is a plant-eater." Let y represent Danny. We can represent the argument symbolically as follows.

$$\begin{array}{l} \exists x[d(x) \wedge p(x)] \\ \underline{p(y) \qquad\qquad} \\ d(y) \end{array}$$

(b) Draw an Euler diagram where the region representing "dinosaurs" intersects the region representing "plant-eaters." This keeps the first premise true.



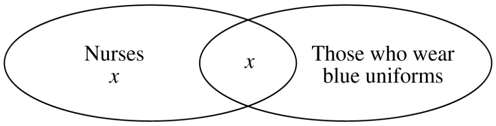
x represents Danny

Let x represent "Danny". By the second premise, x must lie in the region representing "plant-eaters." Thus, he could be inside or outside the region "dinosaurs." Since this allows for a false conclusion, the argument is invalid.

18. **(a)** Let $n(x)$ represent "x is a nurse" and $b(x)$ represent "x wears a blue uniform." Let k represent Becky Finnerty. We can represent the argument symbolically as follows.

$$\begin{array}{l} \exists x[n(x) \wedge b(x)] \\ \underline{n(k) \qquad\qquad} \\ b(k) \end{array}$$

(b) Draw an Euler diagram where the region representing "nurses" intersects the region representing "those who wear blue uniforms." This keeps the first premise true.



x represents Becky Finnerty

Let x represent "Becky Finnerty". By the second premise, x must lie in the region representing "nurses." Thus, he could be inside or outside the region "those who wear blue uniforms." Since this allows for a false conclusion, the argument is invalid.

20. The (valid) argument of Example 5 (in the text) is

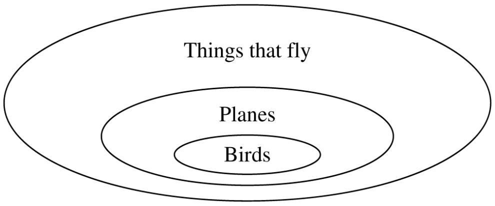
All expensive things are desirable.
All desirable things make you feel good.
All things that make you feel good make you live longer.
All expensive things make you live longer.

Another possible conclusion, which will keep the argument valid, is "All expensive things make you feel good." The argument remains valid since the premises diagrammed (Figure 19, in the text) force this conclusion to be true also.

22. The following is a valid argument which can be constructed from the given Euler diagram.

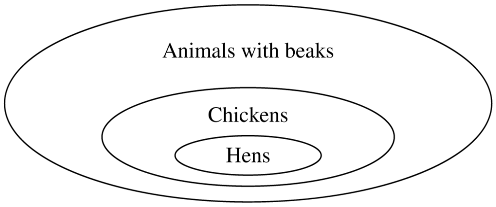
All people with blue eyes have blond hair.
Julie Ward does not have blond hair.
Julie Ward does not have blue eyes.

24. The following represents one way to diagram the premises so that they are true but does not lead to a true conclusion.



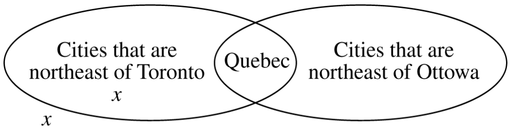
Thus, the argument in <u>invalid</u>.

26. The following Euler diagram yields true premises. It also forces the conclusion to be true.



Thus, the argument is <u>valid</u>. Observe that the diagram is the only way to show true premises.

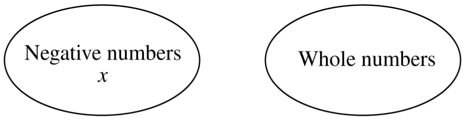
28. The following Euler diagram represents true premises.



x represents Ottowa

But x can reside inside or outside of the "Cities that are northeast of Toronto" diagram. In the one case (x inside) the conclusion is true. In the other case (x outside) the conclusion is false. Since true premises must always give a true conclusion, the argument is <u>invalid</u>.

30. The following Euler diagram represents the two premises as being true and we are forced into a true conclusion.



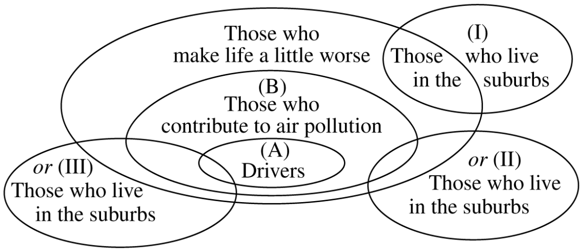
x represents –4

Thus, the argument is <u>valid</u>.

The premises marked A, B, and C are followed by several possible conclusions (Exercises 34–38). Take each conclusion in turn, and check whether the resulting argument is valid or invlaid.

A. All people who drive contribute to air pollution.
B. All people who contribute to air pollution make life a little worse.
C. Some people who live in a suburb make life a little worse.

There are three possibilities for diagramming premise C; labelled I, II, and III.



34. We are not forced into the conclusion, "Some people who live in a suburb contribute to air pollution" since option (I) represents a true premise and a false conclusion. Thus, the argument is <u>invalid</u>.

36. We are not forced into the conclusion, "Suburban residents never drive" since diagram (III) represents a true premise where this conclusion is false. Thus, the argument is <u>invalid</u>.

38. The conclusion, "Some people who make life a little worse live in a suburb" yields a <u>valid</u> argument since all three options (I–III) represent true premises and force this conclusion to be true.

42. We might write the statement "There is no one here who has not done that at one time or another" as "Everyone here has done that at one time or another."

44. The condition that "No picture has a frame" is satisfied by group B.

46. The condition that "Not every picture has a frame" is satisfied by groups A and B. Observe that this statement is equivalent to "At least one picture does not have a frame."

48. The condition that "No picture does not have a frame" is satisfied by group C. Observe that this statement is equivalent to "All pictures have a frame."

50. The condition that "Not every picture does not have a frame" is satisfied by groups A and C. Observe that this statement is equivalent to "At least one picture does have a frame."

Chapter 6 Review Exercises

2. The negation of "She passed GO and collected \$200" is "She did not pass GO or did not collect \$200." Remember that $\sim(p \wedge q) \equiv (\sim p \vee \sim q)$.

4. The symbolic form of "I will love you if you will love me" (or equivalently, "If you will love me, then I will love you") is "$y \to i$."

6. Writing the symbolic form "$\sim y \wedge i$" in words, we get "You won't love me and I will love you."

8. Replacing q and r with the given truth values, we have

$$\begin{gathered}\sim\mathrm{F} \wedge \sim\mathrm{F} \\ \mathrm{T} \wedge \mathrm{T} \\ \mathrm{T}\end{gathered}$$

The compound statement $\sim q \wedge \sim r$ is true.

10. Replacing r with the given truth value (s not known), we have

$$\begin{gathered}\mathrm{F} \to (\mathrm{s} \wedge \mathrm{F}) \\ \mathrm{T}\end{gathered}$$

since a conditional statement with a false antecedent is true.
The compound statement $r \to (s \vee r)$ is true.

14.

p	q	p	$\wedge$	$(\sim p$	$\vee$	$q)$
T	T	T	T	F	T	T
T	F	T	F	F	F	F
F	T	F	F	T	T	T
F	F	F	F	T	T	F
		1	4	2	3	2

The statement is not a tautology.

The wording may vary in the answers in Exercises 16–24.

16. "All integers are rational numbers" can be stated as "If the number is an integer, then it is a rational number."

18. "Being divisible by 3 is necessary for a number to be divisible by 9" can be stated as "If a number is divisible by 9, then it is divisible by 3." Remember that the "necessary" part of the statement becomes the consequent.

20. The direct statement: If a picture paints a thousand words, the graph will help me understand it.

(a) *Converse:* If the graph will help me understand it, then a picture paints a thousand words.

(b) *Inverse:* If a picture doesn't paint a thousand words, the graph won't help me understand it.

(c) *Contrapositive:* If the graph doesn't help me understand it, then a picture doesn't paint a thousand words.

22. **(a)** Let p represent "he eats liver" and q represent "he will eat anything." The argument is then represented symbolically by:

$$\begin{array}{l} p \to q \\ \underline{p\qquad} \\ q. \end{array}$$

This is the valid argument form Modus Ponens, hence the answer is A.

(b) Let p represent "you use your seat belt" and q represent "you will be safer." The argument is then represented symbolically by:

$$\begin{array}{l} \ \ p \to q \\ \underline{\sim p\qquad} \\ \sim q. \end{array}$$

The answer is F, a Fallacy of the Inverse.

(c) Let p represent "I hear *Come Saturday Morning*," q represent "I think of her," and r represent "I get depressed." The argument is then represented symbolically by:

$$\begin{array}{l} p \to q \\ \underline{q \to r} \\ p \to r. \end{array}$$

This is the valid argument form Reasoning by Transitivity, hence the answer is C.

(d) Let p represent "she sings" and q represent "she dances." The argument is then represented symbolically by:

$$\begin{array}{l} p \vee q \\ \underline{\sim p \qquad} \\ q. \end{array}$$

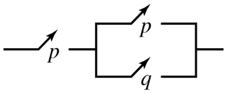
This is the valid argument form Disjunctive Syllogism, hence the answer is D.

24. In the diagram, two series circuits represented by $p \wedge p$ and $\sim p \wedge q$ are in parallel. The logical statement is $(p \wedge p) \vee (\sim p \wedge q)$. This statement is equivalent to $p \vee q$.

p	q	$(p \wedge q)$	$\vee$	$(\sim p$	$\wedge$	$q)$	$p \vee q$
T	T	T	T	F	F	T	T
T	F	T	T	F	F	F	T
F	T	F	T	T	T	T	T
F	F	F	F	T	F	F	F
		1	4	2	3	2	5

Columns 4 and 5 are identical.

26. The logical statement $p \wedge (p \vee q)$ can be represented by the following circuit.



The statement simplifies to $p \wedge (p \vee q)$ simplifies to p by the Absorption Law.

28.

p	q	$(p \veebar q)$	$\sim$	$[(p \vee q)$	$\rightarrow$	$(p \wedge q)]$
T	T	F	F	T	T	T
T	F	T	T	T	F	F
F	T	T	T	T	F	F
F	F	F	F	F	T	F
		1	5	2	4	3

Columns 1 and 5 are identical.

32. Let s represent "Smith received enough votes to qualify," j represent "Jones received enough votes to qualify," and e represent "the election was rigged." The argument is then represented symbolically by:

$$\begin{array}{ll} 1. \sim(s \vee j) & \text{T} \\ 2. \ \underline{\ s \rightarrow \ e\ } & \text{T} \\ \quad \sim e. & \text{F} \end{array}$$

The argument is invalid. The premises are true but the conclusion is false when $s = F$, $j = F$, and $e = T$.

34. The argument is valid.

1. $p \rightarrow q$ Premise
2. $r \rightarrow \sim q$ Premise
3. $q \rightarrow \sim r$ 2, Contrapositive
4. $p \rightarrow \sim r$ 1, 3, Transitivity

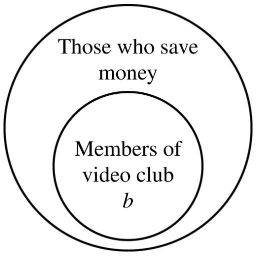
36. Let $m(x)$ represent "x is a member of the class" and $f(x)$ represent "x went on the field trip."

(a) $\exists x[m(x) \wedge f(x)]$
(b) $\forall x[m(x) \rightarrow \sim f(x)]$
(c) No members of this class went on the field trip.

38. **(a)** Let $v(x)$ represent "x is a member of that video club," $m(x)$ represent "x saves money" and b represent Bill Hoffman.

$$\begin{array}{l} \exists x[v(x) \wedge \sim m(x)] \\ \underline{m(b) \qquad\qquad\qquad} \\ \sim v(b) \end{array}$$

(b) Complete an Euler diagram as follows:



The conclusion is not forced to be true. The argument is invalid.

40. **(a)** This is a compound statement, formed using the statements "You adopt a child who is not a U.S. citizen or resident at the time the adoption effort begins" and "A credit may not be claimed until the year the adoption becomes final."

(b) This is a compound statement, formed using the statements "A foster child who lives with you for the whole year qualifies," "The child was placed with you by an authorized placement agency," and "The child is your sibling or a descendant of a sibling."

(c) This is a compound statement, formed using the statements "An individual who is a nonresident alien for any part of the year who is not eligible for the credit," "An individual is married," and "An election is made up by the couple to have all of their worldwide income subject to U.S. tax."

(d) This is not a statement and therefore not a compound statement.

42. The contrapositive of $p \to q$ ("p only if q") is $\sim q \to \sim p$: "If a foster child who lives with you the whole year was not placed with you by an authorized placement agency and is not your sibling or a descendant of a sibling, then the child does not qualify."

44. The negation of $p \wedge q$ is $\sim p \vee \sim q$.

(a) Regulations do not have both costs and benefits or rules that are passed to solve a problem cannot make it worse.

(b) Shooters do not overwhelmingly have problems with alcoholism or they do not have long criminal histories, particularly arrests for violent acts.

(c) They are not disproportionately involved in automobile crashes or they are not much more likely to have their driver's license suspended or revoked.

46. **(a), (b), (d)** These are commands or requests, not declarative sentences and therefore not statements.

(c) This is a declarative sentence that is true or false and therefore a statement.

48. **(a)** $\sim s \to g$
(b) $l \to \sim g$
(c) $w \to l \equiv \sim l \sim w$
(d) If the puppy does not lie still, it does not care to do worsted work. In Lewis Carroll's words, "Puppies that will not lie still never care to do worsted work."

50. **(a)** $f \to t \equiv \sim t \to \sim f$
(b) $\sim a \to \sim g \equiv g \to a$
(c) $w \to f$
(d) $t \to \sim g \equiv g \to \sim t$
(e) $d \to w \equiv \sim w \to \sim a$
(f) If the kitten will play with a gorilla, it does not have green eyes. In Lewis Carroll's words, "No kitten with green eyes will play with a gorilla."

Extended Application: Logic Puzzles

1. Draw charts as indicated and complete using the initial information given. Use "•" for Yes and "x" for No. For any cell that is assigned "•", mark "x" in the appropriate nearby unmarked cells in that row and column.

(a) *Neither Lauren nor Zach was the child accompanied by Ms. Reed. Tara was more interested in shopping at the country store than in picking pumpkins.* Mark (x)'s in the boxes for Lauren and Zach under Ms. Reed's column. Place (•) into the intersection of Tara and country store since that was her interest. Note: remember to always mark (x)'s in the appropriate nearby cells of a row or column when marking a cell with (•).

(b) *Xander and his father, Mr. Morgan, didn't go on the hay ride. Ms. Fedor's child (who isn't Zach or Lauren) was fascinated by the cider-making process.* Place (•) into cell representing the intersection of Xander and Mr. Morgan since you know that they are father and son. Place (x) into cell representing the intersection of Xander and hay ride as well as the cell for his father, Mr Morgan, and hay ride. Place (•) into the cell the intersection of Ms. Fedor and cider-making. Place (x)'s in the cells representing Lauren and Zach in Ms. Fedor's column. Since neither Lauren nor Zach is Ms. Fedor's child neither can be interested in cider-making. Mark (x)'s in the cells for cider-making in Zach and Lauren's rows.

(c) *Zach is neither the child who went on the hay ride nor the one who wanted to go apple picking. Mr. Hanson's child didn't go on the hay ride.* Place an (x) in the aspect cells for apple picking and hay ride in Zach's row. This leaves feeding animals as Zach's favorite aspect. Place (•) in the feeding animal cell for Zach. Mark (x) in the aspect cell for the hay ride in Mr. Hanson's column.

Tara and Raven are the only two children that could still belong to Ms. Fedor. Since Ms. Fedor's child liked cider-making check the aspect columlns and notice Tara liked the country store. This leaves Raven as Ms. Fedor's child. Place (•) in the cell for Raven in Ms Fedor's column and (•) in the cider-making cell in Raven's row. The only aspect now available to Xander is apple picking.

Place (•) in that cell. The last aspect, the hay ride, is now Lauren's only possible aspect. Place (•) in the appropriate cell.

At this point Chart I will have been completed.

Chart 1

		PARENT					ASPECT				
		MS. FEDOR	MR. HANSON	MR. MAIER	MR. MORGAN	MS. REED	APPLE PICKING	CIDER MAKING	COUNTRY STR.	FEEDING ANM.	HAY RIDE
CHILD	LAUREN	X			X	X	X	X	X	X	•
	RAVEN	•	X	X	X	X	X	•	X	X	X
	TARA	X			X		X	X	•	X	X
	XANDER	X	X	X	•	X	•	X	X	X	X
	ZACH	X			X	X	X	X	X	•	X
ASPECT	APPLE PICK.	X									
	CIDER MAK.	•	X	X	X	X					
	COUNTRY	X									
	FEEDING	X									
	HAY RIDE	X	X		X						

Tara is now the only child left in Ms. Reed's column. Place (•) at the intersection of Lauren and Ms. Reed. Notice that Xander's favorite aspect is apple picking and we know that Mr. Morgan is Xander's father. Thus we can place (•) in Mr. Morgan's column in the apple picking cell. This leaves Lauren and Zach with the possibility of Mr. Hanson and Mr. Maier for parents. By comparing Lauren's favorite aspect, the hay ride, to Mr. Hanson and Mr. Maier's aspect columns we find that Mr. Hanson's child did NOT like the hay ride. Thus Mr. Maier must be Lauren's father. Mr. Hanson is left as Zach's father. Now it is a simple matter of matching each child's interest with the aspect cell for his or her parent. Zach likes feeding animals; mark (•) in the feeding animals cell for Mr. Hanson. Lauren likes the hay ride; mark (•) in the hay ride cell for Mr. Maier. The country store is now the only available interest in Ms. Reed's column. Check to see that Ms. Reed's child, Tara is, in fact, the child that likes the country store. Place (•) in the appropriate cell and the logic puzzle is complete (Chart 2).

Chart 2

		PARENT					ASPECT				
		MS. FEDOR	MR. HANSON	MR. MAIER	MR. MORGAN	MS. REED	APPLE PICKING	CIDER MAKING	COUNTRY STR.	FEEDING ANM.	HAY RIDE
CHILD	LAUREN	X	X	•	X	X	X	X	X	X	•
	RAVEN	•	X	X	X	X	X	•	X	X	X
	TARA	X	X	X	X	•	X	X	•	X	X
	XANDER	X	X	X	•	X	•	X	X	X	X
	ZACH	X	•	X	X	X	X	X	X	•	X
ASPECT	APPLE PICK.	X	X	X	•	X					
	CIDER MAK.	•	X	X	X	X					
	COUNTRY	X	X	X	X	•					
	FEEDING	X	•	X	X	X					
	HAY RIDE	X	X	•	X	X					

Thus, Lauren, Mr. Maier, hay ride; Raven, Ms. Fedor, cider making; Tara, Ms. Reed, country store; Xander, Mr. Morgan, apple picking; Zach, Mr. Hanson, feeding animals.

2. Draw charts as indicated and complete using the initial information given. Use "•" for Yes and "x" for No. For any cell that is assigned "•", mark "x" in the appropriate nearby unmarked cells in that row and column.

(a) *Arlene had her name imprinted on her new orange bowling ball. Devon bought a bowling bag for his new ball, which is lighter than Silas's.* Arlene's ball is orange and therefore, can't be another color. Devon's ball is lighter than Silas's. Thus Devon's ball cannot weigh the most, 18 lb. Mark (x) appropriately.

(b) *Tina bowls with a 14-pound ball. The pink bowling ball is exactly 6 pounds lighter than the turquoise one.* Tina's ball is 14 lbs. and therefore, can't be any other weight (all other cells x).

(c) *The red bowling ball, which isn't Rosetta's weighs the most. Rosetta didn't buy the 16-pound ball.* The red ball weighs the most so it must be 18 lbs. Place (•) in the 18 lb. cell in the red column. The red ball (which is 18 lbs.) is not Rosetta's so (x) the 18 lb. and red columns in Rosetta's row. Also mark (x) in the 16 lb. cell in Rosetta's row.

The smallest ball is 10 lbs. and the largest ball is 18 lbs. Having already determined that the 18 lb. ball is red, the turquoise ball (which weighs 6 pounds heavier than the pink ball) must be 16 lb. and the pink ball 10 lb. Put (x)'s in all rows and columns that contain a known value (•). At this point you will have completed the following chart.

Chart 1

		COLOR					WEIGHT/POUNDS				
		GRAY	ORANGE	PINK	RED	TURQUOISE	10	12	14	16	18
BOWLER	ARLENE	X	•	X	X	X			X		
	DEVON		X						X		X
	ROSETTA		X		X				X	X	X
	SILAS		X						X		
	TINA		X		X	X	X	X	•	X	X
WEIGHT	10	X	X	•	X	X					
	12			X	X	X					
	14			X	X	X					
	16	X	X	X	X	•					
	18	X	X	X	•	X					

The orange ball cannot be 10, 16, or 18 lbs. By transferring these (x)'s to Arlene's row (Arlene owns the orange ball), we determine that Arleme's ball is 12 lb. Place (•) in the cell at the intersection of 12 lb. and Arlene's ball. The only weight cell open in Rosetta's row is 10 lbs. and we know that the 10 lb. ball is pink. Thus Rosetta has the 10 lb. pink ball. This leaves the 16 lb. ball as the only weight available to Devon. Silas's ball must be 18 lb. to be heavier than Devon's. The 16 lb. ball is turquoise therefore Devon has the turquoise ball. We know the 18 lb. ball, which belongs to Silas, is red. Thus, Tina must own the gray ball. Place (•) in the appropriate cell. Since Tina's (gray) ball is 14 lbs. mark (•) in the 14 lb. cell in the gray column. This leaves the orange ball at 12 lbs. We can check by noting the orange ball belongs to Arlene and Arlene's ball is 12 lbs.

Your chart should now be completed as in Chart 2 below.

Chart 2

		COLOR					WEIGHT/POUNDS				
		GRAY	ORANGE	PINK	RED	TURQUOISE	10	12	14	16	18
BOWLER	ARLENE	X	•	X	X	X	X	•	X	X	X
	DEVON	X	X	X	X	•	X	X	X	•	X
	ROSETTA	X	X	•	X	X	•	X	X	X	X
	SILAS	X	X	X	•	X	X	X	X	X	•
	TINA	•	X	X	X	X	X	X	•	X	X
WEIGHT	10	X	X	•	X	X					
	12	X	•	X	X	X					
	14	•	X	X	X	X					
	16	X	X	X	X	•					
	18	X	X	X	•	X					

Thus, Arlene, orange, 12; Devon, turquoise, 16; Rosetta, pink, 10; Silas, red, 18; Tina, gray, 14.

3. Draw charts as indicated and complete using the initial information given. Use "•" for Yes and "x" for No. For any cell that is assigned "•", mark "x" in the appropriate nearby cells in that row and column.

(a) *The tie with the grinning leprechauns wasn't a present from a daughter.* Mark (x) in the cell at the intersection of daughter and grinning leaprechauns.

(b) *Mr. Crow's tie features neither the dancing reindeer nor the yellow happy faces.* (*x*) the dancing reindeer and the yellow happy faces from Mr. Crow's row.

(c) *Mr. Speigler's tie wasn't a present from his uncle.* Place (x) in the uncle cell in Mr. Speigler's row.

(d) *The tie with the yellow happy faces wasn't a gift from a sister.* Mark (x) in the cell at the intersection of yellow happy faces and sister.

(e) *Mr. Evans and Mr. Speigler own the tie with the grinning leprechauns and the tie was a present from a father-in-law, in some order.* The leprechaun tie could not have come from the father-in-law so mark (x) at that intersection. For either Mr. Evans or Mr. Speigler to receive the tie from his father-in-law no one else could have received a tie from his father-in-law so (x) the father-in-law cells for Mr. Crow and Mr. Hurley. The same logic applies to the leprechaun; neither Mr. Crow nor Mr. Hurley could have received the leprechaun tie. (x) the appropriate cells. Mr. Crow's only tie option is now the tie with the cupids so place (•) in the cell at the intersection of cupids and Mr. Crow.

(f) *Mr. Hurley received his flamboyant tie from his sister.* Place (•) for the cell for sister in Mr. Hurley's row. Since Mr. Hurley received the tie from his sister and the sister did NOT give the happy faces tie (x) the happy faces cell in Mr. Hurley's row. This leaves reindeer as the only choice for Mr. Hurley's row. (•) the cell for reindeer in Mr. Hurley's row and (•) the cell for reindeer in the sister row because Mr. Hurley received his tie from his sister.

Since Mr. Crow did NOT receive his tie from his father-in-law or sister (x) those cells in the column under cupids. The father-in-law could now only have given the happy faces tie so (•) the appropriate cell. This leaves the cupids tie for the daughter and the leprechaun tie for the uncle. Since the cupid tie belongs to Mr. Crow, place (•) in the daughter cell in Mr. Crow's row.

Mr. Speigler's tie could now only have come from his father-in-law, leaving the uncle's tie for Mr. Evans. Now notice that Mr. Evan's tie came from his uncle and the uncle purchased the leprechaun tie so Mr. Evans received the tie with leprechauns and Mr. Speigler received the only remaining tie, the tie with happy faces.

Your completed chart should look like the following chart.

Chart

	Cupids	Happy faces	Leprechauns	Reindeer	Daughter	Father-in-law	Sister	Uncle
Mr. Crow	•	X	X	X	•	X	X	X
Mr. Evans	X	X	•	X	X	X	X	•
Mr. Hurley	X	X	X	•	X	X	•	X
Mr. Speigler	X	•	X	X	X	•	X	X
Daughter	•	X	X	X				
Father-in-law	X	•	X	X				
Sister	X	X	X	•				
Uncle	X	X	•	X				

Thus, Mr. Crow, cupids, daughter; Mr. Evans, leprechauns, uncle; Mr. Hurley, reindeer, sister; Mr. Speigler, happy faces, father-in-law.

4. Draw charts as indicated and complete using the initial information given. Use "•" for Yes and "x" for No. For any cell that is assigned "•", mark "x" in the appropriate nearby cells in that row and column.

(a) *The seal (who isn't the creation of either Joanne or Lou) neither rode to the moon in a spaceship nor took a trip around the world on a magic train.* Place (x) in the cells for train and spaceship in the column under seal. Also (x) that seal cell for Joanne and Lou.

(b) *Joanne's imaginary friend (who isn't the grizzly bear) went to the circus.* Place (x) in the cell for grizzly bear in Joanne's row. Mark (•) at the intersection of Joanne and Circus. Since the grizzly bear did not go to the circus (x) the appropriate cell in the grizzly bear column.

(c) *Winnie's imaginary friend is a zebra.* Place (•) for zebra in Winnie's row. Joanne is left with the moose as her character so place (•) in that cell. This leaves the grizzly bear for Lou and once that is marked the seal lis left for Ralph. Joanne's moose went to the circus so place (•) in the cell for circus in the moose's column. The seal's only adventure choice is now the rock band so place (•) appropriately. Since Ralph created the seal place (•) in the rock band cell in Ralph's row.

(d) *The grizzly bear didn't board the spaceship to the moon.* Mark (x) in the cell at the intersection of grizzly bear and spaceship. The grizzly bear must have taken the train so (•) the train cell in the grizzly bear column. This leaves the spaceship for the zebra. Lou's character, the grizzly bear rode the train so mark (•) in the train cell in Lou's row. This leaves the spaceship for Winnie's character, the zebra. The completed chart would look like the following chart.

Chart	Grizzly bear	Moose	Seal	Zebra	Circus	Rock band	Spaceship	Train
Joanne	X	•	X	X	•	X	X	X
Lou	•	X	X	X	X	X	X	•
Ralph	X	X	•	X	X	•	X	X
Winnie	X	X	X	•	X	X	•	X
Circus	X	•	X	X				
Rock band	X	X	•	X				
Spaceship	X	X	X	•				
Train	•	X	X	X				

Thus, Joanne, moose, circus; Lou, grizzly bear, train; Ralph, seal, rock band; Winnie, zebra, spaceship.

5. Draw charts as indicated and complete using the initial information given. Use "•" for Yes and "x" for No. For any cell that is assigned "•", mark "x" in the appropriate nearby cells in that row and column.

(a) *The right fielder took three more throws than Josh (who wasn't the last dunker).* Either the right fielder took five throws and Josh took two, or else the right fielder took nine throws and Josh took six. Mark (x) in the row for Right Field under Josh, Two, Four, and Six. Also put (x) under Josh in the rows for Fitfh, Four, Five, and Nine.

(b) *The shortstop threw immediately before Mr. Belford, who threw immediately before Geoffrey.* The shortstop threw third at the latest, Mr. Belford did not throw first or last, and Geoffrey threw third at the earliest. Mark (x) under Geoffrey in the rows for First, Second, Shortstop, and Belford. Mark (x) under Belford in the rows for First, Fifth, and Shortstop. Mark (x) under Shortstop in the rows for Fourth and Fifth.

(c) *Mr. Cavallo dunked the coach with his sixth pitch.* Put (•) at the intersection of Six and Cavallo and (x) the appropriate nearby cells.

(d) *The players who took an even number of throws are Scott, Mr. Hepler, and the fifth person to dunk the coach, in some order.* Mark (x) under Scott in the rows for Fifth, Five, Nine, and Hepler. Put (x)'s under Hepler in the rows for Fifth, Five, and Nine. Mark (x) under Five and Nine in the row for Fifth.

(e) *The pitcher (who wasn't the fourth dunker) took more throws than Oliver, but fewer throws than Mr. Janney.* The pitcher did not take the fewest or the most throws, Oliver took at most five throws, and Mr. Janney took at least five throws. Place (x) under Oliver in the rows for Pitcher, Six, Nine, and Janney. Mark (x) under Janney in the rows for Pitcher, Two, and Four. Mark (x) in the row for Pitcher under Two and Nine. Put an (x) in the row for Fourth under Pitcher.

(f) *The players who took an odd number of throws are Edgar and the catcher, in some order.* Write (x) under Edgar in the rows for Catcher, Two, Four, and Six. Write (x) in the row for Catcher under Two, Four, and Six

You have now completed this chart:

	Edgar	Geoffry	Josh	Oliver	Scott	Belford	Cavallo	Hepler	Janney	Winslow	Two	Four	Five	Six	Nine	Catcher	First Base	Pitcher	Right Field	Shortstop
First		X				X														
Second		X																		
Third																				
Fourth																		X		X
Fifth			X		X	X		X					X		X					X
Catcher	X										X	X		X						
First Base																				
Pitcher				X					X		X				X					
Right Field			X								X	X		X						
Shortstop		X				X														
Two	X						X		X											
Four	X		X				X		X											
Five			X		X		X	X												
Six	X			X		X	•	X	X	X										
Nine			X	X	X		X	X												
Belford		X																		
Cavallo																				
Hepler					X															
Janney				X																
Winslow																				

The catcher and right fielder each took five or nine throws, so one of them took five and the other took nine. Edgar took five or nine throws, so he is either the catcher or the right fielder. But his is not the catcher, so he is the right fielder. Mark (•) at the intersection of Right Field and Edgar.

Josh took two or six throws and thus took an even number. But he can't be Scott and he didn't throw last, so he is Mr. Hepler. Mark (•) at the intersection of Hepler and Josh. And because Josh is not Mr. Cavallo, he took two throws. Mark (•) at the intersection of Two and Josh.

Since Josh took two throws, the right fielder, namely Edgar, took five. Place an (•) at the intersection of Five and Edgar and another (•) at the intersection of Right Field and Five. It follows that the catcher took nine throws. Put an (•) at the intersection of Catcher and Nine.

Oliver can only have taken four throws, so mark an (•) at the intersection of Four and Oliver. And because Goeffrey is the only player left who can have nine throws, he must be the Catcher. Mark an (•) at the intersection of Nine and Geoffrey and another (•) at the intersection of Catcher and Geoffrey.

This leaves Scott to have taken six throws, and thus he must be Mr. Cavallo. Write an (•) at the intersection of Six and Scott and an (•) at the intersection of Cavallo and Scott. Since neither Scott Cavallo nor Josh Hepler threw fifth, the last person to throw must have been Oliver, the only other player who took an even number of throws. Write an (•) at the intersection of Fifth and Oliver.

The pitcher threw more than Oliver and so must have thrown six and must be Scott Cavallo. Mark (•) at the intersections of Pitcher and Scott, Pitcher and Cavallo, and Pitcher and Six. Mr. Janney, who threw more than the pitcher, must have thrown nine and thus must be Geoffrey, the catcher. Mark (•) at the intersection of Janney and Geoffrey and at the intersection of Catcher and Janney.

The shortstop, Mr. Belford, and Geoffrey threw in succession. They cannot have thrown third, fourth and fifth, since Oliver threw fifth. But they also cannot have thrown first, second, and third—since that would leave no room for the pitcher, Scott Cavallo, who didn't throw fifth and didn't throw fourth. So the shortstop, Mr. Belford, and catcher Geoffrey Janney threw second, third, and fourth. Then the pitcher threw first. Mark (•) for the following intersections: First and Scott, First and Cavallo, First and Pitcher, Second and Shortstop, Third and Belford, Fourth and Geoffrey, Fourth and Janney, and Fourth and Catcher.

It is now apparent that Mr. Hepler, i.e., Josh can only have thrown second. Write (•) at the intersection of Second and Josh and of Second and Hepler. The remaining open cells reveal that Edgar went third and is Mr. Belford, and that Oliver, who went last, is Mr. Winslow. Write (•) at these intersections: Third and Edgar, Belford and Edgar, Fifth and Winslow, and Winslow and Oliver.

From the first five columns (the first names), it is evident that the first dunker (Scott) took six throws, the second dunker (Josh) took two throws, the third dunker (Edgar) took five throws, the fourth dunker (Geoffrey) took nine throws, and the fifth dunker (Oliver) took four throws. Mark (•) in the appropriate cells under Two, Four, Five, Six, and Nine.

The first five columns also show that two throws were taken by Mr. (Josh) Helper, four by Mr. (Oliver) Winslow, five by Mr. (Edgar) Belford, and nine by Mr. (Geoffrey) Janney. Mark (•) in the appropriate cells under Belford, Hepler, Janney, and Winslow.

Enough is known about the right fielder (Edgar Belford, who went third and took five throws) to fill in the rest of both the row and the column labeled "Right Field." Mark (•) and (x) as appropriate. The shortstop, who went second and took two throws, is Josh Hepler; mark (•) and (x) in the row labeled "Shortstop" as appropriate. The first baseman is Oliver Winslow who took four throws. Mark (•) and (x) in the row labeled "First Base" as appropriate.

Finish filling in the chart by placing (x) in any remaining blank cells. The completed chart should look like this.

	Edgar	Geofrry	Josh	Oliver	Scott	Belford	Cavallo	Hepler	Janney	Winslow	Two	Four	Five	Six	Nine	Catcher	First Base	Pitcher	Right Field	Shortstop
First	X	X	X	X	•	X	•	X	X	X	X	X	X	•	X	X	X	•	X	X
Second	X	X	•	X	X	X	X	•	X	X	•	X	X	X	X	X	X	X	X	•
Third	•	X	X	X	X	•	X	X	X	X	X	X	•	X	X	X	X	X	•	X
Fourth	X	•	X	X	X	X	X	X	•	X	X	X	X	X	•	•	X	X	X	X
Fifth	X	X	X	•	X	X	X	X	X	•	X	•	X	X	X	X	•	X	X	X
Catcher	X	•	X	X	X	X	X	X	•	X	X	X	X	X	•					
First Base	X	X	X	•	X	X	X	X	X	•	X	•	X	X	X					
Pitcher	X	X	X	X	•	X	•	X	X	X	X	X	X	•	X					
Right Field	•	X	X	X	X	•	X	X	X	X	X	X	•	X	X					
Shortstop	X	X	•	X	X	X	X	•	X	X	•	X	X	X	X					
Two	X	X	•	X	X	X	X	•	X	X										
Four	X	X	•	X	X	X	X	X	X	•										
Five	•	X	X	X	X	•	X	X	X	X										
Six	X	X	X	X	•	X	•	X	X	X										
Nine	X	•	X	X	X	X	X	X	•	X										
Belford	•	X	X	X	X															
Cavallo	X	X	X	X	•															
Hepler	X	X	•	X	X															
Janney	X	•	X	X	X															
Winslow	X	X	X	•	X															

Here is the final answer:

Order	First name	Last name	Throws	Position
1	Scott	Cavallo	6	Pitcher
2	Josh	Hepler	2	Shortstop
3	Edgar	Belford	5	Right Field
4	Geoffrey	Janney	9	Catcher
5	Oliver	Winslow	4	1st Base

Chapter 7

SETS AND PROBABILITY

7.1 Sets

2. $6 \in \{-2, 6, 9, 5\}$

Since 6 is an element of the given set, the statement is true.

4. $3 \notin \{7, 6, 5, 4\}$

Since 3 is not an element of the given set, the statement is true.

6. $\{3, 7, 12, 14\} = \{3, 7, 12, 14, 0\}$

Two sets are equal only if they contain exactly the same elements. Since 0 is an element of the second set but not the first, the statement is false.

8. $\{x|x \text{ is an odd integer, } 6 \leq x \leq 18\}$
$= \{7, 9, 11, 15, 17\}$

The number 13 should be included in the set so the statement is false.

In Exercises 12-22,

$$\begin{aligned} A &= \{2, 4, 6, 8, 10, 12\}, \\ B &= \{2, 4, 8, 10\}, \\ C &= \{4, 8, 12\}, \\ D &= \{2, 10\}, \\ E &= \{6\}, \\ \text{and} \quad U &= \{2, 4, 6, 8, 10, 12, 14\}. \end{aligned}$$

12. Since every element of E is also an element of A, E is a subset of A, written $E \subseteq A$.

14. Since 10 is an element of B but is not an element of C, B is not a subset of C, written $B \nsubseteq C$.

16. Since 0 is an element of $\{0, 2\}$, but is not an element of D, $\{0, 2\} \nsubseteq D$.

18. Since 2, 6, and 10 are elements of A, but are not elements of C, $A \nsubseteq C$.

20. A set with n distinct elements has 2^n subsets. A has $n = 6$ elements, so there are exactly $2^6 = 64$ subsets of A.

22. A set with n distinct elements has 2^n subsets, and C has $n = 3$ elements. Therefore, there are exactly $2^3 = 8$ subsets of C.

26. $\{8, 11, 15\} \cap \{8, 11, 19, 20\} = \{8, 11\}$

$\{8, 11\}$ is the set of all elements belonging to both of the first two sets, so it is the intersection of those sets.

28. $\{6, 12, 14, 16\} \cap \{6, 14, 19\} = \{6, 14\}$

$\{6, 14\}$ is the set of all elements belonging to both of the first two sets, so it is the intersection of those sets.

30. $\{3, 5, 9, 10\} \cup \emptyset = \{3, 5, 9, 10\}$

The empty set contains no elements, so the union of any set with the empty set will result in an answer set that is identical to the original set. (On the other hand, $\{3, 5, 9, 10\} \cap \emptyset = \emptyset$.)

32. It is possible for two nonempty sets to have the same intersection and union only if they are equal, as in Exercise 31.

In Exercises 34-42,

$$\begin{aligned} U &= \{2, 3, 4, 5, 7, 9\}, \\ X &= \{2, 3, 4, 5\}, \\ Y &= \{3, 5, 7, 9\}, \\ \text{and} \quad Z &= \{2, 4, 5, 7, 9\}. \end{aligned}$$

34. $X \cup Y$, the union of X and Y, is the set of all elements belonging to X or Y or both. Thus,

$$X \cup Y = \{2, 3, 4, 5, 7, 9\}.$$

36. Y', the complement of Y, is the set of all elements of U that do not belong to Y. Thus,

$$Y' = \{2, 4\}.$$

38. $X' \cap Z = \{7, 9\} \cap \{2, 4, 5, 7, 9\}$
$= \{7, 9\}$

40. From Exercise 35, $X' = \{7, 9\}$; from Exercise 36, $Y' = \{2, 4\}$.

$$\begin{aligned} X' \cap (Y' \cup Z) &= \{7, 9\} \cap (\{2, 4\} \cup \{2, 4, 5, 7, 9\}) \\ &= \{7, 9\} \cap \{2, 4, 5, 7, 9\} \\ &= \{7, 9\} \end{aligned}$$

42. **(a)** $(A \cap B) \cup (A \cap B')$

$$\begin{aligned} &= (\{3, 6, 9\} \cap \{2, 4, 6, 8\}) \cup (\{3, 6, 9\} \\ &\quad \cap \{0, 1, 3, 5, 7, 9, 10\}) \\ &= \{6\} \cup \{3, 9\} \\ &= \{3, 6, 9\} \\ &= A \end{aligned}$$

44. $M \cup N$ is the set of all students in this school taking this course or taking accounting.

46. $N' \cap P'$ is the set of all students in this school not taking accounting and not taking zoology.

48. Disjoint sets have no elements in common. Since each pair of sets has at least one element in common, none of the pairs are disjoint.

50. $A \cap B$ is the set of all stocks with a high price greater than \$80 and a closing price between \$60 and \$90. $A \cap B = \{\text{Chevron}\}$.

52. $(A \cup C)'$ is the set of all stocks on the list that do not have a high price greater than \$80 or a positive price change.

$$(A \cup C)' = \{\text{Hershey, GnMill, PepsiCo}\}$$

54. $B = \{a, b, c, \{d\}, \{e, f\}\}$

(a) $a \in B$ is true.

(b) $\{b, c, d\} \subset B$ is false. ($d \notin B$)

(c) $\{d\} \in B$ is true.

(d) $\{d\} \subseteq B$ is false. ($\{d\} \in B$)

(e) $\{e, f\} \in B$ is true.

(f) $\{a, \{e, f\}\} \subset B$ is true.

(g) $\{e, f\} \subset B$ is false. ($\{e, f\} \in B$)

56. $V \cap (F \cup T) = \{$General Electric Co., Microsoft Corp., Pfizer, Inc., Wal-Mart Stores, Inc., Exxon Mobil Corp.$\} \cap (\{$Citigroup Inc., General Electric Inc., Viacom International, American International Group, Inc., Microsoft Corp.$\} \cup \{$Pfizer, Inc., Citigroup, Inc., UnitedHealth Group, Microsoft Corp., First Data$\})$

$= \{$General Electric Co., Microsoft Corp., Pfizer, Inc., Wal-Mart Stores, Inc., Exxon Mobil Corp.$\} \cap \{$Citigroup, Inc., General Electric Co., Viacom International, American International Group, Inc., Microsoft Corp., Pfizer, Inc., UnitedHealth Group, First Data$\}$

$= \{$General Electric Co., Microsoft Corp., Pfizer, Inc.$\}$

58. $J' \cap T' = \{$Citigroup, Inc., Roche Holding A.G., Pfizer, Inc., Microsoft Corp., Samsung Electronics Co., Ltd.$\}' \cap \{$Pfizer, Inc., Citigroup, Inc., UnitedHealth Group, Microsoft Corp., First Data$\}'$

$= \{$General Electric Co., Wal-Mart Stores, Inc., Exxon Mobil Corp., Viacom International, American International Group, Inc., UnitedHealth Group, First Data$\} \cap \{$General Electric Co., Wal-Mart Stores, Inc., Exxon Mobil Corp., Roche Holding A.G., Samsung Electronics Co., Ltd., Viacom International, American International Group, Inc.$\}$

$= \{$General Electric Co., Wal-Mart Stores, Inc., Exxon Mobil Corp., Viacom International, American International Group, Inc.$\}$

60. $U = \{s, d, c, g, i, m, h\}$ and $N = \{s, d, c, g\}$, so

$$N' = \{i, m, h\}.$$

62. $$\begin{aligned} N \cup O &= \{s, d, c, g\} \cup \{i, m, h, g\} \\ &= \{s, d, c, g, i, m, h\} \\ &= U \end{aligned}$$

64. The number of subsets of a set with 51 elements (50 states plus the District of Columbia) is

$$2^{51} \approx 2.522 \times 10^{15}.$$

66. The number of subsets of a set containing 9 elements is $2^9 = 512$.

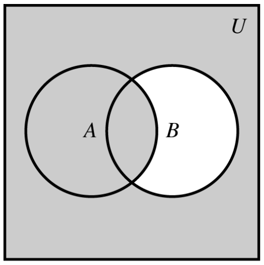
68. $G = \{$TBS, ESPN$\}$

70. $F \cap G = \{$TBS, Discovery Channel, ESPN$\}$
$\cap \{$TBS, ESPN$\}$
$= \{$TBS, ESPN$\}$

72. $F' = \{$TBS, Discovery Channel, ESPN$\}'$
$= \{$USA, C-SPAN$\}$

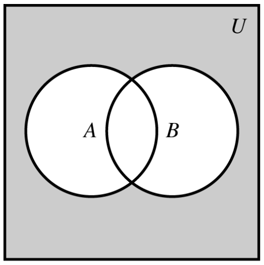
7.2 Applications of Venn Diagrams

2. $A \cup B'$ is the set of all elements in A *or* not in B, or both.



$A \cup B'$

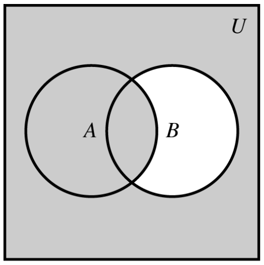
4. $A' \cap B'$ is the set of all elements not in A *and* not in B.



$A' \cap B'$

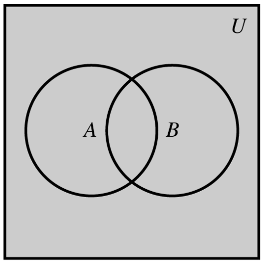
6. $(A \cap B) \cup B'$

First find $A \cap B$, the set of elements in A *and* in B. Now combine this region with B', the set of all elements not in B. For the union, we want those elements in $A \cap B$ *or* B', or both.



$(A \cap B) \cup B'$

8. $\emptyset' = U$, so the entire region is shaded.

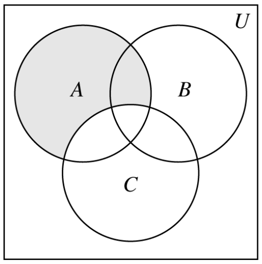


$\emptyset' = U$

10. The notation $n(A)$ represents the number of elements in set A.

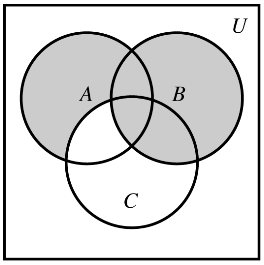
12. $(A \cap C') \cup B$

First find $A \cap C'$, the region in A *and* not in C.



$A \cap C'$

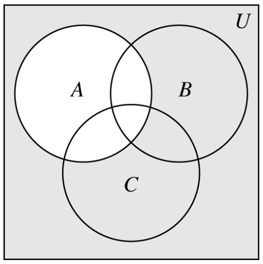
For the union, we want the region in $(A \cap C')$ *or* in B, or both.



$(A \cap C') \cup B$

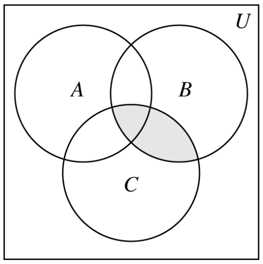
14. $A' \cap (B \cap C)$

First find A', the region not in A.



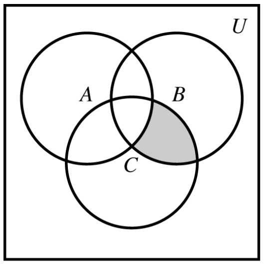
A'

Then find $B \cap C$, the region where B and C overlap.



$B \cap C$

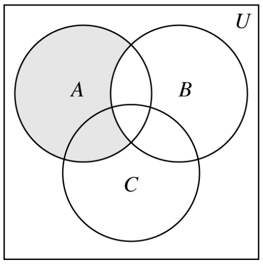
Now intersect these regions.



$A' \cap (B \cap C)$

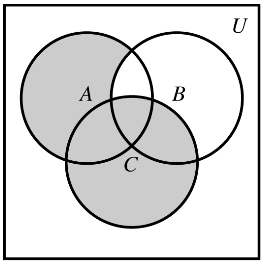
16. $(A \cap B') \cup C$

First find $A \cap B'$, the region in A and not in B.



$A \cap B'$

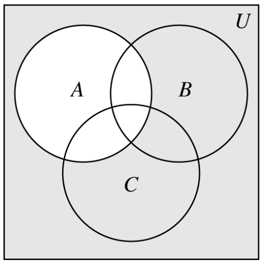
For the union, we want the region in $(A \cap B')$ or in C, or both.



$(A \cap B') \cup C$

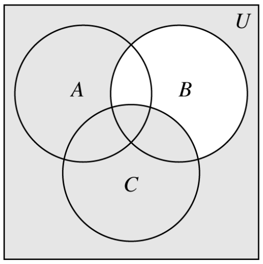
18. $A' \cap (B' \cup C)$

First find A'.



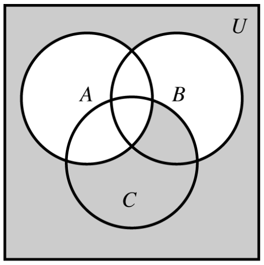
A'

Then find $B' \cup C$, the region not in B *or* in C, or both.



$B' \cup C$

Now intersect these regions.



$A' \cap (B' \cup C)$

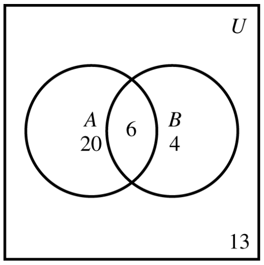
20.
$$\begin{aligned} n(A \cup B) &= n(A) + n(B) - n(A \cap B) \\ 30 &= 12 + 27 - n(A \cap B) \\ 30 &= 39 - n(A \cap B) \\ n(A \cap B) &= 9 \end{aligned}$$

22. $$\begin{aligned} n(A \cup B) &= n(A) + n(B) - n(A \cap B) \\ 35 &= 13 + n(B) - 5 \\ 35 &= 8 + n(B) \\ n(B) &= 27 \end{aligned}$$

24. $n(A) = 26$
$n(B) = 10$
$n(A \cup B) = 30$
$n(A') = 17$

$$\begin{aligned} n(A \cup B) &= n(A) + n(B) - n(A \cap B) \\ 30 &= 26 + 10 - n(A \cap B) \\ 30 &= 36 - n(A \cap B) \\ n(A \cap B) &= 6 \end{aligned}$$

To fill in the regions, start with $A \cap B$. $n(A) = 26$ and $n(A \cap B) = 6$, so $n(A \cap B') = 20$. $n(B) = 10$ and $n(A \cap B) = 6$, so $n(B \cap A') = 4$. Since $n(A') = 17$, 4 of which are accounted for in $B \cap A'$, 13 elements remain in $A' \cap B'$.

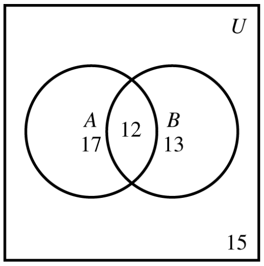


26. $n(A') = 28$
$n(B) = 25$
$n(A' \cup B') = 45$
$n(A \cap B) = 12$

$n(B) = 25$ and $n(A \cap B) = 12$, so $n(B \cap A') = 13$. Since $n(A') = 28$, of which 13 elements are accounted for, 15 elements are in $A' \cap B'$.

$$\begin{aligned} n(A' \cup B') &= n(A') + n(B') - n(A' \cap B') \\ 45 &= 28 + n(B') - 15 \\ 45 &= 13 + n(B') \\ 32 &= n(B') \end{aligned}$$

15 elements are in $A' \cap B'$, so the rest are in $A \cap B'$, and $n(A \cap B') = 17$.



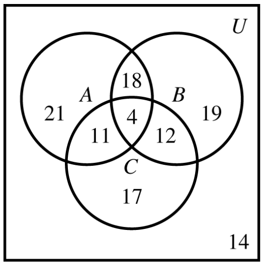
28. $n(A) = 54$
$n(A \cap B) = 22$
$n(A \cup B) = 85$
$n(A \cap B \cap C) = 4$
$n(A \cap C) = 15$
$n(B \cap C) = 16$
$n(C) = 44$
$n(B') = 63$

Start with $A \cap B \cap C$. We have $n(A \cap C) = 15$, of which 4 elements are in $A \cap B \cap C$, so $n(A \cap B' \cap C) = 11$. $n(B \cap C) = 16$, of which 4 elements are in $A \cap B \cap C$, so $n(B \cap C \cap A') = 12$. $n(C) = 44$, so 17 elements are in $C \cap A' \cap B'$. $n(A \cap B) = 22$, so 18 elements are in $A \cap B \cap C'$. $n(A) = 54$, so $54 - 11 - 18 - 4 = 21$ elements are in $A \cap B' \cap C'$.

Now use the union rule.

$$\begin{aligned} n(A \cup B) &= n(A) + n(B) - n(A \cap B) \\ 85 &= 54 + n(B) - 22 \\ 53 &= n(B) \end{aligned}$$

This leaves 19 elements in $B \cap A' \cap C'$. $n(B') = 63$, of which $21 + 11 + 17 = 49$ are accounted for, leaving 14 elements in $A' \cap B' \cap C'$.



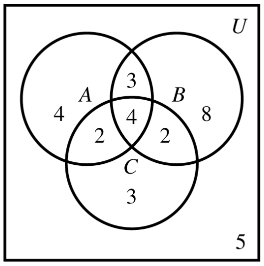
30. $n(A) = 13$
$n(A \cap B \cap C) = 4$
$n(A \cap C) = 6$
$n(A \cap B') = 6$
$n(B \cap C) = 6$
$n(B \cap C') = 11$
$n(B \cup C) = 22$
$n(A' \cap B' \cap C') = 5$

Start with the regions $A \cap B \cap C$ and $A' \cap B' \cap C'$.
$n(B \cap C) = 6$, leaving 2 elements in $B \cap C \cap A'$.
$n(A \cap C) = 6$, leaving 2 elements in $A \cap C \cap B'$.
$n(A \cap B') = 6$, leaving 4 elements in $A \cap B' \cap C'$.
$n(A) = 13$, leaving 3 elements in $A \cap B \cap C'$.
$n(B \cap C') = 11$, leaving 8 elements in $B \cap C' \cap A'$.

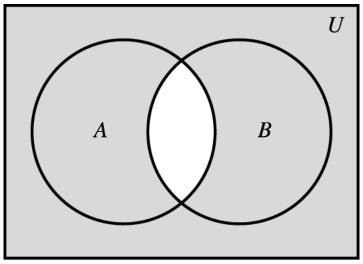
Now use the union rule.

$$\begin{aligned} n(B \cup C) &= n(B) + n(C) - n(B \cap C) \\ 22 &= 17 + n(C) - 6 \\ 11 &= n(C) \end{aligned}$$

This leaves 3 elements in $C \cap A' \cap B'$.



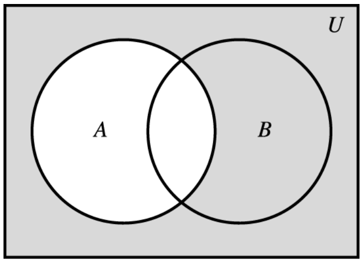
32. $(A \cap B)'$ is the complement of the intersection of A and B; hence it contains all elements not in $A \cap B$.



$(A \cap B)'$

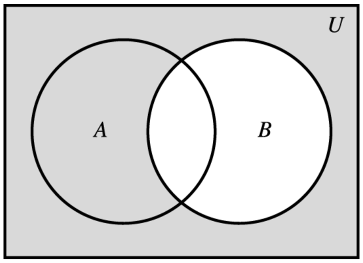
$A' \cup B'$ is the union of the complements of A and B; hence it contains any element that is either not in A *or* not in B.

A' is the set of all elements in U that are not in A.



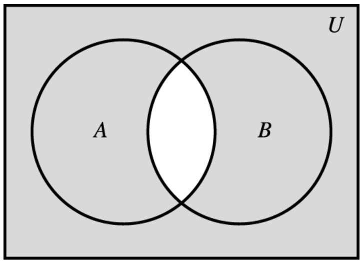
A'

B' is the set of all elements in U that are not in B.



B'

Form the union of A' and B'.



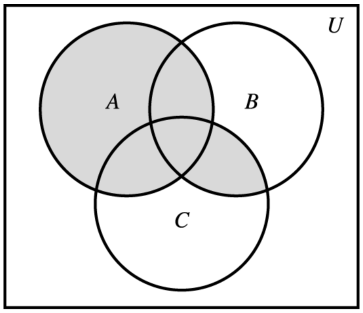
$A' \cup B'$

The Venn diagrams show that

$$(A \cap B)' = A' \cup B',$$

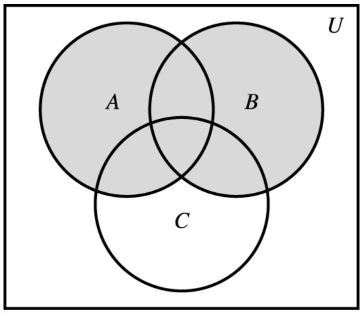
as claimed.

34. $A \cup (B \cap C)$ contains all the elements in A or in both B and C.

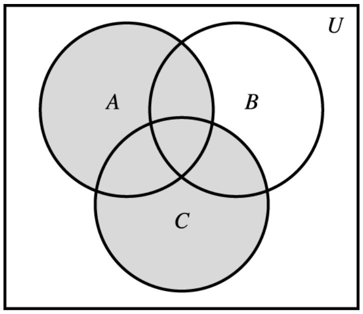


$A \cup (B \cap C)$

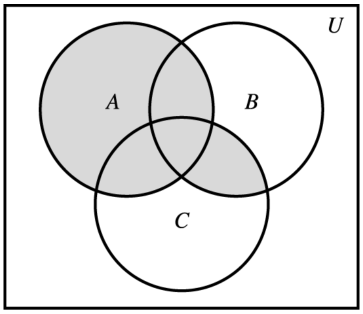
$(A \cup B) \cap (A \cup C)$ contains the intersection $A \cup B$ and $A \cup C$.



$A \cup B$



$A \cup C$



$(A \cup B) \cap (A \cup C)$

Comparing the Venn diagrams, we see that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$$

as claimed.

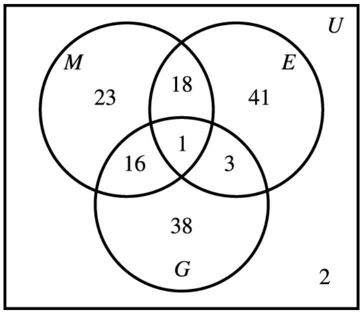
36. Let M be the set of those who use a microwave oven, E be the set of those who use an electric range, and G be the set of those who use a gas range. We are given the following information.

$n(U) = 140$
$n(M) = 58$
$n(E) = 63$
$n(G) = 58$
$n(M \cap E) = 19$
$n(M \cap G) = 17$
$n(G \cap E) = 4$
$n(M \cap G \cap E) = 1$
$n(M' \cap G' \cap E') = 2$

Since $n(M \cap G \cap E) = 1$, there is 1 element in the region where the three sets overlap.
Since $n(M \cap E) = 19$, there are $19 - 1 = 18$ elements in $M \cap E$ but not in $M \cap G \cap E$.
Since $n(M \cap G) = 17$, there are $17 - 1 = 16$ elements in $M \cap G$ but not in $M \cap G \cap E$.
Since $n(G \cap E) = 4$, there are $4 - 1 = 3$ elements in $G \cap E$ but not in $M \cap G \cap E$.
Now consider $n(M) = 58$. So far we have $16 + 1 + 18 = 35$ in M; there must be another $58 - 35 = 23$ in M not yet counted.
Similarly, $n(E) = 63$; we have $18 + 1 + 3 = 22$ counted so far. There must be $63 - 22 = 41$ more in E not yet counted.

Also, $n(G) = 58$; we have $16 + 1 + 3 = 20$ counted so far. There must be $58 - 20 = 38$ more in G not yet counted.

Lastly, $n(M' \cap G' \cap E') = 2$ indicates that there are 2 elements outside of all three sets.



Note that the numbers in the Venn diagram add up to 142 even though $n(U) = 140$. Jeff has made some error, and he should definitely be reassigned.

38. (a) $n(Y \cap R) = 40$ since 40 is the number in the table where the Y row and the R column meet.

(b) $n(M \cap D) = 30$ since 30 is the number in the table where the M row and the D column meet.

(c) $n(D \cap Y) = 15$ and $n(M) = 80$ since that is the total in the M row. $n(M \cap (D \cap Y)) = 0$ since no person can simultaneously have an age in the range 21-25 *and* have an age in the range 26-35. By the union rule for sets,

$$\begin{aligned} &n(M \cup (D \cap Y)) \\ &\quad = n(M) + n(D \cap Y) - n(M \cap (D \cap Y)) \\ &\quad = 80 + 15 - 0 \\ &\quad = 95. \end{aligned}$$

(d) $Y' \cap (D \cup N)$ consists of all people in the D column or in the N column who are at the same time not in the Y row. Therefore,

$$\begin{aligned} n(Y' \cap (D \cup N)) &= 30 + 50 + 20 + 10 \\ &= 110. \end{aligned}$$

(e) $n(N) = 45$
$n(O) = 70$
$n(O') = 220 - 70 = 150$
$n(O' \cap N) = 15 + 20 = 35$

By the union rule,

$$\begin{aligned} n(O' \cup N) &= n(O') + n(N) - n(O' \cap N) \\ &= 150 + 45 - 35 \\ &= 160. \end{aligned}$$

(f) $M' \cap (R' \cap N')$ consists of all people who are not in the R column and not in the N column and who are at the same time not in the M row. Therefore,

$$n(M' \cap (R' \cap N')) = 15 + 50 = 65.$$

(g) $M \cup (D \cap Y)$ consists of all people age 21-25 who drink diet cola *or* anyone age 26-35.

40. We start with the innermost region, which is the number of professors who invested in stock and bonds and certificates of deposit (CDs). Since this is our unknown, place an x in this region. If 80 invested in stocks and bonds, then $80-x$ invested in only stocks and bonds.

If 83 invested in bonds and CDs, then $83 - x$ invested in only bonds and CDs. If 85 invested in stocks and CDs, then $85-x$ invested in only stocks and CDs. If 111 invested in stocks, then the number who invested in only stocks is:

$$111 - [(80 - x) + (85 - x) + x] = x - 54.$$

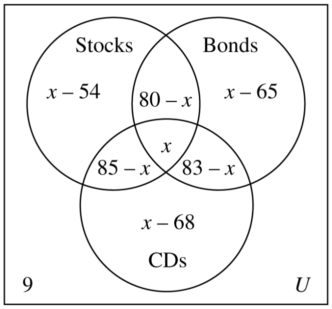
If 98 invested in bonds, then the number who invested in only bonds is;

$$98 - [(80 - x) + (83 - x) + x] = x - 65.$$

If 100 invested in CDs, then the number who invested in only CDs is:

$$100 - [(85 - x) + (83 - x) + x] = x - 68.$$

There are 9 who did not invest in any of the three, so place a 9 outside all three circles. Now, the sum of all the regions is 150, so



$$\begin{aligned}150 &= (x - 54) + (80 - x) + (x - 65) + x \\ &\quad + (85 - x) + (83 - x) + (x - 68) + 9 \\ 150 &= 70 + x \\ x &= 80\end{aligned}$$

80 professors invested in stocks and bonds and certificates of deposits.

42. **(a)** The blood has the A antigen but is Rh negative and has no B antigen. This blood type is A-negative.

(b) Both A and B antigens are present and the blood is Rh negative. This blood type is AB-negative.

(c) Only the B antigen is present. This blood type is B-negative.

(d) Both A and Rh antigens are present. This is blood type is A-positive.

(e) All antigens are present. This blood type is AB-positive.

(f) Both B and Rh antigens are present. This blood type is B-positive.

(g) Only the Rh antigen is present. This blood type is O-positive.

(h) No antigens at all are present. This blood type is 0-negative.

44. Extend the table to include totals for each row and each column.

	W	B	I	A	Total
F	1,009,509	132,410	4591	13,696	1,160,206
M	986,884	144,110	5985	17,060	1,154,039
Total	1,996,393	276,520	10,576	30,756	2,314,245

(a) $n(F)$ is the total for the first row in the table. Thus, there are 1,160,206 people in the set F.

(b) $n(F \cap (I \cup A)) = n(F \cap I) + n(F \cap A) = 4591 + 13{,}696 = 18{,}287$

There are 18,287 people in the set $F \cap (I \cup A)$.

(c) $$\begin{aligned}n(M \cup B) &= n(M) + n(B) - n(M \cap B) \\ &= 1{,}154{,}039 + 276{,}520 - 144{,}110 \\ &= 1{,}286{,}449\end{aligned}$$

There are 1,286,449 people in the set $M \cup B$.

(d) $W' \cup I' \cup A'$ is the universe since each person is either *not* white, *not* American Indian, or *not* Asian or Pacific Islander. Thus, there are 2,314,245 in the set $W' \cup I' \cup' A'$.

(e) The set $F \cap (I \cup A)$ consists of females who are either American Indian or Asian or Pacific Islander.

46. **(a)** $n(A \cup B) = n(A) + n(B) = 71{,}778 + 66{,}166 = 137{,}944$, since A and B are disjoint sets. Thus, there are 137,944 female military personnel in the set $A \cup B$.

(b)
$$\begin{aligned} n(E \cup (C \cup D)) &= n(E) + n(C \cup D) - n(E \cap (C \cup D)) \\ &= n(E) + n(C) + n(D) - n(E \cap C) - n(E \cap D) \\ &= 165{,}518 + 50{,}694 + 9782 - 42{,}261 - 8928 \\ &= 174{,}805 \end{aligned}$$

There are 174,805 female military personnel in the set $E \cup (C \cup D)$.

(c) $O' \cap M' = E$, since the enlisted are the only group who are both *not* officers and *not* cadets or midshipmen. Thus, $n(O' \cap M') = m(E) = 165{,}518$. There are 165,518 female military personnel in the set $O' \cap M'$.

48. Of the 3842 thousand children living with grandparents, 531 + 1732 + 220 = 2483 thousand have at least one parent living with them. Thus, 3842 − 2483 = 1359 thousand live with grandparents only.

50. $n(G \cup B) = n(G) + n(B) - n(G \cap B) = 61.3 + 32.5 - 4.9 = 88.9$

There are 88.9 million people in the set $G \cap B$.

52. $n(F \cap (B \cup H)) = n(F \cap B) = 25.7$

There are 25.7 million people in the set $F \cap (B \cup H)$.

54. $A' \cap C' = B \cup D \cup E$ since the people who are both *not* in the set A and *not* in the set C are the people who are in one of the sets B, D, or E. The set $G' \cap (A' \cap C')$ consists of people in either F or H and also in either B, D, or E. Thus,

$$\begin{aligned} n(G' \cap (A' \cap C')) &= n(F \cap B) + n(F \cap D) + n(F \cap E) + n(H \cap B) + n(H \cap D) + n(H \cap E) \\ &= 25.7 + 7.6 + 1.5 + 1.9 + .8 + .2 \\ &= 37.7. \end{aligned}$$

There are 37.7 million people in the set $G' \cap (A' \cap C')$.

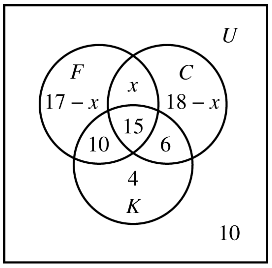
56. Let F be the set of people who brought food, C be the set of those who brought costumes, and K be the set of those who brought crafts. We are given the following information.

$n(U) = 75$
$n(F \cap C \cap K) = 15$
$n(F \cap C) = 25$
$n(F) = 42$
$n(C \cap K \cap F') = 6$
$n(K \cap F' \cap C') = 4$
$n(F' \cap C' \cap K') = 10$
$n(C \cap K') = 18$

Start by putting 15 in the Venn diagram for $F \cap C \cap K$, 6 for $C \cap K \cap F'$, 4 for $K \cap F' \cap C'$, and 10 for $F' \cap C' \cap K'$, as shown. With $n(F \cap C) = 25$ this leaves $n(F \cap C' \cap K') = 25 - 15 = 10$.

With no other regions that we can calculate, denote by x the number in $F \cap C \cap K'$. Then $n(C \cap K' \cap F') = 18 - x$, and $n(F \cap C' \cap K') = 42 - 10 - 15 - x = 17 - x$, as shown. Summing the values for all eight regions,

$$\begin{aligned} (17 - x) + x + (18 - x) + 10 + 15 + 6 + 4 + 10 &= 75 \\ 80 - x &= 75 \\ x &= 5 \end{aligned}$$



(a) $n(C \cap F) = 15 + x = 15 + 5 = 20$

(b) $n(C) = x + (180 - x) + 15 + 6 = 39$

(c) $n(K \cap C') = 10 + 4 = 14$

(d)$n(K') = (17 - x) + x + (18 - x) + 10$
$= 12 + 5 + 13 + 10 = 40$

(e) $n(F \cup C) = (17 - x) + x + (18 - x) + 10 + 15 + 6$
$= 12 + 5 + 13 + 10 + 15 + 6$
$= 61$

7.3 Introduction to Probability

4. List the days in April.

$$S = \{1, 2, 3, 4, \ldots, 29, 30\}$$

6. List the possible number of hours of TV watching in a day.

$$S = \{0, 1, 2, 3, \ \ldots, 23, 24\}$$

8. Let u = up and d = down. There are $2^3 = 8$ possible 3-day outcomes, so

$$S = \{uuu, uud, udu, duu, ddu, dud, udd, ddd\}.$$

10. There are $5 \cdot 5 = 25$ possible outcomes.

$$\begin{aligned} S =\{&(1,1), (1,2), (1,3), (1,4), (1,5), \\ &(2,1), (2,2), (2,3), (2,4), (2,5), \\ &(3,1), (3,2), (3,3), (3,4), (3,5), \\ &(4,1), (4,2), (4,3), (4,4), (4,5), \\ &(5,1), (5,2), (5,3), (5,4), (5,5)\} \end{aligned}$$

14. $S = \{Y, W, B\}$

(a) A yellow marble is drawn. This event is written $\{Y\}$.

(b) A blue marble is drawn. This event is written $\{B\}$.

(c) A white marble is drawn. This event is written $\{W\}$.

(d) A black marble is drawn. There are no black marbles, so this event is written $\emptyset$.

16. Let w = wrong; c = correct.

$S = \{www, wwc, wcw, cww, ccw, cwc, wcc, ccc\}$

(a) The student gets three answers wrong. This event is written $\{www\}$.

(b) The student gets exactly two answers correct. Since either the first, second, or third answer can be wrong, this can happen in three ways. The event is written $\{ccw, cwc, wcc\}$.

(c) The student gets only the first answer correct. The second and third answers must be wrong. This event is written $\{cww\}$.

18. There are 4 possibilities for the first choice and 5 for the second choice. The sample space is

$$\begin{aligned} S = \{&(1,1), (1,2), (1,3), (1,4), (1,5), \\ &(2,1), (2,2), (2,3), (2,4), (2,5), \\ &(3,1), (3,2), (3,3), (3,4), (3,5), \\ &(4,1), (4,2), (4,3), (4,4), (4,5)\}. \end{aligned}$$

(a) The first choice must be 2 or 4; the second can range from 1 to 5:

$$\begin{aligned} \{&(2,1), (2,2), (2,3), (2,4), (2,5), \\ &(4,1), (4,2), (4,3), (4,4), (4,5)\}. \end{aligned}$$

(b) The first choice can range from 1 to 4; the second must be 2 or 4:

$$\{(1,2), (1,4), (2,2), (2,4), (3,2), (3,4), (4,2), (4,4)\}.$$

(c) The choices must add up to 5:

$$\{(1,4), (2,3), (3,2), (4,1)\}.$$

(d) It is not possible for the sum to be 1: $\emptyset$.

For Exercises 20-24, use the sample space

$$S = \{1, 2, 3, 4, 5, 6\}.$$

20. Let O be the event "getting an odd number."

$$O = \{1, 3, 5\}$$
$$P(O) = \frac{3}{6} = \frac{1}{2}$$

22. Let A be the event "getting a number greater than 2."

$$A = \{3, 4, 5, 6\}$$
$$P(A) = \frac{4}{6} = \frac{2}{3}$$

24. Let B be the event "getting any number except 3."

$$B = \{1, 2, 4, 5, 6\}$$
$$P(B) = \frac{5}{6}$$

For Exercises 26-34, the sample space contains all 52 cards in the deck, so $n(S) = 52$.

26. Let B be the event "drawing a black card." There are 26 black cards in the deck, 13 spades and 13 clubs.

$$n(B) = 26$$
$$P(B) = \frac{26}{52} = \frac{1}{2}$$

28. Let H be the event "a heart is drawn." There are 13 hearts in the deck.

$$n(H) = 13$$
$$P(H) = \frac{13}{52} = \frac{1}{4}$$

30. Let F be the event "drawing a face card." The face cards are the jack, queen, and king of each of the four suits.

$$n(F) = 12$$
$$P(F) = \frac{12}{52} = \frac{3}{13}$$

32. Let R be the event "drawing a black 7 or a red 8." There are two black 7's and two red 8's.

$$n(R) = 4$$
$$P(R) = \frac{4}{52} = \frac{1}{13}$$

34. Let H be the event "drawing a heart or drawing a spade." There are 13 hearts and 13 spades.

$$n(H) = 26$$
$$P(H) = \frac{26}{52} = \frac{1}{2}$$

For Exercises 36-40, the sample space consists of all the marbles in the jar. There are $2 + 3 + 5 + 8 = 18$ marbles, so $n(S) = 18$.

36. There are 3 orange marbles.

$$P(\text{orange}) = \frac{3}{18} = \frac{1}{6}$$

38. There are 8 black marbles.

$$P(\text{black}) = \frac{8}{18} = \frac{4}{9}$$

40. There are 8 marbles which are orange or yellow.

$$P(\text{orange or yellow}) = \frac{8}{18} = \frac{4}{9}$$

42. Let W_1 be the event "win on the first draw" and W_2 be the event "win on the second draw."
First of all, $P(W_1) = \frac{1}{3}$, a simple choice from three slips of paper.
To determine $P(W_2)$, the first slip of paper is drawn (from the three available) and thrown away. There are two slips left which may be chosen in any way. So $P(W_2) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$. Altogether, the probability of winning using this strategy is

$$P(W_1) + P(W_2) = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}.$$

44. **(a)** $P(\text{Federal government}) = \dfrac{15.558}{26.343} \approx .5906$

(b) $P(\text{Industry}) = \dfrac{1.896}{26.343} \approx .07197$

(c) $P(\text{Academic institutions}) = \dfrac{4.979}{26.343} \approx .1890$

46. **(a)** $P(\text{invested in stocks and bonds}) = \dfrac{80}{150}$

$$= \frac{8}{15}$$

(b) $P(\text{invested in stocks and bonds and CDs})$

$$= \frac{80}{150}$$

$$= \frac{8}{15}$$

48. E: person smokes

F: person has a family history of heart disease

G: person is overweight

(a) $E \cup F$ occurs when E or F or both occur, so $E \cup F$ is the event "person smokes or has a family history of heart disease, or both."

(b) $E' \cap F$ occurs when E does not occur and F does occur, so $E' \cap F$ is the event "person does not smoke and has a family history of heart disease."

(c) $F' \cup G'$ is the event "person does not have a family history of heart disease or is not overweight, or both."

50. The total population for 2000 is 275,400, and the total for 2025 is 338,300.

(a) $P(\text{Hispanic in 2000}) = \dfrac{32{,}500}{275{,}400}$

$$\approx .118$$

(b) $P(\text{Hispanic in 2025}) = \dfrac{56{,}900}{338{,}300}$

$$\approx .168$$

(c) $P(\text{African-American in 2000}) = \dfrac{33,500}{275{,}400}$

$$\approx .122$$

(d) $P(\text{African-American in 2025}) = \dfrac{44{,}700}{338{,}300}$

$$\approx .132$$

52. **(a)** $P(\text{III Corps}) = \dfrac{22{,}083}{70{,}076} \approx .32$

(b) $P(\text{lost in battle}) = \dfrac{22{,}557}{70{,}076} \approx .32$

(c) $P(\text{I Corps lost in battle}) = \dfrac{7661}{20{,}706}$

$$\approx .37$$

(d) $P(\text{I Corps not lost in battle}) = \dfrac{20{,}706 - 7661}{20{,}706}$

$$\approx .63$$

$$P(\text{II Corps not lost in battle}) = \frac{20{,}666 - 6603}{20{,}666}$$

$$\approx .68$$

$$P(\text{III Corps not lost in battle}) = \frac{22{,}083 - 8007}{22{,}083}$$

$$\approx .64$$

$$P(\text{Calvary not lost in battle}) = \frac{6621 - 286}{6621}$$

$$\approx .96$$

The Calvary had the highest probability of not being lost in battle.

(e) $P(\text{I Corps loss}) = \dfrac{7661}{20{,}706} \approx .37$

$$P(\text{II Corps loss}) = \frac{6603}{20{,}666} \approx .32$$

$$P(\text{III Corps loss}) = \frac{8007}{22{,}083} \approx .36$$

$$P(\text{Calvary loss}) = \frac{286}{6621} \approx .04$$

I Corps had the highest probability of loss.

54. There were 342 in attendance.

(a) $P(\text{speaks Cantonese}) = \dfrac{150}{342}$

$$= \frac{25}{57}$$

(b) $P(\text{does not speaks Cantonese}) = \dfrac{192}{342}$

$$= \frac{32}{57}$$

(c) $P(\text{woman who did not light firecracker})$.

$$= \frac{72}{342} = \frac{4}{19}$$

7.4 Basic Concepts of Probability

2. A person can own a car and a truck at the same time. No, these events are not mutually exclusive.

4. A person can be married and over 30 at the same time. No, these events are not mutually exclusive.

6. In one roll of a die, it is impossible to get both a 4 and an odd number at the same time. Yes, these events are mutually exclusive.

8. When two dice are rolled, there are 36 equally likely outcomes.

(a) Of the 36 ordered pairs, there is only one for which the sum is 2, namely $\{(1,1)\}$. Thus,

$$P(\text{sum is 2}) = \frac{1}{36}.$$

(b) $\{(1,3),(2,2),(3,1)\}$ comprise the ways of getting a sum of 4. Thus,

$$P(\text{sum is 4}) = \frac{3}{36} = \frac{1}{12}.$$

(c) $\{(1,4),(2,3),(3,2),(4,1)\}$ comprise the ways of getting a sum of 5. Thus,

$$P(\text{sum is 5}) = \frac{4}{36} = \frac{1}{9}.$$

(d) $\{(1,5),(2,4),(3,3),(4,2),(5,1)\}$ comprise the ways of getting a sum of 6. Thus,

$$P(\text{sum is 6}) = \frac{5}{36}.$$

10. Again, when two dice are rolled there are 36 equally likely outcomes.

(a) Here, the event is the union of four mutually exclusive events, namely, the sum is 9, the sum is 10, the sum is 11, and the sum is 12. Hence,

$$\begin{aligned}P(\text{sum is 9 or more}) &= P(\text{sum is 9}) + P(\text{sum is 10})\\ &\quad + P(\text{sum is 11}) + (\text{sum is 12})\\ &= \frac{4}{36}+\frac{3}{36}+\frac{2}{36}+\frac{1}{36}\\ &= \frac{10}{36} = \frac{5}{18}.\end{aligned}$$

(b) $P(\text{sum is less than 7})$

$$\begin{aligned}&= P(\text{sum is 1}) + P(\text{sum is 2}) + P(\text{sum is 3})\\ &\quad + P(\text{sum is 4}) + P(\text{sum is 5}) + P(\text{sum is 6})\\ &= \frac{0}{36}+\frac{1}{36}+\frac{2}{36}+\frac{3}{36}+\frac{4}{36}+\frac{5}{36}\\ &= \frac{15}{36}\\ &= \frac{5}{12}\end{aligned}$$

(c) $P(\text{sum is between 5 and 8})$

$$\begin{aligned}&= P(\text{sum is 6}) + P(\text{sum is 7})\\ &= \frac{5}{36}+\frac{6}{36}\\ &= \frac{11}{36}\end{aligned}$$

12. $P(\text{first die is 3 or sum is 8})$

$$\begin{aligned}&= P(\text{first die is 3}) + P(\text{sum is 8})\\ &\quad - P(\text{first die is 3 and sum is 8})\\ &= \frac{6}{36}+\frac{5}{36}-\frac{1}{36}\\ &= \frac{10}{36}\\ &= \frac{5}{18}\end{aligned}$$

14. **(a)** The sample space is

$$\begin{array}{cccccc}3-1 & 3-1 & 3-5 & 3-5 & 3-9 & 3-9\\ 3-1 & 3-1 & 3-5 & 3-5 & 3-9 & 3-9\\ 4-1 & 4-1 & 4-5 & 4-5 & 4-9 & 4-9\\ 4-1 & 4-1 & 4-5 & 4-5 & 4-9 & 4-9\\ 8-1 & 8-1 & 8-5 & 8-5 & 8-9 & 8-9\\ 8-1 & 8-1 & 8-5 & 8-5 & 8-9 & 8-9\end{array}$$

where the first number in each pair is the number that appears on A and the second the number that appears on B. B beats A in 20 of 36 possible outcomes. Thus,

$$P(B \text{ beats } A) = \frac{20}{36} = \frac{5}{9}.$$

(b) The sample space is

$$\begin{array}{cccccc}1-2 & 1-2 & 1-6 & 1-6 & 1-7 & 1-7\\ 1-2 & 1-2 & 1-6 & 1-6 & 1-7 & 1-7\\ 5-2 & 5-2 & 5-6 & 5-6 & 5-7 & 5-7\\ 5-2 & 5-2 & 5-6 & 5-6 & 5-7 & 5-7\\ 9-2 & 9-2 & 9-6 & 9-6 & 9-7 & 9-7\\ 9-2 & 9-2 & 9-6 & 9-6 & 9-7 & 9-7\end{array}$$

where the first number in each pair is the number that appears on B and the second the number that appears on C. C beats B in 20 of 36 possible outcomes. Thus,

$$P(C \text{ beats } B) = \frac{20}{36} = \frac{5}{9}.$$

(c) The sample space is

$$\begin{array}{cccccc}
3-2 & 3-2 & 3-6 & 3-6 & 3-7 & 3-7 \\
3-2 & 3-2 & 3-6 & 3-6 & 3-7 & 3-7 \\
4-2 & 4-2 & 4-6 & 4-6 & 4-7 & 4-7 \\
4-2 & 4-2 & 4-6 & 4-6 & 4-7 & 4-7 \\
8-2 & 8-2 & 8-6 & 8-6 & 8-7 & 8-7 \\
8-2 & 8-2 & 8-6 & 8-6 & 8-7 & 8-7
\end{array}$$

where the first number in each pair is the number that appears on A and the second the number that appears on C. A beats C in 20 of 36 possible outcomes. Thus,

$$P(A \text{ beats } C) = \frac{20}{36} = \frac{5}{9}.$$

16. **(a)** Less than a 4 would be an ace, a 2, or a 3. There are a total of 12 aces, 2's, and 3's in a deck of 52, so

$$P(\text{ace or 2 or 3}) = \frac{12}{52} = \frac{3}{13}.$$

(b) There are 13 diamonds plus three 7's in other suits, so

$$P(\text{diamond or 7}) = \frac{16}{52} = \frac{4}{13}.$$

Alternatively, using the union rule for probability,

$$\begin{aligned}
&P(\text{diamond}) + P(7) - P(7 \text{ of diamonds}) \\
&\quad = \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}.
\end{aligned}$$

(c) There are 26 black cards plus 2 red aces, so

$$P(\text{black or ace}) = \frac{28}{52} = \frac{7}{13}.$$

(d) $P(\text{heart or black}) = \frac{13}{52} + \frac{26}{52} = \frac{39}{52} = \frac{3}{4}.$

(e) There are 26 red cards plus 6 black face cards, so

$$P(\text{red or face card}) = \frac{32}{52} = \frac{8}{13}.$$

18. **(a)** There are 3 uncles plus 2 cousins out of 10, so

$$P(\text{uncle or cousin}) = \frac{5}{10} = \frac{1}{2}.$$

(b) There are 3 uncles, 2 brothers, and 2 cousins, for a total of 7 out of 10, so

$$P(\text{male or cousin}) = \frac{7}{10}.$$

(c) There are 2 aunts, 2 cousins, and 1 mother, for a total of 5 out of 10, so

$$P(\text{female or cousin}) = \frac{5}{10} = \frac{1}{2}.$$

20. **(a)** The sample space for this experiment is listed in part (b) below. The only outcomes in which both numbers are even are $(2,4)$ and $(4,2)$, so

$$P(\text{even}) = \frac{2}{20} = \frac{1}{10}.$$

(b) The sample space is

$\{\underline{(1,2)}, (1,3), \underline{(1,4)}, \underline{(1,5)}, \underline{(2,1)}, \underline{(2,3)}, \underline{(2,4)}, \underline{(2,5)}, (3,1), \underline{(3,2)}, \underline{(3,4)}, \underline{(3,5)}, \underline{(4,1)}, \underline{(4,2)}, \underline{(4,3)}, \underline{(4,5)}, \underline{(5,1)}, \underline{(5,2)}, \underline{(5,3)}, \underline{(5,4)}\}.$

The 18 underlined pairs are the outcomes in which one number is even or greater than 3, so

$$P(\text{even or } > 3) = \frac{18}{20} = \frac{9}{10}.$$

(c) The sum is 5 in the outcomes $(1,4)$, $(2,3)$, $(3,2)$, and $(4,1)$. The second draw is 2 in the outcomes $(1,2)$, $(3,2)$, $(4,2)$, and $(5,2)$. There are 7 distinct outcomes out of 20, so

$$P(\text{sum is 5 or second number is 2}) = \frac{7}{20}.$$

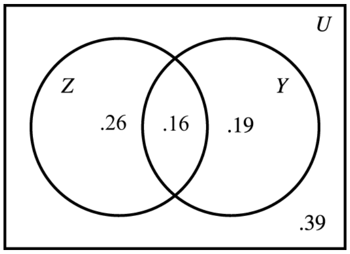
22. $P(Z) = .42,\ P(Y) = .35,\ P(Z \cup Y) = .61$

Begin by using the union rule for probability.

$$\begin{aligned}
P(Z \cup Y) &= P(Z) + P(Y) - P(Z \cap Y) \\
.61 &= .42 + .35 - P(Z \cap Y) \\
.61 &= .77 - P(Z \cap Y) \\
-.16 &= -P(Z \cap Y) \\
.16 &= P(Z \cap Y)
\end{aligned}$$

This gives the first value to be labeled in the Venn diagram. Then the part of Z outside Y must contain $.42 - .16 = .26$, and the part of Y outside Z must contain $.35 - .16 = .19$.

Observe that $.26 + .16 + .19 = .61$, which agrees with the given information that $P(Z \cup Y) = .61$. The part of U outside both Y and Z must contain $1 - .61 = .39$.



This Venn diagram may now be used to find the following probabilities.

(a) $Z' \cap Y'$ is the event presented by the part of the Venn diagram that is outside Z and outside Y.

$$P(Z' \cap Y') = .39$$

(b) $Z' \cup Y'$ is everything outside Z or outside Y or both, which is all of U except $Z \cap Y$.

$$P(Z' \cup Y') = 1 - .16 = .84$$

(c) $Z' \cup Y$ is everything outside Z or inside Y or both.

$$P(Z' \cup Y) = .19 + .39 + .16 = .74$$

(d) $Z \cap Y'$ is everything inside Z and outside Y.

$$P(Z \cap Y') = .26$$

24. Let E be the event "a 5 is rolled."

$$P(E) = \frac{1}{6} \text{ and } P(E') = \frac{5}{6}.$$

The odds in favor of rolling a 5 are

$$\frac{P(E)}{P(E')} = \frac{\frac{1}{6}}{\frac{5}{6}} = \frac{1}{5},$$

which is written "1 to 5."

26. Let E be the event "a 1, 2, 3, or 4 is rolled." Here $P(E) = \frac{4}{6} = \frac{2}{3}$ and $P(E') = \frac{1}{3}$. The odds in favor of E are

$$\frac{P(E)}{P(E')} = \frac{\frac{2}{3}}{\frac{1}{3}} = \frac{2}{1},$$

which is written "2 to 1."

28. **(a)** Yellow: There are 3 ways to win and 12 ways to lose. The odds in favor of drawing yellow are 3 to 12, or 1 to 4.

(b) Blue: There are 8 ways to win and 7 ways to lose; the odds in favor of drawing blue are 8 to 7.

(c) White: There are 4 ways to win and 11 ways to lose; the odds in favor of drawing white are 4 to 11.

30. When two dice are rolled, there are 36 equally likely outcomes.

$$\begin{aligned} P(7 \text{ or } 11) &= P(7) + P(11) \\ &= \frac{6}{36} + \frac{2}{36} \\ &= \frac{8}{36} \\ &= \frac{2}{9} \end{aligned}$$

The odds of rolling a 7 or 11 are

$$\frac{P(E)}{P(E')} = \frac{\frac{2}{9}}{1 - \frac{2}{9}} = \frac{\frac{2}{9}}{\frac{7}{9}} = \frac{2}{7},$$

which is written "2 to 7."

34. It is possible to establish an exact probability for this event, so this is not an empirical probability.

36. It is not possible to establish an exact probability for this event, so this is an empirical probability.

38. It is not possible to establish an exact probability for this event, so this is an empirical probability.

40. The gambler's claim is a mathematical fact, so this is not an empirical probability.

44. The probability assignment is possible because the probability of each outcome is a number between 0 and 1, and the sum of the probabilities of all the outcomes is

$$.92 + .03 + 0 + .02 + .03 = 1.$$

46. The probability assignment is not possible. All of the probabilities are between 0 and 1, but the sum of the probabilities is

$$\frac{1}{5} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{10} = \frac{13}{12}$$

which is greater than 1.

48. The probability assignment is not possible. One of the probabilities is negative instead of being between 0 and 1, and the sum of the probabilities is not 1.

50. The answers that are given are theoretical. Using the Monte Carlo method with at least 50 repetitions on a graphing calculator should give values close to these.

(a) .2778 **(b)** .4167

52. The answers that are given are theoretical. Using the Monte Carlo method with at least 50 repetitions on a graphing calculator should give values close to these.

(a) .15625 **(b)** .3125

54. **(a)**
$$\begin{aligned} P(\$500 \text{ or more}) &= 1 - P(\text{less than } \$500) \\ &= 1 - (.31 + .18) \\ &= 1 - .49 \\ &= .51 \end{aligned}$$

(b)
$$\begin{aligned} P(\text{less than } \$1000) &= .31 + .18 + .18 \\ &= .67 \end{aligned}$$

(c)
$$\begin{aligned} P(\$500 \text{ to } \$2999) &= .18 + .13 + .08 \\ &= .39 \end{aligned}$$

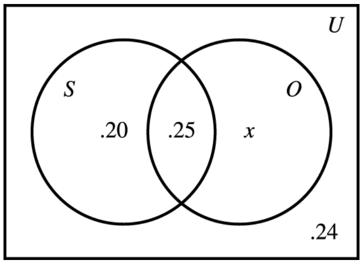
(d)
$$\begin{aligned} P(\$3000 \text{ or more}) &= .05 + .06 + .01 \\ &= .12 \end{aligned}$$

56. **(a)**
$$\begin{aligned} &P(\text{less than } \$350) \\ &= 1 - P(\$350 \text{ or more}) \\ &= 1 - (.08 + .03) \\ &= 1 - .11 \\ &= .89 \end{aligned}$$

(b)
$$\begin{aligned} &P(\$75 \text{ or more}) \\ &= P(\$75\text{-}\$99.99) + P(\$100\text{-}\$199.99) \\ &\quad + P(\$200\text{-}\$349.99) + P(\$350\text{-}\$499.99) \\ &\quad + P(\$500 \text{ or more}) \\ &= .11 + .09 + .07 + .08 + .03 \\ &= .38 \end{aligned}$$

(c)
$$\begin{aligned} &P(\$200 \text{ or more}) \\ &= P(\$200\text{-}\$349.99) + P(\$350\text{-}\$499.99) \\ &\quad + P(\$500 \text{ or more}) \\ &= .07 + .08 + .03 \\ &= .18 \end{aligned}$$

58. Let S be the event "the person is short," and let O be the event "the person is overweight."



From the Venn diagram,

$$\begin{aligned} .20 + .25 + x + .24 &= 1 \\ .69 + x &= 1 \\ x &= .31. \end{aligned}$$

The probability that a person is

(a) overweight is $.25 + .31 = .56$;

(b) short, but not overweight is .20;

(c) tall (not short) and overweight is .31.

60. **(a)** Since red is dominant, the event "plant has red flowers"

$$= \{RR, RW, WR\};\ P(\text{red}) = \frac{3}{4}.$$

(b) $P(\text{white}) = 1 - P(\text{red}) = \frac{1}{4}$

62. **(a)** $P(\text{no more than 4 good toes})$

$$= .77 + .13 = .90$$

(b) $P(5 \text{ toes}) = .13 + .10 = .23$

64. Since 55 of the workers were women, $130 - 55 = 75$ were men. Since 3 of the women earned more than \$40,000, $55 - 3 = 52$ of them earned \$40,000 or less. Since 62 of the men earned \$40,000 or less, $75 - 62 = 13$ earned more than \$40,000.
These data for the 130 workers can be summarized in the following table.

	Men	Women
\$40,000 or less	62	52
Over \$40,000	13	3

(a) $P(\text{a woman earning } \$40,000 \text{ or less})$

$$= \frac{52}{130} = .4$$

(b) P(a man earning more than \$40,000)

$$= \frac{13}{130} = .1$$

(c) P(a man or is earning more than \$40,000)

$$= \frac{62+13+3}{130}$$
$$= \frac{78}{130} = .6$$

(d) P(a woman or is earning \$40,000 or less)

$$= \frac{52+3+62}{130}$$
$$= \frac{117}{130} = .9$$

66. Let A be the set of refugees who came to escape abject poverty and B be the set of refugees who came to escape political oppression. Then $P(A) = .80$, $P(B) = .90$, and $P(A \cap B) = .70$.

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= .80 + .90 - .70 = 1 \\ P(A' \cap B') &= 1 - P(A \cap B) \\ &= 1 - 1 = 0 \end{aligned}$$

The probability that a refugee in the camp was neither poor nor seeking political asylum is 0.

68. The odds of winning are 3 to 2; this means there are 3 ways to win and 2 ways to lose, out of a total of 2 + 3 = 5 ways altogether. Hence, the probability of losing is $\frac{2}{5}$.

70. **(a)** P(somewhat or extremely intolerant of Facists)

$= P$(somewhat intolerant of Fascists)

$+ P$(extremely intolerant of Facists)

$$= \frac{27.1}{100} + \frac{59.5}{100} = \frac{86.6}{100} = .866$$

(b) P(completely tolerant of Communists)

$= P$(no intolerance at all of Communists)

$$= \frac{47.8}{100} = .478$$

72. Since the odds are 4 to 7, the probability of rain is

$$\frac{4}{4+7} = \frac{4}{11}.$$

7.5 Conditional Probability; Independent Events

2. $$P(4|\text{even}) = \frac{P(4 \cap \text{even})}{P(\text{even})} = \frac{n(4 \cap \text{even})}{n(\text{even})} = \frac{1}{3}$$

4. P(sum of 8|greater than 7)

$$= \frac{P(8 \cap \text{greater than } 7)}{P(\text{greater than } 7)}$$
$$= \frac{n(8 \cap \text{greater than } 7)}{n(\text{greater than } 7)}$$
$$= \frac{5}{15} = \frac{1}{3}$$

6. The event of getting a double given that 9 was rolled is impossible; hence,

$$P(\text{double}|\text{sum of } 9) = 0.$$

8. P(second is black|first is a spade) $= \frac{25}{51}$, since there are 25 black cards left out of 51 cards. Note that the sample space is reduced from 52 cards to 51 cards after the first card is drawn.

10. P(second is an ace|first is not an ace) $= \frac{4}{51}$, since there are 4 aces left out of 51 cards.

12. The probability of drawing an ace and then drawing a 4 is

$$\frac{4}{52} \cdot \frac{4}{51} = \frac{4}{663}.$$

The probability of drawing a 4 and then drawing an ace is

$$\frac{4}{52} \cdot \frac{4}{51} = \frac{4}{663}.$$

Since these are mutually exclusive, the probability of drawing an ace and a 4 is

$$\frac{4}{663} + \frac{4}{663} \approx .012.$$

14. The probability that the first card is a heart is $\frac{13}{52} = \frac{1}{4}$. The probability that the second is a heart, given that the first is a heart, is $\frac{12}{51}$. Thus, the probability that both are hearts is

$$\frac{1}{4} \cdot \frac{12}{51} \approx .059.$$

16. **(a)** The events that correspond to "sum is 7" are $(2,5),(3,4),(4,3)$, and $(5,2)$, where the first number is the number on the first slip of paper and the second number is the number on the second. Of these, only $(3,4)$ corresponds to "first is 3," so

$$P(\text{first is }3|\text{sum is }7)=\frac{1}{4}.$$

(b) The events that correspond to "sum is 8" are $(3,5)$ and $(5,3)$. Of these, only $(3,5)$ corresponds to "first is 3," so

$$P(\text{first is }3|\text{sum is }8)=\frac{1}{2}.$$

18. **(a)** Many answers are possible; for example, let B be the event that the first die is a 5. Then

$$P(A\cap B)=P(\text{sum is 7 and first is 5})=\frac{1}{36}$$

$$\begin{aligned}P(A)\cdot P(B)&=P(\text{sum is 7})\cdot P(\text{first is 5})\\&=\frac{6}{36}\cdot\frac{1}{6}=\frac{1}{36}\end{aligned}$$

so, $P(A\cap B)=P(A)\cdot P(B)$.

(b) Many answers are possible; for example, let B be the event that at least one die is a 5.

$$\begin{aligned}P(A\cap B)&=P(\text{sum is 7 and at least one is a 5})\\&=\frac{2}{36}\end{aligned}$$

$$\begin{aligned}P(A)\cdot P(B)&=P(\text{sum is 7})\cdot P(\text{at least one is a 5})\\&=\frac{6}{36}\cdot\frac{11}{6}\end{aligned}$$

so, $P(A\cap B)\neq P(A)\cdot P(B)$.

22. At the first booth, there are three possibilities:
shaker 1 has heads and shaker 2 has heads;
shaker 1 has tails and shaker 2 has heads;
shaker 1 has heads and shaker 2 has tails.
We restrict ourselves to the condition that at least one head has appeared. These three possibilities are equally likely so the probability of two heads is $\frac{1}{3}$.

At the second booth we are given the condition of one head in one shaker. The probability that the second shaker has one head is $\frac{1}{2}$.

Therefore, you stand the best chance at the second booth.

24. No, these events are not independent.

26. Since A and B are independent events.

$$P(A\cap B)=P(A)\cdot P(B)=\frac{1}{4}\cdot\frac{1}{5}=\frac{1}{20}.$$

Thus,

$$\begin{aligned}P(A\cup B)&=P(A)+P(B)-P(A\cap B)\\&=\frac{1}{4}+\frac{1}{5}-\frac{1}{20}\\&=\frac{2}{5}.\end{aligned}$$

28. Assume that each box is equally likely to be drawn from and that within each box each marble is equally likely to be drawn. If Laura does not redistribute the marbles, then the probability of winning the Porsche is $\frac{1}{2}$, since the event of a pink marble being drawn is equivalent to the event of choosing the first of the two boxes.

If however, Laura puts 49 of the pink marbles into the second box with the 50 blue marbles, the probability of a pink marble being drawn increases to $\frac{74}{99}$. The probability of the first box being chosen is $\frac{1}{2}$, and the probability of drawing a pink marble from this box is 1. The probability of the second box being chosen is $\frac{1}{2}$, and the probability of drawing a pink marble from this box is $\frac{49}{99}$. Thus, the probability of drawing a pink marble is $\frac{1}{2}\cdot 1+\frac{1}{2}\cdot\frac{49}{99}=\frac{74}{99}$.

30. The probability that a customer cashing a check will fail to make a deposit is

$$P(D'|C)=\frac{n(D'\cap C)}{n(C)}=\frac{30}{80}=\frac{3}{8}.$$

32. The probability that a customer making a deposit will not cash a check is

$$P(C'|D)=\frac{n(C'\cap D)}{n(D)}=\frac{20}{70}=\frac{2}{7}.$$

34. **(a)** Since the separate flights are independent, the probability of 3 flights in a row is

$$.98(.98)(.98)\approx .94.$$

(b) Independence is not very realistic.

36. Let W be the event "withdraw cash from ATM" and C be the event "check account balance at ATM."

$$\begin{aligned} P(C \cup W) &= P(C) + P(W) - P(C \cap W) \\ .96 &= .32 + .92 - P(C \cap W) \\ -.28 &= -P(C \cap W) \\ P(C \cap W) &= .28 \end{aligned}$$

$$\begin{aligned} P(W|C) &= \frac{P(C \cap W)}{P(C)} \\ &= \frac{.28}{.32} \\ &\approx .875 \end{aligned}$$

The probability that she uses an ATM to get cash given that she checked her account balance is .875.

38. Let I be the event "the bike passes the inspection" and B be the event "the bike came off assembly line B." The fact that 90% of line B's products pass inspection gives $P(I|B) = .9$. Thus,

$$\begin{aligned} P(I' \cap B) &= P(B) \cdot P(I'|B) \\ &= .6(.1) \\ &= .06. \end{aligned}$$

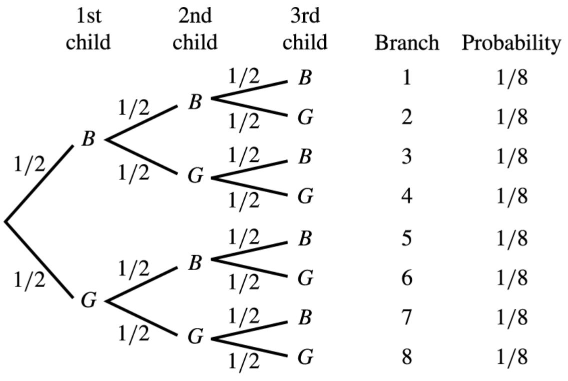
40. The sample space is

$$\{RW, WR, RR, WW\}.$$

The event "red" is $\{RW, WR, RR\}$, and the event "mixed" is $\{RW, WR\}$.

$$\begin{aligned} P(\text{mixed}|\text{red}) &= \frac{n(\text{mixed and red})}{n(\text{red})} \\ &= \frac{2}{3}. \end{aligned}$$

Use the following tree diagram for Exercises 42 and 44.



42. $P(3 \text{ girls}|3\text{rd is a girl})$

$$\begin{aligned} &= \frac{P(3 \text{ girls and 3rd is a girl})}{P(3\text{rd is a girl})} \\ &= \frac{\frac{1}{8}}{\frac{1}{2}} \\ &= \frac{1}{4} \end{aligned}$$

44. $P(3 \text{ girls}|\text{at least 2 girls})$

$$\begin{aligned} &= \frac{P(3 \text{ girls and at least 2 girls})}{P(\text{at least 2 girls})} \\ &= \frac{P(3 \text{ girls})}{P(\text{at least 2 girls})} \\ &= \frac{P(3 \text{ girls})}{P(2 \text{ girls}) + P(3 \text{ girls})} \\ &= \frac{\frac{1}{8}}{\frac{3}{8} + \frac{1}{8}} \\ &= \frac{\frac{1}{8}}{\frac{4}{8}} \\ &= \frac{1}{4} \end{aligned}$$

(Note that

$$P(3 \text{ girls}) = P(GGG) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

and

$$\begin{aligned} P(2 \text{ girls}) &= P(GGB) + P(BGG) + P(GBG) \\ &= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \\ &= \frac{3}{8}.) \end{aligned}$$

46. **(a)** $P(\text{homosexual contact}|\text{male})$

$$= \frac{8388}{18{,}423} \approx .46$$

(b) $P(\text{intervenous drug use}|\text{female})$

$$= \frac{1541}{5401} \approx .29$$

(c) $P(\text{female}) = \dfrac{5401}{23{,}824} \approx .23$

(d) $P(\text{female}|\text{heterosexual contact})$

$$= \frac{1806}{2952} \approx .61$$

(e) No, the probabilities in (c) and (d) are not equal.

48. $P(M) = .487$, the total of the M column.

50. $P(M \cap C) = .035$, the entry in the M column and C row.

52. $$P(M|C) = \frac{P(M \cap C)}{P(C)} = \frac{.035}{.039} \approx .897$$

54. $$P(M'|C) = \frac{P(M' \cap C)}{P(C)} = \frac{.004}{.039} \approx .103$$

56. **(a)** From the table,

$$P(C \cap D) = .0008 \text{ and}$$
$$P(C) \cdot P(D) = .0400(.0200) = .0008.$$

Since $P(C \cap D) = P(C) \cdot P(D)$, C and D are independent events; color blindness and deafness are independent events.

58. **(a)** $P(\text{letter is in drawer 1}) = .1$

(b) $P(\text{letter is in drawer 2}|\text{letter is not in drawer 1})$

$$= \frac{.1}{.9} \approx .1111$$

(c) $P(\text{letter is in drawer 3}|\text{letter is not in drawer 1 or 2})$

$$= \frac{.1}{.8} = .125$$

(d) $P(\text{letter is in drawer 8}|\text{letter is not in drawers 1 through 7})$

$$= \frac{.1}{.3} \approx .3333$$

(e) The probability is increasing.

(f) $P(\text{letter is in some drawer}) = .8$

(g) $P(\text{letter is in some drawer}|\text{not in drawer 1})$

$$= \frac{.7}{.9} \approx .7777$$

(h) $P(\text{letter is in some drawer}|\text{not in drawer 1 or 2})$

$$= \frac{.6}{.8} = .75$$

(i) $P(\text{letter is in some drawer}|\text{not in drawers 1 through 7})$

$$= \frac{.1}{.3} \approx .3333$$

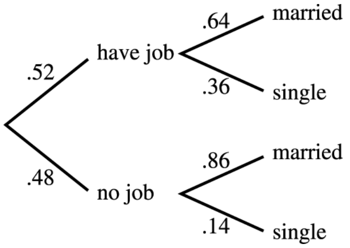
(j) The probability is decreasing.

60. $P(A) = \dfrac{129}{576} \approx .224$

62. $P(A|H) = \dfrac{95}{347} \approx .274$

64. No, $P(A) \neq P(A|H)$.

66. Draw the tree diagram.



(a)
$$\begin{aligned} P(\text{married}) &= P(\text{job and married}) \\ &\quad + P(\text{no job and married}) \\ &= .52(.64) + .48(.86) \\ &= .3328 + .4128 \\ &= .7456 \end{aligned}$$

(b) $P(\text{job and single}) = .52(.36) = .1872$

68. $P(\text{component fails}) = .03$

(a) Let n represent the number of these components to be connected in parallel; that is, the component has $n - 1$ backup components.

$P(\text{at least one component works})$

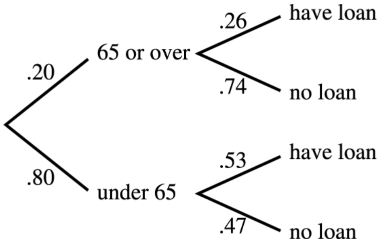
$$\begin{aligned} &= 1 - P(\text{no component works}) \\ &= 1 - P(\text{all } n \text{ components fail}) \\ &= 1 - (.03)^n \end{aligned}$$

If this probability is to be at least .999999, then

$$\begin{aligned} 1 - (.03)^n &\geq .999999 \\ -(.03)^n &\geq -.000001 \\ (.03)^n &\leq .000001, \end{aligned}$$

and the smallest whole number value of n for which this inequality holds true is $n = 4$. Therefore, $4 - 1 = 3$ backup components must be used.

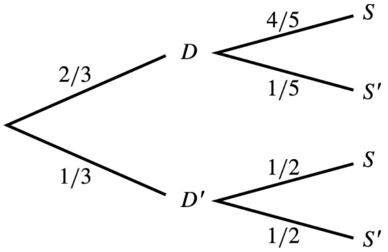
70. First draw the tree diagram.



(a) P(person is 65 or over and has a loan)
$= P(\text{65 or over}) \cdot P(\text{has loan}|\text{65 or over})$
$= .20(.26) = .052$

(b) P(person has a loan)
$= P(\text{65 or over and has loan})$
$+ P(\text{under 65 and has loan})$
$= .20(.26) + .80(.53)$
$= .052 + .424$
$= .476$

72.



From the tree diagram, we see that the probability that a person

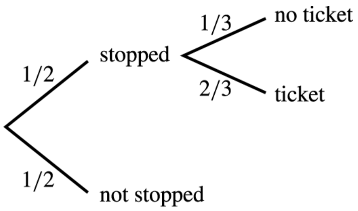
(a) drinks diet soft drinks is

$$\frac{2}{3}\left(\frac{4}{5}\right)+\frac{1}{3}\left(\frac{1}{2}\right)=\frac{8}{15}+\frac{1}{6}$$
$$=\frac{21}{30}=\frac{7}{10};$$

(b) diets, but does not drink diet soft drinks is

$$\frac{2}{3}\left(\frac{1}{5}\right)=\frac{2}{15}.$$

74. First draw a tree diagram.



(a) $P(\text{no ticket}) = \frac{1}{2}\left(\frac{1}{3}\right)+\frac{1}{2}=\frac{2}{3}$

(b) P(no ticket on three consecutive weekends)

$$=\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}=\frac{8}{27}\approx .296$$

76. Let A be the event "student studies" and B be the event "student gets a good grade." We are told that $P(A) = .6$, $P(B) = .7$, and $P(A \cap B) = .52$.

$$P(A) \cdot P(B) = .6(.7) = .42$$

(a) Since $P(A) \cdot P(B)$ is not equal to $P(A \cap B)$, A and B are not independent. Rather, they are dependent events.

(b) Let A be the event "a student studies" and B be the event "the student gets a good grade." Then

$$P(B|A)=\frac{P(B\cap A)}{P(A)}=\frac{.52}{.6}\approx .87.$$

(c) Let A be the event "a student gets a good grade" and B be the event "the student studied." Then

$$P(B|A)=\frac{P(B\cap A)}{P(A)}=\frac{.52}{.7}\approx .74.$$

7.6 Bayes' Theorem

2. By Bayes' theorem,

$$P(M'|N) = 1 - P(M|N)$$
$$= 1 - \frac{P(M)\cdot P(N|M)}{P(M)\cdot P(N|M) + P(M')\cdot P(N|M')}$$
$$= 1 - \frac{.4(.3)}{.4(.3)+.6(.4)}$$
$$= 1 - \frac{.12}{.12+.24}$$
$$= 1 - \frac{1}{3} = \frac{2}{3}.$$

4. $P(R_2|Q)$

$$= \frac{P(R_2) \cdot P(Q|R_2)}{P(R_1)P(Q|R_1) + P(R_2)P(Q|R_2) + P(R_3)P(Q|R_3)}$$

$$= \frac{.6(.3)}{.05(.4) + .6(.3) + .35(.6)}$$

$$= \frac{.18}{.41} = \frac{18}{41}$$

6. This is the complement of the event in Exercise 3, so

$$P(R_1{}'|Q) = 1 - P(R_1|Q)$$
$$= 1 - \frac{2}{41} = \frac{39}{41}.$$

8. $P(J_3|W)$

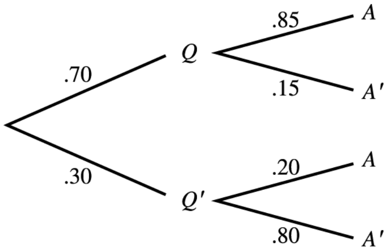
$$= \frac{P(J_3) \cdot P(W|J_3)}{P(J_1)P(W|J_1) + P(J_2)P(W|J_2) + P(J_3)P(W|J_3)}$$

$$= \frac{\frac{1}{6} \cdot \frac{1}{2}}{\frac{1}{2} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{2}{3} + \frac{1}{6} \cdot \frac{1}{2}}$$

$$= \frac{\frac{1}{12}}{\frac{1}{6} + \frac{2}{9} + \frac{1}{12}}$$

$$= \frac{3}{17}$$

10. Let Q be the event "person is qualified" and A be the event "person was approved." Set up a tree diagram.



Using Bayes' theorem,

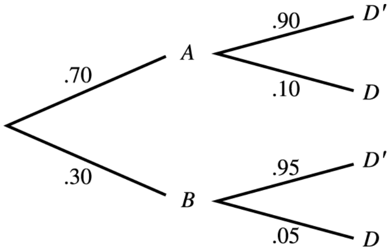
$$P(Q|A) = \frac{P(Q) \cdot P(A|Q)}{P(Q) \cdot P(A|Q) + P(Q') \cdot P(A|Q')}$$

$$= \frac{.70(.85)}{.70(.85) + .30(.20)}$$

$$= \frac{.595}{.655} = \frac{595}{655}$$

$$= \frac{119}{131} \approx .908.$$

12. Let A be the event "the bag came from supplier A," B be the event "the bag came from supplier B," and D be the event "the bag is damaged." Set up a tree diagram.

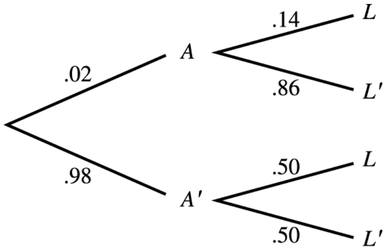


$$P(A|D) = \frac{P(A) \cdot P(D|A)}{P(A) \cdot P(D|A) + P(B) \cdot P(D|B)}$$

$$= \frac{.70(.10)}{.70(.10) + .30(.05)}$$

$$= \frac{.070}{.085} \approx .824$$

14. Let A represent the event "slow pay" and L represent the event "large down payment." Set up a tree diagram.



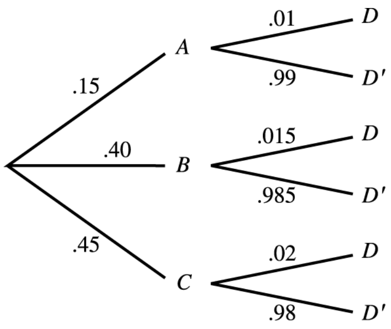
Use Bayes' therorem.

$$P(A|L) = \frac{P(A) \cdot P(L|A)}{P(A) \cdot P(L|A) + P(A') \cdot P(L|A')}$$

$$= \frac{.02(.14)}{.02(.14) + .98(.50)}$$

$$= \frac{.0028}{.4928} = \frac{1}{176} \approx .006$$

16. Let D be the event "a defective appliance" and A be the event "appliance manufactured by company A." Start with a tree diagram, where the first stage refers to the companies and the second to defective appliances.



From Bayes' theorem, we have

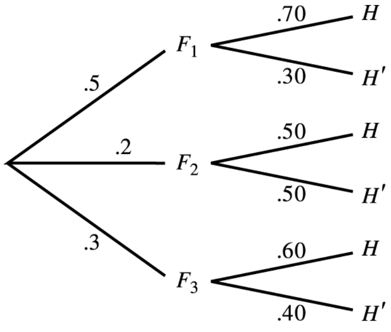
$$P(A|D)$$

$$= \frac{P(A) \cdot P(D|A)}{P(A) \cdot P(D|A) + P(B) \cdot P(D|B) + P(C) \cdot P(D|C)}$$

$$= \frac{.0015}{.0015 + .006 + .009}$$

$$\approx .091.$$

18. Let H be the event "high rating," F_1 be the event "sponsors college game," F_2 be the event "sponsors baseball game," and F_3 be the event "sponsors pro football game." First set up a tree diagram.



Now use Bayes' theorem.

$$P(F_1|H)$$

$$= \frac{P(F_1) \cdot P(H|F_1)}{P(F_1)P(H|F_1)+P(F_2)P(H|F_2)+P(F_3)P(H|F_3)}$$

$$= \frac{.5(.70)}{.5(.70) + .2(.50) + .3(.60)}$$

$$= \frac{.35}{.63} = \frac{5}{9}$$

20. Let L be the event "the object was shipped by land," A be the event "the object was shipped by air," S be the event "the object was shipped by sea," and E be the event "an error occurred."

$$P(L|E)$$

$$= \frac{P(L) \cdot P(E|L)}{P(L) \cdot P(E|L) + P(A) \cdot P(E|A) + P(S) \cdot P(E|S)}$$

$$= \frac{.50(.02)}{.50(.02) + .40(.04) + .10(.14)}$$

$$= \frac{.0100}{.0400} = .25$$

The correct response is (c).

22. Let E represent the event "hemoccult test is positive," and let F represent the event "has colorectal cancer." We are given

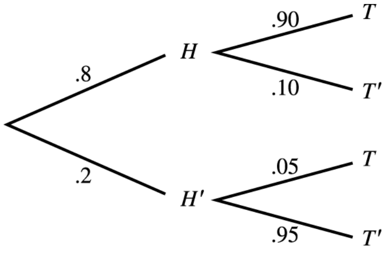
$$P(F) = .003, P(E|F) = .5, \text{ and } P(E|F') = .03$$

and we want to find $P(F|E)$. Since $P(F) = .003$, $P(F') = .997$. Therefore,

$$P(F|E) = \frac{P(F) \cdot P(E|F)}{P(F) \cdot P(E|F) + P(F') \cdot P(E|F')}$$

$$= \frac{.003 \cdot .5}{.003 \cdot .5 + .997 \cdot .03} \approx .0478.$$

24. Let H be the event "has hepatitis" and T be the event "positive test." First set up a tree diagram.



Using Bayes' theorem,

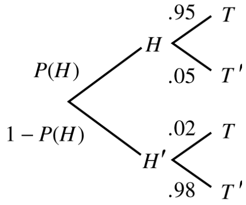
$$P(H|T) = \frac{P(H) \cdot P(T|H)}{P(H) \cdot P(T|H) + P(H') \cdot P(T|H')}$$

$$= \frac{.8(.90)}{.8(.90) + .2(.05)}$$

$$= \frac{.72}{.73} = \frac{72}{73} \approx .986.$$

26. Let H be the event "has HIV" and T be the event "positive test." The problem asks for $P(H|T)$. Using the given values

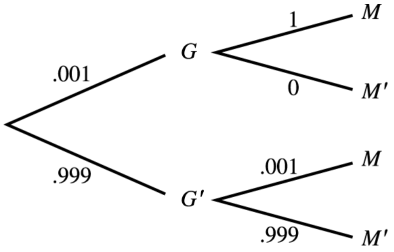
$$P(H) = \frac{950{,}000}{491{,}000{,}000}$$



Using Bayes' Theorem,

$$P(H|T) = \frac{P(H)\cdot P(T|H)}{P(H)\cdot P(T|H) + P(H')\cdot P(T|H')} = \frac{\left(\frac{950{,}000}{491{,}000{,}000}\right)(.95)}{\left(\frac{950{,}000}{491{,}000{,}000}\right)(.95) + \left(1 - \frac{950{,}000}{491{,}000{,}000}\right)(.02)} \approx .0843$$

28. Let M be the event "wife was murdered" and G be the event "husband is guilty." Set up a tree diagram.



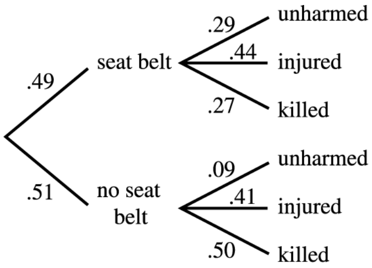
$$P(G|M) = \frac{P(G)\cdot P(M|G)}{P(G)\cdot P(M|G) + P(G')\cdot P(M|G')} = \frac{.001(1)}{.001(1) + .999(.001)} \approx .500$$

30. $P(\text{between 18 and 24}|\text{woman who has been married})$

$$= \frac{.123(1-.775)}{.123(1-.775) + .194(1-.298) + .219(1-.121) + .284(1-.062) + .181(1-.047)}$$

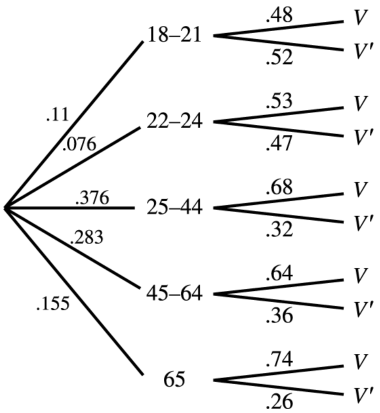
$$= \frac{.027675}{.795249} \approx .035$$

32. Set up a tree diagram.



$$P(\text{killed}|\text{wore seat belt}) = \frac{.49(.27)}{.49(.27) + .51(.50)} \approx .342$$

34. Let V represent the event "person votes" and "65" represent "65 or older." Look at the tree diagram.



$P(\text{18-21}|V)$

$$= \frac{.11(.48)}{.11(.48)+.076(.53)+.376(.68)+.283(.64)+.155(.74)}$$

$$= \frac{.0528}{.64458} \approx .082$$

36. Refer to the tree diagram for Exercise 32.

$P(\text{45-64})|V')$

$$= \frac{.283(.36)}{.11(.52)+.076(.47)+.376(.32)+.283(.36)+.155(.26)}$$

$$= \frac{.10188}{.35542} \approx .287$$

Chapter 7 Review Exercises

2. $4 \notin \{3, 9, 7\}$

Since 4 is not an element of the given set, the statement is true.

4. $0 \in \{0, 1, 2, 3, 4\}$

Since 0 is an element of the given set, the statement is true.

6. $\{1, 2, 5, 8\} \subseteq \{1, 2, 5, 10, 11\}$

Since 8 is an element of the first set but not of the second set, the first set cannot be a subset of the second. The statement is false.

8. $\emptyset \subseteq \{1\}$

The empty set is a subset of every set, so the statement is true.

10. $0 \subseteq \emptyset$

The empty set contains no elements and has no subsets except itself. Therefore, the statement is false.

In Exercises 12-20,

$$U = \{\text{a, b, c, d, e, f, g}\},$$
$$K = \{\text{c, d, f, g}\}, \text{ and}$$
$$R = \{\text{a, c, d, e, g}\}.$$

12. $n(R) = 5$, so R has $2^5 = 32$ subsets.

14. $R' = \{\text{b, f}\}$ since these elements are in U but not in R.

16. $K \cup R = \{\text{a, c, d, e, f, g}\}$ since these elements are in K or R, or both.

18. $(K \cup R)' = \{\text{b}\}$ since this element is in U but not in $K \cup R$. (*See* Exercise 16.)

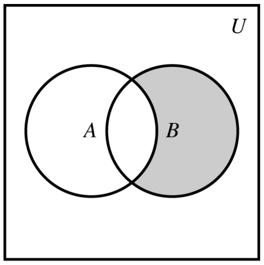
20. $U' = \emptyset$, which is always true.

22. $B \cap D$ is the set of all sales employees who have MBA degrees.

24. $A' \cap D$ is the set of all employees with MBA degrees who are not in the accounting department.

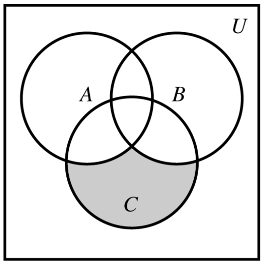
26. $(B \cup C)'$ is the set of all employees who are not either in the sales department or female, that is, all male employees not in the sales department.

28. $A' \cap B$ contains all elements in B and not in A.



$A' \cap B$

30. $(A \cup B)' \cap C$ includes those elements in C and not in either A or B.



$(A \cup B)' \cap C$

32. $S = \{\text{ace}, 2, 3, 4, 5, 6, 7, 8, 9, 10, \text{J}, \text{Q}, \text{K}\}$

34. There are 16 possibilities.

$$\begin{aligned}S = \{&\text{hhhh, hhht, hhth, hthh, thhh,}\\ &\text{hhtt, htht, htth, thht, tthh,}\\ &\text{thth, httt, thtt, ttht, ttth, tttt}\}\end{aligned}$$

36. Let R = red and G = green.

$E = \{(7, R), (7, G), (9, R), (9, G), (11, R), (11, G)\}$

38. The outcomes are not equally likely since there are more red than green balls. For example, $(7, R)$ is twice as likely as $(7, G)$.

40. There are 2 red queens out of 52 cards, so

$$P(\text{red queen}) = \frac{2}{52} = \frac{1}{26}.$$

42. There are 26 black cards plus 6 red face cards, so

$$P(\text{black or a face card}) = \frac{32}{52} = \frac{8}{13}.$$

44. There are 12 face cards, of which 4 are jacks, so

$$P(\text{jack}|\text{face card}) = \frac{4}{12} = \frac{1}{3}.$$

50. Independent events are never mutually exclusive. If A and B are nonempty and independent, then

$$P(A \cap B) = P(A) \cdot P(B).$$

For mutually exclusive events, $P(A \cap B) = 0$, which would mean $P(A) = 0$ or $P(B) = 0$. This is impossible.

52. Let C be the event "a club is drawn." There are 13 clubs in the deck, so $n(C) = 13$, $P(C) = \frac{13}{52} = \frac{1}{4}$, and $P(C') = 1 - P(C) = \frac{3}{4}$. The odds in favor of drawing a club are

$$\frac{P(C)}{P(C')} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3},$$

which is written "1 to 3."

54. Let R be the event "a red face card is drawn" and Q be the event "a queen is drawn." Use the union rule for probability to find $P(R \cup Q)$.

$$\begin{aligned}P(R \cup Q) &= P(R) + P(Q) - P(R \cap Q)\\ &= \frac{6}{52} + \frac{4}{52} - \frac{2}{52}\\ &= \frac{8}{52} = \frac{2}{13}\end{aligned}$$

$$\begin{aligned}P(R \cup Q)' &= 1 - P(R \cup Q)\\ &= 1 - \frac{2}{13} = \frac{11}{13}\end{aligned}$$

The odds in favor of drawing a red face card or a queen are

$$\frac{P(R \cup Q)}{P(R \cup Q)'} = \frac{\frac{2}{13}}{\frac{11}{13}} = \frac{2}{11},$$

which is written "2 to 11."

56. A sum of 0 is impossible, so

$$P(\text{sum is } 0) = 0.$$

58. $P(\text{sum is no more than } 5)$

$$\begin{aligned}&= P(2) + P(3) + P(4) + P(5)\\ &= \frac{1}{36} + \frac{2}{36} + \frac{3}{36} + \frac{4}{36}\\ &= \frac{10}{36} = \frac{5}{18} \approx .278\end{aligned}$$

60. A roll greater than 10 means 11 or 12. There are 3 ways to get 11 or 12, 2 for 11 and 1 for 12. Hence,

$$P(12|\text{sum greater than } 10) = \frac{1}{3}.$$

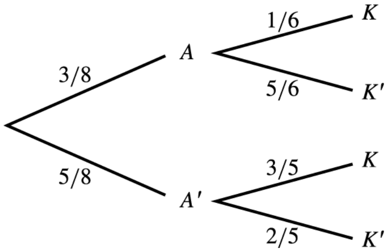
62. For $P(\text{at least } 9|\text{one die is a } 5)$, the sample space is reduced to

$$\begin{aligned}\{&(5, 1), (5, 2), (5, 3), (5, 4), (5, 5),\\ &(5, 6), (1, 5), (2, 5), (3, 5), (4, 5), (6, 5)\};\end{aligned}$$

of these 11 outcomes, 5 give a sum of 9 or more, so

$$P(\text{at least } 9|\text{at least one die is } 5) = \frac{5}{11} \approx .455.$$

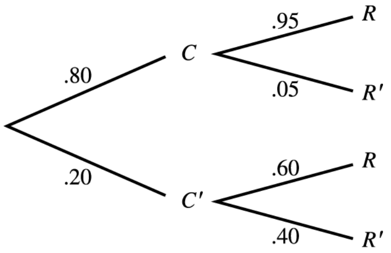
64. First make a tree diagram. Let A represent "box A" and K represent "black ball."



Use Bayes' theorem.

$$\begin{aligned}P(A|K) &= \frac{P(A) \cdot P(K|A)}{P(A) \cdot P(K|A) + P(A') \cdot P(K|A')}\\ &= \frac{\frac{3}{8} \cdot \frac{1}{6}}{\frac{3}{8} \cdot \frac{1}{6} + \frac{5}{8} \cdot \frac{3}{5}}\\ &= \frac{\frac{1}{16}}{\frac{7}{16}} = \frac{1}{7} \approx .143\end{aligned}$$

66. First make a tree diagram letting C represent "a competent shop" and R represent "an appliance is repaired correctly."



To obtain $P(C|R)$, use Bayes' theorem.

$$\begin{aligned}P(C|R) &= \frac{P(C)\cdot P(R|C)}{P(C)\cdot P(R|C)+P(C')\cdot P(R|C')}\\ &= \frac{.80(.95)}{.80(.95)+.20(.60)}\\ &= \frac{.76}{.88}=\frac{19}{22}\approx .864\end{aligned}$$

68. Refer to the tree diagram for Exercise 66. Use Bayes' theorem.

$$\begin{aligned}P(C|R') &= \frac{P(C)\cdot P(R'|C)}{P(C)\cdot P(R'|C)+P(C')\cdot P(R'|C')}\\ &= \frac{.80(.05)}{.80(.05)+.20(.40)}\\ &= \frac{.04}{.12}=\frac{1}{3}\end{aligned}$$

70. To find $P(R)$, use

$$\begin{aligned}P(R) &= P(C)\cdot P(R|C)+P(C')\cdot P(R|C')\\ &= .80(.95)+.20(.60)=.88\end{aligned}$$

72. **(a)** "A customer buys neither machine" may be written $(E\cup F)'$ or $E'\cap F'$.

(b) "A customer buys at least one of the machines" is written $E\cup F$.

74. Use Bayes' theorem to find the required probabilities.

(a) Let D be the event "item is defective" and E_k be the event "item came from supplier k," $k=1,2,3,4$.

$$\begin{aligned}P(D) &= P(E_1)\cdot P(D|E_1)+P(E_2)\cdot P(D|E_2)\\ &\qquad +P(E_3)\cdot P(D|E_3)+P(E_4)\cdot P(D|E_4)\\ &= .17(.04)+.39(.02)+.35(.07)+.09(.03)\\ &= .0418\end{aligned}$$

(b) Find $P(E_4|D)$. Using Bayes' theorem, the numerator is

$$P(E_4)\cdot P(D|E_4)=.09(.03)=.0027.$$

The denominator is $P(E_1)\cdot P(D|E_1)+P(E_2)\cdot P(D|E_2)+P(E_3)\cdot P(D|E_3)+P(E_4)\cdot P(D|E_4)$, which equals

$$.17(.04)+.39(.02)+.35(.07)+.09(.03)=.0418.$$

Therefore,

$$P(E_4|D)=\frac{.0027}{.0418}\approx .065.$$

(c) Find $P(E_2|D)$. Using Bayes' theorem with the same denominator as in part (a),

$$\begin{aligned}P(E_2|D) &= \frac{P(E_2)\cdot P(D|E_2)}{.0418}\\ &= \frac{.39(.02)}{.0418}\\ &= \frac{.0078}{.0418}\\ &\approx .187.\end{aligned}$$

(d) Since $P(D)=.0418$ and $P(D|E_4)=.03$,

$$P(D)\neq P(D|E_4)$$

Therfore, the events are not independent.

76. **(a)**

	N_2	T_2
N_1	N_1N_2	N_1T_2
T_1	T_1N_2	T_1T_2

Since the four combinations are equally likely, each has probability $\frac{1}{4}$.

(b) $P(\text{two trait cells})=P(T_1T_2)=\dfrac{1}{4}$

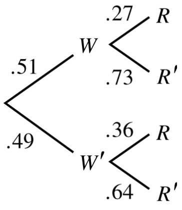
(c) $P(\text{one normal cell and one trait cell})$

$$\begin{aligned}&= P(N_1T_2)+P(T_1N_2)\\ &= \frac{1}{4}+\frac{1}{4}\\ &= \frac{1}{2}\end{aligned}$$

(d) $P(\text{not a carrier and does not have the disease})$

$$\begin{aligned}&= P(N_1N_2)\\ &= \frac{1}{4}\end{aligned}$$

78. Let W be the event "voter was a woman" and R be the event "voter was a Republican." Complete a tree diagram.



To obtain $P(W|R)$, use Bayes' Theorem.

$$\begin{aligned} P(W|R) &= \frac{P(W)\cdot P(R|W)}{P(W)\cdot P(R|W)+P(W')\cdot P(R|W')} \\ &= \frac{.51(.27)}{.51(.27)+.49(.36)} \\ &= \frac{.1377}{.3141} \approx .44 \end{aligned}$$

80. (a) $P(\text{answer yes})$

$$= P(\text{answer } B)\cdot P(\text{answer yes}|\text{answer } B) + P(\text{answer } A)\cdot P(\text{answer yes}|\text{answer } A)$$

Divide by $P(\text{answer } B)$.

$$\frac{P(\text{answer yes})}{P(\text{answer } B)} = P(\text{answer yes}|\text{answer } B) + \frac{P(\text{answer } A)\cdot P(\text{answer yes}|\text{answer } A)}{P(\text{answer } B)}$$

Solve for $P(\text{answer yes}|\text{answer } B)$.

$$P(\text{answer yes}|\text{answer } B) = \frac{P(\text{answer yes})-P(\text{answer } A)\cdot P(\text{answer yes}|\text{ answer } A)}{P(\text{answer } B)}$$

(b) Using the formula from part (a),

$$\frac{.6-\frac{1}{2}\left(\frac{1}{2}\right)}{\frac{1}{2}} = \frac{7}{10}.$$

82. In calculating the probability of two babies in a family would die of SIDS is $(1/8500)^2$, he assumed that the events that either infant died of SIDS are independent. There may be a genetic factor, in which case the events are dependent.

84. (a) $P(\text{making a 1st down with } n \text{ yards to go})$

$$= \frac{\text{number of successes}}{\text{number of trials}}$$

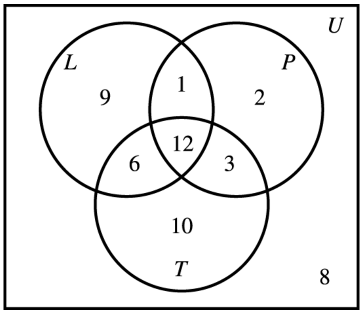
n	Trials	Successes	Probability of Making First Down with n Yards to Go
1	543	388	$\frac{388}{543} \approx .715$
2	327	186	$\frac{186}{327} \approx .569$
3	356	146	$\frac{146}{356} \approx .410$
4	302	97	$\frac{97}{302} \approx .321$
5	336	91	$\frac{91}{336} \approx .271$

86. Let $L =$ the set of songs about love;
$P =$ the set of songs about prison;
and $I =$ the set of songs about trucks.

We are given the following information.

$n(L \cap P \cap T) = 12$
$n(L \cap P) = 13$
$n(L) = 28$
$n(L \cap T) = 18$
$n(P \cap T \cap L') = 3$
$n(P \cap L' \cap T') = 2$
$n(L' \cap P' \cap T') = 8$
$n(T \cap P') = 16$

Start with $L \cap P \cap T$. $n(L \cap P \cap T) = 12$
Complete $L \cap P$ with 1 for a total of 13.
Complete $L \cap T$ with 6 for a total of 18.
Complete $P \cap T$ with 3 since $n(P \cap T \cap L') = 3$.
Complete L with $28 - (1 + 12 + 6) = 9$.
Complete P with 2 since $n(P \cap L' \cap T') = 2$.
Since $n(T \cap P') = 16$, complete T with $16 - 6 = 10$.
Put 8 in $(L' \cap P' \cap T')$.



(a) The total number of songs surveyed is

$$9 + 1 + 12 + 6 + 2 + 3 + 10 + 8 = 51.$$

(b) The number of songs about truck drivers is

$$n(T) = 6 + 12 + 3 + 10 = 31.$$

(c) The number of songs about prisoners is

$$n(P) = 1 + 2 + 3 + 12 = 18.$$

(d) The number of songs about truck drivers in prison is

$$n(T \cap P) = 12 + 3 = 15.$$

(e) The number of songs about people not in prison is

$$n(P') = 9 + 6 + 10 + 8 = 33.$$

(f) The number of songs about people not in love is

$$n(L') = 2 + 3 + 10 + 8 = 23.$$

88. **(a)** $P(\text{double miss}) = .05(.05) = .0025$

(b) $P(\text{specific silo destroyed})$

$$\begin{aligned} &= 1 - P(\text{double miss}) \\ &= 1 - .0025 \\ &= .9975 \end{aligned}$$

(c) $P(\text{all ten destroyed}) = (.9975)^{10} \approx .9753$

(d) $P(\text{at least one survived})$

$$\begin{aligned} &= 1 - P(\text{none survived}) \\ &= 1 - P(\text{all ten destroyed}) \\ &= 1 - .9753 \\ &= .0247 \quad \text{or} \quad 2.47\% \end{aligned}$$

This does not agree with the quote of a 5% chance that at least one would survive.

(e) The events that each of the two bombs hit their targets are assumed to be independent. The events that each silo is destroyed are assumed to be independent.

Extended Application: Medical Diagnosis

1. Using Bayes' theorem,

$$\begin{aligned} &P(H_2|C_1) \\ &= \frac{P(C_1|H_2) \cdot P(H_2)}{P(C_1|H_1)P(H_1) + P(C_1|H_2)P(H_2) + P(C_1|H_3)P(H_3)} \\ &= \frac{.4(.15)}{.9(.8) + .4(.15) + .1(.05)} \\ &= \frac{.06}{.785} \approx .076. \end{aligned}$$

2. Using Bayes' theorem,

$$\begin{aligned} &P(H_1|C_2) \\ &= \frac{P(C_2|H_1) \cdot P(H_1)}{P(C_2|H_1)P(H_1) + P(C_2|H_2)P(H_2) + P(C_2|H_3)P(H_3)} \\ &= \frac{.2(.8)}{.2(.8) + .8(.15) + .3(.05)} \\ &= \frac{.16}{.295} \approx .542. \end{aligned}$$

3. Using Bayes' theorem,

$$\begin{aligned} &P(H_3|C_2) \\ &= \frac{P(C_2|H_3) \cdot P(H_3)}{P(C_2|H_1)P(H_1) + P(C_2|H_2)P(H_2) + P(C_2|H_3)P(H_3)} \\ &= \frac{.3(.05)}{.2(.8) + .8(.15) + .3(.05)} \\ &= \frac{.015}{.295} \approx .051. \end{aligned}$$

Chapter 8

COUNTING PRINCIPLES; FURTHER PROBABILITY TOPICS

8.1 The Multiplication Principle; Permutations

2. $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$

4. $$\begin{aligned} 16! &= 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \\ &\quad \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\ &\approx 2.092 \cdot 10^{13} \end{aligned}$$

6. $$\begin{aligned} P(12,3) &= \frac{12!}{(12-3)!} = \frac{12!}{9!} \\ &= \frac{12 \cdot 11 \cdot 10 \cdot 9!}{9!} \\ &= 1320 \end{aligned}$$

8. $$\begin{aligned} P(33,19) &= \frac{33!}{(33-19)!} = \frac{33!}{14!} \\ &\approx 9.960 \cdot 10^{25} \end{aligned}$$

10. $$P(n,n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!$$

12. $$\begin{aligned} P(n,n-1) &= \frac{n!}{[n-(n-1)]!} = \frac{n!}{(n-n+1)!} \\ &= \frac{n!}{1!} = n! \end{aligned}$$

14. By the multiplication principle, there will be

$$3 \cdot 8 \cdot 5 = 120$$

different meals possible.

16. The number of ways to choose a slate of 3 officers is

$$\begin{aligned} P(15,3) &= \frac{15!}{(15-3)!} = \frac{15!}{12!} \\ &= \frac{15 \cdot 14 \cdot 13 \cdot 12!}{12!} \\ &= 2730. \end{aligned}$$

18. There is exactly one 3-letter subset of the letters A, B, and C, namely A, B, and C.

20. **(a)** initial

This word contains 3 i's, 1 n, 1 t, 1 a, and 1 ℓ. Use the formula for distinguishable permutations with $n = 7$, $n_1 = 3$, $n_2 = 1$, $n_3 = 1$, $n_4 = 1$, and $n_5 = 1$.

$$\begin{aligned} \frac{n!}{n_1!n_2!n_3!n_4!n_5!} &= \frac{7!}{3!1!1!1!1!} \\ &= \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3!}{3!} \\ &= 840 \end{aligned}$$

There are 840 distinguishable permutations of the letters.

(b) little

Use the formula for distinguishable permutations with $n = 6$, $n_1 = 2$, $n_2 = 1$, $n_3 = 2$, and $n_4 = 1$.

$$\frac{6!}{2!1!2!1!} = \frac{6!}{2!2!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 2 \cdot 1} = 180$$

There are 180 distinguishable permutations.

(c) decreed

Use the formula for distinguishable permutations with $n = 7$, $n_1 = 2$, $n_2 = 3$, $n_3 = 1$, and $n_4 = 1$.

$$\frac{7!}{2!3!1!1!} = \frac{7!}{2!3!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3!}{2 \cdot 1 \cdot 3!} = 420$$

There are 420 distinguishable permutations.

22. **(a)** The 9 books can be arranged in

$$P(9,9) = 9! = 362{,}880 \text{ ways.}$$

(b) The blue books can be arranged in 4! ways, the green books can be arranged in 3! ways, and the red books can be arranged in 2! ways. There are 3! ways to choose the order of the 3 groups of books. Therefore, using the multiplication principle, the number of possible arrangements is

$$4!3!2!3! = 24 \cdot 6 \cdot 2 \cdot 6 = 1728.$$

(c) Use the formula for distinguishable permutations with $n = 9$, $n_1 = 4$, $n_2 = 3$, and $n_3 = 2$. The number of distinguishable arrangements is

$$\frac{9!}{4!3!2!} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4!}{4! \cdot 6 \cdot 2} = 1260.$$

(d) There are 4 choices for the blue book, 3 for the green book, and 2 for the red book. The total number of arrangements is

$$4 \cdot 3 \cdot 2 = 24.$$

24. $P(4,4) = \frac{4!}{(4-4)!} = \frac{4!}{0!}$

If $0! = 0$, then $P(4,4)$ would be undefined.

26. **(a)** The number 13! has 2 factors of five so there must be 2 ending zeros in the answer.

(b) The number 27! has 6 factors of five (one each in 5, 10, 15, and 20 and two factors in 25), so there must be 6 ending zeros in the answer.

(c) The number 75! has $15 + 3 = 18$ factors of five (one each in 5, 10,…, 75 and two factors each in 25, 50, and 75), so there must be 18 ending zeros in the answer.

28. Use the multiplication principle. There are

$$7 \cdot 6 \cdot 4 \cdot 5 = 840$$

varieties of automobile available.

30. If each species were to be assigned 3 initials, since there are 26 different letters in the alphabet, there could be $26^3 = 17{,}576$ different 3-letter designations. This would not be enough. If 4 initials were used, the biologist could represent $26^4 = 456{,}976$ different species, which is more than enough. Therefore, the biologist should use at least 4 initials.

32. The number of ways to seat the people is

$$\begin{aligned} P(6,6) &= \frac{6!}{0!} = \frac{6!}{1} \\ &= 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\ &= 720. \end{aligned}$$

34. The number of ways to arrange a schedule of 3 classes is

$$\begin{aligned} P(6,3) &= \frac{6!}{(6-3)!} = \frac{6!}{3!} \\ &= \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3!} \\ &= 120. \end{aligned}$$

36. The number of possible batting orders is

$$\begin{aligned} P(20,9) = \frac{20!}{(20-9)!} &= \frac{20!}{11!} \\ &= 60{,}949{,}324{,}800 \\ &\approx 6.09493 \cdot 10^{10}. \end{aligned}$$

38. **(a)** The number of ways 5 works can be arranged is

$$P(5,5) = 5! = 120.$$

(b) If one of the 2 overtures must be chosen first, followed by arrangements of the 4 remaining pieces, then

$$P(2,1) \cdot P(4,4) = 2 \cdot 24 = 48$$

is the number of ways the program can be arranged.

40. By the multiplication principle, a person could schedule the evening of television viewing in

$$8 \cdot 5 \cdot 6 = 240$$

different ways.

42. **(a)** There are 5 odd digits: 1, 3, 5, 7, and 9. There are 7 decisions to be made, one for each digit; there are 5 choices for each digit. Thus, $5^7 = 78{,}125$ phone numbers are possible.

(b) The first digit has 9 possibilities, since 0 is not allowed; the middle 5 digits each have 10 choices; the last digit must be 0. Thus, there are

$$9 \cdot 10^5 \cdot 1 = 900{,}000$$

possible phone numbers.

(c) Solve as in part (b), except that the last *two* digits must be 0; therefore there are

$$9 \cdot 10^4 \cdot 1 \cdot 1 = 90{,}000$$

possible phone numbers.

(d) There are no choices for the first three digits; thus,

$$1^3 \cdot 10^4 = 10{,}000$$

phone numbers are possible.

(e) The first digit cannot be 0; in the absence of repetitions there are 9 choices for the second digit, and the choices decrease by one for each subsequent digit. The result is

$$9 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 544{,}320$$

phone numbers.

44. There are 8 choices for the first digit, since it cannot be 0 or 1. Since restrictions are eliminated for the second digit, there are 10 possibilities for each of the second and third digits. Thus, the total number of area codes would be

$$8 \cdot 10 \cdot 10 = 800.$$

46. Since a social security number has 9 digits with no restrictions, there are

$$10^9 = 1,000,000,000 \text{ (1 billion)}$$

different social security numbers. Yes, this is enough for every one of the 281 million people in the United States to have a social security number.

48. Since a zip code has nine digits with no restrictions, there are

$$10^9 = 1{,}000{,}000{,}000$$

different 9-digit zip codes.

50. Since a 20-sided die is rolled 12 times, the number of possible games is

$$20^{12} \quad \text{or} \quad 4.096 \cdot 10^{15} \text{ games.}$$

52. **(a)** The number of different circuits is $P(9, 9)$ since we do not count the city he is starting in.

$$P(9, 9) = 9! = 362, 880$$

is the number of different circuits.

(b) He must check half of the circuits since, for each circuit, there is a corresponding one in the reverse order. Therefore,

$$\frac{1}{2}(362, 880) = 181, 440$$

circuits should be checked.

(c) No, it would not be feasible.

8.2 Combinations

2. To evaluate $\binom{8}{3}$, use the formula

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

with $n = 8$ and $r = 3$.

$$\begin{aligned}\binom{8}{3} &= \frac{8!}{(8-3)!3!} \\ &= \frac{8!}{5!3!} \\ &= \frac{8 \cdot 7 \cdot 6 \cdot 5!}{5! \cdot 3 \cdot 2 \cdot 1} = 56\end{aligned}$$

4. To evaluate $\binom{44}{20}$, use the formula

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

with $n = 44$ and $r = 20$.

$$\begin{aligned}\binom{44}{20} &= \frac{44!}{(44-20)!20!} \\ &= \frac{44!}{24!20!} \\ &= 1.761 \cdot 10^{12}\end{aligned}$$

6.
$$\begin{aligned}\binom{n}{0} &= \frac{n!}{(n-0)!0!} \\ &= \frac{n!}{n! \cdot 1} \\ &= 1\end{aligned}$$

8.
$$\begin{aligned}\binom{n}{1} &= \frac{n!}{(n-1)!1!} \\ &= \frac{n(n-1)!}{(n-1)! \cdot 1} \\ &= n\end{aligned}$$

10. There are 13 clubs, from which 6 are to be chosen. The number of ways in which a hand of 6 clubs can be chosen is

$$\binom{13}{6} = \frac{13!}{7!6!} = 1716.$$

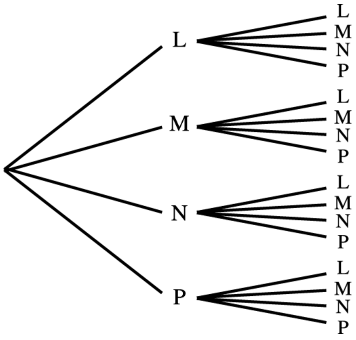
12. **(a)** The number of ways to select a committee of 4 from a club with 30 members is

$$\binom{30}{4} = 27, 405.$$

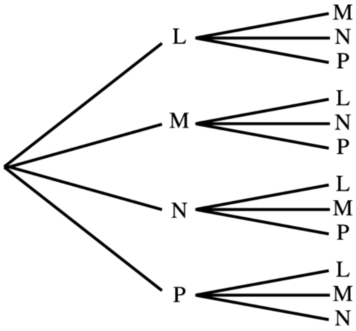
(b) If the committee must have at least 1 member and at most 3 members, it must have 1, 2, or 3 members. The number of committees is

$$\begin{aligned}\binom{30}{1} + \binom{30}{2} + \binom{30}{3} &= 30 + 435 + 4060 \\ &= 4525.\end{aligned}$$

14. **(a)** With repetition permitted, the tree diagram shows 16 different pairs.



(b) If repetition is not permitted, one branch is missing from each of the clusters of second branches, for a total of 12 different pairs.



(c) Find the number of combinations of 4 elements taken 2 at a time.

$$\binom{4}{2} = 6$$

No repetitions are allowed, so the answer cannot equal that for part (a). However, since order does not matter, our answer is only half of the answer for part (b). For example, LM and ML are distinct in (b) but not in (c). Thus, the answer differs from both (a) and (b).

18. Since order is not important, the answers are combinations.

(a) If there are at least 4 women, there will be either 4 women and 1 man or 5 women and no men. The number of such committees is

$$\binom{11}{4}\binom{8}{1} + \binom{11}{5}\binom{8}{0} = 2640 + 462$$
$$= 3102.$$

(b) If there are no more than 2 men, there will be either no men and 5 women, 1 man and 4 women, or 2 men and 3 women. The number of such committees is

$$\binom{8}{0}\binom{11}{5} + \binom{8}{1}\binom{11}{4} + \binom{8}{2}\binom{11}{3}$$
$$= 462 + 2640 + 4620$$
$$= 7722.$$

20. Order does not matter, so use combinations.

(a) The 3 students who will take part in the course can be chosen in

$$\binom{12}{3} = \frac{12!}{9!3!} = \frac{12 \cdot 11 \cdot 10 \cdot 9!}{9! \cdot 3 \cdot 2 \cdot 1} = 220 \text{ ways.}$$

(b) The 9 students who will not take part in the course can be chosen in

$$\binom{12}{9} = \frac{12!}{3!9!} = 220 \text{ ways.}$$

22. Since order does not matter, use combinations.

(a) There are

$$\binom{25}{3} = 2300$$

possible samples of 3 apples.

(b) There are

$$\binom{5}{3} = 10$$

possible samples of 3 rotten apples.

(c) There are

$$\binom{5}{1}\binom{20}{2} = 950$$

possible samples with exactly 1 rotten apple.

24. Since order is important, use a permutation. The plants can be arranged in

$$P(9,5) = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 15,120$$

different ways.

26. Use combinations since order does not matter.

(a) First consider how many pairs of circles there are. This number is

$$\binom{6}{2} = \frac{6!}{2!4!} = 15.$$

Each pair intersects in two points. The total number of intersection points is $2 \cdot 15 = 30$.

(b) The number of pairs of circles is

$$\binom{n}{2} = \frac{n!}{(n-2)!2!} = \frac{n(n-1)(n-2)!}{(n-2)!\cdot 2} = \frac{1}{2}n(n-1).$$

Each pair intersects in two points. The total number of points is

$$2\cdot\frac{1}{2}n(n-1) = n(n-1).$$

28. Since order is important, use permutations. (Each secretary is being assigned to a manager, which is essentially the same as putting them in numbered slots.) The secretaries can be selected in

$$P(7,3) = 7\cdot 6\cdot 5 = 210$$

different ways.

30. Since order is not important, use combinations.

(a) Since 2 workers are to be chosen from a group of 7, the number of possible delegations is

$$\binom{7}{2} = 21.$$

(b) Since a particular worker must be in the delegation, the first person can only be chosen in 1 way. The second person must be selected from the 6 workers who are not the worker who must be included. The number of different delegations is

$$1\cdot\binom{6}{1} = 6.$$

(c) We must count those delegations with exactly 1 woman (1 woman and 1 man) and those with 2 women. The number of delegations including at least 1 woman is

$$\binom{2}{1}\binom{5}{1} + \binom{2}{2} = 2\cdot 5 + 1 = 11.$$

32. Since order is not important, use combinations.

$$\binom{50}{5} = \frac{50!}{45!5!} = 2,118,760$$

34. Since the plants are selected at random, that is, order does not matter, the answers are combinations.

(a) She is selecting 4 plants out of 11 plants. The number of ways in which this can be done is

$$\binom{11}{4} = 330.$$

(b) She is selecting 2 of the 6 wheat plants and 2 of the 5 other plants. The number of ways in which this can be done is

$$\binom{6}{2}\binom{5}{2} = 150.$$

36. Since order is important, use permutations.

$$P(10,4) = \frac{10!}{(10-4)!} = \frac{10!}{6!} = 5040$$

different committees are possible.

38. Since order does not matter, use combinations.

$$\binom{52}{13} = \frac{52!}{(52-13)!13!} = \frac{52!}{39!13!} = 635,013,559,600$$

40. Since order does not matter, use combinations.

2 good hitters: $\binom{5}{2}\binom{4}{1} = 10\cdot 4 = 40$

3 good hitters: $\binom{5}{3}\binom{4}{0} = 10\cdot 1 = 10$

The total number of ways is 40 + 10 = 50.

42. Since order does not matter, use combinations.

(a) There are

$$\binom{20}{5} = 15,504$$

different ways to select 5 of the orchids.

(b) If 2 special orchids must be included in the show, that leaves 18 orchids from which the other 3 orchids for the show must be chosen. This can be done in

$$\binom{18}{3} = 816$$

different ways.

44. In the lottery, 6 different numbers are to be chosen from the 99 numbers.

(a) There are

$$\binom{99}{6} = \frac{99!}{93!6!} = 1,120,529,256$$

different ways to choose 6 numbers if order is not important.

(b) There are

$$P(99,6) = \frac{99!}{93!} = 806,781,064,320$$

different ways to choose 6 numbers if order matters.

46. **(a)** There can be 5, 4, 3, 2, 1, or no toppings. The total number of possibilities for the first pizza is

$$\binom{11}{5} + \binom{11}{4} + \binom{11}{3} + \binom{11}{2} + \binom{11}{1} + \binom{11}{0}$$
$$= 462 + 330 + 165 + 55 + 11 + 1$$
$$= 1024.$$

The total number of possibilities for the toppings on two pizzas is

$$1024 \cdot 1024 = 1,048,576.$$

(b) In part (a), we found that if the order of the two pizzas matters, there are

$$1024^2 = 1,048,576$$

possibilities. If we had a list of all of these possibilities and if the order of the pizzas doesn't matter, we must eliminate all of the possibilities that involve the same two pizzas. There are 1024 such items on the list, one of each of the possibilities for one pizza. Therefore, the number of items on the list that have a duplicate is

$$1{,}048{,}576 - 1024 = 1{,}047{,}552.$$

To eliminate duplicates, we eliminate the second listing of each of these, that is,

$$\frac{1{,}047{,}552}{2} = 523{,}776.$$

Subtracting this from the number of possibilities on the list, we see that if the order of the two pizzas doesn't matter, the number of possibilities is

$$1{,}048{,}576 - 523{,}776 = 524{,}800.$$

48. Consider the first conference with five teams in each of the three divisions. The three winners from the three divisions can result in $5 \cdot 5 \cdot 5$, or 5^3, ways. Then, of the remaining $15 - 3 = 12$ teams, the three wild card teams can be chosen in $12 \cdot 11 \cdot 10$ ways. And since the order of the teams is not relevant, there are $\frac{5^3 \cdot 12 \cdot 11 \cdot 10}{6}$ ways to choose the teams from the first conference.
In the second conference, the situation is the same with the single exception being that one division has six, not five, teams. Therefore, there are $6 \cdot 5 \cdot 5 = 6 \cdot 5^2$ ways to choose the three division winners and $13 \cdot 12 \cdot 11$ ways to choose the wild card teams. Therefore, there are $\frac{6 \cdot 5^2 \cdot 13 \cdot 12 \cdot 11}{6}$ ways to choose the six teams from the second conference.
Finally, the number of ways the six teams can be selected from the first conference and the six teams can be selected from the second conference is

$$\frac{5^3 \cdot 12 \cdot 11 \cdot 10}{6} \cdot \frac{6 \cdot 5^2 \cdot 13 \cdot 12 \cdot 11}{6} = 1{,}179{,}750{,}000.$$

8.3 Probability Applications of Counting Principles

2. There are $\binom{10}{3} = 120$ ways to select 3 of the 10 apples, while there are $\binom{4}{3} = 4$ ways to select 3 yellow ones. Hence,

$$P(3 \text{ yellow}) = \frac{\binom{4}{3}}{\binom{10}{3}} = \frac{4}{120} = \frac{1}{30}.$$

4. "More red than yellow" means 2 or 3 red. There are $\binom{6}{2}$ ways to choose 2 red apples and $\binom{4}{1}$ ways to pick a yellow; hence, there are $\binom{6}{2}\binom{4}{1} = 60$ ways to choose 2 red. Since there are $\binom{6}{3} = 20$ ways to pick 3 red, we have $60 + 20 = 80$ ways to have more red than yellow. Therefore,

$$P(\text{more red}) = \frac{80}{\binom{10}{3}} = \frac{80}{120} = \frac{2}{3}.$$

6. There are $\binom{4}{2} = 6$ ways to pick 2 aces out of $\binom{52}{2}$ ways to pick 2 cards; hence,

$$P(2 \text{ aces}) = \frac{\binom{4}{2}}{\binom{52}{2}} = \frac{6}{1326} = \frac{1}{221} \approx .0045.$$

8. There are $\binom{13}{2} = 78$ ways to pick 2 spades; hence,

$$P(2 \text{ spades}) = \frac{78}{1326} = \frac{1}{17} \approx .059.$$

10. There are $\binom{12}{2} = 66$ ways to pick 2 face cards; hence,

$$P(2 \text{ face cards}) = \frac{66}{1326} = \frac{11}{221} \approx .0498.$$

12. Ace, 2, 3, 4, 5, 6, 7, and 8 are the cards in each suit that are "not higher than 8," for a total of 32, so

$$P(\text{no card higher than } 8)$$
$$= \frac{\binom{32}{2}}{\binom{52}{2}} = \frac{496}{1326} = \frac{248}{663} \approx .374.$$

14. Only the first letter is specified; the other 4 can be any letter. The probability of starting with the letter p is

$$\frac{1}{26} \approx .038.$$

16. There are 26^5 possible 5-letter "words," and 23^5 words" that do not contain x, y, or z. Hence,

$$P(\text{no x, y, or z})$$
$$= \frac{23^5}{26^5} = \frac{6,436,343}{11,881,376} \approx .5417.$$

20. Using the result from Example 6, the probability that at least 2 people in a group of n people have the same birthday is

$$1 - \frac{P(365, n)}{(365)^n}.$$

Therefore, the probability that at least 2 of the 100 U.S. Senators have the same birthday is

$$1 - \frac{P(365, 100)}{(365)^{100}}.$$

22. There are $\binom{n}{2}$ ways to pick which pair is to have the same birthday. One member of the pair has 365 choices of a birthday, the other only 1. The other $n-2$ people have 364, 363, 362, etc., choices. Thus, the probability is

$$\binom{n}{2} \cdot \frac{P(365, n-1)}{(365)^n}.$$

24. Let $x =$ the total number of balls. Since the probability of picking 5 balls which all are blue is $\frac{1}{2}$, we can see that $x > 5$. (If $x = 5$, the probability would be 1.) Let's look at the number of blue balls needed. If there were 6 blue balls, $\binom{6}{5} = 6$ and $\binom{x}{5} = 12$, since the probability is $\frac{1}{2}$. Since x must be larger than the number of blue balls, $x \geq 7$. But since $\binom{7}{5} = 21$,

$$\binom{x}{5} \geq 21 \neq 12.$$

If there were 7 blue balls, $\binom{7}{5} = 21$ and $\binom{x}{5} = 42$. Since $x \geq 8$,

$$\binom{x}{5} \geq 56 \neq 42.$$

If there were 8 blue balls, $\binom{8}{5} = 56$ and $\binom{x}{5} = 112$. Since $x \geq 9$,

$$\binom{x}{5} \geq 126 \neq 112.$$

If there were 9 blue balls, $\binom{9}{5} = 126$ and $\binom{x}{5} = 252$. Since $x \geq 10$, $\binom{x}{5} \geq 252$, and x must be 10.

Therefore, there were 10 balls, 9 of them blue,

$$P(\text{all 5 blue}) = \frac{\binom{9}{5}}{\binom{10}{5}} = \frac{1}{2}.$$

26. $P(\text{matched pair})$

$$= P(2 \text{ black or } 2 \text{ brown or } 2 \text{ blue})$$
$$= P(2 \text{ black}) + P(2 \text{ brown}) + P(2 \text{ blue})$$
$$= \frac{\binom{9}{2}}{\binom{17}{2}} + \frac{\binom{6}{2}}{\binom{17}{2}} + \frac{\binom{2}{2}}{\binom{17}{2}}$$
$$= \frac{36}{136} + \frac{15}{136} + \frac{1}{136}$$
$$= \frac{52}{136}$$
$$= \frac{13}{34}$$

28. There are 6 letters so the number of possible spellings (counting duplicates) is $6! = 720$. Since the letter l is repeated 2 times and the letter t is repeated 2 times, the spelling little will occur $2!2! = 4$ times. The probability that little will be spelled is $\frac{4}{720} = \frac{1}{180}$.

30. There are 9 ways to choose 1 typewriter from the shipment of 9. Since 2 of the 9 are defective, there are 7 ways to choose 1 nondefective typewriter. Thus,

$$P(\text{1 drawn from the 9 is not defective}) = \frac{7}{9}.$$

32. There are $\binom{9}{3}$ ways to choose 3 typewriters.

$$\binom{9}{3} = \frac{9!}{3!6!} = \frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = 84$$

There are $\binom{7}{3}$ ways to choose 3 nondefective typewriters.

$$\binom{7}{3} = \frac{7!}{3!4!} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$$

Thus,

$$P(\text{3 drawn from the 9 are nondefective})$$
$$= \frac{35}{84} = \frac{5}{12}.$$

34. There are $\binom{12}{4} = 495$ different ways to choose 4 engines for testing from the crate of 12. A crate will not be shipped if any one of the 4 in the sample is defective. If there are 2 defectives in the crate, then there are $\binom{10}{4} = 210$ ways of choosing a sample with no defectives. Thus,

$$P(\text{shipping a crate with 2 defectives})$$
$$= \frac{210}{495} \approx .424.$$

36. There are $P(5,5)$ different orders of the names. Only one of these would be in alphabetical order. Therefore, $P(5,5) - 1$ are not in alphabetical order. Thus,

$$P(\text{not in alphabetical order})$$
$$= \frac{P(5,5) - 1}{P(5,5)} = \frac{120 - 1}{120} = \frac{119}{120}.$$

38. **(a)** $P(\text{first person}) = \frac{5}{40} = \frac{1}{8}$

(b) $P(\text{last person}) = \frac{5(39!)}{40!}$

$$= \frac{5(39!)}{40(39!)}$$
$$= \frac{5}{40} = \frac{1}{8}$$

(c) No, everybody has the same chance.

40. A flush could start with an ace, 2, 3, 4, …, 7, 8, or 9. This gives 9 choices in each of 4 suits, so there are 36 choices in all. Thus,

$$P(\text{straight flush}) = \frac{36}{\binom{52}{5}} = \frac{36}{2,598,960}$$
$$\approx .00001385$$
$$= 1.385 \cdot 10^{-5}.$$

42. A straight could start with an ace, 2, 3, 4, 5, 6, 7, 8, 9, or 10 as the low card, giving 40 choices. For each succeeding card, only the suit may be chosen. Thus, the number of straights is

$$40 \cdot 4^4 = 10,240.$$

But this also counts the straight flushes, of which there are 36 (see Exercise 37), and the 4 royal flushes. There are thus 10,200 straights that are not also flushes, so

$$P(\text{straight}) = \frac{10,200}{2,598,960} \approx .00392.$$

44. There are 13 different values of cards and 4 cards of each value. Choose 2 values out of the 13 for the values of the pairs. The number of ways to select the 2 values is $\binom{13}{2}$. The number of ways to select a pair for each value is $\binom{4}{2}$. There are $52 - 8 = 44$ cards that are neither of these 2 values, so the number of ways to select the fifth card is $\binom{44}{1}$. Thus,

$$P(\text{two pairs}) = \frac{\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{44}{1}}{\binom{52}{5}}$$
$$= \frac{123,552}{2,598,960} \approx .04754.$$

46. There are $\binom{52}{13}$ different 13-card bridge hands. Since there are only 13 hearts, there is exactly one way to get a bridge hand containing only hearts. Thus,

$$P(\text{only hearts}) = \frac{1}{\binom{52}{13}} \approx 1.575 \cdot 10^{-12}.$$

48. There are $\binom{4}{3}$ ways to obtain 3 aces, $\binom{4}{3}$ ways to obtain 3 kings, and $\binom{44}{7}$ ways to obtain the 7 remaining cards. Thus,

$$P(\text{exactly 3 aces and exactly 3 kings})$$
$$= \frac{\binom{4}{3}\binom{4}{3}\binom{44}{7}}{\binom{52}{13}} \approx 9.655 \cdot 10^{-4}.$$

50. There are 21 books, so the number of selections of any 6 books is

$$\binom{21}{6} = 54,264.$$

(a) The probability that the selection consisted of 3 Hughes and 3 Morrison books is

$$\frac{\binom{9}{3}\binom{7}{3}}{\binom{21}{6}} = \frac{85 \cdot 35}{54,264} = \frac{2940}{54,264} \approx .054.$$

(b) A selection containing exactly 4 Baldwin books will contain 2 of the 16 books by the other authors, so the probability is

$$\frac{\binom{5}{4}\binom{16}{2}}{\binom{21}{6}} = \frac{5 \cdot 120}{54,264} = \frac{600}{54,264} \approx .011.$$

(c) The probability of a selection consisting of 2 Hughes, 3 Baldwin, and 1 Morrison book is

$$\frac{\binom{9}{2}\binom{5}{3}\binom{7}{1}}{\binom{21}{6}} = \frac{36 \cdot 10 \cdot 7}{54,264} = \frac{2520}{54,264} \approx .046.$$

(d) A selection consisting of at least 4 Hughes books may contain 4, 5, or 6 Hughes books, with any remaining books by the other authors. Therefore, the probability is

$$\frac{\binom{9}{4}\binom{12}{2} + \binom{9}{5}\binom{12}{1} + \binom{9}{6}\binom{12}{0}}{\binom{21}{6}}$$

$$= \frac{126 \cdot 66 + 126 \cdot 12 + 84}{54,264}$$

$$= \frac{8316 + 1512 + 84}{54,264}$$

$$= \frac{9912}{54,264} \approx .183.$$

(e) Since there are 9 Hughes books and 5 Baldwin books, there are 14 books written by males. The probability of a selection with exactly 4 books written by males is

$$\frac{\binom{14}{4}\binom{7}{2}}{\binom{21}{6}} = \frac{1001 \cdot 21}{54,264} = \frac{21,021}{54,264} \approx .387.$$

(f) A selection with no more than 2 books written by Baldwin may contain 0, 1, or 2 books by Baldwin, with the remaining books by the other authors. Therefore, the probability is

$$\frac{\binom{5}{0}\binom{16}{6} + \binom{5}{1}\binom{16}{5} + \binom{5}{2}\binom{16}{4}}{\binom{21}{6}}$$

$$= \frac{8008 + 5 \cdot 4368 + 10 \cdot 1820}{54,264}$$

$$= \frac{8008 + 21,840 + 18,200}{54,264}$$

$$= \frac{48,048}{54,264} \approx .885.$$

52. To find the probability of picking 5 of the 6 lottery numbers correctly, we must recall that the total number of ways to pick the 6 lottery numbers is $\binom{99}{6} = 1,120,529,256$. To pick 5 of the 6 winning numbers, we must also pick 1 of the 93 losing numbers. Therefore, the number of ways of picking 5 of the 6 winning numbers is

$$\binom{6}{5}\binom{93}{1} = 558.$$

Thus, the probability of picking 5 of the 6 numbers correctly is

$$\frac{\binom{6}{5}\binom{93}{1}}{\binom{99}{6}} \approx 5.0 \cdot 10^{-7}.$$

54. P(saying "Math class is tough.")

$$= \frac{\binom{1}{1}\binom{269}{3}}{\binom{270}{4}} \approx .0148$$

No, it is not correct. The correct figure is 1.48%.

56. **(a)** There are only 4 ways to win in just 4 calls: the 2 diagonals, the center column, and the center row. There are $\binom{75}{4}$ combinations of 4 numbers that can occur. The probability that a person will win bingo after just 4 numbers are called is $\frac{4}{\binom{75}{4}} \approx 3.29 \times 10^{-6}$.

(b) There is only 1 way to get an L. It can occur in as few as 9 calls. There are $\binom{75}{9}$ combinations of 9 numbers that can occur in 9 calls is $\frac{1}{\binom{75}{9}} \approx 7.96 \times 10^{-12}$.

(c) There is only 1 way to get an X-out. It can ocur in as few as 8 calls. There are $\binom{75}{8}$ combinations of 8 numbers that can occur. The probability that an X-out occurs in 8 calls is $\frac{1}{\binom{75}{8}} \approx 5.93 \times 10^{-11}$.

(d) Four columns contain a permutation of 15 numbers taken 5 at a time. One column contains a permutation of 15 numbers taken 4 at a time. The number of distinct cards is $P(15,5)^4 \cdot P(15,4) \approx 5.52 \times 10^{26}$.

8.4 Binomial Probability

2. This is a Bernoulli trial problem with $P(\text{success}) = P(\text{girl}) = \frac{1}{2}$. The probability of exactly x successes in n trials is

$$\binom{n}{x} p^x (1-p)^{n-x},$$

where p is the probability of success in a single trial. Here $n = 5,\ x = 3$, and $p = \frac{1}{2}$.

$$P(\text{exactly 3 girls and 2 boys})$$
$$= \binom{5}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 = 10\left(\frac{1}{32}\right) = \frac{5}{16} \approx .313$$

4. We have $n = 5,\ x = 0$ is the number of boys, and $p = \frac{1}{2}$ is the probability of having a boy.

$$P(\text{no boys}) = \binom{5}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^5 = \frac{1}{32} \approx .031$$

6. We have 3, 4, or 5 boys, so

$$P(\text{at least 3 boys})$$
$$= \binom{5}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 + \binom{5}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^1$$
$$+ \binom{5}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^0$$
$$= \frac{10}{32} + \frac{5}{32} + \frac{1}{32} = \frac{16}{32} = \frac{1}{2}.$$

8. $P(\text{no more than 4 girls})$
$$= 1 - P(\text{5 girls})$$
$$= 1 - \binom{5}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^0$$
$$= 1 - \frac{1}{32} = \frac{31}{32} \approx .969$$

10. We have $n = 12,\ x = 6$, and $p = \frac{1}{6}$, so

$$P(\text{exactly 6 ones}) = \binom{12}{6}\left(\frac{1}{6}\right)^6\left(\frac{5}{6}\right)^6 \approx .0066.$$

12. We have $n = 12,\ x = 2$, and $p = \frac{1}{6}$, so

$$P(\text{exactly 2 ones}) = \binom{12}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{10} \approx .296.$$

14. "No more than 1 one" means 0 one or 1 one. Thus,

$$P(\text{no more than 1 one})$$
$$= P(\text{0 one}) + P(\text{1 one})$$
$$= \binom{12}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^{12} + \binom{12}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^{11}$$
$$\approx .381.$$

16. We have $n = 6,\ x = 3$, and $p = \frac{1}{2}$, so

$$P(\text{exactly 3 heads}) = \binom{6}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^3 = \frac{20}{64} = \frac{5}{16} \approx .313.$$

18. $P(\text{at least 3 heads})$
$$= P(\text{3 heads}) + P(\text{4 heads}) + P(\text{5 heads}) + P(\text{6 heads})$$
$$= \binom{6}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^3 + \binom{6}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^2$$
$$+ \binom{6}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^1 + \binom{6}{6}\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^0$$
$$= \frac{20}{64} + \frac{15}{64} + \frac{6}{64} + \frac{1}{64}$$
$$= \frac{42}{64} = \frac{21}{32} \approx .656$$

22. Since these two crib deaths cannot be assumed to be independent events, the use of binomial probabilities is not applicable and thus the probabilities that are computed are not correct.

24. We define a success to be the event that a customer overpays. In this situation, $n = 15,\ x = 3,\ p = \frac{1}{10}$, and $1 - p = \frac{9}{10}$.

$$P(\text{customer overpays on 3 items})$$
$$= \binom{15}{3}\left(\frac{1}{10}\right)^3\left(\frac{9}{10}\right)^{12} \approx .1285$$

26. As in Exercise 24, we define a success to be the event that a customer overpays. In this situation, $n = 15$; $x = 1, 2, 3, \ldots, 15$; $p = \frac{1}{10}$; and $1 - p = \frac{9}{10}$.

$$\begin{aligned}&P(\text{customer overpays on at least one item})\\&\quad = 1 - P(\text{customer overpays on no items})\\&\quad = 1 - \binom{15}{0}\left(\frac{1}{10}\right)^0\left(\frac{9}{10}\right)^{15}\\&\quad \approx .7941\end{aligned}$$

28. Again, we define a success to be the event that a customer overpays. In this situation, $n = 15$; $x = 0$, 1, or 2; $p = \frac{1}{10}$; and $1 - p = \frac{9}{10}$.

$$\begin{aligned}&P(\text{customer overpays on at most 2 items})\\&\quad = \binom{15}{0}\left(\frac{1}{10}\right)^0\left(\frac{9}{10}\right)^{15} + \binom{15}{1}\left(\frac{1}{10}\right)^1\left(\frac{9}{10}\right)^{14}\\&\qquad + \binom{15}{2}\left(\frac{1}{10}\right)^2\left(\frac{9}{10}\right)^{13}\\&\quad \approx .8159\end{aligned}$$

30. $n = 10,\ p = .9, x = 9$

$$P(\text{exactly } 9) = \binom{10}{9}(.9)^9(.1)^1 \approx .387$$

32. $n = 10,\ p = .9, x = 8, 9, \text{ or } 10$

$$\begin{aligned}&P(\text{less than } 8)\\&\quad = 1 - P(8,\ 9,\ \text{or } 10)\\&\quad = 1 - \left[\binom{10}{8}(.9)^8(.1)^2 + \binom{10}{9}(.9)^9(.1)^1\right.\\&\qquad \left. + \binom{10}{10}(.9)^{10}(.1)^0\right]\\&\quad \approx 1 - .930\\&\quad = .070\end{aligned}$$

34. $n = 6,\ p = \frac{1}{5},\ x = 0$

$$\begin{aligned}P(0 \text{ correct}) &= \binom{6}{0}\left(\frac{1}{5}\right)^0\left(\frac{4}{5}\right)^6\\&= \frac{4096}{15,625}\\&\approx .262\end{aligned}$$

36. "No more than 3 correct answers" means 0, 1, 2, or 3 correct answers.

$$\begin{aligned}&P(\text{no more than 3 correct})\\&\quad = P(0 \text{ correct}) + P(1 \text{ correct})\\&\qquad + P(2 \text{ correct}) + P(3 \text{ correct})\\&\quad = \binom{6}{0}\left(\frac{1}{5}\right)^0\left(\frac{4}{5}\right)^6 + \binom{6}{1}\left(\frac{1}{5}\right)^1\left(\frac{4}{5}\right)^5\\&\qquad + \binom{6}{2}\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^4 + \binom{6}{3}\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^3\\&\quad \approx .983\end{aligned}$$

38. $n = 20,\ p = .05,\ x = 0$

$$\begin{aligned}&P(0 \text{ defective transistors})\\&\quad = \binom{20}{0}(.05)^0(.95)^{20} \approx .358\end{aligned}$$

40. Let success mean producing a defective item. Then we have $n = 75,\ p = .05$, and $1 - p = .95$.

(a) If there are exactly 5 defective items, then $x = 5$. Thus,

$$\begin{aligned}P(\text{exactly 5 defective}) &= \binom{75}{5}(.05)^5(.95)^{70}\\&\approx .149.\end{aligned}$$

(b) If there are no defective items, then $x = 0$. Thus,

$$\begin{aligned}P(\text{none defective}) &= \binom{75}{0}(.05)^0(.95)^{75}\\&\approx .021.\end{aligned}$$

(c) If there is at least 1 defective item, then we are interested in $x \geq 1$. We have

$$\begin{aligned}P(\text{at least one defective}) &= 1 - P(x = 0)\\&\approx 1 - .021\\&= .979.\end{aligned}$$

42. (a) Since 80% of the "good nuts" are good, 20% of the "good nuts" are bad. Let's let success represent "getting a bad nut." Then .2 is the probability of success in a single trial. The probability of 8 successes in 20 trials is

$$\begin{aligned}\tbinom{20}{8}(.2)^8(1 - .2)^{20-8} &= \tbinom{20}{8}(.2)^8(.8)^{12}\\&\approx .0222\end{aligned}$$

(b) Since 60% of the "blowouts" are good, 40% of the "blowouts" are bad. Let's let success represent "getting a bad nut." The .4 is the probability of success in a single trial. The probability of 8 successes in 20 trials is

$$\binom{20}{8}(.4)^8(1-.4)^{20-8} = \binom{20}{8}(.4)^8(.6)^{12} \approx .1797$$

(c) The probability that the nuts are "blowouts" is

$$\frac{\text{Probability of "Blowouts" having 8 bad nuts out of 20}}{\text{Probability of "Good Nuts" or "Blowouts" having 8 bad nuts out of 20}}$$

$$= \frac{.3\left[\binom{20}{8}(.4)^8(.6)^{12}\right]}{.7\left[\binom{20}{8}(.2)^8(.8)^{12}\right] + .3\left[\binom{20}{8}(.4)^8(.6)^{12}\right]} \approx .7766$$

44. We have $n = 20,\ x = 17,\ p = .7$, and $1 - p = .3$. Thus,

$$\begin{aligned} P(\text{exactly 17 cured}) &= \binom{20}{17}(.7)^{17}(.3)^3 \\ &= 1140(.7)^{17}(.3)^3 \\ &\approx .072. \end{aligned}$$

46. $P(\text{at least 18 cured})$

$$\begin{aligned} &= P(\text{exactly 18 cured}) + P(\text{exactly 19 cured}) + P(\text{20 cured}) \\ &= \binom{20}{18}(.7)^{18}(.3)^2 + \binom{20}{19}(.7)^{19}(.3)^1 + (.7)^{20} + \binom{20}{20}(.7)^{20}(.3)^0 \\ &= 190(.7)^{18}(.3)^2 + 20(.7)^{19}(.3) + (.7)^{20} \\ &\approx .035 \end{aligned}$$

48. We have $n = 100,\ p = .012$, and $1 - p = .988$. Thus,

$$\begin{aligned} &P(\text{at most 2 sets}) \\ &= P(x=0) + P(x=1) + P(x=2) \\ &= \binom{100}{0}(.012)^0(.988)^{100} + \binom{100}{1}(.012)^1(.988)^{99} + \binom{100}{2}(.012)^2(.988)^{98} \\ &= (.012)^0(.988)^{100} + 100(.012)^1(.988)^{99} + 4950(.012)^2(.988)^{98} \approx .881. \end{aligned}$$

50. We have $n = 6,\ x = 3,\ p = .70$, and $1 - p = .30$. Thus,

$$P(\text{exactly 3 recover}) = \binom{6}{3}(.7)^3(.3)^3 \approx .185.$$

52. $P(\text{no more than 3 recover})$

$$\begin{aligned} &= P(\text{0 recover}) + P(\text{1 recovers}) + P(\text{2 recover}) + P(\text{3 recover}) \\ &= \binom{6}{0}(.7)^0(.3)^6 + \binom{6}{1}(.7)^1(.3)^5 + \binom{6}{2}(.7)^2(.3)^4 + \binom{6}{3}(.7)^3(.3)^3 \\ &\approx .256 \end{aligned}$$

54. We have $n = 10,000,\ p = 2.5 \cdot 10^{-7} = .00000025$, and $1 - p = .99999975$. Thus,

$$\begin{aligned} &P(\text{at least 1 mutation occurs}) \\ &= 1 - P(\text{none occurs}) \\ &= 1 - \binom{10,000}{0} p^0(1-p)^{10,000} \\ &= 1 - (.99999975)^{10,000} \\ &\approx .0025. \end{aligned}$$

56. We define a success to be the event that an inoculated person gets the flu. In this situation, $n = 83,\ p = .2$, and $1 - p = .8$.

(a) $$P(\text{10 successes}) = \binom{83}{10}(.2)^{10}(.8)^{73} \approx .0210$$

(b) $P(\text{no more than 4 succeses})$

$$\begin{aligned} &= P(\text{0 successes}) + P(\text{1 success}) + P(\text{2 successes}) + P(\text{3 successes}) + P(\text{4 successes}) \\ &= \binom{83}{0}(.2)^0(.8)^{83} + \binom{83}{1}(.2)^1(.8)^{82} + \binom{83}{2}(.2)^2(.8)^{81} + \binom{83}{3}(.2)^3(.8)^{80} + \binom{83}{4}(.2)^4(.83)^{79} \\ &\approx 8.004 \times 10^{-5} \end{aligned}$$

(c) P(none of the inoculated people get the flu)

$$= P(\text{no successes})$$

$$= \binom{83}{0}(.2)^0(.8)^{83} \approx 9.046 \times 10^{-9}$$

58. (a) P(10 or more)

$$= 1 - P(\text{less than } 10)$$
$$= 1 - [P(0) + P(1) + P(2) + \ldots + P(9)]$$
$$= 1 - \left[\binom{100}{0}(.073)^0(.927)^{100} + \binom{100}{1}(.073)^1(.927)^{99} + \binom{100}{2}(.073)^2(.927)^{98} + \binom{100}{3}(.073)^3(.927)^{97} + \binom{100}{4}(.073)^4(.927)^{96} + \binom{100}{5}(.073)^5(.927)^{95} + \binom{100}{6}(.073)^6(.927)^{94} + \binom{100}{7}(.073)^7(.927)^{93} + \binom{100}{8}(.073)^8(.927)^{92} + \binom{100}{9}(.073)^9(.927)^{91}\right]$$
$$\approx 1 - [.00051 + .00402 + .01567 + .04031 + .07698 + .11639 + .14512 + .15346 + .14049 + .11309]$$
$$= 1 - .80604$$
$$= .19396$$

The probability that 10 or more will experience nausea/vomiting is about .194.

(b) P(10 or more)

$$= 1 - P(\text{less than } 10)$$
$$= 1 - [P(0) + P(1) + P(2) + \ldots + P(9)]$$
$$= 1 - \left[\binom{100}{0}(.071)^0(.929)^{100} + \binom{100}{1}(.071)^1(.929)^{99} + \binom{100}{2}(.071)^2(.929)^{98} + \binom{100}{3}(.071)^3(.929)^{97} + \binom{100}{4}(.071)^4(.929)^{96} + \binom{100}{5}(.071)^5(.929)^{95} + \binom{100}{6}(.071)^6(.929)^{94} + \binom{100}{7}(.071)^7(.929)^{93} + \binom{100}{8}(.071)^8(.929)^{92} + \binom{100}{9}(.071)^9(.929)^{91}\right]$$
$$= 1 - .82765$$
$$= .17235$$

The probability that 10 or more will experience nausea/vomiting is about .172.

60. We define a success to be the event that a woman would prefer to work part-time rather than full-time. In this situation, $n = 10$; $x = 3, 4, 5, \ldots, 10$; $p = .33$; and $1 - p = .67$.

P(at least 3)

$$= 1 - P(x = 0,\ 1, \text{or } 2)$$

$$= 1 - [P(x = 0) + P(x = 1) + P(x = 2)]$$
$$= 1 - P(x = 0) - P(x = 1) - P(x = 2)$$

$$= 1 - \binom{10}{0}(.33)^0(.67)^{10} - \binom{10}{1}(.33)^1(.67)^9 - \binom{10}{2}(.33)^2(.67)^8$$

$$\approx .6930$$

62. We have $n = 10$, $x = 7$, $p = .2$, and $1 - p = .8$. Thus,

$$P(\text{exactly 7 correct}) = \binom{10}{7}(.2)^7(.8)^3 \approx .00079.$$

64. We have

$$\begin{aligned}
&P(\text{fewer than 8 correct})\\
&\quad = 1 - P(\text{8 or more correct})\\
&\quad = 1 - [P(x=8) + P(x=9) + P(x=10)]\\
&\quad = 1 - P(x=8) - P(x=9) - P(x=10)\\
&\quad = 1 - \binom{10}{8}(.2)^8(.8)^2 - \binom{10}{9}(.2)^9(.8)^1\\
&\qquad - \binom{10}{10}(.2)^{10}(.8)^0\\
&\quad = 1 - .000074 - .000004 - .0000001\\
&\quad = .999922.
\end{aligned}$$

66. **(a)** No, the results only indicated that 84% of college students believe they need to cheat to get ahead in the world today. It says nothing about whether or not they cheat.

(b)

$$\begin{aligned}
&P(\text{90 or more})\\
&= P(90) + P(91) + \ldots + P(100)\\
&= \binom{100}{90}(.84)^{90}(.16)^{10} + \binom{100}{91}(.84)^{91}(.16)^{9}\\
&= \binom{100}{92}(.84)^{92}(.16)^{8} + \binom{100}{93}(.84)^{93}(.16)^{7}\\
&= \binom{100}{94}(.84)^{94}(.16)^{6} + \binom{100}{95}(.84)^{95}(.16)^{5}\\
&= \binom{100}{96}(.84)^{96}(.16)^{4} + \binom{100}{97}(.84)^{97}(.16)^{3}\\
&= \binom{100}{98}(.84)^{98}(.16)^{2} + \binom{100}{99}(.84)^{99}(.16)^{1}\\
&= \binom{100}{100}(.84)^{100}(.16)^{0}\\
&\approx .02915 + .01682 + .00864 + .00390\\
&\quad + .00152 + .00051 + .00014\\
&\quad + .00003 + 0 + 0 + 0\\
&= .06071
\end{aligned}$$

The probability that 90 or more will answer affirmatively to the question is about .06.

8.5 Probability Distributions; Expected Value

2. There are 36 outcomes. We count the number of ways each sum can be obtained, and then divide by 36 to get each probability.

Number of Points	2	3	4	5	6	7	8	9	10	11	12
Ways to Get This Total	1	2	3	4	5	6	5	4	3	2	1
Probability	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

4. Use combinations to find the probabilities of drawing 0, 1, and 2 black balls.

$$P(0) = \frac{\binom{2}{0}\binom{4}{2}}{\binom{6}{2}} = \frac{6}{15} = \frac{2}{5}$$

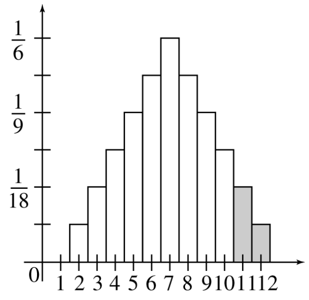
$$P(1) = \frac{\binom{2}{1}\binom{4}{1}}{\binom{6}{2}} = \frac{8}{15}$$

$$P(2) = \frac{\binom{2}{2}\binom{4}{0}}{\binom{6}{2}} = \frac{1}{15}$$

Number of Black Balls	0	1	2
Probability	$\frac{2}{5}$	$\frac{8}{15}$	$\frac{1}{15}$

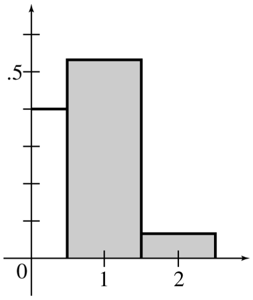
6. $P(x \geq 11)$

Use the probabilities that were calculated in Exercise 2, and shade the regions corresponding to $x = 11$ and $x = 12$.



8. P(at least one black ball)

Use the probabilities that were calculated in Exercise 4, and shade the regions corresponding to $x = 1$ and $x = 2$.



10.
$$\begin{aligned} E(y) &= 4(.4) + 6(.4) + 8(.05) + 10(.15) \\ &= 5.9 \end{aligned}$$

12.
$$\begin{aligned} E(x) &= 30(.31) + 32(.30) + 36(.29) \\ &\quad + 38(.06) + 44(.04) \\ &= 33.38 \end{aligned}$$

14. $P(2) = .2,\ P(4) = .3,\ P(6) = .2,$ $P(8) = .1,$ and $P(10) = .2.$

$$\begin{aligned} E(x) &= 2(.2) + 4(.3) + 6(.2) + 8(.1) \\ &\quad + 10(.2) \\ &= 5.6 \end{aligned}$$

16. Notice that the probability of all values is .2.

$$\begin{aligned} E(x) &= .2(10 + 20 + 30 + 40 + 50) \\ &= .2(150) \\ &= 30 \end{aligned}$$

18.

	Possible Results	
Result of toss	H	H
Call	H	T
Caller wins?	Yes	No
Probability	$\frac{1}{2}$	$\frac{1}{2}$

(a) Yes, this is still a fair game, since the probability of Donna matching is still $\frac{1}{2}$.

(b) If Donna calls heads, her expected gain (since she will match with probability = 1) is

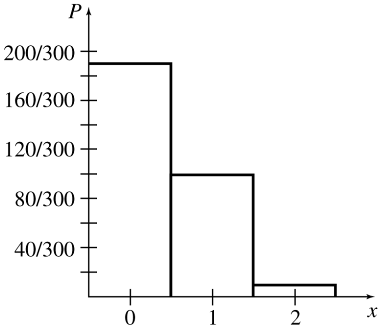
$$1(80¢) = 80¢.$$

(c) If Donna calls tails, her expected gain (since she will lose with probability = 1) is

$$1(-80¢) = -80¢.$$

20. **(a)** Use combinations to set up the probability distribution. In each case, there are 5 rotten apples and 20 good apples to pick from. There are a total of $\binom{25}{2}$ ways to choose any two apples.

Number of Rotten Apples	0	1	2
Probability	$\frac{\binom{5}{0}\binom{20}{2}}{\binom{25}{2}}$	$\frac{\binom{5}{1}\binom{20}{1}}{\binom{25}{2}}$	$\frac{\binom{5}{2}\binom{20}{0}}{\binom{25}{2}}$
Simplified	$\frac{190}{300}$	$\frac{100}{300}$	$\frac{10}{300}$



(b) We therefore have

$$\begin{aligned} E(x) &= 0\left(\frac{190}{300}\right) + 1\left(\frac{100}{300}\right) + 2\left(\frac{10}{300}\right) \\ &= \frac{120}{300} = .4. \end{aligned}$$

22. The probability that the delegation contains no liberals and 3 conservatives is

$$\frac{\binom{5}{0}\binom{4}{3}}{\binom{9}{3}} = \frac{1 \cdot 4}{84} = \frac{4}{84}.$$

Similarly, use combinations to calculate the remaining probabilities for the probability distribution.

(a) Let x represent the number of liberals on the delegation. The probability distribution of x is as follows.

x	0	1	2	3
$P(x)$	$\frac{4}{84}$	$\frac{30}{84}$	$\frac{40}{84}$	$\frac{10}{84}$

The expected value is

$$\begin{aligned} E(x) &= 0\left(\frac{4}{84}\right) + 1\left(\frac{30}{84}\right) + 2\left(\frac{40}{84}\right) + 3\left(\frac{10}{84}\right) \\ &= \frac{140}{84} = \frac{5}{3} \approx 1.667 \text{ liberals.} \end{aligned}$$

(b) Let y represent the number of conservatives on the committee. The probability distribution of y is as follows.

y	0	1	2	3
$P(y)$	$\frac{10}{84}$	$\frac{40}{84}$	$\frac{30}{84}$	$\frac{4}{84}$

The expected value is

$$E(y) = 0\left(\frac{10}{84}\right) + 1\left(\frac{40}{84}\right) + 2\left(\frac{30}{84}\right) + 3\left(\frac{4}{84}\right)$$
$$= \frac{112}{84} = \frac{4}{3} \approx 1.333 \text{ conservatives.}$$

24. Let x represent the number of junior members on the committee. Use combinations to find the probabilities of 0, 1, 2, and 3 junior members. The probability distribution of x is as follows.

x	0	1	2	3
$P(x)$	$\frac{57}{203}$	$\frac{95}{203}$	$\frac{45}{203}$	$\frac{6}{203}$

The expected value is

$$E(x) = 0\left(\frac{57}{203}\right) + 1\left(\frac{95}{203}\right) + 2\left(\frac{45}{203}\right) + 3\left(\frac{6}{203}\right)$$
$$= 1 \text{ junior member.}$$

26. The probability of drawing 2 diamonds is

$$\frac{\binom{13}{2}}{\binom{52}{2}} = \frac{78}{1326},$$

and the probability of not drawing 2 diamonds is

$$1 - \frac{78}{1326} = \frac{1248}{1326}.$$

Let x represent your net winnings. Then the expected value of the game is

$$E(x) = 4.5\left(\frac{78}{1326}\right) + (-.5)\left(\frac{1248}{1326}\right)$$
$$= -\frac{273}{1326} \approx -\$.21 \quad \text{or} \quad -21¢.$$

No, the game is not fair since your expected winnings are not zero.

30. We first compute the amount of money the company can expect to pay out for each kind of policy. The sum of these amounts will be the total amount the company can expect to pay out. For a single \$10,000 policy, we have the following probability distribution.

	Pay	Don't Pay
Outcome	\$10,000	\$10,000
Probability	.001	.999

$$E(\text{payoff}) = 10,000(.001) + 0(.999)$$
$$= \$10$$

For all 100 such policies, the company can expect to pay out

$$100(10) = \$1000.$$

For a single \$5000 policy,

$$E(\text{payoff}) = 5000(.001) + 0(.999)$$
$$= \$5.$$

For all 500 such policies, the company can expect to pay out

$$500(5) = \$2500.$$

Similarly, for all 1000 policies of \$1000, the company can expect to pay out

$$1000(1) = \$1000.$$

Thus, the total amount the company can expect to pay out is

$$\$1000 + \$2500 + \$1000 = \$4500.$$

32. **(a)** Expected number of good nuts in 50 "blow outs" is

$$E(x) = 50(.60) = 30.$$

(b) Since 80% of the "good nuts" are good, 20% are bad. Expected number of bad nuts in 50 "good nuts" is

$$E(x) = 50(.20) = 10.$$

34. **(a)** Expected cost of Amoxicillin:

$$E(x) = .75(\$59.30) + .25(\$96.15)$$
$$= \$68.51$$

Expected cost of Cefaclor:

$$E(x) = .90(\$69.15) + .10(\$106.00)$$
$$= \$72.84$$

(b) Amoxicillin should be used to minimize total expected cost.

36. Expected number of a group of 500 college students that say they need to cheat is

$$E(x) = 500(.84) = 420.$$

38. **(a)** If the two players are equally skilled, the old pro's expected winnings are

$$\frac{1}{2}(80,000) + \frac{1}{2}(20,000) = \$50,000.$$

(b) If the pro's chance of winning is $\frac{3}{4}$, then his expected winnings are

$$\frac{3}{4}(80,000) + \frac{1}{4}(20,000) = \$65,000.$$

40. **(a)** We define a success to be the event that a letter was delivered the next day. In this situation, $n = 10$; $x = 0, 1, 2, 3, \ldots, 10$; $p = .83$, and $1 - p = .17$.

Number of Letters Delivered the Next Day	Probability
0	$\binom{10}{0}(.83)^0(.17)^{10} \approx .0000$
1	$\binom{10}{1}(.83)^1(.17)^9 \approx .0000$
2	$\binom{10}{2}(.83)^2(.17)^8 \approx .0000$
3	$\binom{10}{3}(.83)^3(.17)^7 \approx .0003$
4	$\binom{10}{4}(.83)^4(.17)^6 \approx .0024$
5	$\binom{10}{5}(.83)^5(.17)^5 \approx .0141$
6	$\binom{10}{6}(.83)^6(.17)^4 \approx .0573$
7	$\binom{10}{7}(.83)^7(.17)^3 \approx .1600$
8	$\binom{10}{8}(.83)^8(.17)^2 \approx .2929$
9	$\binom{10}{9}(.83)^9(.17)^1 \approx .3178$
10	$\binom{10}{10}(.83)^{10}(.17)^0 \approx .1552$

(b) $P(4 \text{ or fewer letters would be delivered})$

$$\begin{aligned} &= P(x=0) + P(x=1) + P(x=2) \\ &\quad + P(x=3) + P(x=4) \\ &\approx .0027 \end{aligned}$$

(d) Expected number of letters delivered next day

$$\begin{aligned} &\approx 0(.0000) + 1(.0000) + 2(.0000) \\ &\quad + 3(.0003) + 4(.0024) + 5(.0141) \\ &\quad + 6(.0573) + 7(.1600) + 8(.2929) \\ &\quad + 9(.3178) + 10(.1552) \\ &\approx 8.3 \end{aligned}$$

42. Reduce each price by the 50¢ cost of the raffle ticket, and multiply by the corresponding probability.

$$\begin{aligned} E(x) &= 999.50\left(\frac{1}{10,000}\right) + 299.50\left(\frac{2}{10,000}\right) \\ &\quad + 9.50\left(\frac{20}{10,000}\right) + (-.50)\left(\frac{9977}{10,000}\right) \\ &= \frac{-3200}{10,000} = -\$.32 \quad \text{or} \quad -32¢ \end{aligned}$$

No, this is not a fair game. In a fair game the expected value is 0.

44. The probability of getting exactly 3 of the 4 selections correct and winning this game is

$$\binom{4}{3}\left(\frac{1}{13}\right)^3\left(\frac{12}{13}\right)^1 \approx .001681.$$

The probability of losing is .998319. If you win, your winnings are \$199. Otherwise, you lose \$1 (win −\$1). If x represents your winnings, then the expected value is

$$\begin{aligned} E(x) &= 199(.001681) + (-1)(.998319) \\ &= .334519 - .998319 \\ &= -.6638 \approx -\$.66 \quad \text{or} \quad -66¢. \end{aligned}$$

46. In this form of roulette,

$$P(\text{even}) = \frac{18}{37} \text{ and } P(\text{noneven}) = \frac{19}{37}.$$

If an even number comes up, you win \$1. Otherwise, you lose \$1 (win −\$1). If x represents your winnings, then the expected value is

$$\begin{aligned} E(x) &= 1\left(\frac{18}{37}\right) + (-1)\left(\frac{19}{37}\right) \\ &= -\frac{1}{37} \approx -\$.027 \text{ or } -2.7¢. \end{aligned}$$

48. In this form of the game Keno,

$$P(\text{your number comes up}) = \frac{20}{80} = \frac{1}{4}$$

and

$$P(\text{your number doesn't come up}) = \frac{60}{80} = \frac{3}{4}.$$

If your number comes up, you win \$2.20. Otherwise, you lose \$1 (win −\$1). If x represents your winnings, then the expected value is

$$\begin{aligned} E(x) &= 2.20\left(\frac{1}{4}\right) - 1\left(\frac{3}{4}\right) \\ &= .55 - .75 = -\$.20 \quad \text{or} \quad -20¢. \end{aligned}$$

50. At any one restaurant, your expected winnings are

$$\begin{aligned} E(x) &= 100,000\left(\frac{1}{176,402,500}\right)+25,000\left(\frac{1}{39,200,556}\right) \\ &+ 5000\left(\frac{1}{17,640,250}\right)+1000\left(\frac{1}{1,568,022}\right) \\ &+ 100\left(\frac{1}{288,244}\right)+5\left(\frac{1}{7056}\right)+1\left(\frac{1}{588}\right) \\ &= .00488. \end{aligned}$$

Going to 25 restaurants gives you expected earnings of 25(.00488) = .122. Since you spent \$1, you lose 87.8¢ on the average, so your expected value is −87.8¢.

52. **(a)** Expected value of a two-point conversion:

$$E(x) = 2(.37) = .74$$

Expected value of an extra-point kick:

$$E(x) = 1(.94) = .94$$

(b) Since the expected value of an extra-point kick is greater than the expected value of a two-point conversion, the extra-point kick will maximize the number of points scored over the long run.

Chapter 8 Review Exercises

2. Since order makes a difference, use permutations.

$$\begin{aligned} P(6,3) &= \frac{6!}{(6-3)!} = \frac{6!}{3!} \\ &= 6 \cdot 5 \cdot 4 = 120 \end{aligned}$$

There are 120 variations in first-, second- and third-place finishes.

4. **(a)** The sample will include 1 of the 2 rotten oranges and 2 of the 10 good oranges. Using the multiplication principle, this can be done in

$$\binom{2}{1}\binom{10}{2} = 2 \cdot 45 = 90 \text{ ways.}$$

(b) The sample will include both of the 2 rotten oranges and 1 of the 10 good oranges. This can be done in

$$\binom{2}{2}\binom{10}{1} = 1 \cdot 10 = 10 \text{ ways.}$$

(c) The sample will include 0 of the 2 rotten oranges and 3 of the 10 good oranges. This can be done in

$$\binom{2}{0}\binom{10}{3} = 1 \cdot 120 = 120 \text{ ways.}$$

(d) If the sample contains at most 2 rotten oranges, it must contain 0, 1, or 2 rotten oranges. Adding the results from parts (a), (b), and (c), this can be done in

$$90 + 10 + 120 = 220 \text{ ways.}$$

6. Since a certain picture must be first, the number of ways to arrange the pictures is

$$1 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 24.$$

8. **(a)** The order within each list is not important. Use combinations and the multiplication principle. The choice of three items from column A can be made in $\binom{8}{3}$ ways, and the choice of two from column B can be made in $\binom{6}{2}$ ways. Thus, the number of possible dinners is

$$\binom{8}{3}\binom{6}{2} = 56 \cdot 15 = 840.$$

(b) There are

$$\binom{8}{0}+\binom{8}{1}+\binom{8}{2}+\binom{8}{3}$$

ways to pick up to 3 items from column A. Likewise, there are

$$\binom{6}{0}+\binom{6}{1}+\binom{6}{2}$$

ways to pick up to 2 items from column B. We use the multiplication principle to obtain

$$\begin{aligned} &\left[\binom{8}{0}+\binom{8}{1}+\binom{8}{2}+\binom{8}{3}\right]\left[\binom{6}{0}+\binom{6}{1}+\binom{6}{2}\right] \\ &= (1+8+28+56)(1+6+15) \\ &= 93(22) \\ &= 2046. \end{aligned}$$

Since we are assuming that the diner will order at least one item, subtract 1 to exclude the dinner that would contain no items. Thus, the number of possible dinners is 2045.

12. There are $\binom{11}{3}$ ways to choose the 3 balls and $\binom{4}{3}$ ways to get all black balls. Thus,

$$P(\text{all black}) = \frac{\binom{4}{3}}{\binom{11}{3}} = \frac{4}{165} \approx .024.$$

14. There are $\binom{4}{2}$ ways to get 2 black balls and $\binom{5}{1}$ ways to get 1 green ball. Thus,

$$\begin{aligned} &P(\text{2 black and 1 green}) \\ &= \frac{\binom{4}{2}\binom{5}{1}}{\binom{11}{3}} = \frac{6 \cdot 5}{165} = \frac{30}{165} = \frac{2}{11} \approx .182. \end{aligned}$$

16. There are $\binom{2}{1}$ ways to get 1 blue ball and $\binom{9}{2}$ ways to get 2 nonblue balls. Thus,

$$\begin{aligned} &P(\text{exactly 1 blue}) \\ &= \frac{\binom{2}{1}\binom{9}{2}}{\binom{11}{3}} = \frac{2 \cdot 36}{165} = \frac{72}{165} = \frac{24}{55} \approx .436. \end{aligned}$$

18. This is a Bernoulli trial problem with $P(\text{success}) = P(\text{girl}) = \frac{1}{2}$. Here, $n = 6$, $p = \frac{1}{2}$, and $x = 3$.

$$\begin{aligned} &P(\text{exactly 3 girls}) \\ &= \binom{6}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^3 \\ &= \frac{20}{64} = \frac{5}{16} \\ &\approx .313 \end{aligned}$$

20.
$$\begin{aligned} &P(\text{at least 4 girls}) \\ &\quad = P(\text{4 girls}) + P(\text{5 girls}) + P(\text{6 girls}) \\ &\quad = \binom{6}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^2 + \binom{6}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^1 \\ &\qquad + \binom{6}{6}\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^0 \\ &\quad = \frac{22}{64} = \frac{11}{32} \\ &\quad \approx .344 \end{aligned}$$

22.
$$\begin{aligned} &P(\text{both red}) \\ &= \frac{\binom{26}{2}}{\binom{52}{2}} = \frac{325}{1326} = \frac{25}{102} \approx .245 \end{aligned}$$

24.
$$\begin{aligned} &P(\text{at least 1 card is a spade}) \\ &\quad = 1 - P(\text{neither is a spade}) \\ &\quad = 1 - \frac{\binom{39}{2}}{\binom{52}{2}} = 1 - \frac{741}{1326} \\ &\quad = \frac{585}{1326} = \frac{15}{34} \approx .441 \end{aligned}$$

26. There are 12 face cards and 40 nonface cards in an ordinary deck.

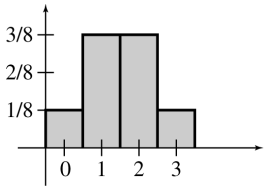
$$\begin{aligned} &P(\text{at least 1 face card}) \\ &\quad = P(\text{1 face card}) + P(\text{2 face cards}) \\ &\quad = \frac{\binom{12}{1}\binom{40}{1}}{\binom{52}{2}} + \frac{\binom{12}{2}}{\binom{52}{2}} \\ &\quad = \frac{480}{1326} + \frac{66}{1326} \\ &\quad = \frac{546}{1326} \approx .4118 \end{aligned}$$

28. This is a Bernoulli trial problem.

(a) $P(\text{success}) = P(\text{head}) = \frac{1}{2}$. Hence, $n = 3$ and $p = \frac{1}{2}$.

Number of Heads	Probability
0	$\binom{3}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^3 = .125$
1	$\binom{3}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^2 = .375$
2	$\binom{3}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^1 = .375$
3	$\binom{3}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^0 = .125$

(b)



(c)
$$\begin{aligned} E(x) &= 0(.125) + 1(.375) + 2(.375) + 3(.125) \\ &= 1.5 \end{aligned}$$

30. The probability that corresponds to the shaded region of the histogram is the total of the shaded areas, that is,

$$1(.3) + 1(.2) + 1(.1) = .6.$$

32. The probability of rolling a 6 is $\frac{1}{6}$, and your net winnings would be \$2. The probability of rolling a 5 is $\frac{1}{6}$, and your net winnings would be \$1. The probability of rolling something else is $\frac{4}{6}$, and your net winnings would be $-\$2$. Let x represent your winnings. The expected value is

$$\begin{aligned} E(x) &= 2\left(\frac{1}{6}\right) + 1\left(\frac{1}{6}\right) + (-2)\left(\frac{4}{6}\right) \\ &= -\frac{5}{6} \\ &\approx -\$.833 \quad \text{or} \quad -83.3¢. \end{aligned}$$

This is not a fair game since the expected value is not 0.

34. (a)

Number of Aces	Probability
0	$\dfrac{\binom{4}{0}\binom{48}{3}}{\binom{52}{3}} = \dfrac{17,296}{22,100}$
1	$\dfrac{\binom{4}{1}\binom{48}{2}}{\binom{52}{3}} = \dfrac{4512}{22,100}$
2	$\dfrac{\binom{4}{2}\binom{48}{1}}{\binom{52}{3}} = \dfrac{288}{22,100}$
3	$\dfrac{\binom{4}{3}\binom{48}{0}}{\binom{52}{3}} = \dfrac{4}{22,100}$

$$\begin{aligned} E(x) &= 0\left(\frac{17,296}{22,100}\right) + 1\left(\frac{4512}{22,100}\right) \\ &\quad + 2\left(\frac{288}{22,100}\right) + 3\left(\frac{4}{22,100}\right) \\ &= \frac{5100}{22,100} = \frac{51}{221} \\ &= \frac{3}{13} \approx .231 \end{aligned}$$

(b)

Number of Clubs	Probability
0	$\dfrac{\binom{13}{0}\binom{39}{3}}{\binom{52}{3}} = \dfrac{9139}{22,100}$
1	$\dfrac{\binom{13}{1}\binom{39}{2}}{\binom{52}{3}} = \dfrac{9633}{22,100}$
2	$\dfrac{\binom{13}{2}\binom{39}{1}}{\binom{52}{3}} = \dfrac{3042}{22,100}$
3	$\dfrac{\binom{13}{3}\binom{39}{0}}{\binom{52}{3}} = \dfrac{286}{22,100}$

$$\begin{aligned} E(x) &= 0\left(\frac{9139}{22,100}\right) + 1\left(\frac{9633}{22,100}\right) \\ &\quad + 2\left(\frac{3042}{22,100}\right) + 3\left(\frac{286}{22,100}\right) \\ &= \frac{16,575}{22,100} \\ &= \frac{3}{4} = .75 \end{aligned}$$

36. We define a success to be the event that a student flips heads and is on the committee. In this situation, $n = 6$; $x = 1, 2, 3, 4,$ or 5; $p = \frac{1}{2}$; and $1 - p = \frac{1}{2}$.

$$\begin{aligned} &P(x = 1, 2, 3, 4, \text{ or } 5) \\ &\quad = 1 - P(x = 6) - P(x = 0) \\ &\quad = 1 - \binom{6}{6}\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^0 - \binom{6}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^6 \\ &\quad = 1 - \frac{1}{64} - \frac{1}{64} \\ &\quad = \frac{62}{64} = \frac{31}{32} \end{aligned}$$

38. (a)

First Card	Second Card	Number of Possibilities for Third Card
1	2	7
1	3	6
1	4	5
1	5	4
1	6	3
1	7	2
1	8	1
2	3	6
2	4	5
2	5	4
2	6	3
2	7	2
2	8	1
3	4	5
3	5	4
3	6	3
3	7	2
3	8	1
4	5	4
4	6	3
4	7	2
4	8	1
5	6	3
5	7	2
5	8	1
6	7	2
6	8	1
7	8	1

The sum of the numbers in the third column is 84.

(b) There are 4 even digits and 5 odd digits.

$$P(\text{all even}) = \frac{\binom{4}{3}\binom{5}{0}}{\binom{9}{3}} = \frac{4}{84} = \frac{1}{21}$$

(c) There are 7 possibilities for three consecutive digits:

$$\begin{gathered}1,2,3\\2,3,4\\3,4,5\\4,5,6\\5,6,7\\6,7,8\\7,8,9.\end{gathered}$$

$$P(\text{consecutive integers}) = \frac{7}{\binom{9}{3}} = \frac{7}{84} = \frac{1}{12}$$

(d) Refer to column 3 from part (a). The sum of the numbers when the first card is 4 is 1 + 2 + 3 + 4 = 10.

$$P(x=4) = \frac{10}{84} = \frac{5}{42}$$

(e) From part (a), we see that the first card x ranges from 1 to 7. If k is an integer such that $1 \le k \le 7$, the number of possibilities is

$$\begin{aligned}&1+2+\cdots+[9-(k+1)]\\&\quad=\frac{[9-(k+1)][9-(k+1)+1]}{2}\\&\quad=\frac{(8-k)(9-k)}{2}.\end{aligned}$$

The probability of $x = k$ is given by

$$\begin{aligned}P(x=k) &= \frac{\frac{(9-k)(8-k)}{2}}{\binom{9}{3}}\\&=\frac{(9-k)(8-k)}{2}\cdot\frac{1}{84}\\&=\frac{(9-k)(8-k)}{168}.\end{aligned}$$

The expected value of x is

$$\begin{aligned}E(x) &= 1\left(\frac{8\cdot 7}{168}\right)+2\left(\frac{7\cdot 6}{168}\right)+3\left(\frac{6\cdot 5}{168}\right)\\&\quad+4\left(\frac{5\cdot 4}{168}\right)+5\left(\frac{4\cdot 3}{168}\right)+6\left(\frac{3\cdot 2}{168}\right)\\&\quad+7\left(\frac{2\cdot 1}{168}\right)\\&=\frac{420}{168}=\frac{5}{2}.\end{aligned}$$

40. We have $n = 20$, $x = 3$, $p = .01$, and $1 - p = .99$, so

$$P(x=3) = \binom{20}{3}(.01)^3(.99)^{17} \approx .00096.$$

42. $P(x \ge 12)$

$$\begin{aligned}&= P(x=12)+P(x=13)+\cdots+P(x=20)\\&=\binom{20}{12}(.01)^{12}(.99)^8+\binom{20}{13}(.01)^{13}(.99)^7\\&\quad+\cdots+\binom{20}{20}(.01)^{20}(.99)^0\end{aligned}$$

44. Denote by S the event that a product is successful.
Denote by U the event that a product is unsuccessful.
Denote by Q the event of passing quality control.
We must calculate the conditional probabilities $P(S|Q)$ and $P(U|Q)$ using Bayes' Theorem in order to calculate the expected net profit (in millions)
$E = 40P(S|Q) - 15P(U|Q)$.
$P(S) = P(U) = .5$
$P(Q|S) = .8, P(Q|U) = .25$

$$\begin{aligned}P(S|Q) &= \frac{P(S)\cdot P(Q|S)}{P(S)\cdot P(Q|S)+P(U)\cdot P(Q|U)}\\&=\frac{.5(.8)}{.5(.8)+.5(.25)}\\&=\frac{.4}{.4+.125}=.762\end{aligned}$$

$$\begin{aligned}P(U|Q) &= \frac{P(U)\cdot P(Q|U)}{P(U)\cdot P(Q|U)+P(S)\cdot P(Q|S)}\\&=\frac{.125}{.525}=.238\end{aligned}$$

Therefore,

$$\begin{aligned}E &= 40P(S|Q)-15P(U|Q)\\&=40(.762)-15(.238)\\&\approx 27, \text{ so}\end{aligned}$$

the expected net profit is \$27 million, or choice (e).

46. Let $I(x)$ represent the airline's net income if x people show up.

$I(0) = 0$
$I(1) = 100$
$I(2) = 2(100) = 200$
$I(3) = 3(100) = 300$
$I(4) = 3(100) - 100 = 200$
$I(5) = 3(100) - 2(100) = 100$
$I(6) = 3(100) - 3(100) = 0$

Let $P(x)$ represent the probability that x people will show up. Use the binomial probability formula to find the values of $P(x)$.

$$P(0) = \binom{6}{0}(.6)^0(.4)^6 = .004$$

$$P(1) = \binom{6}{1}(.6)^1(.4)^5 = .037$$

$$P(2) = \binom{6}{2}(.6)^2(.4)^4 = .138$$

$$P(3) = \binom{6}{3}(.6)^3(.4)^3 = .276$$

$$P(4) = \binom{6}{4}(.6)^4(.4)^2 = .311$$

$$P(5) = \binom{6}{5}(.6)^5(.4)^1 = .187$$

$$P(6) = \binom{6}{6}(.6)^6(.4)^0 = .047$$

(a) $$\begin{aligned} E(I) &= 0(.004) + 100(.037) + 200(.138) \\ &\quad + 300(.276) + 200(.311) + 100(.187) \\ &\quad + 0(.047) \\ &= \$195 \end{aligned}$$

(b) $n = 3$

x	0	1	2	3
Income	0	100	200	300
$P(x)$	.064	.288	.432	.216

$$\begin{aligned} E(I) &= 0(.064) + 100(.288) + 200(.432) \\ &\quad + 300(.216) \\ &= \$180 \end{aligned}$$

On the basis of all the calculations, the table given in the exercise is completed as follows.

x	0	1	2	3	4	5	6
Income	0	100	200	300	200	100	0
$P(x)$	.004	.037	.138	.276	.311	.187	.047

$n = 4$

x	1	1	2	3	4
Income	0	100	200	300	200
$P(x)$	.0256	.1536	.3456	.3456	.1296

$$\begin{aligned} E(I) &= 0(.0256) + 100(.1536) + 200(.3456) \\ &\quad + 300(.3456) + 200(.1296) \\ &= \$214.08 \end{aligned}$$

$n = 5$

x	0	1	2	3	4	5
Income	0	100	200	300	200	100
$P(x)$	.01024	.0768	.2304	.3456	.2592	.07776

$$\begin{aligned} E(I) &= 0(.01024) + 100(.0768) + 200(.2304) \\ &\quad + 300(.3456) + 200(.2592) + 100(.07776) \\ &= \$217.06 \end{aligned}$$

Since $E(I)$ is greatest when $n = 5$, the airlines should book 5 reservations to maximize revenue.

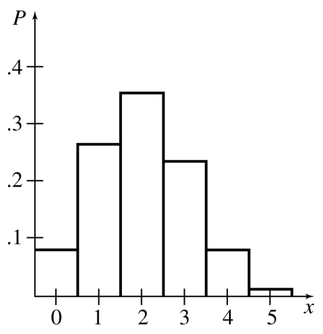
48. $$P(\text{exactly } 5) = \binom{40}{5}\left(\frac{1}{8}\right)^5\left(\frac{7}{8}\right)^{35} \approx .1875$$

The probability that exactly 5 will choose a green piece of paper is about .1875.

50. **(a)** We define a success to be the event that a woman athlete is selected. In this situation, $n = 5$; $x = 0, 1, 2, 3, 4,$ or 5; $p = .4$; and $1 - p = .6$.

Number of Women	Probability
0	$\binom{5}{0}(.4)^0(.6)^5 = .07776$
1	$\binom{5}{1}(.4)^1(.6)^4 = .2592$
2	$\binom{5}{2}(.4)^2(.6)^3 = .3456$
3	$\binom{5}{3}(.4)^3(.6)^2 = .2304$
4	$\binom{5}{4}(.4)^4(.6)^1 = .0768$
5	$\binom{5}{5}(.4)^5(.6)^0 = .01024$

(b) Sketch the histogram with 6 rectangles.

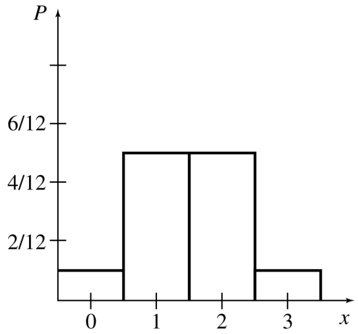


(c) Expected value $= np = 5(.4) = 2$

52. **(a)**

Number Who Did Not Do Homework	Probability
0	$\frac{\binom{3}{0}\binom{7}{5}}{\binom{10}{5}} = \frac{21}{252} = \frac{1}{12}$
1	$\frac{\binom{3}{1}\binom{7}{4}}{\binom{10}{5}} = \frac{105}{252} = \frac{5}{12}$
2	$\frac{\binom{3}{2}\binom{7}{3}}{\binom{10}{5}} = \frac{105}{252} = \frac{5}{12}$
3	$\frac{\binom{3}{3}\binom{7}{2}}{\binom{10}{5}} = \frac{21}{252} = \frac{1}{12}$

(b) Draw a histogram with four rectangles.



(c) Expected number who did not do homework

$$= 0\left(\frac{1}{12}\right) + 1\left(\frac{5}{12}\right) + 2\left(\frac{5}{12}\right) + 3\left(\frac{1}{12}\right)$$

$$= \frac{18}{12} = \frac{3}{2}$$

54. It costs $2(.37 + .04) = .82$ to play the game.

x	\$1999.18	−\$.82
$P(x)$	$\frac{1}{8000}$	$\frac{7999}{8000}$

$$E(x) = 1999.18\left(\frac{1}{8000}\right) - \$.82\left(\frac{7999}{8000}\right)$$

$$= -\$.57$$

56. If the game was played 365 times a year for 26 years, it was played 9490 times. About $\frac{1}{1000}$ of those times, any one outcome–specifically, 000–would result. So, 000 would result $\frac{1}{1000}(9490) = 9.49$, or about 9.5 times.

58. **(a)** (i) When 5 socks are selected, we could get 1 matching pair and 3 odd socks or 2 matching pairs and 1 odd sock.

First consider 1 matching pair and 3 odd socks. The number of ways this could be done is

$$\binom{10}{1}\left[\binom{18}{3} - \binom{9}{1}\binom{16}{1}\right] = 6720.$$

$\binom{10}{1}$ gives the number of ways for 1 pair, while $\left[\binom{18}{3} - \binom{9}{1}\binom{16}{1}\right]$ gives the number of ways for the remaining 3 socks from the 18 socks left. We must subtract the number of ways the last 3 socks could contain a pair from the 9 pairs remaining.

Next consider 2 matching pairs and 1 odd sock. The number of ways this could be done is

$$\binom{10}{2}\binom{16}{1} = 720.$$

$\binom{10}{2}$ gives the number of ways for 2 pairs, while $\binom{16}{1}$ gives the number of ways for the 1 odd sock.

The total number of ways is

$$6720 + 720 = 7440.$$

Then

$$P(\text{matching pair}) = \frac{7440}{\binom{20}{5}} \approx .4799.$$

(ii) When 6 socks are selected, we could get 3 matching pairs and no odd socks or 2 matching pairs and 2 odd socks or 1 matching pair and 4 odd socks. The number of ways of obtaining 3 matching pairs is $\binom{10}{3} = 120$. The number of ways of obtaining 2 matching pairs and 2 odd socks is

$$\binom{10}{2}\left[\binom{16}{2} - \binom{8}{1}\right] = 5040.$$

The 2 odd socks must come from the 16 socks remaining but cannot be one of the 8 remaining pairs.

The number of ways of obtaining 1 matching pair and 4 odd socks is

$$\binom{10}{1}\left[\binom{18}{4}-\binom{9}{2}-\binom{9}{1}\left[\binom{16}{2}-8\right]\right]=20,160.$$

The 4 odd socks must come from the 18 socks remaining but can be 2 pairs and cannot be 1 pair and 2 odd socks.

The total number of ways is

$$120+5040+20,160=25,320.$$

Thus,

$$P(\text{matching pair}) = \frac{25,320}{\binom{20}{6}} \approx .6533.$$

(c) Suppose 6 socks are lost at random. The worst case is they are 6 odd socks. The best case is they are 3 matching pairs.

First find the number of ways of selecting 6 odd socks. This is

$$\binom{10}{6}\binom{2}{1}\binom{2}{1}\binom{2}{1}\binom{2}{1}\binom{2}{1}\binom{2}{1}=13,440.$$

The $\binom{10}{6}$ gives the number of ways of choosing 6 different socks from the 10 pairs. But with each pair, $\binom{2}{1}$ gives the number of ways of selecting 1 sock. Then

$$P(6\text{ odd socks})=\frac{13,440}{\binom{20}{6}}\approx .3467.$$

Next find the number of ways of selecting three matching pairs. This is $\binom{10}{3}=120$. Then

$$P(3\text{ matching pairs})=\frac{120}{\binom{20}{6}}\approx .003096.$$

Extended Application: Optimal Inventory For a Service Truck

1. **(a)** $C(M_0)$

$$\begin{aligned}&= NL[1-(1-p_1)(1-p_2)(1-p_3)]\\&= 3(54)[1-(.91)(.76)(.83)]\\&= \$69.01\end{aligned}$$

(b) $C(M_2)$

$$\begin{aligned}&= H_2+NL[1-(1-p_1)(1-p_3)]\\&= 40+3(54)[1-(.91)(.83)]\\&= \$79.64\end{aligned}$$

(c) $C(M_3)$

$$\begin{aligned}&= H_3+NL[1-(1-p_1)(1-p_2)]\\&= 9+3(54)[1-(.91)(.76)]\\&= \$58.96\end{aligned}$$

(d) $C(M_{12})$

$$\begin{aligned}&= H_1+H_2+NL[1-(1-p_3)]\\&= 15+40+3(54)[1-.83]\\&= \$82.54\end{aligned}$$

(e) $C(M_{13})$

$$\begin{aligned}&= H_1+H_3+NL[1-(1-p_2)]\\&= 15+9+3(54)[1-.76]\\&= \$62.88\end{aligned}$$

(f) $C(M_{123})$

$$\begin{aligned}&= H_1+H_2+H_3+NL[1-1]\\&= 15+40+9\\&= \$64.00\end{aligned}$$

2. Policy M_3, stocking only part 3 on the truck, leads to the lowest expected cost.

3. It is not necessary for the probabilities to add up to 1 because it is possible that no parts will be needed. That is, the events of needing parts 1, 2, and 3 are not the only events in the sample space.

4. For 3 different parts we have 8 different policies: 1 with no parts, 3 with 1 part, 3 with 2 parts, and 1 with 3 parts. The number of different policies, $8=2^3$, is the number of subsets of a set containing 3 distinct elements. If there are n different parts, the number of policies is the number of subsets of a set containing n distinct elements which is 2^n.

Chapter 9

STATISTICS

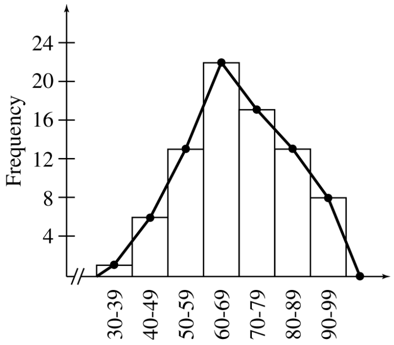
9.1 Frequency Distributions; Measures of Central Tendency

2. **(a)-(b)** Use seven intervals, beginning with 30-39 and ending with 90-99. Making a tally of how many data values lie in each interval leads to the following frequency distribution.

Interval	Frequency
30-39	1
40-49	6
50-59	13
60-69	22
70-79	17
80-89	13
90-99	8

(c)-(d) The histogram consists of 7 bars of equal width having heights as determined by the frequency of each interval.

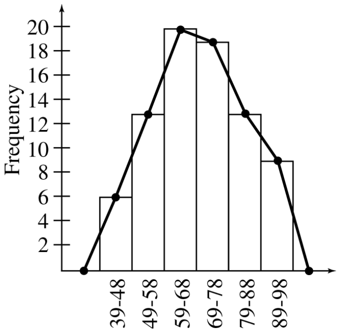
For the frequency polygon, join consecutive midpoints of the tops of the histogram bars with straight line segments.



4. **(a)-(b)**

Interval	Frequency
39-48	6
49-58	13
59-68	20
69-78	19
79-88	13
89-98	9

(c)-(d)



8.
$$\begin{aligned}\bar{x} &= \frac{\sum x}{n}\\ &= \frac{44+41+25+36+67+51}{6}\\ &= \frac{264}{6} = 44\end{aligned}$$

10.
$$\begin{aligned}\sum x &= 38{,}500 + 39{,}720 + 42{,}183\\ &\quad + 21{,}982 + 43{,}250\\ &= 185{,}635\end{aligned}$$

The mean of the 5 numbers is

$$\bar{x} = \frac{\sum x}{n} = \frac{185{,}635}{5} = 37,127.$$

12.
$$\begin{aligned}\sum x &= 30.1 + 42.8 + 91.6 + 51.2 + 88.3\\ &\quad + 21.9 + 43.7 + 51.2\\ &= 420.8\end{aligned}$$

The mean of the 8 numbers is

$$\bar{x} = \frac{\sum x}{n} = \frac{420.8}{8} = 52.6.$$

14.

Value	Frequency	Value × Frequency
9	3	$9 \cdot 3 = 27$
12	5	$12 \cdot 5 = 60$
15	1	$15 \cdot 1 = 15$
18	1	$18 \cdot 1 = 18$
Totals:	10	120

$$\bar{x} = \frac{120}{10} = 12$$

16. $596, 604, 612, 683, 719$

The median is the middle number, in this case, 612.

18. $1072, 1068, 1093, 1042, 1056, 1005, 1009$

First arrange the numbers in numerical order, from smallest to largest.

$1005, 1009, 1042, 1056, 1068, 1072, 1093$

The median is the middle number, in this case, 1056.

20. $.6, .4, .9, 1.2, .3, 4.1, 2.2, .4, .7, .1$

First arrange the numbers in numerical order, from smallest to largest.

$.1, .3, .4, .4, .6, .7, .9, 1.2, 2.2, 4.1$

Since there is an even number of entries, the median is the mean of the two middle numbers.

$$\text{median} = \frac{.6 + .7}{2} = .65$$

22. Using a graphing calculator, $\bar{x} = 69.475$ and the median is 69.

24. $21, 32, 46, 32, 49, 32, 49$

The mode is the most frequent entry. In this case the mode is 32, which occurs 3 times.

26. $158, 162, 165, 162, 165, 157, 163$

The mode is the most frequent entry. In this case there are two modes, 162 and 165, each of which occurs twice.

28. $12.75, 18.32, 19.41, 12.75, 18.30, 19.45, 18.33$

The mode is the most frequent entry. In this case the mode is 12.75, which occurs twice.

32.

Interval	Midpoint, x	Frequency, f	Product, xf
30-39	34.5	1	34.5
40-49	44.5	6	267
50-59	54.5	13	708.5
60-69	64.5	22	1419
70-79	74.5	17	1266.5
80-89	84.5	13	1098.5
90-99	94.5	8	756
Totals:		80	5550

Use the formula for the mean of a grouped frequency distribution.

$$\overline{x} = \frac{\sum xf}{n} = \frac{5550}{80} \approx 69.4$$

The modal class is the interval containing the most data values. The above table shows that the highest frequency, 22, occurs in the interval 60-69.

34. Find the mean of the numbers in the "Price" column.

$$\overline{x} = \frac{\sum x}{n} = \frac{32.73}{10} = 3.273$$

The mean price per bushel of wheat is \$3.27.
To find the median, list the ten values in the "Price" column from smallest to largest.

$$2.48, 2.62, 2.65, 2.80, 3.24,$$
$$3.26, 3.38, 3.45, 4.30, 4.55$$

The median is the mean of the two middle entries.

$$\frac{3.24 + 3.26}{2} = 3.25$$

The median price per bushel of wheat is \$3.25.

36. **(a)** Find the mean.

$$\overline{x} = \frac{\sum x}{n} = \frac{1{,}963{,}217}{15} = 130{,}881.13$$

Since the total pay is given in thousands of dollars, the mean total pay for this group of people is \$130,881,133.

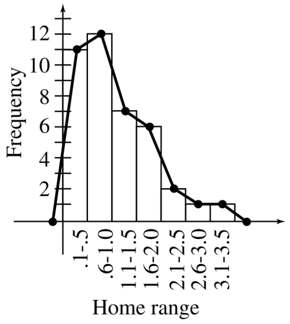
(b) The values are listed in order, so the median is the middle value, 86,534.
The median total pay for this group of people is \$86,534,000.

38. **(a)** Find the mean for grouped data.

$$\begin{aligned}\overline{x} &= \frac{\sum xf}{n}\\ &= \frac{(2500)(2037) + (7500)(4870) + \ldots + (87{,}500)(9740)}{2037 + 4870 + \ldots + 9740}\\ &= \frac{3{,}185{,}002{,}500}{75{,}973} \approx 41{,}923\end{aligned}$$

The estimated mean household income for full-time white Americans in 2000 is \$41,923.

40.



42. **(a)** From the histogram, the height of the bar representing 10-19 is 14.
Therefore, 14% of the population is estimated to be in the 10-19 age group.

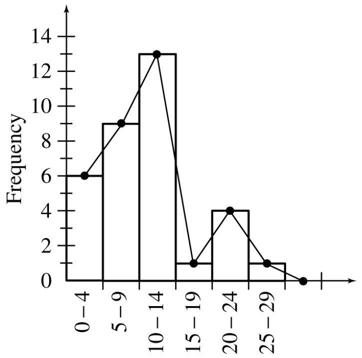
(b) From the histogram, the height of the bar representing 60-69 is 7.
Therefore, 7% of the population is estimated to be in the 60-69 age group.

(c) The tallest bars have height of 15.
These represent age groups 30-39 and 40-49.
Therefore, the age range 30-49 has the largest percent of the population.

44. **(a)**

Interval	Frequency
0-4	6
5-9	9
10-14	13
15-19	1
20-24	4
25-29	1

(b)



(c) For the original data, $\sum x = 355$, $n = 34$, and

$$\bar{x} = \frac{\sum x}{n} = \frac{355}{34} \approx 10.44.$$

(d)

Interval	Midpoint, x	Frequency, f	Product, xf
0-4	2	6	12
5-9	7	9	63
10-14	12	13	156
15-19	17	1	17
20-24	22	4	88
25-29	27	1	27
Totals:		34	363

The mean of this collection of grouped data is

$$\bar{x} = \frac{\sum xf}{n} = \frac{363}{34} \approx 10.68.$$

(f) Arrange the data in increasing order.

0, 3, 3, 4, 4, 4, 5, 5, 5, 6, 7, 7, 8 ,8,
8, 10, 10, 11, 11, 11, 11,11, 11, 12,
12, 13, 14, 14, 16, 20, 21, 21, 24, 25

There are 34 items. The mean of the two middle items is 10.5, so the median is 10.5.

The item with the greatest frequency is 11, so the mode is 11.

46. **(a)** $$\bar{x} = \frac{\sum x}{n}$$

$$= \frac{35+37+37+37+49+57+64+67+72+77}{10}$$

$$= \frac{532}{10} \approx 53$$

The mean is 53.

(b) The median is $\dfrac{49+57}{2} = 53.$

(c) The mode is the most frequent number, which is 37.

48. **(a)** Counting Gates, there would be 10,001 residents of the town. 10,000 residents have no personal wealth; 1 resident is worth \$10,000,000,000.

$$\bar{x} = \frac{10{,}000(0) + 1(10{,}000{,}000{,}000)}{10{,}001}$$

$$\bar{x} = 999{,}900.01$$

The mean wealth is roughly \$1 million.

(b) The median is \$0 since 0 is the middle value in the list of 10,001 values.

(c) The mode is \$0 since it occurs with the greatest frequency.

(d) The median or mode is most representative of the wealth of the town's population.

50. $a^0 = 1$ and $a^1 = a$, so the sequence can be written:

$$a^0, a^1, a^2, a^3, \ldots, a^n.$$

There are $n+1$ values and n is even. Therefore, there is an odd number of values. Since there is an odd number of values, the median is the value in the middle position. The middle value is in position $\frac{n-0}{2} = \frac{n}{2}$, which is a whole number since n is even.
The median is equal to $a^{n/2}$, which is choice (c).

9.2 Measures of Variation

2. For every set of numbers, the sum of the deviations from the mean equals zero.

4. $122, 132, 141, 158, 162, 169, 180$

The range is the difference between the largest and smallest numbers. Here the largest number is 180 and the smallest is 122. The range is

$$180 - 122 = 58.$$

To find the standard deviation, the first step is to find the mean.

$$\begin{aligned}\bar{x} &= \frac{\sum x}{n}\\ &= \frac{122+132+141+158+162+169+180}{7}\\ &= \frac{1064}{7} = 152\end{aligned}$$

Now complete the following table.

x	x^2
122	14,884
132	17,424
141	19,881
158	24,964
162	26,244
169	28,561
180	32,400
Total:	164,358

The total of the second column gives $\sum x^2 =$ 164,358. The variance is

$$\begin{aligned}s^2 &= \frac{\sum x^2 - n\bar{x}^2}{n-1}\\ &= \frac{164{,}358 - 7(152)^2}{7-1}\\ &= \frac{164{,}358 - 161{,}728}{6}\\ &= \frac{2630}{6} \approx 438.3,\end{aligned}$$

and the standard deviation is

$$s = \sqrt{438.3} \approx 20.9.$$

6. $51, 58, 62, 64, 67, 71, 74, 78, 82, 93$

The range is $93 - 51 = 42$. The mean is

$$\bar{x} = \frac{\sum x}{n} = \frac{700}{10} = 70.$$

x	x^2
51	2601
58	3364
62	3844
64	4096
67	4489
71	5041
74	5476
78	6084
82	6724
93	8649
Total:	50,368

The standard deviation is

$$\begin{aligned}s &= \sqrt{\frac{\sum x^2 - n\bar{x}^2}{n-1}}\\ &= \sqrt{\frac{50{,}368 - 10(70)^2}{9}}\\ &= \sqrt{152} \approx 12.3.\end{aligned}$$

8. $15, 42, 53, 7, 9, 12, 28, 47, 63, 14$

The range is $63 - 7 = 56$. The mean is

$$\bar{x} = \frac{290}{10} = 29.$$

x	x^2
15	225
42	1764
53	2809
7	49
9	81
12	144
28	784
47	2209
63	3969
14	196
Total:	12,230

The standard deviation is

$$\begin{aligned}s &= \sqrt{\frac{\sum x^2 - n\bar{x}^2}{n-1}}\\ &= \sqrt{\frac{12{,}230 - 10(29)^2}{9}}\\ &= \sqrt{424.4} \approx 20.6.\end{aligned}$$

10. Recall that when working with grouped data, x represents the midpoint of each interval. Complete the following table.

Interval	f	x	xf	x^2	fx^2
30-39	1	34.5	34.5	1190.25	1190.25
40-49	6	44.5	267.0	1980.25	11,881.50
50-59	13	54.5	708.5	2970.25	38,613.25
60-69	22	64.5	1419.0	4160.24	91,525.50
70-79	17	74.5	1266.5	5550.25	94,354.25
80-89	13	84.5	1098.5	7140.25	92,823.25
90-99	8	94.5	756.0	8930.25	71,442.00
Totals:	80		5550.0		401,830.00

Use the formulas for grouped frequency distributions to find the mean and then the standard deviation.

$$\bar{x} = \frac{\sum xf}{n} = \frac{5550}{80} = 69.375$$

$$\begin{aligned} s &= \sqrt{\frac{\sum fx^2 - n\bar{x}^2}{n-1}} \\ &= \sqrt{\frac{401{,}830 - 80(69.375)^2}{79}} \\ &= \sqrt{\frac{16{,}798.75}{79}} \\ &\approx \sqrt{212.64} \approx 14.6 \end{aligned}$$

12. Use $k = 4$ in Chebyshev's theorem.

$$1 - \frac{1}{k^2} = 1 - \frac{1}{4^2} = \frac{15}{16},$$

so at least $\frac{15}{16}$ of the numbers in the data set lie within 4 standard deviations of the mean.

14. We have $38 = 50 - 2 \cdot 6 = \mu - 2\sigma$, and $62 = 50 + 2 \cdot 6 = \mu + 2\sigma$, so Chebyshev's theorem applies with $k = 2$; hence, at least $1 - \frac{1}{k^2} = .75 = 75\%$ of the numbers lie between 38 and 62.

16. The answer here is the complement of the answer to Exercise 14. It was found that at least 75% of the distribution of numbers are between 38 and 62, so at most $100\% - 75\% = 25\%$ of the numbers are less than 38 or more than 62.

20.
$$\begin{aligned} s^2 &= \frac{\sum(x-\bar{x})^2}{n-1} \\ &= \frac{\sum(x^2 - 2x\bar{x} + \bar{x}^2)}{n-1} \\ &= \frac{\sum x^2 - \sum 2x\bar{x} + \sum \bar{x}^2}{n-1} \\ &= \frac{\sum x^2 - 2\bar{x}\sum x + n\bar{x}^2}{n-1} \\ &= \frac{\sum x^2 - 2\bar{x}(n\frac{\sum x}{n}) + n\bar{x}^2}{n-1} \\ &= \frac{\sum x^2 - 2\bar{x}(n\bar{x}) + n\bar{x}^2}{n-1} \\ &= \frac{\sum x^2 - 2n\bar{x}^2 + n\bar{x}^2}{n-1} \\ &= \frac{\sum x^2 - n\bar{x}^2}{n-1} \end{aligned}$$

22. **(a)** $\bar{x} = \frac{\sum x}{n} = \frac{18 + 15 + 7 + 10}{4} = 12.5$

(b) $\bar{x} = \frac{\sum x}{n} = \frac{1 + (-8) + (-5) + 0}{4} = -3.0$

(c)

x	x^2
+18	324
+15	225
+7	49
+10	100
Total:	698

$$\begin{aligned} s &= \sqrt{\frac{\sum x^2 - n\bar{x}^2}{n-1}} \\ &= \sqrt{\frac{698 - 4(12.5)^2}{3}} \\ &\approx \sqrt{24.33} \approx 4.9 \end{aligned}$$

(d)

x	x^2
+1	1
−8	64
−5	25
0	0
Total:	90

$$\begin{aligned} s &= \sqrt{\frac{\sum x^2 - n\bar{x}^2}{n-1}} \\ &= \sqrt{\frac{90 - 4(-3.0)^2}{3}} \\ &= \sqrt{18} \approx 4.2 \end{aligned}$$

(e) $12.5 - (-3) = 15.5$

(f) Low: $15.5 - 7.95 = 7.55$
High: $15.5 + 7.95 = 23.45$

24.

Sample Number	Sample Mean	Sample Standard Deviation
1	0	4.36
2	−.33	3.21
3	1.00	1.00
4	.67	4.51
5	.33	3.21
6	−1.67	3.79

Use the results from Exercise 23(e) and (f). The upper and lower control limits for the standard means are 4.41 and −2.51, while the upper and lower control limits for the sample deviations are 4.31 and 0. The table shows that for the data given in this exercise, all sample means fall within the control limits.

However, two of the sample standard deviations, those for samples 1 and 4, exceed the upper control limit of 4.31. Therefore, the process is out of control.

26. (a) $\bar{x} = \frac{\sum x}{n} = \frac{72.42}{10} = 7.242$

Unemployment is closest to the mean 7.242 in 1996 when it was 7.24.

(b) Use a graphing calculator with statistical keys to help find the standard deviation.

$$\begin{aligned} s &= \sqrt{\frac{\sum x^2 - n\bar{x}^2}{n-1}} \\ &= \sqrt{\frac{539.4826 - 10(7.242)^2}{10-1}} \\ &= \sqrt{\frac{15.01696}{9}} \approx 1.292 \end{aligned}$$

(c) $\bar{x} + s = 7.242 + 1.292 = 8.534$
$\bar{x} - s = 7.242 - 1.292 = 5.95$

Six of the years fall between these two values, that is, are within 1 standard deviation of the mean.

(d) $\bar{x} + 3s = 7.242 + 3(1.292) = 11.118$
$\bar{x} - 3s = 7.242 - 3(1.292) = 3.366$

All ten of the years fall between these two values, that is, are within 3 standard deviations of the mean.

28. (a) Use a graphing calculator or a spreadsheet to find the variance and standard deviation.

$$\begin{aligned} s^2 &= 14.76 \\ s &= \sqrt{14.76} = 3.84 \end{aligned}$$

(b) $\overline{x} + s = 7.38 + 3.84 = 11.22$
$\overline{x} - s = 7.38 - 3.84 = 3.54$

Ten of the animals have blood types that are within 1 standard deviation of the mean.

30. (a) $\bar{x} = \frac{\sum x}{n} = \frac{1627}{11} \approx 148$

The mean salary is $148,000. The salary of governors of Illinois and New Mexico is $150,000, so Illinois and New Mexico have governors with a salary closest to the mean.

(b) Use a graphing calculator or spreadsheet to find the standard deviation.

$$s \approx 20.7$$

The standard deviation is $20,700.

(c) $\bar{x} + s = 148 + 20.7 = 168.7$
$\bar{x} - s = 148 - 20.7 = 127.3$

Five of the 11 salaries fall between these two values. Thus,

$$\frac{5}{11}(100) \approx 45\%$$

of the governors have salaries within 1 standard deviation of the mean.

(d) $\bar{x} + 3s = 148 + 3(20.7) = 210.1$
$\bar{x} - 3s = 148 - 3(20.7) = 85.9$

All 11 salaries fall between these two values, so 100% of the governors have salaries within 3 standard deviations of the mean.

32. Use a graphing calculator to find values.

(a) $\overline{x} = 2.898$ g
maximum = 3.8 g
minimum = 2.1 g
$s = .374$ g

(b) $\overline{x} = 5.756$ g
maximum = 6.9 g
minimum = 3.3 g
$s = .709$ g

(c) $\overline{x} + 25 = 5.756 + 2(.709) = 7.174$
$\overline{x} - 25 = 5.756 - 2(.709) = 4.338$

There are no data values, 0%, of traditional Oreo cookies within 2 standard deviations of the Double Stuf Oreo mean.

(d) There are 46 data values, $\frac{46}{49} \approx 94\%$, of the traditional Oreo cookies (when multiplied by 2) that are within 2 standard deviations of the Double Stuf Oreo mean.

9.3 The Normal Distribution

2. The total area under the normal curve (above the horizontal axis) is 1.

6. Use the table "Area Under a Normal Curve to the Left of z" in the Appendix. To find the percent of the area under a normal curve between the mean and .81 standard deviation from the mean, subtract the table entry for $z = 0$ (representing the mean) from the table entry for $z = .81$.

$$.7910 - .5000 = .2910$$

Therefore, 29.10% of the area lies between μ and $\mu + .81\sigma$.

8. By the symmetry of the normal curve, the area between $z = -2.04$ and the mean is the same as the area between the mean and $z = 2.04$. Proceeding as before, we obtain

$$.9793 - .5000 = .4793 = 47.93\%.$$

Note: Since the table in the text gives areas for the negative values of z (often omitted), this answer could also have been found by subtracting the area below $z = -2.04$ from the area below the mean:

$$.5000 - .0207 = .4793 = 47.93\%.$$

10. Between $z = .64$ and $z = 2.11$

$$\begin{aligned} &P(.64 \le z \le 2.11)\\ &\quad = P(z \le 2.11) - P(z \le .64)\\ &\quad = (\text{area to the left of } z = 2.11)\\ &\qquad - (\text{area to the left of } z = .64)\\ &\quad = .9826 - .7389\\ &\quad = .2437 \quad \text{or} \quad 24.37\% \end{aligned}$$

12. Between $z = -1.74$ and $z = -1.02$

$$\begin{aligned} &P(-1.74 \le z \le -1.02)\\ &\quad = P(z \le -1.02) - P(z \le -1.74)\\ &\quad = (\text{area to the left of } z = -1.02)\\ &\qquad - (\text{area to the left of } z = -1.74)\\ &\quad = .1539 - .0409\\ &\quad = .1130 \quad \text{or} \quad 11.3\% \end{aligned}$$

14. Between $z = -2.94$ and $z = .43$

$$\begin{aligned} &P(-2.94 \le z \le .43)\\ &\quad = P(z \le .43) - P(z \le -2.94)\\ &\quad = (\text{area to the left of } .43)\\ &\qquad - (\text{area to the left of } -2.94)\\ &\quad = .6664 - .0016\\ &\quad = .6648 \quad \text{or} \quad 66.48\% \end{aligned}$$

16. 1% of the total area is to the left of z.

Use the table backwards, looking for the value closest to .0100, and find the corresponding z using the left column and top column of the table.

The closest value to .0100 in the body of the table is .0099, which corresponds to $z = -2.33$.

18. 25% of the total area is to the right of z.

Having 25% of the area to the right of z means that 75% is to the left. The closest value to .75 in the body of the table is .7486, which corresponds to $z = .67$.

20. According to Chebyshev's theorem, the probability that a number will lie within 2 standard deviations of the mean of a probability distribution is at least

$$1 - \frac{1}{2^2} = 1 - \frac{1}{4} = \frac{3}{4} = .75.$$

Using the normal distribution, the probability that a number will lie within 2 standard deviations of the mean is .954.
These values are not contradictory, since "at least .75" means .75 or more. For the normal distribution, the value is more.

In Exercises 22-28, let x represent the life of a light bulb.

$$\mu = 500,\ \sigma = 100$$

22. At least 500 hr means $x \geq 500$.

$$P(x \geq 500) = 1 - P(x < 500)$$

Find the z-score that corresponds to $x = 500$.

$$z = \frac{500 - 500}{100} = 0$$

$$\begin{aligned} P(x \geq 500) &= P(z \geq 0) \\ &= 1 - .5000 = .5000 \end{aligned}$$

Thus, 10,000(.5) = 5000 bulbs can be expected to last at least 500 hr.

24. Between 650 and 780 hr

For $x = 650$,

$$z = \frac{650 - 500}{100} = 1.5.$$

For $x = 780$,

$$z = \frac{780 - 500}{100} = 2.8.$$

$$\begin{aligned} P(650 \leq x \leq 780) &= P(1.5 \leq z \leq 2.8) \\ &= P(z \leq 2.8) - P(z \leq 1.5) \\ &= .9974 - .9332 \\ &= .0642 \end{aligned}$$

Thus, 10,000(.0642) = 642 bulbs can be expected to last between 650 and 780 hr.

26. Less than 740 hr

For $x = 740$,

$$z = \frac{740 - 500}{100} = 2.4.$$

$$P(x < 740) = P(z < 2.4) = .9918$$

Thus, 10,000(.9918) = 9918 bulbs can be expected to last less than 740 hr.

28. In the standard normal distribution, we must first find the z-values which bound the middle 80%, that is, we must find z_1 and z_2 such that

$$P(z < z_1) = .1000 \text{ and } P(z < z_2) = .9000.$$

Read backwards in the table, finding the closest values,

$$z_1 = -1.28 \text{ and } z_2 = 1.28.$$

Now find x_1 and x_2.

$$\begin{aligned} \frac{x_1 - 500}{100} &= -1.28 \quad \text{and} \quad & \frac{x_2 - 500}{100} &= 1.28 \\ x_1 - 500 &= -128 & x_2 - 500 &= 128 \\ x_1 &= 372 \quad \text{and} & x_2 &= 628 \end{aligned}$$

Thus, the shortest and longest lengths of life for the middle 80% of the bulbs are 372 hr and 628 hr.

In Exercises 30 and 32, let x represent the weight of a package.

30. For $x = 16$, $\mu = 16.5$, and $\sigma = .3$,

$$z = \frac{16 - 16.5}{.3} = -1.67.$$

Hence,

$$\begin{aligned} &P(x < 16) \\ &\quad = P(z < -1.67) \\ &\quad = \text{area to the left of } z = -1.67 \\ &\quad = .0475. \end{aligned}$$

The fraction of the boxes that are underweight is .048.

32. This is the same as Exercise 30 except that $\sigma = .1$, so

$$z = \frac{16 - 16.5}{.1} = -5.$$

Hence,

$$\begin{aligned} &P(x < 16) \\ &\quad = P(z < -5) \\ &\quad = \text{area to the left of } z = -5 \\ &\quad = .0000. \end{aligned}$$

None of the boxes are underweight.

In Exercises 34-38, let x represent the weight of a chicken.

34. Less than 1800 g

For $x = 1800$, we have

$$z = \frac{1800 - 1850}{150} = -.33.$$

Thus,

$$\begin{aligned} &P(x < 1800) \\ &\quad = P(z < -.33) \\ &\quad = \text{area to the left of } z = -.33 \\ &\quad = .3707. \end{aligned}$$

Therefore, 37.07% of the chickens will weigh less than 1800 g.

36. Between 1600 and 2000 g

For $x = 1600$,

$$z = \frac{1600 - 1850}{150} = -1.67,$$

and for $x = 2000$,

$$z = \frac{2000 - 1850}{150} = 1.$$

Thus,

$$\begin{aligned} &P(1600 \le x \le 2000) \\ &\quad = P(-1.67 \le z \le 1) \\ &\quad = \text{area between } z = -1.67 \text{ and } z = 1 \\ &\quad = .8413 - .0475 \\ &\quad = .7938. \end{aligned}$$

Therefore, 79.38% of the chickens will weigh between 1600 and 2000 g.

38. Let $-k$ and k be the two z-values such that the middle 95% of the distribution falls between them.

$$P(z < -k) = .0250 \text{ and } P(z < k) = .9750.$$

Reading the table backwards, $k = 1.96$. Find the corresponding x-values.

$$\begin{aligned} \frac{x - 1850}{150} &= -1.96 \quad \text{or} \quad & \frac{x - 1850}{150} &= 1.96 \\ x - 1850 &= -294 & x - 1850 &= 294 \\ x &= 1556 \quad \text{or} & x &= 2144 \end{aligned}$$

Therefore, the middle 95% of the chickens have weights between 1556 g and 2144 g.

40. Let x represent the number of ounces of milk in a carton.

$$\mu = 32.2, \ \sigma = 1.2$$

Find the z-score for $x = 32$.

$$z = \frac{x - \mu}{\sigma} = \frac{32 - 32.2}{1.2} \approx -.17$$

$$\begin{aligned} P(x < 32) &= P(z < -.17) \\ &= \text{ area to the left of } z = -\,.17 \\ &= .4325 \end{aligned}$$

The probability that a filled carton will contain less than 32 oz is .4325.

42. Let x represent the weight of an egg (in ounces).

$$\mu = 1.5, \ \sigma = .4$$

Find the z-score for $x = 2.2$.

$$z = \frac{x - \mu}{\sigma} = \frac{2.2 - 1.5}{.4} = 1.75,$$

so

$$\begin{aligned} &P(x > 2.2) \\ &\quad = P(z > 1.75) \\ &\quad = \text{area to the right of } z = 1.75 \\ &\quad = 1 - (\text{area to the left of } z = 1.75) \\ &\quad = 1 - .9599 \\ &\quad = .0401. \end{aligned}$$

Thus, out of 5 dozen eggs, we expect $.0401(60) = 2.406$ eggs, or about 2, to be graded extra large.

44. The Recommended Daily Allowance is

$$\begin{aligned} \mu + 2.5\sigma &= 1800 + 2.5(140) \\ &= 2150 \text{ units.} \end{aligned}$$

46. The Recommended Daily Allowance is

$$\begin{aligned} \mu + 2.5\sigma &= 1200 + 2.5(92) \\ &= 1430 \text{ units.} \end{aligned}$$

48. Let x represent the length of a fish.

$$\mu = 12.3, \ \sigma = 4.1$$

Find the z-score for $x = 18$.

$$z = \frac{18 - 12.3}{4.1} \approx 1.39$$

$$\begin{aligned} P(x > 18) &= 1 - P(x \le 18) \\ &= 1 - P(z \le 1.39) \\ &= 1 - .9177 \\ &= .0823 \end{aligned}$$

50. Let x represent a driving speed.

$$\mu = 30, \ \sigma = 5$$

Read the table backwards. In the body of the table, the closest area to .85 is .8508, which corresponds to $z = 1.04$. Find the value of x that corresponds to $z = 1.04$.

$$\begin{aligned} z &= \frac{x - \mu}{\sigma} \\ 1.04 &= \frac{x - 30}{5} \\ 5.2 &= x - 30 \\ 35.2 &= x \end{aligned}$$

The 85th percentile speed is 35.2 mph.

52. For a grade of B, we have

$$\frac{1}{2} \leq z \leq \frac{3}{2}.$$

The area between $z = .5$ and $z = 1.5$ is

$$.9332 - .6915 = .2417 \text{ or } 24.17\%,$$

which should be the percent of students to receive B's.

54. This system would be more likely to be fair for the freshman psychology class since a large group of students is more likely to produce a normal distribution of total points.

In Exercises 56 and 58, let x represent a student's test score.

56. Since the top 8% of the students get A's and the next 15% get B's, we want to find the number b for which

$$\begin{aligned} &P(x \geq b) = .23 \\ \text{or} \quad &P(x \leq b) = .77. \end{aligned}$$

Read the table backwards to find the z-score for an area of .77, which is $z = .74$. Find the value of x that corresponds to $z = .74$.

$$\begin{aligned} z &= \frac{x - \mu}{\sigma} \\ .74 &= \frac{x - 74}{6} \\ x &= 78.44 \end{aligned}$$

The bottom cutoff score for a B should be 78.

58. Since 8% of the students get F's, the minimum passing score (cutoff for a D) would be the number d for which

$$P(x \leq d) = .08.$$

Read the table backwards to find the z-score for an area of .08, which is $z = -1.41$. Find the value of x that corresponds to $z = -1.41$.

$$\begin{aligned} z &= \frac{x - \mu}{\sigma} \\ -1.41 &= \frac{x - 74}{6} \\ x &= 65.54 \end{aligned}$$

The bottom cutoff score for a D should be 66.

60. Let x represent the height of a man.

$$\mu = 69.60, \ \sigma = 3.20$$

For $x = 66.27$,

$$\begin{aligned} z &= \frac{66.27 - 69.60}{3.20} \\ &= -1.04. \end{aligned}$$

$$P(x < 66.27) = P(z < -1.04) = .1492$$

62. **(a)** $\mu = 6.9, \sigma = 4.6$

Find the z-score for $x = 6.0$.

$$\begin{aligned} z &= \frac{x - \mu}{\sigma} = \frac{6.0 - 6.9}{4.6} \\ &\approx -.20 \end{aligned}$$

Thus,

$$\begin{aligned} P(x \geq 6.0) &= P(z \geq -.20) \\ &= \text{area to the right of } z = -.20. \\ &= 1 - .4207 \\ &= .5793 \approx .58 \end{aligned}$$

(c) $\mu = .6, \sigma = .3$

Find the z-score for $x = 6.0$.

$$z = \frac{6.0 - .6}{.3} = 18$$

Thus,

$$\begin{aligned} P(x \geq 6.0) &= P(z \geq 18) \\ &= \text{area to the right of } z = 18 \\ &\approx 0 \end{aligned}$$

No, the probability is about 0.

64.

	Reference	Football
a. head size:	$z = \dfrac{53.0 - 53.7}{2.9} = -.24$ $P(x \geq 53.0) = P(z \geq -.24) = 1 - .4052 = .5948$	$z = \dfrac{53.0 - 52.1}{2.3} = .39$ $P(x \geq 53.0) = P(z \geq .39) = 1 - .6517 = .3483$
b. neck size:	$z = \dfrac{32.1 - 34.2}{1.9} = -1.11$ $P(x \leq 32.1) = P(z \leq -1.11) = .1335$	$z = \dfrac{32.1 - 34.6}{1.8} = -1.39$ $P(x \leq 32.1) = P(z \leq -1.39) = .0823$
c. chest size:	$z = \dfrac{75.0 - 91.2}{4.8} = -3.38$ $P(x \leq 75.0) = P(z \leq -3.38) = .0004$	$z = \dfrac{75.0 - 92.3}{3.5} = -4.94$ $P(x \leq 75.0) = P(z \leq -4.94) \approx 0$
d. upper arm size:	$z = \dfrac{27.1 - 28.8}{2.2} = -.77$ $P(x \leq 27.1) = P(z \leq -.77) = .2206$	$z = \dfrac{27.1 - 29.9}{1.9} = -1.47$ $P(x \leq 27.1) = P(z \leq -1.47) = .0708$
e. waist size:	$z = \dfrac{56.5 - 80.9}{9.8} = -2.49$ $P(x \leq 56.5) = P(z \leq -2.49) = .0064$	$z = \dfrac{56.5 - 75.1}{3.6} = -5.17$ $P(x \leq 56.5) = P(z \leq -5.17) \approx 0$

9.4 Normal Approximation to the Binomial Distribution

2. The normal distribution can be used to approximate a binomial distribution as long as $np \geq 5$ and $n(1-p) \geq 5$.

4. Let x represent the number of heads tossed.
For this experiment, $n = 16, x = 7$, and $p = \frac{1}{2}$.

(a) $P(x = 7) = \binom{16}{7}\left(\frac{1}{2}\right)^7\left(1 - \frac{1}{2}\right)^9 \approx .1746$

(b) $\mu = np = 16\left(\frac{1}{2}\right) = 8$

$\sigma = \sqrt{np(1-p)} = \sqrt{16\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)} = \sqrt{4} = 2$

For $x = 6.5$,

$$z = \frac{6.5 - 8}{2} = -.75.$$

For $x = 7.5$,

$$z = \frac{7.5 - 8}{2} = -.25.$$

$P(-.75 < z < -.25) = P(z < -.25) - P(z < -.75) = .4013 - .2266 = .1747$

6. Let x represent the number of tails tossed. For this experiment, $n = 16; x = 0, 1, 2, 3,$ or 4; and $p = \frac{1}{2}$.

(a) $P(x = 0, 1, 2, 3, \text{ or } 4)$

$$\begin{aligned}
&= \binom{16}{0}\left(\frac{1}{2}\right)^0\left(1-\frac{1}{2}\right)^{16} \\
&\quad + \binom{16}{1}\left(\frac{1}{2}\right)^1\left(1-\frac{1}{2}\right)^{15} \\
&\quad + \binom{16}{2}\left(\frac{1}{2}\right)^2\left(1-\frac{1}{2}\right)^{14} \\
&\quad + \binom{16}{3}\left(\frac{1}{2}\right)^3\left(1-\frac{1}{2}\right)^{13} \\
&\quad + \binom{16}{4}\left(\frac{1}{2}\right)^4\left(1-\frac{1}{2}\right)^{12} \\
&\approx .00001 + .00024 + .00183 + .00854 \\
&\quad + .02777 \\
&\approx .0384
\end{aligned}$$

(b) $\mu = np = 16\left(\frac{1}{2}\right) = 8$

$$\sigma = \sqrt{np(1-p)} = \sqrt{16\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)} = \sqrt{4} = 2$$

For $x = 4.5$,

$$z = \frac{4.5 - 8}{2} = -1.75.$$

$P(x < -1.75) = .0401$

8. Let x represent the number of heads tossed.

$n = 1000, p = \frac{1}{2}$

We have

$$\mu = np = 1000\left(\frac{1}{2}\right) = 500$$

and

$$\begin{aligned}
\sigma = \sqrt{np(1-p)} &= \sqrt{1000\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)} \\
&= \sqrt{250} \approx 15.81.
\end{aligned}$$

Exactly 510 heads corresponds to the area under the normal curve between $x = 509.5$ and $x = 510.5$. The corresponding z-scores are

$$z = \frac{509.5 - 500}{15.81} \approx .60$$

and

$$z = \frac{510.5 - 500}{15.81} \approx .66.$$

$$\begin{aligned}
P(\text{exactly 510 heads}) &= P(509.5 \le x \le 510.5) \\
&= P(.60 \le z \le .66) \\
&= P(z \le .66) - P(z \le .60) \\
&= .7454 - .7257 \\
&= .0197
\end{aligned}$$

10. Let x represent the number of tails tossed.

$n = 1000, p = \frac{1}{2}$

As in Exercise 8, $\mu = 500$ and $\sigma = 15.81$.

Less than 470 tails corresponds to the area under the normal curve to the left of $x = 469.5$. The corresponding z-score is

$$z = \frac{469.5 - 500}{15.81} \approx -1.93.$$

$$\begin{aligned}
P(\text{less than 470 tails}) &= P(x \le 469.5) \\
&= P(z \le -1.93) \\
&= .0268
\end{aligned}$$

12. Let x represent the number of 6's tossed.

$n = 120, p = \frac{1}{6}$

$$\mu = np = 120\left(\frac{1}{6}\right) = 20$$

$$\sigma = \sqrt{np(1-p)} = \sqrt{120\left(\tfrac{1}{6}\right)\left(\tfrac{5}{6}\right)} \approx 4.08$$

We want the area between $x = 23.5$ and $x = 24.5$. The corresponding z-scores are .86 and 1.103.

$$\begin{aligned}
&P(\text{exactly twenty-four 6's}) \\
&\quad = P(23.5 \le x \le 24.5) \\
&\quad = P(.86 \le z \le 1.103) \\
&\quad = P(z \le 1.103) - P(z \le .86) \\
&\quad = .8643 - .8051 \\
&\quad = .0592
\end{aligned}$$

14. Let x represent the number of 6's tossed.

$n = 120, p = \frac{1}{6}$

As in Exercise 12, $\mu = 20$ and $\sigma \approx 4.08$.
We want the area to the left of $x = 21.5$. The corresponding z-score is .37.

$$\begin{aligned}
&P(\text{fewer than twenty-two 6's}) \\
&\quad = P(x \le 21.5) \\
&\quad = P(z \le .37) \\
&\quad = .6443
\end{aligned}$$

16. **(a)** Let x represent the number of heaters that are defective.

$$n = 10{,}000,\ p = .02$$

$$\mu = np = 10{,}000(.02) = 200$$
$$\sigma = \sqrt{np(1-p)} = \sqrt{10{,}000(.02)(.98)} = 14$$

To find P(fewer than 170), find the z-score for $x = 169.5$.

$$z = \frac{169.5 - 200}{14} \approx -2.18$$

P(fewer than 170) $= .0146$

(b) Let x represent the number of heaters that are defective.

$n = 10{,}000,\ p = .02,\ \mu = np = 200$

$$\sigma = \sqrt{np(1-p)} = \sqrt{10{,}000(.02)(.98)} = 14$$

We want the area to the right of $x = 222.5$. For $x = 222.5$,

$$z = \frac{222.5 - 200}{14} \approx 1.61.$$

$$\begin{aligned} P(\text{more than 222 defects}) &= P(x \geq 222.5) \\ &= P(z \geq 1.61) \\ &= 1 - P(z \leq 1.61) \\ &= 1 - .9463 \\ &= .0537 \end{aligned}$$

18. Use a calculator or computer to complete this exercise. The answers are given.

(a) P(all 58 like it) $= 1.04 \times 10^{-9} \approx 0$

(b) P(exactly 28, 29, or 30 like it) $= .0018$

20. Let x be the number of nests escaping predation.

$n = 26, p = .3$

$$\mu = np = 26(.3) = 7.8$$

$$\begin{aligned} \sigma &= \sqrt{np(1-p)} = \sqrt{26(.3)(.7)} \\ &= \sqrt{5.46} \approx 2.337 \end{aligned}$$

To find P(at least half escape predation), find the z-score for $x = 12.5$.

$$z = \frac{12.5 - 7.8}{2.337} \approx 2.01$$

$P(z > 2.01) = 1 - .9778 = .0222$

22. Let x represent the number of hospital patients struck by falling coconuts.

(a) $n = 20; x = 0$ or 1; and $p = .025$

$$\begin{aligned} P(x = 0 \text{ or } 1) &= \binom{20}{0}(.025)^0(.975)^{20} \\ &\quad + \binom{20}{1}(.025)^1(.975)^{19} \\ &\approx .60269 + .30907 \\ &\approx .9118 \end{aligned}$$

(b) $n = 2000; x = 0, 1, 2, \ldots,$ or 70; $p = .025$

$$\mu = np = 2000(.025) = 50$$

$$\sigma = \sqrt{np(1-p)} = \sqrt{2000(.025)(.975)} = \sqrt{48.75}$$

To find P(70 or less), find the z-score for $x = 70.5$.

$$z = \frac{70.5 - 50}{\sqrt{48.75}} \approx 2.94$$

$P(x < 2.94) = .9984$

24. **(a)** Let x represent the number of people cured.

$n = 25, p = .80$

$$\mu = np = 25(.80) = 20$$

$$\sigma = \sqrt{np(1-p)} = \sqrt{25(.80)(.20)} = 2$$

To find

$$P(\text{exactly 20 cured}) = P(19.5 \leq x \leq 20.5),$$

find the z-scores for $x = 19.5$ and $x = 20.5$.

For $x = 19.5$,

$$z = \frac{19.5 - 20}{2} = -.25.$$

For $x = 20.5$,

$$z = \frac{20.5 - 20}{2} = .25.$$

Using the table,

$$\begin{aligned} P(\text{exactly 20 cured}) &= .5987 - .4013 \\ &= .1974. \end{aligned}$$

(b) Let x represent the number of people who are cured.

$n = 25, p = .8$

$$\mu = np = 25(.8) = 20$$
$$\sigma = \sqrt{np(1-p)} = \sqrt{25(.8)(.2)} = \sqrt{4} = 2$$

To find P(all are cured) = P(25 are cured), find the z-scores for $x = 24.5$ and $x = 25.5$.

$$z = \frac{24.5 - 20}{2} = 2.25 \qquad z = \frac{25.5 - 20}{2} = 2.75.$$

Using the table,

$$\begin{aligned} P(\text{all are cured}) &= .9970 - .9878 \\ &= .0092. \end{aligned}$$

(c) $P(x = 0) = \binom{25}{0}(.80)^0(.20)^{25}$

$$\begin{aligned} &= (.20)^{25} \\ &= 3.36 \times 10^{-18} \\ &\approx 0 \end{aligned}$$

(d) From parts a and b, $\mu = 20$ and $\sigma = 2$.

To find P(12 or fewer are cured) = $P(x \le 12)$, find the z-score for $x = 12.5$.

$$z = \frac{12.5 - 20}{2} = -3.75$$

Since the table does not go out to $z = -3.75$, we must extrapolate, that is, read beyond the values in the table.

$$P(x \le 12) \approx .0001$$

26. **(a)** Let x represent the number that are AB—.
$n = 1000$; $x = 10, 11, 12, \ldots, 1000$; $p = .006$
$\mu = np = 1000(.006) = 6$

$$\begin{aligned} \sigma &= \sqrt{np(1-p)} \\ &= \sqrt{1000(.006)(.994)} \\ &\approx 2.442 \end{aligned}$$

To find P(10 or more), find the z-score for $x = 9.5$.

$$z = \frac{9.5 - 6}{2.442} = 1.43$$

$P(z > 1.43) = 1 - .9236 = .0764$

(b) Let x represent the number that are B−.
$n = 1000$; $x = 20, 21, \ldots, 39, 40$; $p = .015$
$\mu = np = 1000(.015) = 15$

$$\begin{aligned} \sigma &= \sqrt{np(1-p)} \\ &= \sqrt{1000(.015)(.985)} \\ &\approx 3.844 \end{aligned}$$

To find P(between 20 to 40 inclusive), find the z-scores for $x = 19.5$ and $x = 40.5$.

$$z = \frac{19.5 - 15}{3.844} = 1.17, z = \frac{40.5 - 15}{3.844} = 6.63$$

$$\begin{aligned} P(1.17 < z < 6.63) &= P(z < 6.63) - P(z < 1.17) \\ &= 1 - .8790 \\ &= .1210 \end{aligned}$$

(c) $\mu = np = 500(.015) = 7.5$

$$\begin{aligned} \sigma &= \sqrt{np(1-p)} \\ &= \sqrt{500(.015)(.985)} \\ &\approx 2.718 \end{aligned}$$

To find P(15 or more donors being B−), find the z-score for $x = 14.5$.

$$z = \frac{14.5 - 7.5}{2.718} = 2.57$$

$$P(z > 2.57) = 1 - .9949 = .0051$$

The probability that 15 or more donors are B− is only .0051.

28. $n = 1400, p = .55$

$$\mu = np = 1400(.55) = 770$$
$$\sigma = \sqrt{np(1-p)} = \sqrt{1400(.55)(.45)} \approx 18.6$$

To find P(at least 750 people) = $P(x \ge 749.5)$, find the z-score for $x = 749.5$.

$$z = \frac{749.5 - 770}{18.6} \approx -1.10$$

$$\begin{aligned} P(z \ge -1.10) &= 1 - P(z \le -1.10) \\ &= 1 - .1357 \\ &= .8643 \end{aligned}$$

30. **(a)** The numbers are too large for the calculator to handle.

(b) $n = 5{,}825{,}043$, $p = .5$
$\mu = np = 5{,}825{,}043(.5) = 2{,}912{,}522$

$$\begin{aligned} \sigma &= \sqrt{np(1-p)} \\ &= \sqrt{5{,}825{,}043(.5)(.5)} \\ &\approx 1206.8 \end{aligned}$$

$$z = \frac{2{,}912{,}253 - 2{,}912{,}522}{1206.8} = -.22$$
$$z = \frac{2{,}912{,}790 - 2{,}912{,}522}{1206.8} = .22$$

$$\begin{aligned} &P(2{,}912{,}253 \le x \le 2{,}912{,}790) \\ &\quad = P(-.22 \le z \le .22) \\ &\quad = P(z \le .22) - P(z \le -.22) \\ &\quad = .5871 - .4129 \\ &\quad = .1742 \end{aligned}$$

32. Let x represent the number of questions.

$n = 100; x = 60, 61, 62, \ldots,$ or 100; $p = \frac{1}{2}$

$$\mu = np = 180\left(\frac{1}{2}\right) = 50$$

$$\sigma = \sqrt{np(1-p)} = \sqrt{100\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}$$

$$= \sqrt{25} = 5$$

To find $P(60 \text{ or more correct})$, find the z-score for $x = 59.5$.

$$z = \frac{59.5 - 50}{5} = 1.90$$

$$\begin{aligned} P(z > 1.90) &= 1 - P(z \leq 1.90) \\ &= 1 - .9713 \\ &= .0287 \end{aligned}$$

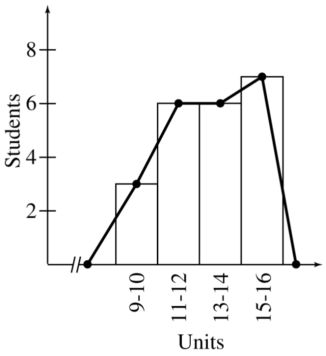
Chapter 9 Review Exercises

2. In a grouped frequency distribution, there should be from 6 to 15 intervals.

4. (a)

Interval	Frequency
9-10	3
11-12	6
13-14	6
15-16	7

(b)-(c)



6. $105, 108, 110, 115, 106, 110, 104, 113, 117$

$$\bar{x} = \frac{\sum x}{n} = \frac{988}{9} \approx 109.8$$

8.

Interval	Midpoint, x	Frequency, f	Product, xf
40-44	42	2	84
45-49	47	5	235
50-54	52	7	364
55-59	57	10	570
60-64	62	4	248
65-69	67	1	67
Totals:		29	1568

Use the formula for the mean of a grouped frequency distribution.

$$\bar{x} = \frac{\sum xf}{n} = \frac{1568}{29} \approx 54.1$$

10. $32, 35, 36, 44, 46, 46, 59$

The median is the middle number; in this case, it is 44.

The mode is the most frequent number; in this case, it is 46.

12. The modal class is the interval with the greatest frequency; in this case, the modal class is 30-39.

14. The range of a distribution is the difference between the largest and smallest data items.

16. $14, 17, 18, 19, 32$

The range is the difference between the largest and smallest numbers. For this distribution, the range is

$$32 - 14 = 18.$$

To find the standard deviation, the first step is to find the mean.

$$\bar{x} = \frac{\sum x}{n} = \frac{14 + 17 + 18 + 19 + 32}{5} = \frac{100}{5} = 20$$

Now complete the following table.

x	x^2
14	196
17	289
18	324
19	361
32	1024
Total:	2194

$$s = \sqrt{\frac{\sum x^2 - n\bar{x}^2}{n-1}}$$

$$= \sqrt{\frac{2194 - 5(20)^2}{4}}$$

$$= \sqrt{48.5} \approx 7.0$$

18. Recall that when working with grouped data, x represents the midpoint of each interval. Complete the following table.

Interval	f	x	xf	x^2	fx^2
10-19	6	14.5	87.0	210.25	1261.50
20-29	12	24.5	294.0	600.25	7203.00
30-39	14	34.5	483.0	1190.25	16,663.50
40-49	10	44.5	445.0	1980.25	19,802.50
50-59	8	54.5	436.0	2970.25	23,762.00
Totals:	50		1745.0		68,692.50

Use the formulas for grouped frequency distributions to find the mean and then the standard deviation. (The mean was also calculated in Exercise 7.)

$$\bar{x} = \frac{\sum xf}{n} = \frac{1745}{50} = 34.9$$

$$\begin{aligned} s &= \sqrt{\frac{\sum fx^2 - n\bar{x}^2}{n-1}} \\ &= \sqrt{\frac{68{,}692.5 - 50(34.9)^2}{49}} \\ &\approx 12.6 \end{aligned}$$

22. Between $z = 0$ and $z = 1.27$

$$\begin{aligned} P(0 \le z \le 1.27) &= .8980 - .5000 \\ &= .3980 \end{aligned}$$

24. Between $z = -1.88$ and $z = 2.10$

$$\begin{aligned} P(-1.88 \le z \le 2.10) &= .9821 - .0301 \\ &= .9520 \end{aligned}$$

26. If 8% of the area under the curve is to the right of z, then 92% of the area is to the left of z. Find the value closest to .92 in the body of the table, and read the table backwards to find the corresponding z-score, which is $z = 1.41$.

28. **(a)** Let x represent the number of hearts drawn.

$n = 1{,}000{,}000$; $x = 249{,}500, 249{,}501, \ldots, 251{,}000$;

$p = \frac{13}{52} = \frac{1}{4}$

$$\mu = np = 1{,}000{,}000\,(1/4) = 250{,}000$$

$$\begin{aligned} \sigma &= \sqrt{np(1-p)} = \sqrt{1{,}000{,}000\left(\tfrac{1}{4}\right)\left(\tfrac{3}{4}\right)} \\ &= \sqrt{\frac{3{,}000{,}000}{16}} = \frac{1000\sqrt{3}}{4} = 250\sqrt{3} \end{aligned}$$

To find P(between 249,500 and 251,000), find the z-scores for $x = 249{,}499.5$ and $x = 251{,}000.5$.

For $x = 249{,}499.5$,

$$z = \frac{249{,}499.5 - 250{,}000}{250\sqrt{3}} \approx -1.16.$$

For $x = 251{,}000.5$,

$$z = \frac{251{,}000.5 - 250{,}000}{250\sqrt{3}} \approx 2.31.$$

$$\begin{aligned} P(-1.16 < z < 2.31) &= P(2.31) - P(-1.16) \\ &= .9896 - .1230 \\ &= .8666 \end{aligned}$$

30.

Interval	x	Tally	f	xf
1-3	2	𝍸 |	6	12
4-6	5	𝍸	5	25
7-9	8	𝍸 𝍸 |	11	88
10-12	11	𝍸 𝍸 𝍸 𝍸	20	220
13-15	14	𝍸 |	6	84
16-18	17	||	2	34
Totals:			50	463

x	$P(x)$	$x \cdot P(x)$
0	.001	.000
1	.010	.010
2	.044	.088
3	.117	.351
4	.205	.820
5	.246	1.230
6	.205	1.230
7	.117	.819
8	.044	.352
9	.010	.090
10	.001	.010
Totals:	1.000	5.000

(a) The mean of the frequency distribution is

$$\bar{x} = \frac{\sum xf}{n} = \frac{463}{50} = 9.26 \approx 9.3.$$

The expected value of the probability distribution is

$$E(x) = \sum (x \cdot P(x)) = 5.$$

(b) The standard deviation of the frequency distribution is

$$\begin{aligned} s &= \sqrt{\frac{\sum fx^2 - n\bar{x}^2}{n-1}} \\ &= \sqrt{\frac{5027 - 50(9.26)^2}{50-1}} \approx 3.9. \end{aligned}$$

The standard deviation of the probability distribution is

$$s = \sqrt{np(1-p)} = \sqrt{10(.5)(.5)} \approx 1.58.$$

(c) 95.44% of the area under the normal approximation of the binomial probability distribution will lie between $z = -2$ and $z = 2$.

$$\begin{aligned} &z = -2 \text{ means } -2 = \frac{x-5}{1.58} \quad \text{or} \quad 1.84 = x. \\ &z = 2 \text{ means } 2 = \frac{x-5}{1.58} \quad \text{or} \quad 8.16 = x. \end{aligned}$$

The interval is $1.84 \leq x \leq 8.16$.

(d) The normal distribution cannot be used to answer probability questions about the frequency distribution because the histogram of the frequency distribution is not close enough to the shape of a normal curve.

32. $P(x < 32) = P\left(z < \frac{32 - 32.1}{.1}\right)$

$$\begin{aligned} &= P(z < -1) \\ &= .1587 \text{ or } 15.87\% \end{aligned}$$

34. Let x represent the number of businesses that go bankrupt in their first year.

$n = 50, p = .21$

$$\begin{aligned} \mu &= np = 50(.21) = 10.5 \\ \sigma &= \sqrt{np(1-p)} = \sqrt{50(.21)(.79)} \approx 2.88 \end{aligned}$$

(a) Use the binomial probability formula.

$$P(x = 8) = \binom{50}{8}(.21)^8(.79)^{42}$$

$$\approx .1019$$

Now use the normal approximation. To find P(8 go bankrupt), find the z-scores for $x = 7.5$ and $x = 8.5$.

For $x = 7.5$,

$$z = \frac{7.5 - 10.5}{2.88} \approx -1.04.$$

For $x = 8.5$,

$$z = \frac{8.5 - 10.5}{2.88} \approx -.69.$$

$$\begin{aligned} P(8 \text{ go bankrupt}) &= .2451 - .1492 \\ &= .0959 \end{aligned}$$

(b) Use the binomial probability formula.

$$\begin{aligned} &P(x = 0, 1, \text{ or } 2) \\ &= \binom{50}{0}(.21)^0(.79)^{50} + \binom{50}{1}(.21)^1(.79)^{49} \\ &\quad + \binom{50}{2}(.21)^2(.79)^{48} \\ &\approx .00001 + .00010 + .00066 \\ &\approx .0008 \end{aligned}$$

Now use the normal curve approximation. To find $P(x \leq 2)$, find the z-score for $x = 2.5$.

$$z = \frac{2.5 - 10.5}{2.88} \approx -2.78$$

$$P(x \leq 2) = .0027$$

36. Let x represent the number of flies that are killed.

$n = 1000, x = 980, p = .98$

$$\begin{aligned} \mu &= np = 1000(.98) = 980 \\ \sigma &= \sqrt{np(1-p)} \\ &= \sqrt{1000(.98)(.02)} \\ &= \sqrt{19.6} \end{aligned}$$

To find P(exactly 980), find the z-scores for $x = 979.5$ and $x = 980.5$.

For $x = 979.5$,

$$z = \frac{979.5 - 980}{\sqrt{19.6}} \approx -.11.$$

For $x = 980.5$,

$$z = \frac{980.5 - 980}{\sqrt{19.6}} \approx .11.$$

$$\begin{aligned} P(-.11 < z < .11) &= P(z < .11) - P(z < -.11) \\ &= .5438 - .4562 \\ &= .0876 \end{aligned}$$

38. Again, let x represent the number of flies that are killed.

$n = 100; x = 975, 976, \ldots, 980; p = .98$

As in Exercise 36, $\mu = 980$ and $\sigma = \sqrt{19.6}$. To find $P(\text{at least } 975)$, find the z-score for $x = 974.5$.

$$z = \frac{974.5 - 980}{\sqrt{19.6}} \approx -1.24$$

$$\begin{aligned} P(z > -1.24) &= 1 - P(z < -1.24) \\ &= 1 - .1075 \\ &= .8925 \end{aligned}$$

40. **(a)** Find the mean.

$$\overline{x} = \frac{\sum x}{n} = \frac{126}{10} = 12.6$$

The mean is 12.6.
To find the median, list the values from smallest to largest.

$$5, 9, 11, 11, 12, 12, 13, 15, 17, 21$$

The median is the mean of the two middle values.

$$\frac{12 + 12}{2} = 12.$$

The median is 12.
There are two values that appear twice in the list, 11 and 12, so the modes are 11 and 12.

(b) Use a graphing calculator or spreadsheet to find the standard deviation.

$$s \approx 4.3767$$

The standard deviation is 4.3767.

(c) $\overline{x} + s = 12.6 + 4.3767 = 16.9767$
$\overline{x} - s = 12.6 - 4.3767 = 8.2233$

Seven of the 10 entries fall between these two values.
Thus $\frac{7}{10}(100) = 70\%$ of the data are within one standard deviation of the mean.

(d) $\overline{x} + 3s = 12.6 + 3(4.3767) = 25.7301$
$\overline{x} - 3s = 12.6 - 3(4.3767) = -.5301$

All of the data fall between these two values, so 100% of the data are within three standard deviations of the mean.

42. No more than 35 min/day

$\mu = 42, \ \sigma = 12$

Find the z-score for $x = 35$.

$$z = \frac{35 - 42}{12} \approx -.58$$

$P(x \leq 35) = P(z \leq -.58) = .2810$

28.10% of the residents commute no more than 35 min/day.

44. Between 38 and 60 min/day

$\mu = 42, \ \sigma = 12$

Find the z-scores for $x = 38$ and $x = 60$.

For $x = 38$,

$$z = \frac{38 - 42}{12} \approx -.33.$$

For $x = 60$,

$$z = \frac{60 - 42}{12} = 1.5.$$

$$\begin{aligned} P(38 \leq x \leq 60) &= P(-.33 \leq z \leq 1.5) \\ &= P(z \leq 1.5) - P(z \leq -.33) \\ &= .9332 - .3707 \\ &= .5625 \end{aligned}$$

56.25% of the residents commute between 38 and 60 min/day.

46. $n = 500, p = .555$
$\mu = np = 500(.555) = 277.5$
$\sigma = \sqrt{np(1-p)}$
$\quad = \sqrt{500(.555)(.445)}$
$\quad \approx 11.11$

$$\begin{aligned} P(x > 300) &= P\left(z > \frac{300.5 - 277.5}{11.11}\right) \\ &= P(z > 2.07) \\ &= 1 - .9808 \\ &= .0192 \end{aligned}$$

Extended Application: Statistics in the Law-The Castaneda Decision

1. $z = \dfrac{688 - 339}{\sqrt{870 \cdot 0.791 \cdot 0.209}} \approx 29.1$

2. The court used the normal approximation to compute this (tiny) probability.

3. **(a)** $220(.791) = 174.02$

The expected number of Mexican-American is 174.

(b)
$$\begin{aligned}\sigma &= \sqrt{np(1-p)}\\ &= \sqrt{220(.791)(1-.791)}\\ &\approx 6.03\end{aligned}$$

(c) $z = \dfrac{100 - 174}{6.03} \approx -12.3$

The actual number is 12.3 standard deviations from the expected number.

(d) The table only goes to -3.4 standard deviations; a result this far below the mean, or farther, has probability 0.003, so the pool with 100 Mexican-Americans has probability less than 0.003. Using the normal approximations to the binomial distribution, we can show that the true probability is *much* smaller, in fact less than 10^{-30}.

4. **(a)** $z = \dfrac{6 - \frac{88}{294} \cdot 112}{\sqrt{112 \cdot \frac{88}{294} \cdot \frac{206}{294}}} \approx -5.7$

The expected number is about 6 standard deviations from the actual number.

(b) Yes; The court should have decided this *was* evidence of discrimination.

Chapter 10

MARKOV CHAINS

10.1 Basic Properties of Markov Chains

2. $\begin{bmatrix} \frac{1}{2} & 1 \end{bmatrix}$ could not be a probability vector because the sum of the entries is not equal to 1.

4. $\begin{bmatrix} .1 & .1 \end{bmatrix}$ could not be a probability vector because the sum of the entries is not equal to 1.

6. $\begin{bmatrix} \frac{1}{4} & \frac{1}{8} & \frac{5}{8} \end{bmatrix}$ could be a probability vector since it is a matrix of only one row, having nonnegative entries whose sum is 1.

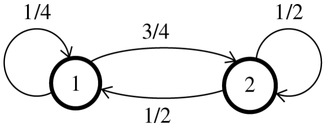
8. $\begin{bmatrix} .3 & -.1 & .8 \end{bmatrix}$ could not be a probability vector because it has a negative entry.

10. $\begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix}$

The entries in each row do not add up to 1, so this could not be a transition matrix.

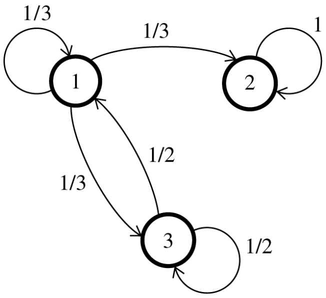
12. $\begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

This could be a transition matrix since it is a square matrix, all entries are between 0 and 1, inclusive, and the sum of the entries in each row is 1.



14. $\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$

This could be a transition matrix since it meets all the requirements given in the solution for Exercise 12.



16. The given diagram is not a transition diagram because the sum of the probabilities from each of the three states is not 1.

18.

$$\begin{array}{c|ccc} & A & B & C \\ \hline A & .6 & .2 & .2 \\ B & .9 & .02 & .08 \\ C & .4 & 0 & .6 \end{array}$$

All entries in this square matrix are between 0 and 1, inclusive, and the sum of the entries in each row is 1, so this is a transition matrix.

20. $B = \begin{bmatrix} .7 & .3 \\ 0 & 1 \end{bmatrix}$

$$B^2 = \begin{bmatrix} .7 & .3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} .7 & .3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} .49 & .51 \\ 0 & 1 \end{bmatrix}$$

$$B^3 = B \cdot B^2 = \begin{bmatrix} .7 & .3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} .49 & .51 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} .343 & .657 \\ 0 & 1 \end{bmatrix}$$

The entry in row 1, column 2 of B^3 gives the probability that state 1 changes to state 2 after three repetitions of the experiment. This probability is .657.

22. $D = \begin{bmatrix} .3 & .2 & .5 \\ 0 & 0 & 1 \\ .6 & .1 & .3 \end{bmatrix}$

$$D^2 = \begin{bmatrix} .39 & .11 & .5 \\ .6 & .1 & .3 \\ .36 & .15 & .49 \end{bmatrix}$$

$$D^3 = D \cdot D^2 = \begin{bmatrix} .417 & .128 & .455 \\ .36 & .15 & .49 \\ .402 & .121 & .477 \end{bmatrix}$$

The probability that state 1 changes to state 2 after three repetitions is .128.

24. $F = \begin{bmatrix} .01 & .9 & .09 \\ .72 & .1 & .18 \\ .34 & 0 & .66 \end{bmatrix}$

$$F^2 = \begin{bmatrix} .6787 & .099 & .2223 \\ .1404 & .658 & .2016 \\ .2278 & .306 & .4662 \end{bmatrix}$$

$$F^3 = \begin{bmatrix} .1536 & .6207 & .2256 \\ .5437 & .1922 & .2641 \\ .3811 & .2356 & .3833 \end{bmatrix}$$

The probability that state 1 changes to state 2 after three repetitions is .6207.

26. This exercise should be solved by graphing calculator methods. Let A be the given transition matrix. The first power, A^1, is the given matrix. The other powers of the transition matrix are

$$A^2 = \begin{bmatrix} .23 & .21 & .24 & .17 & .15 \\ .26 & .18 & .26 & .16 & .14 \\ .23 & .18 & .24 & .19 & .16 \\ .19 & .19 & .27 & .18 & .17 \\ .17 & .20 & .26 & .19 & .18 \end{bmatrix},$$

$$A^3 = \begin{bmatrix} .226 & .192 & .249 & .177 & .156 \\ .222 & .196 & .252 & .174 & .156 \\ .219 & .189 & .256 & .177 & .159 \\ .213 & .192 & .252 & .181 & .162 \\ .213 & .189 & .252 & .183 & .163 \end{bmatrix},$$

$$A^4 = \begin{bmatrix} .2205 & .1916 & .2523 & .1774 & .1582 \\ .2206 & .1922 & .2512 & .1778 & .1582 \\ .2182 & .1920 & .2525 & .1781 & .1592 \\ .2183 & .1909 & .2526 & .1787 & .1595 \\ .2176 & .1906 & .2533 & .1787 & .1598 \end{bmatrix},$$

$$A^5 = \begin{bmatrix} .21932 & .19167 & .25227 & .17795 & .15879 \\ .21956 & .19152 & .25226 & .17794 & .15872 \\ .21905 & .19152 & .25227 & .17818 & .15898 \\ .21880 & .19144 & .25251 & .17817 & .15908 \\ .21857 & .19148 & .25253 & .17824 & .15918 \end{bmatrix}.$$

The probability that state 2 changes to state 4 after 5 repetitions of the experiment is found by looking at the entry in row 2, column 4 of A^5. This probability is .17794.

28. The given matrix is

$$A = \begin{bmatrix} .8 & .2 \\ .35 & .65 \end{bmatrix}.$$

The one-week matrix is just A.

The two-week matrix is

$$A^2 = \begin{bmatrix} .71 & .29 \\ .5075 & .4925 \end{bmatrix}.$$

The three-week matrix is

$$A^3 = \begin{bmatrix} .6695 & .3305 \\ .578375 & .421625 \end{bmatrix}.$$

The four-week matrix is

$$A^4 = \begin{bmatrix} .651275 & .348725 \\ .61026875 & .38973125 \end{bmatrix}.$$

(a) The required probability is given by the first row, first column entry in A, .8.

(b) Use A^2; the probability is .71.

(c) Use A^3; the probability is .6695.

(d) Use A^4; the probability is $.651275 \approx .6513$.

(e) Use A^2; the required probability is given in row 2, column 1. This probability is .5075.

30. The probability of a G_1 becoming a G_2 must be .20 so that the row sum is 1. All of the other entries are given directly in the exercise. All of the probabilities are given in the following transition matrix.

$$\begin{array}{c} \\ G_0 \\ G_1 \\ G_2 \end{array} \begin{array}{c} \begin{array}{ccc} G_0 & G_1 & G_2 \end{array} \\ \begin{bmatrix} .85 & .10 & .05 \\ 0 & .80 & .20 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

32. $P = \begin{bmatrix} .85 & .10 & .05 \\ .15 & .75 & .10 \\ .10 & .30 & .60 \end{bmatrix}$

is the transition matrix, and the entries of

$$X_0 = \begin{bmatrix} 50{,}000 & 0 & 0 \end{bmatrix}$$

gives the number of policyholders in the initial group. To obtain the numbers for each subsequent year, multiply by P.

(a) $X_0P = \begin{bmatrix} 42{,}500 & 5000 & 2500 \end{bmatrix}$

After 1 yr, there will be 42,500 policyholders in G_0, 5000 in G_1, and 2500 in G_2.

(b) $$X_0P^2 = (X_0P)P$$
$$= \begin{bmatrix} 37{,}125 & 8750 & 4125 \end{bmatrix}$$

After 2 yr, there will be 37,125 policyholders in G_0, 8750 in G_1, and 4125 in G_2.

(c) $$X_0P^3 = (X_0P^2)P$$
$$= \begin{bmatrix} 33{,}281 & 11{,}513 & 5206 \end{bmatrix}$$

After 3 yr, there will be 33,281 policyholders in G_0, 11,513 in G_1, and 5206 in G_2.

(d) The transition matrix for a 2-yr period is

$$P^2 = \begin{bmatrix} .7425 & .175 & .0825 \\ .25 & .6075 & .1425 \\ .19 & .415 & .395 \end{bmatrix}.$$

(e) The probability that a G_0 will be a G_0 after 2 yr is given by the entry of P^2 in the first row, first column, that is .7425.

34. **(a)** Let A denote agricultural, U denote urban, and I denote idle. Then the transition matrix P is given by

$$\begin{array}{c} \\ \text{A} \\ \text{U} \\ \text{I} \end{array} \begin{array}{c} \begin{array}{ccc} \text{A} & \text{U} & \text{I} \end{array} \\ \begin{bmatrix} .80 & .15 & .05 \\ 0 & .90 & .10 \\ .10 & .20 & .70 \end{bmatrix} \end{array}.$$

(b) The initial categories are

$$X_0 = \begin{bmatrix} .35 & .10 & .55 \end{bmatrix}.$$

(c) Multiply by P to obtain the results after 10 yr:

$$X_0P = \begin{bmatrix} .335 & .2525 & .4125 \end{bmatrix}.$$

(d) Multiply again by P to obtain the results after 20 yr:

$$X_0P^2 = \begin{bmatrix} .30925 & .36 & .33075 \end{bmatrix}.$$

(e) The transition matrix for a 20-yr period is

$$P^2 = \begin{bmatrix} .645 & .265 & .09 \\ .01 & .83 & .16 \\ .15 & .335 & .515 \end{bmatrix}.$$

(f) The probability that an idle plot of land is still idle after 20 yr is given by the entry of P^2 in the third row, third column; that is, .515.

36. **(a)** Since the transition matrix

$$P = \begin{bmatrix} \frac{5}{7} & \frac{2}{7} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

describes the proportional change of rabbits among various immune response classifications from one week to the next, we need to consider the entry in the first row, first column of P^5. Because P has zero entries in rows two through four of column one, the proportion of the rabbits in group 1 that were still in group 1 five weeks later was $\frac{5^5}{7^5} \approx .1859$.

(b) Find the product X_0P^4, where $X_0 = [9 \quad 4 \quad 0 \quad 0]$.

$X_0P^4 = [2.34 \quad 2.62 \quad 3.47 \quad 4.56]$, where the entries are approximate. Therefore, after 4 weeks there are approximately 2.34 rabbits in group 1, 2.62 rabbits in group 2, 3.47 in group 3, and 4.56 in group 4.

(c) Using a graphing calculator, when P is raised to a larger and larger power, the entries in the first row, first two columns, and the entries in the second row, first two columns are either zero or positive numbers that are getting smaller—that is, they are approaching zero. This leads to the conclusion that the long-range probability of rabbits in group 1 or 2 staying in group 1 or 2 is zero.

38. **(a)** Use L for liberal, C for conservative, and I for independent. The transition matrix P is given by

$$\begin{array}{c} \\ \text{L} \\ \text{C} \\ \text{I} \end{array} \begin{array}{c} \begin{array}{ccc} \text{L} & \text{C} & \text{I} \end{array} \\ \begin{bmatrix} .80 & .15 & .05 \\ .20 & .70 & .10 \\ .20 & .20 & .60 \end{bmatrix} \end{array}.$$

(b) The initial distribution is given by

$$X_0 = \begin{bmatrix} .40 & .45 & .15 \end{bmatrix}.$$

(c) One month later in July,

$$X_0P = \begin{bmatrix} .44 & .405 & .155 \end{bmatrix},$$

so there will be 44% liberals, 40.5% conservatives, and 15.5% independents.

(d) Two months later in August,

$$X_0P^2 = (X_0P)P = \begin{bmatrix} .464 & .3805 & .1555 \end{bmatrix},$$

so there will be 46.4% liberals, 38.05% conservatives, and 15.55% independents.

(e) Three months later in September,

$$X_0P^3 = (X_0P^2)P = \begin{bmatrix} .4784 & .36705 & .15455 \end{bmatrix},$$

so there will be 47.84% liberals, 36.705% conservatives, and 15.455% independents.

(f) Four months later in October,

$$X_0P^4 = (X_0P^3)P = \begin{bmatrix} .48704 & .359605 & .153355 \end{bmatrix},$$

so there will be 48.704% liberals, 35.9605% conservatives, and 15.3355% independents.

(g) The transition matrix for a two-month period is given by

$$\begin{aligned} P^2 &= \begin{bmatrix} .80 & .15 & .05 \\ .20 & .70 & .10 \\ .20 & .20 & .60 \end{bmatrix} \begin{bmatrix} .80 & .15 & .05 \\ .20 & .70 & .10 \\ .20 & .20 & .60 \end{bmatrix} \\ &= \begin{bmatrix} .68 & .235 & .085 \\ .32 & .54 & .14 \\ .32 & .29 & .39 \end{bmatrix} \end{aligned}$$

10.2 Regular Markov Chains

2. Let $A = \begin{bmatrix} .22 & .78 \\ .43 & .57 \end{bmatrix}$.

A is a regular transition matrix since $A^1 = A$ contains all positive entries.

4. Let $B = \begin{bmatrix} .55 & .45 \\ 0 & 1 \end{bmatrix}$.

$$B^2 = \begin{bmatrix} .30250 & .69750 \\ 0 & 1 \end{bmatrix}$$

$$B^3 = \begin{bmatrix} .16638 & .83363 \\ 0 & 1 \end{bmatrix}$$

B is not regular since any power of B will have $\begin{bmatrix} 0 & 1 \end{bmatrix}$ as its second row and thus cannot have all positive entries.

6. Let $C = \begin{bmatrix} .3 & .5 & .2 \\ 1 & 0 & 0 \\ .5 & .1 & .4 \end{bmatrix}$.

$$C^2 = \begin{bmatrix} .69 & .17 & .14 \\ .3 & .5 & .2 \\ .45 & .29 & .26 \end{bmatrix}$$

C is a regular transition matrix since C^2 contains all positive entries.

8. Let $P = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{8} & \frac{7}{8} \end{bmatrix}$, and let V be the probability vector $\begin{bmatrix} v_1 & v_2 \end{bmatrix}$. We want to find V such that

$$VP = V,$$

or $\begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{8} & \frac{7}{8} \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$.

Use matrix multiplication on the left.

$$\begin{bmatrix} \frac{2}{3}v_1 + \frac{1}{8}v_2 & \frac{1}{3}v_1 + \frac{7}{8}v_2 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$$

Set corresponding entries from the two matrices equal to get

$$\begin{aligned} \tfrac{2}{3}v_1 + \tfrac{1}{8}v_2 &= v_1 \\ \tfrac{1}{3}v_1 + \tfrac{7}{8}v_2 &= v_2. \end{aligned}$$

Multiply both equations by 24 to eliminate fractions.

$$\begin{aligned} 16v_1 + 3v_2 &= 24v_1 \\ 8v_1 + 21v_2 &= 24v_2 \end{aligned}$$

Simplify both equations.

$$\begin{aligned} -8v_1 + 3v_2 &= 0 \\ 8v_1 - 3v_2 &= 0 \end{aligned}$$

These equations are dependent. To find the values of v_1 and v_2, an additional equation is needed. Since $V = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$ is a probability vector

$$v_1 + v_2 = 1.$$

To find v_1 and v_2, solve the system

$$\begin{aligned} -8v_1 + 3v_2 &= 0 \quad (1) \\ v_1 + v_2 &= 1. \quad (2) \end{aligned}$$

From equation (2), $v_1 = 1 - v_2$. Substitute $1 - v_2$ for v_1 in equation (1) to get

$$\begin{aligned} -8(1 - v_2) + 3v_2 &= 0 \\ -8 + 8v_2 + 3v_2 &= 0 \\ 11v_2 &= 8 \\ v_2 &= \tfrac{8}{11} \\ \text{and} \qquad v_1 &= 1 - v_2 = \tfrac{3}{11}. \end{aligned}$$

The equilibrium vector is

$$\begin{bmatrix} \frac{3}{11} & \frac{8}{11} \end{bmatrix}.$$

10. Let $P = \begin{bmatrix} .8 & .2 \\ .1 & .9 \end{bmatrix}$, and let V be the probability vector $\begin{bmatrix} v_1 & v_2 \end{bmatrix}$.

$$\begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} .8 & .2 \\ .1 & .9 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$$

$$\begin{aligned} .8v_1 + .1v_2 &= v_1 \\ .2v_1 + .9v_2 &= v_2 \end{aligned}$$

Simplify these equations to get the system

$$\begin{aligned} -.2v_1 + .1v_2 &= 0 \\ .2v_1 - .1v_2 &= 0. \end{aligned}$$

These equations are dependent. Since V is a probability vector,

$$v_1 + v_2 = 1.$$

Solve the system

$$\begin{aligned} -.2v_1 + .1v_2 &= 0 \\ v_1 + v_2 &= 1 \end{aligned}$$

by the substitution method.

$$\begin{aligned} -.2(1 - v_2) + .1v_2 &= 0 \\ -.2 + .2v_2 + .1v_2 &= 0 \\ .3v_2 &= .2 \\ v_2 &= \tfrac{2}{3} \\ v_1 &= 1 - \tfrac{2}{3} = \tfrac{1}{3} \end{aligned}$$

Thus, the equilibrium vector is

$$\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

12. Let $P = \begin{bmatrix} .5 & .2 & .3 \\ .1 & .4 & .5 \\ .2 & .2 & .6 \end{bmatrix}$, and let V be the probability vector $\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$.

$$\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} .5 & .2 & .3 \\ .1 & .4 & .5 \\ .2 & .2 & .6 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$$

$$\begin{aligned} .5v_1 + .1v_2 + .2v_3 &= v_1 \\ .2v_1 + .4v_2 + .2v_3 &= v_2 \\ .3v_1 + .5v_2 + .6v_3 &= v_3 \end{aligned}$$

Simplify these equations to get the system

$$\begin{aligned} -.5v_1 + .1v_2 + .2v_3 &= 0 \\ .2v_1 - .6v_2 + .2v_3 &= 0 \\ .3v_1 + .5v_2 - .4v_3 &= 0. \end{aligned}$$

Since V is a probability vector,

$$v_1 + v_2 + v_3 = 1.$$

This gives us a system of four equations in three variables.

$$\begin{aligned} v_1 + v_2 + v_3 &= 1 \\ -.5v_1 + .1v_2 + .2v_3 &= 0 \\ .2v_1 - .6v_2 + .2v_3 &= 0 \\ .3v_1 + .5v_2 - .4v_3 &= 0. \end{aligned}$$

This system can be solved by the Gauss-Jordan method. Start with the augmented matrix

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ -.5 & .1 & .2 & 0 \\ .2 & -.6 & .2 & 0 \\ .3 & .5 & -.4 & 0 \end{array}\right].$$

The solution of this system is $v_1 = \frac{1}{4}$, $v_2 = \frac{1}{4}$, and $v_3 = \frac{1}{2}$, so the equilibrium vector is

$$\begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$

14. Let $P = \begin{bmatrix} .16 & .28 & .56 \\ .43 & .12 & .45 \\ .86 & .05 & .09 \end{bmatrix}$, and let V be the probability vector $\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$.

$$\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} .16 & .28 & .56 \\ .43 & .12 & .45 \\ .86 & .05 & .09 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$$

$$\begin{aligned} .16v_1 + .43v_2 + .86v_3 &= v_1 \\ .28v_1 + .12v_2 + .05v_3 &= v_2 \\ .56v_1 + .45v_2 + .09v_3 &= v_3 \end{aligned}$$

Simplify these equations and also use the equation $v_1 + v_2 + v_3 = 1$ to get the system

$$\begin{aligned} v_1 + v_2 + v_3 &= 1 \\ -.84v_1 + .43v_2 + .86v_3 &= 0 \\ .28v_1 - .88v_2 + .05v_3 &= 0 \\ .56v_1 + .45v_2 - .91v_3 &= 0. \end{aligned}$$

Solve this system by the Gauss-Jordan method to obtain $v_1 = \frac{7783}{16{,}799}$, $v_2 = \frac{2828}{16{,}799}$, and $v_3 = \frac{6188}{16{,}799}$. The equilibrium vector is

$$\begin{bmatrix} \frac{7783}{16{,}799} & \frac{2828}{16{,}799} & \frac{6188}{16{,}799} \end{bmatrix}.$$

16. Let $P = \begin{bmatrix} .85 & .10 & .05 \\ .15 & .75 & .10 \\ .10 & .30 & .60 \end{bmatrix}$, and let V be the probability vector $\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$.

$$\begin{aligned} .85v_1 + .15v_2 + .10v_3 &= v_1 \\ .10v_1 + .75v_2 + .30v_3 &= v_2 \\ .05v_1 + .10v_2 + .60v_3 &= v_3 \end{aligned}$$

Simplify these equations and also use the equation $v_1 + v_2 + v_3 = 1$ to get the system

$$\begin{aligned} v_1 + \quad v_2 + \quad v_3 &= 1 \\ -.15v_1 + .15v_2 + .10v_3 &= 0 \\ .10v_1 - .25v_2 + .30v_3 &= 0 \\ .05v_1 + .10v_2 - .40v_3 &= 0. \end{aligned}$$

Solve this system by the Gauss-Jordan method to obtain $v_1 = \frac{28}{59}$, $v_2 = \frac{22}{59}$, and $v_3 = \frac{9}{59}$. The equilibrium vector is

$$\begin{bmatrix} \frac{28}{59} & \frac{22}{59} & \frac{9}{59} \end{bmatrix}.$$

18. $\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} .80 & .15 & .05 \\ 0 & .90 & .10 \\ .10 & .20 & .70 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$

$$\begin{aligned} .80v_1 \qquad\quad + .10v_3 &= v_1 \\ .15v_1 + .90v_2 + .20v_3 &= v_2 \\ .05v_1 + .10v_2 + .70v_3 &= v_3 \end{aligned}$$

Simplify these equations and also use the equation $v_1 + v_2 + v_3 = 1$ to get the system

$$\begin{aligned} v_1 + \quad v_2 + \quad v_3 &= 1 \\ -.20v_1 \qquad\quad + .10v_3 &= 0 \\ .15v_1 - .10v_2 + .20v_3 &= 0 \\ .05v_1 + .10v_2 - .30v_3 &= 0. \end{aligned}$$

Solving this system by the Gauss-Jordan method, we obtain $v_1 = \frac{2}{17}$, $v_2 = \frac{11}{17}$, and $v_3 = \frac{4}{17}$. The equilibrium vector is

$$\begin{bmatrix} \frac{2}{17} & \frac{11}{17} & \frac{4}{17} \end{bmatrix}.$$

20. $\begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} .90 & .10 \\ .05 & .95 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$

$$\begin{aligned} .90v_1 + .05v_2 &= v_1 \\ .10v_1 + .95v_2 &= v_2 \end{aligned}$$

Simplify these equations to obtain the system

$$\begin{aligned} -.10v_1 + .05v_2 &= 0 \\ .10v_1 - .05v_2 &= 0. \end{aligned}$$

These equations are dependent. Use the substitution method to solve the system

$$\begin{aligned} -.10v_1 + .05v_2 &= 0 \\ v_1 + \quad v_2 &= 1, \end{aligned}$$

obtaining $v_1 = \frac{1}{3}$ and $v_2 = \frac{2}{3}$. The equilibrium vector is

$$\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

24. There are an infinite number of solutions to the system $VK = V$. A few examples would be $(0, 0, 0)$, $(5, 7, 11)$, and $(\frac{5}{11}, \frac{7}{11}, 1)$. In fact, any vector of the form $(\frac{5}{11}x_3, \frac{7}{11}x_3, x_3)$ would be a solution to the system.

26. $\begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} .95 & .05 \\ .80 & .20 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$

$$\begin{aligned} .95v_1 + .80v_2 &= v_1 \\ .05v_1 + .20v_2 &= v_2 \end{aligned}$$

Simplify these equations to obtain the system

$$\begin{aligned} -.05v_1 + .80v_2 &= 0 \\ .05v_1 - .80v_2 &= 0. \end{aligned}$$

These equations are dependent. Use the substitution method to solve the system

$$\begin{aligned} -.05v_1 + .80v_2 &= 0 \\ v_1 + \quad v_2 &= 1, \end{aligned}$$

obtaining $v_1 = \frac{16}{17}$ and $v_2 = \frac{1}{17}$. The long-run probability that the line will work properly is $\frac{16}{17}$.

28. This exercise should be solved by graphing calculator methods. The solution may vary. The answers are as follows.

(a) $\begin{bmatrix} .4 & .6 \end{bmatrix}$; $\begin{bmatrix} .53 & .47 \end{bmatrix}$; $\begin{bmatrix} .5885 & .4115 \end{bmatrix}$; $\begin{bmatrix} .614825 & .385175 \end{bmatrix}$; $\begin{bmatrix} .626671 & .373329 \end{bmatrix}$; $\begin{bmatrix} .632002 & .367998 \end{bmatrix}$; $\begin{bmatrix} .634401 & .365599 \end{bmatrix}$; $\begin{bmatrix} .635480 & .364520 \end{bmatrix}$; $\begin{bmatrix} .635966 & .364034 \end{bmatrix}$; $\begin{bmatrix} .636185 & .363815 \end{bmatrix}$

(b) .236364; .106364; .047864; .021539; .009693; .004362; .001963; .000884; .000398; .000179

(c) The ratio is roughly .45 for each week.

(d) Each week, the difference between the probability vector and the equilibrium vector is slightly less than half of what it was the previous week.

(e) $\begin{bmatrix} .75 & .25 \end{bmatrix}; \begin{bmatrix} .6875 & .3125 \end{bmatrix};$

$\begin{bmatrix} .659375 & .340625 \end{bmatrix}; \begin{bmatrix} .646719 & .353281 \end{bmatrix};$

$\begin{bmatrix} .641023 & .358977 \end{bmatrix}; \begin{bmatrix} .638461 & .361539 \end{bmatrix};$

$\begin{bmatrix} .637307 & .362693 \end{bmatrix}; \begin{bmatrix} .636788 & .363212 \end{bmatrix};$

$\begin{bmatrix} .636555 & .363445 \end{bmatrix}; \begin{bmatrix} .636450 & .363550 \end{bmatrix}$

$.113636; .051136; .023011; .010355; .004659;$
$.002097; .000943; .000424; .000191; .000086$

The ratio is roughly .45 for each week, which is the same conclusion as before.

30. The transition matrix is

$$\begin{array}{c} \\ \text{A} \\ \text{B} \\ \text{C} \end{array} \begin{array}{c} \begin{array}{ccc} \text{A} & \text{B} & \text{C} \end{array} \\ \begin{bmatrix} .3 & .3 & .4 \\ .15 & .3 & .55 \\ .3 & .6 & .1 \end{bmatrix} \end{array}.$$

To find the long-range distribution, use the system

$$\begin{aligned} v_1 + \quad v_2 + \quad v_3 &= 1 \\ .3v_1 + .15v_2 + .3v_3 &= v_1 \\ .3v_1 + \quad .3v_2 + .6v_3 &= v_2 \\ .4v_1 + .55v_2 + .1v_3 &= v_3. \end{aligned}$$

Simplify these equations to obtain the system

$$\begin{aligned} v_1 + \quad v_2 + \quad v_3 &= 1 \\ -.7v_1 + .15v_2 + .3v_3 &= 0 \\ .3v_1 - \quad .7v_2 + .6v_3 &= 0 \\ .4v_1 + .55v_2 - .9v_3 &= 0. \end{aligned}$$

Solve this system by the Gauss-Jordan method to obtain $v_1 = \frac{60}{251}$, $v_2 = \frac{102}{251}$, and $v_3 = \frac{89}{251}$. The long-range prediction is

$$\begin{bmatrix} \frac{60}{251} & \frac{102}{251} & \frac{89}{251} \end{bmatrix}.$$

32. **(a)** Look at the probability in the third row, third column of

$$P = \begin{bmatrix} .645 & .099 & .152 & .033 & .071 \\ .611 & .138 & .128 & .033 & .090 \\ .514 & .067 & .271 & .030 & .118 \\ .609 & .107 & .178 & .064 & .042 \\ .523 & .093 & .183 & .022 & .179 \end{bmatrix}$$

which is .271.

(b) Use a graphing calculator to find

$$P^2 = \begin{bmatrix} .612 & .098 & .171 & .033 & .087 \\ .611 & .100 & .168 & .033 & .088 \\ .592 & .092 & .187 & .032 & .097 \\ .611 & .098 & .174 & .034 & .084 \\ .595 & .096 & 178 & .031 & .100 \end{bmatrix}$$

where entries are rounded to three decimal places. The probability that a criminal's second crime after committing theft is also theft is the entry in the third row, third column of P^2, which is .187.

(c) Since all of the entries of P are positive, then P is a regular transition matrix. Therefore, there is a unique equilibrium probability vector $V = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 \end{bmatrix}$. The equation $VP = V$ yields the system of equations

$$\begin{aligned} .645v_1 + .611v_2 + .514v_3 + .609v_4 + .523v_5 &= v_1 \\ .099v_1 + .138v_2 + .067v_3 + .107v_4 + .093v_5 &= v_2 \\ .152v_1 + .128v_2 + .271v_3 + .178v_4 + .183v_5 &= v_3 \\ .033v_1 + .033v_2 + .030v_3 + .064v_4 + .022v_5 &= v_4 \\ .071v_1 + .090v_2 + .118v_3 + .042v_4 + .179v_5 &= v_5, \end{aligned}$$

which along with the equation $v_1 + v_2 + v_3 + v_4 + v_5 = 1$ simplifies to

$$\begin{aligned} -.355v_1 + .611v_2 + .514v_3 + .609v_4 + .523v_5 &= 0 \\ .099v_1 - .862v_2 + .067v_3 + .107v_4 + .093v_5 &= 0 \\ .152v_1 + .128v_2 - .729v_3 + .178v_4 + .183v_5 &= 0 \\ .033v_1 + .033v_2 + .030v_3 - .936v_4 + .022v_5 &= 0 \\ .071v_1 + .090v_2 + .118v_3 + .042v_4 - .821v_5 &= 0. \end{aligned}$$

$$v_1 + v_2 + v_3 + v_4 + v_5 = 1.$$

Use matrices and a graphing calculator to solve this system by the Gauss-Jordan method. This yields the solution

$$V = \begin{bmatrix} .607 & .097 & .174 & .032 & .090 \end{bmatrix},$$

where entries are rounded to three decimal places. Therefore, the long-term probabilities for each type of crime are .607 for nonindex, .097 for injury, .174 for theft, .032 for damage, and .090 for combination.

34. The transition matrix is

$$\begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}.$$

The columns sum to $p+(1-p)=1$, so by Exercise 23 the equilibrium vector is $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

The long-range prediction for the fraction of the people who will hear the decision correctly is $\frac{1}{2}$.

36. Let F, C, and R represent fair, cloudy, and rainy, respectively, and let $\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$ be a probability vector. The transition matrix is

$$\begin{array}{c} \\ \text{F} \\ \text{C} \\ \text{R} \end{array} \begin{array}{c} \begin{array}{ccc} \text{F} & \text{C} & \text{R} \end{array} \\ \begin{bmatrix} .60 & .25 & .15 \\ .40 & .35 & .25 \\ .35 & .40 & .25 \end{bmatrix} \end{array},$$

and the resulting system of equations is

$$\begin{aligned} .60v_1 + .40v_2 + .35v_3 &= v_1 \\ .25v_1 + .35v_2 + .40v_3 &= v_2 \\ .15v_1 + .25v_2 + .25v_3 &= v_3. \end{aligned}$$

Also, $v_1 + v_2 + v_3 = 1$.

Solving this system, we obtain

$$v_1 = \tfrac{155}{318}, v_2 = \tfrac{99}{318}, v_3 = \tfrac{32}{159}.$$

The equilibrium vector is

$$\begin{bmatrix} \frac{155}{318} & \frac{99}{318} & \frac{32}{159} \end{bmatrix}.$$

Over the long term, the proportion of days that are expected to be fair, cloudy, and rainy are $\frac{155}{318} \approx 48.7\%$, $\frac{99}{318} \approx 31.1\%$, and $\frac{32}{159} \approx 20.1\%$, respectively.

38. The transition matrix is

$$P = \begin{bmatrix} .12 & .88 \\ .54 & .46 \end{bmatrix}.$$

Let V be the probability vector $\begin{bmatrix} v_1 & v_2 \end{bmatrix}$.

$$\begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} .12 & .88 \\ .54 & .46 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$$

$$\begin{aligned} .12v_1 + .54v_2 &= v_1 \\ .88v_1 + .46v_2 &= v_2 \end{aligned}$$

Simplify these equations to get the dependent system

$$\begin{aligned} -.88v_1 + .54v_2 &= 0 \\ .88v_1 - .54v_2 &= 0. \end{aligned}$$

Also, $v_1 + v_2 = 1$.

Solving this system, we obtain

$$v_1 = \frac{27}{71} \text{ and } v_2 = \frac{44}{71},$$

and note that

$$\frac{27}{71} \approx .38 = 38\%.$$

About 38% of letters in English text are expected to be vowels.

10.3 Absorbing Markov Chains

2.

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ 1 & .1 & .5 & .4 \\ 2 & .2 & .2 & .6 \\ 3 & 0 & 0 & 1 \end{array}$$

Since $p_{33} = 1$, state 3 is an absorbing state. Since $p_{13} \neq 0$ and $p_{23} \neq 0$, it is possible to go from the nonabsorbing states (states 1 and 2) to the absorbing state. Thus, this is the transition matrix for an absorbing Markov chain.

4.

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ 1 & .5 & .5 & 0 \\ 2 & .8 & .2 & 0 \\ 3 & 0 & 0 & 1 \end{array}$$

Since $p_{33} = 1$, state 3 is absorbing. Since $p_{13} = 0$ and $p_{23} = 0$, it is not possible to go from either of the nonabsorbing states (states 1 and 2) to the absorbing state. Thus, this is not the transition matrix of an absorbing Markov chain.

6.

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ 1 & .32 & .41 & .16 & .11 \\ 2 & .42 & .30 & 0 & .28 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 1 & 0 & 0 & 0 \end{array}$$

No states are absorbing, so this matrix is not that of an absorbing Markov chain.

8.

$$\begin{array}{c|ccc|} & 1 & 2 & 3 \\ \hline 1 & 1 & 0 & 0 \\ 2 & .6 & .1 & .3 \\ 3 & 0 & 0 & 1 \\ \hline \end{array}$$

Rearrange the rows and columns so that absorbing states 1 and 3 come first.

$$\begin{array}{c|cc|c|} & 1 & 3 & 2 \\ 1 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ \hline 2 & .6 & .3 & .1 \end{array}$$

Here $R = \begin{bmatrix} .6 & .3 \end{bmatrix}$ and $Q = [.1]$. Find the fundamental matrix F.

$$F = (I_1 - Q)^{-1} = \left[\tfrac{9}{10}\right]^{-1} = \left[\tfrac{10}{9}\right]$$

The product matrix FR is

$$FR = \left[\tfrac{10}{9}\right]\begin{bmatrix} \frac{6}{10} & \frac{3}{10} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix}.$$

10.

$$\begin{array}{c|ccc|} & 1 & 2 & 3 \\ \hline 1 & 1 & 0 & 0 \\ 2 & \frac{3}{8} & \frac{1}{8} & \frac{1}{2} \\ 3 & 0 & 0 & 1 \\ \hline \end{array}$$

The rearranged matrix is

$$\begin{array}{c|cc|c|} & 1 & 3 & 2 \\ 1 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ \hline 2 & \frac{3}{8} & \frac{1}{2} & \frac{1}{8} \end{array}.$$

Here $R = \begin{bmatrix} \frac{3}{8} & \frac{1}{2} \end{bmatrix}$ and $Q = \left[\frac{1}{8}\right]$.

Then

$$F = \left[\tfrac{7}{8}\right]^{-1} = \left[\tfrac{8}{7}\right],$$

and

$$FR = \left[\tfrac{8}{7}\right]\begin{bmatrix} \frac{3}{8} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{7} & \frac{4}{7} \end{bmatrix}.$$

12.

$$\begin{array}{c|cccc|} & 1 & 2 & 3 & 4 \\ \hline 1 & \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} \\ 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \\ 4 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \hline \end{array}$$

Note that states 2 and 3 are absorbing states. Rearranging, we obtain

$$\begin{array}{c|cc|cc|} & 2 & 3 & 1 & 4 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 \\ \hline 1 & \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ 4 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{array}.$$

Here $R = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix}$ and $Q = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

Therefore,

$$F = (I_2 - Q)^{-1} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\right)^{-1}$$

$$= \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix},$$

and

$$FR = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

14.

$$\begin{array}{c|ccccc|} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & .4 & .2 & .3 & 0 & .1 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 & 0 \\ 4 & .1 & .5 & .1 & .1 & .2 \\ 5 & 0 & 0 & 0 & 0 & 1 \\ \hline \end{array}$$

Observe that states 2, 3, and 5 are absorbing. The rearranged matrix is

$$\begin{array}{c|ccc|cc|} & 2 & 3 & 5 & 1 & 4 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 & 0 \\ 5 & 0 & 0 & 1 & 0 & 0 \\ \hline 1 & .2 & .3 & .1 & .4 & 0 \\ 4 & .5 & .1 & .2 & .1 & .1 \end{array}.$$

Then

$$F = (I_2 - Q)^{-1} = \begin{bmatrix} \frac{3}{5} & 0 \\ -\frac{1}{10} & \frac{9}{10} \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \frac{5}{3} & 0 \\ \frac{5}{27} & \frac{10}{9} \end{bmatrix},$$

and

$$FR = \begin{bmatrix} \frac{5}{3} & 0 \\ \frac{5}{27} & \frac{10}{9} \end{bmatrix}\begin{bmatrix} .2 & .3 & .1 \\ .5 & .1 & .2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{16}{27} & \frac{1}{6} & \frac{13}{54} \end{bmatrix}.$$

16. The transition matrix is

$$\begin{array}{ccccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{3}{5} & 0 & \frac{2}{5} & 0 & 0 & 0 \\ 2 & 0 & \frac{3}{5} & 0 & \frac{2}{5} & 0 & 0 \\ 3 & 0 & 0 & \frac{3}{5} & 0 & \frac{2}{5} & 0 \\ 4 & 0 & 0 & 0 & \frac{3}{5} & 0 & \frac{2}{5} \\ 5 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}.$$

States 0 and 5 are absorbing. We arrange to obtain

$$\begin{array}{ccc|cccc} & 0 & 5 & 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & \frac{3}{5} & 0 & 0 & \frac{2}{5} & 0 & 0 \\ 2 & 0 & 0 & \frac{3}{5} & 0 & \frac{2}{5} & 0 \\ 3 & 0 & 0 & 0 & \frac{3}{5} & 0 & \frac{2}{5} \\ 4 & 0 & \frac{2}{5} & 0 & 0 & \frac{3}{5} & 0 \end{array}.$$

Then,

$$F = \begin{bmatrix} 1 & -\frac{2}{5} & 0 & 0 \\ -\frac{3}{5} & 1 & -\frac{2}{5} & 0 \\ 0 & -\frac{3}{5} & 1 & -\frac{2}{5} \\ 0 & 0 & -\frac{3}{5} & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1.540284 & .9004739 & .4739336 & .1895735 \\ 1.350711 & 2.251185 & 1.184834 & .4739336 \\ 1.066351 & 1.777251 & 2.251185 & .9004739 \\ .6398104 & 1.066351 & 1.350711 & 1.540284 \end{bmatrix}$$

and

$$FR = \begin{array}{ccc} & 0 & 5 \\ 1 & .9242 & .0758 \\ 2 & .8104 & .1896 \\ 3 & .6398 & .3602 \\ 4 & .3839 & .6161 \end{array}.$$

(a) If B has \$3, then A has \$2 and the probability of ruin is .810.

(b) If B has \$1, then A has \$4 and the probability of ruin is .384.

18. $p = .50$, so

$$r = \frac{1-p}{p} = \frac{1-.50}{.50} = 1.$$

Hence,

$$x_a = \frac{b}{a+b} = \frac{30}{10+30} = \frac{30}{40} = \frac{3}{4}.$$

20. To calculate the expected total number of times a Markov chain will visit state j before absorption, regardless of the current state, simply sum the elements in column j of the fundamental matrix.

22. This exercise should be solved by graphing calculator methods. The solution may vary. The answers are as follows.

(a) $F = \begin{bmatrix} 1.66667 & .606061 \\ 0 & 1.81818 \end{bmatrix}$

$$FR = \begin{bmatrix} .113636 & .886364 \\ .090909 & .909091 \end{bmatrix}$$

(b) .090909

(c) .886364

24. **(a)** The transition matrix is

$$\begin{array}{cccc} & 1 & 2 & 3 \\ 1 & .05 & .15 & .8 \\ 2 & .05 & .15 & .8 \\ 3 & 0 & 0 & 1 \end{array}.$$

Rearranging, we obtain the matrix

$$\begin{array}{cc|cc} & 3 & 1 & 2 \\ 3 & 1 & 0 & 0 \\ \hline 1 & .8 & .05 & .15 \\ 2 & .8 & .05 & .15 \end{array}.$$

$$R = \begin{bmatrix} .8 \\ .8 \end{bmatrix}; Q = \begin{bmatrix} .05 & .15 \\ .05 & .15 \end{bmatrix}$$

(b) $F = [I_2 - Q]^{-1}$

$$= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} .05 & .15 \\ .05 & .15 \end{bmatrix} \right)^{-1}$$

$$= \begin{bmatrix} .95 & -.15 \\ -.05 & .85 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1.0625 & .1875 \\ .0625 & 1.1875 \end{bmatrix}$$

$$FR = \begin{bmatrix} 1.0625 & .1875 \\ .0625 & 1.1875 \end{bmatrix} \begin{bmatrix} .8 \\ .8 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(c) The probability that the disease eventually disappears is 1, since that is the entry in row 2, column 1 of FR.

(d) The expected number of people is 1.25, since that is the sum of the entries in row 2 of F.

26. **(a)** The transition matrix is

$$\begin{array}{cc} & \begin{array}{ccc} \text{A} & \text{LR} & \text{SO} \end{array} \\ \begin{array}{c} \text{A} \\ \text{LR} \\ \text{SO} \end{array} & \begin{bmatrix} .80 & .15 & .05 \\ .05 & .80 & .15 \\ 0 & 0 & 1 \end{bmatrix} \end{array}.$$

Rearranging, we obtain the matrix

$$\begin{array}{cc} & \begin{array}{ccc} \text{SO} & \text{A} & \text{LR} \end{array} \\ \begin{array}{c} \text{SO} \\ \text{A} \\ \text{LR} \end{array} & \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline .05 & .80 & .15 \\ .15 & .05 & .80 \end{array}\right] \end{array}.$$

$$R = \begin{bmatrix} .05 \\ .15 \end{bmatrix}; Q = \begin{bmatrix} .80 & .15 \\ .05 & .80 \end{bmatrix}$$

$$\begin{aligned} F &= [I_2 - Q]^{-1} \\ &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} .80 & .15 \\ .05 & .80 \end{bmatrix} \right)^{-1} \\ &= \begin{bmatrix} .2 & -.15 \\ -.05 & .2 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 6.154 & 4.615 \\ 1.538 & 6.154 \end{bmatrix} \end{aligned}$$

$$FR = \begin{bmatrix} 6.154 & 4.615 \\ 1.538 & 6.154 \end{bmatrix} \begin{bmatrix} .05 \\ .15 \end{bmatrix} = \begin{bmatrix} 1.000 \\ 1.000 \end{bmatrix}$$

(b) The probability that a person who commuted by car ends up avoiding the downtown area is 1, since that is the entry in row 1, column 1 of FR.

(c) The expected number of years until a person who commutes by automobile this year ends up avoiding the downtown area is $10.769 \approx 10.77$ yr since that is the sum of the entries in row 1 of F.

28. **(a)**

$$\begin{array}{cc} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} .4 & .3 & .2 & .1 & 0 \\ .2 & .1 & 0 & .6 & .1 \\ 0 & 0 & 1 & 0 & 0 \\ .1 & .1 & .4 & .1 & .3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

is the transition matrix. States 3 and 5 are absorbing. Upon rearranging, we obtain

$$\begin{array}{cc} & \begin{array}{ccccc} 3 & 5 & 1 & 2 & 4 \end{array} \\ \begin{array}{c} 3 \\ 5 \\ 1 \\ 2 \\ 4 \end{array} & \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline .2 & 0 & .4 & .3 & .1 \\ 0 & .1 & .2 & .1 & .6 \\ .4 & .3 & .1 & .1 & .1 \end{array}\right] \end{array}.$$

Then

$$\begin{aligned} F &= \begin{bmatrix} .6 & -.3 & -.1 \\ -.2 & .9 & -.6 \\ -.1 & -.1 & .9 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 2.0436 & .7629 & .7357 \\ .6540 & 1.4441 & 1.0354 \\ .2997 & .2452 & 1.3079 \end{bmatrix}, \end{aligned}$$

and

$$FR = \begin{array}{cc} & \begin{array}{cc} 3 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 4 \end{array} & \begin{bmatrix} .703 & .297 \\ .545 & .455 \\ .583 & .417 \end{bmatrix} \end{array}.$$

The second column of FR gives the probability of ending up in compartment 5 given the initial compartment.

(b) From compartment 1, the probability is .297.

(c) From compartment 2, the probability is .455.

(d) From compartment 3, the probability is 0 since state 3 is absorbing.

(e) From compartment 4, the probability is .417.

(f) To find the expected number of times that a rat in compartment 1 will be in compartment 1 before ending up in compartment 3 or 5, look at the entry in row 1, column 1 of F, $2.0436 \approx 2.04$.

(g) To find the expected number of times that a rat in compartment 4 will be in compartment 4 before ending up in compartment 3 or 5, look at the entry in row 3, column 3 of F, $1.3079 \approx 1.31$.

Chapter 10 Review Exercises

4. $\begin{bmatrix} -.2 & 1.2 \\ .8 & .2 \end{bmatrix}$

This could not be a transition matrix because it contains a negative entry.

6. $\begin{bmatrix} .6 & .2 & .3 \\ .1 & .5 & .4 \\ .3 & .3 & .4 \end{bmatrix}$

This could not be a transition matrix because the sum of the entries in the first row is not 1.

8. (a) $D = \begin{bmatrix} .3 & .7 \\ .5 & .5 \end{bmatrix}$

$$D^2 = \begin{bmatrix} .3 & .7 \\ .5 & .5 \end{bmatrix} \begin{bmatrix} .3 & .7 \\ .5 & .5 \end{bmatrix} = \begin{bmatrix} .44 & .56 \\ .4 & .6 \end{bmatrix}$$

$$\begin{aligned} D^3 = D \cdot D^2 &= \begin{bmatrix} .3 & .7 \\ .5 & .5 \end{bmatrix} \begin{bmatrix} .44 & .56 \\ .4 & .6 \end{bmatrix} \\ &= \begin{bmatrix} .412 & .588 \\ .42 & .58 \end{bmatrix} \end{aligned}$$

(b) The entry in row 2, column 1 of D^3 gives the probability that state 2 changes to state 1 after three repetitions of the experiment. This probability is .42.

10. (a) $F = \begin{bmatrix} .14 & .12 & .74 \\ .35 & .28 & .37 \\ .71 & .24 & .05 \end{bmatrix}$

$$F^2 = \begin{bmatrix} .587 & .228 & .185 \\ .4097 & .2092 & .3811 \\ .2189 & .1644 & .6167 \end{bmatrix}$$

$$F^3 = F \cdot F^2 = \begin{bmatrix} .2933 & .1787 & .5280 \\ .4012 & .1992 & .3996 \\ .5260 & .2203 & .2536 \end{bmatrix}$$

(The entries of F^3 are rounded to 4 places.)

(b) The probability that state 2 changes to state 1 after three repetitions of the experiment is .4012.

12. $D = \begin{bmatrix} .8 & .2 \end{bmatrix}$; $T = \begin{bmatrix} .7 & .3 \\ .2 & .8 \end{bmatrix}$

$$T^2 = \begin{bmatrix} .7 & .3 \\ .2 & .8 \end{bmatrix} \begin{bmatrix} .7 & .3 \\ .2 & .8 \end{bmatrix} = \begin{bmatrix} .55 & .45 \\ .3 & .7 \end{bmatrix}$$

Then

$$DT^2 = \begin{bmatrix} .8 & .2 \end{bmatrix} \begin{bmatrix} .55 & .45 \\ .3 & .7 \end{bmatrix} = \begin{bmatrix} .5 & .5 \end{bmatrix}$$

gives the distribution after two repetitions. To find the long-range distribution, let V be the probability vector $\begin{bmatrix} v_1 & v_2 \end{bmatrix}$. We want to find V such that

$$VT = V,$$

or $\begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} .7 & .3 \\ .2 & .8 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$.

This matrix equation is equivalent to the system

$$\begin{aligned} .7v_1 + .2v_2 &= v_1 \\ .3v_1 + .8v_2 &= v_2. \end{aligned}$$

Simplify these equations to get the system

$$\begin{aligned} -.3v_1 + .2v_2 &= 0 \\ .3v_1 - .2v_2 &= 0. \end{aligned}$$

These equations are dependent, so we need an additional equation in order to find the values of v_1 and v_2. Since $V = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$ is a probability vector,

$$v_1 + v_2 = 1.$$

To find v_1 and v_2, solve the system

$$\begin{aligned} -.3v_1 + .2v_2 &= 0 \quad (1) \\ v_1 + v_2 &= 1. \quad (2) \end{aligned}$$

From equation (2), $v_1 = 1 - v_2$. Substitute $1 - v_2$ for v_1 in equation (1) to get

$$\begin{aligned} -.3(1 - v_2) + .2v_2 &= 0 \\ -.3 + .3v_2 + .2v_2 &= 0 \\ .5v_2 &= .3 \\ v_2 &= \tfrac{3}{5}, \\ \text{and} \qquad v_1 &= 1 - \tfrac{3}{5} = \tfrac{2}{5}. \end{aligned}$$

The long-range distribution is

$$\begin{bmatrix} \frac{2}{5} & \frac{3}{5} \end{bmatrix}.$$

14. $D = \begin{bmatrix} .1 & .1 & .8 \end{bmatrix}$; $T = \begin{bmatrix} .2 & .3 & .5 \\ .1 & .1 & .8 \\ .7 & .1 & .2 \end{bmatrix}$

$$T^2 = \begin{bmatrix} .42 & .14 & .44 \\ .59 & .12 & .29 \\ .29 & .24 & .47 \end{bmatrix},$$

so

$$DT^2 = \begin{bmatrix} .333 & .218 & .449 \end{bmatrix}$$

gives the distribution after two repetitions. The long-range distribution comes from the solution of the system

$$\begin{aligned} .2v_1 + .1v_2 + .7v_3 &= v_1 \\ .3v_1 + .1v_2 + .1v_3 &= v_2 \\ .5v_1 + .8v_2 + .2v_3 &= v_3. \end{aligned}$$

Simplify these equations and also use the equation $v_1 + v_2 + v_3 = 1$ to get the system

$$\begin{aligned} v_1 + v_2 + v_3 &= 1 \\ -.8v_1 + .1v_2 + .7v_3 &= 0 \\ .3v_1 - .9v_2 + .1v_3 &= 0 \\ .5v_1 + .8v_2 - .8v_3 &= 0. \end{aligned}$$

Solve this system by the Gauss-Jordan method to obtain $v_1 = \frac{32}{81}$, $v_2 = \frac{29}{162}$, and $v_3 = \frac{23}{54}$. The long-term distribution is

$$\begin{bmatrix} \frac{32}{81} & \frac{29}{162} & \frac{23}{54} \end{bmatrix}.$$

16. $A = \begin{bmatrix} 0 & 1 \\ .2 & .8 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 0 & 1 \\ .2 & .8 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ .2 & .8 \end{bmatrix}$$
$$= \begin{bmatrix} .2 & .8 \\ .16 & .84 \end{bmatrix}$$

A^2 has all positive entries, so A is regular.

18. Let $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ .3 & .5 & .2 \end{bmatrix}$.

B is not regular since any power of B will have $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ as its first row and thus cannot have all positive entries.

22. $\begin{bmatrix} 1 & 0 & 0 \\ .5 & .1 & .4 \\ 0 & 1 & 0 \end{bmatrix}$

Since $p_{11} = 1$, state 1 is absorbing. Since $p_{21} \neq 0$, it is possible to go from state 2 to the absorbing state. Since $p_{32} = 1$ and $p_{21} = .5$, it is possible to go from state 3 to the absorbing state in two steps. This is the transition matrix of an absorbing Markov chain.

24. $\begin{bmatrix} .5 & .1 & .1 & .3 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ .1 & .8 & .05 & .05 \end{bmatrix}$

There are no absorbing states. Hence, this is not the transition matrix for an absorbing Markov chain.

26.
$$\begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ .3 & .1 & .6 \end{bmatrix} \end{array}$$

States 1 and 2 are absorbing. Since the absorbing states already come first, we do not have to rearrange the rows and columns. We have

$$P = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline .3 & .1 & .6 \end{array}\right]$$

$$R = \begin{bmatrix} .3 & .1 \end{bmatrix} \text{ and } Q = [.6].$$

$$F = (I_1 - Q)^{-1} = [1 - .6]^{-1} = [.4]^{-1} = \begin{bmatrix} \frac{5}{2} \end{bmatrix},$$

and

$$FR = \begin{bmatrix} \frac{5}{2} \end{bmatrix} \begin{bmatrix} \frac{3}{10} & \frac{1}{10} \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \end{bmatrix}.$$

28.
$$\begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} .3 & .5 & .1 & .1 \\ .4 & .1 & .3 & .2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Rearrange the rows and columns so that absorbing states 3 and 4 come first.

$$\begin{array}{cc} & \begin{array}{cccc} 3 & 4 & 1 & 2 \end{array} \\ \begin{array}{c} 3 \\ 4 \\ 1 \\ 2 \end{array} & \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline .1 & .1 & .3 & .5 \\ .3 & .2 & .4 & .1 \end{array}\right] \end{array}$$

We have

$$R = \begin{bmatrix} .1 & .1 \\ .3 & .2 \end{bmatrix} \text{ and } Q = \begin{bmatrix} .3 & .5 \\ .4 & .1 \end{bmatrix}.$$

$$F = (I_2 - Q)^{-1} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} .3 & .5 \\ .4 & .1 \end{bmatrix} \right)^{-1}$$
$$= \begin{bmatrix} .7 & -.5 \\ -.4 & .9 \end{bmatrix}^{-1} = \begin{bmatrix} 2.0930 & 1.1628 \\ .9302 & 1.6279 \end{bmatrix}$$

and

$$FR = \begin{bmatrix} 2.0930 & 1.1628 \\ .9302 & 1.6279 \end{bmatrix} \begin{bmatrix} .1 & .1 \\ .3 & .2 \end{bmatrix}$$
$$= \begin{bmatrix} .5581 & .4419 \\ .5814 & .4186 \end{bmatrix}.$$

30. Let $P = \begin{bmatrix} .8 & .2 \\ .4 & .6 \end{bmatrix}$, and let V be the probability vector $\begin{bmatrix} v_1 & v_2 \end{bmatrix}$.

$$\begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} .8 & .2 \\ .4 & .6 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$$

$$\begin{aligned} .8v_1 + .4v_2 &= v_1 \\ .2v_1 + .6v_2 &= v_2 \end{aligned}$$

Simplify these equations to get the system

$$\begin{aligned} -.2v_1 + .4v_2 &= 0 \\ .2v_1 - .4v_2 &= 0. \end{aligned}$$

These equations are dependent. Use the substitution method to solve the system

$$\begin{aligned} -.2v_1 + .4v_2 &= 0 \\ v_1 + \quad v_2 &= 1, \end{aligned}$$

obtaining $v_1 = \frac{2}{3}$ and $v_2 = \frac{1}{3}$. The long-range market share for Dogkins is $v_1 = \frac{2}{3}$.

32. $X_0 = \begin{bmatrix} .4 & .4 & .2 \end{bmatrix}$

$$A = \begin{bmatrix} .8 & .15 & .05 \\ .25 & .55 & .2 \\ .04 & .21 & .75 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} .6795 & .213 & .1075 \\ .3455 & .382 & .2725 \\ .1145 & .279 & .6065 \end{bmatrix}$$

The distribution after 2 mo is given by

$$X_0A^2 = \begin{bmatrix} .4329 & .2938 & .2733 \end{bmatrix}.$$

34. To find the long-range distribution, use the system

$$\begin{aligned} v_1 + \quad v_2 + \quad v_3 &= 1 \\ .8v_1 + .25v_2 + .04v_3 &= v_1 \\ .15v_1 + .55v_2 + .21v_3 &= v_2 \\ .05v_1 + \ .2v_2 + .75v_3 &= v_3. \end{aligned}$$

Simplify these equations to obtain the system

$$\begin{aligned} v_1 + \quad v_2 + \quad v_3 &= 1 \\ -.2v_1 + .25v_2 + .04v_3 &= 0 \\ .15v_1 - .45v_2 + .21v_3 &= 0 \\ .05v_1 + \ .2v_2 - .25v_3 &= 0. \end{aligned}$$

Solve this system by the Gauss-Jordan method to obtain $v_1 = \frac{47}{114}$, $v_2 = \frac{32}{114}$, and $v_3 = \frac{35}{114}$. The long-range distribution is

$$\begin{bmatrix} \frac{47}{114} & \frac{32}{114} & \frac{35}{114} \end{bmatrix}.$$

In Exercises 36-44, the original transition matrix is

$$P = \begin{bmatrix} .3 & .5 & .2 \\ .2 & .6 & .2 \\ .1 & .5 & .4 \end{bmatrix}.$$

36. The probability that a man of normal weight will have a thin son is given by the entry in row 2, column 1 of P, which is .2.

38. $P^2 = \begin{bmatrix} .3 & .5 & .2 \\ .2 & .6 & .2 \\ .1 & .5 & .4 \end{bmatrix} \begin{bmatrix} .3 & .5 & .2 \\ .2 & .6 & .2 \\ .1 & .5 & .4 \end{bmatrix}$

$$= \begin{bmatrix} .21 & .55 & .24 \\ .2 & .56 & .24 \\ .17 & .55 & .28 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} .3 & .5 & .2 \\ .2 & .6 & .2 \\ .1 & .5 & .4 \end{bmatrix} \begin{bmatrix} .21 & .55 & .24 \\ .2 & .56 & .24 \\ .17 & .55 & .28 \end{bmatrix}$$

$$= \begin{bmatrix} .197 & .555 & .248 \\ .196 & .556 & .248 \\ .189 & .555 & .256 \end{bmatrix}$$

The probability that a man of normal weight will have a thin great-grandson is given by the entry in row 2, column 1 of P^3, which is .196.

40. $P^2 = \begin{bmatrix} .21 & .55 & .24 \\ .2 & .56 & .24 \\ .17 & .55 & .28 \end{bmatrix}$

The probability that an overweight man will have an overweight grandson is given by the entry in row 3, column 3 of P^2, which is .28.

42. The distribution of men by weight after 1 generation is

$$\begin{bmatrix} .2 & .55 & .25 \end{bmatrix} \begin{bmatrix} .3 & .5 & .2 \\ .2 & .6 & .2 \\ .1 & .5 & .4 \end{bmatrix}$$

$$= \begin{bmatrix} .195 & .555 & .25 \end{bmatrix}.$$

44. The distribution of men by weight after 3 generations is

$$\begin{bmatrix} .2 & .55 & .25 \end{bmatrix} \begin{bmatrix} .197 & .555 & .248 \\ .196 & .556 & .248 \\ .189 & .555 & .256 \end{bmatrix}$$

$$= \begin{bmatrix} .194 & .556 & .25 \end{bmatrix}.$$

46. If the offspring both carry genes AA, then so must their offspring; hence, state 1 ends up in state 1 with probability 1. If the offspring both carry genes a, then so must their offspring; hence, state 6 ends up in state 6 with probability 1. If AA mates with aa, then the offspring will carry genes Aa; hence, state 3 ends up in state 4 with probability 1. If AA mates with Aa, then there are four possible outcomes for a pair of offspring: AA and AA is one of the outcomes, so state 2 ends up in state 1 with probability $\frac{1}{4}$; AA and Aa can happen two ways, so state 2 ends up in state 2 with probability $\frac{2}{4}$ or $\frac{1}{2}$; and Aa and Aa is the last possible outcome, so state 2 ends up in state 4 with probability $\frac{1}{4}$. If Aa mates with Aa, then there are sixteen possible outcomes for a pair of offspring: state 4 ends up in states 1, 2, 3, 4, 5, 6 with respective probabilities $\frac{1}{16}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{4}$, $\frac{1}{4}$, and $\frac{1}{16}$. If Aa mates with aa, then there are four possible outcomes for a pair of offspring, corresponding to three of the possible states: state 5 ends up in states 4, 5, 6 with respective probabilities $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{1}{4}$.

This verifies that the transition matrix for this mating experiment is

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{1}{16} & \frac{1}{4} & \frac{1}{8} & \frac{1}{4} & \frac{1}{4} & \frac{1}{16} \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

48. Rearrange the rows and columns of the transition matrix so that the absorbing states come first.

$$\begin{array}{c|cc|cccc}
 & 1 & 6 & 2 & 3 & 4 & 5 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
6 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline
2 & \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} & 0 \\
3 & 0 & 0 & 0 & 0 & 1 & 0 \\
4 & \frac{1}{16} & \frac{1}{16} & \frac{1}{4} & \frac{1}{8} & \frac{1}{4} & \frac{1}{4} \\
5 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{2}
\end{array}$$

From this rearranged matrix, observe that

$$Q = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{8} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$

50. In Exercise 49, it was shown that the fundamental matrix for this absorbing Markov chain is

$$F = \begin{bmatrix} \frac{8}{3} & \frac{1}{6} & \frac{4}{3} & \frac{2}{3} \\ \frac{4}{3} & \frac{4}{3} & \frac{8}{3} & \frac{4}{3} \\ \frac{4}{3} & \frac{1}{3} & \frac{8}{3} & \frac{4}{3} \\ \frac{2}{3} & \frac{1}{6} & \frac{4}{3} & \frac{8}{3} \end{bmatrix}.$$

If Aa mates with Aa (which corresponds to state 4, which in turn corresponds to row 3 of F), $\frac{8}{3}$ pairs of offspring with these genes can be expected before ending up in one of the two absorbing states. This is because $\frac{8}{3}$ is the entry in row 3, column 3 of F.

52. **(a)** After the duplication, there are $2n$ genes and n of them are being selected; this can be done in $\binom{2n}{n}$ different ways. Suppose there are i mutant genes before the duplication and j mutant genes in the next generation. After the duplication, there will be $2i$ mutant genes, of which j will be selected; this can be done in $\binom{2i}{j}$ different ways. Also, there are $2n - 2i$ nonmutant genes, of which $n - j$ will be selected; this can be done in $\binom{2n-2i}{n-j}$ different ways.

Therefore, the probability of a generation with i mutant genes being followed by a generation with j mutant genes, which is the transition probability from state i to state j, is

$$p_{ij} = \frac{\binom{2i}{j}\binom{2n-2i}{n-j}}{\binom{2n}{n}}.$$

(b) The absorbing states are state 0 and state n. If a generation has no mutant genes, then after duplication there will still be none, and if a generation consists entirely of mutant genes, its successor will also.

(c) Use

$$p_{ij} = \frac{\binom{2i}{j}\binom{2n-2i}{n-j}}{\binom{2n}{n}}$$

with $n = 3$ and $i = 0, 1, 2, 3$ and $j = 0, 1, 2, 3$ to calculate the entries of the transition matrix. Let $\binom{n}{r} = 0$ when $n < r$.

$$p_{00} = \frac{\binom{0}{0}\binom{6}{3}}{\binom{6}{3}} = 1,\ p_{01} = \frac{\binom{0}{1}\binom{6}{2}}{\binom{6}{3}} = 0, p_{02} = 0,\ p_{03} = 0, p_{10} = \frac{\binom{2}{0}\binom{4}{3}}{\binom{6}{3}} = \frac{1}{5}, p_{11} = \frac{\binom{2}{1}\binom{4}{2}}{\binom{6}{3}} = \frac{3}{5},$$

$$p_{12} = \frac{\binom{2}{2}\binom{4}{1}}{\binom{6}{3}} = \frac{1}{5},\ p_{13} = \frac{\binom{2}{3}\binom{4}{0}}{\binom{6}{3}} = 0, p_{20} = \frac{\binom{4}{0}\binom{2}{3}}{\binom{6}{3}} = 0,\ p_{21} = \frac{\binom{4}{1}\binom{2}{2}}{\binom{6}{3}} = \frac{1}{5}, p_{22} = \frac{\binom{4}{2}\binom{2}{1}}{\binom{6}{3}} = \frac{3}{5},$$

$$p_{23} = \frac{\binom{4}{3}\binom{2}{0}}{\binom{6}{3}} = \frac{1}{5}, p_{30} = \frac{\binom{6}{0}\binom{0}{3}}{\binom{6}{3}} = 0,\ p_{31} = 0,\ p_{32} = 0, p_{33} = \frac{\binom{6}{3}\binom{0}{0}}{\binom{6}{3}} = 1$$

The transition matrix is

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{5} & \frac{3}{5} & \frac{1}{5} & 0 \\ 0 & \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} = P.$$

(d) Rearrange the rows and columns of P.

$$\begin{array}{c|cc|cc} & 0 & 3 & 1 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 \\ \hline 1 & \frac{1}{5} & 0 & \frac{3}{5} & \frac{1}{5} \\ 2 & 0 & \frac{1}{5} & \frac{1}{5} & \frac{3}{5} \end{array}$$

$$R = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{5} \end{bmatrix},\ Q = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{3}{5} \end{bmatrix}, F = [I_2 - Q]^{-1} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{3}{5} \end{bmatrix} \right)^{-1} = \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{10}{3} & \frac{5}{3} \\ \frac{5}{3} & \frac{10}{3} \end{bmatrix}$$

$$FR = \begin{bmatrix} \frac{10}{3} & \frac{5}{3} \\ \frac{5}{3} & \frac{10}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

(e) If a set of 3 genes has 1 mutant gene, the probability that the mutant gene will disappear is $\frac{2}{3}$, since that is the entry in row 1, column 1 of FR.

(f) If a set of 3 genes has 1 mutant gene, 5 generations would be expected to have 1 mutant gene before either the mutant genes or the nonmutant genes disappear, since that is the sum of the entries in row 1 of F.

Extended Application: A Markov Chain Model for Teacher Rentention

1. 0.139; 0.055.
2. They will spend, on average, 13 years in the system.
3. In the transition matrix, the only possible transition from "On Sabbatical" is to "Continuing." This means that every teacher who was on sabbatical in year 1 or year 2 of the study came back to teach the following year. In general, sabbaticals are granted at most every 7 years, on the assumption that the teacher will teach at least one more year, so this outcome is not a surprise. Since in this sample the future of a teacher on sabbatical is exactly like that of a continuing teacher, the only difference in rows 2 and 4 of F is that row 4 is 1 higher in the sabbatical column (we know this teacher will spend at least one year on sabbatical, since he or she is currently on sabbatical). However, if a teacher had died on sabbatical, or become seriously ill in the following year, the transition matrix would show other possible transitions from the sabbatical state, and rows 2 and 4 would look different.
4. The expected number of years spent on sabbatical by a currently active teacher is 0.091, so sabbaticals are rare in this school system.

5. The product FR is

$$F \cdot R = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{cccc} \text{Resigned} & \text{Retired} & \text{Decreased} & \\ 0.785 & 0.198 & 0.021 & \text{New} \\ 0.734 & 0.240 & 0.026 & \text{Continuing} \\ 0.933 & 0.060 & 0.007 & \text{On Leave} \\ 0.734 & 0.240 & 0.026 & \text{On Sabbatical} \\ 0.626 & 0.352 & 0.022 & \text{Ill} \end{array}$$

(a) 0.933; 0.533

(b) The teacher with at least one year in the system (probability 0.24 versus 0.197 for the new teacher).

(c) 0.021

6. The Markov model assumes the transition probabilities are constant over time. With two sets of transition data, the experimenters could check this by seeing if the transition frequencies for year 1 to year 2 were approximately the same as those for year 2 to 3.

Chapter 11

GAME THEORY

11.1 Strictly Determined Games

For Exercises 2-8, use the following game.

$$\begin{array}{cc} & \begin{array}{ccc} & B & \\ 1 & 2 & 3 \end{array} \\ A\ \begin{array}{c}1\\2\\3\end{array} & \begin{bmatrix} 6 & -4 & 0 \\ 3 & -2 & 6 \\ -1 & 5 & 11 \end{bmatrix} \end{array}$$

2. The strategy $(1, 2)$ means that player A chooses row 1, and player B chooses column 2. A negative number represents a payoff from A to B. Row 1 and column 2 lead to the number -4, which represents a payoff of \$4 from A to B.

4. The strategy $(2, 3)$ means that player A chooses row 2, and player B chooses column 3. A positive number represents a payoff from B to A. Row 2 and column 3 lead to the number 6, which represents a payoff of \$6 from B to A.

6. The strategy $(3, 2)$ means that player A chooses row 3 and player B chooses column 2. Row 3 and column 2 lead to the number 5, which represents a payoff of \$5 from B to A.

8. Underline the smallest number in each row, and draw a box around the largest number in each column.

$$\begin{bmatrix} \boxed{6} & \underline{-4} & 0 \\ 3 & \underline{-2} & 6 \\ \underline{-1} & \boxed{5} & \boxed{11} \end{bmatrix}$$

There is no number that is both the smallest number in its row and the largest number in its column, so the game has no saddle point.

10. $\begin{bmatrix} 6 & 5 \\ 3 & 8 \\ -1 & -4 \end{bmatrix}$

Row 3 is dominated by both rows 1 and 2, so remove row 3 to obtain

$$\begin{bmatrix} 6 & 5 \\ 3 & 8 \end{bmatrix}.$$

12. $\begin{bmatrix} 2 & 3 & 1 & -5 \\ -1 & 5 & 4 & 1 \\ 1 & 0 & 2 & -3 \end{bmatrix}$

Column 4 dominates columns 2 and 3, so remove columns 2 and 3 to obtain

$$\begin{bmatrix} 2 & -5 \\ -1 & 1 \\ 1 & -3 \end{bmatrix}.$$

14. $\begin{bmatrix} 6 & 2 \\ -1 & 10 \\ 3 & 5 \end{bmatrix}$

There are no dominated rows or columns, so there are no dominated strategies.

16. $\begin{bmatrix} \boxed{\underline{7}} & 8 \\ \underline{-2} & \boxed{15} \end{bmatrix}$

Underline the smallest number in each row, and box the largest number in each column. The number that is both boxed and underlined is 7. Thus, the saddle point is 7 at the strategy $(1, 1)$, and the game has a value of 7. Since the game has a saddle point, it is strictly determined.

18. $\begin{bmatrix} -4 & 2 & -3 & \boxed{\underline{-7}} \\ \boxed{4} & \boxed{3} & \boxed{5} & \underline{-9} \end{bmatrix}$

-7 is both the smallest number in its row and the largest number in its column. The saddle point is -7 at the strategy $(1, 4)$; the game has value -7 and is strictly determined.

20. $\begin{bmatrix} 1 & 4 & \underline{-3} & 1 & -1 \\ \boxed{2} & \boxed{5} & \underline{0} & 4 & \boxed{10} \\ 1 & \underline{-3} & \boxed{2} & \boxed{5} & 2 \end{bmatrix}$

There is no number that is both the smallest in its row and the largest in its column. Therefore, there is no saddle point, and the game is not strictly determined.

22. $$\begin{bmatrix} \boxed{3} & \boxed{8} & -4 & \underline{-9} \\ -1 & -2 & \boxed{\underline{-3}} & 0 \\ -2 & 6 & \underline{-4} & \boxed{5} \end{bmatrix}$$

-3 is both the smallest number in its row and the largest number in its column. The saddle point is -3 at the strategy $(2, 3)$; the game has value -3 and is strictly determined.

24. $$\begin{bmatrix} -3 & -2 & \boxed{6} \\ 2 & \boxed{0} & 2 \\ \boxed{5} & -2 & \underline{-4} \end{bmatrix}$$

The entry 0 is both the smallest number in its row and the largest number in its column. The saddle point is 0 at the strategy $(2, 2)$; the game has a value of 0 and is strictly determined.

26. $$\begin{bmatrix} 3 & 8 & -4 & -9 \\ -1 & -2 & -3 & 0 \\ -2 & 6 & -4 & 5 \end{bmatrix}$$

(a) Every entry in column 3 is smaller than the corresponding entries in columns 1 and 2. Thus, column 3 dominates column 1 and column 2. So, remove columns 1 and 2 to obtain

$$\begin{bmatrix} -4 & -9 \\ -3 & 0 \\ -4 & 5 \end{bmatrix}.$$

(b) Every entry in row 2 is greater than the corresponding entry in row 1. Thus, row 2 dominates row 1. So, remove row 1 to obtain

$$\begin{bmatrix} -3 & 0 \\ -4 & 5 \end{bmatrix}.$$

(c) Every entry in column 1 is smaller than the corresponding entry in column 2. This, column 1 dominates column 2. So, remove column 2 to obtain

$$\begin{bmatrix} -3 \\ -4 \end{bmatrix}.$$

(d) The entry in row 1 is larger than the entry in row 2. Thus, row 1 dominates row 2. So, remove row 2 to obtain

$$[-3].$$

To verify that this is a saddle point, underline the smallest number in each row and box the largest number in each column.

$$\begin{bmatrix} \boxed{3} & \boxed{8} & -4 & \underline{-9} \\ -1 & -2 & \boxed{\underline{-3}} & 0 \\ -2 & 6 & \underline{-4} & \boxed{5} \end{bmatrix}$$

Thus, -3 is the saddle point.

28.
$$\begin{array}{l} \\ \text{Repair} \\ \text{No Repair} \end{array} \begin{array}{c} \begin{array}{ccc} .01 & .10 & .20 \end{array} \\ \begin{bmatrix} -\$130 & -\$130 & -\$130 \\ -\ \ \$25 & -\$200 & -\$500 \end{bmatrix} \end{array}$$

(a) An optimist should make no repairs; minimum cost is \$25.

(b) A pessimist should make repairs; a worst case of \$130 is better than a possible cost of \$500 if no repairs are made.

(c) Find the expected cost of each strategy.

Make repairs:

$$.7(-130) + .2(-130) + .1(-130) = \$130$$

Make no repairs:

$$.7(-25) + .2(-200) + .1(-500) = -\$107.50$$

He should make no repairs. The expected cost to the company if this strategy is chosen is $-\$107.50$.

32. **(a)**
$$\begin{array}{l} \\ \text{Overhaul} \\ \text{Don't Overhaul} \end{array} \begin{array}{c} \begin{array}{cc} \text{Fails} & \text{Doesn't Fail} \end{array} \\ \begin{bmatrix} -\$8600 & -\$2600 \\ -\$6000 & \$0 \end{bmatrix} \end{array}$$

(b) Find the expected cost under each strategy.

Overhaul:

$$.1(-8600) + .9(-2600) = -\$3200$$

Don't overhaul:

$$.55(-6000) + .7(0) = -\$3300$$

To minimize his expected costs, the businessman should overhaul the machine before shipping.

(c) Column 1 dominates column 2, and row 2 dominates row 1. The saddle point is $-\$6000$.

34.

$$\begin{array}{cc|ccc} & & & \text{B} & \\ & & \text{City 1} & \text{City 2} & \text{City 3} \\ & \text{City 1} & 5 & -2 & 6 \\ \text{A} & \text{City 2} & 7 & 5 & 9 \\ & \text{City 3} & 3 & -3 & 5 \end{array}$$

To get the entries in the above matrix, look, for example, at the entry in row 2, column 1. If merchant A locates in city 2 and merchant B in city 1, then merchant A will get 80% of the business in city 2, 20% in city 3, and 60% in city 1. Taking into account the fraction of the population living in each city, we get

$$.80(.45) + .20(.30) + .60(.25) = .57.$$

Thus, merchant A gets 57% of the total business. Now 57% is 7 percentage points above 50%, so the entry in row 2, column 1 is +7.

Likewise for row 3, column 1, we get

$$.80(.25) + .20(.30) + .60(.45) = .53 = 53\%,$$

which is 3 percentage points above 50%. The other entries are found in a similar manner. (Note that all diagonal entries are 5 since 55% is 5 percentage points above 50%.)

The 5 at $(2, 2)$ is the smallest entry in its row and the largest in its column, so the saddle point is the 5 at $(2, 2)$. The value of the game is 5.

36. $$\begin{bmatrix} \boxed{3} & -8 & \boxed{\underline{-9}} \\ 0 & \boxed{6} & \underline{-12} \\ -8 & 4 & \underline{-10} \end{bmatrix}$$

Underline the smallest number in each row, and box the largest number in each column. -9 is the smallest entry in its row and the largest in its column. The saddle point is -9 at $(1, 3)$, and the value of the game is -9.

38. The payoff matrix is as follows.

$$\begin{array}{c|ccc} & \text{Rock} & \text{Paper} & \text{Scissors} \\ \text{Rock} & 0 & \underline{-1} & \boxed{1} \\ \text{Paper} & \boxed{1} & 0 & \underline{-1} \\ \text{Scissors} & \underline{-1} & \boxed{1} & 0 \end{array}$$

Underline the smallest number in each row, and box the largest number in each column; in this matrix, the two categorizations do not overlap. The game is not strictly determined since it does not have a saddle point.

11.2 Mixed Strategies

2. (a) $$AMB = \begin{bmatrix} .1 & .4 & .5 \end{bmatrix} \begin{bmatrix} 0 & -4 & 1 \\ 3 & 2 & -4 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} .2 \\ .4 \\ .4 \end{bmatrix}$$

$$= \begin{bmatrix} 1.7 & -.1 & -1.5 \end{bmatrix} \begin{bmatrix} .2 \\ .4 \\ .4 \end{bmatrix}$$

$$= [-.3]$$

The expected value of the game is $-.3$.

(b) $$AMB = \begin{bmatrix} .3 & .4 & .3 \end{bmatrix} \begin{bmatrix} 0 & -4 & 1 \\ 3 & 2 & -4 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} .8 \\ .1 \\ .1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.5 & -.7 & -1.3 \end{bmatrix} \begin{bmatrix} .8 \\ .1 \\ .1 \end{bmatrix}$$

$$= [1]$$

The expected value of the game is 1.

4. $$\begin{bmatrix} -4 & 5 \\ 3 & -4 \end{bmatrix}$$

This game has no saddle point, so it is not strictly determined and mixed strategies must be used. Here $a_{11} = -4, a_{12} = 5, a_{21} = 3$, and $a_{22} = -4$. To find the optimum strategy for player A, first find p_1.

$$\begin{aligned} p_1 &= \frac{a_{22} - a_{21}}{a_{11} - a_{21} - a_{12} + a_{22}} \\ &= \frac{-4-3}{-4-3-5+(-4)} = \frac{-7}{-16} = \frac{7}{16}, \\ p_2 &= 1 - p_1 = 1 - \frac{7}{16} = \frac{9}{16} \end{aligned}$$

Now find the optimum strategy for player B.

$$\begin{aligned} q_1 &= \frac{a_{22} - a_{12}}{a_{11} - a_{21} - a_{12} + a_{22}} \\ &= \frac{-4-5}{-4-3-5+(-4)} \\ &= \frac{-9}{-16} = \frac{9}{16}, \\ q_2 &= 1 - q_1 = 1 - \frac{9}{16} \\ &= \frac{7}{16} \end{aligned}$$

Thus, player A should choose row 1 with probability $\frac{7}{16}$ and row 2 with probability $\frac{9}{16}$. Player B should choose column 1 with probability $\frac{9}{16}$ and column 2 with probability $\frac{7}{16}$. The value of the game is

$$\begin{aligned}\frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11} - a_{21} - a_{12} + a_{22}} &= \frac{-4(-4) - 5(3)}{-4 - 3 - 5 + (-4)} \\ &= -\frac{1}{16}.\end{aligned}$$

6. $\begin{bmatrix} 6 & 2 \\ -1 & 10 \end{bmatrix}$

There are no saddle points. For player A, the optimum strategy is

$$\begin{aligned}p_1 &= \frac{10 - (-1)}{6 - (-1) - 2 + 10} = \frac{11}{15}, \\ p_2 &= 1 - p_1 = \frac{4}{15}.\end{aligned}$$

For player B, the optimum strategy is

$$\begin{aligned}q_1 &= \frac{10 - 2}{6 - (-1) - 2 + 10} = \frac{8}{15}, \\ q_2 &= 1 - q_1 = \frac{7}{15}.\end{aligned}$$

The value of the game is

$$\frac{6(10) - 2(-1)}{6 - (-1) - 2 + 10} = \frac{62}{15}.$$

8. $\begin{bmatrix} 0 & 6 \\ 4 & 0 \end{bmatrix}$

There are no saddle points. For player A, the optimum strategy is

$$\begin{aligned}p_1 &= \frac{0 - 4}{0 - 4 - 6 + 0} = \frac{2}{5}, \\ p_2 &= 1 - p_1 = \frac{3}{5}.\end{aligned}$$

For player B, the optimum strategy is

$$\begin{aligned}q_1 &= \frac{0 - 6}{0 - 4 - 6 + 0} = \frac{3}{5}, \\ q_2 &= 1 - q_1 = \frac{2}{5}.\end{aligned}$$

The value of the game is

$$\frac{0(0) - 6(4)}{0 - 4 - 6 + 0} = \frac{12}{5}.$$

10. $\begin{bmatrix} 6 & \frac{3}{4} \\ \frac{2}{3} & -1 \end{bmatrix}$

The game is strictly determined since it has a saddle point. Thus, pure strategies can be used. The value of the game is $\frac{3}{4}$, which is the saddle point and occurs at the strategy $(1, 2)$.

12. $\begin{bmatrix} -\frac{1}{2} & \frac{2}{3} \\ \frac{7}{8} & -\frac{3}{4} \end{bmatrix}$

There are no saddle points. For player A, the optimum strategy is

$$\begin{aligned}p_1 &= \frac{-\frac{3}{4} - \frac{7}{8}}{-\frac{1}{2} - \frac{7}{8} - \frac{2}{3} + \left(-\frac{3}{4}\right)} = \frac{-\frac{13}{8}}{-\frac{67}{24}} = \frac{39}{67}, \\ p_2 &= 1 - p_1 = \frac{28}{67}.\end{aligned}$$

For player B, the optimum strategy is

$$\begin{aligned}q_1 &= \frac{-\frac{3}{4} - \frac{2}{3}}{-\frac{1}{2} - \frac{7}{8} - \frac{2}{3} + \left(-\frac{3}{4}\right)} = \frac{-\frac{17}{12}}{-\frac{67}{24}} = \frac{34}{67}, \\ q_2 &= 1 - q_1 = \frac{33}{67}.\end{aligned}$$

The value of the game is

$$\frac{-\frac{1}{2}\left(-\frac{3}{4}\right) - \frac{2}{3}\left(\frac{7}{8}\right)}{-\frac{1}{2} - \frac{7}{8} - \frac{2}{3} + \left(-\frac{3}{4}\right)} = \frac{-\frac{5}{24}}{-\frac{67}{24}} = \frac{5}{67}.$$

14. $\begin{bmatrix} 8 & 18 \\ -4 & 2 \end{bmatrix}$

The game is strictly determined since there is a saddle point. The value of the game is 8, which is the saddle point and occurs at the strategy $(1, 1)$.

16. $\begin{bmatrix} 3 & 4 & -1 \\ -2 & 1 & 0 \end{bmatrix}$

Column 1 dominates column 2, so remove column 2. This gives the payoff matrix

$$\begin{bmatrix} 3 & -1 \\ -2 & 0 \end{bmatrix}.$$

For player A, the optimum strategy is

$$\begin{aligned}p_1 &= \frac{0 - (-2)}{3 - (-2) - (-1) + 0} = \frac{1}{3}, \\ p_2 &= 1 - p_1 = \frac{2}{3}.\end{aligned}$$

For player B, the optimum strategy is

$$\begin{aligned}q_1 &= \frac{0 - (-1)}{3 - (-2) - (-1) + 0} = \frac{1}{6}, \\ q_2 &= 0 \text{ (column 2 was removed)}, \\ q_3 &= 1 - (q_1 + q_2) = \frac{5}{6}.\end{aligned}$$

The value of the game is

$$\frac{3(0)-(-1)(-2)}{3-(-2)-(-1)+0} = -\frac{1}{3}.$$

18. $\begin{bmatrix} -1 & 6 \\ 8 & 3 \\ -2 & 5 \end{bmatrix}$

Row 1 dominates row 3, so remove row 3. This gives the matrix

$$\begin{bmatrix} -1 & 6 \\ 8 & 3 \end{bmatrix}.$$

For player A, the optimum strategy is

$$p_1 = \frac{3-8}{-1-8-6+3} = \frac{5}{12},$$
$$p_2 = 1 - p_1 = \frac{7}{12},$$
$$p_3 = 0 \text{ (row 3 was removed).}$$

For player B, the optimum strategy is

$$q_1 = \frac{3-6}{-1-8-6+3} = \frac{1}{4},$$
$$q_2 = 1 - q_1 = \frac{3}{4}.$$

The value of the game is

$$\frac{-1(3)-6(8)}{-1-8-6+3} = \frac{17}{4}.$$

20. $\begin{bmatrix} 4 & 8 & -3 \\ 2 & -1 & 1 \\ 7 & 9 & 0 \end{bmatrix}$

Remove row 1 (which is dominated by row 3) and column 1 (which is dominated by column 3). This gives the matrix

$$\begin{bmatrix} -1 & 1 \\ 9 & 0 \end{bmatrix}.$$

For player A, the optimum strategy is

$$p_1 = 0 \text{ (row 1 was removed),}$$
$$p_2 = \frac{0-9}{-1-9-1+0} = \frac{9}{11},$$
$$p_3 = 1 - (p_1 + p_2) = \frac{2}{11}.$$

For player B, the optimum strategy is

$$q_1 = 0 \text{ (column 1 was removed),}$$
$$q_2 = \frac{0-1}{-1-9-1+0} = \frac{1}{11},$$
$$q_3 = 1 - (q_1 + q_2) = \frac{10}{11}.$$

The value of the game is

$$\frac{-1(0)-1(9)}{-1-9-1+0} = \frac{9}{11}.$$

22. Let the non-strictly-determined game have payoff matrix

$$M = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

The optimum strategy for player A is $\begin{bmatrix} p_1 & p_2 \end{bmatrix}$, where

$$p_1 = \frac{a_{22}-a_{21}}{a_{11}-a_{21}-a_{12}+a_{22}},$$
$$p_2 = \frac{a_{11}-a_{12}}{a_{11}-a_{21}-a_{12}+a_{22}}.$$

The optimum strategy for player B is $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$, where

$$q_1 = \frac{a_{22}-a_{12}}{a_{11}-a_{21}-a_{12}+a_{22}},$$
$$q_2 = \frac{a_{11}-a_{21}}{a_{11}-a_{21}-a_{12}+a_{22}}.$$

The value of the game is AMB.

$$AMB = \begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$
$$= \begin{bmatrix} p_1a_{11}+p_2a_{21} & p_1a_{12}+p_2a_{22} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$
$$AMB = [p_1a_{11}q_1 + p_2a_{21}q_1 + p_1a_{12}q_2 + p_2a_{22}q_2]$$

Since this is a 1×1 matrix, we will drop the matrix notation.

$$\begin{aligned} AMB = {} & \frac{a_{22}-a_{21}}{a_{11}-a_{21}-a_{12}+a_{22}} \cdot a_{11} \cdot \frac{a_{22}-a_{12}}{a_{11}-a_{21}-a_{12}+a_{22}} \\ & + \frac{a_{11}-a_{12}}{a_{11}-a_{21}-a_{12}+a_{22}} \cdot a_{21} \cdot \frac{a_{22}-a_{21}}{a_{11}-a_{21}-a_{12}+a_{22}} \\ & + \frac{a_{22}-a_{21}}{a_{11}-a_{21}-a_{12}+a_{22}} \cdot a_{12} \cdot \frac{a_{11}-a_{21}}{a_{11}-a_{21}-a_{12}+a_{22}} \\ & + \frac{a_{11}-a_{12}}{a_{11}-a_{21}-a_{12}+a_{22}} \cdot a_{22} \cdot \frac{a_{11}-a_{21}}{a_{11}-a_{21}-a_{12}+a_{22}} \\ = {} & \frac{a_{11}(a_{22}-a_{21})(a_{22}-a_{12})}{(a_{11}-a_{21}-a_{12}+a_{22})^2} \\ & + \frac{a_{21}(a_{11}-a_{12})(a_{22}-a_{12})}{(a_{11}-a_{21}-a_{12}+a_{22})^2} \\ & + \frac{a_{12}(a_{22}-a_{21})(a_{11}-a_{21})}{(a_{11}-a_{21}-a_{12}+a_{22})^2} \\ & + \frac{a_{22}(a_{11}-a_{12})(a_{11}-a_{21})}{(a_{11}-a_{21}-a_{12}+a_{22})^2} \end{aligned}$$

$$= \frac{(a_{22}-a_{12})[a_{11}(a_{22}-a_{21})+a_{21}(a_{11}-a_{12})]}{(a_{11}-a_{21}-a_{12}+a_{22})^2} + \frac{(a_{11}-a_{21})[a_{12}(a_{22}-a_{21})+a_{22}(a_{11}-a_{12})]}{(a_{11}-a_{21}-a_{12}+a_{22})^2}$$

$$= \frac{(a_{22}-a_{12})(a_{11}a_{22}-a_{21}a_{12})}{(a_{11}-a_{21}-a_{12}+a_{22})^2} + \frac{(a_{11}-a_{21})(-a_{12}a_{21}+a_{22}a_{11})}{(a_{11}-a_{21}-a_{12}+a_{22})^2}$$

$$= \frac{(a_{11}a_{22}-a_{21}a_{12})(a_{22}-a_{12}+a_{11}-a_{21})}{(a_{11}-a_{21}-a_{12}+a_{22})^2}$$

$$= \frac{a_{11}a_{22}-a_{21}a_{12}}{a_{11}-a_{21}-a_{12}+a_{22}}$$

This is the expression for the value of a non-strictly-determined game that is given in the text.

26. Let

$$M = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

be the payoff matrix of the game.

(a) Let $A = [p_1 \quad p_2]$ be the optimum strategy for player A. Thus,

$$p_1 = \frac{a_{22}-a_{21}}{d} \text{ and } p_2 = \frac{a_{11}-a_{12}}{d}$$

where $d = a_{11} - a_{21} - a_{12} + a_{22}$.

$$AM = [p_1 \quad p_2] \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$
$$= [p_1a_{11}+p_2a_{21} \quad p_1a_{12}+p_2a_{22}]$$
$$= [g \quad g]$$

Since

$$p_1a_{11}+p_2a_{21}$$
$$= \frac{a_{22}-a_{21}}{d}\cdot a_{11} + \frac{a_{11}-a_{12}}{d}\cdot a_{21}$$
$$= \frac{a_{11}a_{22}-a_{11}a_{21}}{d} + \frac{a_{11}a_{21}-a_{12}a_{21}}{d}$$
$$= \frac{a_{11}a_{22}-a_{11}a_{21}+a_{11}a_{21}-a_{12}a_{21}}{d}$$
$$= \frac{a_{11}a_{22}-a_{12}a_{21}}{d}$$
$$= g$$

and

$$p_1a_{12}+p_2a_{22}$$
$$= \frac{a_{22}-a_{21}}{d}\cdot a_{12} + \frac{a_{11}-a_{12}}{d}\cdot a_{22}$$
$$= \frac{a_{12}a_{22}-a_{12}a_{21}}{d} + \frac{a_{11}a_{22}-a_{12}a_{22}}{d}$$
$$= \frac{a_{12}a_{22}-a_{12}a_{21}+a_{11}a_{22}-a_{12}a_{22}}{d}$$
$$= \frac{a_{11}a_{22}-a_{12}a_{21}}{d}$$
$$= g,$$

then $AM = [g \quad g]$, where g is the value of the game.

(b) Let $B = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$ be the optimum strategy for player B. Thus,

$$q_1 = \frac{a_{22}-a_{12}}{d} \text{ and } q_2 = \frac{a_{11}-a_{21}}{d}$$

where $d = a_{11} - a_{21} - a_{12} + a_{22}$.

$$MB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$
$$= \begin{bmatrix} a_{11}q_1 + a_{12}q_2 \\ a_{21}q_1 + a_{22}q_2 \end{bmatrix}$$
$$= \begin{bmatrix} g \\ g \end{bmatrix}$$

Since

$$a_{11}q_1 + a_{12}q_2$$
$$= a_{11}\cdot\frac{a_{22}-a_{12}}{d} + a_{12}\cdot\frac{a_{11}-a_{21}}{d}$$
$$= \frac{a_{11}a_{22}-a_{11}a_{12}}{d} + \frac{a_{11}a_{12}-a_{12}a_{21}}{d}$$
$$= \frac{a_{11}a_{22}-a_{11}a_{12}+a_{11}a_{12}-a_{12}a_{21}}{d}$$
$$= \frac{a_{11}a_{22}-a_{12}a_{21}}{d} = g$$

and

$$a_{21}q_1 + a_{22}q_2$$
$$= a_{21}\cdot\frac{a_{22}-a_{12}}{d} + a_{22}\cdot\frac{a_{11}-a_{21}}{d}$$
$$= \frac{a_{21}a_{22}-a_{12}a_{21}}{d} + \frac{a_{11}a_{22}-a_{21}a_{22}}{d}$$
$$= \frac{a_{21}a_{22}-a_{12}a_{21}+a_{11}a_{22}-a_{21}a_{22}}{d}$$
$$= \frac{a_{11}a_{22}-a_{12}a_{21}}{d} = g,$$

then $MB = \begin{bmatrix} g \\ g \end{bmatrix}$, where g is the value of the game.

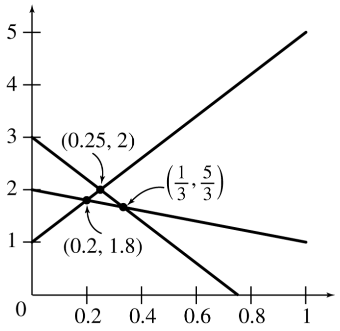
28. $\begin{bmatrix} -1 & 5 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

(a)
$$\begin{aligned} E_1 &= -1p_1 + 3(1-p_1) \\ &= -p_1 + 3 - 3p_1 \\ &= 3 - 4p_1 \end{aligned}$$

$$\begin{aligned} E_2 &= 5p_1 + 1(1-p_1) \\ &= 5p_1 + 1 - p_1 \\ &= 1 + 4p_1 \end{aligned}$$

$$\begin{aligned} E_3 &= 1p_1 + 2(1-p_1) \\ &= p_1 + 2 - 2p_1 \\ &= 2 - p_1 \end{aligned}$$

(b)



(c) From the graph, $p_1 = \frac{1}{4}$ maximizes the minimum expected value the row player receives.

30.

$$\begin{array}{ll|cc} & & \textit{Competitor's Strategy} & \\ & & 4.9 & 4.75 \\ \hline \textit{Boeing's} & 4.9 & 2 & -4 \\ \textit{Strategy} & 4.75 & 0 & 2 \end{array}$$

There is no saddle point. The optimum strategy for pricing is

$$p_1 = \frac{0-2}{-4-2-2+0} = \frac{-2}{-8} = \frac{1}{4},$$
$$p_2 = 1 - p_1 = \frac{3}{4}.$$

(q_1 and q_2 are not of interest here.) This means that Boeing's price strategy should be to aim for the \$4.9 million profit $\frac{1}{4}$ of the time and the \$4.75 million profit $\frac{3}{4}$ of the time. The value is

$$\frac{-4(0)-2(2)}{-4-2-2+0} = \frac{-4}{-8} = \frac{1}{2},$$

which means this strategy will increase Boeing's profit by $\frac{1}{2}$ million dollars.

32. (a)

$$\begin{array}{ll|cc} & & \textit{Strain} & \\ & & 1 & 2 \\ \hline \textit{Medicine} & 1 & .6 & .4 \\ & 2 & 0 & 1 \end{array}$$

(b) There is no saddle point. The optimum strategy for prescribing medicine is

$$p_1 = \frac{1-0}{.6-0-.4+1} = \frac{1}{1.2} = \frac{10}{12} = \frac{5}{6},$$
$$p_2 = 1 - p_1 = \frac{1}{6}.$$

(q_1 and q_2 are not of interest here.) This means that Dr. Goedeker should prescribe medicine 1 about $\frac{5}{6}$ of the time and medicine 2 about $\frac{1}{6}$ of the time. The result (value) will be

$$\frac{.6(1)-.4(0)}{.6-0-.4+1} = \frac{.6}{1.2} = \frac{6}{12} = \frac{1}{2},$$

which indicates an effectiveness of 50%.

34. (a) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

(b) There is no saddle point. For player A, the optimum strategy is

$$p_1 = \frac{1-(-1)}{1-(-1)-(-1)+1} = \frac{2}{4} = \frac{1}{2},$$
$$p_2 = 1 - p_1 = \frac{1}{2}.$$

For player B, the optimum strategy is

$$q_1 = \frac{1-(-1)}{1-(-1)-(-1)+1} = \frac{2}{4} = \frac{1}{2},$$
$$q_2 = 1 - q_1 = \frac{1}{2}.$$

The value of the game is

$$\frac{1(1)-(-1)(-1)}{1-(-1)-(-1)+1} = \frac{0}{4} = 0,$$

so this is a fair game.

36. (a) The playoff matrix for the game is

$$\begin{array}{ll|cc} & & \textit{Number of Fingers} & \\ & & 0 & 2 \\ \hline \textit{Number of} & 0 & 0 & -2 \\ \textit{Fingers} & 2 & -2 & 4 \end{array}.$$

(b) For player A, the optimum strategy is

$$p_1 = \frac{4-(-2)}{0-(-2)-(-2)+4} = \frac{6}{8} = \frac{3}{4},$$
$$p_2 = 1 - p_1 = \frac{1}{4}.$$

For player B, the optimum strategy is

$$q_1 = \frac{4-(-2)}{0-(-2)-(-2)+4} = \frac{6}{8} = \frac{3}{4},$$

$$q_2 = 1 - q_1 = \frac{1}{4}.$$

Both players should choose 0 fingers with probability $\frac{3}{4}$ and 2 fingers with probability $\frac{1}{4}$.

The value of the game is

$$\frac{0(4)-(-2)(-2)}{0-(-2)-(-2)+4} = \frac{-4}{8} = -\frac{1}{2}.$$

11.3 Game Theory and Linear Programming

2. $\begin{bmatrix} 5 & 2 \\ 0 & 3 \end{bmatrix}$

Let player A choose row 1 with probability p_1 and row 2 with probability p_2. Then

$$\begin{aligned} & E_1 = 5p_1 \\ \text{and} \quad & E_2 = 2p_1 + 3p_2. \end{aligned}$$

Let g represent the minimum of the expected gains, so that

$$\begin{aligned} E_1 &= 5p_1 & &\geq g \\ E_2 &= 2p_1 + 3p_2 & &\geq g. \end{aligned}$$

Dividing by g yields

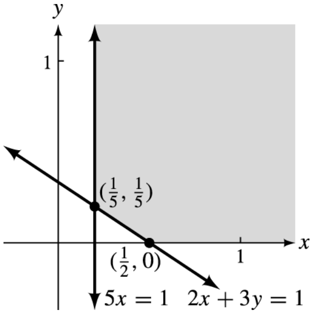
$$\begin{aligned} 5\left(\frac{p_1}{g}\right) & & &\geq 1 \\ 2\left(\frac{p_1}{g}\right) &+ 3\left(\frac{p_2}{g}\right) & &\geq 1. \end{aligned}$$

Let $x = \dfrac{p_1}{g}$ and $y = \dfrac{p_2}{g}$.

We can rewrite the problem as the following linear programming problem:

$$\begin{aligned} \text{Minimize} \quad & w = x + y \\ \text{subject to:} \quad & 5x \quad\;\; \geq 1 \\ & 2x + 3y \geq 1 \\ \text{with} \quad & x \geq 0, y \geq 0. \end{aligned}$$

This linear programming problem can be solved by the graphical method. Graph the feasible region.



The corner points are $(\frac{1}{5}, \frac{1}{5})$ and $(\frac{1}{2}, 0)$. The minimum value of $w = x + y$ is $\frac{2}{5}$ at $(\frac{1}{5}, \frac{1}{5})$. Thus, the value of the game is $g = \frac{1}{w} = \frac{5}{2}$. The optimum strategy for A is

$$p_1 = gx = \frac{5}{2}\left(\frac{1}{5}\right) = \frac{1}{2},$$

$$p_2 = gy = \frac{5}{2}\left(\frac{1}{5}\right) = \frac{1}{2}.$$

Let player B choose column 1 with probability q_1 and column 2 with probability q_2. Then

$$\begin{aligned} & E_1 = 5q_1 + 2q_2 \\ \text{and} \quad & E_2 = \qquad\;\; 3q_2. \end{aligned}$$

Let g represent the maximum of the expected gains, so that

$$\begin{aligned} E_1 &= 5q_1 + 2q_2 \leq g \\ E_2 &= \qquad\;\; 3q_2 \leq g. \end{aligned}$$

Dividing by g yields

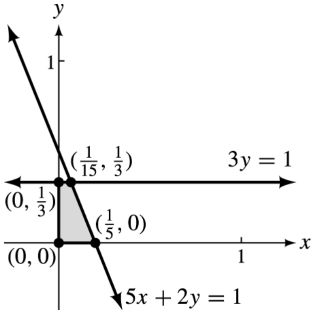
$$\begin{aligned} 5\left(\frac{q_1}{g}\right) + 2\left(\frac{q_2}{g}\right) &\leq 1 \\ 3\left(\frac{q_2}{g}\right) &\leq 1. \end{aligned}$$

Let $x = \dfrac{q_1}{g}$ and $y = \dfrac{q_2}{g}$.

We now have the linear programming problem.

$$\begin{aligned} \text{Maximize} \quad & z = x + y \\ \text{subject to:} \quad & 5x + 2y \leq 1 \\ & \qquad\;\; 3y \leq 1 \\ \text{with} \quad & x \geq 0, y \geq 0. \end{aligned}$$

Graph the feasible region.



There are four corner points to consider: $(0,0), (0,\frac{1}{3})$, $(\frac{1}{15},\frac{1}{3})$, and $(\frac{1}{5},0)$. The maximum value of $z = x+y$ is $\frac{1}{15}+\frac{1}{3} = \frac{6}{15} = \frac{2}{5}$ at $(\frac{1}{15},\frac{1}{3})$. Thus, as before, the value of the game is $g = \frac{1}{z} = \frac{5}{2}$. The optimum strategy for B is

$$q_1 = gx = \frac{5}{2}\left(\frac{1}{15}\right) = \frac{1}{6},$$
$$q_2 = gy = \frac{5}{2}\left(\frac{1}{3}\right) = \frac{5}{6}.$$

To summarize, player A should choose row 1 with probability $\frac{1}{2}$ and row 2 with probability $\frac{1}{2}$, while player B should choose column 1 with probability $\frac{1}{6}$ and column 2 with probability $\frac{5}{6}$. When these optimum strategies are used, the value of the game is $\frac{5}{2}$.

4. $\begin{bmatrix} -1 & 5 \\ 1 & -6 \end{bmatrix}$

To guarantee that the value of the game is positive, we add 6 to all entries in the matrix to obtain

$$\begin{bmatrix} 5 & 11 \\ 7 & 0 \end{bmatrix}.$$

Let player A choose row 1 with probability p_1 and row 2 with probability p_2. Then,

$$E_1 = 5p_1 + 7p_2$$
and $$E_2 = 11p_1.$$

Let g represent the minimum of the expected gains, so that

$$\begin{aligned} E_1 &= 5p_1 + 7p_2 \geq g \\ E_2 &= 11p_1 \qquad \geq g. \end{aligned}$$

Dividing by g yields

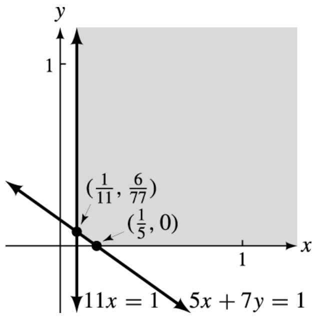
$$\begin{aligned} 5\left(\frac{p_1}{g}\right) + 7\left(\frac{p_2}{g}\right) &\geq 1 \\ 11\left(\frac{p_1}{g}\right) \qquad\qquad &\geq 1. \end{aligned}$$

Let $x = \dfrac{p_1}{g}$ and $y = \dfrac{p_2}{g}$.

We have the following linear programming problem:

$$\begin{array}{ll} \text{Minimize} & w = x + y \\ \text{subject to:} & 5x + 7y \geq 1 \\ & 11x \qquad \geq 1. \\ \text{with} & x \geq 0, y \geq 0. \end{array}$$

Graph the feasible region.



The corner points are $(\frac{1}{11},\frac{6}{77})$ and $(\frac{1}{5},0)$. The minimum value of $w = x+y$ is $\frac{13}{77}$ at $(\frac{1}{11},\frac{6}{77})$. Thus, the value of the game is $g = \frac{1}{w} = \frac{77}{13}$. To find the value of the original game, we must subtract 6.

$$\frac{77}{13} - 6 = \frac{77}{13} - \frac{78}{13} = -\frac{1}{13}.$$

The value of the original game is $-\frac{1}{13}$. The optimum strategy for A is

$$p_1 = gx = \frac{77}{13}\left(\frac{1}{11}\right) = \frac{7}{13},$$
$$p_2 = gy = \frac{77}{13}\left(\frac{6}{77}\right) = \frac{6}{13}.$$

Let player B choose column 1 with probability q_1 and column 2 with probability q_2. Then

$$E_1 = 5q_1 + 11q_2$$
and $$E_2 = 7q_1.$$

Let g represent the maximum of the expected gains, so that

$$\begin{aligned} E_1 &= 5q_1 + 11q_2 \leq g \\ E_2 &= 7q_1 \qquad\quad \leq g. \end{aligned}$$

Dividing by g yields

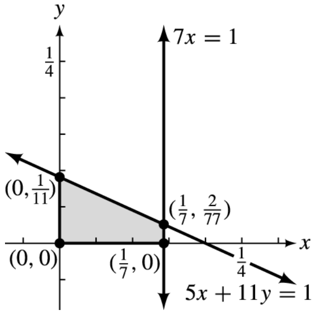
$$\begin{aligned} 5\left(\frac{q_1}{g}\right) + 11\left(\frac{q_2}{g}\right) &\le 1 \\ 7\left(\frac{q_1}{g}\right) \qquad\qquad &\le 1. \end{aligned}$$

Let $x = \dfrac{q_1}{g}$ and $y = \dfrac{q_2}{g}$.

We have the following linear programming problem:

$$\begin{array}{ll} \text{Maximize} & z = x + y \\ \text{subject to:} & 5x + 11y \le 1 \\ & 7x \qquad\quad \le 1 \\ \text{with} & x \ge 0, y \ge 0. \end{array}$$

Graph the feasible region.



There are four corner points to consider: $(0,0)$, $(0, \frac{1}{11})$, $(\frac{1}{7}, \frac{2}{77})$, and $(\frac{1}{7}, 0)$. The maximum value of $z = x + y$ is $\frac{13}{77}$ at $(\frac{1}{7}, \frac{2}{77})$. As before, the value of the game is $g = \frac{1}{z} = \frac{77}{13}$. The value of the original game again is $\frac{77}{13} - 6 = -\frac{1}{13}$. The optimum strategy for B is

$$q_1 = gx = \frac{77}{13}\left(\frac{1}{7}\right) = \frac{11}{13},$$

$$q_2 = gy = \frac{77}{13}\left(\frac{2}{77}\right) = \frac{2}{13}.$$

To summarize, player A should choose row 1 with probability $\frac{7}{13}$ and row 2 with probability $\frac{6}{13}$, while player B should choose column 1 with probability $\frac{11}{13}$ and column 2 with probability $\frac{2}{13}$. When these optimum strategies are used, the value of the game is $-\frac{1}{13}$.

6. $\begin{bmatrix} -4 & 1 \\ 5 & 0 \end{bmatrix}$

To guarantee that the value of the game is positive, we add 4 to all the entries in the matrix to obtain

$$\begin{bmatrix} 0 & 5 \\ 9 & 4 \end{bmatrix}.$$

Let player A choose row 1 with probability p_1 and row 2 with probability p_2. Then

$$\begin{aligned} E_1 &= \qquad 9p_2 \\ E_2 &= 5p_1 + 4p_2. \end{aligned}$$

Let g represent the minimum of the expected gains, so that

$$\begin{aligned} E_1 &= \qquad 9p_2 \ge g \\ E_2 &= 5p_1 + 4p_2 \ge g. \end{aligned}$$

Dividing by g yields

$$9\left(\frac{p_2}{g}\right) \ge 1$$

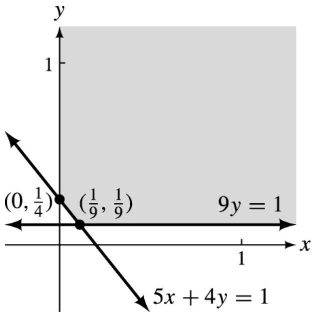
$$5\left(\frac{p_1}{g}\right) + 4\left(\frac{p_2}{g}\right) \ge 1.$$

Let $x = \dfrac{p_1}{g}$ and $y = \dfrac{p_2}{g}$.

We have the following linear programming problem:

$$\begin{array}{ll} \text{Minimize} & w = x + y \\ \text{subject to:} & \qquad 9y \ge 1 \\ & 5x + 4y \ge 1 \\ \text{with} & x \ge 0, y \ge 0. \end{array}$$

Graph the feasible region.



The corner points are $(0, \frac{1}{4})$ and $(\frac{1}{9}, \frac{1}{9})$. The minimum value of $w = x + y$ is $\frac{2}{9}$ at $(\frac{1}{9}, \frac{1}{9})$. Thus, the value of the game is $g = \frac{1}{w} = \frac{9}{2}$. To find the value of the original game, we must subtract 4.

$$\frac{9}{2} - 4 = \frac{9}{2} - \frac{8}{2} = \frac{1}{2}$$

The value of the original game is $\frac{1}{2}$. The optimum strategy for A is

$$\begin{aligned} p_1 &= gx = \frac{9}{2}\left(\frac{1}{9}\right) = \frac{1}{2}, \\ p_2 &= gy = \frac{9}{2}\left(\frac{1}{9}\right) = \frac{1}{2}. \end{aligned}$$

Let player B choose column 1 with probability q_1 and column 2 with probability q_2. Then

$$\begin{aligned} E_1 &= 5q_2 \\ E_2 &= 9q_1 + 4q_2. \end{aligned}$$

Let g represent the maximum of the expected gains, so that

$$\begin{aligned} E_1 &= 5q_2 \le g \\ E_2 &= 9q_1 + 4q_2 \le g. \end{aligned}$$

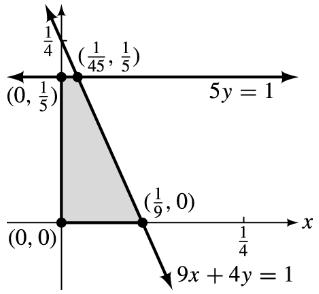
Dividing by g yields

$$\begin{aligned} 5\left(\frac{q_2}{g}\right) &\le 1 \\ 9\left(\frac{q_1}{g}\right) + 4\left(\frac{q_2}{g}\right) &\le 1. \end{aligned}$$

We have the following linear programming problem:

$$\begin{array}{ll} \text{Maximize} & z = x + y \\ \text{subject to:} & 5y \le 1 \\ & 9x + 4y \le 1 \\ \text{with} & x \ge 0, y \ge 0. \end{array}$$

Graph the feasible region.



There are four corner points to consider: $(0,0), (0,\frac{1}{5})$, $(\frac{1}{45},\frac{1}{5})$, and $(\frac{1}{9},0)$. The maximum value of $z = x + y$ is $\frac{2}{9}$ at $(\frac{1}{45},\frac{1}{5})$. As before, the value of the game is $g = \frac{1}{z} = \frac{9}{2}$. The value of the original game again is $\frac{9}{2} - 4 = \frac{1}{2}$. The optimum strategy for B is

$$\begin{aligned} q_1 &= gx = \frac{9}{2}\left(\frac{1}{45}\right) = \frac{1}{10}, \\ q_2 &= gy = \frac{9}{2}\left(\frac{1}{5}\right) = \frac{9}{10}. \end{aligned}$$

To summarize, player A should choose row 1 with probability $\frac{1}{2}$ and row 2 with probability $\frac{1}{2}$, while player B should choose column 1 with probability $\frac{1}{10}$ and column 2 with probability $\frac{9}{10}$. When these optimum strategies are used, the value of the game is $\frac{1}{2}$.

8. $\begin{bmatrix} -6 & 1 & 4 & 2 \\ 9 & 3 & -8 & -7 \end{bmatrix}$

Because of negative entries, we will add 8 to all entries. The resulting payoff matrix is

$$\begin{bmatrix} 2 & 9 & 12 & 10 \\ 17 & 11 & 0 & 1 \end{bmatrix}.$$

The linear programming problem to be solved is:

$$\begin{array}{ll}
\text{Maximize} & z = x_1 + x_2 + x_3 + x_4 \\
\text{subject to:} & \;\;2x_1 + \;\;9x_2 + 12x_3 + 10x_4 \le 1 \\
 & 17x_1 + 11x_2 \qquad\quad + \quad x_4 \le 1. \\
\text{with} & x_1 \ge 0, x_2 \ge 0, x_3 \ge 0, x_4 \ge 0.
\end{array}$$

We will solve this problem by the simplex method. The initial tableau is

$$\begin{array}{ccccccc|c}
x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & z & \\
2 & 9 & 12 & 10 & 1 & 0 & 0 & 1 \\
\boxed{17} & 11 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline
-1 & -1 & -1 & -1 & 0 & 0 & 1 & 0
\end{array}.$$

We arbitrarily choose the first column. The smallest ratio is formed by the 17 in row 2. We make this the pivot and arrive at the following matrix.

$$\begin{array}{r|ccccccc|c}
 & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & z & \\
-2\mathrm{R}_2+17\mathrm{R}_1 \to \mathrm{R}_1 & 0 & 131 & \boxed{204} & 168 & 17 & -2 & 0 & 15 \\
 & 17 & 11 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline
\mathrm{R}_2+17\mathrm{R}_3 \to \mathrm{R}_3 & 0 & -6 & -17 & -16 & 0 & 1 & 17 & 1
\end{array}$$

The next pivot is the 204 in row 1, column 3. We arrive at the following matrix.

$$\begin{array}{r|ccccccc|c}
 & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & z & \\
 & 0 & 131 & 204 & \boxed{168} & 17 & -2 & 0 & 15 \\
 & 17 & 11 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline
\mathrm{R}_1+12\mathrm{R}_3 \to \mathrm{R}_3 & 0 & 59 & 0 & -24 & 17 & 10 & 204 & 27
\end{array}$$

The next pivot is the 168 in row 1, column 4. We arrive at the following matrix.

$$\begin{array}{r|ccccccc|c}
 & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & z & \\
 & 0 & 131 & 204 & 168 & 17 & -2 & 0 & 15 \\
-\mathrm{R}_1+168\mathrm{R}_2 \to \mathrm{R}_2 & 2856 & 1717 & -204 & 0 & -17 & 170 & 0 & 153 \\ \hline
\mathrm{R}_1+\;\;7\mathrm{R}_3 \to \mathrm{R}_3 & 0 & 544 & 204 & 0 & 136 & 68 & 1428 & 204
\end{array}$$

Dividing the bottom row by 1428 gives a z-value of $\frac{204}{1428} = \frac{1}{7}$, so $g = \frac{1}{z} = 7$. The values of y_1 and y_2 are read from the bottom of the columns for the two slack variables after dividing the bottom row by 1428:

$$y_1 = \frac{136}{1428} = \frac{2}{21}, y_2 = \frac{68}{1428} = \frac{1}{21}.$$

We find the values of p_1 and p_2 by multiplying the values of y_1 and y_2 by g:

$$p_1 = 7\left(\frac{2}{21}\right) = \frac{2}{3},$$

$$p_2 = 7\left(\frac{1}{21}\right) = \frac{1}{3}.$$

Next, we find the values of x_1, x_2, x_3, and x_4 by using the first four columns combined with the last column:

$$x_1 = \frac{153}{2856} = \frac{3}{56}, x_2 = 0, x_3 = 0, x_4 = \frac{5}{56}.$$

We find the values of q_1, q_2, q_3, and q_4 by multiplying these by g:

$$\begin{aligned} q_1 &= 7\left(\frac{3}{56}\right) = \frac{3}{8}, \\ q_2 &= 7(0) = 0, \\ q_3 &= 7(0) = 0, \\ q_4 &= 7\left(\frac{5}{56}\right) = \frac{5}{8}. \end{aligned}$$

Finally, the value of the game is found by subtracting from g the 8 that was added at the beginning, yielding $7 - 8 = -1$.

To summarize, the optimum strategy for player A is $p_1 = \frac{2}{3}$ and $p_2 = \frac{1}{3}$. The optimum strategy for player B is $q_1 = \frac{3}{8}, q_2 = 0, q_3 = 0$, and $q_4 = \frac{5}{8}$. When these strategies are used, the value of the game is -1.

10. $\begin{bmatrix} 1 & 0 \\ -2 & 4 \\ -1 & -1 \end{bmatrix}$

Row 1 dominates row 3. Removing row 3 gives us the following game:

$$\begin{bmatrix} 1 & 0 \\ -2 & 4 \end{bmatrix}.$$

Because of the negative entry, we will add 2 to all entries. The resulting payoff matrix is

$$\begin{bmatrix} 3 & 2 \\ 0 & 6 \end{bmatrix}.$$

The linear programming problem to be solved is:

$$\begin{array}{ll} \text{Maximize} & z = x_1 + x_2 \\ \text{subject to:} & 3x_1 + 2x_2 \le 1 \\ & \qquad 6x_2 \le 1 \\ \text{with} & x_1 \ge 0, x_2 \ge 0. \end{array}$$

The initial tableau is

$$\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ 3 & 2 & 1 & 0 & 0 & 1 \\ 0 & \boxed{6} & 0 & 1 & 0 & 1 \\ \hline -1 & -1 & 0 & 0 & 1 & 0 \end{array}.$$

We arbitrarily choose the second column. The smallest ratio is formed by the 6 in row 2. We make this the first pivot and arrive at the following matrix.

$$\begin{array}{c|ccccc|c} & x_1 & x_2 & s_1 & s_2 & z & \\ -R_2 + 3R_1 \rightarrow R_1 & \boxed{9} & 0 & 3 & -1 & 0 & 2 \\ & 0 & 6 & 0 & 1 & 0 & 1 \\ \hline R_2 + 6R_3 \rightarrow R_3 & -6 & 0 & 0 & 1 & 6 & 1 \end{array}$$

The next pivot is the 9 in row 1, column 1.

$$\begin{array}{c|ccccc|c} & x_1 & x_2 & s_1 & s_2 & z & \\ & 9 & 0 & 3 & -1 & 0 & 2 \\ & 0 & 6 & 0 & 1 & 0 & 1 \\ \hline 2R_1 + 3R_3 \rightarrow R_3 & 0 & 0 & 6 & 1 & 18 & 7 \end{array}$$

Dividing the bottom row by 18 gives a z-value of $\frac{7}{18}$, so $g = \frac{1}{z} = \frac{18}{7}$. The values of y_1 and y_2 are read from the bottom of the columns for the two slack variables after dividing the bottom row by 18:

$$y_1 = \frac{6}{18} = \frac{1}{3}, y_2 = \frac{1}{18}.$$

We find the values of p_1 and p_2 by multiplying the values of y_1 and y_2 by g:

$$\begin{aligned} p_1 &= \frac{18}{7}\left(\frac{1}{3}\right) = \frac{6}{7}, \\ p_2 &= \frac{18}{7}\left(\frac{1}{18}\right) = \frac{1}{7}, \\ p_3 &= 0 \text{ (row 3 was removed).} \end{aligned}$$

We next find the values of x_1 and x_2 by using the first two columns combined with the last column:

$$x_1 = \frac{2}{9}, x_2 = \frac{1}{6}.$$

We find the values of q_1 and q_2 by multiplying these by g:

$$\begin{aligned} q_1 &= \frac{18}{7}\left(\frac{2}{9}\right) = \frac{4}{7}, \\ q_2 &= \frac{18}{7}\left(\frac{1}{6}\right) = \frac{3}{7}. \end{aligned}$$

Finally, the value of the game is found by subtracting from g the 2 that was added at the beginning, yielding $\frac{18}{7} - 2 = \frac{4}{7}$.

To summarize, the optimum strategy for player A is $p_1 = \frac{6}{7}, p_2 = \frac{1}{7}$, and $p_3 = 0$ (since row 3 was removed). The optimum strategy for player B is $q_1 = \frac{4}{7}$ and $q_2 = \frac{3}{7}$. When these strategies are used, the value of the game is $\frac{4}{7}$.

12. $\begin{bmatrix} 2 & -1 & 1 \\ 0 & 2 & 3 \\ 4 & 1 & 0 \end{bmatrix}$

Because of the negative entry, we will add 1 to all the entries. The resulting payoff matrix is

$$\begin{bmatrix} 3 & 0 & 2 \\ 1 & 3 & 4 \\ 5 & 2 & 1 \end{bmatrix}.$$

The linear programming problem to be solved is:

$$\begin{array}{ll} \text{Maximize} & z = x_1 + x_2 + x_3 \\ \text{subject to:} & 3x_1 \phantom{{}+3x_2} + 2x_3 \le 1 \\ & x_1 + 3x_2 + 4x_3 \le 1 \\ & 5x_1 + 2x_2 + x_3 \le 1 \\ \text{with} & x_1 \ge 0, x_2 \ge 0, x_3 \ge 0. \end{array}$$

The initial tableau is

$$\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ 3 & 0 & 2 & 1 & 0 & 0 & 0 & 1 \\ 1 & 3 & 4 & 0 & 1 & 0 & 0 & 1 \\ \boxed{5} & 2 & 1 & 0 & 0 & 1 & 0 & 1 \\ \hline -1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}.$$

We arbitrarily choose the first column. The smallest ratio is formed by the 5 in row 3. We make this the first pivot and arrive at the following matrix.

$$\begin{array}{r|ccccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ -3\text{R}_3 + 5\text{R}_1 \to \text{R}_1 & 0 & -6 & 7 & 5 & 0 & -3 & 0 & 2 \\ -\text{R}_3 + 5\text{R}_2 \to \text{R}_2 & 0 & 13 & \boxed{19} & 0 & 5 & -1 & 0 & 4 \\ & 5 & 2 & 1 & 0 & 0 & 1 & 0 & 1 \\ \hline \text{R}_3 + 5\text{R}_4 \to \text{R}_4 & 0 & -3 & -4 & 0 & 0 & 1 & 5 & 1 \end{array}$$

The next pivot is the 19 in row 2, column 3.

$$\begin{array}{r|ccccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ -7\text{R}_2 + 19\text{R}_1 \to \text{R}_1 & 0 & -205 & 0 & 95 & -35 & -50 & 0 & 10 \\ & 0 & \boxed{13} & 19 & 0 & 5 & -1 & 0 & 4 \\ -\text{R}_2 + 19\text{R}_3 \to \text{R}_3 & 95 & 25 & 0 & 0 & -5 & 20 & 0 & 15 \\ \hline 4\text{R}_2 + 19\text{R}_4 \to \text{R}_4 & 0 & -5 & 0 & 0 & 20 & 15 & 95 & 35 \end{array}.$$

The next pivot is the 13 in row 2, column 2.

$$\begin{array}{r|ccccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ 205\text{R}_2 + 13\text{R}_1 \to \text{R}_1 & 0 & 0 & 3895 & 1235 & 570 & -855 & 0 & 950 \\ & 0 & 13 & 19 & 0 & 5 & -1 & 0 & 4 \\ -25\text{R}_2 + 13\text{R}_3 \to \text{R}_3 & 1235 & 0 & -475 & 0 & -190 & 285 & 0 & 95 \\ \hline 5\text{R}_2 + 13\text{R}_4 \to \text{R}_4 & 0 & 0 & 95 & 0 & 285 & 190 & 1235 & 475 \end{array}$$

Dividing the bottom row by 1235 gives a z-value of $\frac{475}{1235} = \frac{5}{13}$, so $g = \frac{1}{z} = \frac{13}{5}$. The values of y_1, y_2, and y_3 are read from the bottoms of the columns for the three slack variables after multiplying the bottom row by $\frac{1}{1235}$:

$$y_1 = \frac{0}{1235} = 0, y_2 = \frac{285}{1235} = \frac{3}{13}, y_3 = \frac{190}{1235} = \frac{2}{13}.$$

We find the values of p_1, p_2, and p_3 by multiplying the values of y_1, y_2, and y_3 by g:

$$p_1 = \frac{13}{5}(0) = 0, p_2 = \frac{13}{5}\left(\frac{3}{13}\right) = \frac{3}{5}, p_3 = \frac{13}{5}\left(\frac{2}{13}\right) = \frac{2}{5}.$$

We next find the values of x_1, x_2, and x_3 by using the first three columns combined with the last column:

$$x_1 = \frac{95}{1235} = \frac{1}{13}, x_2 = \frac{4}{13}, x_3 = 0.$$

We find the values of q_1, q_2, and q_3 by multiplying the values of x_1, x_2, and x_3 by g:

$$q_1 = \frac{13}{5}\left(\frac{1}{13}\right) = \frac{1}{5}, q_2 = \frac{13}{5}\left(\frac{4}{13}\right) = \frac{4}{5}, q_3 = 0.$$

Finally, the value of the game is found by subtracting from g the 1 that was added at the beginning, yielding $\frac{13}{5} - 1 = \frac{8}{5}$.

To summarize, the optimum strategy for player A is $p_1 = 0, p_2 = \frac{3}{5}$, and $p_3 = \frac{2}{5}$. The optimum strategy for player B is $q_1 = \frac{1}{5}, q_2 = \frac{4}{5}$, and $q_3 = 0$. When these strategies are used, the value of the game is $\frac{8}{5}$.

14. $$\begin{bmatrix} -1 & -1 & 1 \\ -1 & 0 & 0 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$$

Because of the negative entries, we will add 1 to all the entries. The resulting payoff matrix is

$$\begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 0 & 0 \\ 2 & 2 & 0 \end{bmatrix}.$$

The linear programming problem to be solved is:

$$\begin{array}{ll} \text{Maximize} & z = x_1 + x_2 + x_3 \\ \text{subject to:} & 2x_3 \le 1 \\ & x_2 + x_3 \le 1 \\ & 2x_1 \le 1 \\ & 2x_1 + 2x_2 \le 1 \\ \text{with} & x_1 \ge 0, x_2 \ge 0, x_3 \ge 0. \end{array}$$

The initial tableau is

$$\begin{array}{cccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & s_4 & z & \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ \boxed{2} & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ \hline -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}.$$

We arbitrarily choose the first column. Both ratios are $\frac{1}{2}$, so we will use the 2 in row 4 as the first pivot. We arrive at the following matrix.

$$\begin{array}{r|cccccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & s_4 & z & \\ & 0 & 0 & \boxed{2} & 1 & 0 & 0 & 0 & 0 & 1 \\ & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ -R_4 + R_3 \to R_3 & 0 & -2 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ & 2 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ \hline R_4 + 2R_5 \to R_5 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 2 & 1 \end{array}$$

The next pivot is the 2 in row 1, column 3.

$$\begin{array}{r|cccccccc|c|}
 & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & s_4 & z & \\
 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 1 \\
-\mathrm{R}_1+2\mathrm{R}_2 \rightarrow \mathrm{R}_2 & 0 & 2 & 0 & -1 & 2 & 0 & 0 & 0 & 1 \\
 & 0 & -2 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
 & 2 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
\hline
\mathrm{R}_1+\mathrm{R}_5 \rightarrow \mathrm{R}_5 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 2 & 2
\end{array}$$

Dividing the bottom row by 2 gives a z-value of 1, so $g = \frac{1}{z} = 1$. The values of y_1, y_2, y_3, and y_4 are read from the bottom of the columns for the four slack variables after dividing the bottom row by 2:

$$y_1 = \frac{1}{2}, y_2 = 0, y_3 = 0, y_4 = \frac{1}{2}.$$

We find the values of p_1, p_2, p_3, and p_4 by multiplying the values of y_1, y_2, y_3, and y_4 by g:

$$p_1 = 1\left(\frac{1}{2}\right) = \frac{1}{2}, p_2 = 0, p_3 = 0, p_4 = 1\left(\frac{1}{2}\right) = \frac{1}{2}.$$

We next find the values of x_1, x_2, and x_3 by using the first three columns combined with the last column:

$$x_1 = \frac{1}{2}, x_2 = 0, x_3 = \frac{1}{2}.$$

We find the values of q_1, q_2, and q_3 by multiplying the values of x_1, x_2, and x_3 by g:

$$q_1 = 1\left(\frac{1}{2}\right) = \frac{1}{2}, q_2 = 0, q_3 = 1\left(\frac{1}{2}\right) = \frac{1}{2}.$$

Finally, the value of the game is found by subtracting from g the 1 that was added at the beginning, yielding $1 - 1 = 0$.

Thus, labor should use strategies 1 and 4 with probabilities $\frac{1}{2}$ each and should never use strategies 2 or 3. Management should use strategies 1 and 3 with probabilities $\frac{1}{2}$ each and should never use strategy 2. The value of the game is 0.

16. This exercise should be solved by graphing calculator or computer methods. The solution may vary slightly. The answer is that the manufacturer should emphasize modern cards with probability .088, old-fashioned cards with probability .418, and a mixture with probability .495, giving a value of \$78.87 for the game.

18. **(a)** Row 3 dominates row 4. So, remove row 4 to obtain

$$\begin{bmatrix} 5 & 8 & 10 \\ 8 & 10 & 6 \\ 10 & 6 & 2 \end{bmatrix}.$$

(b) Using the matrix from part (a), the linear programming problem to be solved is:

Maximize: $z = x_1 + x_2 + x_3$

Subject to: $5x_1 + 8x_2 + 10x_3 \leq 1$

$8x_1 + 10x_2 + 6x_3 \leq 1$

$10x_1 + 6x_2 + 2x_3 \leq 1$

The initial tableau is

$$\left[\begin{array}{ccccccc|c}
x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\
5 & 8 & 10 & 1 & 0 & 0 & 0 & 1 \\
8 & 10 & 6 & 0 & 1 & 0 & 0 & 1 \\
\boxed{10} & 6 & 2 & 0 & 0 & 1 & 0 & 1 \\
\hline
-1 & -1 & -1 & 0 & 0 & 0 & 1 & 0
\end{array}\right].$$

We arbitrarily choose the first column. The smallest ratio is formed by the 10 in row 3. We use this as the first pivot and arrive at the following matrix.

$$\begin{array}{r|rrrrrrr|r}
 & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\
-R_3+2R_1\rightarrow R_1 & 0 & 10 & 18 & 2 & 0 & -1 & 0 & 1 \\
-4R_3+5R_2\rightarrow R_2 & 0 & 26 & \boxed{22} & 0 & 5 & -4 & 0 & 1 \\
 & 10 & 6 & 2 & 0 & 0 & 1 & 0 & 1 \\
\hline
R_3+10R_4\rightarrow R_4 & 0 & -4 & -8 & 0 & 0 & 1 & 10 & 1
\end{array}$$

The next pivot is the 22 in row 2, column 3.

$$\begin{array}{r|rrrrrrr|r}
 & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\
-9R_2+11R_1\rightarrow R_1 & 0 & -124 & 0 & 22 & -45 & \boxed{25} & 0 & 2 \\
 & 0 & 26 & 22 & 0 & 5 & -4 & 0 & 1 \\
-R_2+11R_3\rightarrow R_3 & 110 & 40 & 0 & 0 & -5 & 15 & 0 & 10 \\
\hline
4R_2+11R_4\rightarrow R_4 & 0 & 60 & 0 & 0 & 20 & -5 & 110 & 15
\end{array}$$

The next pivot is the 25 in row 1, column 6.

$$\begin{array}{r|rrrrrrr|r}
 & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\
 & 0 & -124 & 0 & 22 & -45 & 25 & 0 & 2 \\
4R_1+25R_2\rightarrow R_2 & 0 & 154 & 550 & 88 & -55 & 0 & 0 & 33 \\
-3R_1+5R_3\rightarrow R_3 & 550 & 572 & 0 & -66 & 110 & 0 & 0 & 44 \\
\hline
R_1+5R_4\rightarrow R_4 & 0 & 176 & 0 & 22 & 55 & 0 & 550 & 77
\end{array}$$

Create a 1 in the columns corresponding to x_1, x_3, s_3, and z.

$$\begin{array}{r|rrrrrrr|r}
 & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\
\frac{1}{25}R_1\rightarrow R_1 & 0 & -\frac{124}{25} & 0 & \frac{22}{25} & -\frac{9}{5} & 1 & 0 & \frac{2}{25} \\
\frac{1}{550}R_2\rightarrow R_2 & 0 & \frac{7}{25} & 1 & \frac{4}{25} & -\frac{1}{10} & 0 & 0 & \frac{3}{50} \\
\frac{1}{550}R_3\rightarrow R_3 & 1 & \frac{26}{25} & 0 & -\frac{3}{25} & \frac{1}{5} & 0 & 0 & \frac{2}{25} \\
\hline
\frac{1}{550}R_4\rightarrow R_4 & 0 & \frac{8}{25} & 0 & \frac{1}{25} & \frac{1}{10} & 0 & 1 & \frac{7}{50}
\end{array}$$

From the final tableau, we have

$x_1 = \dfrac{2}{25}, x_2 = 0, x_3 = \dfrac{3}{50}, y_1 = \dfrac{1}{25}, y_2 = \dfrac{1}{10},$

$y_3 = 0$ and $z = \dfrac{7}{50}$.

Note that

$$g = \frac{1}{z} = \frac{50}{7}.$$

The optimum strategy for the hospital is

$$p_1 = gy_1 = \frac{50}{7}\left(\frac{1}{25}\right) = \frac{2}{7}$$

$$p_2 = gy_2 = \frac{50}{7}\left(\frac{1}{10}\right) = \frac{5}{7}$$

$$p_3 = gy_3 = \frac{50}{7}(0) = 0$$

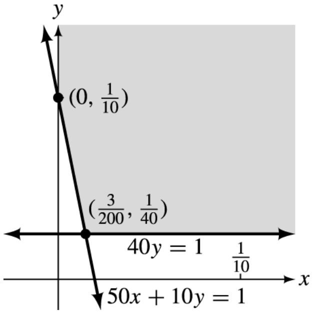
$p_4 = 0$ (row 4 was removed).

That is, the hospital should assign one nurse with probability $\frac{2}{7}$ and two nurses with probability $\frac{5}{7}$. Never assign three or four nurses.

20. $\begin{bmatrix} 50 & 0 \\ 10 & 40 \end{bmatrix}$

To find the student's optimum strategy,

$$\begin{array}{ll} \text{Minimize} & w = x + y \\ \text{subject to:} & 50x + 10y \geq 1 \\ & \qquad\quad 40y \geq 1 \\ \text{with} & x \geq 0, y \geq 0. \end{array}$$



Corner Point	Value of $w = x + y$
$(0, \frac{1}{10})$	$0 + \frac{1}{10} = \frac{1}{10}$
$(\frac{3}{200}, \frac{1}{40})$	$\frac{3}{200} + \frac{1}{40} = \frac{1}{25}$

The minimum value is $w = \frac{1}{25}$ at $(\frac{3}{200}, \frac{1}{40})$. Thus, the value of the game is $g = \frac{1}{w} = 25$ points, and the optimum strategy for the student is

$$p_1 = gx = 25\left(\frac{3}{200}\right) = \frac{3}{8},$$

$$p_2 = gy = 25\left(\frac{1}{40}\right) = \frac{5}{8}.$$

That is, the student should choose the calculator with probability $\frac{3}{8}$ and the book with probability $\frac{5}{8}$.

22. **(a)** For player A, choice 1 is to believe B when B says "ace," and choice 2 is to ask B to show his card when B says "ace." For player B, there are four choices. Choice 1 is to always tell the truth.

Choice 2 is to lie only if the card is a queen. Choice 3 is to lie only if the card is a king. Choice 4 is to lie if the card is a queen or king. These lead to the following payoff matrix:

$$\begin{array}{cc} & \text{B} \\ \text{A} & \begin{bmatrix} 0 & -\frac{2}{3} & -\frac{1}{3} & -1 \\ -\frac{1}{3} & 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \end{array}.$$

(b) Because of the negative entries, we will add 1 to all the entries, giving the matrix

$$\begin{bmatrix} 1 & \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{2}{3} & 1 & \frac{4}{3} & \frac{5}{3} \end{bmatrix}.$$

The linear programming problem to be solved is:

$$\begin{array}{ll} \text{Maximize} & z = x_1 + x_2 + x_3 + x_4 \\ \text{subject to:} & \quad x_1 + \frac{1}{3}x_2 + \frac{2}{3}x_3 \qquad\quad \leq 1 \\ & \frac{2}{3}x_1 + \ \ x_2 + \frac{4}{3}x_3 + \frac{5}{3}x_4 \leq 1 \\ \text{with} & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0. \end{array}$$

The initial tableau is

$$\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & z & \\ \boxed{1} & \frac{1}{3} & \frac{2}{3} & 0 & 1 & 0 & 0 & 1 \\ \frac{2}{3} & 1 & \frac{4}{3} & \frac{5}{3} & 0 & 1 & 0 & 1 \\ \hline -1 & -1 & -1 & -1 & 0 & 0 & 1 & 0 \end{array}.$$

We arbitrarily choose the first column. The smallest ratio is formed by the 1 in row 1. We use this as the first pivot and arrive at the following matrix.

$$\begin{array}{r ccccccc|c} & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & z & \\ & 1 & \frac{1}{3} & \frac{2}{3} & 0 & 1 & 0 & 0 & 1 \\ -\frac{2}{3}\mathrm{R}_1 + \mathrm{R}_2 \rightarrow \mathrm{R}_2 & 0 & \frac{7}{9} & \frac{8}{9} & \boxed{\frac{5}{3}} & -\frac{2}{3} & 1 & 0 & \frac{1}{3} \\ \hline \mathrm{R}_1 + \mathrm{R}_3 \rightarrow \mathrm{R}_3 & 0 & -\frac{2}{3} & -\frac{1}{3} & -1 & 1 & 0 & 1 & 1 \end{array}$$

The next pivot is the $\frac{5}{3}$ in row 2, column 4.

$$\begin{array}{r ccccccc|c} & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & z & \\ & 1 & \frac{1}{3} & \frac{2}{3} & 0 & 1 & 0 & 0 & 1 \\ & 0 & \boxed{\frac{7}{9}} & \frac{8}{9} & \frac{5}{3} & -\frac{2}{3} & 1 & 0 & \frac{1}{3} \\ \hline \mathrm{R}_2 + \frac{5}{3}\mathrm{R}_3 \rightarrow \mathrm{R}_3 & 0 & -\frac{1}{3} & \frac{1}{3} & 0 & 1 & 1 & \frac{5}{3} & 2 \end{array}$$

The next pivot is the $\frac{7}{9}$ in row 2, column 2.

$$\begin{array}{r ccccccc|c} & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & z & \\ -\mathrm{R}_2 + \frac{7}{3}\mathrm{R}_1 \rightarrow \mathrm{R}_1 & \frac{7}{3} & 0 & \frac{2}{3} & -\frac{5}{3} & 3 & -1 & 0 & 2 \\ & 0 & \frac{7}{9} & \frac{8}{9} & \frac{5}{3} & -\frac{2}{3} & 1 & 0 & \frac{1}{3} \\ \hline \mathrm{R}_2 + \frac{7}{3}\mathrm{R}_3 \rightarrow \mathrm{R}_3 & 0 & 0 & \frac{5}{3} & \frac{5}{3} & \frac{5}{3} & \frac{10}{3} & \frac{35}{9} & 5 \end{array}$$

Dividing the bottom row by $\frac{35}{9}$ gives a z-value of $\frac{45}{35} = \frac{9}{7}$, so $g = \frac{1}{z} = \frac{7}{9}$. The values of y_1 and y_2 are read from the bottom of the columns for the two slack variables after dividing the bottom row by $\frac{35}{9}$: $y_1 = \frac{3}{7}, y_2 = \frac{6}{7}$. We find the values of p_1 and p_2 by multiplying these values by g:

$$p_1 = \frac{7}{9}\left(\frac{3}{7}\right) = \frac{1}{3},$$

$$p_2 = \frac{7}{9}\left(\frac{6}{7}\right) = \frac{2}{3}.$$

Next, we find the values of x_1, x_2, x_3, and x_4 by using the first four columns combined with the last column: $x_1 = \frac{6}{7}, x_2 = \frac{3}{7}, x_3 = 0, x_4 = 0$. We find the values of q_1, q_2, q_3, and q_4 by multiplying the values of x_1, x_2, x_3, and x_4 by g:

$$p_1 = \frac{7}{9}\left(\frac{6}{7}\right) = \frac{2}{3},$$

$$p_2 = \frac{7}{9}\left(\frac{3}{7}\right) = \frac{1}{3},$$

$$p_3 = 0, p_4 = 0.$$

Finally, the value of the game is found by subtracting from g the 1 that was added at the beginning, yielding $\frac{7}{9} - 1 = -\frac{2}{9}$.

Thus, player A should use strategy 1 with probability $\frac{1}{3}$ and strategy 2 with probability $\frac{2}{3}$. Player B should use strategy 1 with probability $\frac{2}{3}$, strategy 2 with probability $\frac{1}{3}$, and should never use strategies 3 and 4. The value of the game is $-\frac{2}{9}$.

Chapter 11 Review Exercises

For Exercises 4-8, use the following payoff matrix.

$$\begin{bmatrix} -2 & 5 & -6 & 3 \\ 0 & -1 & 7 & 5 \\ 2 & 6 & -4 & 4 \end{bmatrix}$$

4. The strategy $(1, 4)$ means that player A chooses row 1 and player B chooses column 4. A positive number represents a payoff from B to A. Row 1 and column 4 lead to the number 3, which represents a payoff of \$3 from B to A.

6. The strategy $(3, 4)$ means that player A chooses row 3 and player B chooses column 4. Row 3 and column 4 lead to the number 4, which represents a payoff of \$4 from B to A.

8. Underline the smallest number in each row, and box the largest number in each column.

$$\begin{bmatrix} -2 & 5 & \underline{-6} & 3 \\ 0 & \underline{-1} & \boxed{7} & \boxed{5} \\ \boxed{2} & \boxed{6} & \underline{-4} & 4 \end{bmatrix}$$

There is no number that is both the smallest in its row and the largest in its column, so the game has no saddle point.

10. $\begin{bmatrix} -1 & 9 & 0 \\ 4 & -10 & 6 \\ 8 & -6 & 7 \end{bmatrix}$

Row 3 dominates row 2, so remove row 2 to obtain

$$\begin{bmatrix} -1 & 9 & 0 \\ 8 & -6 & 7 \end{bmatrix}.$$

12. $\begin{bmatrix} 3 & -1 & 4 \\ 0 & 4 & -1 \\ 1 & 2 & -3 \\ 0 & 0 & 2 \end{bmatrix}$

There are no dominated strategies.

14. $\begin{bmatrix} -4 & 0 & 2 & \underline{-5} \\ \boxed{6} & \boxed{9} & \boxed{\underline{3}} & \boxed{8} \end{bmatrix}$

3 is both the smallest number in its row and the largest in its column. The saddle point is 3 at $(2, 3)$, and the value of the game is 3.

16. $\begin{bmatrix} \boxed{4} & \boxed{\underline{-1}} & \boxed{6} \\ \underline{-3} & -2 & 0 \\ -1 & \underline{-4} & 3 \end{bmatrix}$

-1 is both the smallest number in its row and the largest number in its column. The saddle point is -1 at $(1, 2)$, and the value of the game is -1.

18. $\begin{bmatrix} 2 & \underline{-9} \\ \boxed{7} & \underline{1} \\ 4 & \boxed{\underline{2}} \end{bmatrix}$

The 2 in row 3, column 2 is both the smallest number in its row and the largest number in its column. The saddle point is 2 at $(3, 2)$, and the value of the game is 2.

20. $\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$

There is no saddle point, so the game is not strictly determined, and mixed strategies must be used.

For player A, the optimum strategy is

$$p_1 = \frac{5 - (-3)}{2 - (-3) - (-3) + 5} = \frac{8}{13},$$
$$p_2 = 1 - p_1 = \frac{5}{13}.$$

For player B, the optimum strategy is

$$q_1 = \frac{5 - (-3)}{2 - (-3) - (-3) + 5} = \frac{8}{13},$$
$$q_2 = 1 - q_1 = \frac{5}{13}.$$

The value of the game is

$$\frac{2(5) - (-3)(-3)}{2 - (-3) - (-3) + 5} = \frac{1}{13}.$$

22. $\begin{bmatrix} 8 & -3 \\ -6 & 2 \end{bmatrix}$

There is no saddle point.
For player A, the optimum strategy is

$$p_1 = \frac{2-(-6)}{8-(-6)-(-3)+2} = \frac{8}{19},$$
$$p_2 = 1 - p_1 = \frac{11}{19}.$$

For player B, the optimum strategy is

$$q_1 = \frac{2-(-3)}{8-(-6)-(-3)+2} = \frac{5}{19},$$
$$q_2 = 1 - q_1 = \frac{14}{19}.$$

The value of the game is

$$\frac{8(2)-(-3)(-6)}{8-(-6)-(-3)+2} = -\frac{2}{19}.$$

24. $\begin{bmatrix} 1 & 0 & 3 & -3 \\ 4 & -2 & 4 & -1 \end{bmatrix}$

Column 2 dominates columns 1 and 3, so remove columns 1 and 3 to obtain

$$\begin{bmatrix} 0 & -3 \\ -2 & -1 \end{bmatrix}.$$

For player A, the optimum strategy is

$$p_1 = \frac{-1-(-2)}{0-(-2)-(-3)+(-1)} = \frac{1}{4},$$
$$p_2 = 1 - p_1 = \frac{3}{4}.$$

For player B, the optimum strategy is

$$q_1 = 0 \text{ (column 1 was removed)},$$
$$q_2 = \frac{-1-(-3)}{0-(-2)-(-3)+(-1)} = \frac{1}{2},$$
$$q_3 = 0 \text{ (column 3 was removed)},$$
$$q_4 = 1 - (0 + \frac{1}{2} + 0) = \frac{1}{2}.$$

The value of the game is

$$\frac{0(-1)-(-3)(-2)}{0-(-2)-(-3)+(-1)} = \frac{-6}{4} = -\frac{3}{2}.$$

26. $\begin{bmatrix} 8 & -6 \\ 4 & -8 \\ -9 & 9 \end{bmatrix}$

Row 1 dominates row 2, so remove row 2 to obtain

$$\begin{bmatrix} 8 & -6 \\ -9 & 9 \end{bmatrix}.$$

For player A, the optimum strategy is

$$p_1 = \frac{9-(-9)}{8-(-9)-(-6)+9} = \frac{18}{32} = \frac{9}{16},$$
$$p_2 = 0 \text{ (row 2 was removed)},$$
$$p_3 = 1 - (p_1 + p_2) = \frac{7}{16}.$$

For player B, the optimum strategy is

$$q_1 = \frac{9-(-6)}{8-(-9)-(-6)+9} = \frac{15}{32},$$
$$q_2 = 1 - q_1 = \frac{17}{32}.$$

The value of the game is

$$\frac{8(9)-(-6)(-9)}{8-(-9)-(-6)+9} = \frac{18}{32} = \frac{9}{16}.$$

28. $\begin{bmatrix} -2 & 2 \\ 3 & 1 \end{bmatrix}$

To guarantee that the value of the game is positive, we add 2 to all entries in the matrix to obtain

$$\begin{bmatrix} 0 & 4 \\ 5 & 3 \end{bmatrix}.$$

Let player A choose row 1 with probability p_1 and row 2 with probability p_2. Then

$$\begin{aligned} E_1 &= 5p_2 \\ \text{and} \quad E_2 &= 4p_1 + 3p_2. \end{aligned}$$

Let g represent the minimum of the expected gains, so that

$$\begin{aligned} E_1 &= 5p_2 \geq g \\ E_2 &= 4p_1 + 3p_2 \geq g. \end{aligned}$$

Dividing by g yields

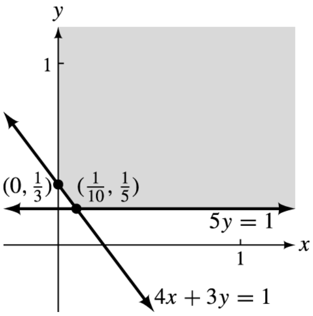
$$5\left(\frac{p_2}{g}\right) \geq 1$$
$$4\left(\frac{p_1}{g}\right) + 3\left(\frac{p_2}{g}\right) \geq 1.$$

Let $x = \dfrac{p_1}{g}$ and $y = \dfrac{p_2}{g}$.

We have the following linear programming problem:

$$\begin{array}{ll} \text{Minimize} & w = x + y \\ \text{subject to:} & \quad 5y \geq 1 \\ & 4x + 3y \geq 1 \\ \text{with} & x \geq 0, y \geq 0. \end{array}$$

Graph the feasible region.



The corner points are $(0,\frac{1}{3})$ and $(\frac{1}{10}, \frac{1}{5})$. The minimum value of $w = x + y$ is $\frac{3}{10}$ at $(\frac{1}{10}, \frac{1}{5})$. Thus, the value of the game is $g = \frac{1}{w} = \frac{10}{3}$. To find the value of the original game we must subtract 2:

$$\frac{10}{3} - 2 = \frac{4}{3}.$$

The optimum strategy for A is

$$p_1 = gx = \frac{10}{3}\left(\frac{1}{10}\right) = \frac{1}{3},$$

$$p_2 = gy = \frac{10}{3}\left(\frac{1}{5}\right) = \frac{2}{3}.$$

Let player B choose column 1 with probability q_1 and column 2 with probability q_2. Then

$$\begin{array}{rl} E_1 = & \quad 4q_2 \\ \text{and} \quad E_2 = & 5q_1 + 3q_2. \end{array}$$

Let g represent the maximum of the expected gains, so that

$$\begin{array}{rl} E_1 = & \quad 4q_2 \leq g \\ E_2 = & 5q_1 + 3q_2 \leq g. \end{array}$$

Dividing by g yields

$$4\left(\frac{q_2}{g}\right) \leq 1$$

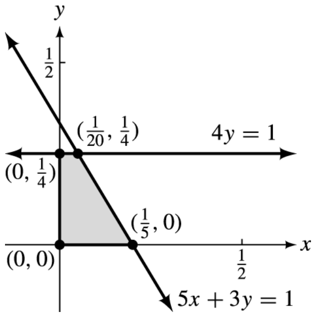
$$5\left(\frac{q_1}{g}\right) + 3\left(\frac{q_2}{g}\right) \leq 1.$$

Let $x = \dfrac{q_1}{g}$ and $y = \dfrac{q_2}{g}$.

We have the following linear programming problem:

$$\begin{array}{ll} \text{Maximize} & z = x + y \\ \text{subject to:} & \quad 4y \leq 1 \\ & 5x + 3y \leq 1 \\ \text{with} & x \geq 0, y \geq 0. \end{array}$$

Graph the feasible region.



There are four corner points to consider: $(0, 0)$, $(0, \frac{1}{4})$, $(\frac{1}{20}, \frac{1}{4})$, and $(\frac{1}{5}, 0)$. The maximum value of $z = x + y$ is $\frac{3}{10}$ at $(\frac{1}{20}, \frac{1}{4})$. As before, the value of the game is $g = \frac{1}{z} = \frac{10}{3}$. The value of the original game again is $\frac{10}{3} - 2 = \frac{4}{3}$.

The optimum strategy for B is

$$q_1 = gx = \frac{10}{3}\left(\frac{1}{20}\right) = \frac{1}{6},$$

$$q_2 = gy = \frac{10}{3}\left(\frac{1}{4}\right) = \frac{5}{6}.$$

To summarize, player A should choose row 1 with probability $\frac{1}{3}$ and row 2 with probability $\frac{2}{3}$, while player B should choose column 1 with probability $\frac{1}{6}$ and column 2 with probability $\frac{5}{6}$. When these optimum strategies are used, the value of the game is $\frac{4}{3}$.

30. $\begin{bmatrix} 0 & -2 \\ -1 & 3 \end{bmatrix}$

To guarantee that the value of the game is positive, we add 2 to all entries in the matrix to obtain

$$\begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix}.$$

Let player A choose row 1 with probability p_1 and row 2 with probability p_2. Then

$$\begin{array}{rl} E_1 = & 2p_1 + \quad p_2 \\ \text{and} \quad E_2 = & \qquad 5p_2. \end{array}$$

Let g represent the minimum of the expected gains, so that

$$\begin{array}{rl} E_1 = & 2p_1 + \quad p_2 \geq g \\ E_2 = & \qquad 5p_2 \geq g. \end{array}$$

Dividing by g yields

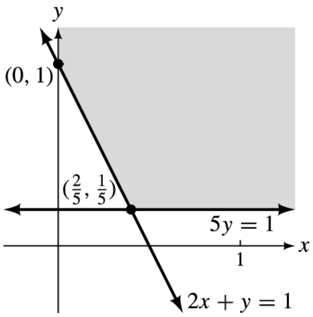
$$\begin{aligned} 2\left(\frac{p_1}{g}\right) + \left(\frac{p_2}{g}\right) &\geq 1 \\ 5\left(\frac{p_2}{g}\right) &\geq 1. \end{aligned}$$

Let $x = \dfrac{p_1}{g}$ and $y = \dfrac{p_2}{g}$.

We have the following linear programming problem:

$$\begin{array}{ll} \text{Minimize} & w = x + y \\ \text{subject to:} & 2x + y \geq 1 \\ & \quad\;\; 5y \geq 1 \\ \text{with} & x \geq 0, y \geq 0. \end{array}$$

Graph the feasible region.



The corner points are $(0, 1)$ and $(\frac{2}{5}, \frac{1}{5})$. The minimum value of $w = x + y$ is $\frac{3}{5}$ at $(\frac{2}{5}, \frac{1}{5})$. Thus, the value of the game is $g = \frac{1}{w} = \frac{5}{3}$. To find the value of the original game, we must subtract 2:

$$\frac{5}{3} - 2 = -\frac{1}{3}.$$

The value of the original game is $-\frac{1}{3}$. The optimum strategy for A is

$$\begin{aligned} p_1 &= gx = \frac{5}{3}\left(\frac{2}{5}\right) = \frac{2}{3}, \\ p_2 &= gy = \frac{5}{3}\left(\frac{1}{5}\right) = \frac{1}{3}. \end{aligned}$$

Let player B choose column 1 with probability q_1 and column 2 with probability q_2. Then

$$\begin{aligned} & E_1 = 2q_1 \\ \text{and}\quad & E_2 = \;\; q_1 + 5q_2. \end{aligned}$$

Let g represent the maximum of the expected gains, so that

$$\begin{aligned} E_1 &= 2q_1 \qquad\quad \leq g \\ E_2 &= \;\; q_1 + 5q_2 \leq g. \end{aligned}$$

Dividing by g yields

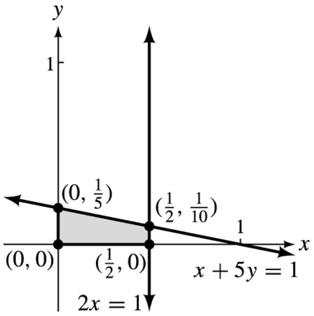
$$\begin{aligned} 2\left(\frac{q_1}{g}\right) \qquad\qquad & \leq 1 \\ \frac{q_1}{g} + 5\left(\frac{q_2}{g}\right) & \leq 1. \end{aligned}$$

Let $x = \dfrac{q_1}{g}$ and $y = \dfrac{q_2}{g}$.

We have the following linear programming problem:

$$\begin{array}{ll} \text{Maximize} & z = x + y \\ \text{subject to:} & 2x \qquad\; \leq 1 \\ & \;\; x + 5y \leq 1 \\ \text{with} & x \geq 0, y \geq 0. \end{array}$$

Graph the feasible region.



There are four corner points to consider: $(0, 0)$, $(0, \frac{1}{5})$, $(\frac{1}{2}, \frac{1}{10})$, and $(\frac{1}{2}, 0)$. The maximum of $z = x + y$ is $\frac{3}{5}$ at $(\frac{1}{2}, \frac{1}{10})$. As before, the value of the game is $g = \frac{1}{z} = \frac{5}{3}$. The value of the original game again is $\frac{5}{3} - 2 = -\frac{1}{3}$.

The optimum strategy for B is

$$\begin{aligned} q_1 &= gx = \frac{5}{3}\left(\frac{1}{2}\right) = \frac{5}{6}, \\ q_2 &= gy = \frac{5}{3}\left(\frac{1}{10}\right) = \frac{1}{6}. \end{aligned}$$

To summarize, player A should choose row 1 with probability $\frac{2}{3}$ and row 2 with probability $\frac{1}{3}$, while player B should choose column 1 with probability $\frac{5}{6}$ and column 2 with probability $\frac{1}{6}$. When these optimum strategies are used, the value of the game is $-\frac{1}{3}$.

32. $\begin{bmatrix} 1 & -3 \\ -4 & 2 \\ -2 & 1 \end{bmatrix}$

Because of the negative entries, we will add 4 to all the entries. The resulting payoff matrix is

$$\begin{bmatrix} 5 & 1 \\ 0 & 6 \\ 2 & 5 \end{bmatrix}.$$

The linear programming problem to be solved is:

$$\begin{array}{ll} \text{Maximize} & z = x_1 + x_2 \\ \text{subject to:} & 5x_1 + x_2 \le 1 \\ & 6x_2 \le 1 \\ & 2x_1 + 5x_2 \le 1 \\ \text{with} & x_1 \ge 0, x_2 \ge 0. \end{array}$$

The initial tableau is

$$\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ \boxed{5} & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 6 & 0 & 1 & 0 & 0 & 1 \\ 2 & 5 & 0 & 0 & 1 & 0 & 1 \\ \hline -1 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}.$$

We arbitrarily choose the first column. The smallest ratio is formed by the 5 in row 1. We use this entry as the first pivot and arrive at the following matrix.

$$\begin{array}{r|cccccc|c} & x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ & 5 & 1 & 1 & 0 & 0 & 0 & 1 \\ & 0 & 6 & 0 & 1 & 0 & 0 & 1 \\ -2\text{R}_1 + 5\text{R}_3 \rightarrow \text{R}_3 & 0 & \boxed{23} & -2 & 0 & 5 & 0 & 3 \\ \hline \text{R}_1 + 5\text{R}_4 \rightarrow \text{R}_4 & 0 & -4 & 1 & 0 & 0 & 5 & 1 \end{array}$$

The next pivot is the 23 in row 3, column 2.

$$\begin{array}{r|cccccc|c} & x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ -\text{R}_3 + 23\text{R}_1 \rightarrow \text{R}_1 & 115 & 0 & 25 & 0 & -5 & 0 & 20 \\ -6\text{R}_3 + 23\text{R}_2 \rightarrow \text{R}_2 & 0 & 0 & 12 & 23 & -30 & 0 & 5 \\ & 0 & 23 & -2 & 0 & 5 & 0 & 3 \\ \hline 4\text{R}_3 + 23\text{R}_4 \rightarrow \text{R}_4 & 0 & 0 & 15 & 0 & 20 & 115 & 35 \end{array}$$

Dividing the bottom row by 115 gives a z-value of $\frac{7}{23}$, so $g = \frac{1}{z} = \frac{23}{7}$. The values of y_1, y_2, and y_3 are read from the bottom of the columns for the slack variables after dividing the bottom row by 115:

$$y_1 = \frac{15}{115} = \frac{3}{23}, y_2 = 0, y_3 = \frac{20}{115} = \frac{4}{23}.$$

We find the values of p_1, p_2, and p_3 by multiplying the values of y_1, y_2, and y_3 by g:

$$\begin{aligned} p_1 &= \frac{23}{7}\left(\frac{3}{23}\right) = \frac{3}{7}, \\ p_2 &= 0, \\ p_3 &= \frac{23}{7}\left(\frac{4}{23}\right) = \frac{4}{7}. \end{aligned}$$

Next, we find the values of x_1 and x_2 by using the first two columns combined with the last column:

$$x_1 = \frac{20}{115} = \frac{4}{23} \text{ and } x_2 = \frac{3}{23}.$$

We find the values of q_1 and q_2 by multiplying these by g:

$$\begin{aligned} q_1 &= \frac{23}{7}\left(\frac{4}{23}\right) = \frac{4}{7} \\ q_2 &= \frac{23}{7}\left(\frac{3}{23}\right) = \frac{3}{7}. \end{aligned}$$

Finally, the value of the game is found by subtracting from g the 2 that was added at the beginning, yielding

$$\frac{23}{7} - 2 = \frac{9}{7}.$$

To summarize, the optimum strategy for player A is $p_1 = \frac{3}{7}, p_2 = 0$, and $p_3 = \frac{4}{7}$. The optimum strategy for player B is $q_1 = \frac{4}{7}$ and $q_2 = \frac{3}{7}$. When these strategies are used, the value of the game is $\frac{9}{7}$.

34. $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 2 \\ -1 & 2 & 0 \end{bmatrix}$

Because of the negative entries, we will add 1 to all the entries. The resulting payoff matrix is

$$\begin{bmatrix} 3 & 2 & 0 \\ 1 & 2 & 3 \\ 0 & 3 & 1 \end{bmatrix}.$$

The linear programming problem to be solved is:

$$\begin{array}{ll} \text{Maximize} & z = x_1 + x_2 + x_3 \\ \text{subject to:} & 3x_1 + 2x_2 \le 1 \\ & x_1 + 2x_2 + 3x_3 \le 1 \\ & 3x_2 + x_3 \le 1 \\ \text{with} & x_1 \ge 0, x_2 \ge 0, x_3 \ge 0. \end{array}$$

The initial tableau is

$$\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ \hline \boxed{3} & 2 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 2 & 3 & 0 & 1 & 0 & 0 & 1 \\ 0 & 3 & 1 & 0 & 0 & 1 & 0 & 1 \\ \hline -1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}.$$

We arbitrarily choose the first column. The smallest ratio is formed by the 3 in row 1. We make this the first pivot and arrive at the following matrix.

$$\begin{array}{r|ccccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ \hline & 3 & 2 & 0 & 1 & 0 & 0 & 0 & 1 \\ -\mathrm{R}_1+3\mathrm{R}_2 \rightarrow \mathrm{R}_2 & 0 & 4 & \boxed{9} & -1 & 3 & 0 & 0 & 2 \\ & 0 & 3 & 1 & 0 & 0 & 1 & 0 & 1 \\ \hline \mathrm{R}_1+3\mathrm{R}_4 \rightarrow \mathrm{R}_4 & 0 & -1 & -3 & 1 & 0 & 0 & 3 & 1 \end{array}$$

The next pivot is the 9 in row 2, column 3.

$$\begin{array}{r|ccccccc|c} & x_1 & x_1 & x_3 & s_1 & s_2 & s_3 & z & \\ \hline & 3 & 2 & 0 & 1 & 0 & 0 & 0 & 1 \\ & 0 & 4 & 9 & -1 & 3 & 0 & 0 & 2 \\ -\mathrm{R}_2+9\mathrm{R}_3 \rightarrow \mathrm{R}_3 & 0 & 23 & 0 & 1 & -3 & 9 & 0 & 7 \\ \hline \mathrm{R}_2+3\mathrm{R}_4 \rightarrow \mathrm{R}_4 & 0 & 1 & 0 & 2 & 3 & 0 & 9 & 5 \end{array}$$

Dividing the bottom row by 9 gives a z-value of $\frac{5}{9}$, so $g = \frac{1}{z} = \frac{9}{5}$. The values of y_1, y_2, and y_3 are read from the bottom of the columns for the three slack variables after dividing the bottom row by 9:

$$y_1 = \frac{2}{9}, y_2 = \frac{3}{9}, y_3 = 0.$$

We find the values of p_1, p_2, and p_3 by multiplying the values of y_1, y_2, and y_3 by g:

$$p_1 = \frac{9}{5}\left(\frac{2}{9}\right) = \frac{2}{5},$$

$$p_2 = \frac{9}{5}\left(\frac{3}{9}\right) = \frac{3}{5},$$

$$p_3 = 0.$$

Next, we find the values of x_1, x_2, and x_3 by using the first three columns combined with the last column:

$$x_1 = \frac{1}{3}, x_2 = 0,\ x_3 = \frac{2}{9}.$$

We find the values of q_1, q_2, and q_3 by multiplying these values by g:

$$q_1 = \frac{9}{5}\left(\frac{1}{3}\right) = \frac{3}{5},$$

$$q_2 = 0,$$

$$q_3 = \frac{9}{5}\left(\frac{2}{9}\right) = \frac{2}{5}.$$

Finally, the value of the game is found by subtracting from g the 1 that was added at the beginning, yielding

$$\frac{9}{5} - 1 = \frac{4}{5}.$$

To summarize, the optimum strategy for player A is $p_1 = \frac{2}{5}, p_2 = \frac{3}{5}$, and $p_3 = 0$. The optimum strategy for player B is $q_1 = \frac{3}{5}, q_2 = 0$, and $q_3 = \frac{2}{5}$. When these strategies are used, the value of the game is $\frac{4}{5}$.

36. If the chance of rain is 90% every day, carry an umbrella every day. For "player A," we will say strategy 1 is to carry an umbrella, and strategy 2 is to not carry an umbrella. For "player B," strategy 1 is that it rains, and strategy 2 is that it does not rain.

The payoff matrix is

$$\begin{bmatrix} 0 & -1 \\ -100 & 0 \end{bmatrix}.$$

$$\begin{aligned} p_1 &= \frac{a_{22} - a_{21}}{a_{11} - a_{21} - a_{12} + a_{22}} \\ &= \frac{0 - (-100)}{0 - (-100) - (-1) + 0} \\ p_1 &= \frac{100}{101} \end{aligned}$$

According to game theory, carry an umbrella with a probability of $\frac{100}{101}$.

In the first situation, there was no penalty for carrying an umbrella when there was no rain.

In Exercises 38 and 40, use the following payoff matrix.

$$\begin{array}{ccc|cc} & & & \textit{Management} & \\ & & & \text{Friendly} & \text{Hostile} \\ \hline \textit{Labor} & \text{Friendly} & & \$600 & \$800 \\ & \text{Hostile} & & \$400 & \$950 \end{array}$$

38. "Friendly" has a worst payoff of \$600 and "hostile" has a worst payoff of \$400, so the absolute worst payoff of \$400 can be avoided by labor being friendly. Therefore, a pessimist should choose the strategy of being friendly.

40. Find the expected payoff from each strategy.

Friendly: .6(600) + .4(800) = \$680;

Hostile: .6(400) + .4(950) = \$620

Labor should adopt a friendly strategy.

42.

$$\begin{array}{ll|cc|} & & \multicolumn{2}{c}{\textit{Economy}} \\ & & \text{Inflationary} & \text{Stable} \\ \textit{Stocks} & \text{Blue-Chip} & 2800 & 3200 \\ & \text{Growth} & 5000 & -2000 \end{array}$$

The optimum strategy for Victor is

$$p_1 = \frac{-2000 - 5000}{2800 - 5000 - 3200 + (-2000)} = \frac{35}{37},$$

$$p_2 = 1 - \frac{35}{37} = \frac{2}{37}.$$

That is, Victor should invest in blue-chip stocks with probability $\frac{35}{37}$ and growth stocks with probability $\frac{2}{37}$.

The value of the game is

$$\frac{2800(-2000) - 5000(3200)}{2800 - 5000 - 3200 + (-2000)} = \frac{-21{,}600{,}000}{-7400}$$
$$\approx \$2918.92.$$

44. $\begin{bmatrix} 2800 & 3200 \\ 5000 & -2000 \end{bmatrix}$

Add 2000 to each entry to obtain

$$\begin{bmatrix} 4800 & 5200 \\ 7000 & 0 \end{bmatrix}.$$

The linear programming problem to be solved is:

$$\begin{array}{ll} \text{Maximize} & z = x_1 + x_2 \\ \text{subject to:} & 4800x_1 + 5200x_2 \le 1 \\ & 7000x_1 \qquad\qquad\;\; \le 1 \\ \text{with} & x_1 \ge 0, x_2 \ge 0. \end{array}$$

The initial tableau is as follows.

$$\begin{array}{c} \begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc|c} 4800 & \boxed{5200} & 1 & 0 & 0 & 1 \\ 7000 & 0 & 0 & 1 & 0 & 1 \\ \hline -1 & -1 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Pivot on each indicated entry.

$$\begin{array}{r} \\ \\ \\ R_1 + 5200R_3 \to R_3 \end{array} \begin{array}{c} \begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc|c} 4800 & 5200 & 1 & 0 & 0 & 1 \\ \boxed{7000} & 0 & 0 & 1 & 0 & 1 \\ \hline -400 & 0 & 1 & 0 & 5200 & 1 \end{array}\right] \end{array}$$

$$\begin{array}{r|ccccc|c|} & x_1 & x_2 & s_1 & s_2 & z & \\ -24R_2+35R_1 \to R_1 & 0 & 182{,}000 & 35 & -24 & 0 & 11 \\ & 7000 & 0 & 0 & 1 & 0 & 1 \\ \hline 2R_2+35R_3 \to R_3 & 0 & 0 & 35 & 2 & 182{,}000 & 37 \end{array}$$

Create a 1 in the columns corresponding to x_1, x_2, and z.

$$\begin{array}{r|ccccc|c|} & x_1 & x_2 & s_1 & s_2 & z & \\ \frac{1}{182{,}000}R_1 \to R_1 & 0 & 1 & \frac{1}{5200} & -\frac{3}{22{,}750} & 0 & \frac{11}{182{,}000} \\ \frac{1}{7000}R_2 \to R_2 & 1 & 0 & 0 & \frac{1}{7000} & 0 & \frac{1}{7000} \\ \hline \frac{1}{182{,}000}R_3 \to R_3 & 0 & 0 & \frac{1}{5200} & \frac{1}{91{,}000} & 1 & \frac{37}{182{,}000} \end{array}$$

From this final tableau, we have

$$x_1 = \frac{1}{7000}, x_2 = \frac{11}{182{,}000},$$
$$y_1 = \frac{1}{5200}, y_2 = \frac{1}{91{,}000}, \text{ and } z = \frac{37}{182{,}000}.$$

Note that $g = \frac{1}{z} = \frac{182{,}000}{37}$. The optimum strategy for Victor is

$$p_1 = gy_1 = \frac{182{,}000}{37}\left(\frac{1}{5200}\right) = \frac{35}{37},$$
$$p_2 = gy_2 = \frac{182{,}000}{37}\left(\frac{1}{91{,}000}\right) = \frac{2}{37}.$$

That is, he should invest in blue-chip stocks with probability $\frac{35}{37}$ and growth stocks with probability $\frac{2}{37}$.

The value of the game is

$$\frac{182{,}000}{37} - 2000 \approx \$2918.92.$$

For Exercises 46 and 48, use the following payoff matrix.

$$\begin{array}{ll|ccc|} & & & \textit{Opponent} & \\ & & \text{Favors} & \text{Waffles} & \text{Opposes} \\ & \text{Favors} & 0 & -1000 & -4000 \\ \textit{Candidate} & \text{Waffles} & 1000 & 0 & -500 \\ & \text{Opposes} & 5000 & 2000 & 0 \end{array}$$

46. As a pessimist, Martha would look at the worst possible outcome from each strategy, and choose the best of them. If she favors the factory, she could lose 4000 votes; if she waffles, she could lose 500 votes; and if she opposes the issue, she could lose 0 votes. The best of these options is to lose 0 votes, so she should oppose the factory.

48. Find the expected payoffs (in vote changes) for each strategy.

Favors:

$$0(0) + .7(-1000) + .3(-4000) = -1900$$

Waffles:

$$0(1000) + .7(0) + .3(-500) = -1500$$

Opposes:

$$0(5000) + .7(2000) + .3(0) = 1400$$

Martha should oppose the factory for an expected gain of 1400 votes.

For Exercises 50 and 52, use the following payoff matrix.

$$\begin{array}{llcc} & & \multicolumn{2}{c}{\text{Rontovia}} \\ & & \text{Attack 1} & \text{Attack 2} \\ \text{Ravogna} & \text{Defend } 1 & 4 & 1 \\ & \text{Defend } 2 & 3 & 4 \end{array}$$

50.
$$\begin{aligned} p_1 &= \frac{a_{22} - a_{21}}{a_{11} - a_{21} - a_{12} + a_{22}} \\ &= \frac{4-3}{4-3-1+4} = \frac{1}{4} \\ p_2 &= \frac{a_{11} - a_{12}}{a_{11} - a_{21} - a_{12} + a_{22}} \\ &= \frac{4-1}{4} = \frac{3}{4} \\ q_1 &= \frac{a_{22} - a_{12}}{a_{11} - a_{21} - a_{12} + a_{22}} \\ &= \frac{4-1}{4} = \frac{3}{4} \\ q_2 &= \frac{a_{11} - a_{21}}{a_{11} - a_{21} - a_{12} + a_{22}} \\ &= \frac{4-3}{4} = \frac{1}{4} \end{aligned}$$

The value of the game is

$$\begin{aligned} \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11} - a_{21} - a_{12} + a_{22}} &= \frac{4(4) - 1(3)}{4} \\ &= \frac{13}{4}. \end{aligned}$$

Ravogna should defend installation #1 with probability $\frac{1}{4}$ and installation #2 with probability $\frac{3}{4}$. Rontovia should attack installation #1 with probability $\frac{3}{4}$ and installation #2 with probability $\frac{1}{4}$. The value of the game is $\frac{13}{4}$.

52. The linear programming problem to be solved is:

$$\begin{array}{ll} \text{Maximize} & z = x_1 + x_2 \\ \text{subject to:} & 4x_1 + x_2 \le 1 \\ & 3x_1 + 4x_2 \le 1 \\ \text{with} & x_1 \ge 0, x_2 \ge 0. \end{array}$$

The initial tableau is

$$\left[\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ \boxed{4} & 1 & 1 & 0 & 0 & 1 \\ 3 & 4 & 0 & 1 & 0 & 1 \\ \hline -1 & -1 & 0 & 0 & 1 & 0 \end{array}\right].$$

We arbitrarily choose the first column. The smallest ratio is formed by the 4 in row 1. We make this entry the first pivot and arrive at the following matrix.

$$\begin{array}{r|ccccc|c} & x_1 & x_2 & s_1 & s_2 & z & \\ & 4 & 1 & 1 & 0 & 0 & 1 \\ -3\text{R}_1 + 4\text{R}_2 \to \text{R}_2 & 0 & \boxed{13} & -3 & 4 & 0 & 1 \\ \hline \text{R}_1 + 4\text{R}_3 \to \text{R}_3 & 0 & -3 & 1 & 0 & 4 & 1 \end{array}$$

The next pivot is the 13 in row 2, column 2.

$$\begin{array}{r|ccccc|c} & x_1 & x_2 & s_1 & s_2 & z & \\ -\text{R}_2 + 13\text{R}_1 \to \text{R}_1 & 52 & 0 & 16 & -4 & 0 & 12 \\ & 0 & 13 & -3 & 4 & 0 & 1 \\ \hline 3\text{R}_2 + 13\text{R}_3 \to \text{R}_3 & 0 & 0 & 4 & 12 & 52 & 16 \end{array}$$

Dividing the bottom row by 52 gives a z-value of $\frac{16}{52} = \frac{4}{13}$, so $g = \frac{1}{z} = \frac{13}{4}$. The value of the game is $\frac{13}{4}$. The values of y_1 and y_2 are read from the bottom of the columns for the two slack variables after dividing the bottom row by 52:

$$y_1 = \frac{4}{52} = \frac{1}{13}, y_2 = \frac{12}{52} = \frac{3}{13}.$$

We find the values of p_1 and p_2 by multiplying the values of y_1 and y_2 by g:

$$\begin{aligned} p_1 &= \frac{13}{4}\left(\frac{1}{13}\right) = \frac{1}{4}, \\ p_2 &= \frac{13}{4}\left(\frac{3}{13}\right) = \frac{3}{4}. \end{aligned}$$

Next, we find the values of x_1 and x_2 by using the first two columns combined with the last column:

$$x_1 = \frac{12}{52} = \frac{3}{13}, x_2 = \frac{-1}{-13} = \frac{1}{13}.$$

We find the values of q_1 and q_2 by multiplying these values by g:

$$q_1 = \frac{13}{4}\left(\frac{3}{13}\right) = \frac{3}{4},$$
$$\text{and} \quad q_2 = \frac{13}{4}\left(\frac{1}{13}\right) = \frac{1}{4}.$$

Thus, Ravogna should defend installation #1 with probability $\frac{1}{4}$ and installation #2 with probability $\frac{3}{4}$. Rontovia should attack installation #1 with probability $\frac{3}{4}$ and installation #2 with probability $\frac{1}{4}$. The value of the game is $\frac{13}{4}$.

Extended Application: The Prisoner's Dilemma-Non-Zero-Sum Games in Economics

1. Since neither row dominates the other, Linda does not have a dominated strategy. Similarly, since neither column dominates the other, Mel does not have a dominated strategy. Linda and Mel should consider cooperating.

2. If they decide to cooperate, they will either both pick Chinese or both pick French, since disagreement leads to the worst payoff for each player. If they both pick Chinese, their combined degree of enjoyment is 6, whereas if they both pick French, their combined degree of enjoyment is 8. They maximize their combined enjoyment by both picking French.

3. Linda's expected payoff:

$$\begin{aligned}[.8 \quad .2]\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} .1 \\ .9 \end{bmatrix} &= [4 \quad .6]\begin{bmatrix} .1 \\ .9 \end{bmatrix} \\ &= [.94]\end{aligned}$$

Mel's expected payoff:

$$\begin{aligned}[.8 \quad .2]\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} .1 \\ .9 \end{bmatrix} &= [.8 \quad 1]\begin{bmatrix} .1 \\ .9 \end{bmatrix} \\ &= [.98]\end{aligned}$$

Mel does better.

4. Let $[p_1 \quad p_2]$ represent Linda's strategy which maximizes her expected payoff. Then, her expected payoff is given by

$$\begin{aligned}E &= [p_1 \quad p_2]\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} .1 \\ .9 \end{bmatrix} \\ &= [5p_1 \quad 3p_2]\begin{bmatrix} .1 \\ .9 \end{bmatrix} \\ &= .5p_1 + 2.7p_2.\end{aligned}$$

Substituting $1 - p_1$ for p_2, we find

$$E = .5p_1 + 2.7(1 - p_1) = -2.2p_1 + 2.7.$$

Since this is maximized when $p_1 = 0$, we have $p_2 = 1 - 0 = 1$. Thus, Linda should choose the strategy $[0, 1]$. That is, Linda should always choose French for an expected payoff of 2.7. If she does this, Mel might always choose French in order to maximize her enjoyment.

5. If Linda picks Chinese and Mel picks French, her degree of hapiness will be zero, however, Mel may not continue with his "always French" strategy since his degree of happiness would also be zero.

6. Since their combined enjoyment is 6 when choosing Chinese and 8 when choosing French, they may opt to make their decisions based on the following spinner. If the arrow of the spinner lands in the unshaded region, they will eat French. Otherwise, they will eat Chinese.