Preface

This Instructor's Solutions Manual contains solutions for essentially all of the exercises in the text that are intended to be done by hand. Solutions to Matlab exercises are not included. The Student's Solutions Manual that accompanies this text contains solutions for only selected odd-numbered exercises, including those exercises whose answers appear in the answer key. The solutions that appear in the students' manual are identical to those provided in this manual, and generally provide a more detailed solution than is available in the answer key. Although no pattern is strictly adhered to throughout the student manual, the solutions provided there are primarily to the computational exercises, whereas solutions that involve proof are generally not included. None of the solutions to the supplementary end-of-chapter exercises are included in the student manual.

Contents

Chapter 1

Matrices and Systems of Equations

1.1 Introduction to Matrices and Systems of Linear Equations

1. Linear.
2. Nonlinear.
3. Linear.
4. Nonlinear.
5. Nonlinear.
6. Linear.
7. $$\begin{array}{rcl} x_1+3x_2 & = & 7 \\ 4x_1-x_2 & = & 2 \end{array} \quad \begin{array}{rcl} 1+3\cdot 2 & = & 7 \\ 4\cdot 1-2 & = & 2 \end{array}$$
8. $$\begin{array}{rcr} 6x_1-x_2+x_3 & = & 14 \\ x_1+2x_2+4x_3 & = & 4 \end{array} \quad \begin{array}{rcr} 6\cdot 2-(-1)+1 & = & 14 \\ 2+2\cdot(-1)+4\cdot 1 & = & 4 \end{array}$$
9. $$\begin{array}{rcr} x_1+x_2 & = & 0 \\ 3x_1+4x_2 & = & -1 \\ -x_1+2x_2 & = & -3 \end{array} \quad \begin{array}{rcr} 1+(-1) & = & 0 \\ 3\cdot 1+4\cdot(-1) & = & -1 \\ -1+2\cdot(-1) & = & -3 \end{array}$$
10. $$\begin{array}{rcl} 3x_2 & = & 9, \\ 4x_1 & = & 8, \end{array} \quad \begin{array}{rcl} 3\cdot 3 & = & 9 \\ 4\cdot 2 & = & 8 \end{array}$$
11. Unique solution.
12. No Solution
13. Infinitely many solutions.

14. No solution.

15. (a) The planes do not intersect; that is, the planes are parallel.

    (b) The planes intersect in a line or the planes are coincident.

16. The planes intersect in the line $x = (1-t)/2, y = 2, z = t$.

17. The planes intersect in the line $x = 4 - 3t, y = 2t - 1, z = t$.

18. Coincident planes.

19. $A = \begin{bmatrix} 2 & 1 & 6 \\ 4 & 3 & 8 \end{bmatrix}$.

20. $C = \begin{bmatrix} 1 & 2 & 7 & 1 \\ 2 & 2 & 4 & 3 \end{bmatrix}$.

21. $Q = \begin{bmatrix} 1 & 4 & -3 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$.

22. $\begin{array}{rrrcr} x_1 & +2x_2 & +7x_3 & = & 1 \\ 2x_1 & +2x_2 & +4x_3 & = & 3 \end{array}$

23. $\begin{array}{rcrcr} 2x_1 & + & x_2 & = & 6 \\ 4x_1 & + & 3x_2 & = & 8 \end{array}$; $\begin{array}{rcrcr} x_1 & + & 4x_2 & = & -3 \\ 2x_1 & + & x_2 & = & 1 \\ 3x_1 & + & 2x_2 & = & 1 \end{array}$

24. $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 3 \end{bmatrix}$.

25. $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 2 & 0 & -1 & 1 \end{bmatrix}$.

26. $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 5 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 2 & 5 & 1 & 5 \\ 1 & 1 & 1 & 3 \end{bmatrix}$.

27. $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & -1 \\ -1 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 2 & 6 \\ 3 & 4 & -1 & 5 \\ -1 & 1 & 1 & 2 \end{bmatrix}$.

28. $A = \begin{bmatrix} 1 & 1 & -3 \\ 1 & 2 & -5 \\ -1 & -3 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & -3 & -1 \\ 1 & 2 & -5 & -2 \\ -1 & -3 & 7 & 3 \end{bmatrix}$.

29. $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 1 & 2 \\ 1 & -1 & 3 & 2 \end{bmatrix}$.

30. Elementary operations on equations: $E_2 - 2E_1$.

    Reduced system of equations: $\begin{aligned} 2x_1 + 3x_2 &= 6 \\ -7x_2 &= -5 \end{aligned}$.

    Elementary row operations: $R_2 - 2R_1$.

    Reduced augmented matrix: $\begin{bmatrix} 2 & 3 & 6 \\ 0 & -7 & -5 \end{bmatrix}$.

31. Elementary operations on equations: $E_2 - E_1,\ E_3 + 2E_1$.

    Reduced system of equations: $\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ -x_2 + 3x_3 &= 1 \\ 5x_2 - 2x_3 &= 6 \end{aligned}$.

    Elementary row operations: $R_2 - R_1,\ R_3 + 2R_1$.

    Reduced augmented matrix: $\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 5 & -2 & 6 \end{bmatrix}$.

32. Elementary operations on equations: $E_1 \leftrightarrow E_2,\ E_3 - 2E_1$.

    Reduced system of equations: $\begin{aligned} x_1 - x_2 + 2x_3 &= 1 \\ x_2 + x_3 &= 4 \\ 3x_2 - 5x_3 &= 4 \end{aligned}$.

    Elementary row operations: $R_1 \leftrightarrow R_2,\ R_3 - 2R_1$.

    Reduced augmented matrix: $\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 3 & -5 & 4 \end{bmatrix}$.

33. Elementary operations on equations: $E_2 - E_1,\ E_3 - 3E_1$.

    Reduced system of equations: $\begin{aligned} x_1 + x_2 &= 9 \\ -2x_2 &= -2 \\ -2x_2 &= -21 \end{aligned}$.

    Elementary row operations: $R_2 - R_1,\ R_3 - 3R_1$.

    Reduced augmented matrix: $\begin{bmatrix} 1 & 1 & 9 \\ 0 & -2 & -2 \\ 0 & -2 & -21 \end{bmatrix}$.

34. Elementary operations on equations: $E_2 + E_1,\ E_3 + 2E_1$.

Reduced system of equations:
$$\begin{array}{rcl} x_1 + x_2 + x_3 - x_4 & = & 1 \\ 2x_2 & = & 4 \\ 3x_2 + 3x_3 - 3x_4 & = & 4 \end{array} .$$

Elementary row operations: $R_2 + R_1,\ R_3 + 2R_1$.

Reduced augmented matrix:
$$\begin{bmatrix} 1 & 1 & 1 & -1 & 1 \\ 0 & 2 & 0 & 0 & 4 \\ 0 & 3 & 3 & -3 & 4 \end{bmatrix}.$$

35. Elementary operations on equations: $E_2 \leftrightarrow E_1,\ E_3 + E_1$.

Reduced system of equations:
$$\begin{array}{rcl} x_1 + 2x_2 - x_3 + x_4 & = & 1 \\ x_2 + x_3 - x_4 & = & 3 \\ 3x_2 + 6x_3 & = & 1 \end{array} .$$

Elementary row operations: $R_2 \leftrightarrow R_1,\ R_3 + R_1$.

Reduced augmented matrix:
$$\begin{bmatrix} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & 1 & -1 & 3 \\ 0 & 3 & 6 & 0 & 1 \end{bmatrix}.$$

36. Elementary operations on equations: $E_2 - E_1,\ E_3 - 3E_1$.

Reduced system of equations:
$$\begin{array}{rcl} x_1 + x_2 & = & 0 \\ -2x_2 & = & 0 \\ -2x_2 & = & 0 \end{array} .$$

Elementary row operations: $R_2 - R_1,\ R_3 - 3R_1$.

Reduced augmented matrix:
$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & -2 & 0 \end{bmatrix}.$$

37. (b) In each case, the graph of the resulting equation is a line.

38. Now if $a_{11} = 0$ we easily obtain the equivalent system

$$\begin{array}{rcl} a_{21}x_1 + a_{22}x_2 & = & b_2 \\ a_{12}x_2 & = & b_1 \end{array}$$

Thus we may suppose that $a_{11} \neq 0$. Then :

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 & = & b_1 \\ a_{21}x_1 + a_{22}x_2 & = & b_2 \end{array} \left\{ \begin{array}{c} E_2 - (a_{21}/a_{11})E_1 \\ \Longrightarrow \end{array} \right\}$$

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 & = & b_1 \\ ((-a_{21}/a_{11})a_{12} + a_{22})x_2 & = & (-a_{21}/a_{11})b_1 + b_2 \end{array} \quad \left\{ \begin{array}{c} a_{11}E_2 \\ \Longrightarrow \end{array} \right\}$$

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 & = & b_1 \\ (a_{11}a_{22} - a_{12}a_{21})x_2 & = & -a_{21}b_1 + a_{11}b_2 \end{array}$$

Each of a_{11} and $(a_{11}a_{22} - a_{12}a_{21})$ is non-zero. ■

39. Let

$$\mathcal{A} = \left\{ \begin{array}{c} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{array} \right\}$$

and let

$$\mathcal{B} = \left\{ \begin{array}{c} a_{11}x_1 + a_{12}x_2 = b_1 \\ ca_{21}x_1 + ca_{22}x_2 = cb_2 \end{array} \right\}$$

Suppose that $x_1 = s_1$, $x_2 = s_2$ is a solution to $\mathcal{A}$. Then $a_{11}s_1 + a_{12}s_2 = b_1$, and $a_{21}s_1 + a_{22}s_2 = b_2$. But this means that $ca_{21}s_1 + ca_{22}s_2 = cb_2$ and so $x_1 = s_1$, $x_2 = s_2$ is also a solution to $\mathcal{B}$. Now suppose that $x_1 = t_1$, $x_2 = t_2$ is a solution to $\mathcal{B}$. Then $a_{11}t_1 + a_{12}t_2 = b_1$ and $ca_{21}t_1 + ca_{22}t_2 = cb_2$. Since $c \neq 0$, $a_{21}x_1 + a_{22}x_2 = b_2$. ■

40. Let

$$\mathcal{A} = \left\{ \begin{array}{rcl} a_{11}x_1 + a_{12}x_2 & = & b_1 \\ a_{21}x_1 + a_{22}x_2 & = & b_2 \end{array} \right\}$$

and let

$$\mathcal{B} = \left\{ \begin{array}{r} a_{11}x_1 + a_{12}x_2 = b_1 \\ (a_{21} + ca_{11})x_1 + (a_{22} + ca_{12})x_2 = b_2 + cb_1 \end{array} . \right\}$$

Let $x_1 = s_1$ and $x_2 = s_2$ be a solution to $\mathcal{A}$. Then $a_{11}s_1 + a_{12}s_2 = b_1$ and $a_{21}s_1 + a_{22}s_2 = b_2$ so $a_{11}s_1 + a_{12}s_2 = b_1$ and $(a_{21} + ca_{11})s_1 + (a_{22} + ca_{12})s_2 = b_2 + cb_1$ as required. Now if $x_1 = t_1$ and $x_2 = t_2$ is a solution to $\mathcal{B}$ then $a_{11}t_1 + a_{12}t_2 = b_1$ and $(a_{21} + ca_{11})t_1 + (a_{22} + ca_{12})t_2 = b_2 + cb_1$, so $a_{11}t_1 + a_{12}t_2 = b_1$ and $a_{21}t_1 + a_{12}t_2 = b_2$ as required. ■

41. The proof is very similar to that of 45 and 46.

42. By adding the two equations we obtain: $2x_1^2 - 2x_1 = 4$. Then $x_1 = 2$ or $x_1 = -1$ and substituting these values in the second equation we find that there are three solutions: $x_1 = -1, x_2 = 0$; $x_1 = 2, x_2 = \sqrt{3},$; $x_1 = 2, x_2 = -\sqrt{3}$.

1.2 Echelon Form and Gauss-Jordan Elimination

1. The matrix is in echelon form. The row operation $R_2 - 2R_1$ transforms the matrix to reduced echelon form $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

2. Echelon form. $R_2 - 2R_1$ yields reduced row echelon form $\begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & 3 \end{bmatrix}$.

3. Not in echelon form. $(1/2)R_1,\ R_2 - 4R_1,\ (-1/5)R_2$ yields echelon form $\begin{bmatrix} 1 & 3/2 & 1/2 \\ 0 & 1 & 2/5 \end{bmatrix}$.

4. Not in echelon form. $R_1 \leftrightarrow R_2$ yields echelon form $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$.

5. Not in echelon form.
   $R_1 \leftrightarrow R_2,\ (1/2)R_1,\ (1/2)R_2$ yields the echelon form $\begin{bmatrix} 1 & 0 & 1/2 & 2 \\ 0 & 0 & 1 & 3/2 \end{bmatrix}$.

6. Not in echelon form.
   $(1/2)R_1$ yields the echelon form $\begin{bmatrix} 1 & 0 & 3/2 & 1/2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$.

7. Not in echelon form. $R_2 - 4R_3,\ R_1 - 2R_3,\ R_1 - 3R_2$ yields the reduced echelon form $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

8. Not in echelon form. $(1/2)R_1,\ (-1/3)R_3$ yields the echelon form $\begin{bmatrix} 1 & -1/2 & 3/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

9. Not in echelon form. $(1/2)R_2$ yields the echelon form $\begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & -1 & -3/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

10. Not in echelon form $-R_1,\ (1/2)R_2$ yields the echelon form $\begin{bmatrix} 1 & -4 & 3 & -4 & -6 \\ 0 & 1 & 1/2 & -3/2 & -3/2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$.

11. $x_1 = 0$, $x_2 = 0$.

12. The system is inconsistent.

13. $x_1 = -2 + 5x_3$, $x_2 = 1 - 3x_3$, x_3 is arbitrary.

14. $x_1 = 1 - 2x_3$, $x_2 = 0$.

15. $x_1 = 0$, $x_2 = 0$, $x_3 = 0$.

16. $x_1 = 0$, $x_2 = 0$, $x_3 = 0$.

17. $x_1 = x_3 = x_4 = 0$, x_2 is arbitrary.

18. The system is inconsistent.

19. The system is inconsistent.

20. $x_1 = 3x_4 - 5x_5 - 2$, $x_2 = x_4 + x_5 - 2$, $x_3 = -2x_4 - x_5 + 2$, x_4 and x_5 are arbitrary.

21. $x_1 = -1 - (1/2)x_2 + (1/2)x_4$, $x_3 = 1 - x_4$, x_2 and x_4 arbitrary, $x_5 = 0$.

22. $x_1 = (5 + 3x_2)/2$, x_2 arbitrary.

23. The system is inconsistent.

24. $x_1 = x_3$, $x_2 = -3 + 2x_3$, x_3 arbitrary.

25. $x_1 = 2 - x_2$, x_2 arbitrary.

26. $x_1 = 10 + x_2$, x_2 arbitrary, $x_3 = -6$.

27. $x_1 = 2 - x_2 + x_3$, x_2 and x_3 arbitrary.

28. $x_1 = 2x_3$, $x_2 = 1$, x_3 arbitrary.

29. $x_1 = 3 - 2x_3$, $x_2 = -2 + 3x_3$, x_3 arbitrary.

30. $x_1 = -3x_4 - 6x_5$, $x_2 = 1 + 3x_4 + 7x_5$, $x_3 = -2x_4 - 5x_5$, x_4 and x_5 arbitrary.

31. $x_1 = 3 - (7x_4 - 16x_5)/2$, $x_2 = (x_4 + 2x_5)/2$, $x_3 = -2 + (5x_4 - 12x_5)/2$, x_4 and x_5 arbitrary.

32. $x_1 = 2$, $x_2 = -1$.

33. The system is inconsistent.

34. $x_1 = 1 - 2x_2$, x_2 arbitrary.

35. The system is inconsistent.

36. $$\begin{array}{rcr} x_1 + 2x_2 & = & -3 \\ ax_1 - 2x_2 & = & 5 \end{array} \left\{ \begin{array}{c} E_1 + E_2 \\ \Longrightarrow \end{array} \right\} \begin{array}{rcr} x_1 + 2x_2 & = & -3 \\ (a+1)x_1 & = & 2 \end{array}$$

$\boxed{\text{Hence if } a = -1 \text{ there is no solution.}}$

37. $$\begin{array}{rcl} x_1 + 3x_2 & = & 4 \\ 2x_1 + 6x_2 & = & a \end{array} \left\{ \begin{array}{c} E_2 - 2E_1 \\ \Longrightarrow \end{array} \right\} \begin{array}{rcr} x_1 + 3x_2 & = & 4 \\ 0 & = & a - 8 \end{array}$$

$\boxed{\text{Thus, if } a \neq 8 \text{ there is no solution.}}$

38. $\begin{array}{rcl} 2x_1+4x_2 &=& a \\ 3x_1+6x_2 &=& 5 \end{array} \left\{ \begin{array}{c} E_2-(3/2)E_1 \\ \Longrightarrow \end{array} \right\} \begin{array}{rcr} 2x_1+4x_2 &=& a \\ 0 &=& 5-(3/2)a \end{array}$

$\boxed{\text{Thus, if } a \neq 10/3 \text{ there is no solution.}}$

39. $\begin{array}{rcl} 3x_1+ax_2 &=& 3 \\ ax_1+3x_2 &=& 5 \end{array} \left\{ \begin{array}{c} E_2-(a/3)E_1 \\ \Longrightarrow \end{array} \right\} \begin{array}{rcr} 3x_1+ax_2 &=& 3 \\ (a^2/3-3)x_2 &=& 5-a \end{array}$

$\boxed{\text{Thus, if } a = \pm 3 \text{ there is no solution.}}$

40. $\begin{array}{rcl} x_1+ax_2 &=& 6 \\ ax_1+2ax_2 &=& 4 \end{array} \left\{ \begin{array}{c} E_2-aE_1 \\ \Longrightarrow \end{array} \right\} \begin{array}{rcr} x_1+ax_2 &=& 6 \\ (2a-a^2)x_2 &=& 4-6a \end{array}$

41. $\cos\alpha = 1/2$ and $\sin\beta = 1/2$, so $\alpha = \pi/3$ or $\alpha = 5\pi/3$ and $\beta = \pi/6$ or $\beta = 5\pi/6$.

42. $\cos^2\alpha = 3/4$ and $\sin^2\beta = 1/2$. The choices for α are $\pi/6$, $5\pi/6$, $7\pi/6$, and $11\pi/6$. The choices for β are $\pi/4$, $3\pi/4$, $5\pi/4$, and $7\pi/4$.

43. $x_1 = 1-2x_3$, $x_2 = 2+x_3, x_3$ arbitrary. (a) $x_3 = 1/2$. (b) In order for $x_1 \geq 0, x_2 \geq 0,$ we must have $-2 \leq x_3 \leq 1/2$; for a given x_1 and $x_2, y = -6-7x_3$, so the minimum value is $y = 8$ at $x_3 = -2$. (c) The minimum value is 20.

44. $\begin{bmatrix} 1 & d \\ c & b \end{bmatrix} \left\{ \begin{array}{c} R_2-cR_1 \\ \Longrightarrow \end{array} \right\} \begin{bmatrix} 1 & d \\ 0 & b-cd \end{bmatrix} \left\{ \begin{array}{c} R_1-(d/(b-cd))R_2 \\ (\text{ recall } b-cd \neq 0 \\ \Longrightarrow \end{array} \right\}$

$\begin{bmatrix} 1 & 0 \\ 0 & b-cd \end{bmatrix} \left\{ \begin{array}{c} 1/(b-cd)R_2 \\ \Longrightarrow \end{array} \right\} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$

45. $\begin{bmatrix} 1 & x & x \\ 0 & 1 & x \end{bmatrix}, \begin{bmatrix} 1 & x & x \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & x & x \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 & x \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$

46. (a) $\begin{bmatrix} 1 & x \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & x \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$

(b) $\begin{bmatrix} 1 & x & x \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & x & x \\ 0 & 1 & x \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & x & x \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & x & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$

$\begin{bmatrix} 0 & 1 & x \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$

(c) $\begin{bmatrix} 1 & x & x & x \\ 0 & 1 & x & x \\ 0 & 0 & 1 & x \end{bmatrix}$, $\begin{bmatrix} 1 & x & x & x \\ 0 & 1 & x & x \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & x & x & x \\ 0 & 1 & x & x \\ 0 & 0 & 0 & 0 \end{bmatrix}$,

$\begin{bmatrix} 1 & x & x & x \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & x & x & x \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & x & x & x \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

$\begin{bmatrix} 1 & x & x & x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & x & x \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & x & x \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$,

$\begin{bmatrix} 0 & 1 & x & x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 & x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$,

$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

47. $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \left\{ \begin{matrix} 2R_2 \\ \Longrightarrow \end{matrix} \right\} \begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix} \left\{ \begin{matrix} R_2 - R_1 \\ \Longrightarrow \end{matrix} \right\} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

48. $\begin{bmatrix} 1 & 4 \\ 3 & 7 \end{bmatrix} \left\{ \begin{matrix} R_2 - 3R_1 \\ \Longrightarrow \end{matrix} \right\} \begin{bmatrix} 1 & 4 \\ 0 & -5 \end{bmatrix} \left\{ \begin{matrix} R_1 + (2/5)R_2 \\ \Longrightarrow \end{matrix} \right\}$

$\begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix} \left\{ \begin{matrix} (3/5)R_2 \\ \Longrightarrow \end{matrix} \right\} \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix} \left\{ \begin{matrix} R_2 + 2R_1 \\ \Longrightarrow \end{matrix} \right\} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

49. $$\begin{aligned} 100x_1 + 10x_2 + x_3 &= 15(x_1 + x_2 + x_3) \\ 100x_3 + 10x_2 + x_1 &= 100x_1 + 10x_2 + x_3 + 396 \\ x_3 &= x_1 + x_2 + 1 \end{aligned}$$

$x_1 = 1$, $x_2 = 3$, and $x_3 = 5$, so $N = 135$.

50. $$\begin{aligned} a - b + c &= 6 \\ a + b + c &= 4 \\ 4a + 2b + c &= 9 \end{aligned}$$

$a = 2$, $b = -1$, $c = 3$. So $y = 2x^2 - x + 3$.

51. Let x_1, x_2, x_3 be the amounts initially held by players one, two and three, respectively. Also assume that player one loses the first game, player two loses the second game, and player three loses the third game. Then after three games, the amount of money held by each player is given by the following table

Player	**Amount of money**
1	$4x_1 - 4x_2 - 4x_3 = 24$
2	$-2x_1 + 6x_2 - 2x_3 = 24$
3	$-x_1 - x_2 + 7x_3 = 24$

Solving yields $x_1 = 39$, $x_2 = 21$, and $x_3 = 12$.

52. The resulting system of equations is

$$\begin{aligned} x_1 + x_2 + x_3 &= 34 \\ x_1 + x_2 &= 7 \\ x_2 + x_3 &= 22 \end{aligned}$$

The solution is $x_1 = 12$, $x_2 = -5$, $x_3 = 27$.

53. If x_1 is the number of adults, x_2 the number of students, and x_3 the number of children, then $x_1 + x_2 + x_3 = 79$, $6x_1 + 3x_2 + (1/2)x_3 = 207$, and for $j = 1, 2, 3$, x_j is an integer such that $0 \le x_j \le 79$. Following is a list of possiblities

Number of Adults	0	5	10	15	20	25	30
Number of Students	67	56	45	34	23	12	1
Number of Children	12	18	24	30	36	42	48

54. The resulting system of equations is

$$\begin{aligned} a + b + c + d &= 5 \\ b + 2c + 3d &= 5 \\ a + 2b + 4c + 8d &= 17 \\ b + 4c + 12d &= 21. \end{aligned}$$

The solution is $a = 3$, $b = 1$, $c = -1$, $d = 2$. So $p(x) = 3 + x - x^2 + 2x^3$.

55. By (7), $1 + 2 + 3 + \cdots + n = a_1 n + a_2 n^2$. Setting $n = 1$ and $n = 2$ gives

$$\begin{aligned} a_1 + a_2 &= 1 \\ 2a_1 + 4a_2 &= 3 \end{aligned}$$

The solution is $a_1 = a_2 = 1/2$, so $1 + 2 + 3 + \ldots + n = n(n+1)/2$.

56. By (7), $1^2 + 2^2 + 3^2 + \cdots + n^2 = a_1 n + a_2 n^2 + a_3 n^3$. Setting $n = 1$, $n = 2$, $n = 3$, gives

$$\begin{aligned} a_1 + a_2 + a_3 &= 1 \\ 2a_1 + 4a_2 + 8a_3 &= 5 \\ 3a_1 + 9a_2 + 27a_3 &= 14 \end{aligned}$$

The solution is $a_1 = 1/6$, $a_2 = 1/2$ and $a_3 = 1/3$, so $1^2 + 2^2 + 3^2 + \ldots + n^2 = n(n+1)(2n+1)/6$.

57. The system of equations obtained from (7) is

$$\begin{aligned} a_1 + a_2 + a_3 + a_4 + a_5 &= 1 \\ 2a_1 + 4a_2 + 8a_3 + 16a_4 + 32a_5 &= 17 \\ 3a_1 + 9a_2 + 27a_3 + 81a_4 + 242a_5 &= 98 \\ 4a_1 + 16a_2 + 64a_3 + 256a_4 + 1024a_5 &= 354 \\ 5a_1 + 25a_2 + 125a_3 + 625a_4 + 3125a_5 &= 979 \end{aligned}$$

The solution is $a_1 = -1/30$, $a_2 = 0$, $a_3 = 1/3$, $a_4 = 1/2$, $a_5 = 1/5$. Therefore, $1^4 + 2^4 + 3^4 + \cdots + n^4 = n(n+1)(2n+1)(3n^2+3n-1)/30$.

58. $1^5 + 2^5 + 3^5 + \cdots + n^5 = n^2(n+1)^2(2n^2+2n-1)/12$.

1.3 Consistent Systems of Linear Equations

1. The augmented matrix reduces to $\begin{bmatrix} 1 & 1 & 0 & 5/6 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.
   n = 3, r = 2, x_2 is independent.

2. The augmented matrix reduces to $\begin{bmatrix} 1 & 0 & -3/2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.
   n = 2, r = 2.

3. The augmented matrix reduces to $\begin{bmatrix} 1 & 0 & 4 & 0 & 13/2 \\ 0 & 1 & -1 & 0 & -3/2 \\ 0 & 0 & 0 & 1 & 1/2 \end{bmatrix}$.
   n = 4, r = 3, x_3 is independent.

4. The augmented matrix reduces to $\begin{bmatrix} 1 & 2 & 3 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.
   n = 4, r = 2, x_2 and x_3 are independent.

5. $n = 2$ and $r \leq 2$ so $r = 0$, $n - r = 2$; $r = 1$, $n - r = 1$; $r = 2$, $n - r = 0$. There could be a unique solution.

6. $n = 4$ and $r \leq 3$ so $r = 0$, $n - r = 4$; $r = 1$, $n - r = 3$; $r = 2$, $n - r = 2$; $r = 3$, $n - r = 1$. By the corollary to Theorem 3, there are infinitely many solutions.

7. Infinitely many solutions.

8. Infinitely many solutions.

9. Infinitely many solutions, a unique solution or no solution.

10. Infinitely many solutions, a unique solution, or no solution.

11. A unique solution or infinitely many solutions.

12. Infinitely many solutions or a unique solution.

13. Infinitely many solutions.

14. Infinitely many solutions.

15. Infinitely many solutions or a unique solution.

16. Infinitely many solutions or a unique solution.

17. Infinitely many solutions.

18. Infinitely many solutions.

19. There are nontrivial solutions.

20. There are nontrivial solutions.

21. There is only the trivial solution.

22. There is only the trivial solution.

23. If $a = -1$ then when we reduce the augmented matrix we obtain a row of zeroes and hence infinitely many nontrivial solutions.

24. (a) Reduced row echelon form of the augmented matrix is $\begin{bmatrix} 1 & 0 & 2 & -2b_1 + 3b_2 \\ 0 & 1 & -1 & b_1 - b_2 \\ 0 & 0 & 0 & b_3 - b_1 - 2b_2 \end{bmatrix}$.
    Hence, if $b_3 - b_1 - 2b_2 \neq 0$ then the system is inconsistent. Therefore, the system of equations is consistent if and only if $b_3 - b_1 - 2b_2 = 0$.

    (b) (i) The system is consistent. For example, a solution is $x_1 = -1$, $x_2 = 1$ and $x_3 = 1$. (ii) The system is inconsistent by part (a). (iii) The system is consistent. For example, a solution is $x_1 = 1$, $x_2 = 0$ and $x_3 = 1$.

25. (a) $B = \begin{bmatrix} * & x & x \\ 0 & * & x \\ 0 & 0 & * \\ 0 & 0 & 0 \end{bmatrix}$.

(b) In the third row of the matrix of 25(a) for B, we need $0 \cdot x_1 + 0 \cdot x_2 = *$ and, in general, this can't be.

26. The resulting system of equations is $\begin{array}{rcl} 3a+b+c & = & 0 \\ 7a+2b+c & = & 0 \end{array}.$
The general solution is a = c, b = -4c.
Thus $x - 4y + 1 \ = \ 0$ is an equation for the line.

27. The resulting system of equations is $\begin{array}{rcl} 2a+8b+c & = & 0 \\ 4a+b+c & = & 0 \end{array}.$
The general solution is a = (-7/30)c, b = (-1/15)c.
Thus $-7x - 2y + 30 \ = \ 0$ is an equation for the line.

28. The resulting system of equations is
$$\begin{array}{rcl} 16a-4d+f & = & 0 \\ 4a+4b+4c-2d-2e+f & = & 0 \\ 9c+3e+f & = & 0\ . \\ a+b+c+d+e+f & = & 0 \\ 16a+4d+f & = & 0 \end{array}$$
The general solution is:
a = (-1/16)f, b = (-71/144)f, c = (1/18)f, d = 0, e = (-1/2)f.
An equation is $9x^2 + 71xy - 8y^2 + 72y - 144 \ = \ 0$.

29. The resulting system of equations is
$$\begin{array}{rcl} 16a-4b+c-4d+e+f & = & 0 \\ a-2b+4c-d+2e+f & = & 0 \\ 9a+6b+4c+3d+2e+f & = & 0\ . \\ 25a+5b+c+5d+e+f & = & 0 \\ 49a-7b+c+7d-e+f & = & 0 \end{array}$$
The general solution is:
a = (-3/113)f, b = (3/113)f, c = (1/113)f, d = 0, e = (-54/113)f.
An equation is $-3x^2 + 3xy + y^2 - 54y + 113 \ = \ 0$.

30. Using equation (4), the given points result in a system of 9 equations in 10 unknowns, with the solution:
$$\begin{array}{lll} a = (-15/16)j, & b = (-1/16)j, & c = (7/8)j, \\ d = (15/16)j, & e = (-15/16)j, & f = (1/8)j, \\ g = (15/8)j, & h = (-15/16)j, & i = (-15/8)j. \end{array}$$
An equation is: $-15x^2 - y^2 + 14z^2 + 15xy - 15xz + 2yz + 30x - 15y - 15z + 16 \ = \ 0$.

31. Omitted

32. The resulting system of equations is: $\begin{array}{rcl} 2a+b+c+d & = & 0 \\ 5a+2b+c+d & = & 0\ . \\ 13a+3b+2c+d & = & 0 \end{array}$

The general solution is: $a = (1/6)d, b = (-1/2)d, c = (-5/6)d$.
Thus, $x^2 + y^2 - 3x - 5y + 6 \ = \ 0$, is an equation for the circle.

33. The resulting system of equations is: $\begin{aligned} 25a + 4b + 3c + d &= 0 \\ 5a + b + 2c + d &= 0 \\ 4a + 2b + d &= 0 \end{aligned}$.

The general solution is: $a = (7/50)d, b = (-39/50)d, c = (-23/50)d$.
Thus, $7x^2 + 7y^2 - 39x - 23y + 50 \ = \ 0$,

is an equation for the circle.

1.4 Applications

1. (a) $\begin{aligned} x_1 + x_4 &= 1200 \\ x_1 + x_2 &= 1000 \\ x_3 + x_4 &= 600 \\ x_2 + x_4 &= 400 \end{aligned}$

   The solution is $x_1 = 1200 - x_4$, $x_2 = -200 + x_4$, $x_3 = 600 - x_4$.

   (b) $x_1 = 1100$, $x_2 = -100$, $x_3 = 500$.

   (c) $200 \leq x_4 \leq 600$ so $600 \leq x_1 \leq 1000$

2. (a) $\begin{aligned} x_1 &= 1200 \\ x_1 + x_2 &= 1000 \\ x_3 + x_4 &= 1000 \\ x_2 + x_3 &= 800 \end{aligned}$

   The solution is $x_1 = 1200 - x_4$, $x_2 = -200 + x_4$, $x_3 = 1000 - x_4$.

   (b) $x_1 = 1100$, $x_2 = -100$, $x_3 = 900$.

   (c) $200 \leq x_4 \leq 1000$ so $200 \leq x_1 \leq 1000$.

3. $x_2 = 800$, $x_3 = 400$, $x_4 = 200$.

4. $x_2 = 400$, $x_3 = 700$, $x_4 = 300$, $x_5 = 500$, $x_6 = 100$.

5. $4I_1 + 3I_2 = 2$, $3I_2 + 4I_3 = 4$, and $I_1 + I_3 = I_2$. Therefore, $I_1 = 1/20$, $I_2 = 3/5$, and $I_3 = 11/20$.

6. $I_1 + I_2 = 7$, $I_1 + 2I_3 = 3$, and $I_1 + I_3 = I_2$. Therefore, $I_1 = 18/5$, $I_2 = 17/5$, and $I_3 = -1/5$.

7. $5/7, 20/7, 15/7$

8. $7/4, 15/8, -13/8, 1/8, 1/8, 27/8$.

9. (a)
$$\begin{aligned} x_1 - x_4 &= a_1 - a_2 \\ x_1 - x_2 &= -b_1 + b_2 \\ -x_3 + x_4 &= d_1 - d_2 \\ x_2 - x_3 &= -c_1 + c_2 \end{aligned}$$

10. Let $I_1, I_2, \ldots, I_5$ be the currents flowing through $R_1, R_2, \ldots, R_5$, respectively. If $I_5 = 0$ then $I_1 = I_2$, $I_3 = I_4$, $I_1R_1 - I_3R_3 = 0$, and $I_2R_2 - I_4R_4 = 0$. It follows that either all currents are zero or $R_1R_4 = R_2R_3$.

1.5 Matrix Operations

1. (a) $\begin{bmatrix} 2 & 0 \\ 2 & 6 \end{bmatrix}$, (b) $\begin{bmatrix} 0 & 4 \\ 2 & 4 \end{bmatrix}$, (c) $\begin{bmatrix} 0 & -6 \\ 6 & 18 \end{bmatrix}$, (d) $\begin{bmatrix} -6 & 8 \\ 4 & 6 \end{bmatrix}$.

2. (a) $\begin{bmatrix} -2 & 2 \\ 2 & 4 \end{bmatrix}$, (b) $\begin{bmatrix} 6 & 3 \\ 3 & 9 \end{bmatrix}$, (c) $\begin{bmatrix} -2 & 7 \\ 3 & 5 \end{bmatrix}$, (d) $\begin{bmatrix} -2 & 3 \\ 1 & 1 \end{bmatrix}$.

3. $\begin{bmatrix} -2 & -2 \\ 0 & 0 \end{bmatrix}$.

4. $\begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$.

5. $\begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}$.

6. $\begin{bmatrix} 2 & 4 \\ -2 & -6 \end{bmatrix}$.

7. (a) $\begin{bmatrix} 3 \\ -3 \end{bmatrix}$, (b) $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$, (c) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

8. (a) $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, (b) $\begin{bmatrix} -11 \\ 18 \end{bmatrix}$, (c) $\begin{bmatrix} -5 \\ 24 \end{bmatrix}$.

9. (a) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, (b) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, (c) $\begin{bmatrix} 17 \\ 14 \end{bmatrix}$.

10. (a) $\begin{bmatrix} -4 \\ 13 \end{bmatrix}$, (b) $\begin{bmatrix} -15 \\ 0 \end{bmatrix}$, (c) $\begin{bmatrix} 3 \\ -6 \end{bmatrix}$.

11. (a) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$, (b) $\begin{bmatrix} 20 \\ 16 \end{bmatrix}$.

12. (a) $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$, (b) $\begin{bmatrix} -33 \\ -17 \end{bmatrix}$.

13. $a_1 = 11/3$, $a_2 = -(4/3)$.

14. $a_1 = 0$, $a_2 = -2$.

15. $a_1 = -2$, $a_2 = 0$.

16. $a_1 = 4/11$, $a_2 = 14/11$.

17. The equation has no solution.

18. The equation has no solution.

19. $a_1 = 4$, $a_2 = -(3/2)$.

20. $a_1 = 9/11$, $a_2 = -(17/11)$.

21. $\mathbf{w_1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{w_2} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $AB = \begin{bmatrix} 1 & 1 \\ 3 & 8 \end{bmatrix}$, $(AB)\mathbf{r} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

22. $\mathbf{w_1} = \begin{bmatrix} -13 \\ -1 \end{bmatrix}$, $\mathbf{w_2} = \begin{bmatrix} -27 \\ -16 \end{bmatrix}$, $Q = \begin{bmatrix} -3 & 7 \\ 1 & 6 \end{bmatrix}$, $Q\mathbf{s} = \begin{bmatrix} -27 \\ -16 \end{bmatrix}$.

23. $\mathbf{w_1} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $\mathbf{w_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\mathbf{w_3} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $Q = \begin{bmatrix} -1 & 4 \\ 2 & 17 \end{bmatrix}$,

$Q\,\mathbf{r} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

24. $\mathbf{w_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{w_2} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, $\mathbf{w_3} = \begin{bmatrix} -3 \\ 8 \end{bmatrix}$.

$Q = \begin{bmatrix} -3 & -4 \\ 8 & 19 \end{bmatrix}$, $Q\,\mathbf{r} = \begin{bmatrix} -3 \\ 8 \end{bmatrix}$.

25. $\begin{bmatrix} -4 & 6 \\ 2 & 12 \end{bmatrix}$.

26. $\begin{bmatrix} 3 & -1 \\ 15 & 30 \end{bmatrix}$.

27. $\begin{bmatrix} 4 & 12 \\ 4 & 10 \end{bmatrix}$.

28. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

29. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

30. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

31. $AB = \begin{bmatrix} 5 & 16 \\ 5 & 18 \end{bmatrix}$, $BA = \begin{bmatrix} 4 & 11 \\ 6 & 19 \end{bmatrix}$.

32. $\begin{bmatrix} 50 & 11 \\ 16 & 10 \\ 3 & -2 \\ 28 & 4 \end{bmatrix}$.

33. $A\,\mathbf{u} = \begin{bmatrix} 11 \\ 13 \end{bmatrix}$, $\mathbf{v}\,A = [8, 22]$.

34. $\mathbf{uv} = \begin{bmatrix} 2 & 4 \\ 6 & 12 \end{bmatrix}$, $\mathbf{vu} = 14$.

35. $\mathbf{v}\,B\mathbf{u} = 66$.

36. $B\mathbf{u} = \begin{bmatrix} 7 \\ 13 \end{bmatrix}$.

37. $CA = \begin{bmatrix} 5 & 10 \\ 8 & 12 \\ 15 & 20 \\ 8 & 17 \end{bmatrix}$.

38. $CB = \begin{bmatrix} 3 & 8 \\ 4 & 8 \\ 7 & 12 \\ 5 & 14 \end{bmatrix}$.

39. $C(B)\,\mathbf{u}\,) = \begin{bmatrix} 27 \\ 28 \\ 43 \\ 47 \end{bmatrix}$.

40. $(AB)\mathbf{u} = \begin{bmatrix} 53 \\ 59 \end{bmatrix}$, $A(B\mathbf{u}\,) = \begin{bmatrix} 53 \\ 59 \end{bmatrix}$.

41. $(BA)\mathbf{u} = \begin{bmatrix} 37 \\ 63 \end{bmatrix}$, $B(A\mathbf{u}\,) = \begin{bmatrix} 37 \\ 63 \end{bmatrix}$.

42. $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 + 2x_4 \\ -2x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

43. $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 + x_3 \\ 3 - 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

44 $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 + x_3 \\ 1 - 2x_3 \\ x_3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}.$$

45. $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_3 + x_5 \\ -2x_3 - x_5 \\ x_3 \\ -x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

46. $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 + x_3 + 2x_4 + 3x_5 \\ -2x_3 - 3x_4 - 4x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} =$$
$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

47. $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_3 + 2x_4 + 3x_5 \\ -2x_3 - 3x_4 - 4x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

48. $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} x_3 + x_5 + 2x_6 \\ -2x_3 - x_5 - 2x_6 \\ x_3 \\ -x_5 - x_6 \\ x_6 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 2 \\ -2 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

49. $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_2 + 2x_4 \\ x_2 \\ -2x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}.$$

50. $A(B\mathbf{u})$ has 8 multiplications while $(AB)\mathbf{u}$ has 12 multiplications.

51. $C(A(B\,\mathbf{u}))$ has 12 multiplications, $(CA)(B\mathbf{u})$ has 16 multiplications, $[C(AB)](\mathbf{u})$ has 20 multiplications, and $C[(AB)\mathbf{u}]$ has 16 multiplications.

52. (a) $\mathbf{A_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{A_2} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $\mathbf{D_1} = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{D_2} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 3 \end{bmatrix}$,

$\mathbf{D_3} = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{D_4} = \begin{bmatrix} 6 \\ 4 \\ -1 \\ 2 \end{bmatrix}$.

(b) $\mathbf{A_1}$ is in $R^2, \mathbf{D_1}$ is in R^4.

(c) $A\mathbf{B_1} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$, $A\mathbf{B_2} = \begin{bmatrix} 16 \\ 18 \end{bmatrix}$, $AB = \begin{bmatrix} 5 & 16 \\ 5 & 18 \end{bmatrix}$.

53. (a) AB is a 2 x 4 matrix, BA is not defined.

(b) AB is not defined, BA is not defined.

(c) AB is not defined. BA is a 6 x 7 matrix.

(d) AB is a 2 x 2 matrix, BA is a 3 x 3 matrix.

(e) AB is a 3 x 1 matrix, BA is not defined.

(f) $A(BC)$ and $(AB)C$ are 2 x 4 matrices.

(g) AB is a 4 x 4 matrix. BA is a 1 x 1 matrix.

54. $(AB)(CD)$ is a 2 x 2 matrix, $A(B(CD))$ and $((AB)C)D$ are 2 x 2 matrices.

55. $A^2 = AA$ provided A is a square matrix.

56. Since $b \neq 0$ is arbitrary in B, the equation has infinitely many solutions.

57. (a) $P\mathbf{x} = \begin{bmatrix} 135{,}000 \\ 120{,}000 \\ 45{,}000 \end{bmatrix}$ is the state vector after one year and $P^2\mathbf{x} = \begin{bmatrix} 126{,}000 \\ 132{,}000 \\ 42{,}000 \end{bmatrix}$ is the state vectore after two years.

(b) $P^n\mathbf{x}$

58. (a) Setting $AB = BA$ yields the system of equations $3b - 2c = 0$, $2a + 3b - 2d = 0$, and $3a + 3c - 3d = 0$. The solution is $a = -c + d$ and $b = 2c/3$, so $B = \begin{bmatrix} -c+d & 2c/3 \\ c & d \end{bmatrix}$.

(b) $B = \begin{bmatrix} -2 & 2 \\ 3 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ are possible choices for B and C.

59. Let A be an $(m \times n)$ matrix and B be a $(p \times r)$ matrix. Since AB is defined, $n = p$ and AB is an $(m \times r)$ matrix. But AB is a square matrix, so $m = r$. Thus, B is an $(n \times m)$ matrix, so BA is defined and is an $(n \times n)$ matrix.

60. Let $B = [\mathbf{B_1}, \mathbf{B_2}, \ldots, \mathbf{B_s}]$. Then $AB = [A\mathbf{B_1}, A\mathbf{B_2}, \ldots, A\mathbf{B_s}]$.

(a) If $\mathbf{B_j} = \boldsymbol{\theta}$ then the j^{th} column of AB is $A\mathbf{B_j} = \boldsymbol{\theta}$.

(b) If $\mathbf{B_i} = \mathbf{B_j}$ then $A\mathbf{B_i} = A\mathbf{B_j}$.

61. (a) (i) $A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$.

(ii) $A = \begin{bmatrix} 1 & -3 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

(b) (i) $x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$.

(ii) $x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

(c) (i) $x_1 = 2$, $x_2 = 1$, $2\mathbf{A_1} + \mathbf{A_2} = \mathbf{b}$.

(ii) $x_1 = 2$, $x_2 = 1$, $x_3 = 2$, $2\mathbf{A_1} + \mathbf{A_2} + 2\mathbf{A_3} = \mathbf{b}$.

62. $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \left\{ \begin{array}{c} R_2 - R_1 \\ \Longrightarrow \end{array} \right\} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$.

Thus $x_1 = 1$ and $x_2 = 1$.

63. (a) We solve each of the systems

(i) $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$,

(ii)) $A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(i) $\mathbf{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$; (ii) $\mathbf{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

(b) $B = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ and $AB = I = BA$.

64. The i^{th} component of $A\mathbf{x}$ is the $\sum_{j=1}^{n} a_{ij}x_j$. Now the i^{th} components of $x_1\mathbf{A_1}$, $x_2\mathbf{A_2}$, $\ldots$, $x_n\mathbf{A_n}$ are $x_1a_{i1}, x_2a_{i2} \ldots, x_na_{in}$, respectively. Thus the i^{th} component of $x_1\mathbf{A_1} + x_2\mathbf{A_2} + \cdots + x_n\mathbf{A_n}$ is $\sum_{j=1}^{n} a_{ij}x_j$

as required. ■

65. (a) $B = \begin{bmatrix} -1 & 6 \\ 1 & 0 \end{bmatrix}$. (b) No B exists. (c) $B = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$.

66. If $A = (a_{ij})$ and $B = (b_{ij})$ then

$$AB = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ 0 & a_{22}b_{22} & a_{22}b_{23} + a_{23}b_{33} \\ 0 & 0 & a_{33}b_{33} \end{bmatrix}.$$

67. Let $A = (a_{ij})$ and $B = (b_{ij})$ be upper triangular (n x n) matrices. Then the ij^{th} entry of AB equals $\sum_{k=1}^{n} a_{ik}b_{kj}$. Suppose $i > j$. If $k > j$ then $b_{kj} = 0$. If $j \geq k$ then $i > k$ so $a_{ik} = 0$. Thus the ij^{th} component of AB equals zero.

68. $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4 - 3x_2 - 22x_5 \\ x_2 \\ 6 - 9x_5 \\ 5 - x_5 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 6 \\ 5 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -22 \\ 0 \\ -9 \\ -1 \\ 1 \end{bmatrix}.$$

69. $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 + 2x_4 - 3x_5 \\ 4 - 3x_4 - 2x_5 \\ 2 - x_4 - x_5 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 2 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

70. $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 2 - x_2 - 2x_6 \\ x_2 \\ 3 - x_6 \\ 2 - 2x_6 \\ 3 - 5x_6 \\ x_6 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \\ 2 \\ 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -2 \\ 0 \\ -1 \\ -2 \\ -5 \\ 1 \end{bmatrix}.$$

1.6 Algebraic Properties of Matrix Operations

1. $DE = \begin{bmatrix} 8 & 15 \\ 11 & 18 \end{bmatrix}$, $EF = \begin{bmatrix} 9 & 9 \\ 5 & 5 \end{bmatrix}$, $(DE)F = D(EF) =$

$\begin{bmatrix} 23 & 23 \\ 29 & 29 \end{bmatrix}$.

2. $FE = \begin{bmatrix} 5 & 9 \\ 5 & 9 \end{bmatrix}$, $ED = \begin{bmatrix} 12 & 27 \\ 7 & 14 \end{bmatrix}$, $F(ED) = (FE)D =$ $\begin{bmatrix} 19 & 41 \\ 19 & 41 \end{bmatrix}$.

3. $DE = \begin{bmatrix} 8 & 15 \\ 11 & 18 \end{bmatrix}$, $ED = \begin{bmatrix} 12 & 27 \\ 7 & 14 \end{bmatrix}$.

4. $EF = \begin{bmatrix} 9 & 9 \\ 5 & 5 \end{bmatrix}$, $FE = \begin{bmatrix} 5 & 9 \\ 5 & 9 \end{bmatrix}$.

5. $F\mathbf{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $F\mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

6. $3F\mathbf{u} = 3\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $7F\mathbf{v} = 7\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

7. $A^{\mathrm{T}} = \begin{bmatrix} 3 & 4 & 2 \\ 1 & 7 & 6 \end{bmatrix}$.

8. $D^{\mathrm{T}} = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$.

9. $E^{\mathrm{T}}F = \begin{bmatrix} 5 & 5 \\ 9 & 9 \end{bmatrix}$.

10. $A^{\mathrm{T}}C = \begin{bmatrix} 34 & 15 & 28 & 20 \\ 56 & 32 & 37 & 35 \end{bmatrix}$.

11. $(F\,\mathbf{v})^{\mathrm{T}} = \begin{bmatrix} 0 & 0 \end{bmatrix}$.

12. $(EF)\,\mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

13. -6.

14. 0.

15. 36.

16. 0.

17. 2.

18. 18.

19. $\sqrt{2}$.

20. $3\sqrt{10}$.

21. $\sqrt{29}$.

22. $4\sqrt{2}$.

23. 0.

24. 0.

25. $2\sqrt{5}$

26. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and let $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $(A-B)(A+B) = \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}$ and $A^2 - B^2 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$.

27. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and let $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $A^2 = AB$ and $A \neq B$.

28. The argument depends upon the "fact" that if the product of two matrices is $\mathcal{O}$ then one of the factors must be $\mathcal{O}$. This is not true. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $A^2 = \mathcal{O} = AB$ and neither of A or B is $\mathcal{O}$.

29. D and F are symmetric.

30. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and let $B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Then each of A and B are symmetric and $AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is not symmetric.

31. If each of A and B are symmetric, then a necessary and sufficient condition that AB be symmetric is that $AB = BA$.

32. $\mathbf{x}^{\mathrm{T}}\, G\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + (x_1 + x_2)^2$. This term is always greater than zero whenever x_1 and x_2 are not simultaneously zero.

33. $\mathbf{x}^{\mathrm{T}}\, D\mathbf{x} = [x_2, x_2] \begin{bmatrix} 2 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + 3x_2^2 + (x_1 + x_2)^2$. This term is always greater than zero whenever x_1 and x_2 are not simultaneously zero.

34. $\mathbf{x}^{\mathrm{T}}F\mathbf{x} = [x_1, x_2]\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (x_1+x_2)^2$. Then $\mathbf{x}^{\mathrm{T}}F\mathbf{x} = 0$ if and only if $x_1 + x_2 = 0$.

35. $\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}$.

36. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

37. $\begin{bmatrix} -27 & -9 \\ 27 & 9 \end{bmatrix}$.

38. $\begin{bmatrix} 9 & 3 \\ -9 & -3 \end{bmatrix}$.

39. $\begin{bmatrix} -12 & 18 & 24 \\ 18 & -27 & -36 \\ 24 & 36 & -48 \end{bmatrix}$.

40. $\begin{bmatrix} -12 & 18 & 24 \\ 12 & -18 & -24 \\ 24 & -36 & -48 \end{bmatrix}$.

41. (a) $\mathbf{x}^{\mathrm{T}}\mathbf{a} = 6$ means that $x_1 + 2x_2 = 6$ and $\mathbf{x}^{\mathrm{T}}\mathbf{b} = 2$ means that $3x_1 + 4x_2 = 2$. Thus $x_1 = -10$, $x_2 = 8$ and $\mathbf{x} = \begin{bmatrix} -10 \\ 8 \end{bmatrix}$.

(b) $\mathbf{x}^{\mathrm{T}}(\mathbf{a}+\mathbf{b}) = 12$ and $\mathbf{x}^{\mathrm{T}}\mathbf{a} = 2$ yields $4x_1 + 6x_2 = 12$ and $x_1 + 2x_2 = 2$. Thus $x_1 = 6$, $x_2 = -2$ and $\mathbf{x} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$.

42. (a) $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$.

(b) $\begin{bmatrix} 5 & 11 \\ -3 & -7 \end{bmatrix}$.

(c) $B\mathbf{C_1} = \begin{bmatrix} 14 \\ 18 \end{bmatrix}$, $\mathbf{B_1^{\mathrm{T}}}C = \begin{bmatrix} 6 & 8 \end{bmatrix}$, $(B\mathbf{C_1})^{\mathrm{T}}\mathbf{C_2} = 132$, $\| C\mathbf{B_2} \| = 2\sqrt{337}$.

43. (b) $A^5\mathbf{u} = 2^5\mathbf{u} = 32\mathbf{u} = \begin{bmatrix} 32 \\ 96 \\ 64 \end{bmatrix}$

(c) $A^n\mathbf{u} = 2^n\mathbf{u}$. Property 3 is required. For example, $A^2\mathbf{u} = A(A\mathbf{u}) = A(2\mathbf{u}) = 2(A\mathbf{u}) = 2(2\mathbf{u}) = 2^2\mathbf{u}$.

44. (a) By property (3) there exists an $(m \times n)$ matrix $\mathcal{O}$ such that $A + \mathcal{O} = A$.

(b) By property (4) there exists an $(m \times n)$ matrix D such that $C + D = \mathcal{O}$. Thus, $A = A + \mathcal{O} = A + (C + D)$.

(c) Since matrix addition is associative (property 2), $A = A + (C + D) = (A + C) + D$. Now $A + C = B + C$ by assumption so, by substitution, $A = (B + C) + D$.

(d) Since matrix addition is associative, this becomes $A = B + (C + D)$.

(e) By choice of D, $C + D = \mathcal{O}$, so $A = B + \mathcal{O}$.

(f) But $B + \mathcal{O} = B$ so $A = B$.

45. (a) Theorem 9, part(2)

(b) Theorem 8, part(3)

(c) Theorem 9, part(3)

46. Using Theorem 10, it can be seen that $\mathbf{y}^T\mathbf{x} = (\mathbf{x}^T\mathbf{y})^T = 0^T = 0$. Thus $\| \mathbf{x} - \mathbf{y} \| = \sqrt{(\mathbf{x}-\mathbf{y})^T(\mathbf{x}-\mathbf{y})} = \sqrt{(\mathbf{x}^T - \mathbf{y}^T)(\mathbf{x}-\mathbf{y})} = \sqrt{\mathbf{x}^T\mathbf{x} - \mathbf{x}^T\mathbf{y} - \mathbf{y}^T\mathbf{x} + \mathbf{y}^T\mathbf{y}} = \sqrt{\|\mathbf{x}\| + \|\mathbf{y}\|} = \sqrt{2}$.

47. $(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$.

49. (a) Q^{T} is a (n x m) matrix, $Q^{\mathrm{T}}Q$ is a n x n matrix and QQ^{T} is a m x m matrix. Now $(Q^{\mathrm{T}}Q)^{\mathrm{T}} = Q^{\mathrm{T}}(Q^{\mathrm{T}})^{\mathrm{T}} = Q^{\mathrm{T}}Q$ so $Q^{\mathrm{T}}Q$ is symmetric. A similar argument shows that QQ^{T} is symmetric.

(b) $(ABC)^{\mathrm{T}} = ((AB)C)^{\mathrm{T}} = C^{\mathrm{T}}(AB)^{\mathrm{T}} = C^{\mathrm{T}}(B^{\mathrm{T}}A^{\mathrm{T}}) = C^{\mathrm{T}}B^{\mathrm{T}}A^{\mathrm{T}}$.

50. $0 \le \|Q\mathbf{x}\|^{\mathbf{2}} = (Q\mathbf{x})^{\mathrm{T}}(Q\mathbf{x}) = \mathbf{x}^{\mathrm{T}}Q^{\mathrm{T}}Q\mathbf{x}$.

51. Property 2. Let $A = (a_{ij}), B = (b_{ij})$ and $C = (c_{ij})$. The $(ij)^{th}$ com-

ponent of $(A + B) + C$ is $(a_{ij} + b_{ij}) + c_{ij}$ whereas the $(ij)^{th}$ component of $A + (B + C)$ is $a_{ij} + (b_{ij} + c_{ij})$. The two are clearly equal.

Property 3. Let $\mathcal{O}$ denote the (m x n) matrix with all zero entries. Clearly $A + \mathcal{O} = A$ for every (m x n) matrix A.

Property 4. If $A = (a_{ij})$ then set $P = (-a_{ij})$. Clearly $A + P = \mathcal{O}$.

52. Let $A = (a_{ij}), B = (b_{ij}), C = (c_{ij}), AB = (d_{ij})$, and $BC = (e_{ij})$. The $(rs)^{th}$ entry of $(AB)C$ is $\sum_{k=1}^{p} d_{rk}c_{ks}$, where $d_{rk} = \sum_{j=1}^{n} a_{rj}b_{jk}$. Thus the $(rs)^{th}$ entry of $(AB)C$ is $\sum_{k=1}^{p}(\sum_{j=1}^{n} a_{rj}b_{jk})c_{ks} =$

$\sum_{k=1}^{p}\sum_{j=1}^{n} a_{rj}b_{jk}c_{ks} = \sum_{j=1}^{n} a_{rj}(\sum_{k=1}^{p} b_{jk}c_{ks}) = \sum_{j=1}^{n} a_{rj}e_{js}$. The last sum is the $(rs)^{th}$ entry of $A(BC)$ so it follows that $(AB)C = A(BC)$.

53. Property 2: If $A = (a_{ij})$ then the $(ij)^{th}$ entry of $r(sA)$ is $r(sa_{ij})$.

Similarly the $(ij)^{th}$ entry of $(rs)A$ is $(rs)a_{ij}$. The two are clearly equal.

Property 3: Let $A = (a_{ij})$ and $B = (b_{ij})$. The $(ij)^{th}$ entry of $r(AB)$ is $r\sum_{k=1}^{n} a_{ik}b_{kj}$. The $(ij)^{th}$ entry of $(rA)B$ is $\sum_{k=1}^{n}(ra_{ik})b_{kj}$. Finally, the $(ij)^{th}$ entry of $A(rB)$ is $\sum_{k=1}^{n} a_{ik}(rb_{kj})$. The three are equal so $r(AB) = (rA)B = A(rB)$.

54. Property 2: Let $A = (a_{ij}), B = (b_{ij})$ and $C = (c_{ij})$. The $(rs)^{th}$ entry of $A(B + C)$ is $\sum_{k=1}^{n} a_{rk}(b_{ks} + c_{ks}) = \sum_{k=1}^{n} a_{rk}b_{ks} + \sum_{k=1}^{n} a_{rk}c_{ks}$. The last expression in the $(rs)^{th}$ entry of $AB + AC$ so $A(B + C) = AB + AC$.

Property 3: The $(ij)^{th}$ entry of $(r + s)A$ is $(r + s)a_{ij}$. The $(ij)^{th}$ entry of $rA + sA$ is $ra_{ij} + sa_{ij}$. The entries are equal so $(r + s)A = rA + sA$.

Property 4: The $(ij)^{th}$ entry of $r(A + B)$ is $r(a_{ij} + b_{ij})$. The $(ij)^{th}$

entry of $rA + rB$ is $ra_{ij} + rb_{ij}$. Since the entries are equal $r(A + B) = rA + rB$.

55. Property 1: Let $A = (a_{ij}), B = (b_{ij})$, and $A + B = (c_{ij})$, where $c_{ij} = a_{ij} + b_{ij}$. The $(rs)^{th}$ entry of $(A + B)^{\mathrm{T}}$ is $c_{sr} = a_{sr} + b_{sr}$. But a_{sr} is the $(rs)^{th}$ entry of A^{T} and b_{sr} is the $(rs)^{th}$ entry of B^{T}. Thus $a_{sr} + b_{sr}$ is the $(rs)^{th}$ entry of $A^{\mathrm{T}} + B^{\mathrm{T}}$.

Property 3: Let $A = (a_{ij}), A^{\mathrm{T}} = (d_{ij})$. and $(A^{\mathrm{T}})^{\mathrm{T}} = (e_{ij})$. Thus $e_{rs} = d_{sr} = a_{rs}$; that is, $(A^{\mathrm{T}})^{\mathrm{T}} = A$.

56. n = 2, m = 3.

57. n = 5, m = 7.

58. n = m = 4.

59. n = 4, m = 6.

60. n = 4, m = 2.

61. n = m = 5.

1.7 Linear Independence and Nonsingular Matrices

1. $x_1\mathbf{v_1} + x_2\mathbf{v_2} = \theta$ has only the trivial solution so $\{\mathbf{v_1}, \mathbf{v_2}\}$ is linearly independent.

2. Linearly dependent. $\mathbf{v_3} = 2\mathbf{v_1}$.

3. Linearly dependent. $\mathbf{v_5} = 3\mathbf{v_1}$.

4. $x_1\mathbf{v_2} + x_3\mathbf{v_3} = \theta$ has only the trivial solution so $\{\mathbf{v_2}, \mathbf{v_3}\}$ is linearly independent.

5. Linearly dependent. $\mathbf{v_3} = 2\mathbf{v_1}$.

6. Linearly dependent. $\mathbf{v_3} = 2\,\mathbf{v_1} - 2\,\mathbf{v_4}$.

7. Linearly dependent. $\mathbf{u_4} = 4\,\mathbf{u_5}$.

8. $x_1\mathbf{u_3} + x_2\mathbf{u_4} = \theta$ has only the trivial solution. So $\{\mathbf{u_3}\,,\ \mathbf{u_4}\}$ is linearly independent.

9. $x_1\mathbf{u_1} + x_2\mathbf{u_2} + x_3\mathbf{u_5} = \theta$ has only the trivial solution so $\{\mathbf{u_1},\ \mathbf{u_2},\ \mathbf{u_5}\}$ is linearly independent.

10. Linearly dependent. $\mathbf{u_4} = 4\,\mathbf{u_5}$.

11. Linearly dependent. $\mathbf{u_4} = 4\,\mathbf{u_5}$.

12. $x_1\mathbf{u_1} + x_2\mathbf{u_2} + x_3\mathbf{u_4} = \theta$ has only the trivial solution so $\{\mathbf{u_1},\ \mathbf{u_2},\ \mathbf{u_4}\}$ is linearly independent.

13. Linearly dependent. $\mathbf{u_4} = (16/5)\mathbf{u_0} + (12/5)\mathbf{u_1} - (4/5)\mathbf{u_2}$.

14. Linearly dependent. $\mathbf{u_4} = (16/5)\mathbf{u_0} + (4/5)\mathbf{u_2} + (4/5)\mathbf{u_3}$.

15. Sets 5, 6, 13, and 14 are linearly dependent by inspection.

16. A is nonsingular.

17. B is singular, $x_1 = -2x_2$.

18. C is nonsingular.

19. AB is singular, $x_1 = -2x_2$.

20. BA is singular, $7x_1 = -10x_2$.

21. D is singular, $x_1 = x_2 = 0$, x_3 arbitrary.

22. F is nonsingular.

23. $D + F$ is nonsingular.

24. E is singular, x_1 arbitrary, $x_2 = 0 = x_3$.

25. EF is singular, x_1 arbitrary, $x_2 = 0 = x_3$.

26. DE is singular, x_1 arbitrary, $x_2 = 0 = x_3$.

27. F^{T} is nonsingular.

28. $\{\mathbf{v_1}\,,\ \mathbf{v_2}\}$ is linearly dependent if $a = 3/2$.

29. $a = 6$.

30. $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is linearly dependent if $a = 1$.

31. $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is linearly dependent if $b(a-2) = 4$.

32. $\{\mathbf{v_1}, \mathbf{v_2}\}$ is linearly dependent if $3a = b$.

33. $\{\mathbf{v_1}, \mathbf{v_2}\}$ is linearly dependent if $c = ab$.

34. $\mathbf{x} = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$, $\mathbf{v_1} = (1/2)\mathbf{A_2}$.

35. $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{v_3} = \mathbf{A_2}$.

36. $\mathbf{x} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$, $\mathbf{v_4} = (-1/2)\mathbf{C_1} + (1/2)\mathbf{C_2}$.

37. $\mathbf{x} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$, $\mathbf{v_2} = (1/2)(\mathbf{C_1} + \mathbf{C_2})$.

38. $\mathbf{x} = \begin{bmatrix} -2/3 \\ 4/3 \\ -1 \end{bmatrix}$, $\mathbf{u_1} = (-2/3)\mathbf{F_1} + (4/3)\mathbf{F_2} - \mathbf{F_3}$.

39. $\mathbf{x} = \begin{bmatrix} -8/3 \\ -2/3 \\ 3 \end{bmatrix}$ $\mathbf{u_3} = (-8\mathbf{F_1} - 2\mathbf{F_2} + 9\mathbf{F_3})/3$.

40. $8\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 3\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$.

41. $-11\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 7\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

42. $8\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 4\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$.

43. $0\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

44. $1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

45. $-3\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

46. (a) Since $\mathbf{v_2} = -2\mathbf{v_1}$ the set S is linearly dependent for any value of a.

    (b) If $a = -3$ then $\mathbf{v_3} = \mathbf{v_1} - \mathbf{v_2}$.

47. (a) The set S is linearly dependent for any value of a.

    (b) The vector $\mathbf{v_3}$ can be written as a linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$ for any value of a.

48. A nontrivial solution is: $1\mathbf{v_1} + 0\mathbf{v_2} + 0\mathbf{v_3} = \theta$.

49. $0 = \boldsymbol{\theta}^T\boldsymbol{\theta} = (a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3)^T(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3) = a_1^2 \parallel \mathbf{v}_1 \parallel^2 + a_2^2 \parallel \mathbf{v}_2 \parallel^2 + a_3^2 \parallel \mathbf{v}_3 \parallel^2$, so $a_i = 0, i = 1, 2, 3$

50. If $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \boldsymbol{\theta}$, where some $a_i \neq 0$, then $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + 0\mathbf{v}_4 = \boldsymbol{\theta}$.

51. If $\boldsymbol{\theta} = a_1\mathbf{v}_1 + a_2(\mathbf{v}_1 + \mathbf{v}_2) + a_3(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3)$ then $\boldsymbol{\theta} = (a_1 + a_2 + a_3)\mathbf{v}_1 + (a_2 + a_3)\mathbf{v}_2 + a_3\mathbf{v}_3$. Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly iindependent, $a_1 + a_2 + a_3 = 0$, $a_2 + a_3 = 0$, and $a_3 = 0$. It follows that $a_1 = a_2 = a_3 = 0$.

52. $AB = [A\mathbf{B}_1, \ldots, A\mathbf{B}_n] = \mathcal{O}$, so $A\mathbf{B}_i = \boldsymbol{\theta}$ for $1 \leq i \leq n$. Since A is nonsingular, $\mathbf{B}_i = \boldsymbol{\theta}$ for $1 \leq i \leq n$, so $B = \mathcal{O}$.

53. If $AB = AC$ then $A(B - C) = \mathcal{O}$. By Exercise 50, $B - C = \mathcal{O}$. Therefore, $B = C$.

54. Suppose, $\mathbf{b} = [b_1, \ldots, b_{n-1}]^T$. If $\mathbf{c} = [b_1, \ldots, b_{n-1}, -1]^T$ then $B\mathbf{c} = b_1\mathbf{A}_1 + \cdots + b_{n-1}\mathbf{A}_{n-1} - A\mathbf{b} = A\mathbf{b} - A\mathbf{b} = \boldsymbol{\theta}$.

55. If $\mathbf{x_1}$ is a nontrivial vector such that $B\mathbf{x_1} = \theta$, then $AB\mathbf{x_1} = A\theta = \theta$.

56. By Theorem 12, $A = [\mathbf{w_1}, \mathbf{w_2}]$ is a nonsingular matrix. By Theorem 13, $A\mathbf{x} = \mathbf{b}$ has a (unique) solution.

57. Let $\mathbf{v}$ be any vector such that $A^{\mathrm{T}}\mathbf{v} = \theta$. By Theorem 13, there exists a vector $\mathbf{w}$, such that $A\mathbf{w} = \mathbf{v}$. Then $A^{\mathrm{T}}(A\mathbf{w}) = A^{\mathrm{T}}\mathbf{v} = \theta$, and so $\mathbf{w}^{\mathrm{T}}(A^{\mathrm{T}}A\mathbf{w}) = \mathbf{w}^{\mathrm{T}}\,\theta = 0$. Then $(A\mathbf{w})^{\mathrm{T}}(A\mathbf{w}) = \mathbf{w}^{\mathrm{T}}A^{\mathrm{T}}A\mathbf{w} = 0$ and $\|A\mathbf{w}\| = 0$. Thus $A\mathbf{w} = \theta$, and since A is nonsingular, $\mathbf{w} = \theta$. Thus $A\mathbf{w} = A\,\theta = \theta$. But then $\mathbf{v}$ had to be θ and A^{T} is nonsingular.

1.8 Data Fitting, Numerical Integration & Differentiation

1. $p(t) = (-1/2)t^2 + (9/2)t - 1$.

2. $p(t) = t^2 - 4t + 1$.

3. $p(t) = 2t + 3$.

4. $p(t) = 3t + 2.$
5. $p(t) = 2t^3 - 2t^2 + 3t + 1.$
6. $p(t) = t^3 + t^2 + 1.$
7. $y = 2e^{2x} + e^{3x}.$
8. $y = -e^{x-1} + 2e^{3(x-1)}.$
9. $y = 3e^{-x} + 4e^{x} + e^{2x}.$
10. $y = 2e^{x} - 4e^{2x} + e^{3x}.$
11. $\int_0^{3h} f(t)dt \approx \frac{3h}{2}[f(h) + f(2h)].$
12. $\int_0^{h} f(t)dt \approx \frac{h}{2}[f(0) + f(h)].$
13. $\int_0^{3h} f(t)dt \approx \frac{3h}{8}[f(0) + 3f(h) + 3f(2h) + f(3h)].$
14. $\int_0^{4h} \approx \frac{8h}{3}f(h) - \frac{4h}{3}f(2h) + \frac{8h}{3}f(3h).$
15. $\int_0^{h} f(t)dt \approx \frac{h}{2}[-f(-h) + 3f(0)].$
16. $\int_0^{h} f(t)dt \approx -\frac{h}{12}f(-h) + \frac{2h}{3}f(0) + \frac{5h}{12}f(h).$
17. $f'(0) \approx \frac{-1}{h}f(0) + \frac{1}{h}f(h).$
18. $f'(0) \approx \frac{-1}{h}f(-h) + \frac{1}{h}f(0).$
19. $f'(0) \approx -\frac{3}{2h}f(0) + \frac{2}{h}f(h) + \frac{-1}{2h}f(2h).$
20. $f'(0) \approx \frac{-11}{6h}f(0) + \frac{3}{h}f(h) - \frac{3}{2h}f(2h) + \frac{1}{3h}f(3h).$
21. $f''(0) \approx \frac{1}{h^2}[f(-h) - 2f(0) + f(h)].$
22. $f''(0) \approx \frac{1}{h^2}[f(0) - 2f(h) + f(2h)].$
23. $\begin{bmatrix} 1 & 1 & 1 & b-a \\ a & t & b & (b^2-a^2)/2 \\ a^2 & t^2 & b^2 & (b^3-a^3)/3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 1 & b-a \\ 0 & t-a & b-a & (b^2+a^2-2ab)/2 \\ 0 & t^2-a^2 & b^2-a^2 & (b^3-a^3-3(b-a)a^2)/3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & b-a \\ 0 & 1 & 2 & b-a \\ 0 & t^2-a^2 & b^2-a^2 & (b^3-a^3-3(b-a)a^2)/3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & b-a \\ 0 & 1 & 2 & b-a \\ 0 & 0 & (b-a)^2/2 & (b-a)^3/12 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & b-a \\ 0 & 1 & 2 & b-a \\ 0 & 0 & 6 & b-a \end{bmatrix}$$

24. $$\begin{bmatrix} 1 & 1 & 1 & 0 \\ a-h & a & a+h & 1 \\ (a-h)^2 & a^2 & (a+h)^2 & 2a \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & h & 2h & 1 \\ 0 & h(2a-h) & 4ah & 2a \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & h & 2h & 1 \\ 0 & 0 & 2h^2 & h \end{bmatrix}.$$

25. By Rolle's Theorem there exist u_1 and u_2 such that $t_0 < u_1 < t_1 < u_2 < t_2$ and $p\prime(u_1) = p\prime(u_2) = 0$. Since $p\prime(u_1) = p\prime(u_2) = 0$, $u_1 < u_2$, and $p\prime(t) = 2at + b$, it follows that $b = 0 = a$. Finally, $p(t_0) = 0$ means $c = 0$.

26. Suppose we have seen that a nonzero polynomial of degree $n-1$ can have at most $n-1$ distinct real zeros. Now assume that $p(t)$ has $n+1$ zeros; that is there exist real numbers $t_0, t_1, \ldots, t_n$ such that $t_0 < t_1 < \cdots < t_n$ and $p(t_i) = 0$ for $0 \le i \le n$. By Rolle's Theorem there are real numbers $u_1, \ldots, u_n$ such that $t_{i-1} < u_i < t_i$, for $1 \le i \le n$, and such that $p\prime(u_i) = 0$ for each i. Now $p\prime(t) = na_nt^{n-1} + \cdots + a_1$ and $p'(t)$ has n zeros. By assumption $p'(t)$ is the zero polynomial. Thus $0 = a_1 = \cdots = a_n$. This leaves $p(t) = a_0$; but $p(t_0) = 0$ so $a_0 = 0$. Therefore $p(t)$ is the zero polynomial.

27. We must solve the system $L\mathbf{x} = \mathbf{b}$ where $L = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}$

$\mathbf{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ 8 \\ 10 \end{bmatrix}$. $p(t) = t^3 + 2t^2 + 3t + 2$.

28. $p(t) = t^2 + 2t + 1$.

29. $p(t) = t^3 + t^2 + 4t + 3$.

30. $p(t) = 2t^3 - 2t + 3$.

31. By Rolle's Theorem there exists a number s such that $t_0 < s < t_1$ and $p'(s) = 0$. Thus $p'(t)$ has three zeros. By Exercise 25, $p'(s)$ is the zero polynomial. It follows that $p(t)$ is the zero polynomial.

32. The coefficient matrix of $p(t) = at^3 + bt^2 + ct + d$ must satisfy
$L\mathbf{x} = \mathbf{b}$

where $L = \begin{bmatrix} t_0^3 & t_0^2 & t_0 & 1 \\ 3t_0^2 & 2t_0 & 1 & 0 \\ t_1^3 & t_1^2 & t_1 & 1 \\ 3t_1^2 & 2t_1 & 1 & 0 \end{bmatrix}$ $\mathbf{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ $\mathbf{b} = \begin{bmatrix} y_0 \\ s_0 \\ y_1 \\ s_1 \end{bmatrix}$.

Suppose $L\mathbf{x_0} = \theta$, where $\mathbf{x_0} = [a, b, c, d]^{\mathrm{T}}$. If $p(t) = at^3 + bt^2 + ct + d$ then $p(t_0) = p(t_1) = 0$ and $p'(t_0) = p'(t_1) = 0$. By Exercise 31 $a = b = c = d = 0$; that is $\mathbf{x_0} = \theta$. This proves that L is nonsingular so by Theorem 13, $L\mathbf{x} = \mathbf{b}$ has a unique solution.

33. First supppose that $p(t_i) = p'(t_i) = 0$ for $0 \le i \le n$. Since $p(t_{i-1}) = 0 = p(t_i)$, it follows from Rolle's Theorem that there is a real number u_i such that $t_{i-1} < u_i < t_i$ and $p'(u_i) = 0$. Therefore $p'(t)$ has $2n+1$ zeros, $t_0, t_1, \ldots, t_n, u_1, \ldots, u_n$. By Exercise 26, $p'(t)$ is the zero polynomial and it follows that $p(t)$ is the zero polynomial.

Now set $p(t) = \sum_{k=0}^{2n+1} a_k t^k$ and assume that $p(t_i) = y_i$ and $p'(t_i) = s_i$ for $0 \le i \le n$. These constraints yield a system of equations $L\mathbf{x}=\mathbf{b}$, where

$$L = \begin{bmatrix} t_0^{2n+1} & t_0^{2n} & \ldots & t_0 & 1 \\ (2n+1)t_0^{2n} & 2nt_0^{2n-1} & \ldots & 1 & 0 \\ \vdots & & & & \\ t_n^{2n+1} & t_n^{2n} & \ldots & t_n & 1 \\ (2n+1)t_n^{2n} & 2nt_n^{2n-1} & \ldots & 1 & 0 \end{bmatrix},$$

$\mathbf{x} = [a_{2n+1}, a_{2n}, \ldots, a_1, a_0]^{\mathrm{T}}$ and $\mathbf{b} = [y_0, s_0, \ldots, y_n, s_n]^{\mathrm{T}}$. Suppose $L\mathbf{x_0} = \theta$, where $\mathbf{x_0} = [b_{2n+1}, b_{2n}, \ldots, b_1, b_0]^{\mathrm{T}}$. If we set $q(t) = \sum_{k=0}^{2n+1} b_k t^k$ then it follows that $q(t_i) = q'(t_i) = 0$ for $0 \le i \le n$. As we have shown above, this implies that $b_{2n+1} = b_{2n} = \cdots = b_1 = b_0 = 0$. In particular $\mathbf{x_0} = \theta$ and it follows that L is nonsingular. By Theorem 13 the system $L\mathbf{x}=\mathbf{b}$ has a unique solution.

34. $\int_0^{5h} f(x)dx \approx \frac{5h}{24}[\frac{19}{12}f(0) + \frac{25}{4}f(h) + \frac{25}{6}f(2h) + \frac{25}{6}f(3h) + \frac{25}{4}f(4h) + \frac{19}{12}f(h)]$.

35. $f'(a) \approx \frac{1}{12h}[f(a-2h) - 8f(a-h) + 8f(a+h) - f(a+2h)]$

1.9 Matrix Inverses and their Properties

5. (cf. Ex. 2) $\mathbf{x} = A^{-1}\mathbf{b} = B\mathbf{b} = \begin{bmatrix} 1 & -1 \\ -.2 & .3 \end{bmatrix} \begin{bmatrix} 6 \\ 9 \end{bmatrix} = \begin{bmatrix} -3 \\ 1.5 \end{bmatrix}$.

6. (cf. Ex. 1) $\mathbf{x} = A^{-1}\mathbf{b} = B\mathbf{b} = \begin{bmatrix} 3 & -4 \\ -5 & 7 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ -11 \end{bmatrix}$.

7. (cf. Ex. 3) $\mathbf{x} = B^{-1}\mathbf{b} = A\mathbf{b} = \begin{bmatrix} -1 & -2 & 11 \\ 1 & 3 & -15 \\ 0 & -1 & 5 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 14 \\ -20 \\ 8 \end{bmatrix}$.

8. (cf. Ex. 4) $\mathbf{x} = B^{-1}\mathbf{b} = A\mathbf{b} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 20 \end{bmatrix}$.

9. If B is any 3 x 3 matrix, then the $(1,1)^{th}$ entry of AB is zero and so $AB \neq I$.

10. The $(1,1)^{th}$ entry of BA is zero.

11. Let $B = (x_{ij})$ be a (3 x 3) matrix and suppose that $BA = I$. Then the $(1,1)^{th}$ entry of BA must be one and the $(1,2)^{th}$ entry of BA be must be zero. But each of these entries equals $2x_{11} + x_{12} + 3x_{13}$ and cannot simultaneously be one and zero.

12. The $(1,1)^{th}$ and the $(2,1)^{th}$ entry of AB cannot be simultaneously be zero and one.

13. $\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$.

14. $\begin{bmatrix} -7/4 & 3/4 \\ 3/2 & -1/2 \end{bmatrix}$.

15. $\begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix}$.

16. $\begin{bmatrix} 0 & 1 & 3 \\ 5 & 5 & 4 \\ 1 & 1 & 1 \end{bmatrix}$.

17. $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -4 & 1 \end{bmatrix}$.

18. $\begin{bmatrix} 1 & 11 & -7 \\ 0 & -7 & 4 \\ 0 & 2 & -1 \end{bmatrix}$.

19. $\begin{bmatrix} 1 & -2 & 0 \\ 3 & -3 & -1 \\ -6 & 7 & 2 \end{bmatrix}$.

20. $\begin{bmatrix} -35 & -16 & 26 & -17 \\ -15 & -7 & 11 & -7 \\ 4 & 2 & -3 & 2 \\ -2 & -2 & 2 & -1 \end{bmatrix}$.

21. $\begin{bmatrix} -1/2 & -2/3 & -1/6 & 7/6 \\ 1 & 1/3 & 1/3 & -4/3 \\ 0 & -1/3 & -1/3 & 1/3 \\ -1/2 & 1 & 1/2 & 1/2 \end{bmatrix}$.

22. $\frac{-1}{5}\begin{bmatrix} 1 & -2 \\ -1 & -3 \end{bmatrix}$.

23. $\Delta = 10$ so $A^{-1} = \frac{1}{10}\begin{bmatrix} 3 & 2 \\ -2 & 2 \end{bmatrix}$.

24. $\frac{-1}{7}\begin{bmatrix} 1 & -3 \\ -2 & -1 \end{bmatrix}$.

25. $\Delta = 0$ so A^{-1} does not exist.

26. A^{-1} does not exist.

27. $\lambda \neq 2,\ -2$

28. $\lambda \neq 2$

29. $\mathbf{x} = A^{-1}\,\mathbf{b} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \end{bmatrix}$.

30. $\begin{bmatrix} -4 \\ 4 \end{bmatrix}$.

31. $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 4 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 18 \\ 13 \end{bmatrix}$.

32. $\begin{bmatrix} 17 \\ -11 \end{bmatrix}$.

33. $\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{10}\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 5/2 \end{bmatrix}$.

34. $\mathbf{x} = \begin{bmatrix} 6.8 \\ -3.2 \end{bmatrix}$.

35. $Q^{-1} = C^{-1}A^{-1} = \begin{bmatrix} -3 & 1 \\ 3 & 5 \end{bmatrix}$.

36. $Q^{-1} = A^{-1}C^{-1} = \begin{bmatrix} -2 & 5 \\ 2 & 4 \end{bmatrix}$.

37. $Q^{-1} = (A^{-1})^{\mathrm{T}} = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$.

38. $Q^{-1} = C^{-1}(A^{-1})^{\mathrm{T}} = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 5 & 4 \end{bmatrix}$.

39. $Q^{-1} = (A^{-1})^{\mathrm{T}}(C^{-1})^{\mathrm{T}} = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 1 & 5 \end{bmatrix}$.

40. $Q^{-1} = A^{-1}B = \begin{bmatrix} 5 & 7 \\ 4 & 2 \end{bmatrix}$.

41. $Q^{-1} = BC^{-1} = \begin{bmatrix} 1 & 5 \\ -1 & 4 \end{bmatrix}$.

42. $Q^{-1} = B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

43. $Q^{-1} = \frac{1}{2}A^{-1} = \begin{bmatrix} 3/2 & 1/2 \\ 0 & 1 \end{bmatrix}$.

44. $Q^{-1} = \frac{1}{10}C^{-1} = \begin{bmatrix} -1/10 & 1/10 \\ 1/10 & 1/5 \end{bmatrix}$.

45. $Q^{-1} = B(C^{-1}A^{-1}) = \begin{bmatrix} 3 & 11 \\ -3 & 7 \end{bmatrix}$.

46. $B = A^{-1}D = \begin{bmatrix} -4 & 6 & 7 \\ 3 & -4 & -4 \end{bmatrix}$, $C = EA^{-1} = \begin{bmatrix} 8 & -3 \\ 1 & 0 \\ -6 & 3 \end{bmatrix}$.

47. $B = A^{-1}D = \begin{bmatrix} 1 & 10 \\ 15 & 12 \\ 3 & 3 \end{bmatrix}$, $C = EA^{-1}, = \begin{bmatrix} 13 & 12 & 8 \\ 2 & 3 & 5 \end{bmatrix}$.

48. $a \neq -1$.

49. $(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} 2 & 35 & 1 \\ 14 & 35 & 34 \\ 23 & 12 & 70 \end{bmatrix}$

$(3A)^{-1} = (1/3)A^{-1} = \begin{bmatrix} 1/3 & 2/3 & 5/3 \\ 1 & 1/3 & 2 \\ 2/3 & 8/3 & 1/3 \end{bmatrix}$

$$(A^T)^{-1} = (A^{-1})^T = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 8 \\ 5 & 6 & 1 \end{bmatrix}$$

50. $A^2 = AB + 2A = A(B + 2I)$ so $A = B + 2I = \begin{bmatrix} 4 & 1 & -1 \\ 0 & 5 & 2 \\ -1 & 4 & 3 \end{bmatrix}$

51. I

52. (a) $X = I$ and $X = -I$

    (d) The equation $(X-I)(X+I) = \mathcal{O}$ does not require that either $X-I = \mathcal{O}$ or $X+I = \mathcal{O}$.

53. $A^{\mathrm{T}}A = \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} = \begin{bmatrix} \mathbf{u}^{\mathrm{T}}\mathbf{u} & \mathbf{u}^{\mathrm{T}}\mathbf{v} \\ \mathbf{u}^{\mathrm{T}}\mathbf{v} & \mathbf{v}^{\mathrm{T}}\mathbf{v} \end{bmatrix}$.

54. $AA = (I - \mathbf{u}\mathbf{u}^{\mathrm{T}})(I - \mathbf{u}\mathbf{u}^{\mathrm{T}}) = I^2 - 2\mathbf{u}\mathbf{u}^{\mathrm{T}} + (\mathbf{u}\mathbf{u}^{\mathrm{T}})(\mathbf{u}\mathbf{u}^{\mathrm{T}}) = (I - 2\mathbf{u}\mathbf{u}^{\mathrm{T}} + \mathbf{u}(\mathbf{u}^{\mathrm{T}}\mathbf{u})\mathbf{u}^{\mathrm{T}} = I - 2\mathbf{u}\mathbf{u}^{\mathrm{T}}$ $+\mathbf{u}\mathbf{u}^{\mathrm{T}} =$

    $I - \mathbf{u}\mathbf{u}^{\mathrm{T}} = A$.

55. $I = AA^{-1} = (AA)A^{-1} = A(AA^{-1}) = AI = A$.

56. Symmetry: $A^{\mathrm{T}} = (I - a\mathbf{v}\mathbf{v}^{\mathrm{T}})^{\mathrm{T}} = I^{\mathrm{T}} - a(\mathbf{v}\mathbf{v}^{\mathrm{T}})^{\mathrm{T}} =$

    $I - a(\mathbf{v}^{\mathrm{T}\,\mathrm{T}}\mathbf{v}^{\mathrm{T}}) = I - a\mathbf{v}\mathbf{v}^{\mathrm{T}} = A$.

    $AA = (I - a\mathbf{v}\mathbf{v}^{\mathrm{T}})(I - a\mathbf{v}\mathbf{v}^{\mathrm{T}}) = I - 2\mathrm{a}\mathbf{v}\mathbf{v}^{\mathrm{T}} + a^2(\mathbf{v}\mathbf{v}^{\mathrm{T}})(\mathbf{v}\mathbf{v}^{\mathrm{T}}) =$

    $I - 2a\mathbf{v}\mathbf{v}^{\mathrm{T}} + a[2/\mathbf{v}\mathbf{v}^{\mathrm{T}}](\mathbf{v}\mathbf{v}^{\mathrm{T}}\,\mathbf{v}\mathbf{v}^{\mathrm{T}}) =$

    $I - 2a\mathbf{v}\mathbf{v}^{\mathrm{T}} + (2a\mathbf{v}\mathbf{v}^{\mathrm{T}}) = I$.

57. $A\mathbf{x} = I\mathbf{x} - a(\mathbf{v}\mathbf{v}^{\mathrm{T}})\mathbf{x} = \mathbf{x} - a\mathbf{v}\,(\mathbf{v}^{\mathrm{T}}\mathbf{x}) = \mathbf{x} - a(\mathbf{v}^{\mathrm{T}}\mathbf{x})\mathbf{v} = \mathbf{x} - \lambda\mathbf{v}$ where $\lambda = a(\mathbf{v}^{\mathrm{T}}\mathbf{x})$.

58. $\| A\mathbf{x} \| = \sqrt{(A\mathbf{x})^{\mathrm{T}}(A\mathbf{x})} = \sqrt{(\mathbf{x}^{\mathrm{T}}A^{\mathrm{T}})(A\mathbf{x})} = \sqrt{\mathbf{x}^{\mathrm{T}}(A^{\mathrm{T}}A)\mathbf{x}} =$
    $\sqrt{\mathbf{x}^{\mathrm{T}}I\mathbf{x}} = \sqrt{\mathbf{x}^{\mathrm{T}}\mathbf{x}} = \| \mathbf{x} \|$.

59. $A(I - a\mathbf{u}\mathbf{v}^{\mathrm{T}}) = (I + \mathbf{u}\mathbf{v}^{\mathrm{T}})(I - a\mathbf{u}\mathbf{v}^{\mathrm{T}}) = I^2 + \mathbf{u}\mathbf{v}^{\mathrm{T}} - a\mathbf{u}\mathbf{v}^{\mathrm{T}} -$

    $a(\mathbf{u}\mathbf{v}^{\mathrm{T}})(\mathbf{u}\mathbf{v}^{\mathrm{T}}) = I + \mathbf{u}\mathbf{v}^{\mathrm{T}} - a\mathbf{u}\mathbf{v}^{\mathrm{T}} - a\mathbf{u}(\mathbf{v}^{\mathrm{T}}\mathbf{u})\mathbf{v}^{\mathrm{T}} =$

    $I + \mathbf{u}\mathbf{v}^{\mathrm{T}} - a\mathbf{u}\mathbf{v}^{\mathrm{T}}(1 + \mathbf{v}^{\mathrm{T}}\mathbf{u}) = I + \mathbf{u}\mathbf{v}^{\mathrm{T}} - \mathbf{u}\mathbf{v}^{\mathrm{T}} = I$.

60. $AB = A(A^2 - 2A + 3I) = A^3 - 2A^2 + 3A - I + I = \theta + I = I$.

61. $AB = A(-1/b_0)[A + b_1 I] = (-1/b_0)(A^2 + b_1 A) + (-I + I) =$
    $(-1/b_0)(A^2 + b_1 A + b_0 I) + I = \theta + I = I$.

62. $\theta = A^2 + b_1A + b_0I =$
$\begin{bmatrix} 9 & 5 \\ 25 & 14 \end{bmatrix} + \begin{bmatrix} 2b_1 & b_1 \\ 5b_1 & 3b_1 \end{bmatrix} + \begin{bmatrix} b_0 & 0 \\ 0 & b_0 \end{bmatrix} = \begin{bmatrix} 9+2b_1+b_0 & 5+b_1 \\ 25+5b_1 & 14+3b_1+b_0 \end{bmatrix}$.
$\mathrm{b}_0 = 1, \mathrm{b}_1 = -5$.
$(-1/\mathrm{b}_0)[A + b_1 I] = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = A^{-1}$.

63. $\theta = A^2 + b_1A + b_0I = \begin{bmatrix} 11 & -4 \\ -2 & 3 \end{bmatrix} + \begin{bmatrix} -3b_1 & 2b_1 \\ b_1 & b_1 \end{bmatrix} + \begin{bmatrix} b_0 & 0 \\ 0 & b_0 \end{bmatrix}$
$= \begin{bmatrix} 11-3b_1+b_0 & -4+2b_1 \\ -2+b_1 & 3+b_1+b_0 \end{bmatrix}.\mathrm{b}_0 = -5, \mathrm{b}_1 = 2$.
$(-1/\mathrm{b}_0)[A + b_1 I] = -(1/5)\begin{bmatrix} 1 & -2 \\ -1 & -3 \end{bmatrix} = A^{-1}$.

64. $\theta = A^2 + b_1A + b_0I = \begin{bmatrix} 0 & -10 \\ 10 & 5 \end{bmatrix} + \begin{bmatrix} 2b_1 & -2b_1 \\ 2b_1 & 3b_1 \end{bmatrix} +$
$\begin{bmatrix} b_0 & 0 \\ 0 & b_0 \end{bmatrix} = \begin{bmatrix} 2b_1+b_0 & -10-2b_1 \\ 10+2b_1 & 5+3b_1+b_0 \end{bmatrix}.\mathrm{b}_0 = 10, \mathrm{b}_1 = -5$.
$(-1/\mathrm{b}_0)[A + b_1 I] = \frac{1}{10}\begin{bmatrix} 3 & 2 \\ -2 & 2 \end{bmatrix} = A^{-1}$.

65. $\theta = A^2 + b_1A + b_0I = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} + \begin{bmatrix} -b_1 & +3b_1 \\ 2b_1 & b_1 \end{bmatrix} + \begin{bmatrix} b_0 & 0 \\ 0 & b_0 \end{bmatrix} = \begin{bmatrix} 7-b_1+b_0 & 3b_1 \\ 2b_1 & 7+b_1+b_0 \end{bmatrix}$.
$\mathrm{b}_0 = -7, \mathrm{b}_1 = 0$
$(-1/b_0)[A + b_1 I] = \frac{1}{7}\begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix} = A^{-1}$.

66. (a) $A\mathbf{x} = \mathbf{b}_1$, has solution $\mathbf{x} \approx \begin{bmatrix} 106.395 \\ -4909.194 \\ 4979.886 \end{bmatrix}$. $A\mathbf{x} = \mathbf{b}_2$ has solution $\mathbf{x} \approx \begin{bmatrix} 107.459 \\ -4958.286 \\ 5029.685 \end{bmatrix}$

(b) $A^{-1} = \begin{bmatrix} 25.315 & 21.316 & -59.764 \\ -1100.002 & -1028.428 & 2780.764 \\ 1114.405 & 1044.910 & -2820.572 \end{bmatrix}$

67. (a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

68. $(A^{-1})^T = (A^T)^{-1} = A^{-1}$.

69. (a) $(AB = \mathcal{O}) \implies B = A^{-1}AB = A^{-1}\mathcal{O} = \mathcal{O}$.

(b) $B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$.

70. $(AB = AC) \Longrightarrow B = IB = (A^{-1}A)B = A^{-1}(AB) =$
$A^{-1}(AC) = C$.

71. If either $d \neq 0$ or $c \neq 0$ then
$A \begin{bmatrix} d \\ -c \end{bmatrix} = \theta$ and so A is singular.
If $c = 0 = d$, then $A = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$, and $A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ does not have a solution so A is singular.

72. $(AB)^{-1} = B^{-1}A^{-1}$.

73. The hypothesis of Theorem 17 is not satisfied; that is, Theorem 17 assumes that A and B have inverses.

74. (a) $(B\mathbf{v} = \theta) \Longrightarrow (A(B\mathbf{v}) = \theta \Longrightarrow (AB)\mathbf{v} = \theta)$
$(\Longrightarrow \mathbf{v} = \theta)$ so B is singular.
(b) AB and B^{-1} are nonsingular so $A = (AB)B^{-1}$ is nonsingular.

75. Suppose each of the systems $A\mathbf{x} = \mathbf{e_k}$ is consistent and let $\mathbf{b_k}$ be a solution. If $B = [\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_n}]$ then $AB = [A\mathbf{b_1}, A\mathbf{b_2}, \ldots, A\mathbf{b_n}] = [\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}] = I$. Thus $B = A^{-1}$ and A is nonsingular.

76. Clearly $I^{-1} = I$, so by Theorem 1.5, I is non-singular.

77. Since AB is defined, $q = r$ and AB is a $(p \times s)$ matrix. Since BA is defined, $s = p$ and BA is a $(r \times q)$ matrix. But $AB = BA$, so $p = r$ and $q = s$.

79. Suppose B and C are inverses for A.
Then $B = BI = B(AC) = (BA)C = IC = C$.

1.10 Supplementary Exercises

1. The augmented matrix for the system reduces to

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & (a-1)(a+2) & (a-3)(a+2) \end{bmatrix}$$

There are infinitely many solutions if $a = -2$, no solution if $a = 1$, and a unique solution in which $x_2 = 0$ if $a = 3$.

2. (a) The system is consistent if and only if $-b_1 + 2b_2 + b_3 = 0$ and in this case the solution is $x_1 = b_2 - b_1 - 2x_3$, $x_2 = b_2 - 2b_1 - 3x_3$, x_3 arbitrary.
   (b) (i) inconsistent
   (ii) $x_1 = -3 - 2x_3$, $x_2 = -8 - 3x_2$, x_3 arbitrary
   (iii) $x_1 = -4 - 2x_3$, $x_2 = -11 - 3x_2$, x_3 arbitrary
   (iv) inconsistent

3. (a) Reducing the matrix $[A, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3]$ yields: for $\mathbf{b}_1$, $x_1 = -48$, $x_2 = 11$, $x_3 = 18$; for $\mathbf{b}_2$, $x_1 = 4$, $x_2 = 2$, $x_3 = 1$; and for $\mathbf{b}_3$, $x_1 = -3$, $x_2 = 2$, $x_3 = 2$.
   (b) $C = \begin{bmatrix} -48 & 4 & -3 \\ 11 & 2 & 2 \\ 18 & 1 & 2 \end{bmatrix}$

4. Set $C = [\mathbf{c}_1, \mathbf{c}_2]$. Reducing the matrix $[A, C]$ yields solution $x_1 = 2 - x_3$, $x_2 = 1 + 2x_3$, x_3 arbitrary, for the system $A\mathbf{x} = \mathbf{c}_1$. Similarly, the system $A\mathbf{x} = \mathbf{c}_2$ has solution $x_1 = -1 - x_3$, $x_2 = -3 + 2x_3$, x_3 arbitrary. Therefore, if $\mathbf{b}_1 = \begin{bmatrix} 2-a \\ 1+2a \\ a \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} -1-b \\ -3+2b \\ b \end{bmatrix}$ for arbitrary a, b, then $A\mathbf{b}_1 = \mathbf{c}_1$ and $A\mathbf{b}_2 = \mathbf{c}_2$, $B = [\mathbf{b}_1, \mathbf{b}_2]$ is the desired matrix since $AB = C$.

5. By assumption, $\mathbf{A}_5 + 3\mathbf{A}_1 + 5\mathbf{A}_4 + 7\mathbf{A}_2 + 9\mathbf{A}_3 = \mathbf{b}$. Reordering the terms yield $3\mathbf{A}_1 + 7\mathbf{A}_2 + 9\mathbf{A}_3 + 5\mathbf{A}_4 + \mathbf{A}_5 = \mathbf{b}$ so $A\mathbf{x} = \mathbf{b}$ has solution $[3, 7, 9, 5, 1]^T$.

6. (a) $x_1 = 2 + x_3$, $x_2 = 3 - 3x_3$, x_3 arbitrary.
   (b) $x_1 = x_3$, $x_2 = -3x_3$, x_3 nonzero.

7. (a) $\mathbf{x} = [2, 1, 0]^T$ is the unique solution.
   (b) $\mathbf{x} = \boldsymbol{\theta}$ is the unique solution.

8. $\mathbf{x} = [-1, 0, 1]^T$

9. (a) $A^{-1} = \begin{bmatrix} -4 & 1 & 3 \\ 4 & -1 & -2 \\ -3 & 1 & 1 \end{bmatrix}$
   (b) $A^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

10. $\Delta = (\lambda - 4)(\lambda - 1) + 2 = (\lambda - 2)(\lambda - 3)$. A is singular when $\Delta = 0$; that is, when $\lambda = 2$ or $\lambda = 3$. When A is nonsingular

$$A^{-1} = \frac{1}{(\lambda-2)(\lambda-3)} \begin{bmatrix} \lambda - 1 & 1 \\ -2 & \lambda - 4 \end{bmatrix}$$

11. $A = \begin{bmatrix} 1/2 & -1/4 \\ -5/4 & 3/4 \end{bmatrix}$

12. $A = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$

13. $A^{99} = A$; $A^{100} = I$

14. $\mathbf{x} = A^{-1}\mathbf{b} = [3 - 6 - 1]^T$

15. $(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} 28 & -22 & -17 \\ 27 & -1 & 49 \\ 29 & 8 & 16 \end{bmatrix}$

16. $(3A)^{-1} = (1/3)A^{-1} = \begin{bmatrix} 2/3 & 1 & 5/3 \\ 7/3 & 2/3 & 1/3 \\ 4/3 & -4/3 & 1 \end{bmatrix}$

17. $(A^TB)^{-1} = B^{-1}(A^{-1})^T = \begin{bmatrix} 15 & -31 & -31 \\ 36 & 52 & 47 \\ 18 & 21 & -1 \end{bmatrix}$

18. $[(A^{-1}B^{-1})^{-1}A^{-1}B]^{-1} = (B^{-1})^2 = \begin{bmatrix} 70 & -19 & 5 \\ -39 & 44 & 21 \\ 11 & 8 & 22 \end{bmatrix}$

1.11 Conceptual Exercises

1. False. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$ then A and B are symmetric but $AB = \begin{bmatrix} 3 & 9 \\ 5 & 14 \end{bmatrix}$ is not symmetric.

2. True. $(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$.

3. True. $A^{-1} = A$ and $B^{-1} = B$ so $AB^{-1} = B^{-1}A^{-1} = BA$.

4. False. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ then A and B are nonsingular, but $A + B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ is singular.

5. False. The system

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 2 \\ x_1 + x_2 &= 3 \end{aligned}$$

clearly has a unique solution $x_1 = 1$, $x_2 = 2$.

6. True. Suppose $A = [\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n]$. Then for $1 \le j \le n$, $\boldsymbol{\theta} = A\mathbf{e}_j = \mathbf{A}_j$.

7. False. If $\{\mathbf{u}_1, \mathbf{u}_2\}$ is linearly dependent then so is $\{A\mathbf{u}_1, A\mathbf{u}_2\}$. (cf. Exercise 12).

8. True. Since AB is defined, $n = p$ and AB is an $(m \times q)$ matrix. But AB is square, so $m = q$. Thus BA is defined and is an $(n \times n)$ matrix.

9. $Q^{-1} = RP$.

10. $AB = (AB)^T = B^T A^T = BA$.

11. First note that $\mathbf{u}_i^T \mathbf{u}_j = (\mathbf{u}_j^T \mathbf{u}_i)^T$, since $\mathbf{u}_j^T \mathbf{u}_i$ is an (1×1) matrix. If $\boldsymbol{\theta} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$ then $0 = \boldsymbol{\theta}^T \boldsymbol{\theta} = (c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3)^T (c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3) = c_1^2 \parallel \mathbf{u}_1 \parallel^2 + c_2^2 \parallel \mathbf{u}_2 \parallel^2 + c_3^2 \parallel \mathbf{u}_3 \parallel^2$. If follows that $c_1 = c_2 = c_3 = 0$.

12. Suppose $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 = \boldsymbol{\theta}$, where $c_i \ne 0$ for $i = 1$ or $i = 2$. Then $\boldsymbol{\theta} = A\boldsymbol{\theta} = A(c_1\mathbf{u}_1 + c_2\mathbf{u}_2) = c_1 A\mathbf{u}_1 + c_2 A\mathbf{u}_2$.

13. $\parallel A\mathbf{x} \parallel^2 = (A\mathbf{x})^T (A\mathbf{x}) = \mathbf{x}^T A^T A\mathbf{x} = \mathbf{x}^T I\mathbf{x} = \mathbf{x}^T\mathbf{x} = \parallel \mathbf{x} \parallel^2$.

14. $A^2 = AI$ so $A = I$.

15. $(AB)^2 = (AB)(AB) = A(BA)B = A(AB)B = A^2B^2 = AB$.

16.(b) Since $A^{k-1} \ne \mathcal{O}$, there exists a vector $\mathbf{b}$ such that $A^{k-1}\mathbf{b} \ne \boldsymbol{\theta}$ (cf. Exercise 6). If $\mathbf{c} = A^{k-1}\mathbf{b}$ then $A\mathbf{c} = A(A^{k-1}\mathbf{b}) = A^k\mathbf{b} = \mathcal{O}\mathbf{b} = \boldsymbol{\theta}$. It follows that A is singular.

Chapter 2

Vectors in 2-Space and 3-Space

2.1 Vectors in the Plane

1. For vector $\overrightarrow{AB}$ the $x-$component is $-4-0=4$ and the $y-$component is $3-(-2)=5$. For vector $\overrightarrow{CD}$ the $x-$component is $1-5=-4$ and the $y-$component is $4-(-1)=5$. The vectors are equal.

2. For vector $\overrightarrow{AB}$ the $x-$component is $3-(-1)=4$ and the $y-$component is $-2-3=-5$. For vector $\overrightarrow{CD}$ the $x-$component is $1-5=-4$ and the $y-$component is $4-(-1)=5$. The vectors are not equal.

3. For vector $\overrightarrow{AB}$ the $x-$component is $0-(-4)=4$ and the $y-$component is $1-(-2)=3$. For vector $\overrightarrow{CD}$ the $x-$component is $3-0=3$ and the $y-$component is $2-(-2)=4$. The vectors are not equal.

4. For vector $\overrightarrow{AB}$ the $x-$component is $-1-3=-4$ and the $y-$component is $-1-1=-2$. For vector $\overrightarrow{CD}$ the $x-$component is $-6-0=-6$ and the $y-$component is $0-3=-3$. The vectors are not equal.

5. (a) For $\mathbf{u}$: $\|\mathbf{u}\|=\sqrt{(2-(-3))^2+(2-5)^2}=\sqrt{25+9}=\sqrt{34}$.
   For $\mathbf{v}$: $\|\mathbf{v}\|=\sqrt{(-2-3)^2+(7-4)^2}=\sqrt{25+9}=\sqrt{34}$.
   Therefore $\|\mathbf{u}\|=\|\mathbf{v}\|$.

   (b) Segment AB has slope: $(2-5)/(2-(-3))=-3/5$.
   Segment CD has slope: $(7-4)/(-2-3)=3/(-5)$.

   (c) For vector $\overrightarrow{AB}$ the $x-$component is $2-(-3)=5$ and the $y-$component is $2-5=-3$. For vector $\overrightarrow{CD}$ the $x-$component is $-2-3=-5$ and the $y-$component is $7-4=3$. The vectors are not equal.

6. $D=(4,7)$

7. $D = (-2, 5)$

8. $D = (-1, 6)$

9. $D = (-1, 1)$

11. $v_1 = 5$, $v_2 = 3$

12. $v_1 = 5$, $v_2 = -4$

13. $v_1 = -6$, $v_2 = 5$

14. $v_1 = -3$, $v_2 = -4$

15. $B = (3, 3)$

16. $B = (-1, 2)$

17. $A = (2, 4)$

18. $A = (0, 3)$

19. (a) $B = (3, 2)$, $C = (5, 0)$

20. (a) $B = (-2, 3)$, $C = (1, 4)$

21. (a) $Q = (7, 1)$

22. (a) $Q = (4, 1)$

23. (a) $B = (-1, 4)$, $C = (0, -1)$

24. (a) $Q = (3, -2)$

25. (a) $B = (3, 3)$, $C = (6, 1)$

26. (a) $B = (0, 7)$, $C = (-3, -2)$

27. (a) $D = (6, -3)$

28. $\frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

29. $\frac{1}{5} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

30. $\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$

31. $\dfrac{3}{\sqrt{13}}\mathbf{i} - \dfrac{2}{\sqrt{13}}\mathbf{j}$

32. $B = (4, 2)$

33. $B = (1, -2)$

34. $B = (0, -5)$

35. $B = (1/3, -7)$

36. $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\mathbf{u} - 3\mathbf{v} = \begin{bmatrix} -2 \\ -5 \end{bmatrix}$

37. $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $\mathbf{u} - 3\mathbf{v} = \begin{bmatrix} -6 \\ 8 \end{bmatrix}$

38. $\mathbf{u} + \mathbf{v} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{u} - 3\mathbf{v} = -2\mathbf{i} + 5\mathbf{j}$

39. $\mathbf{u} + \mathbf{v} = 4\mathbf{i} + \mathbf{j}$, $\mathbf{u} - 3\mathbf{v} = -4\mathbf{i} - 7\mathbf{j}$

40. Note that given $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ with either $a \neq 0$ or $b \neq 0$, gives $\|\mathbf{u}\| = \sqrt{a^2 + b^2} \neq 0$.

    Then $\mathbf{w} = \dfrac{1}{\|\mathbf{u}\|}\mathbf{u} = \dfrac{1}{\|\mathbf{u}\|}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a/\sqrt{a^2+b^2} \\ b/\sqrt{a^2+b^2} \end{bmatrix}$

    $\|\mathbf{w}\| = \sqrt{\dfrac{a^2}{a^2+b^2} + \dfrac{b^2}{a^2+b^2}} = \sqrt{\dfrac{a^2+b^2}{a^2+b^2}} = \sqrt{1} = 1$

2.2 Vectors in Space

1. $d(P, Q) = \sqrt{(0-1)^2 + (2-2)^2 + (2-1)^2} = \sqrt{2}$
2. $d(P, Q) = \sqrt{(0-1)^2 + (0-1)^2 + (1-0)^2} = \sqrt{3}$
3. $d(P, Q) = \sqrt{(0-1)^2 + (0-0)^2 + (1-0)^2} = \sqrt{2}$
4. $d(P, Q) = \sqrt{(0-1)^2 + (0-1)^2 + (0-1)^2} = \sqrt{3}$
5. $M = (1, 4, 4)$;$d(M, 0) = \sqrt{(0-1)^2 + (0-4)^2 + (0-4)^2} = \sqrt{33}$
6. $M = (2, 1, 4)$;$d(M, 0) = \sqrt{(0-2)^2 + (0-1)^2 + (0-4)^2} = \sqrt{21}$
7. $B = (0, 3/2, -3/2)$, $C = (1, 3, 0)$, $D = (2, 9/2, 3/2)$
8. plane
9. line

10. line

11. plane

12. plane

17. (a) The length of the segment from P to R is $\sqrt{(r_1 - p_1)^2 + (r_2 - p_2)^2 + (r_3 - p_3)^2}$ and the length of the segment from R to S is $\sqrt{(s_1 - r_1)^2 + (s_2 - r_2)^2 + (s_3 - r_3)^2}$. Let a and b be the distances from P to R and R to S respectively and c be the distance from point P to S. Solving the equation $a^2 + b^2 = c^2$ for c yields the desired equality.

    (b) This problem is worked similarly to part (a).

18. (a) $\mathbf{v} = \begin{bmatrix} -2 \\ 3 \\ -3 \end{bmatrix}$

    (b) $D = (-3, 5, -2)$

19. (a) $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}$

    (b) $D = (2, 4, -2)$

20. (a) $\mathbf{v} = \begin{bmatrix} -3 \\ -2 \\ 3 \end{bmatrix}$

    (b) $D = (-4, 0, 4)$

21. (a) $\mathbf{v} = \begin{bmatrix} 0 \\ 5 \\ -7 \end{bmatrix}$

    (b) $D = (-1, 7, -6)$

22. $A = (4, 0, 0)$

23. $A = (3, 2, 1)$

24. $A = (4, 1, 2)$

25. $A = (-2, 3, -1)$

26. (a) $\mathbf{u} + 2\mathbf{v} = \begin{bmatrix} 9 \\ 9 \\ 14 \end{bmatrix}$

    (b) $\|\mathbf{u} - \mathbf{v}\| = 5$

(c) $\mathbf{w} = \begin{bmatrix} 3/2 \\ 0 \\ 2 \end{bmatrix}$

27. (a) $\mathbf{u} + 2\mathbf{v} = \begin{bmatrix} 7 \\ 7 \\ 10 \end{bmatrix}$

(b) $\|\mathbf{u} - \mathbf{v}\| = 3$

(c) $\mathbf{w} = \begin{bmatrix} 1 \\ -1/2 \\ 1 \end{bmatrix}$

28. (a) $\mathbf{u} + 2\mathbf{v} = \begin{bmatrix} 11 \\ -3 \\ 4 \end{bmatrix}$

(b) $\|\mathbf{u} - \mathbf{v}\| = \sqrt{74}$

(c) $\mathbf{w} = \begin{bmatrix} -4 \\ 3/2 \\ -1/2 \end{bmatrix}$

29. (a) $\mathbf{u} + 2\mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

(b) $\|\mathbf{u} - \mathbf{v}\| = \sqrt{150}$

(c) $\mathbf{w} = \begin{bmatrix} 7/2 \\ -5 \\ 1/2 \end{bmatrix}$

30. $\mathbf{u} = 2\mathbf{v} = 2\mathbf{i} + 2\mathbf{j}$

31. $\mathbf{u} = 2\mathbf{k}$

32. $\mathbf{u} = -4\mathbf{v} = \begin{bmatrix} -4 \\ 0 \\ -4 \end{bmatrix}$

33. $\mathbf{u} = -\dfrac{5}{3}\mathbf{v} = \begin{bmatrix} 5/3 \\ -10/3 \\ -10/3 \end{bmatrix}$

34. $\mathbf{u} = \begin{bmatrix} -1/2 \\ -1 \\ 0 \end{bmatrix}$

35. $\mathbf{u} = \begin{bmatrix} -2 \\ -4 \\ -1 \end{bmatrix}$

2.3 The Dot Product and the Cross Product

1. -2

2. 0

3. 7

4. 0

5. $\cos(\theta) = \dfrac{11}{\sqrt{290}}$

6. $\cos(\theta) = \dfrac{-3}{\sqrt{130}}$

7. $\cos(\theta) = \dfrac{1}{6}$

8. $\cos(\theta) = \dfrac{11}{14}$

9. $\theta = \dfrac{\pi}{6}$

10. $\theta = \dfrac{2\pi}{3}$

11. $\theta = \dfrac{\pi}{2}$

12. $\theta = 0$

13. $\mathbf{u} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$

14. $\mathbf{u} = 0\mathbf{i} + 0\mathbf{j} + 4\mathbf{k}$

15. $\mathbf{u} = 3\mathbf{i} + 0\mathbf{j} + 4\mathbf{k}$

16. $\mathbf{u} = 12\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ or $\mathbf{u} = 12\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$

17. $\mathbf{u} = -1\mathbf{i} + 3\mathbf{j} + 1\mathbf{k}$

18. $\mathbf{u} = 1\mathbf{i} + 1\mathbf{j} + 2\mathbf{k}$

19. $R = (33/10, 11/10)$

20. $R = (8, 2)$

21. $R = (-3, -1)$

22. $R = (0, 0)$

    $\mathbf{u}$ and $\mathbf{q}$ are perpendicular so $\mathbf{w}$ is the zero vector.

23. $\mathbf{u_1} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$, $\mathbf{u_2} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

24. $\mathbf{u_1} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$, $\mathbf{u_2} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$

25. $\mathbf{u_1} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$, $\mathbf{u_2} = \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix}$

26. $\mathbf{u_1} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$, $\mathbf{u_2} = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$

27. If $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, then $\mathbf{u} \cdot \mathbf{u} = u_1^2 + u_2^2 + u_3^2$. Each term of this sum is a real number squared and therefore greater than or equal to zero, hence the sum is greater than or equal to zero.

28. Note that $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$ and $\mathbf{v} \cdot \mathbf{u} = v_1u_1 + v_2u_2 + v_3u_3$. By the commutative property of multiplication, $u_iv_i = v_iu_i$ for each value of i and hence $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.

29. Note that
$$\begin{aligned}
\mathbf{u} \cdot (c\mathbf{v}) &= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \cdot \begin{bmatrix} c\ v_1 \\ c\ v_2 \\ c\ v_3 \end{bmatrix} \\
&= u_1\ c\ v_1 + u_2\ c\ v_2 + u_3\ c\ v_3 \\
&= c(u_1v_1 + u_2v_2 + u_3v_3) \\
&= c(\mathbf{u} \cdot \mathbf{v})
\end{aligned}$$

30.
$$\begin{aligned}
\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) &= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \cdot \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{bmatrix} \\
&= u_1(v_1 + w_1) + u_2(v_2 + w_2) + u_3(v_3 + w_3) \\
&= u_1v_1 + u_1w_1 + u_2v_2 + u_2w_2 + u_3v_3 + u_3w_3 \\
&= (u_1v_1 + u_2v_2 + u_3v_3) + (u_1w_1 + u_2w_2 + u_3w_3) \\
&= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}
\end{aligned}$$

31. With the given $\mathbf{u}$ and $\mathbf{v}$ the left-hand-side of equation (1) yields:

$\|\mathbf{u} - \mathbf{v}\|^2 = (u_1 - v_1)^2 + (u_2 - v_2)^2 = u_1^2 - 2u_1v_1 + v_1^2 + u_2^2 - 2u_2v_2 + v_2^2$

The right-hand-side of equation (1) yields:

$\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos(\theta) = u_1^2 + u_2^2 + v_1^2 + v_2^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos(\theta)$

Setting these expanded left and right hand sides equal to each other and then cancelling terms yields:

$-2u_1v_1 - 2u_2v_2 = -2\|\mathbf{u}\|\|\mathbf{v}\|\cos(\theta)$

Dividing the last equation by -2 gives the desired result, equation (2a).

32. $\begin{bmatrix} 0 \\ 0 \\ -11 \end{bmatrix}$

33. $\begin{bmatrix} -2 \\ 2 \\ 8 \end{bmatrix}$

34. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

35. $\begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}$

36. $\mathbf{w} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$

37. $\mathbf{w} = \begin{bmatrix} -2 \\ -5 \\ 4 \end{bmatrix}$

38. $\mathbf{w} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

39. $\mathbf{w} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$

40. $\mathbf{w} = \begin{bmatrix} -9 \\ -9 \\ -9 \end{bmatrix}$

41. $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}$

42. $\sqrt{41}$ square units

43. $4\sqrt{6}$ square units

44. $\dfrac{5\sqrt{5}}{2}$ square units

45. $\dfrac{6\sqrt{11}}{2}$ square units

46. 22 cubic units

47. 24 cubic units

48. coplanar

49. NOT coplanar

50. Substitute the given values of x, y, and z into the left-hand-side of each of the two given equations and simplify. This will show that for the given values, the left-hand-side of each equation is zero and therefore the given values are a solution to the system.

51. By finding the two cross products $(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, one can see that they are not equal.

52. (a) Expanding and simplifying the right and left hand sides of the equations, one can see that they are equal.

$$\begin{aligned} \|\mathbf{u} \times \mathbf{v}\|^2 &= (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2 \\ &= u_3^2(v_1^2 + v_2^2) - 2u_1u_3v_1v_3 \\ &\quad -2u_2v_2(u_1v_1 + u_3v_3) + u_2^2(v_1^2 + v_3^2) + u_1^2(v_2^2 + v_3^2) \end{aligned}$$

$$\begin{aligned} \|\mathbf{u}\|^2 \, \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2 &= (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2 \\ &= u_3^2(v_1^2 + v_2^2) - 2u_1u_3v_1v_3 \\ &\quad -2u_2v_2(u_1v_1 + u_3v_3) + u_2^2(v_1^2 + v_3^2) + u_1^2(v_2^2 + v_3^2) \end{aligned}$$

(b) Substitute $\| \mathbf{u} \| \| \mathbf{v} \| \cos(\theta)$ into the equation given in 10.(a) for $\mathbf{u} \cdot \mathbf{v}$:

$$\begin{aligned} \| \mathbf{u} \times \mathbf{v} \|^2 &= \| \mathbf{u} \|^2 \| \mathbf{v} \|^2 - \| \mathbf{u} \|^2 \| \mathbf{v} \|^2 \cos^2(\theta) \\ &= \| \mathbf{u} \|^2 \| \mathbf{v} \|^2 (1 - \cos^2(\theta)) \\ &= \| \mathbf{u} \|^2 \| \mathbf{v} \|^2 \sin^2(\theta) \end{aligned}$$

Therefore, $\| \mathbf{u} \times \mathbf{v} \| = \| \mathbf{u} \| \| \mathbf{v} \| \cos(\theta)$.

53. To see that $\| \mathbf{u} \| \cos(\theta)$ is the height of the parallelepiped, calculate the length of $proj_{(\mathbf{v} \times \mathbf{w})}\mathbf{u}$. Then the volume of the parallelepiped is (area of base)$\times$(height) which is exactly the absolute value of the triple product given.

2.4 Lines and Planes in Space

1. $\begin{aligned} x &= 2 + 3t \\ y &= 4 + 2t \\ z &= -3 + 4t \end{aligned}$

2. $\begin{aligned} x &= 1 + 2t \\ y &= 1 \\ z &= -1 + 3t \end{aligned}$

3. As a direction vector for L we use $\mathbf{u} = \overrightarrow{P_0P_1} = \begin{bmatrix} 1-0 \\ 2-4 \\ 4-1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$. Therefore, one set of parametric equations for L are $x = t$, $y = 4 - 2t$, $z = 1 + 3t$.

4. As a direction vector for L we use $\mathbf{u} = \overrightarrow{P_0P_1} = \begin{bmatrix} 6-5 \\ 6-1 \\ 4-(-3) \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}$. Therefore, one set of parametric equations for L are $x = 5 + t$, $y = 1 + 5t$, $z = -3 + 7t$.

5. The direction vector for the first line is $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$. The direction vector for the second line is $\mathbf{v} = \begin{bmatrix} -4 \\ 2 \\ 6 \end{bmatrix} = -2\mathbf{u}$. Since the two direction vectors are scalar multiples of each other, the lines are parallel.

6. The direction vector for the first line is $\mathbf{u} = \begin{bmatrix} 6 \\ 4 \\ 3 \end{bmatrix}$. The direction vector for the second line is $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$. Since there is no real number k such that $\mathbf{u} = k\mathbf{v}$, the two lines are NOT

parallel.

7. The direction vector for the first line is $\mathbf{u} = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}$. The direction vector for the second line is $\mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$. Since there is no real number k such that $\mathbf{u} = k\mathbf{v}$, the two lines are NOT parallel.

8. The direction vector for the first line is $\mathbf{u} = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}$. The direction vector for the second line is $\mathbf{v} = \begin{bmatrix} 3 \\ -6 \\ 9 \end{bmatrix} = -3\mathbf{u}$. Since the two direction vectors are scalar multiples of each other, the lines are parallel.

9. The normal vector to the plane is $\mathbf{n} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$. Therefore, the equation of the line is $x = 1 + 3t$, $y = 2 + 4t$, $z = 1 - t$.

10. The normal vector to the plane is $\mathbf{n} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$. Therefore, the equation of the line is $x = 2 + t$, $y = 0 - t$, $z = -3 + 2t$.

11. By substituting the parametric equations of the line into the equation for the plane, we find that $t = -1$. Thus $P = (-1, 4, 1)$.

12. When substituting the parametric equations of the line into the equation for the plane, we find a contradiction. Therefore, the line does not intersect the plane at any point. It can also be noted that the normal vector to the plane and the direction vector of the line are perpendicular so that the line is at least parallel to the plane. Then since the point P_0 is not in the plane, the line is not in the plane either.

13. By substituting the parametric equations of the line into the equation for the plane, we find that $t = -4$. Thus $P = (-8, -13, 36)$.

14. By substituting the parametric equations of the line into the equation for the plane, we find that $t = -2$. Thus $P = (-3, 5, -3)$.

15. $6x + y - z = 16$

16. $x - 2y + 3z = 9$

17. A normal vector to the plane can be found by taking the cross product of the vectors $\overrightarrow{PQ} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\overrightarrow{PR} = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}$.

    $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = -7\mathbf{i} - \mathbf{j} + 4\mathbf{k}$.

    Therefore, the equation of the plane is $-7x - y + 4z = 5$.

18. A normal vector to the plane can be found by taking the cross product of the vectors $\overrightarrow{PQ} = \begin{bmatrix} 1 \\ 8 \\ -5 \end{bmatrix}$ and $\overrightarrow{PR} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

    $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = 21\mathbf{i} - 12\mathbf{j} - 15\mathbf{k}$.

    Therefore, the equation of the plane is $21x - 12y - 15z = -12$

    or equivalently $7x - 4y - 5z = -4$.

19. A normal vector to the plane can be found by taking the cross product of the vectors $\overrightarrow{PQ} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$ and $\overrightarrow{PR} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$.

    $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = -2\mathbf{i} + 7\mathbf{j} + 3\mathbf{k}$.

    Therefore, the equation of the plane is $-2x + 7y + 3z = 17$.

20. A normal vector to the plane can be found by taking the cross product of the vectors $\overrightarrow{PQ} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ and $\overrightarrow{PR} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

    $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = 1\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

    Therefore, the equation of the plane is $x + 2y - z = 6$.

21. $\mathbf{v} = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$

22. $\mathbf{v} = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$

23. $x + 2y - 2z = 17$

24. $x + y + 3z = 11$

25. $\begin{aligned} x &= 4 - t \\ y &= 5 + t \\ z &= t \end{aligned}$

26. $\begin{aligned} x &= -3 + t \\ y &= 2 - t \\ z &= t \end{aligned}$

2.5 Supplementary Exercises

1. Solving the system of equations $c_1\mathbf{u}+c_2\mathbf{v} = \mathbf{x}$ for c_1 and c_2 we find that $c_1 = 3$ and $c_2 = -2$. Thus $\mathbf{x_1} = 3\mathbf{u} = \begin{bmatrix} 15 \\ 6 \end{bmatrix}$ and $\mathbf{x_2} = -2\mathbf{v} = \begin{bmatrix} -14 \\ -2 \end{bmatrix}$.

2. The island dock is at $(5, 5\sqrt{3})$. Assuming that the boat travels t miles west and t miles north to get to the buoy, the buoy is at $(5 - t, 5\sqrt{3} + t)$. The distance from the mainland dock to the buoy is $\sqrt{(5-t)^2 + (5\sqrt{3}+t)^2} = \sqrt{2t^2 + (10\sqrt{3} - 2)t + 100}$ miles.

3. $R = (13, -1)$

4. $$\begin{aligned} \|\mathbf{a}\|^2 = \mathbf{a}\cdot\mathbf{a} &= (2\mathbf{u} + 3\mathbf{v}) \cdot (2\mathbf{u} + 3\mathbf{v}) \\ &= 4(\mathbf{u}\cdot\mathbf{u}) + 12(\mathbf{u}\cdot\mathbf{v}) + 9(\mathbf{v}\cdot\mathbf{v}) \\ &= 4\|\mathbf{u}\|^2 + 12(0) + 9\|\mathbf{v}\|^2 \\ &= 4 + 9 = 13 \end{aligned}$$

   Since $\|\mathbf{a}\|^2 = 13$ then $\|\mathbf{a}\| = \sqrt{13}$.

5. $$\begin{aligned} \|\mathbf{a}\|^2 = \mathbf{a}\cdot\mathbf{a} &= (\mathbf{u} + \mathbf{v} + \mathbf{w}) \cdot (\mathbf{u} + \mathbf{v} + \mathbf{w}) \\ &= (\mathbf{u}\cdot\mathbf{u}) + 2(\mathbf{u}\cdot\mathbf{v}) + 2(\mathbf{u}\cdot\mathbf{w}) + 2(\mathbf{v}\cdot\mathbf{w}) + (\mathbf{v}\cdot\mathbf{v}) + (\mathbf{w}\cdot\mathbf{w}) \\ &= 2^2 + 2(0) + 2(0) + 2(0) + 1^2 + 2^2 \\ &= 4 + 1 + 4 = 9 \end{aligned}$$

   Since $\|\mathbf{a}\|^2 = 9$ then $\|\mathbf{a}\| = 3$.

6. $$\begin{aligned} (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) &= (\mathbf{u}\cdot\mathbf{u}) - (\mathbf{v}\cdot\mathbf{v}) \\ &= 4 - 9 = -5 \end{aligned}$$

   Therefore $(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = -5$.

7. Note that since $\mathbf{u}\cdot\mathbf{v} = 0$ then $\mathbf{u}$ and $\mathbf{v}$ are perpendicular. So the angle between $\mathbf{u}$ and $\mathbf{v}$, θ, is $\pi/2$.

   Therefore, $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\|\sin(\theta) = 2\cdot 3\cdot 1 = 6$.

8. Note that since $\mathbf{u}$ and $\mathbf{v}$ are perpendicular then $\| \mathbf{u}-\mathbf{v} \| = \| \mathbf{u}+\mathbf{v} \| = \sqrt{13}$.

   Then $\begin{aligned} \| (\mathbf{u}-\mathbf{v}) \times (\mathbf{u}+\mathbf{v}) \|^2 &= \| \mathbf{u}-\mathbf{v} \|^2 \, \| \mathbf{u}+\mathbf{v} \|^2 \sin^2(\theta) \\ &= 169(1-\sin^2(\theta)) \end{aligned}$

   But $\cos(\theta) = \dfrac{(\mathbf{u}-\mathbf{v}) \cdot (\mathbf{u}+\mathbf{v})}{\| \mathbf{u}-\mathbf{v} \| \| \mathbf{u}+\mathbf{v} \|}$ and from Exercise 6 we have that $(\mathbf{u}-\mathbf{v} \cdot \mathbf{u}+\mathbf{v}) = -5$.

   Therefore, $\| (\mathbf{u}-\mathbf{v}) \times (\mathbf{u}+\mathbf{v}) \|^2 = 169(1-(-5/13)^2) = 169-25 = 144$. Thus $\| (\mathbf{u}-\mathbf{v}) \times (\mathbf{u}+\mathbf{v}) \| = 12$.

9. Unit vectors in the $xz-$plane are of the form $\mathbf{v} = \begin{bmatrix} x \\ 0 \\ y \end{bmatrix}$ where $\sqrt{x^2+z^2} = 1$. For $\mathbf{v}$ to be perpendicular to $\begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$ requires that $\begin{bmatrix} x \\ 0 \\ y \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} = 0$ or $3x+4y=0$. Hence, $\mathbf{v}$ is of the form $\mathbf{v} = k \begin{bmatrix} -4/3 \\ 0 \\ 1 \end{bmatrix}$ for some number k. Since $\mathbf{v}$ is a unit vector then $k = \pm 3/5$.

   Therefore $\mathbf{v} = \begin{bmatrix} -4/5 \\ 0 \\ 3/5 \end{bmatrix}$ or $\mathbf{v} = \begin{bmatrix} 4/5 \\ 0 \\ -3/5 \end{bmatrix}$.

10. Letting $P_a = (a,0,0)$, $P_b = (0,b,0)$ and $P_c = (0,0,c)$ then $\overrightarrow{P_aP_b} = \begin{bmatrix} -a \\ b \\ 0 \end{bmatrix}$ and

    $\overrightarrow{P_aP_c} = \begin{bmatrix} -a \\ 0 \\ c \end{bmatrix}$. Thus a normal vector to the plane is

    $\mathbf{n} = \overrightarrow{P_aP_b} \times \overrightarrow{P_aP_c} = (bc)\mathbf{i} + (ac)\mathbf{j} + (ab)\mathbf{k}$.

    Then the equation of the plane is $(ab)x+(ac)y+(ab)z = abc$. Dividing through by abc the equation of the plane can be rewritten as $\left(\dfrac{1}{a}\right)x + \left(\dfrac{1}{b}\right)y + \left(\dfrac{1}{c}\right)z = 1$.

11. Letting $P_1 = (1,1,2)$, $P_2 = (2,3,9)$, $P_3 = (-2,1,-1)$ and $P_4 = (1,2,5)$, then the equation of the plane determined by P_1, P_2 and P_3 is $-x-3y+z = -2$. Since P_4 satisfies the equation of this plane, i. e. $-1(1)-3(2)+1(5) = -2$ then P_4 is also in the plane. Therefore, all four points lie in the plane $-x-3y+z = -2$.

12. Consider a circle of radius r in the $xy-$plane centered at the origin. Letting $A = (0,r)$, $B = (0,-r)$ and $C = (x,y)$ where $x^2+y^2 = r^2$, then the vector $\overrightarrow{AC} = \begin{bmatrix} x \\ y-r \end{bmatrix}$ and the

vector $\overrightarrow{BC} = \begin{bmatrix} x \\ y+r \end{bmatrix}$. Since $\overrightarrow{AC} \cdot \overrightarrow{BC} = x^2 + (y+r)(y-r) = (x^2+y^2) - r^2 = r^2 - r^2 = 0$ then the vectors $\overrightarrow{AC}$ and $\overrightarrow{BC}$ are orthogonal and therefore, the triangle $\triangle ABC$ is a right triangle.

13. Let the midpoints of the four sides of the quadrilateral $ABCD$ be $P_1 = (\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2})$, $P_1 = (\frac{b_1+c_1}{2}, \frac{b_2+c_2}{2})$, $P_1 = (\frac{c_1+d_1}{2}, \frac{c_2+d_2}{2})$, and $P_1 = (\frac{d_1+a_1}{2}, \frac{d_2+a_2}{2})$. Then note that the line segments P_1P_3 and P_2P_4 share the same midpoint, namely $(\frac{a_1+b_1+c_1+d_1}{4}, \frac{a_2+b_2+c_2+d_2}{4})$.

2.6 Conceptual Exercises

1. False. For example, if $\mathbf{u} = 2\mathbf{i} + 4\mathbf{j}$ and $\mathbf{v} = -2\mathbf{i} + \mathbf{j}$ then $\mathbf{u} \cdot \mathbf{v} = 0$ while both $\mathbf{u}$ and $\mathbf{v}$ are nonzero.

2. False. For example, if $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = 3\mathbf{u} = 6\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$ then $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ while both $\mathbf{u}$ and $\mathbf{v}$ are nonzero.

3. Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$.

   Then
$$\begin{aligned} \|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 &= u_1^2 + 2u_1v_1 + v_1^2 + u_2^2 + 2u_2v_2 + v_2^2 + u_3^2 + 2u_3v_3 + v_3^2 \\ &\quad + u_1^2 - 2u_1v_1 + v_1^2 + u_2^2 - 2u_2v_2 + v_2^2 + u_3^2 - 2u_3v_3 + v_3^2 \\ &= 2(u_1^2 + u_2^2 + u_3^2) + 2(v_1^2 + v_2^2 + v_3^2) \\ &= 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2 \end{aligned}$$

4. The vectors $\mathbf{u}$ and $\mathbf{v}$ form the sides of a parallelogram while the vectors $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ form the diagonals of the parallelogram. Thus, the parallelogram law states that the sum of the squares of the sides of a parallelogram equals the sum of the squares of the diagonals. In other words, the parallelogram law is the application of the law of cosines to a parallelogram.

5. For example, $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, and $\mathbf{w} = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$. Then $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = -6\mathbf{i} + 6\mathbf{j}$ and $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = -7\mathbf{i} - \mathbf{j} - 5\mathbf{k}$. Thus $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.

6. $x_i = \dfrac{A_i \cdot \mathbf{b}}{A_i \cdot A_i}$ for $i = 1, 2, 3$.

7. It is equivalent to show that the columns of A form a linearly independent set (see Section 1.7). That is, if $c_1A_1 + c_2A_2 + c_3A_3 = 0$ then the only solution is $c_1 = c_2 = c_3 = 0$. But due to Exercise 6. we have that $c_i = \dfrac{A_i \cdot \mathbf{0}}{A_i \cdot A_i} = 0$. Therefore, the columns of A are linearly independent and so A is nonsingular.

8. Due to Exercise 7. the matrix A is nonsingular. Thus the solution of $A\mathbf{x} = \mathbf{b}$ can be written as $\mathbf{x} = A^{-1}\mathbf{b}$. From Exercise 6., we have that $x_i = A_i \cdot \mathbf{b}$ (since $A_i \cdot A_i = 1$ for each i). Then let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ so $A_i = \begin{bmatrix} a_{i1} \\ a_{i2} \\ a_{i3} \end{bmatrix}$. Then $x_i = A_i \cdot \mathbf{b} = A_{1i}b_1 + A_{2i}b_2 + A_{3i}b_3$.
   This leads to the following system of equations for x_1, x_2 and x_3:
   $$x_1 = A_{11}b_1 + A_{21}b_2 + A_{31}b_3$$
   $$x_2 = A_{12}b_1 + A_{22}b_2 + A_{32}b_3$$
   $$x_3 = A_{13}b_1 + A_{23}b_2 + A_{33}b_3$$
   In matrix form these equations are written as $\mathbf{x} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \mathbf{b} = A^T\mathbf{b}$.

   Since we already know that $\mathbf{x} = A^{-1}\mathbf{b}$ then it must be true that $A^{-1} = A^T$.

9. It is equivalent to show that $\| A\mathbf{x} \|^2 = \| \mathbf{x} \|^2$. Recall that the dot product of $\mathbf{u}$ and $\mathbf{v}$ can be expressed as $(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u}^T\mathbf{v}$.
   Then
   $$\begin{aligned} \| A\mathbf{x} \|^2 &= (A\mathbf{x}) \cdot (A\mathbf{x}) \\ &= (A\mathbf{x})^T(A\mathbf{x}) \\ &= \mathbf{x}^T A^T A\mathbf{x} \\ &= \mathbf{x}^T\mathbf{x} = \mathbf{x} \cdot \mathbf{x} = \| \mathbf{x} \|^2 \end{aligned}$$

10. $(A\mathbf{u}) \cdot (A\mathbf{v}) = (A\mathbf{u})^T(A\mathbf{v}) = \mathbf{u}^T A^T A\mathbf{v} = \mathbf{u}^T\mathbf{v} = \mathbf{u} \cdot \mathbf{v}$.

11. To show that the angle θ_1 between the vectors $\mathbf{u}$ and $\mathbf{v}$ equals the angle θ_2 between the vectors $A\mathbf{u}$ and $A\mathbf{v}$ consider that
    $$\cos(\theta_1) = \frac{\mathbf{u} \cdot \mathbf{v}}{\| \mathbf{u} \| \| \mathbf{v} \|} = \frac{A\mathbf{u} \cdot A\mathbf{v}}{\| A\mathbf{u} \| \| A\mathbf{v} \|} = \cos(\theta_2).$$
    Since $\cos(\theta_1) = \cos(\theta_2)$ it follows that $\theta_1 = \theta_2$.

12. $(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = (\mathbf{u} \cdot \mathbf{u}) - (\mathbf{v} \cdot \mathbf{v}) = \| \mathbf{u} \|^2 - \| \mathbf{v} \|^2 = 0$ since $\| \mathbf{u} \| = \| \mathbf{v} \|$. Therefore $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

Chapter 3

The Vector Space R^n

3.1 Introduction

12. Geometrically, W consists of the points in the plane that lie on the line with equation $x + y = 1$.

13. Geometrically, W consists of the points in the plane that lie on the line with equation $x = -3y$.

14. Geometrically, W consists of the points in the plane that lie on the $y-$axis.

15. Geometrically, W consists of the points in the plane that have coordinates (x, y) satisfying $x + y \geq 0$.

16. Geometrically, W consists of the points in the plane that lie on the line passing through the origin and the point $(1, 3)$. [Let $t = 0$ and $t = 1$, respectively. If (x, y) is a point on the line then $x = t$ and $y = 3t$, so the equation for the line is $y = 3x$.

17. Geometrically, W consists of the points in the plane that lie on the circle with equation $x^2 + y^2 = 4$.

18. Geometrically, W consists of the points in three-space that lie on positive $x-$axis.

19. Geometrically, W consists of the points in three-space that lie on the plane with equation $x + y + 2z = 0$.

20. Geometrically, W consists of the points in three-space that lie on the line which passes through the origin and the point with coordinates $(2, 0, 1)$. The line can be represented by the equations $x = 2r$, $y = 0$, $z = r$.

21. Geometrically, W consists of the points in three-space that are on or above the $xy-$plane and that lie on the sphere with equation $x^2 + y^2 + z^2 = 1$.

22. $W = \{\mathbf{x} : \mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}, a - 2b = 1\}$.

23. $W = \{\mathbf{x} : \mathbf{x} = \begin{bmatrix} a \\ 0 \end{bmatrix}, a$ any real number $\}$.

24. $W = \{\mathbf{x} : \mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}, b > 0\}$.

25. $W = \{\mathbf{x} : \mathbf{x} = \begin{bmatrix} a \\ 2 \end{bmatrix}, a$ any real number $\}$.

26. $W = \{\mathbf{x} : \mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}, b = a^2\}$.

27. $W = \{\mathbf{x} : \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, x_1 + x_2 - 2x_3 = 0\}$.

28. $W = \{\mathbf{x} : \mathbf{x} = t \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, t$ any real number$\}$.

29. $W = \{\mathbf{x} : \mathbf{x} = \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix}, x_2,\ x_3$ any real number $\}$.

30. $W = \{\mathbf{x} : \mathbf{x} = \begin{bmatrix} x_1 \\ 2 \\ x_3 \end{bmatrix}, x_1,\ x_3$ any real number $\}$.

3.2 Vector Space Properties of R^n

1. Clearly θ is in W. Suppose $\mathbf{u}$ and $\mathbf{v}$ are in W, where $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$.

   Then $u_1 = 2u_2$ and $v_1 = 2v_2$. If a is any scalar then $\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$ and

   $a\mathbf{u} = \begin{bmatrix} au_1 \\ au_2 \end{bmatrix}$. But $u_1 + v_1 = 2u_2 + 2v_2 = 2(u_2 + v_2)$ and $au_1 = a(2u_2) = 2(au_2)$, so $\mathbf{u} + \mathbf{v}$ and $a\mathbf{u}$ are in W. By Theorem 2, W is a subspace of R^2. Geometrically, W consists of the points on the line with equation $x = 2y$.

2. W is not a subspace of R^2. Note for example that θ is not in W. Also if $\mathbf{u} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$, then $\mathbf{u}$ and $\mathbf{v}$ are in W, but $\mathbf{u}+\mathbf{v}$ is not in W.

3. W is not a subspace of R^2 since, for example, $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are in W whereas $\mathbf{u}+\mathbf{v}$ is not in W. Note that W satisfies properties (s1) and (s3) of Theorem 2.

4. W is not a subspace of R^2. For example, $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is in W, but $\sqrt{2}\,\mathbf{u}$ is not in W. Note that W satisfies properties (s1) and (s2) of Theorem 2.

5. Clearly, $\theta = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is in W. If $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ are in W, then $u_1 = v_1 = 0$. Thus $\mathbf{u}+\mathbf{v} = \begin{bmatrix} 0 \\ u_2+v_2 \end{bmatrix}$ is in W. Likewise, if a is any scalar, $a\mathbf{u} = \begin{bmatrix} 0 \\ au_2 \end{bmatrix}$ is in W. By Theorem 2, W is a subspace of R^2. Geometrically, W consists of the points on the $y-$axis .

6. If $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ is in W, then $\mid u_1 \mid + \mid u_2 \mid = 0$ so $u_1 = u_2 = 0$. Thus $W = \{\theta\}$ and W is a subspace of R^2.

7. W satisfies none of the properties (s1) - (s3) of Theorem 2, so W is not a subspace of R^2. Clearly, $\theta = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is not in W. Also, $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are in W whereas $\mathbf{u}+\mathbf{v}$ is not in W. Finally,
   if $a \neq 1$ then $a\mathbf{v}$ is not in W.

8. W is not a subspace of R^2. For example $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
   are in W whereas $\mathbf{u}+\mathbf{v}$ is not in W. Note that W satisfies the properties (s1) and (s3) of Theorem 2.

9. Clearly $\theta = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is in W. If $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ are in W and a is any scalar, then $u_3 = 2u_1 - u_2$ and $v_3 = 2v_1 - v_2$. Now $u_3 + v_3 = (2u_1 - u_2) + (2v_1 - v_2) = 2(u_1 + v_1) - (u_2 + v_2)$ and $au_3 = a(2u_1 - u_2) = 2(au_1) - au_2$. Therefore $\mathbf{u}+\mathbf{v}$ and $a\mathbf{u}$ are in W. By Theorem 2, W is a subspace of R^3. Geometrically, W consists of the points on the plane with equation $2x - y - z = 0$.

10. Clearly $\theta = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is in W. If $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ are in W, it follows that $u_2 + v_2 = (u_3 + v_3) + (u_1 + v_1)$ and for any scalar a, $au_2 = au_3 + au_1$. Thus $\mathbf{u} + \mathbf{v}$ and $a\mathbf{u}$ are in W. By Theorem 2, W is a subspace of R^3. Geometrically, W consists of the points in the plane with equation $x - y + z = 0$.

11. W is not a subspace of R^3. For example, $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ are in W but $\mathbf{u} + \mathbf{v}$ is not in W. Also if $a \neq 1$ and $a \neq 0$ then $a\mathbf{u}$ is not in W.

12. Clearly $\theta = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is in W. If $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ are in W then $u_1 = 2u_3$ and $v_1 = 2v_3$. Therefore $u_1 + v_1 = 2(u_3 + v_3)$ and, for any scalar a, $au_1 = 2au_3$. Thus $\mathbf{u} + \mathbf{v}$ and $a\mathbf{u}$ are in W and by Theorem 2, W is a subspace of R^3. Geometrically, W consists of the points on the plane with equation $x - 2z = 0$.

13. W is not a subspace of R^3. For example $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$ are in W but $\mathbf{u} + \mathbf{v}$ is not in W. Also, if $a \neq 1$ and $a \neq 0$ then $a\mathbf{u}$ is not in W.

14. Clearly θ is in W. Vectors $\mathbf{u}$ and $\mathbf{v}$ in W can be written in the form $\mathbf{u} = \begin{bmatrix} u_1 \\ 0 \\ u_3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ 0 \\ v_3 \end{bmatrix}$. Thus $\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ 0 \\ u_3 + v_3 \end{bmatrix}$ is in W. Likewise, for any scalar a, $a\mathbf{u}$ $= \begin{bmatrix} au_1 \\ 0 \\ au_3 \end{bmatrix}$ is in W. Geometrically, W consists of the points in the $xz-$ plane .

15. Clearly θ is in W. If $\mathbf{u}$ and $\mathbf{v}$ are vectors in W then $\mathbf{u}$ and $\mathbf{v}$ can be expressed in the form $\mathbf{u} = \begin{bmatrix} 2a \\ -a \\ a \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2b \\ -b \\ b \end{bmatrix}$. Then $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 2(a+b) \\ -(a+b) \\ a+b \end{bmatrix}$. Similarly, for any scalar c, $c\mathbf{u} = \begin{bmatrix} 2ca \\ -ca \\ ca \end{bmatrix}$. By Theorem 2, W is a subspace of R^3. Geometrically W consists of the points on the line with parametric equations $x = 2t$, $y = -t$, $z = t$.

16. Clearly θ is in W. Elements $\mathbf{u}$ and $\mathbf{v}$ in W can be expressed in the form $\mathbf{u} = \begin{bmatrix} a \\ 2a \\ 2a \end{bmatrix}$

and $\mathbf{v} = \begin{bmatrix} b \\ 2b \\ 2b \end{bmatrix}$. Therefore $\mathbf{u}+\mathbf{v} = \begin{bmatrix} a+b \\ 2(a+b) \\ 2(a+b) \end{bmatrix}$

is in W and, for any scalar c, $c\mathbf{u} = \begin{bmatrix} ca \\ 2ca \\ 2ca \end{bmatrix}$ is in W.

By Theorem 2, W is a subspace of R^3. Geometrically, W consists of the points on the line with parametric equations $x = t$, $y = 2t$,

$z = 2t$.

17. Clearly θ is in W. Moreover, any two elements $\mathbf{u}$ and $\mathbf{v}$ in W can be written in the form $\mathbf{u} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix}$. Therefore $\mathbf{u}+\mathbf{v} = \begin{bmatrix} a+b \\ 0 \\ 0 \end{bmatrix}$ is in W and for any scalar c, $c\mathbf{u} = \begin{bmatrix} ca \\ 0 \\ 0 \end{bmatrix}$ is in W? By Theorem 2, W is a subspace of R^3. Geometrically, W consists of the points on the $x-$ axis.

18. Clearly $\mathbf{a}^{\mathrm{T}}\theta = 0$ so θ is in W? If $\mathbf{u}$ and $\mathbf{v}$ are in W then $\mathbf{a}^{\mathrm{T}}\mathbf{u} = 0$ and $\mathbf{a}^{\mathrm{T}}\mathbf{v} = 0$. It follows that $\mathbf{a}^{\mathrm{T}}(\mathbf{u}+\mathbf{v}) = \mathbf{a}^{\mathrm{T}}\mathbf{u} + \mathbf{a}^{\mathrm{T}}\mathbf{v} = 0$, so $\mathbf{u}+\mathbf{v}$ is in W. If c is a scalar, then $\mathbf{a}^{\mathrm{T}}(c\mathbf{u}) = c\mathbf{a}^{\mathrm{T}}\mathbf{u} = 0$ and hence $c\mathbf{u}$ is in W. This proves that W is a subspace of R^3.

19. The vector $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ is in W if and only if $0 = \mathbf{a}^{\mathrm{T}}\mathbf{u} = u_1 + 2u_2 + u_3$. Thus geometrically W consists of the points in R^3 which lie in the plane $x + 2y + 3z = 0$.

20. The vector $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ is in W if and only if $0 = \mathbf{a}^{\mathrm{T}}\mathbf{u} = u_1$. Thus geometrically W consists of thepoints in R^3 which lie on the yz- plane.

21. Clearly, $\mathbf{a}^{\mathrm{T}}\theta = \mathbf{b}^{\mathrm{T}}\theta = 0$, so θ is in W. Let $\mathbf{a}^{\mathrm{T}}\mathbf{u} = \mathbf{b}^{\mathrm{T}}\mathbf{u} = \mathbf{a}^{\mathrm{T}}\mathbf{v} = \mathbf{b}^{\mathrm{T}}\mathbf{v} = 0$. It then follows that $\mathbf{a}^{\mathrm{T}}(\mathbf{u}+\mathbf{v}) = 0$ and $\mathbf{b}^{\mathrm{T}}(\mathbf{u}+\mathbf{v}) = 0$. Therefore $\mathbf{u}+\mathbf{v}$ is in W. Likewise, $\mathbf{a}^{\mathrm{T}}(c\mathbf{u}) = c(\mathbf{a}^{\mathrm{T}}\mathbf{u}) = 0$ and $\mathbf{b}^{\mathrm{T}}(c\mathbf{u}) =$

$c(\mathbf{b}^{\mathrm{T}}\mathbf{u}) = 0$ for any scalar c. Therefore $c\mathbf{u}$ is in W. Thus W is a subspace of R^3.

22. The vector $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ is in W if and only if $0 = \mathbf{a}^{\mathrm{T}}\mathbf{u} = u_1 - u_2 + 2u_3$ and $0 = \mathbf{b}^{\mathrm{T}}\mathbf{u}$ $= 2u_1 - u_2 + 3u_3$. Thus W is the set of points in R^3 formed by the intersecting planes $x - y + 2z = 0$ and $2x - y + 3z = 0$. The parametric equations for the line are $x = -t$, $y = t$, $z = t$.

23. The vector $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ is in W if and only if $0 = \mathbf{a}^{\mathrm{T}}\mathbf{u} = u_1 + 2u_2 + 2u_3$ and $0 = \mathbf{b}^{\mathrm{T}}\mathbf{u}$ $= u_1 + 3u_2$. Thus W is the set of points on the line formed by the intersecting planes $x + 2y + 2z = 0$ and $x + 3y = 0$. Solving yields $x = -6z$ and $y = 2z$ so the line has parametric equations $x = -6t$, $y = 2t$, $z = t$.

24. The vector $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ is in W if and only if $0 = \mathbf{a}^{\mathrm{T}}\mathbf{u} = u_1 + u_2 + u_3$ and $0 = \mathbf{b}^{\mathrm{T}}\mathbf{u}$ $= 2u_1 + 2u_2 + 2u_3$. Clearly the latter condition is redundant so W consists of the points on the plane $x + y + z = 0$.

25. The vector $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ is in W if and only if $0 = \mathbf{a}^{\mathrm{T}}\mathbf{u} = u_1 - u_3$ and $0 = \mathbf{b}^{\mathrm{T}}\mathbf{u}$ $= -2u_1 + 2u_3$. Clearly the latter condition is redundant so W consists of the points in the plane $x - z = 0$.

27. Property (m1) is not satisfied. For example $3(2\mathbf{x}) = 3\begin{bmatrix} 4x_1 \\ 4x_2 \end{bmatrix} = \begin{bmatrix} 24x_1 \\ 24x_2 \end{bmatrix}$ where $6\mathbf{x} = \begin{bmatrix} 12x_1 \\ 12x_2 \end{bmatrix}$. Also, (m4) is not satisfied since $1\mathbf{x} = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} \neq \mathbf{x}$.

28. Property (c2) is not satisfied. For example, if $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $a = -1$ then $\mathbf{x}$ is in W but $a\mathbf{x}$ is not in W. This also illustrates that (a4) is not satisfied.

29. The set of points on the line can be expressed as the set $W = \{t \begin{bmatrix} a \\ b \\ c \end{bmatrix} : t$ any real number $\}$. Taking $t = 0$ we see that $\boldsymbol{\theta}$ is in W. If $\mathbf{u} = r\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $\mathbf{v} = s\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ then $\mathbf{u} + \mathbf{v} = (r + s)\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is in W. Likewise, if k is any scalar then $k\mathbf{u} = kr\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is in W. Therefore W is a subspace of R^3.

30. Since θ is in both U and V, $\theta = \theta + \theta$ is in $U+V$. Suppose $\mathbf{x}$ and $\mathbf{y}$ are in $U+V$ and write $\mathbf{x} = \mathbf{u_1} + \mathbf{v_1}$, $\mathbf{y} = \mathbf{u_2} + \mathbf{v_2}$ where $\mathbf{u_1}, \mathbf{u_2}$ are in U and $\mathbf{v_1}, \mathbf{v_2}$ are in V. Then $\mathbf{x} + \mathbf{y} = (\mathbf{u_1} + \mathbf{u_2}) + (\mathbf{v_1} + \mathbf{v_2})$ is in $U+V$. If a is a scalar then $a\mathbf{x} = a\mathbf{u_1} + a\mathbf{v_1}$ is in $U+V$. It follows that $U+V$ is a subspace of R^n.

31. Clearly θ is in $U \cap V$. Suppose $\mathbf{x}$ and $\mathbf{y}$ are in $U \cap V$. Then $\mathbf{x}$ and $\mathbf{y}$ are in U and since U is a subspace, $\mathbf{x} + \mathbf{y}$ is in U. Also for any scalar a, $a\mathbf{x}$ is in U. Similarly, $\mathbf{x} + \mathbf{y}$ and $a\mathbf{x}$ are in V. Therefore $\mathbf{x} + \mathbf{y}$ and $a\mathbf{x}$ are in $U \cap V$. It follows that $U \cap V$ is a subspace of R^n.

32. The vector $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is in U and the vector $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is in V. Thus $\mathbf{u}$ and $\mathbf{v}$ are in $U \cup V$ but $\mathbf{u} + \mathbf{v}$ is in neither U nor V.

33. (a) Clearly θ is in $U \cup V$. Suppose $\mathbf{x}$ is in $U \cup V$ and let a be a scalar. If $\mathbf{x}$ is in U, then $a\mathbf{x}$ is also in U. Similarly, if $\mathbf{x}$ is in V, then $a\mathbf{x}$ is in V. In either case, $a\mathbf{x}$ is in $U \cup V$.

    (b) Assume that $\mathbf{u} + \mathbf{v}$ is in U. Since $-\mathbf{u}$ is in U and U is closed under addition we see that $\mathbf{v} = (\mathbf{u} + \mathbf{v}) + (-\mathbf{u})$ is in U. This contradicts the assumption that $\mathbf{v}$ is not in U. Similarly, $\mathbf{u} + \mathbf{v}$ is not in V.

34. Since W is non-empty, W contains a vector $\mathbf{x}$. By (s3) the vector $0\mathbf{x} = \theta$ is in W. By Theorem 2, W is a subspace of R^n.

3.3 Examples of Subspaces

1. By definition $\mathrm{Sp}(S) = \{t \begin{bmatrix} 1 \\ -1 \end{bmatrix} : t$ any real number $\}$. Thus if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is in R^2, then $\mathbf{x}$ is in $\mathrm{Sp}(S)$ if and only if $x_1 + x_2 = 0$. In particular $\mathrm{Sp}(S)$ is the line with equation $x + y = 0$.

2. By definition $\mathrm{Sp}(S) = \{ t \begin{bmatrix} 2 \\ 3 \end{bmatrix} : t$ any real number $\}$. If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is in R^2, then $\mathbf{x}$ is in $\mathrm{Sp}(S)$ if and only if $3x_1 + 2x_2 = 0$. In particular $\mathrm{Sp}(S)$ is the line with equation $3x + 2y = 0$.

3. $\mathrm{Sp}(S) = \{ t \begin{bmatrix} 0 \\ 0 \end{bmatrix} : t$ any real number $\} = \{\mathbf{e}\}$.$\mathrm{Sp}(S)$ is the point $(0,0)$.

4. $\mathrm{Sp}(S) = \{\mathbf{x}$ in $R^2 : \mathbf{x} = k_1\mathbf{a} + k_2\mathbf{b}$ for scalars k_1 and $k_2\}$. For an arbitrary vector $\mathbf{x}$ in R^2, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\mathbf{x} = k_1\mathbf{a} + k_2\mathbf{b}$ if $k_1 = 3x_1 + 2x_2$ and $k_2 = -x_1 - x_2$. It follows that $\mathrm{Sp}(S) = R^2$.

5. $\mathrm{Sp}(S) = \{\mathbf{x}$ in $R^2 : \mathbf{x} = k_1\mathbf{a} + k_2\mathbf{d}$ for scalars k_1 and $k_2\}$. For an arbitrary vector $\mathbf{x}$ in R^2, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, the equation $k_1\mathbf{a} + k_2\mathbf{d} = \mathbf{x}$ has augmented matrix $\begin{bmatrix} 1 & 1 & x_1 \\ -1 & 0 & x_2 \end{bmatrix}$. This matrix reduces to $\begin{bmatrix} 1 & 1 & x_1 \\ 0 & 1 & x_1 + x_2 \end{bmatrix}$ and backsolving yields $k_1 = -x_2$, and $k_2 = x_1 + x_2$. It follows that $\mathrm{Sp}(S) = R^2$.

6. If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ the equation $k_1\mathbf{a} + k_2\mathbf{c} = \mathbf{x}$ has augmented matrix $\begin{bmatrix} 1 & -2 & x_1 \\ -1 & 2 & x_2 \end{bmatrix}$. This matrix reduces to $\begin{bmatrix} 1 & -2 & x_1 \\ 0 & 0 & x_1 + x_2 \end{bmatrix}$, so the equation has a solution if and only if $x_1 + x_2 = 0$. It follows that $\mathrm{Sp}(S) = \{\mathbf{x} : x_1 + x_2 = 0\}$; that is, $\mathrm{Sp}(S)$ is the line with equation $x + y = 0$.

7. $\mathrm{Sp}(S) = \{\mathbf{x}$ in $R^2 : \mathbf{x} = k_1\mathbf{b} + k_2\mathbf{e}$ for scalars k_1 and $k_2\}$. But $k_1\mathbf{b} + k_2\mathbf{e} = k_1\mathbf{b}$ so $\mathrm{Sp}(S) = \mathrm{Sp}(\{\mathbf{b}\})$. It follows that $\mathrm{Sp}(S) = \{ t \begin{bmatrix} 2 \\ -3 \end{bmatrix} : t$ any real number $\}$. If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is in R^2, $\mathbf{x}$ is in $\mathrm{Sp}(S)$ if and only if $3x_1 + 2x_2 = 0$. Thus, $\mathrm{Sp}(S) = \{\mathbf{x} : 3x_1 + 2x_2 = 0\}$; so $\mathrm{Sp}(S)$ is the line with equation $3x + 2y = 0$.

8. If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ the equation $k_1\mathbf{a} + k_2\mathbf{b} + k_3\mathbf{d} = \mathbf{x}$ has augmented matrix $\begin{bmatrix} 1 & 2 & 1 & x_1 \\ -1 & -3 & 0 & x_2 \end{bmatrix}$. This matrix reduces to $\begin{bmatrix} 1 & 2 & 1 & x_1 \\ 0 & -1 & 1 & x_1 + x_2 \end{bmatrix}$. It follows that the system is consistent for arbitrary $\mathbf{x}$, so $\mathrm{Sp}(S) = R^2$.

9. $\mathrm{Sp}(S) = \{\mathbf{x}$ in $R^2 : \mathbf{x} = k_1\mathbf{b} + k_2\mathbf{c} + k_3\mathbf{d}$ for scalars $k_1, k_2, k_3\}$. With $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, the equation $k_1\mathbf{b} + k_2\mathbf{c} + k_3\mathbf{d} = \mathbf{x}$ has augmented matrix $\begin{bmatrix} 2 & -2 & 1 & x_1 \\ -3 & 2 & 0 & x_2 \end{bmatrix}$. The reduction reveals that the system is consistent for every $\mathbf{x}$, so $\mathrm{Sp}(S) = R^2$.

10. Since $k_1\mathbf{a} + k_2\mathbf{b} + k_3\mathbf{e} = k_1\mathbf{a} + k_2\mathbf{b}$, $\mathrm{Sp}(S) = \mathrm{Sp}\{\mathbf{a}, \mathbf{b}\} = \mathbf{R^2}$.

11. Since $k_1\mathbf{a} + k_2\mathbf{c} + k_3\mathbf{e} = k_1\mathbf{a} + k_2\mathbf{c}$, $\mathrm{Sp}(S) = \mathrm{Sp}\{\mathbf{a}, \mathbf{c}\}$. But $\mathrm{Sp}\{\mathbf{a}, \mathbf{c}\} = \{\mathbf{x} : x_1 + x_2 = 0\}$. Thus $\mathrm{Sp}(S)$ is the line with equation $x + y = 0$. (cf. Exercise 6).

12. $\mathrm{Sp}(S) = \{\mathbf{x} : \mathbf{x} = t\mathbf{v}, t$ a scalar$\}$ so $\mathrm{Sp}(S)$ is the line through $(0, 0, 0)$ and $(1, 2, 0)$. The parametric equations for the line are $x = t$, $y = 2t$, $z = 0$. Equivalently, if $\mathbf{x} = [x_1, x_2, x_3]^{\mathrm{T}}$, then $\mathrm{Sp}(S) = \{\mathbf{x} : -2x_1 + x_2 = 0$ and $x_3 = 0\}$. Thus $\mathrm{Sp}(S)$ the line formed by the intersecting planes $-2x + y = 0$ and $z = 0$.

13. $\mathrm{Sp}(S) = \{\,\mathbf{x}$ in $R^3 : \mathbf{x} = t\mathbf{w}$ for some scalar $t\,\}$.Therefore $\mathrm{Sp}(S)$ is the line through $(0,0,0)$ and $(0,-1,1)$. The parametric equations are $x = 0$, $y = -t$, $z = t$. If $\mathbf{x} = [x_1, x_2, x_3\ \ ]^{\mathrm{T}}$ then $t\mathbf{w} = \mathbf{x}$ has augmented matrix $\begin{bmatrix} 0 & x_1 \\ -1 & x_2 \\ 1 & x_3 \end{bmatrix}$ which reduces to $\begin{bmatrix} 1 & x_3 \\ 0 & x_2 + x_3 \\ 0 & x_1 \end{bmatrix}$.

The system is consistent if $x_2 + x_3 = 0$ and $x_1 = 0$ so we have $\mathrm{Sp}(S) = \{\,\mathbf{x} : x_2 + x_3 = 0$ and $x_1 = 0\}$. Therefore $\mathrm{Sp}(S)$ is the line formed by the the intersecting planes $y + z = 0$ and $x = 0$.

14. For $\mathbf{x} = [x_1, x_2, x_3\ \ ]^{\mathrm{T}}$ the equation $k_1\mathbf{v} + k_2\mathbf{w} = \mathbf{x}$ is consistent if and only if $-2x_1 + x_2 + x_3 = 0$. Thus $\mathrm{Sp}(S) = \{\,\mathbf{x} : -2x_1 + x_2 + x_3 = 0\}$ and is the plane with equation $-2x + y + z = 0$.

15. $\mathrm{Sp}(S) = \{\mathbf{u}$ in $R^3 : \mathbf{u} = k_1\mathbf{v} + k_2\mathbf{x}\}$. The equation $k_1\mathbf{v} + k_2\mathbf{x} = \mathbf{u}$ has augmented matrix $\begin{bmatrix} 1 & 1 & u_1 \\ 2 & 1 & u_2 \\ 0 & -1 & u_3 \end{bmatrix}$, which reduces to

$\begin{bmatrix} 1 & 1 & u_1 \\ 0 & 1 & 2u_1 - u_2 \\ 0 & 0 & 2u_1 - u_2 + u_3 \end{bmatrix}$. A solution exists if and only if $2u_1 - u_2 + u_3 = 0$, so $\mathrm{Sp}(S) = \{\,\mathbf{u}$: $2u_1 - u_2 + u_3 = 0\}$. $\mathrm{Sp}(S)$ is the plane with equation $2x - y + z = 0$.

16. For arbitrary $\mathbf{u}$ in R^3 the equation $k_1\mathbf{v} + k_2\mathbf{w} + k_3\mathbf{x} = \mathbf{u}$ has a solution, so $\mathrm{Sp}(S) = R^3$.

17. $\mathrm{Sp}(S) = \{\,\mathbf{u}$ in $R^3 : \mathbf{u} = k_1\mathbf{w} + k_2\mathbf{x} + k_3\mathbf{z}\}$. For $\mathbf{u} = [u_1, u_2, u_3]^{\mathrm{T}}$ the equation $k_1\mathbf{w} + k_2\mathbf{x} + k_3\mathbf{z} = \mathbf{u}$ has augmented matrix

$\begin{bmatrix} 0 & 1 & 1 & u_1 \\ -1 & 1 & 0 & u_2 \\ 1 & -1 & 2 & u_3 \end{bmatrix}$. This matrix reduces to

$\begin{bmatrix} 1 & -1 & 2 & u_3 \\ 0 & 1 & 1 & u_1 \\ 0 & 0 & 2 & u_2 + u_3 \end{bmatrix}$. Since the system is consistent for every $\mathbf{u}$ in R^3, $\mathrm{Sp}(S) = R^3$.

18. For $\mathbf{u} = [u_1, u_2, u_3]^{\mathrm{T}}$ the system of equations $k_1\mathbf{v} + k_2\mathbf{w} + k_3\mathbf{z} = \mathbf{u}$ is consistent if and only if $-2u_1 + u_2 + u_3 = 0$. Therefore $\mathrm{Sp}(S) =$
$\{\,\mathbf{u} : -\mathbf{2}u_1 + u_2 + u_3 = 0\}$ and $\mathrm{Sp}(S)$ is the plane with equation $-2x + y + z = 0$.

19. The matrix $\begin{bmatrix} 0 & 1 & -2 & u_1 \\ -1 & 1 & -2 & u_2 \\ 1 & -1 & 2 & u_3 \end{bmatrix}$ reduces to

$\begin{bmatrix} 1 & -1 & 2 & u_3 \\ 0 & 1 & -2 & u_1 \\ 0 & 0 & 0 & u_2+u_3 \end{bmatrix}$ so the system of equations $k_1\mathbf{w}+k_2\mathbf{x}+k_3\mathbf{y}=\mathbf{u}$ is consistent if and only if $u_2+u_3=0$. Therefore $\mathrm{Sp}(S)=\{\mathbf{u} : u_2+u_3=0\}$ and $\mathrm{Sp}(S)$ is the plane with equation $y+z=0$.

20. By Exercise 14, $\mathrm{Sp}(S)=\{\mathbf{x} : -2x_1+x_2+x_3=0\}$. Then the vectors given in (a), (c) and (d) are in $\mathrm{Sp}(S)$. Moreover when the system of equations $k_1\mathbf{v}+k_2\mathbf{w}=\mathbf{x}$ is consistent, the unique solution is $k_1=x_1$, and$k_2=x_3$. Thus in (a) $[1,1,1]^{\mathrm{T}}=\mathbf{v}+\mathbf{w}$; in (c) $[1,2,0]^{\mathrm{T}}=\mathbf{v}$; and in (d), $[2,3,1]^{\mathrm{T}}=2\mathbf{v}+\mathbf{w}$.

21. By Exercise 15, $\mathrm{Sp}(S)=\{\mathbf{u}$ in $R^3 : 2u_1-u_2+u_3=0\}$. Thus the vectors given in (b), (c), and (e) are in $\mathrm{Sp}(S)$. From the calculations done in Exercise 15, it follows that when the system of equations $k_1\mathbf{v}+k_2\mathbf{x}=\mathbf{u}$ is consistent, the unique solution is $k_1=-u_1+u_2$ and $k_2=2u_1-u_2$. Thus in (b), $[1,1,-1]^{\mathrm{T}}=\mathbf{x}$; in (c), $[1,2,0]^{\mathrm{T}}=\mathbf{v}$; and in (e), $[-1,2,4]^{\mathrm{T}}= 3\mathbf{v} - 4\mathbf{x}$.

22. The vectors $\mathbf{a},\mathbf{c}$, and $\mathbf{e}$ are in $\mathcal{N}(A)$.

23. The vectors $\mathbf{d}$ and $\mathbf{e}$ are in $\mathcal{N}(A)$ since by direct calculation, $A\mathbf{d}=\theta$ and $A\mathbf{e}=\theta$.

24. The vectors $\mathbf{v},\mathbf{w}$, and $\mathbf{z}$ are in $\mathcal{N}(A)$.

25. The vectors $\mathbf{x}$ and $\mathbf{y}$ are in $\mathcal{N}(A)$ since, by direct calculation $A\mathbf{x}=\theta$ and $A\mathbf{y}=\theta$.

26. The homogeneous system $A\mathbf{x}=\theta$ has solution $x_1=2x_2$, where x_2 is arbitrary. Thus $\mathcal{N}(A)$=$\{\mathbf{x}$ in $R^2 : x_1-2x_2=0\}$. If $\mathbf{b}=\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, then $A\mathbf{x}$=$\mathbf{b}$ is consistent if and only if $3b_1+b_2=0$, so $\mathcal{R}(A)=\{\mathbf{b}$ in $R^2 : 3b_1+b_2=0\}$.

27. The matrix $[A \mid \mathbf{b}]$ is row equivalent to the matrix :

$\begin{bmatrix} -1 & 3 & b_1 \\ 0 & 0 & 2b_1+b_2 \end{bmatrix}$.

It follows that the homogeneous system $A\mathbf{x}=\theta$ has solution $x_1=3x_2$ whereas the system $A\mathbf{x}=\mathbf{b}$ is consistent if and only if $2b_1+b_2=0$. Therefore $\mathcal{N}(A)=\{\mathbf{x} : x_1-3x_2=0\}$ and $\mathcal{R}(A)=$

$\{\mathbf{b} : 2b_1+b_2=0\}$.

28. $\mathcal{N}(A)=\{\theta\}$ and $\mathcal{R}(A)=R^2$.

29. The matrix $[A \mid \mathbf{b}]$ reduces to $\begin{bmatrix} 1 & 1 & b_1 \\ 0 & 3 & -2b_1+b_3 \end{bmatrix}$. Setting $\mathbf{b}$=θ and solving yields $x_1=x_2=0$, so $\mathcal{N}(A)$=$\{\theta\}$. Since the system $A\mathbf{x}$= $\mathbf{b}$ is consistent for every $\mathbf{b}$ in $R^2, \mathcal{R}(A)=R^2$.

30. $\mathcal{N}(A) = \{\mathbf{x}$ in $R^3 : x_1 = -3x_3$ and $x_2 = -x_3\}$ and $\mathcal{R}(A) = R^2$.

31. The matrix $[A \mid \mathbf{b}]$ reduces to $\begin{bmatrix} 1 & 2 & 1 & b_1 \\ 0 & 0 & 1 & -3b_1 + b_2 \end{bmatrix}$. Setting $\mathbf{b} = \theta$ and backsolving yields $x_1 = -2x_2$, $x_3 = 0$ as the solution to $A\mathbf{x} = \theta$. Thus $\mathcal{N}(A)=\{\mathbf{x}$ in $R^3 : x_1 + 2x_2 = 0$ and $x_3 = 0\}$. Since $A\mathbf{x}=\mathbf{b}$ is consistent for arbitrary $\mathbf{b}$ in R^2, $\mathcal{R}(A)=R^2$.

32. The homogeneous system $A\mathbf{x} = \theta$ has only the trivial solution so $\mathcal{N}(A) = \{\theta\}$. The system $A\mathbf{x} = \mathbf{b}$ is consistent precisely when $3b_1 - 2b_2 + b_3 = 0$ so $\mathcal{R}(A) = \{\mathbf{b}$ in $R^3 : 3b_1 - 2b_2 + b_3 = 0\}$.

33. The matrix $[A \mid \mathbf{b}]$ reduces to $\begin{bmatrix} 0 & 1 & b_1 \\ 0 & 0 & -2b_1 + b_2 \\ 0 & 0 & -3b_1 + b_3 \end{bmatrix}$. Setting $\mathbf{b} = \theta$

yields $x_2 = 0$, x_1 arbitrary as the solution to $A\mathbf{x} = \theta$. Thus $\mathcal{N}(\mathcal{A}) = \{\mathbf{x}$ in $R^2 : x_2 = 0\}$. The system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $-2b_1 + b_2 = 0$ and $-3b_1 + b_3 = 0$. Therefore $\mathcal{R}(A) =$

$\{\mathbf{b}$ in $R^3 : b_2 = 2b_1$ and $b_3 = 3b_1\}$.

34. $\mathcal{N}(A) = \{\mathbf{x}$ in $R^3 : x_1 = -7x_3$ and $x_2 = -3x_3\}$ and $\mathcal{R}(A) = \{\mathbf{b}$ in $R^3 : 3b_1 - 2b_2 + b_3 = 0\}$.

35. The matrix $[A \mid \mathbf{b}]$ reduces to $\begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & 1 & -2 & -b_1 + b_2 \\ 0 & 0 & 0 & -4b_1 + 2b_2 + b_3 \end{bmatrix}$.

Setting $\mathbf{b} = \theta$ and backsolving the reduced system yields $\mathcal{N}(A) = \{\mathbf{x}$ in $R^3 : x_1 = -7x_3$ and $x_2 = 2x_3\}$. The system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $-4b_1 + 2b_2 + b_3 = 0$ so $\mathcal{R}(A) = \{\mathbf{b}$ in $R^3 : -4b_1 + 2b_2 + b_3 = 0\}$.

36. $\mathcal{N}(A) = \{\theta\}$ and $\mathcal{R}(A) = R^3$.

37. The matrix $[A \mid \mathbf{b}]$ reduces to $\begin{bmatrix} 1 & 2 & 1 & b_1 \\ 0 & 1 & 2 & -2b_1 + b_2 \\ 0 & 0 & 1 & b_1 - b_2 + b_3 \end{bmatrix}$. Setting $\mathbf{b} = \theta$ and solving yields $\mathcal{N}(A) = \{\theta\}$. The system $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b}$ so $\mathcal{R}(A) = R^3$.

38. (a) The vectors $\mathbf{b}$ in (i), (iv), and (vi) are in $\mathcal{R}(A)$.

(b) For (i), $\mathbf{x} = [1, 0]^{\mathrm{T}}$ is one choice; for (iv), $\mathbf{x} = [-2, 0]^{\mathrm{T}}$ is one choice; for (vi), $\mathbf{x} = [0, 0]^T$ is one choice.

(c) For (i), $\mathbf{b} = \mathbf{A_1}$; for (iv), $\mathbf{b} = -2\mathbf{A_1}$, for (vi), $\mathbf{b} = 0\mathbf{A_1} + 0\mathbf{A_2}$.

39. (a) From the description of $\mathcal{R}(A)$ obtained in Exercise 27 it follows that the vectors $\mathbf{b}$ in (ii),(v), and (vi) are in $\mathcal{R}(A)$.

(b) When the system of equations $A\mathbf{x} = \mathbf{b}$ is consistent, the calculations done in Exercise 27 show that the solution is given by $x_1 = -b_1 + 3x_2$, where x_2 is arbitrary. Thus for (ii), $\mathbf{x} = [1, 0]^{\mathrm{T}}$ is one choice; for (v), $\mathbf{x} = [0, 1]^{\mathrm{T}}$ is one choice; for (vi), $\mathbf{x} = [0, 0]^T$ is one choice.

(c) If $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x} = [x_1, x_2]^{\mathrm{T}}$, then $\mathbf{b} = x_1\mathbf{A_1} + x_2\mathbf{A_2}$. Therefore for (ii), $\mathbf{b} = \mathbf{A_1}$; for (v), $\mathbf{b} = \mathbf{A_2}$; for (vi), $\mathbf{b} = 0\mathbf{A}_1 + 0\mathbf{A}_2$.

40. (a) The vectors $\mathbf{b}$ in (ii), (iii), (iv), and (vi) are in $\mathcal{R}(A)$.

(b) For (ii), $\mathbf{x} = [-1, -1, 0]^{\mathrm{T}}$ is one choice; for (iii), $\mathbf{x} = [2, -1, 0]^{\mathrm{T}}$ is one choice; for (iv), $\mathbf{x} = [2, 1, 0]^T$ is one choice; for (vi), $\mathbf{x} = [0, 0, 0]^T$ is one choice.

(c) For (ii), $\mathbf{b} = -\mathbf{A_1} - \mathbf{A_2}$; for (iii), $\mathbf{b} = 2\mathbf{A_1} - \mathbf{A_2}$; for (iv), $\mathbf{x} = 2\mathbf{A_1} + \mathbf{A_2}$; for (vi), $\mathbf{x} = 0\mathbf{A_1} + 0\mathbf{A_2} + 0\mathbf{A_3}$.

41. (a) From the description of $\mathcal{R}(A)$ obtained in Exercise 35, the vectors $\mathbf{b}$ in (i), (iii), (v), and (vi), are in $\mathcal{R}(A)$.

(b) When the system $A\mathbf{x} = \mathbf{b}$ is consistent, the solution is given by $x_1 = 3b_1 - 2b_2 - 7x_3$ and $x_2 = -b_1 + b_2 + 2x_3$, where x_3 is arbitrary. Thus for (i), $\mathbf{x} = [-1, 1, 0]^{\mathrm{T}}$ is one choice; for (iii), $\mathbf{x} = [-2, 3, 0]^{\mathrm{T}}$ is one choice for (v), $\mathbf{x} = [-2, 1, 0]^{\mathrm{T}}$ is one choice; for (vi), $\mathbf{x} = [0, 0, 0]^{\mathrm{T}}$ is one choice.

(c). If $A\mathbf{x} = \mathbf{b}$, where x= $[x_1, x_2, x_3]^{\mathrm{T}}$, then $\mathbf{b} = x_1\mathbf{A}_1 + x_2\mathbf{A}_2 + x_3\mathbf{A}_3$.
Thus it follows from (b) that for (i), $\mathbf{b} = -\mathbf{A_1} + \mathbf{A_2}$; for (iii), $\mathbf{b} = -2\mathbf{A_1} + 3\mathbf{A_2}$; for (v), $\mathbf{b} = -2\mathbf{A_1} + \mathbf{A_2}$; for (vi), $\mathbf{b} = 0\mathbf{A_1} + 0\mathbf{A_2} + 0\mathbf{A_3}$,

42. $A = \begin{bmatrix} 2 & -3 & 1 \\ -1 & 4 & -2 \\ 2 & 1 & 4 \end{bmatrix}$

43. $A = [\ 3 \quad -4 \quad 2\ ]$

44. $A = [\mathbf{v}, \mathbf{w}, \mathbf{x}] = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

45. $A = [\mathbf{w}, \mathbf{x}, \mathbf{z}] = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$

46. Let A be the (3 x 3) matrix whose columns are the vectors given in S. Then A^{T} reduces to $B^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$. The nonzero columns of B, $\mathbf{w_1} = [1, 0, -1]^{\mathrm{T}}$, and $\mathbf{w_2} = [0, 2, 3]^{\mathrm{T}}$, form a basis for Sp($S$).

47. Let A be the (3 x 3) matrix whose columns are the vectors given in S. Then A^{T} reduces to $B^{\mathrm{T}} = \begin{bmatrix} -2 & 1 & 3 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. The nonzero columns of B, $\mathbf{w_1} = [-2,1,3]^{\mathrm{T}}$ and $\mathbf{w_2} = [0,3,2]^{\mathrm{T}}$ form a basis for $\mathrm{Sp}(S)$.

48. $\mathbf{w_1} = [1,0,1]^T$ and $\mathbf{w_2} = [0,1,1]^T$.

49. Let A be the (3 x 4) matrix whose columns are the vectors given in S. Then A^{T} reduces to $B^{\mathrm{T}} = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The nonzero columns of B, $\mathbf{w_1} = [1,2,2]^{\mathrm{T}}$ and $\mathbf{w_2} = [0,3,1]^{\mathrm{T}}$ form a basis for $\mathrm{Sp}(S)$.

50. (a) $\mathcal{R}(I) = R^n$ and $\mathcal{N}(I) = \{\theta\}$.

(b) $\mathcal{R}(\mathcal{O}) = \{\theta\}$ and $\mathcal{N}(\mathcal{O}) = R^n$.

(c) $\mathcal{R}(A) = R^n$ and $\mathcal{N}(A) = \{\theta\}$.

51. Let $\mathbf{x}$ be in $\mathcal{N}(A) \cap \mathcal{N}(B)$. Then $A\mathbf{x} = \theta$ and $B\mathbf{x} = \theta$. Therefore $(A+B)\mathbf{x} = A\mathbf{x} + B\mathbf{x} = \theta + \theta \doteq \theta$. It follows that $\mathbf{x}$ is in $\mathcal{N}(A+B)$.

52. (a) If $B\mathbf{x} = \theta$ then $(AB)\mathbf{x} = A(B\mathbf{x}) = A\theta \doteq \theta$.

(b) Suppose $\mathbf{b} = (AB)\mathbf{x}$ for some vector $\mathbf{x}$ in R^n. Then $\mathbf{y} = B\mathbf{x}$ is in R^r and $\mathbf{b} = A(B\mathbf{x}) = A\mathbf{y}$.

53. Let θ_m and θ_n denote the zero vectors in R^m and R^n respectively. Then θ_n is in W and $\theta_m = A\theta_n$. Therefore θ_m is in V. Suppose $\mathbf{u}$ and $\mathbf{v}$ are in V. Then there exist vectors $\mathbf{x}$ and $\mathbf{z}$ in W such that $\mathbf{u} = A\mathbf{x}$ and $\mathbf{v} = A\mathbf{z}$. Since $\mathbf{x}+\mathbf{z}$ is in W, $\mathbf{u}+\mathbf{v} = A\mathbf{x} + A\mathbf{z} = A(\mathbf{x}+\mathbf{z})$ is in V. If a is a scalar then $a\mathbf{x}$ is in W so $a\mathbf{u} = a(A\mathbf{x}) = A(a\mathbf{x})$ is in V. Thus V is a subspace of R^n.

54. If A has row vectors $\mathbf{a_1}, \ldots, \mathbf{a_k}, \ldots, \mathbf{a_m}$ then B has row vectors $\mathbf{a_1}, \ldots, c\mathbf{a_k}, \ldots, \mathbf{a_m}$. Note that $d_1\mathbf{a_1} + \cdots + d_k\mathbf{a_k} + \cdots + d_m\mathbf{a_m} = d_1\mathbf{a_1} + \cdots + (d_k/c)c\mathbf{a_k} + \cdots + d_m\mathbf{a_m}$. It follows that :

$$\mathrm{Sp}\{\mathbf{a_1}, \ldots, \mathbf{a_k}, \ldots, \mathbf{a_m}\} = \mathrm{Sp}\{\mathbf{a_1}, \ldots, c\mathbf{a_k}, \ldots, \mathbf{a_m}\}.$$

3.4 Bases for Subspaces

1. Backsolving the given system yields $x_1 = x_3 - x_4$, and $x_2 = x_4$. Thus

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 - x_4 \\ x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

As in Example 5,$\{[1,0,1,0]^{\mathrm{T}}, [-1,1,0,1]^{\mathrm{T}}\}$ is a basis for W.

2. Backsolving yields $x_1 = -x_3 - 2x_4$ and $x_2 = 2x_3 + x_4$.
It follows that $\{[-1,2,1,0]^{\mathrm{T}}, [-2,1,0,1]^{\mathrm{T}}\}$ is a basis for W.

3. Writing $x_1 = x_2 - x_3 + 3x_4$ we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 - x_3 + 3x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus $\{[1,1,0,0]^{\mathrm{T}}, [-1,0,1,0]^{\mathrm{T}}, [3,0,0,1]^{\mathrm{T}}\}$ is the desired basis.

4. Writing $x_1 = x_2 - x_3$ and noting that x_1, x_3 and x_4 are unconstrained variables, we obain $\{[1,1,0,0]^{\mathrm{T}}, [-1,0,1,0]^{\mathrm{T}}, [0,0,0,1]^{\mathrm{T}}\}$ as the desired basis.

5. Since $x_1 = -x_2$ we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

It follows that $\{[-1,1,0,0]^{\mathrm{T}}, [0,0,1,0]^{\mathrm{T}}, [0,0,0,1]^{\mathrm{T}}\}$ is a basis for W.

6. Backsolving yields $x_1 = 2x_4, x_2 = 2x_4, x_3 = x_4$. Thus $\{[2,2,1,1]^{\mathrm{T}}\}$ is a basis for W.

7. Backsolving yields $x_1 = -2x_3 - x_4$ and $x_2 = -x_3$. Thus

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_3 - x_4 \\ -x_3 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore $\{[-2,-1,1,0]^{\mathrm{T}}, [-1,0,0,1]^{\mathrm{T}}\}$ is a basis for W.

8. Backsolving yields $x_1 = -x_4$ and $x_2 = -x_3$. Therefore the set $\{[-1,0,0,1]^{\mathrm{T}}, [0,-1,1,0]^{\mathrm{T}}\}$ is a basis for W.

9. Let $\{\mathbf{w}_1, \mathbf{w}_2\}$ be the basis found in Exercise 1. (a) $\mathbf{x} = 2\mathbf{w}_1 + \mathbf{w}_2$ (b) $\mathbf{x}$ is not in W. (c) $\mathbf{x} = -3\mathbf{w}_2$ (d) $\mathbf{x} = 2\mathbf{w}_1$.

10. Let $\{\mathbf{w}_1, \mathbf{w}_2\}$ be the basis found in Exercise 2. (a) $\mathbf{x} = \mathbf{w}_1 + \mathbf{w}_2$ (b) $\mathbf{x} = 2\mathbf{w}_1 - \mathbf{w}_2$ (c) $\mathbf{x}$ is not in W. (d) $\mathbf{x} = -2\mathbf{w}_2$.

11. (a) $B = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(b) Backsolving the reduced system $B\mathbf{x} = \theta$ yields the solution $x_1 = -x_3 - x_4, x_2 = -x_3 + x_4$ for the homogeneous system $A\mathbf{x} = \theta$. Thus $\mathbf{x} = [x_1, x_2, x_3, x_4]^{\mathrm{T}}$ is in $\mathcal{N}(A)$ if and only if

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 - x_4 \\ -x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

It follows that $\{[-1,-1,1,0]^T, [-1,1,0,1]^T\}$ is a basis for $\mathcal{N}(A)$.

(c) It follows from (b) that $x_1\mathbf{A_1} + x_2\mathbf{A_2} + x_3\mathbf{A_3} + x_4\mathbf{A_4} = \theta$ if and only if $x_1 = -x_3 - x_4$ and $x_2 = -x_3 + x_4$. Since x_3 and x_4 are unconstrained variables $\{\mathbf{A}_1, \mathbf{A}_2\}$ is a basis for $\mathcal{R}(A)$. Setting $x_3 = 1$ and $x_4 = 0$ yields $x_1 = -1$ and $x_2 = -1$ so $-\mathbf{A_1} - \mathbf{A_2} + \mathbf{A_3} = \theta$. Therefore $\mathbf{A_3} = \mathbf{A_1} + \mathbf{A_2}$. Similarly, setting $x_3 = 0$ and $x_4 = 1$ yields $\mathbf{A_4} = \mathbf{A_1} - \mathbf{A_2}$.

(d) The nonzero rows of B form a basis for the row space of A; that is $\{[1,2,3,-1], [0,-1,-1,1]\}$ is the desired basis.

12. (a) $B = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

(b) The system $A\mathbf{x} = \theta$ has solution $x_1 = -x_3$ and $x_2 = -x_3$. Therefore $\{[-1,-1,1]^{\mathrm{T}}\}$ is a basis for $\mathcal{N}(A)$.

(c) $\{\mathbf{A_1}, \mathbf{A_2}\}$ is a basis for $\mathcal{R}(A)$ and $\mathbf{A_3} = \mathbf{A_1} + \mathbf{A_2}$.

(d) $\{[1,1,2], [0,1,1]\}$ is a basis for the row space of A.

13. (a) $B = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(b) The homogeneous system $A\mathbf{x}=\theta$ has solution $x_1 = x_3 - 2x_4, x_2 = -x_3 + x_4$. Thus $\mathbf{x}=[x_1, x_2, x_3, x_4]^{\mathrm{T}}$ is in $\mathcal{N}(A)$ if and only if

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 - 2x_4 \\ -x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

The set $\{[1,-1,1,0]^{\mathrm{T}}, [-2,1,0,1]^{\mathrm{T}}\}$ is a basis for $\mathcal{N}(A)$.

(c) It follows from (b) that in the equation $x_1\mathbf{A_1} + x_2\mathbf{A_2} + x_3\mathbf{A_3} + x_4\mathbf{A_4} = \theta$, x_3 and x_4 are unconstrained variables. Therefore $\{\mathbf{A_1}, \mathbf{A_2}\}$ is a basis for $\mathcal{R}(A)$. Furthermore $\mathbf{A_1} - \mathbf{A_2} + \mathbf{A_3} = \theta$, so $\mathbf{A_3} = -\mathbf{A_1} + \mathbf{A_2}$. Likewise $-2\mathbf{A_1} + \mathbf{A_2} + \mathbf{A_4} = \theta$, so $\mathbf{A_4} = 2\mathbf{A_1} - \mathbf{A_2}$.

(d) The nonzero rows of B, $[1,2,1,0], [0,1,1,-1]$, form a basis for the row space of A.

14. (a) $B = \begin{bmatrix} 2 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) The system $A\mathbf{x}=\theta$ has only the trivial solution so $\mathcal{N}(A) = \{\theta\}$.

(c) It follows from (b) that the columns of A are linearly independent so $\{\mathbf{A_1}, \mathbf{A_2}, \mathbf{A_3}\}$ is a basis for $\mathcal{R}(A)$.

(d) The set $\{[2,2,0], [0,-1,1], [0,0,1]\}$ is a basis for the row space of A.

15. (a) $B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$.

(b) The system $A\mathbf{x}=\theta$ has solution $x_1 = -2x_2, x_3 = 0$.
Thus $\mathcal{N}(A) = \{\mathbf{x} : \mathbf{x} = [-2x_2, x_2, 0]^{\mathrm{T}}\}$ and $\{[-2,1,0]^{\mathrm{T}}\}$ is a basis for $\mathcal{N}(A)$.

(c) In the equation $x_1\mathbf{A_1} + x_2\mathbf{A_2} + x_3\mathbf{A_3} = \theta$, x_2 is an unconstrained variable, so $\{\mathbf{A_1}, \mathbf{A_3}\}$ is a basis for $\mathcal{R}(A)$.
Furthermore, $-2\mathbf{A_1} + \mathbf{A_2} = \theta$, so $\mathbf{A_2} = 2\mathbf{A_1}$.

(d) $\{[1,2,1], [0,0,-1]\}$ is a basis for the row space of A.

16. (a) $B = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$.

(b) $\mathbf{x} = [x_1, x_2, x_3]^{\mathrm{T}}$ is in $\mathcal{N}(A)$ if and only if $x_1 = (-3/2)x_3$ and $x_2 = x_3$. Therefore $\{[-3/2, 1, 1]^{\mathrm{T}}\}$ is a basis for $\mathcal{N}(A)$.

(c) $\{\mathbf{A_1}, \mathbf{A_2}\}$ is a basis for $\mathcal{R}(A)$ and $\mathbf{A_3} = (3/2)\mathbf{A_1} - \mathbf{A_2}$.

(d) $\{[2,1,2],[0,1,-1]\}$ is a basis for the row space of A.

17. The matrix A^{T} is row equivalent to $B^{\mathrm{T}} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The desired basis is $\{[1,3,1]^{\mathrm{T}}, [0,-1,-1]^{\mathrm{T}}\}$, formed by taking the nonzero columns of B.

18. The matrix A^{T} is row equivalent to $B^{\mathrm{T}} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. The desired basis is $\{[1,1,2]^{\mathrm{T}}, [0,0,1]^{\mathrm{T}}\}$, formed by taking the nonzero columns of B.

19. The matrix A^{T} is row equivalent to $B^{\mathrm{T}} = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so $\{[1,2,2,0]^{\mathrm{T}}, [0,1,-2,1]^{\mathrm{T}}\}$ is a basis for $\mathcal{R}(A)$.

20. The matrix A^{T} is row equivalent to $B^{\mathrm{T}} = \begin{bmatrix} 2 & 2 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ so $\{[2,2,2]^{\mathrm{T}}, [0,-1,1]^{\mathrm{T}}, [0,0,1]^{\mathrm{T}}\}$ is a basis for $\mathcal{R}(A)$.

21. (a) For the given vectors $\mathbf{u_1}$ and $\mathbf{u_2}$ the equation $x_1\mathbf{u_1} + x_2\mathbf{u_2} = \theta$ has solution $x_1 = -2x_2$ where x_2 is an unconstrained variable. Therefore $\{\mathbf{u_1}\}$ is a basis for $\mathrm{Sp}(S)$, where $\mathbf{u_1} = [1,2]^{\mathrm{T}}$.

(b) If $A = [\mathbf{u}_1, \mathbf{u}_2]$ then A^T is row equivalent to $B^T = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$. Therefore $\{[1,2]^T\}$ is a basis for $\mathrm{Sp}\,(S)$.

22. (a) For the given vectors $\mathbf{u_1}, \mathbf{u_2}$ and $\mathbf{u_3}$ the equation $x_1\mathbf{u_1} + x_2\mathbf{u_2} + x_3\mathbf{u_3} = \theta$ has solution $x_1 = (-1/3)x_3$ and $x_2 = (-4/3)x_3$, where x_3 is arbitrary. Thus $\{\mathbf{u_1}, \mathbf{u_2}\}$ is a basis for $\mathrm{Sp}(S)$, where $\mathbf{u_1} = [1,2]^{\mathrm{T}}$ and $\mathbf{u_2} = [2,1]^{\mathrm{T}}$.

(b) If $A = [\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}]$ then A^{T} is row equivalent to $B^{\mathrm{T}} = \begin{bmatrix} 1 & 2 \\ 0 & -3 \\ 0 & 0 \end{bmatrix}$. Therefore $\{[1,2]^{\mathrm{T}}, [0,-3]^{\mathrm{T}}\}$ is a basis for $\mathrm{Sp}(S)$.

23. (a) For the given vectors $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}$ the equation $x_1\mathbf{u_1} + x_2\mathbf{u_2} + x_3\mathbf{u_3} + x_4\mathbf{u_4} = \theta$ has solution $x_1 = -x_3 - 3x_4, x_2 = -x_3 + x_4$. Since x_3 and x_4 are unconstrained variables, $\{\mathbf{u_1}, \mathbf{u_2}\}$ is a basis for $\mathrm{Sp}(S)$, where $\mathbf{u_1} = [1,2,1]^{\mathrm{T}}$ and $\mathbf{u_2} = [2,5,0]^{\mathrm{T}}$.

(b) If $A = [\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}]$ then A^{T} is row equivalent to $B^{\mathrm{T}} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Therefore $\{[1,2,1]^{\mathrm{T}}, [0,1,-2]^{\mathrm{T}}\}$ is a basis for $\mathrm{Sp}(S)$.

24. (a) For the given vectors $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}$, in the equation $x_1\mathbf{u_1} + x_2\mathbf{u_2} + x_3\mathbf{u_3} + x_4\mathbf{u_4} = \theta$, x_4 is an unconstrained variable. The desired basis is $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$, where $\mathbf{u_1} = [1,2,-1,3]^{\mathrm{T}}, \mathbf{u_2} = [-2,1,2,-1]^{\mathrm{T}}$, and $\mathbf{u_3} = [-1,-1,1,-3]^{\mathrm{T}}$.

(b) If $A = [\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}]$, then A^{T} reduces to $B^{\mathrm{T}} = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Therefore $\{[1,2,-1,3]^{\mathrm{T}}, [0,1,0,0]^{\mathrm{T}}, [0,0,0,5]^{\mathrm{T}}\}$ is a basis for $\mathrm{Sp}(S)$.

25. (a) Let A denote the given matrix. The homogeneous system $A\mathbf{x} = \theta$ has solution $x_1 = 0, x_2$ is arbitrary, $x_3 = 0$. Thus $\{[0,1,0]^{\mathrm{T}}\}$ is a basis for $\mathcal{N}(A)$.

(b) Let A denote the given matrix. The system $A\mathbf{x} = \theta$ has solution $x_1 = -x_2$, where x_2 and x_3 are arbitrary. Thus $\{[-1,1,0]^{\mathrm{T}}, [0,0,1]^{\mathrm{T}}\}$ is a basis for $\mathcal{N}(A)$.

(c) The system $A\mathbf{x} = \theta$ has solution $x_1 = -x_2, x_3 = 0$, where x_2 is arbitrary. The set $\{[-1,1,0]^{\mathrm{T}}\}$ is a basis for $\mathcal{N}(A)$.

26. (a) $\{[1,1]^{\mathrm{T}}, [0,1]^{\mathrm{T}}\}$. (b) $\{[1,1]^{\mathrm{T}}\}$. (c) $\{[1,1]^{\mathrm{T}}, [0,1]^{\mathrm{T}}\}$.

27. The equation $x_1\mathbf{v_1} + x_2\mathbf{v_2} + x_3\mathbf{v_3} = \theta$ has solution $x_1 = -2x_3$, $x_2 = -3x_3, x_3$ arbitrary. In particular, $x_1 = -2, x_2 = -3, x_3 = 1$ is a nontrivial solution and the set S is linearly dependent. Moreover, from $-2\mathbf{v_1} - 3\mathbf{v_2} + \mathbf{v_3} = \theta$ we obtain $\mathbf{v_3} = 2\mathbf{v_1} + 3\mathbf{v_2}$. If $\mathbf{v}$ is in $\mathrm{Sp}(S)$ then $\mathbf{v} = a_1\mathbf{v_1} + a_2\mathbf{v_2} + a_3\mathbf{v_3} = (a_1 + 2a_3)\mathbf{v_1} + (a_2 + 3a_3)\mathbf{v_2}$, so $\mathbf{v}$ is in $\mathrm{Sp}\{\mathbf{v_1}, \mathbf{v_2}\}$. It follows that $\mathrm{Sp}\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\} = \mathrm{Sp}\{\mathbf{v_1}, \mathbf{v_2}\}$.

28. The subsets $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_3\}$, $\{\mathbf{v}_2, \mathbf{v}_3\}$ are bases for R^2.

29. The subsets are $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, $\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4\}$, and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$. Note that $\mathbf{v}_4 = 3\mathbf{v}_2 - \mathbf{v}_3$.

30. By Theorem 12 of Section 1.8, the matrix $V = [\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}]$ is nonsingular. Thus, by Theorem 13 of Section 1.8, the system of equations $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in R^3; that is each vector $\mathbf{b}$ in R^3 can be written in the form $x_1\mathbf{v_1} + x_2\mathbf{v_2} + x_3\mathbf{v_3} = \mathbf{b}$. This shows that $R^3 = \mathrm{Sp}(B)$ so B is a basis for R^3.

31. Set $V = [\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}]$. By assumption the system $V\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in R^3. By Theorem 13 of Section 1.8, V is a nonsingular matrix. Therefore, by Theorem 12 of Section 1.8, the set $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is linearly independent.

32. The set S is linearly independent so, by Exercise 30, S is a basis for R^3.

33. The set S is linearly dependent so S is not a basis for R^3.

34. The set S is linearly dependent so S is not a basis for R^3.

35. If $\mathbf{u} = [u_1, u_2, u_3]^{\mathrm{T}}$ then $\mathbf{u}$ is in $\mathrm{Sp}(S)$ if and only if $4u_1 - 2u_2 + u_3 = 0$. In particular, $\mathrm{Sp}(S) \neq R^3$ and S is not a basis for R^3.

36. A vector $\mathbf{w} = [w_1, w_2, w_3]^{\mathrm{T}}$ is in $\mathrm{Sp}\{\mathbf{v_1}, \mathbf{v_2}\}$ if and only if $w_1 + w_3 = 0$. In particular $\mathbf{w} = [0, 0, 1]^{\mathrm{T}}$ is not a linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$.

37. (a) By Theorem 11 of Section 1.8, any set of three or more vectors in R^2 is linearly dependent and is not a basis for R^2.

    (b) Suppose $\{\mathbf{v}\}$ is a basis for R^2. Then $\mathbf{e_1} = a_1\mathbf{v}$ and $\mathbf{e_2} = a_2\mathbf{v}$ for some nonzero scalars a_1 and a_2. But then $a_2\mathbf{e_1} - a_1\mathbf{e_2} = \theta$, contradicting the fact that $\{\mathbf{e_1}, \mathbf{e_2}\}$ is a linearly independent set. We conclude that $\{\mathbf{v}\}$ is not a basis for R^2. It follows that every basis for R^2 contains exactly two vectors.

38. If $\mathbf{v}^{\mathrm{T}} = [x_1, x_2, \ldots, x_n]$ then the constraints $\mathbf{v}^{\mathrm{T}}\mathbf{u_i} = 0, 1 \leq i \leq p$, yield a homogeneous system of p equations in the unknowns $x_1, x_2, \ldots, x_n$. By Theorem 4 of Section 1.4 the system has nontrivial solutions.

    Suppose $\mathbf{v} = a_1\mathbf{u_1} + a_2\mathbf{u_2} + \cdots + a_p\mathbf{u_p}$. Then $\|\mathbf{v}\|^2 = \mathbf{v}^{\mathrm{T}}\mathbf{v} = \mathbf{v}^{\mathrm{T}}(a_1\mathbf{u_1} + a_2\mathbf{u_2} + \cdots + a_p\mathbf{u}_p) = a_1\mathbf{v}^{\mathrm{T}}\mathbf{u_1} + a_2\mathbf{v}^{\mathrm{T}}\mathbf{u_2} + \cdots + a_p\mathbf{v}^{\mathrm{T}}\mathbf{u_p} = 0$, contradicting that $\mathbf{v}$ is a nonzero vector.

39. By Theorem 11 of Section 1.8, any set of $n + 1$ or more vectors in R^n is linearly dependent so it is not a basis for R^n. By Exercise 38, any set of less than n vectors cannot span R^n. Therefore a basis for R^n must contain exactly n vectors.

3.5 Dimension

1. S contains only one vector and $\dim(R^2) = 2$, so by property 2 of Theorem 9, S does not span R^2.

2. S does not span R^2 by property 2 of Theorem 9

3. Since S contains three vectors and $\dim(R^2) = 2$, S is linearly dependent by property 1 of Theorem 9.

4. S is linearly dependent by property 1 of Theorem 9.

5. Since $\mathbf{u_4} = \theta$, S is a linearly dependent set; for example $0\mathbf{u_1} + a\mathbf{u_4} = \theta$ for any nonzero scalar a. Also S does not span R^2 since
$\mathrm{Sp}\{\mathbf{u_1}, \mathbf{u_4}\} = \mathrm{Sp}\{\mathbf{u_1}\}$.

6. S is linearly dependent since, for example, $3\mathbf{u_1} - \mathbf{u_2} = \theta$.

7. S contains two vectors and $\dim(R^3) = 3$ so by property 2 of Theorem 9, S does not span R^3.

8. S does not span R^3 by property 2 of Theorem 9.

9. Since S contains four vectors and $\dim(R^3) = 3$, S is linearly dependent by property 1 of Theorem 9.

10. It is easily checked that S is a linearly independent set. Therefore, by property 3 of Theorem 9, S is a basis for R^2.

11. It is easily checked that S is a linearly independent set. Since S
contains two vectors and $\dim(R^2) = 2$ it follows from property 3 of Theorem 9 that S is a basis for R^2.

12. The set S is linearly independent so, by property 3 of Theorem 9, S is a basis for R^3.

13. It is easily shown by direct calculation that S is a linearly dependent set. Therefore S is not a basis for R^3.

14. The set S is linearly independent so, by property 3 of Theorem 9, S is a basis for R^3.

15. If we write $x_1 = 2x_2 - x_3 + x_4$ then the procedure described in Example 5 of Section 2.4 yields a basis $\{[2,1,0,0]^{\mathrm{T}}, [-1,0,1,0]^{\mathrm{T}}, [1,0,0,1]^{\mathrm{T}}\}$ for W. It follows that $\dim(W) = 3$.

16. $\dim(W) = 3$.

17. Following the procedure used in Example 5 of Section 2.4, we obtain a basis $\{[1,-1,0,0]^{\mathrm{T}}, [2,0,-1,0]^{\mathrm{T}}\}$ for W. In particular $\dim(W) = 2$.

18. $\dim(W) = 2$.

19. The set $\{[-1,3,2,1]^{\mathrm{T}}\}$ is a basis for W, so $\dim(W) = 1$.

20. $\dim(W) = 1$.

21. The homogeneous system $A\mathbf{x} = \theta$ has solution $x_1 = -2x_2$.
Therefore $\{[-2,1]^{\mathrm{T}}\}$ is a basis for $\mathcal{N}(A)$ and $\mathrm{nullity}\,(A) = 1$. Since $2 = \mathrm{rank}\,(A) + \mathrm{nullity}\,(A)$, it follows that $\mathrm{rank}\,(A) = 1$.

22. The set $\{[2,1,1]^{\mathrm{T}}\}$ is a basis for $\mathcal{N}(A)$. Therefore nullity $(A) = 1$ and rank $(A) = 2$.

23. The homogeneous system $A\mathbf{x} = \theta$ has solution $x_1 = -5x_3, x_2 = -2x_3$. Thus $\{[-5,-2,1]^{\mathrm{T}}\}$ is a basis for $\mathcal{N}(A)$ and nullity $(A) = 1$. Since $3 = \text{rank}\,(A) + \text{nullity}\,(A)$, it follows that rank $(A) = 2$.

24. The set $\{[2,-1,1,0]^{\mathrm{T}}\}$ is a basis for $\mathcal{N}(A)$. Therefore nullity $(A) = 1$ and rank $(A) = 3$.

25. A^{T} reduces to $B^{\mathrm{T}} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$. It follows that $\{[1,-1,1]^{\mathrm{T}},$ $[0,2,3]^{\mathrm{T}}\}$ is a basis for $\mathcal{R}(A)$. Consequently rank $(A) = 2$. Since $3 = \text{rank}\,(A) + \text{nullity}\,(A)$, it follows that nullity $(A) = 1$.

26. The matrix A^{T} reduces to $B^{\mathrm{T}} = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Therefore $\{[1,2,2]^{\mathrm{T}}, [0,2,-1]^{\mathrm{T}}\}$ is a basis for $\mathcal{R}(A)$, rank $(A) = 2$ and nullity $(A) = 2$.

27. (a) Following the methods of Example 7 in Section 2.4, let $A = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 1 & -2 & 0 & -1 \\ -2 & 3 & -1 & 0 \end{bmatrix}$.
Then A^{T} reduces to $B^{\mathrm{T}} =$
$\begin{bmatrix} 1 & 1 & -2 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. It follows that $\{[1,1,-2]^{\mathrm{T}}, [0,-1,1]^{\mathrm{T}},$ $[0,0,1]^{\mathrm{T}}\}$ is a basis for W. In particular $\dim(W) = 3$.

(b) Following the procedure in (a), we obtain a basis $\{[1,2,-1,1]^{\mathrm{T}},$ $[0,1,-1,1]^{\mathrm{T}}, [0,0,-1,4]^{\mathrm{T}}\}$ for W. In particular, $\dim(W) = 3$.

28. $W = \{\mathbf{x}$ in $R^4 : x_1 + 2x_2 - 3x_3 - x_4 = 0\}$. It follows that $\dim(W) = 3$.

29. The constraints $\mathbf{a}^{\mathrm{T}}\mathbf{x} = 0, \mathbf{b}^{\mathrm{T}}\mathbf{x} = 0$ and $\mathbf{c}^{\mathrm{T}}\mathbf{x} = 0$ yield the homogeneous system of equations $x_1 - x_2 = 0$, $x_1 - x_3 = 0$, and $x_2 - x_3 = 0$. Solving we obtain $x_1 = x_3$ and $x_2 = x_3$ where x_3 and x_4 are arbitrary. Thus $\{[1,1,1,0]^{\mathrm{T}}, [0,0,0,1]^{\mathrm{T}}\}$ is a basis for W and $\dim(W) = 2$.

30. Following the procedure described in the hint, suppose we have obtained a linearly independent subset $S_k = \{\mathbf{w_1}, \ldots, \mathbf{w_k}\}$ of W. If S_k spans W we are done. If not there exists a vector $\mathbf{w_{k+1}}$ in W such that $\mathbf{w_{k+1}}$ is not in $\mathrm{Sp}(S_k)$. Suppose $a_1\mathbf{w_1} + \cdots + a_k\mathbf{w_k} + a_{k+1}\mathbf{w_{k+1}} = \theta$. Now $a_{k+1} = 0$ since otherwise we could solve for $\mathbf{w_{k+1}}$, contradicting that $\mathbf{w_{k+1}}$ is not in $\mathrm{Sp}(S_k)$. Since S_k is linearly independent, it follows

that $a_i = 0$, $1 \leq i \leq k$. This shows that the set $S_{k+1} = \{\mathbf{w_1}, \ldots, \mathbf{w_k}, \mathbf{w_{k+1}}\}$ is linearly independent. A linearly independent subset of R^n contains at most n vectors, so the process must eventually stop. That is, there is a linearly independent subset $S_m = \{\mathbf{w_1}, \ldots, \mathbf{w_m}\}$ such that S_m spans W. Thus S_m is a basis for W.

31. Suppose $\mathbf{x} = a_1\mathbf{u_1} + a_2\mathbf{u_2} + \cdots + a_p\mathbf{u_p}$ and $\mathbf{x} = b_1\mathbf{u_1} + b_2\mathbf{u_2} + \cdots + b_p\mathbf{u_p}$. Then $\theta = \mathbf{x} - \mathbf{x} = (a_1 - b_1)\mathbf{u_1} + (a_2 - b_2)\mathbf{u_2} + \cdots + (a_p - b_p)\mathbf{u_p}$. Since $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_p}\}$ is linearly independent, $a_1 - b_1 = 0, a_2 - b_2 = 0, \ldots, a_p - b_p = 0$. Therefore $a_1 = b_1, a_2 = b_2, \ldots, a_p = b_p$.

32. Let $B = \{\mathbf{u_1}, \ldots, \mathbf{u_m}\}$ be a basis for U. Then B is a linearly independent subset of V so by property 1 of Theorem 9, $m \leq \dim(V)$
Moreover if $m = \dim(V)$ then by property 3 of Theorem 9, B is also a basis for V. It follows that $V = W$.

33. (a) rank $(A) \leq 3$ and nullity $(A) \geq 0$.
    (b) rank $(A) \leq 3$ and nullity $(A) \geq 1$.
    (c) rank $(A) \leq 4$ and nullity $(A) \geq 0$.

34. Use Theorem 9, part(1). The columns of A are vectors in R^3.

35. Use Theorem 9, part(1). The rows of A, when transposed, are vectors in R^3.

36. Since $n = \text{rank}\,(A) + \text{nullity}\,(A)$, it follows that rank $(A) \leq n$. Further, $\mathcal{R}(A)$ is a subset of R^m so, by Exercise 32, rank $(A) \leq m$.

37. Clearly $2 = \text{rank}\,(A) \leq \text{rank}\,([A \mid \mathbf{b}])$. By Exercise 36,
rank $([A \mid \mathbf{b}]) \leq 2$. Therefore rank $([A \mid \mathbf{b}]) = 2 = \text{rank}\,(A)$ and,
by Theorem 11, the system $A\mathbf{x} = \mathbf{b}$ is consistent.

38. $4 = \text{rank}\,(A) + \text{nullity}\,(A)$, so $3 = \text{rank}\,(A) \leq \text{rank}\,([A \mid \mathbf{b}])$. By Exercise 36, rank $([A \mid \mathbf{b}]) \leq 3$. Therefore rank $(A) = 3 =$
rank $([A \mid \mathbf{b}])$ and, by Theorem 11, the system $A\mathbf{x} = \mathbf{b}$ is consistent.

39. The matrix A is, by definition, nonsingular if and only if $\mathcal{N}(A) = \{\boldsymbol{\theta}\}$.

40. If $\mathbf{x}$ is in $\mathcal{N}(B)$ then $(AB)\mathbf{x} = A(B\mathbf{x}) = A\boldsymbol{\theta} = \boldsymbol{\theta}$, so $\mathbf{x}$ is also in $\mathcal{N}(AB)$. Conversely, if $\mathbf{x}$ is in $\mathcal{N}(AB)$ then $\boldsymbol{\theta} = (AB)\mathbf{x} = A(B\mathbf{x})$. Since A is nonsingular, $B\mathbf{x} = \boldsymbol{\theta}$ and $\mathbf{x}$ is in $\mathcal{N}(B)$.

41. Suppose $c_1\mathbf{w_1} + \cdots + c_p\mathbf{w_p} = \theta$ where $c_i \neq 0$. Then $\mathbf{w_i} = a_1\mathbf{w_1} + \cdots + a_{i-1}\mathbf{w_{i-1}} + a_{i+1}\mathbf{w_{i+1}} + \cdots + a_p\mathbf{w_p}$, where $a_j = -c_j/c_i$. If $\mathbf{w}$ is any vector in W then $\mathbf{w} = b_1\mathbf{w_1} + \cdots + b_p\mathbf{w_p}$ for some scalars $b_1, \ldots, b_p$. Substituting for $\mathbf{w_i}$ yields $\mathbf{w} = (b_1 + b_ia_1)\mathbf{w_1} + \cdots + (b_{i-1} + b_ia_{i-1})\mathbf{w_{i-1}} + (b_{i+1} + b_ia_{i+1})\mathbf{w_{i+1}} + \cdots + (b_p + b_ia_p)\mathbf{w_p}$.

It follows that $W = \text{Sp}\{\mathbf{w_1}, \ldots, \mathbf{w_{i-1}}, \mathbf{w_{i+1}}, \ldots, \mathbf{w_p}\}$. By Theorem 8 any set of p vectors in W is linearly dependent. This contradicts the assumption that $\dim(W) = p$. We conclude that $c_i = 0$ for each i so S is a linearly independent set.

42. See the proof given in Exercise 30.

3.6 Orthogonal Bases for Subspaces

1. $\mathbf{u_1}^T\mathbf{u_2} = 1(-1) + 1(0) + 1(1) = 0; \mathbf{u_1}^T\mathbf{u_3} = 1(-1) + 1(2) + 1(-1) = 0; \mathbf{u_2}^T\mathbf{u_3} = -1(-1) + 0(2) + 1(-1) = 0$.

2. $\mathbf{u_1}^T\mathbf{u_2} = \mathbf{u_1}^T\mathbf{u_3} = \mathbf{u_2}^T\mathbf{u_3} = 0$.

3. $\mathbf{u_1}^T\mathbf{u_2} = 1(2) + 1(0) + 2(-1) = 0; \mathbf{u_1}^T\mathbf{u_3} = 1(1) + 1(-5) + 2(2) = 0; \mathbf{u_2}^T\mathbf{u_3} = 2(1) + 0(-5) + (-1)2 = 0$.

4. $\mathbf{u_1}^T\mathbf{u_2} = \mathbf{u_1}^T\mathbf{u_3} = \mathbf{u_2}^T\mathbf{u_3} = 0$.

5. $0 = \mathbf{u_1}^T\mathbf{u_3} = a + b + c$ and $0 = \mathbf{u_2}^T\mathbf{u_3} = 2a + 2b - 4c$. Solving yields $a = -b$, b arbitrary, and $c = 0$.

6. $a = (-1/2)c$, $b = (5/2)c$, c arbitrary.

7. $0 = \mathbf{u_1}^T\mathbf{u_2} = -3 + a; 0 = \mathbf{u_1}^T\mathbf{u_3} = 4 + b + c; 0 = \mathbf{u_2}^T\mathbf{u_3} = -8 - b + ac$. Solving yields $a = 3$, $b = -5$, $c = 1$.

8. $0 = \mathbf{u_1}^T\mathbf{u_2} = 2a + 2$; $0 = \mathbf{u_1}^T\mathbf{u_3} = 2b + 3 - c$; $0 = \mathbf{u_2}^T\mathbf{u_3} = ab + 3 - c$. Solving yields $a = -1, \ , b = 0$, $c = 3$.

9. $\mathbf{v} = a_1\mathbf{u_1} + a_2\mathbf{u_2} + a_3\mathbf{u_3}$ where $a_1 = (\mathbf{u_1}^T\mathbf{v})/(\mathbf{u_1}^T\mathbf{u_{1)}} = 2/3$, $a_2 = (\mathbf{u_2}^T\mathbf{v})/(\mathbf{u_2}^T\mathbf{u_2}) = -1/2$, $a_3 = (\mathbf{u_3}^T\mathbf{v})/(\mathbf{u_3}^T\mathbf{u_3}) = 1/6$.

10. $\mathbf{v} = \mathbf{u_1} + \mathbf{u_2}$.

11. $\mathbf{v} = a_1\mathbf{u_1} + a_2\mathbf{u_2} + a_3\mathbf{u_3}$ where $a_1 = (\mathbf{u_1}^T\mathbf{v})/(\mathbf{u_1}^T\mathbf{u_1}) = 9/3 = 3$, $a_2 = (\mathbf{u_2}^T\mathbf{v})/(\mathbf{u_2}^T\mathbf{u_2}) = 0$, $a_3 = (\mathbf{u_3}^T\mathbf{v})/(\mathbf{u_3}^T\mathbf{u_3}) = 0$.

12. $\mathbf{v} = (4/3)\mathbf{u_1} + (1/3)\mathbf{u_3}$.

13. Denote the given vectors by, $\mathbf{w_1}$, $\mathbf{w_2}$, $\mathbf{w_3}$, respectively. Then $\mathbf{u_1} = \mathbf{w_1} = [0, 0, 1, 0]^T$. $\mathbf{u_2} = \mathbf{w_2} - c_1\mathbf{u_1}$, where $c_1 =$ $(\mathbf{u_1}^T\mathbf{w_2})/(\mathbf{u_1}^T\mathbf{u_1}) = 2$. Then $\mathbf{u_2} = [1, 1, 0, 1]^T$, $\mathbf{u_3} = \mathbf{w_3} - b_1\mathbf{u_1} - b_2\mathbf{u_2}$ where $b_1 = (\mathbf{u_1}^T\mathbf{w_3})/(\mathbf{u_1}^T\mathbf{u_1}) = 1$ and $b_2 = (\mathbf{u_2}^T\mathbf{w_3})/(\mathbf{u_2}^T\mathbf{u_2}) = 2/3$. Therefore $\mathbf{u_3} = [1/3, -2/3, 0, 1/3]^T$.

14. $\mathbf{u_1} = [1, 0, 1, 2]^T$, $\mathbf{u_2} = [1, 1, -1, 0]^T$, $\mathbf{u_3} = [1/2, -1, -1/2, 0]^T$.

15. Denote the given vectors by $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}$, respectively. Then $\mathbf{u_1} = \mathbf{w_1} = [1, 1, 0]^T$, $\mathbf{u_2} = \mathbf{w_2} - c_1\mathbf{u_1}$, where $c_1 = (\mathbf{u_1}^T\mathbf{w_2})/(\mathbf{u_1}^T\mathbf{u_1}) = 2/2 = 1$. Thus $\mathbf{u_2} = [-1, 1, 1]^T$. $\mathbf{u_3} = \mathbf{w_3} - b_1\mathbf{u_1} - b_2\mathbf{u_2}$, where $b_1 = (\mathbf{u_1}^T\mathbf{w_3})/(\mathbf{u_1}^T\mathbf{u_1}) = 2/2 = 1$ and $b_2 = (\mathbf{u_2}^T\mathbf{w_3})/(\mathbf{u_2}^T\mathbf{u_2}) = 6/3 = 2$. Therefore $\mathbf{u_3} = [2, -2, 4]^T$.

16. $\mathbf{u_1} = [0,1,2]^{\mathrm{T}}$, $\mathbf{u_2} = [3,4,-2]^{\mathrm{T}}\mathbf{u_3} = [10,-6,3]^{\mathrm{T}}$.

17. Denote the given system by $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, respectively. Then $\mathbf{u}_1 = \mathbf{w}_1 = [0,1,0,1]^T$, $\mathbf{u}_2 = \mathbf{w}_2 - c_1\mathbf{u}_1$, where $c_1 = (\mathbf{u_1}^{\mathrm{T}}\mathbf{w_2})/(\mathbf{u_1}^{\mathrm{T}}\mathbf{u_1}) = 2/2 = 1$. Thus $\mathbf{u}_2 = [1,1,0,-1]^T$, $\mathbf{u}_3 = \mathbf{w}_3 - b_1 \mathbf{u}_1 - b_2\mathbf{u}_2$, where $b_1 = (\mathbf{u_1}^{\mathrm{T}}\mathbf{w_3})/(\mathbf{u_1}^{\mathrm{T}}\mathbf{u_1}) = 2/2 = 1$ and $b_2 = (\mathbf{u_2}^{\mathrm{T}}\mathbf{w_3})/(\mathbf{u_2}^{\mathrm{T}}\mathbf{u_2}) = 2/3$. Therefore $\mathbf{u}_3 = [-2/3, 1/3, 1, -1/3]^T$.

18. $\mathbf{u_1} = [1,1,0,2]^{\mathrm{T}}, \mathbf{u_2} = [-1,1,1,0]^{\mathrm{T}}, \mathbf{u_3} = [-1/2,-1/6,-1/3,1/3]^{\mathrm{T}}$.

19. If A denotes the given matrix then the homogeneous system $A\mathbf{x} = \theta$ has solution $x_1 = -3x_3 - x_4$, $x_2 = -x_3 - 3x_4$, where x_3 and x_4 are arbitrary. It follows that $\{[-3,-1,1,0]^{\mathrm{T}}, [-1,-3,0,1]^{\mathrm{T}}\}$ is a basis for $\mathcal{N}(A)$ and $\{\mathbf{A_1}, \mathbf{A_2}\}$ is a basis for $\mathcal{R}(A)$, where $\mathbf{A_1} = [1,2,1]^{\mathrm{T}}$ and $\mathbf{A_2} = [-2,1,-1]^{\mathrm{T}}$. The Gram-Schmidt process yields orthogonal bases $\{[-3,-1,1,0]^T,\ [7/11,-27/11,-6/11,1]^T\}$ and $\{[1,2,1]^T,\ [-11/6,8/6,-5/6]^T\}$ for $\mathcal{N}(A)$ and $\mathcal{R}(A)$ respectively.

20. A basis for $\mathcal{N}(A)$ is $\{[-1,-3,1,0,0]^T,\ [-2,-3,0,1,0]^T,\ [-3,-2,0,0,1]^T\}$.
The Gram-Schmidt process yields the orthogonal basis
$\{[-1,-3,1,0,0]^T,\ [-1,0,-1,1,0]^T,\ [-13/11,5/11,2/11,-1,1]^T\}$.
A basis for $\mathcal{R}(A)$ is $\{[1,-1,2]^T,\ [3,2,-1]^T\}$ and the Gram-Schmidt process yields the orthogonal basis $\{[1,-1,2]^T,\ [19/6,11/6,-4/6]T\}$.

21. By Theorem 13 an orthogonal set of nonzero vectors is linearly independent. By property 1 from Theorem 9, Section 2.5, any set of four or more vectors in R^3 is linearly dependent.

22. It follows from Theorem 13 of Section 2.6 and Theorem 12 of Section 1.8 that A is nonsingular. Note that $A^{\mathrm{T}}A$ is the (3 x 3) matrix $[c_{ij}]$ where $c_{ij} = \mathbf{u_i}^{\mathrm{T}}\mathbf{u_j}$. Since S is orthogonal $c_{ij} = \mathbf{u_i}^{\mathrm{T}}\mathbf{u_j} = 0$ if $i \neq j$ and $c_{ii} = \mathbf{u_i}^{\mathrm{T}}\mathbf{u_i} = \| \mathbf{u_i} \|^2$ for $i = 1,2,3$. For the vectors given in Exercise 1, $A^{\mathrm{T}}A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$.

23. Since $\mathbf{v}$ is in W, $0 = \mathbf{v}^{\mathrm{T}}\mathbf{v} = (\| \mathbf{v} \|)^2$. Therefore $\mathbf{v} = \theta$.

24. Suppose $\mathbf{y} \neq \theta$. For any scalar $c, 0 \leq \| \mathbf{x} - c\mathbf{y} \|^2 = (\mathbf{x} - c\mathbf{y})^{\mathrm{T}}$ $(\mathbf{x} - c\mathbf{y}) = \mathbf{x}^{\mathrm{T}}\mathbf{x} - c\mathbf{x}^{\mathrm{T}}\mathbf{y} - c\mathbf{y}^{\mathrm{T}}\mathbf{x} + c^2\mathbf{y}^{\mathrm{T}}\mathbf{y} = (\| \mathbf{x} \|)^2 - 2c\mathbf{x}^{\mathrm{T}}\mathbf{y} + c^2(\| \mathbf{y} \|)^2$. For $c = \mathbf{x}^{\mathrm{T}}\mathbf{y}/\mathbf{y}^{\mathrm{T}}\mathbf{y}$ this implies that $0 \leq \| \mathbf{x} \|^2 - (\mathbf{x}^{\mathrm{T}}\mathbf{y})^2 / \| \mathbf{y} \|^2$. It follows that $| \mathbf{x}^{\mathrm{T}}\mathbf{y} | \leq \| \mathbf{x} \| \, \| \mathbf{y} \|$. If $\mathbf{y} = \theta$ then $| \mathbf{x}^{\mathrm{T}}\mathbf{y} | = \| \mathbf{x} \| \, \| \mathbf{y} \| = 0$.

25. $\| \mathbf{x} + \mathbf{y} \|^2 = (\mathbf{x} + \mathbf{y})^{\mathrm{T}}(\mathbf{x} + \mathbf{y}) = \| \mathbf{x} \|^2 + 2(\mathbf{x}^{\mathrm{T}}\mathbf{y}) + \| \mathbf{y} \|^2 \leq$
$\| \mathbf{x} \|^2 + 2\, | \mathbf{x}^{\mathrm{T}}\mathbf{y} | + \| \mathbf{y} \|^2 \leq \| \mathbf{x} \|^2 + 2\, \| \mathbf{x} \| \, \| \mathbf{y} \| + \| \mathbf{y} \|^2 = (\| \mathbf{x} \| + \| \mathbf{y} \|)^2$.

26. Note that $\| \mathbf{y} \| = \| \mathbf{x} + (\mathbf{y} - \mathbf{x}) \| \leq \| \mathbf{x} \| + \| \mathbf{y} - \mathbf{x} \|$ so
$-(\| \mathbf{x} \| - \| \mathbf{y} \|) = \| \mathbf{y} \| - \| \mathbf{x} \| \leq \| \mathbf{y} - \mathbf{x} \| = \| \mathbf{x} - \mathbf{y} \|$. Similarly, $\| \mathbf{x} \| =$

$\| (\mathbf{x}-\mathbf{y})+\mathbf{y} \| \leq \| \mathbf{x}-\mathbf{y} \| + \| \mathbf{y} \|$. Therefore, $\| \mathbf{x} \| - \| \mathbf{y} \| \leq \| \mathbf{x}-\mathbf{y} \|$. It follows that $| \, \|\mathbf{x}\| - \|\mathbf{y}\| \, | \leq \| \mathbf{x}-\mathbf{y} \|$.

27. If $W = \mathrm{Sp}\{\mathbf{w_i}\}_{i=1}^{p-1}$ then $\{\mathbf{u_i}\}_{i=1}^{p-1}$ is an orthogonal basis for W. Since $\{\mathbf{w_i}\}_{i=1}^{p}$ is linearly dependent, $\mathbf{w_p}$ is in W. Therefore $\mathbf{u_p}$ is in W and $\mathbf{u_p}^{\mathrm{T}}\mathbf{u_i} = 0$ for $1 \leq i \leq p-1$. It follows that $\mathbf{u_p}^{\mathrm{T}}\mathbf{w} = 0$ for every vector $\mathbf{w}$ in W. By Exercise 23, $\mathbf{u_p} = \theta$.

28. $\| \mathbf{v} \|^2 = (a_1\mathbf{u_1} + \cdots + a_p\mathbf{u_p})^{\mathrm{T}}(a_1\mathbf{u_1} + \cdots + a_p\mathbf{u_p}) =$
$\sum_{1 \leq i,j \leq p} a_i a_j \mathbf{u_i}^{\mathrm{T}}\mathbf{u_j} = \sum_{i=1}^{p} a_i^2 \mathbf{u_i}^{\mathrm{T}}\mathbf{u_i} = \sum_{i=1}^{p} a_1^2$ since B is an orthonormal basis.

3.7 Linear Transformations from R^n to R^m

1. (a) $T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
   (b) $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.
   (c) $T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
   (d) $T\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

2. (a) $T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
   (b) $T\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$.
   (c) $T\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$.
   (d) $T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

3. (a), (b), and (d) are in the null space of T.

4. $T(\mathbf{x}) = \mathbf{b}$ requires $2x_1 - 3x_2 = 2$ and $-x_1 + x_2 = -2$. Solving yields $x_1 = 4, x_2 = 2$, so $\mathbf{x} = [4, 2]^{\mathrm{T}}$.

5. If $\mathbf{b} = [b_1, b_2]^{\mathrm{T}}$ then $T(\mathbf{x}) = \mathbf{b}$ requires that $2x_1 - 3x_2 = b_1$ and $-x_1 + x_2 = b_2$. Solving yields $x_1 = -b_1 - 3b_2$ and $x_2 = -b_1 - 2b_2$; that is $\mathbf{x} = [-b_1 - 3b_2, -b_1 - 2b_2]^{\mathrm{T}}$.

6. $T(\mathbf{x}) = \mathbf{b}$ if and only if $A\mathbf{x} = \mathbf{b}$. Solving yields $x_1 = -2 + x_2$, x_2 arbitrary. For example, if $\mathbf{x} = [-2, 0]^{\mathrm{T}}$ then $T(\mathbf{x}) = \mathbf{b}$.

7. The system of equations $A\mathbf{x} = \mathbf{b}$ is easily seen to be inconsistent.

8. Let $\mathbf{u} = [u_1, u_2]^{\mathrm{T}}$ and $\mathbf{v} = [v_1, v_2]^{\mathrm{T}}$. Then

$$F(\mathbf{u}+\mathbf{v}) = \begin{bmatrix} 2(u_1+v_1)-(u_2+v_2) \\ (u_1+v_1)+3(u_2+v_2) \end{bmatrix} =$$

$$\begin{bmatrix} 2u_1 - u_2 \\ u_1 + 3u_2 \end{bmatrix} + \begin{bmatrix} 2v_1 - v_2 \\ v_1 + 3v_2 \end{bmatrix} = F(\mathbf{u}) + \mathbf{F}(\mathbf{v}).$$

Also $F(a\mathbf{u}) = \begin{bmatrix} 2au_1 - au_2 \\ au_1 + 3au_2 \end{bmatrix} = a \begin{bmatrix} 2u_1 - u_2 \\ u_1 + 3u_2 \end{bmatrix} = aF(\mathbf{u})$.

This shows that F is a linear transformation.

9. F is a linear transformation.

10. F is not a linear transformation. For example if $\mathbf{u} = [u_1, u_2]^{\mathrm{T}}$ and $\mathbf{v} = [v_1, v_2]^{\mathrm{T}}$ then $F(\mathbf{u}+\mathbf{v}) = \begin{bmatrix} (u_1+v_1)+(u_2+v_2) \\ 1 \end{bmatrix}$ whereas $F(\mathbf{u})+F(\mathbf{v}) = \begin{bmatrix} (u_1+v_1)+(u_2+v_2) \\ 2 \end{bmatrix}$. Likewise,

$F(a\mathbf{u}) = \begin{bmatrix} au_1 + au_2 \\ 1 \end{bmatrix}$ whereas $aF(\mathbf{u}) = \begin{bmatrix} au_1 + au_2 \\ a \end{bmatrix}$.

11. F is not a linear transformation. For example $F\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = F\left(\begin{bmatrix} 3 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 9 \\ 9 \end{bmatrix}$ whereas $F\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) + F\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$.

12. If $\mathbf{u} = [u_1, u_2, u_3]^{\mathrm{T}}$ and $\mathbf{v} = [v_1, v_2, v_3]^{\mathrm{T}}$ then $F(\mathbf{u}+\mathbf{v}) =$ $\begin{bmatrix} (u_1+v_1)-(u_2+v_2)+(u_3+v_3) \\ -(u_1+v_1)+3(u_2+v_2)-2(u_3+v_3) \end{bmatrix} = \begin{bmatrix} u_1 - u_2 + u_3 \\ -u_1 + 3u_2 - 2u_3 \end{bmatrix} + \begin{bmatrix} v_1 - v_2 + v_3 \\ -v_1 + 3v_2 - 2v_3 \end{bmatrix}$ $= F(\mathbf{u}) + F(\mathbf{v})$. For any scalar a, $F(a\,\mathbf{u}) = \begin{bmatrix} au_1 - au_2 + au_3 \\ -au_1 + 3au_2 - 2au_3 \end{bmatrix} =$ $a \begin{bmatrix} u_1 - u_2 + u_3 \\ -u_1 + 3u_2 - 2u_3 \end{bmatrix} = a\,F(\mathbf{u})$. Thus F is a linear transformation.

13. F is a linear transformation.

14. If $\mathbf{u} = [u_1, u_2]^{\mathrm{T}}$ and $\mathbf{v} = [v_1, v_2]^{\mathrm{T}}$ then $F(\mathbf{u}+\mathbf{v}) =$

$$\begin{bmatrix} (u_1+v_1)-(u_2+v_2) \\ -(u_1+v_1)+(u_2+v_2) \\ u_2+v_2 \end{bmatrix} = \begin{bmatrix} u_1 - u_2 \\ -u_1 + u_2 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 - v_2 \\ -v_1 + v_2 \\ v_2 \end{bmatrix} =$$

$F(\mathbf{u}) + F(\mathbf{v})$. Similarly, $F(a\mathbf{u}) = \begin{bmatrix} au_1 - au_2 \\ -au_1 + au_2 \\ au_2 \end{bmatrix} =$

$a \begin{bmatrix} u_1 - u_2 \\ -u_1 + u_2 \\ u_2 \end{bmatrix} = aF(\mathbf{u})$. It follows that F is a linear transformation.

15. F is a linear transformation.

16. Let $\mathbf{u} = [u_1, u_2]^{\mathrm{T}}$ and $\mathbf{v} = [v_1, v_2]^{\mathrm{T}}$. Then $F(\mathbf{u} + \mathbf{v}) = 2(u_1 + v_1) + 3(u_2 + v_2) = (2u_1 + 3u_2) + (2v_1 + 3v_2) = F(\mathbf{u}) + F(\mathbf{v})$. Likewise

    $F(a\mathbf{u}) = 2au_1 + 3au_2 = a(2u_1 + 3u_2) = a\,F(\mathbf{u})$. This means that F is a linear transformation.

17. F is not a linear transformation. For example note that

    $F(-\mathbf{e_1}) = 1$ whereas $-F(\mathbf{e_1}) = -1$.

18. The set $\{\mathbf{e_1}\}$ is an orthonormal basis for W and $T(\mathbf{v}) = (\mathbf{v}^{\mathrm{T}}\mathbf{e_1})\mathbf{e_1} = [a, 0, 0]^{\mathrm{T}}$.

19. (a) $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = T(\mathbf{e_1} + \mathbf{e_2}) = T(\mathbf{e_1}) + T(\mathbf{e_2}) = \mathbf{u_1} + \mathbf{u_2} = [3, 1, -1]^{\mathrm{T}}$.

    (b) $T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = T(2\mathbf{e_1} - \mathbf{e_2}) = 2T(\mathbf{e_1}) - T(\mathbf{e_2}) = 2\mathbf{u_1} - \mathbf{u_2} = [0, -1, -2]^{\mathrm{T}}$.

    (c) $T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = T(3\mathbf{e_1} + 2\mathbf{e_2}) = 3T(\mathbf{e_1}) + 2T(\mathbf{e_2}) = 3\mathbf{u_1} + 2\mathbf{u_2} = [7, 2, -3]^{\mathrm{T}}$.

20. (a) $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = T(2\mathbf{v_1} - \mathbf{v_2}) = 2\mathbf{u_1} - \mathbf{u_2} = [-3, 3]^{\mathrm{T}}$.

    (b) $T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = T(\mathbf{v_1} - 2\mathbf{v_2}) = \mathbf{u_1} - 2\mathbf{u_2} = [-6, 0]^{\mathrm{T}}$.

    (c) $T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = T(5\mathbf{v_1} - 3\mathbf{v_2}) = 5\mathbf{u_1} - 3\mathbf{u_2} = [-9, 7]^{\mathrm{T}}$.

21. Let $\mathbf{u_1} = [1, 1]^{\mathrm{T}}$ and $\mathbf{u_2} = [1, -1]^{\mathrm{T}}$. If $\mathbf{x} = [x_1, x_2]^{\mathrm{T}}$ then $\mathbf{x} =$

    $[(x_1 + x_2)/2]\mathbf{u_1} + [(x_1 - x_2)/2]\mathbf{u_2}$. Thus $T(\mathbf{x}) =$

    $[(x_1 + x_2)/2] \begin{bmatrix} 2 \\ -1 \end{bmatrix} + [(x_1 - x_2)/2] \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 - 2x_2 \end{bmatrix}$.

22. $T(\mathbf{x}) = \begin{bmatrix} (x_1 + x_2)/2 \\ 2x_1 \\ (3x_1 - x_2)/2 \end{bmatrix}$.

23. Let $\mathbf{u_1} = [1, 0, 1]^{\mathrm{T}}$, $\mathbf{u_2} = [0, -1, 1]^{\mathrm{T}}$, $\mathbf{u_3} = [1, -1, 0]^{\mathrm{T}}$. If $\mathbf{x} = [x_1, x_2, x_3]^{\mathrm{T}}$ then $\mathbf{x} = c_1\mathbf{u_1} + c_2\mathbf{u_2} + c_3\mathbf{u_3}$, where $c_1 = (x_1 + x_2 + x_3)/2$, $c_2 = (-x_1 - x_2 + x_3)/2$ and $c_3 = (x_1 - x_2 - x_3)/2$.

Therefore $T(\mathbf{x}) = c_1[0, 1]^{\mathrm{T}} + c_2[1, 0]^{\mathrm{T}} + c_3[0, 0]^{\mathrm{T}}$; that is,

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} (-x_1 - x_2 + x_3)/2 \\ (x_1 + x_2 + x_3)/2 \end{bmatrix}.$$

24. $$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -x_1 - x_2 + x_3 \\ -x_1 - x_2 \\ x_1 \end{bmatrix}.$$

25. $A = [T(\mathbf{e_1}), T(\mathbf{e_2})] = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$. The homogeneous system of equations $A\mathbf{x} = \theta$ has only the trivial solution so $\mathcal{N}(T) = \mathcal{N}(A) = \{\theta\}$ and nullity $(T) = 0$. Since rank $(T) = 2 -$ nullity $(T) = 2$, it follows that $\mathcal{R}(T) = R^2$.

26. $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$; $\mathcal{N}(T) = \mathcal{N}(A) = \{\theta\}$; $\mathcal{R}(T) = \mathcal{R}(A) = \{[b_1, b_2, b_3]^{\mathrm{T}} : b_1 - b_2 + 2b_3 = 0\}$; nullity (T) $= 0$; rank (T) $= 2$.

27. $A = [T(\mathbf{e_1}), T(\mathbf{e_2})] = [3, 2]$; $\mathcal{N}(T) = \{\mathbf{x} : 3x_1 + 2x_2 = 0\}$; $\mathcal{R}(T) = R^1$; rank $(T) = 1$; nullity $(T) = 1$.

28. $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$; $\mathcal{N}(T) = \mathcal{N}(A) = \{\theta\}$; $\mathcal{R}(T) = \mathcal{R}(A) = R^3$; rank $(T) = 3$; nullity $(T) = 0$.

29. $A = [T(\mathbf{e_1}), T(\mathbf{e_2}), T(\mathbf{e_3})] = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$; $\mathcal{N}(T) = \mathcal{N}(A) = \{\mathbf{x} : x_1 = x_3, x_2 = x_3\}$; $\mathcal{R}(T) = \mathcal{R}(A) = R^2$; rank $(T) = 2$; nullity $(T) = 1$.

30. $A = [2, -1, 4]$; $\mathcal{N}(T) = \{\mathbf{x} : 2x_1 - x_2 + 4x_3 = 0\}$; $\mathcal{R}(T) = R^1$; rank $(T) = 1$; nullity $(T) = 2$.

31. For any x and y in R, $f(x + y) = a(x + y) = ax + ay = f(x) + f(y)$.

If b is any real number then $f(bx) = a(bx) = b(ax) = bf(x)$. Therefore f is a linear transformation.

32. Since T is a linear transformation and x can be viewed as a scalar, $T(x) = T(x \cdot 1) = xT(1) = xa = ax$ for each x in R.

33. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$

34. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$

35. For vectors $\mathbf{u}$ and $\mathbf{v}$ in V, $[F+G](\mathbf{u}+\mathbf{v}) = F(\mathbf{u}+\mathbf{v}) + G(\mathbf{u}+\mathbf{v}) = [F(\mathbf{u}) + F(\mathbf{v})] + [G(\mathbf{u}) + G(\mathbf{v})] = [F(\mathbf{u}) + G(\mathbf{u})] + [F(\mathbf{v}) + G(\mathbf{v})] = [F+G](\mathbf{u}) + [F+G](\mathbf{v})$. Similarly, $[F+G](a\mathbf{u}) = F(a\mathbf{u}) + G(a\mathbf{u}) = aF(\mathbf{u}) + aG(\mathbf{u}) = a[F(\mathbf{u}) + G(\mathbf{u})] = a[F+G](\mathbf{u})$ for every scalar a. This proves that $F+G$ is a linear transformation.

36. (a) $(F+G)\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 + 3x_3 \\ 2x_1 + 5x_2 - 2x_3 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 2 & -5 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 4 & 2 \\ -2 & 3 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 & 3 \\ -2 & 3 & 3 \end{bmatrix}$

37. $[aT](\mathbf{u}+\mathbf{v}) = a[T(\mathbf{u}+\mathbf{v})] = a[T(\mathbf{u}) + T(\mathbf{v})] = aT(\mathbf{u}) + aT(\mathbf{v}) = [aT](\mathbf{u}) + [aT](\mathbf{v})$. Also $[aT](b\mathbf{u}) = a(T(b\mathbf{u})) = a(bT(\mathbf{u})) = b(aT(\mathbf{u})) = b[aT](\mathbf{u})$. This proves that aT is a linear transformation.

38. (a) $[3T]\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 - 3x_2 \\ 3x_2 - 3x_3 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -3 & 0 \\ 0 & 3 & -3 \end{bmatrix}$

39. For $\mathbf{u_1}$ and $\mathbf{u_2}$ in U, $[G \circ F](\mathbf{u_1}+\mathbf{u_2}) = G(F(\mathbf{u_1}+\mathbf{u_2})) = G(F(\mathbf{u_1}) + F(\mathbf{u_2})) = G(F(\mathbf{u_1})) + G(F(\mathbf{u_2})) = [G \circ F](\mathbf{u_1}) + [G \circ F](\mathbf{u_2})$. If $\mathbf{u}$ is in U and a is any scalar then $[G \circ F](a\mathbf{u}) = G(F(a\mathbf{u})) = G(aF(\mathbf{u})) = aG(F(\mathbf{u})) = a[G \circ F](\mathbf{u})$. Thus $G \circ F$ is a linear transformation.

40. (a) $[G \circ F]\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -5x_1 - 8x_2 - 6x_3 \\ x_1 + 16x_2 - 10x_3 \\ 3x_1 + 3x_2 + 5x_3 \end{bmatrix}$

(b) $A = \begin{bmatrix} -1 & 2 & -4 \\ 2 & 5 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 \\ 3 & 2 \\ -1 & 1 \end{bmatrix}$, $\begin{bmatrix} -5 & -8 & -6 \\ 1 & 16 & -10 \\ 3 & 3 & 5 \end{bmatrix}$

41. Write $B = [\mathbf{B_1}, \mathbf{B_2}, \ldots, \mathbf{B_n}]$. Then $T(\mathbf{e_i}) = B\mathbf{e_i} = \mathbf{B_i}$, so $A = [T(\mathbf{e_1}), T(\mathbf{e_2}), \ldots, T(\mathbf{e_n})] = B$.

42. $[G \circ F](\mathbf{x}) = G(F(\mathbf{x})) = G(A\mathbf{x}) = B(A\mathbf{x}) = BA\mathbf{x}$. By Exercise 41, BA is the matrix for $G \circ F$.

43. $A = [\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}] = I$, the $(n\ x\ n)$ identity matrix.

44. $A = [T(\mathbf{e_1}), T(\mathbf{e_2}), \ldots, T(\mathbf{e_n})] = [a\mathbf{e_1}, a\mathbf{e_2}, \ldots, a\mathbf{e_n}]$. Thus $A = \begin{bmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{bmatrix}$.

45. (a) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$.

(c) $A = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$.

46. (a) $\theta = 0$ so $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$; $T(\mathbf{e_1}) = \mathbf{e_1}$; $T(\mathbf{e_2}) = -\mathbf{e_2}$.

(b) $\theta = \pi$ so $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$; $T(\mathbf{e_1}) = -\mathbf{e_1}$; $T(\mathbf{e_2}) = \mathbf{e_2}$.

(c) $\theta = \pi/2$ so $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; $T(\mathbf{e_1}) = \mathbf{e_2}$; $T(\mathbf{e_2}) = \mathbf{e_1}$.

(d) $\theta = 2\pi/3$ so $A = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$; $T(\mathbf{e_1}) = [-1/2, \sqrt{3}/2]^{\mathrm{T}}$;
$T(\mathbf{e_2}) = [\sqrt{3}/2, 1/2]^{\mathrm{T}}$.

47. Set $\mathbf{u_1} = T(\mathbf{e_1})$ and $\mathbf{u_2} = T(\mathbf{e_2})$. By assumption $\| \mathbf{u_1} \| = \| \mathbf{u_2} \| = 1$ and $\mathbf{u_1}^{\mathrm{T}}\mathbf{u_2} = 0$. Moreover $T(\mathbf{v}) = a\mathbf{u_1} + b\mathbf{u_2}$ so $\| T(\mathbf{v}) \|^2 = (a\mathbf{u_1} + b\mathbf{u_2})^{\mathrm{T}}(a\mathbf{u_1} + b\mathbf{u_2}) = a^2 + b^2 = \| \mathbf{v} \|^2$. Thus $\| T(\mathbf{v}) \| = \| \mathbf{v} \|$ and T is orthogonal.

48. Set $\mathbf{u_1} = T(\mathbf{e_1})$ and $\mathbf{u_2} = T(\mathbf{e_2})$. Then $A = [\mathbf{u_1}, \mathbf{u_2}]$ and $A^{\mathrm{T}}A = \begin{bmatrix} \mathbf{u_1}^{\mathrm{T}}\mathbf{u_1} & \mathbf{u_1}^{\mathrm{T}}\mathbf{u_2} \\ \mathbf{u_2}^{\mathrm{T}}\mathbf{u_1} & \mathbf{u_2}^{\mathrm{T}}\mathbf{u_2} \end{bmatrix}$.
It then follows from Theorem 16 that
$A^{\mathrm{T}}A = I$.

49. (a) $A^{\mathrm{T}}A = \begin{bmatrix} \mathbf{A_1}^{\mathrm{T}}\mathbf{A_1} & \mathbf{A_1}^{\mathrm{T}}\mathbf{A_2} \\ \mathbf{A_2}^{\mathrm{T}}\mathbf{A_1} & \mathbf{A_2}^{\mathrm{T}}\mathbf{A_2} \end{bmatrix}$ so it follows that $\mathbf{A_1}^{\mathrm{T}}\mathbf{A_1} = \mathbf{A_2}^{\mathrm{T}}\mathbf{A_2} = 1$ whereas $\mathbf{A_1}^{\mathrm{T}}\mathbf{A_2} = \mathbf{A_2}^{\mathrm{T}}\mathbf{A_1} = 0$. Thus
$\{\mathbf{A_1}, \mathbf{A_2}\}$ is an orthonormal set.

(b) Now $\| T(\mathbf{e_1}) \| = \| \mathbf{A_1} \| = 1, \| T(\mathbf{e_2}) \| = \| \mathbf{A_2} \| = 1$ and
$T(\mathbf{e_1})$ is perpendicular to $T(\mathbf{e_2})$. By Theorem 16, T is orthogonal.

3.8 Least-Squares Solutions to Inconsistent Systems

1. $A^{\mathrm{T}}A = \begin{bmatrix} 3 & 4 \\ 4 & 14 \end{bmatrix}$ and $A^{\mathrm{T}}\mathbf{b} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$. The system of equations $A^{\mathrm{T}}A\mathbf{x} = A^{\mathrm{T}}\mathbf{b}$ has unique solution $\mathbf{x}^* = \begin{bmatrix} -5/13 \\ 7/13 \end{bmatrix}$.

2. $A^{\mathrm{T}}A = \begin{bmatrix} 6 & 11 & 23 \\ 11 & 22 & 44 \\ 23 & 44 & 90 \end{bmatrix}$ and $A^{\mathrm{T}}\mathbf{b} = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}$. The system of equations $A^{\mathrm{T}}A\mathbf{x} = A^{\mathrm{T}}\mathbf{b}$ has solution $\mathbf{x}^* = \begin{bmatrix} -3 \\ 19/11 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$, where x_3 is arbitrary.

3. $A^{\mathrm{T}}A\mathbf{x} = \begin{bmatrix} 11 & 16 & 17 \\ 16 & 30 & 18 \\ 17 & 18 & 33 \end{bmatrix}$ and $A^{\mathrm{T}}\mathbf{b} = \begin{bmatrix} 10 \\ 17 \\ 13 \end{bmatrix}$. The system of equations $A^{\mathrm{T}}A\mathbf{x} = A^{\mathrm{T}}\mathbf{b}$ has solution $\mathbf{x}^* = \begin{bmatrix} 28/74 \\ 27/74 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$ where x_3 is arbitrary.

4. $A^{\mathrm{T}}A = \begin{bmatrix} 15 & 24 & 3 \\ 24 & 39 & 3 \\ 3 & 3 & 6 \end{bmatrix}$ and $A^{\mathrm{T}}\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$. The system of equations $A^{\mathrm{T}}A\mathbf{x} = A^{\mathrm{T}}\mathbf{b}$ has solution $\mathbf{x}^* = \begin{bmatrix} -8/3 \\ 5/3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 3 \\ 1 \end{bmatrix}$ where x_3 is arbitrary.

5. $A^{\mathrm{T}}A = \begin{bmatrix} 14 & 28 \\ 28 & 56 \end{bmatrix}$ and $A^{\mathrm{T}}\mathbf{b} = \begin{bmatrix} 52 \\ 104 \end{bmatrix}$. The system of equations $A^{\mathrm{T}}A\mathbf{x} = A^{\mathrm{T}}\mathbf{b}$ has solution $\mathbf{x}^* = \begin{bmatrix} 26/7 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, where x_2 is arbitrary.

6. $A^{\mathrm{T}}A = \begin{bmatrix} 11 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $A^{\mathrm{T}}\mathbf{b} = \begin{bmatrix} 21 \\ 1 \\ 1 \end{bmatrix}$. The system of equations $A^{\mathrm{T}}A\mathbf{x} = A^{\mathrm{T}}\mathbf{b}$ has solution $\mathbf{x}^* = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ where x_3 is arbitrary.

7. We must obtain the least-squares solution to $A\mathbf{x} = \mathbf{b}$ where $A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} m \\ c \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 4 \end{bmatrix}$. $A^{\mathrm{T}}A = \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}$ and $A^{\mathrm{T}}\mathbf{b} = \begin{bmatrix} 10 \\ 7 \end{bmatrix}$. The system of equations $A^{\mathrm{T}}A\mathbf{x} = A^{\mathrm{T}}b$ has solution $\mathbf{x}^* = \begin{bmatrix} 1.3 \\ 1.1 \end{bmatrix}$. Therefore $y = (1.3)t + 1.1$ is the least-squares linear fit.

8. $y = (-19/35)t + 31/35$ is the least-squares linear fit.

9. We must obtain the least-squares solution to $A\mathbf{x} = \mathbf{b}$ where $A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} m \\ c \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 3 \end{bmatrix}$. In this case $A^{\mathrm{T}}A = \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}$ and $A^{\mathrm{T}}\mathbf{b} = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$. The system of equations $A^{\mathrm{T}}A\mathbf{x} = A^{\mathrm{T}}\mathbf{b}$ has solution $\mathbf{x}^* = \begin{bmatrix} 13/10 \\ 3/5 \end{bmatrix}$ so $y = 2t + 1$ is the least-squares linear fit.

10. $y = 4t - 3/2$ is the least-squares linear fit.

11. We must obtain the least-squares solution to $A\mathbf{x} = \mathbf{b}$ where $A = \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$,

and $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix}$. In this case $A^{\mathrm{T}}A = \begin{bmatrix} 4 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{bmatrix}$ and $A^{\mathrm{T}}\mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ 17 \end{bmatrix}$. The system of equations $A^{\mathrm{T}}A\mathbf{x} = A^{\mathrm{T}}\mathbf{b}$ has solution $\mathbf{x}^* = [0, 1/10, 1/2]^{\mathrm{T}}$ so $y = a_0 + a_1 t + a_2 t^2 = (1/10)t + (1/2)t^2$ is the least-squares quadratic fit.

12. $y = -1/20 - (1/20)t + (1/4)t^2$ is the least-squares quadratic fit.

13. We must obtain the least-squares solution to $A\mathbf{x} = \mathbf{b}$ where $A = \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$,

and $\mathbf{b} = \begin{bmatrix} -3 \\ -1 \\ 0 \\ 3 \end{bmatrix}$. In this case $A^{\mathrm{T}}A = \begin{bmatrix} 4 & -2 & 6 \\ -2 & 6 & -8 \\ 6 & -8 & 18 \end{bmatrix}$ and $A^{\mathrm{T}}\mathbf{b} = \begin{bmatrix} -1 \\ 10 \\ -10 \end{bmatrix}$. The system of equations $A^{\mathrm{T}}A\mathbf{x} = A^{\mathrm{T}}\mathbf{b}$ has solution $\mathbf{x}^* = [9/20, 43/20, 1/4]^{\mathrm{T}}$ so $y = 9/20 + (43/20)t + (1/4)t$ is the least-squares quadratic fit.

14. $y = 31/55 - (4/55)t + (12/11)t^2$ is the least-squares quadratic fit.

15. Note that $A\mathbf{x} - \mathbf{b} = [f(t_1) - y_1, \ldots, f(t_m) - y_m]^{\mathrm{T}}$ so, by definition, $\| A\mathbf{x} - \mathbf{b}\|^2 = \sum_{i=1}^{m} [f(t_i) - y_i]^2$.

16. (a) $A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 3 & -1 \\ 4 & 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 2 \\ 4 \\ 5 \end{bmatrix}$.

(b) $A^{\mathrm{T}}A = \begin{bmatrix} 30 & 2 \\ 2 & 4 \end{bmatrix}$ and $A^{\mathrm{T}}\mathbf{b} = \begin{bmatrix} 36 \\ 3 \end{bmatrix}$. The system of equations $A^{\mathrm{T}}A\mathbf{x} = A^{\mathrm{T}}\mathbf{b}$ has solution $\mathbf{x}^* = [69/58, 9/58]^{\mathrm{T}}$. Therefore $Q(a_1, a_2)$ is minimized if $f(t) = (69/58)\sqrt{t} + (9/58)\cos\pi$t.

17. Suppose that $A\mathbf{x} = \theta$, where $\mathbf{x} = [a_0, a_1, \ldots, a_n]^{\mathrm{T}}$. If $p(t) = a_0 + a_1 t + \cdots + a_n t^n$ then $p(t_i) = 0$ for $0 \le i \le m$; that is, $p(t)$ has $m + 1$ roots and $m + 1 > n$. It follows that $a_0 = a_1 = \cdots = a_n = 0$. Thus nullity $(A) = 0$ and, consequently, rank $(A) = n$.

18. The matrix reduces to: $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

So, rank A = 2.

3.9 Fitting Data and Least Squares Solutions

1. If $\mathbf{u_1} = [2,1,0]^{\mathrm{T}}$ and $\mathbf{u_2} = [-1,0,1]^{\mathrm{T}}$ then $\{\mathbf{u_1}, \mathbf{u_2}\}$ is a basis for W. For $A = [\mathbf{u_1}, \mathbf{u_2}]$ the system of equations $A^{\mathrm{T}}A\mathbf{x} = A^{\mathrm{T}}\mathbf{v}$ is given by $5x_1 - 2x_2 = 4$, $-2x_1 + 2x_2 = 5$. Solving we obtain $x_1 = 3$ and $x_2 = 11/2$. Thus $\mathbf{w}^* = 3\mathbf{u_1} + (11/2)\mathbf{u_2} = [1/2, 3, 11/2]^{\mathrm{T}}$.

2. The system of equations $A^{\mathrm{T}}A\mathbf{x} = A^{\mathrm{T}}\mathbf{v}$ has solution $x_1 = x_2 = 2$. Thus $\mathbf{w}^* = 2\mathbf{u_1} + 2\mathbf{u_2} = [2,2,2]^{\mathrm{T}}$.

3. The basis $\{\mathbf{u_1}, \mathbf{u_2}\}$ is given in Exercise 1. The system $A^{\mathrm{T}}A\mathbf{x} = A^{\mathrm{T}}\mathbf{v}$ is given by $5x_1 - 2x_2 = 3$, $-2x_1 + 2x_2 = 0$. The solution is $x_1 = x_2 = 1$, so $\mathbf{w}^* = \mathbf{u_1} + \mathbf{u_2} = [1,1,1]^{\mathrm{T}} = \mathbf{v}$.

4. $\mathcal{R}(B)$ has basis $\{\mathbf{u_1}, \mathbf{u_2}\}$ where $\mathbf{u_1} = [1,1,0]^{\mathrm{T}}$ and $\mathbf{u_2} = [2,1,1]^{\mathrm{T}}$. If $A = [\mathbf{u_1}, \mathbf{u_2}]$ the system $A^{\mathrm{T}}A\mathbf{x} = A^{\mathrm{T}}\mathbf{v}$ is given by $2x_1 + 3x_2 = 2$, $3x_1 + 6x_2 = 9$. The solution is $x_1 = -5$, $x_2 = 4$. Thus $\mathbf{w}^* = -5\mathbf{u_1} + 4\mathbf{u_2} = [3,-1,4]^{\mathrm{T}}$.

5. $\mathcal{R}(B)$ has basis $\{\mathbf{u_1}, \mathbf{u_2}\}$ where $\mathbf{u_1} = [1,1,0]^{\mathrm{T}}$ and $\mathbf{u_2} = [2,1,1]^{\mathrm{T}}$. If $A = [\mathbf{u_1}, \mathbf{u_2}]$, the system $A^{\mathrm{T}}A\mathbf{x} = A^{\mathrm{T}}\mathbf{v}$ is given by $2x_1 + 3x_2 = 6$, $3x_1 + 6x_2 = 12$. Solving yields $x_1 = 0$, $x_2 = 2$, so $\mathbf{w}^* = 2\mathbf{u_2} = [4,2,2]^{\mathrm{T}}$.

6. The system of equations $A^{\mathrm{T}}A\mathbf{x} = A^{\mathrm{T}}\mathbf{v}$ has solution $x_1 = -3$, $x_2 = 3$. Thus $\mathbf{w}^* = -3\mathbf{u_1} + 3\mathbf{u_2} = [3,0,3]^{\mathrm{T}} = \mathbf{v}$.

7. $\mathcal{R}(B)$ has basis $\{\mathbf{u_1}, \mathbf{u_2}\}$ where $\mathbf{u_1} = [1,-1,1]^{\mathrm{T}}$ and $\mathbf{u_2} = [2,0,1]^{\mathrm{T}}$. If $A = [\mathbf{u_1}, \mathbf{u_2}]$, the system $A^{\mathrm{T}}A\mathbf{x} = A^{\mathrm{T}}\mathbf{v}$ is given by $3x_1 + 3x_2 = 6$, $3x_1 + 5x_2 = 8$. Solving yields $x_1 = x_2 = 1$, so $\mathbf{w}^* = \mathbf{u_1} + \mathbf{u_2} = [3,-1,2]^{\mathrm{T}}$.

8. The system of equations $A^{\mathrm{T}}A\mathbf{x} = A^{\mathrm{T}}\mathbf{v}$ has solution $x_1 = -1$, $x_2 = 2$ so $\mathbf{w}^* = -\mathbf{u_1} + 2\mathbf{u_2} = [3,1,1]^{\mathrm{T}}$.

9. W has basis $\{\mathbf{u}\}$ where $\mathbf{u} = [0,-1,1]^{\mathrm{T}}$. If $A = [\mathbf{u}]$ then the system $A^{\mathrm{T}}A\mathbf{x} = A^{\mathrm{T}}\mathbf{v}$ is given by $2x = -2$. Thus $x = -1$ and $\mathbf{w}^* = -\mathbf{u} = [0,1,-1]^{\mathrm{T}}$.

10. $\mathbf{w}^* = -2\mathbf{u} = [0,2,-2]^{\mathrm{T}}$.

11. An orthogonal basis for W is the set $\{\mathbf{u_1}, \mathbf{u_2}\}$ where $\mathbf{u_1} = [2,1,0]^{\mathrm{T}}$ and $\mathbf{u_2} = [-1/5, 2/5, 1]^{\mathrm{T}}$. The vector $\mathbf{w}^*$ is given by $\mathbf{w}^* = a_1\mathbf{u_1} + a_2\mathbf{u_2}$ where $a_1 = \mathbf{u_1}^{\mathrm{T}}\mathbf{v}/\mathbf{u_1}^{\mathrm{T}}\mathbf{u_1} = 4/5$ and $a_2 = \mathbf{u_2}^{\mathrm{T}}\mathbf{v}/\mathbf{u_2}^{\mathrm{T}}\mathbf{u_2} = 11/2$. Thus $\mathbf{w}^* = [1/2, 3, 11/2]^{\mathrm{T}}$.

12. $\mathbf{w}^* = a_1\mathbf{u_1} + a_2\mathbf{u_2}$ where $\mathbf{u_1}$ and $\mathbf{u_2}$ are given in Exercise 11, $a_1 = \mathbf{u_1}^{\mathrm{T}}\mathbf{v}/\mathbf{u_1}^{\mathrm{T}}\mathbf{u_1} = 6/5$, and $a_2 = \mathbf{u_2}^{\mathrm{T}}\mathbf{v}/\mathbf{u_2}^{\mathrm{T}}\mathbf{u_2} = 2$. Thus $\mathbf{w}^* = [2,2,2]^{\mathrm{T}}$.

13. If $\mathbf{u_1} = [1,1,0]^{\mathrm{T}}$ and $\mathbf{u_2} = [1/2, -1/2, 1]^{\mathrm{T}}$ then $\{\mathbf{u_1}, \mathbf{u_2}\}$ is an orthogonal basis for W. The vector $\mathbf{w}^*$ is given by $\mathbf{w}^* = a_1\mathbf{u_1} + a_2\mathbf{u_2}$ where $a_1 = \mathbf{u_1}^{\mathrm{T}}\mathbf{v}/\mathbf{u_1}^{\mathrm{T}}\mathbf{u_1} = 1$ and $a_2 = \mathbf{u_2}^{\mathrm{T}}\mathbf{v}/\mathbf{u_2}^{\mathrm{T}}\mathbf{u_2} = 4$. Thus $\mathbf{w}^* = [3,-1,4]^{\mathrm{T}}$.

14. $\mathbf{w}^* = a_1\mathbf{u_1} + a_2\mathbf{u_2}$ where $\mathbf{u_1}$ and $\mathbf{u_2}$ are given in Exercise 13, $a_1 = \mathbf{u_1}^{\mathrm{T}}\mathbf{v} / \mathbf{u_1}^{\mathrm{T}}\mathbf{u_1} = 3$ and $a_2 = \mathbf{u_2}^{\mathrm{T}}\mathbf{v} / \mathbf{u_2}^{\mathrm{T}}\mathbf{u_2} = 2$. Thus $\mathbf{w}^* = [4, 2, 2]^{\mathrm{T}}$.

15. If $\mathbf{u_1} = [1, -1, 1]^{\mathrm{T}}$ and $\mathbf{u_2} = [1, 1, 0]^{\mathrm{T}}$ then $\{\mathbf{u_1}, \mathbf{u_2}\}$ is an orthogonal basis for W. The vector $\mathbf{w}^*$ is given by $\mathbf{w}^* = a_1\mathbf{u_1} + a_2\mathbf{u_2}$ where $a_1 = \mathbf{u_1}^{\mathrm{T}}\mathbf{v} / \mathbf{u_1}^{\mathrm{T}}\mathbf{u_1} = 2$ and $a_2 = \mathbf{u_2}^{\mathrm{T}}\mathbf{v} / \mathbf{u_2}^{\mathrm{T}}\mathbf{u_2} = 1$. Therefore $\mathbf{w}^* = [3, -1, 2]^{\mathrm{T}}$.

16. $\mathbf{w}^* = a_1\mathbf{u_1} + a_2\mathbf{u_2}$ where $\mathbf{u_1}$ and $\mathbf{u_2}$ are given in Exercise 15, $a_1 = \mathbf{u_1}^{\mathrm{T}}\mathbf{v} / \mathbf{u_1}^{\mathrm{T}}\mathbf{u_1} = 1$ and $a_2 = \mathbf{u_2}^{\mathrm{T}}\mathbf{v} / \mathbf{u_2}^{\mathrm{T}}\mathbf{u_2} = 2$. Therefore $\mathbf{w}^* = [3, 1, 1]^{\mathrm{T}}$.

3.10 Supplementary Exercises

1. Clearly $\boldsymbol{\theta} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is in W. Suppose $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is in W. Then $x_1x_2 = 0$. If a is any scalar then $a\mathbf{x} = \begin{bmatrix} ax_1 \\ ax_2 \end{bmatrix}$ and $(ax_1)(ax_2) = a^2x_1x_2 = 0$, so $a\mathbf{x}$ is in W.
   Now $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are in W, but $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in W. Therefore, W does not satisfy (S2).

2. Clearly $\boldsymbol{\theta} = [0, 0]^T$ is in W. Let $\mathbf{u}$ and $\mathbf{v}$ be in W, where $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Then $u_1 \geq 0$, $u_2 \geq 0$, $v_1 \geq 0$, and $v_2 \geq 0$. If follows that $u_1 + v_1 \geq 0$ and $u_2 + v_2 \geq 0$. Thus $\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$ is in W. If $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $a = -1$ then $\mathbf{u}$ is in W but $a\mathbf{u}$ is not in W. Therefore W does not satisfy (S3).

3. (a) $A\mathbf{x} = 3\mathbf{x}$ if and only if $(A - 3I)\mathbf{x} = \boldsymbol{\theta}$. Thus, W is the null space of the matrix $A - 3I$.
   (b) The system of equations $(A - 3I)\mathbf{x} = \boldsymbol{\theta}$ has solution $x_1 = -x_2 + x_3$, x_2 and x_3 arbitrary. Therefore $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for W.

4. Let $S = \{\mathbf{u}_1, \mathbf{u}_2\}$, $T = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, and $\mathbf{b} = [b_1, b_2, b_3]^T$. Reducing the matrices $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{b}]$ and $[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{b}]$ yields $\begin{bmatrix} 1 & 0 & 2b_2 - b_1 \\ 0 & 1 & b_1 - b_2 \\ 0 & 0 & -5b_1 + 7b_2 + b_3 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 3 & b_1 \\ 0 & 1 & 2 & b_2 \\ 0 & 0 & 0 & -5b_1 + 7b_2 + b_3 \end{bmatrix}$, respectively. Thus $\mathrm{Sp}\,(S) = \mathrm{Sp}\,(T) = \{\mathbf{b} : -5b_1 + 7b_2 + b_3 = 0\}$. Alternatively, reducing the matrices $[u_1, u_2]^T$ and $[v_1, v_2, v_3]^T$ gives $\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -7 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -7 \\ 0 & 0 & 0 \end{bmatrix}$, respectively. Therefore, $\{[1, 0, 5]^T, [0, 1, -7]^T\}$ is a basis for both $\mathrm{Sp}\,(S)$ and $\mathrm{Sp}\,(T)$.

5. (a) A reduces to $\begin{bmatrix} 1 & -1 & 0 & 7 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so rank $(A) = 2$ and nullity $(A) = 2$.

   (b) $\{[1, -1, 0, 7], [0, 0, 1, -2]\}$.

   (c) $\{[1, 2, 1]^T, [2, 5, 0]^T\}$.

   (d) $\{[1, 2, 1], [2, 5, 0]\}$ is a basis for the row space of A^T and $\{[1, -1, 0, 7]^T, [0, 0, 1, -2]^T\}$ is a basis for the column space of A^T.

   (e) The homogenous system of equations $A\mathbf{x} = \boldsymbol{\theta}$ has solution $x_1 = x_2 - 7x_4$, $x_3 = 2x_4$, x_2 and x_4 arbitrary.A basis for $\mathcal{N}(A)$ is $\{[1, 1, 0, 0]^T, [-7, 0, 2, 1]^T\}$.

6. (a) The matrix $A = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ reduces to $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. If follows that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for Sp (S).

   (b) A^T reduces to $\begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ so $\{[3, 0, 1]^T, [0, 3, -2]^T\}$ is a basis for Sp (S).

   (c) Let $\mathbf{b} = [b_1, b_2, b_3]^T$. The matrix $[A, \mathbf{b}]$ reduces to $\begin{bmatrix} 1 & 0 & 1 & (2b_1 - b_2)/3 \\ 0 & 1 & 2 & (b_1 + b_2)/3 \\ 0 & 0 & 0 & (-b_1 + 2b_2 - 3b_3)/3 \end{bmatrix}$. Therefore, Sp $(S) = \{\mathbf{b} : -b_1 + 2b_2 - 3b_3 = 0\}$. If follows that $\{[2, 1, 0]^T, [3, 0, 1]^T\}$ is a basis for Sp (S).

7. The matrix A is row equivalent to the $(m \times n)$ martrix

$$\begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ 0 & 1 & \dots & n-2 & n-1 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

   Thus, rank $(A) = 2$, nullity $(A) = n - 2$, and $\{[1, 1, \dots, 1, 1], [0, 1, \dots, n-2, n-1]\}$ is a basis for the row space of A.

8. (a) 1 (b) 1 (c) 0

9. (a) 1 (b) 2 (c) 2

10. $T(\mathbf{e}_1) = T(\mathbf{x}_1) - 2T(\mathbf{x}_2) = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$ and $T(\mathbf{e}_2) = 2T(\mathbf{x}_1) + T(\mathbf{x}_2) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ so $\begin{bmatrix} -3 & 4 \\ 3 & 1 \end{bmatrix}$ is the matrix of T.

11. (a) $\mathbf{x}_1 = \begin{bmatrix} -5-8u \\ 2+3u \\ u \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 3-8v \\ -1+3v \\ v \end{bmatrix}$, where u and v are arbitrary.

(b) $\mathbf{x}_3 = \begin{bmatrix} -8w \\ 3w \\ w \end{bmatrix}$ for any nonzero w.

(c) Taking $u = v = 0$ and $w = 1$ in (a) and (b) gives
$B = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\} = \{[-5, 2, 0]^T, [3, -1, 0]^T, [-8, 3, 1]^T\}$.
The set B is linearly independent, so is a basis for R^3.

(d) $\mathbf{e}_1 = \mathbf{x}_1 + 2\mathbf{x}_2$ so $T(\mathbf{e}_1) = T(\mathbf{x}_1) + 2T(\mathbf{x}_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. $\mathbf{e}_2 = 3\mathbf{x}_1 + 5\mathbf{x}_2$ so $T(\mathbf{e}_2) = 3T(\mathbf{x}_1) + 5T(\mathbf{x}_2) = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. $\mathbf{e}_3 = -\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3$ so $T(\mathbf{e}_3) = -T(\mathbf{x}_1) + T(\mathbf{x}_2) + T(\mathbf{x}_3) = -\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. If follows that $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 5 & 1 \end{bmatrix}$.

12. $\{[1, 0, 2, 0, -3, 1], [0, 1, -1, 0, 2, 2], [0, 0, 0, 1, -1, -2]\}$ is a basis for the row space of A. Therefore rank $(A) = 3$ and nullity $(A) = 3$.

13. $\mathbf{b} = [a, b, c, d]^T$ is in $\mathcal{R}(T)$ if and only if $-16a - 7b + 9c + d = 0$. Therefore, $\mathbf{w}_1$, $\mathbf{w}_3$, and $\mathbf{w}_4$ are in $\mathcal{R}(T)$.

14. For $\mathbf{w}_1$, $\mathbf{x}_1 = \begin{bmatrix} 1-2u+3v-w \\ u-2v-2w \\ u \\ v+2w \\ v \\ w \end{bmatrix}$; for $\mathbf{w}_3$, $\mathbf{x}_3 = \begin{bmatrix} 4-2u+3v-w \\ 7+u-2v-2w \\ u \\ -3+v+2w \\ v \\ w \end{bmatrix}$; for $\mathbf{w}_4$, $\mathbf{x}_4 = \begin{bmatrix} 1-2u+3v-w \\ 1+u-2v-2w \\ u \\ v+2w \\ v \\ w \end{bmatrix}$ In each case, u, v, and w are arbitrary.

15. See the solution to Exercise 14. $T(\mathbf{x}_i) = \mathbf{w}_i$ if and only if $A\mathbf{x}_i = \mathbf{w}_i$.

16. (a) In the solutions to Exercise 14, take $u = v = w = 0$. This gives $\mathbf{w}_1 = A\mathbf{x}_1 = \mathbf{A}_1$ $\mathbf{w}_3 = A\mathbf{x}_3 = 4\mathbf{A}_1 + 7\mathbf{A}_2 - 3\mathbf{A}_4$, and $\mathbf{w}_4 = A\mathbf{x}_4 = \mathbf{A}_1 + \mathbf{A}_2$.

(b) $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_4\}$

(c) The homogenous system of equations, $A\mathbf{x} = \boldsymbol{\theta}$, has solution $x_1 = -2x_3 + 3x_5 - x_6$, $x_2 = x_3 - 2x_5 - 2x_6$, $x_4 = x_5 + 2x_6$, x_3, x_5, x_6 arbitrary. Setting $x_3 = 1$, $x_5 = x_6 = 0$ yields $x_1 = -2$, $x_2 = 1$, and $x_4 = 0$. If follows that $-2\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 = \boldsymbol{\theta}$, so $\mathbf{A}_3 = 2\mathbf{A}_1 - \mathbf{A}_2$. Similarly, $\mathbf{A}_5 = -3\mathbf{A}_1 + 2\mathbf{A}_2 - \mathbf{A}_4$ and $\mathbf{A}_6 = \mathbf{A}_1 + 2\mathbf{A}_2 - 2\mathbf{A}_4$.

(d) $\mathbf{b}$ is in the column space of A since, by the condition established in the solution to Exercise 13, $\mathbf{b}$ is in $\mathcal{R}(T)$. The system of equations $A\mathbf{x} = \mathbf{b}$ has solution $x_1 = 0$, $x_2 = -5$, $x_3 = 0$, $x_4 = 2$, $x_5 = 0$, $x_6 = 0$, so $\mathbf{b} = -5\mathbf{A}_2 + 2\mathbf{A}_4$.

(e) $A\mathbf{x} = 2\mathbf{A}_1 + 3\mathbf{A}_2 + \mathbf{A}_3 - \mathbf{A}_4 + \mathbf{A}_5 + \mathbf{A}_6$. Substituting for $\mathbf{A}_3$, $\mathbf{A}_5$, and $\mathbf{A}_6$ the expressions obtained in (c) gives $A\mathbf{x} = 2\mathbf{A}_1 + 6\mathbf{A}_2 - 4\mathbf{A}_4$.

17. (a) If $\mathbf{b} = [a, b, c, d]^T$ then $\mathcal{R}(T) = \{\mathbf{b} : -16a - 7b + 9c + d = 0\}$ (cf. the solution to Exercise 13.). The set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is the basis for $\mathcal{R}(T)$, where $\mathbf{u}_1 = [1, 0, 0, 16]^T$, $\mathbf{u}_2 = [0, 1, 0, 7]^T$, and $\mathbf{u}_3 = [0, 0, 1, -9]^T$. Moreover, if $\mathbf{b}$ is in $\mathcal{R}(T)$ then $\mathbf{b} = [a, b, c, 16a + 7b - 9c]^T = a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3$.

(b) $\mathbf{b} = \mathbf{u}_1 + 2\mathbf{u}_2 + 3\mathbf{u}_3$.

18. (a) If $\mathbf{v}_1 = [-2, 1, 1, 0, 0, 0]^T$, $\mathbf{v}_2 = [3, -2, 0, 1, 1, 0]^T$, and $\mathbf{v}_3 = [-1, -2, 0, 2, 0, 1]^T$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathcal{N}(T)$ (cf. the solution to Exercise 16(c)).
Moreover,if $\mathbf{x} = [x_1, x_2, x_3, x_4, x_5, x_6]^T$ is in $\mathcal{N}(T)$ then
$\mathbf{x} = [-2x_3 + 3x_5 - x_6, x_3 - 2x_5 - 2x_6, x_3, x_5 + 2x_6, x_5, x_6]^T$, where x_3, x_5, x_6 are arbitrary. Thus, $\mathbf{x} = x_3\mathbf{v}_1 + x_5\mathbf{v}_2 + x_6\mathbf{v}_3$.

(b) $\mathbf{x} = \mathbf{v}_1 + 2\mathbf{v}_2 - 2\mathbf{v}_3$.

3.11 Conceptual Exercises

1. False. In R^2 let $W = \{[a, a]^T : a \text{ arbitrary }\}$. Then $\mathbf{e}_1 + \mathbf{e}_2$ is in W but neither $\mathbf{e}_1$ nor $\mathbf{e}_2$ is in W.

2. True. Since $a\mathbf{x}$ is in W, $a^{-1}(a\mathbf{x}) = \mathbf{x}$ is in W.

3. False. $\{\boldsymbol{\theta}\}$ is a linearly dependent subset of R^n.

4. True. cf. Theorem 9(1).

5. True. cf. Theorem 9(2).

6. False. Consider $S = \{[0, 0]^T, [1, 0]^T, [3, 0]^T\}$ in R^2.

7. False. Consider $S_1 = \{[1, 1]^T\}$ and $\{[1, 0]^T, [0, 1]^T\}$ in R^2.

8. False. The sets $\{[1, 0]^T, [0, 1]^T\}$ and $\{[2, 0]^T, [1, 1]^T\}$ are both bases for R^2.

9. False. A basis for W must contain exactly k vectors but if $W \neq \{\boldsymbol{\theta}\}$ then W contains infinitely many vectors.

10. False. Let $B = \{[1,0]^T, [0,1]^T\}$. Then B is a basis for R^2 but no subset of B is a basis for $W = \{[a,a]^T : a \text{ arbitrary } \}$.

11. True. A basis for W must also be a basis for R^n.

12. False. $B_1 = \{[1,1,0]^T, [1,0,1]^T\}$ is a basis for $W_1 = \{[b+c,b,c]^T : b,c \text{ arbitrary } \}$ and $B_2 = \{[2,1,0]^T, [-1,0,1]^T\}$ is a basis for $W_2 = \{[2b-c,b,c]^T : b,c \text{ arbitrary } \}$. $B_1 \cap B_2 = \emptyset$ but $W_1 \cap W_2 = \{[3c,2c,c]^T : c \text{ arbitrary } \}$.

13. No. $\boldsymbol{\theta}$ is not in V.

14. B need not be a subset of W.

Chapter 4

The Eigenvalue Problems

4.1 Introduction

1. The matrix $A - \lambda I = \begin{bmatrix} 1-\lambda & 0 \\ 2 & 3-\lambda \end{bmatrix}$ is singular if and only if $0 = (1-\lambda)(3-\lambda)$. Thus $\lambda = 1$ and $\lambda = 3$ are eigenvalues for A. The eigenvectors corresponding to $\lambda = 1$ $(A-I)\mathbf{x} = \theta$. Solving yields $x_1 = -x_2$, x_2 arbitrary. Therefore any vector of the form $\mathbf{x} = a \begin{bmatrix} -1 \\ 1 \end{bmatrix}, a \neq 0,$ is an eigenvector for $\lambda = 1$. Similarly the eigenvectors corresponding to $\lambda = 3$ are the nontrivial solutions to $(A-3I)\mathbf{x} = \theta$. Solving yields $x_1 = 0, x_2$ arbitrary, so any vector of the form $\mathbf{x} = a \begin{bmatrix} 0 \\ 1 \end{bmatrix}, a \neq 0,$ is an eigenvector for $\lambda = 3$.

2. The matrix $A - \lambda I = \begin{bmatrix} 2-\lambda & 1 \\ 0 & -1-\lambda \end{bmatrix}$ is singular if and only if $0 = (2-\lambda)(-1-\lambda)$. Therefore A has eigenvalues $\lambda = 2$ and $\lambda = -1$. For $\lambda = 2$ the corresponding eigenvectors are $\mathbf{x} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix}, a \neq 0$. For $\lambda = -1$ the corresponding eigenvectors are $\mathbf{x} = a \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, $a \neq 0$.

3. The matrix $A - \lambda I = \begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix}$ is singular if and only if $0 = (2-\lambda)(2-\lambda) - 1 = \lambda^2 - 4\lambda + 3 = (\lambda-1)(\lambda-3)$. Therefore A has eigenvalues $\lambda = 1$ and $\lambda = 3$. Solving $(A-I)\mathbf{x} = \theta$ yields $x_1 = x_2, x_2$ arbitrary, so any vector of the form $\mathbf{x} = a \begin{bmatrix} 1 \\ 1 \end{bmatrix}, a \neq 0,$ is an eigenvector corresponding to $\lambda = 1$. Solving $(A-3I)\mathbf{x} = \theta$ yields $x_1 = -x_2, x_2$ arbitrary, so any vector of the form $\mathbf{x} = a \begin{bmatrix} -1 \\ 1 \end{bmatrix}, a \neq 0,$ is an eigenvector for $\lambda = 3$.

4. $\lambda = 2, \mathbf{x} = a \begin{bmatrix} -2 \\ 1 \end{bmatrix}, a \neq 0$; $\lambda = 3, \mathbf{x} = a \begin{bmatrix} -1 \\ 1 \end{bmatrix}, a \neq 0$.

5. The matrix $A - \lambda I = \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}$ is singular if and only if $0 = (2-\lambda)(2-\lambda) - 1 = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3)$. Therefore A has eigenvalues $\lambda = 1$ and $\lambda = 3$. Solving $(A - I)\mathbf{x} = \theta$ yields $x_1 = -x_2, x_2$ arbitrary, so any vector of the form $\mathbf{x} = a \begin{bmatrix} -1 \\ 1 \end{bmatrix}, a \neq 0,$ is an eigenvector for $\lambda = 1$. Solving $(A - 3I)\mathbf{x} = \theta$ yields $x_1 = x_2, x_2$ arbitrary, so any vector $\mathbf{x} = a \begin{bmatrix} 1 \\ 1 \end{bmatrix}, a \neq 0,$ is an eigenvector for $\lambda = 3$.

6. $\lambda = 2, \mathbf{x} = a \begin{bmatrix} 1 \\ 1 \end{bmatrix}, a \neq 0.$

7. The matrix $A - \lambda I = \begin{bmatrix} 1-\lambda & 0 \\ 2 & 1-\lambda \end{bmatrix}$ is singular if and only if $0 = (1-\lambda)^2$, so $\lambda = 1$ is the only eigenvalue for A. Solving $(A - I)\mathbf{x} = \theta$ yields $x_1 = 0, x_2$ arbitrary, so any vector $\mathbf{x} = a \begin{bmatrix} 0 \\ 1 \end{bmatrix}, a \neq 0,$ is an eigenvector for $\lambda = 1$.

8. $\lambda = 2, \mathbf{x} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix}, a \neq 0.$

9. The matrix $A - \lambda I = \begin{bmatrix} 2-\lambda & 2 \\ 3 & 3-\lambda \end{bmatrix}$ is singular if and only if $0 = (2-\lambda)(3-\lambda) - 6 = \lambda^2 - 5\lambda = \lambda(\lambda - 5)$. Therefore A has eigenvalues $\lambda = 0$ and $\lambda = 5$. Solving $A\mathbf{x} = \theta$ yields $x_1 = -x_2, x_2$ arbitrary, so $\mathbf{x} = a \begin{bmatrix} -1 \\ 1 \end{bmatrix}, a \neq 0,$ is an eigenvector for $\lambda = 0$. Solving

$(A - 5I)\mathbf{x} = \theta$ yields $x_1 = (2/3)x_2, x_2$ arbitrary, so $\mathbf{x} = a \begin{bmatrix} 2 \\ 3 \end{bmatrix},$

$a \neq 0,$ is an eigenvector for $\lambda = 5$.

10. $\lambda = 0, \mathbf{x} = a \begin{bmatrix} -2 \\ 1 \end{bmatrix}, a \neq 0; \lambda = 9, \mathbf{x} = a \begin{bmatrix} 1 \\ 4 \end{bmatrix}, a \neq 0.$

11. The matrix $A - \lambda I = \begin{bmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{bmatrix}$ is singular if and only if $0 = (1-\lambda)(3-\lambda) + 1 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$. Therefore $\lambda = 2$ is the only eigenvalue for A. Solving $(A - 2I)\mathbf{x} = \theta$ yields $x_1 = -x_2$, so $\mathbf{x} = a \begin{bmatrix} -1 \\ 1 \end{bmatrix}, a \neq 0$ is an eigenvector for $\lambda = 2$.

12. $\lambda = 3, \mathbf{x} = a \begin{bmatrix} -1 \\ 1 \end{bmatrix}, a \neq 0.$

13. The matrix $A-\lambda I = \begin{bmatrix} -2-\lambda & -1 \\ 5 & 2-\lambda \end{bmatrix}$ is singular if and only if $0 = (-2-\lambda)(2-\lambda)+5 = \lambda^2+1$. Solving yields $\lambda = \pm i$.

14. The matrix $A-\lambda I$ is singular if and only if $\lambda^2+1=0$. Solving yields $\lambda = \pm i$.

15. The matrix $A-\lambda I = \begin{bmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{bmatrix}$ is singular if and only if $0 = (2-\lambda)(2-\lambda)+1 = \lambda^2-4\lambda+5$. Solving we obtain $\lambda = 2 \pm i$.

16. $A-\lambda I$ is singular if and only if $\lambda^2-2\lambda+2=0$. Solving yields $\lambda = 1 \pm i$.

17. The matrix $A-\lambda I = \begin{bmatrix} a-\lambda & b \\ b & d-\lambda \end{bmatrix}$ is singular if and only if $0 = (a-\lambda)(d-\lambda)-b^2 = \lambda^2-(a+d)\lambda+(ad-b^2)$. Note that $(a+d)^2-4(ad-b^2) = (a-d)^2+4b^2 \geq 0$, so the equation has real roots.

18. The matrix $A-\lambda I$ is singular if and only if $\lambda^2-2a\lambda+(a^2+b^2)=0$. Since $(2a)^2-4(a^2+b^2) = -4b^2 < 0$ the equation has no real roots.

19. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The matrix $A^{\mathrm{T}}-\lambda I = \begin{bmatrix} a-\lambda & c \\ b & d-\lambda \end{bmatrix}$ is singular if and only if $0 = (a-\lambda)(d-\lambda)-bc = \lambda^2-(a+d)\lambda+$
$(ad-bc)$. Therefore the eigenvalues of A^{T} are roots of (5), so
coincide with the eigenvalues of A.

4.2 Determinants and the Eigenvalue Problem

1. $M_{11} = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 4 & 1 \\ 2 & 0 & -2 \end{bmatrix}$. $A_{11} = \det(M_{11}) =$
$\begin{vmatrix} 4 & 1 \\ 0 & -2 \end{vmatrix} - 3\begin{vmatrix} 2 & 1 \\ 2 & -2 \end{vmatrix} - \begin{vmatrix} 2 & 4 \\ 2 & 0 \end{vmatrix} = 18.$

2. $M_{21} = \begin{bmatrix} -1 & 3 & 1 \\ 2 & 4 & 1 \\ 2 & 0 & -2 \end{bmatrix}$. $A_{21} = -\det(M_{21}) = -18.$

3. $M_{31} = \begin{bmatrix} -1 & 3 & 1 \\ 1 & 3 & -1 \\ 2 & 0 & -2 \end{bmatrix}$. $A_{31} = \det(M_{31}) =$
$-\begin{vmatrix} 3 & -1 \\ 0 & -2 \end{vmatrix} - 3\begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = 0.$

4. $M_{41} = \begin{bmatrix} -1 & 3 & 1 \\ 1 & 3 & -1 \\ 2 & 4 & 1 \end{bmatrix}$. $A_{41} = -\det(M_{41}) = 18.$

5. $M_{34} = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 1 & 3 \\ 2 & 2 & 0 \end{bmatrix}$. $A_{34} = -\det(M_{34}) =$
$-2\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} - \begin{vmatrix} 4 & 3 \\ 2 & 0 \end{vmatrix} - 3\begin{vmatrix} 4 & 1 \\ 2 & 2 \end{vmatrix} = 0.$

6. $M_{43} = \begin{bmatrix} 2 & -1 & 1 \\ 4 & 1 & -1 \\ 6 & 2 & 1 \end{bmatrix}$. $A_{43} = -\det(M_{43}) = -18.$

7. $\det(A) = 2A_{11} + 4A_{21} + 6A_{31} + 2A_{41} =$
$2(18) + 4(-18) + 6(0) + 2(18) = 0.$

8. $\det(A) = 5; A$ is nonsingular.

9. $\det(A) = 0; A$ is singular

10. $\det(A) = 0; A$ is singular.

11. $\det(A) = -1; A$ is nonsingular.

12. $\det(A) = 0; A$ is singular.

13. $\det(A) = 2\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} + 3\begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix} + 2\begin{vmatrix} -1 & -2 \\ 3 & 1 \end{vmatrix} = 6;$ A is nonsingular.

14. $\det(A) = 0;$ A is singular.

15. $\det(A) = 2\begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} = 20;$ A is nonsingular.

16. By Theorem 4, $\det(A) = 2(1)(2) = 4;$ A is nonsingular.

17. Expansion along the first column of A yields $\det(A) = \begin{vmatrix} 3 & 0 & 0 \\ 4 & 1 & 2 \\ 3 & 1 & 4 \end{vmatrix}$.

    Now expansion along the first row gives $\det(A) = 3\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = 6.$
    A is nonsingular.

18. $\det(A) = 1; A$ is nonsingular.

19. Expansion along the first column in successive steps yields
$$det(A) = -3\begin{vmatrix} 0 & 0 & 2 \\ 0 & 3 & 1 \\ 2 & 1 & 2 \end{vmatrix} = (-3)(2)\begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} = (-3)(2)(-6) = 36.$$ A is nonsingular.

20. (a) The described algorithm yields $a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$ which equals $\det(A)$.

    (b) Note that for a (4×4) matrix, $A = (a_{ij})$, the definition of $\det(A)$ yields a sum of products with 24 terms, whereas the "basketweave algorithm" yields an expression with only eight summands. For $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$, the basket weave algorithm gives 7, but $\det(A) = 1$.

21. Det $(A) = 4x - 2y - 2$, so A is singular when $4x - 2y - 2 = 0$, that is when $y = 2x - 1$.

22. Det $(A) = (x - 2)(y + 1)$, so A is singular if either $x = 2$ or $y = -1$.

23. For $n = 2$, $\begin{vmatrix} d & 1 \\ 1 & d \end{vmatrix} = d^2 - 1 = (d-1)(d+1)$. For $n = 3$, $\begin{vmatrix} d & 1 & 1 \\ 1 & d & 1 \\ 1 & 1 & d \end{vmatrix} = d\begin{vmatrix} d & 1 \\ 1 & d \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & d \end{vmatrix} + \begin{vmatrix} 1 & d \\ 1 & 1 \end{vmatrix} = d(d-1)(d+1) - (d-1) + (1-d)$
$= (d-1)^2(d+2)$. For $n = 4$, $\begin{vmatrix} d & 1 & 1 & 1 \\ 1 & d & 1 & 1 \\ 1 & 1 & d & 1 \\ 1 & 1 & 1 & d \end{vmatrix} = d\begin{vmatrix} d & 1 & 1 \\ 1 & d & 1 \\ 1 & 1 & d \end{vmatrix} - \begin{vmatrix} 1 & 1 & 1 \\ 1 & d & 1 \\ 1 & 1 & d \end{vmatrix} + \begin{vmatrix} 1 & d & 1 \\ 1 & 1 & 1 \\ 1 & 1 & d \end{vmatrix} -$
$\begin{vmatrix} 1 & d & 1 \\ 1 & 1 & d \\ 1 & 1 & 1 \end{vmatrix} = d(d-1)^2(d+2) - 3(d-1)^2 = (d-1)^3(d+3)$.

24. (a) If A is singular then $\det(A) = 0$ so $\det(AB) = \det(A)\det(B) = 0$. Therefore AB is singular. Similarly if B is singular then so is AB.

    (b) If AB is singular then $0 = \det(AB) = \det(A)\det(B)$. Therefore either $\det(A) = 0$ or $\det(B) = 0$; that is either A or B is singular.

25. $1 = \det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1})$. Therefore $\det(A^{-1}) = 1/\det(A)$.

26. $\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$.

27. Det $(ABA^{-1}) = \det(A)\det(B)/\det(A) = \det(B) = 5$.

28. Det $(A^2B) = [\det(A)]^2 \det(B) = 3^2 5 = 45$.

29. Det $(A^{-1}B^{-1}A^2) = [\det(A)]^2/[\det(A)\det(B)] = \det(A)/\det(B) = 3/5$.

30. Det $(AB^{-1}A^{-1}B) = [\det(A)/\det(B)][\det(B)/\det(A)] = 1$.

31. (a) $H(n) = n!/2$.

    (b) $n = 2$, 3 secs; $n = 5$, 3 min; $n = 10$, 63 days.

32. If $U = [u_{ij}]$ and $V = [v_{ij}]$ then, by Theorem 4, $\det(U) = u_{11}u_{22}\cdots u_{nn}$ and $\det(V) = v_{11}v_{22}\cdots v_{nn}$. By Exercise 59, Section 1.6, UV is an upper triangular matrix. Moreover UV is the (n x n) matrix $[a_{ij}]$ where $a_{11} = u_{11}v_{11}, \ldots, a_{nn} = u_{nn}v_{nn}$. It follow that $\det(UV) = (u_{11}v_{11})(u_{22}v_{22})\cdots(u_{nn}v_{nn}) = \det(U)\det(V)$.

33. Suppose that V is a lower triangular matrix with diagonal entries $t_1, t_2, \ldots, t_n$. Then V^{T} is an upper triangular matrix with the same diagonal entries so $\det(V) = \det(V^{\mathrm{T}}) = t_1t_2\cdots t_n$.

34. For $n = 2, \det(T) = \begin{vmatrix} t_{11} & t_{12} \\ 0 & t_{22} \end{vmatrix} = t_{11}t_{22}$. Assume that if T is a (k x k) matrix then $\det(T) = t_{11}t_{22}\cdots t_{kk}$. Now assume that T is a [(k+1) x (k+1)] matrix. Thus $T = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1,k+1} \\ 0 & t_{22} & \cdots & t_{2,k+1} \\ \vdots & & & \vdots \\ 0 & 0 & \ldots & t_{k+1,k+1} \end{bmatrix}$.

    Expansion along the first column yields $\det(T) = t_{11}\begin{vmatrix} t_{22} & \cdots & t_{2,k+1} \\ \vdots & & \vdots \\ 0 & \cdots & t_{k+1,k+1} \end{vmatrix}$. Since the result holds for (k x k) matrices, we have $\det(T) = t_{11}t_{22}\cdots t_{k+1,k+1}$. It follows by induction that for any integer $n \geq 2, \det(T) = t_{11}t_{22}\cdots t_{nn}$.

4.3 Elementary Operations and Determinants

1. $\det(A) = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & 2 \\ -1 & 1 & 3 \end{vmatrix} \begin{Bmatrix} R_2 - 3R_1 \\ R_3 + R_1 \\ = \end{Bmatrix} \begin{vmatrix} 1 & 2 & 1 \\ 0 & -6 & -1 \\ 0 & 3 & 4 \end{vmatrix} = \begin{vmatrix} -6 & -1 \\ 3 & 4 \end{vmatrix} = -21$.

2. $\det(A) = 20$.

3. $\det(A) = \begin{vmatrix} 3 & 6 & 9 \\ 2 & 0 & 2 \\ 1 & 2 & 0 \end{vmatrix} = (3)(2)\begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{vmatrix} \overset{\left\{\begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix}\right\}}{=}$
$(6)\begin{vmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ 0 & 0 & -3 \end{vmatrix} = 36.$

4. $\det(A) = -24.$

5. $\det(A) = \begin{vmatrix} 2 & 4 & -3 \\ 3 & 2 & 5 \\ 2 & 3 & 4 \end{vmatrix} = (1/2)\begin{vmatrix} 2 & 4 & -3 \\ 6 & 4 & 10 \\ 2 & 3 & 4 \end{vmatrix} \overset{\left\{\begin{matrix} R_2 - 3R_1 \\ R_3 - R_1 \end{matrix}\right\}}{=}$
$(1/2)\begin{vmatrix} 2 & 4 & -3 \\ 0 & -8 & 19 \\ 0 & -1 & 7 \end{vmatrix} = (2)(1/2)\begin{vmatrix} -8 & 19 \\ -1 & 7 \end{vmatrix} = -37.$

6. $\det(A) = -21.$

7. $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 3 \\ 1 & 1 & 0 & 1 \\ 1 & 4 & 2 & 2 \end{vmatrix} \overset{\{C_3 \leftrightarrow C_4\}}{=} (-1)\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 4 & 2 & 2 \end{vmatrix} \overset{\{C_2 \leftrightarrow C_3\}}{=}$

$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 4 & 2 \end{vmatrix} = (1)(3)(1)(2) = 6.$

8. $\begin{vmatrix} 0 & 0 & 3 & 1 \\ 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 2 & 2 & 1 \end{vmatrix} = (-1)\begin{vmatrix} 2 & 1 & 0 & 1 \\ 0 & 2 & 2 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 2 \end{vmatrix} = -24.$

9. $\begin{vmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 4 & 1 & 3 \\ 2 & 1 & 5 & 6 \end{vmatrix} \overset{\left\{\begin{matrix} C_1 \leftrightarrow C_4 \\ C_2 \leftrightarrow C_3 \end{matrix}\right\}}{=} \begin{vmatrix} 0 & 2 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 3 & 1 & 4 & 0 \\ 6 & 5 & 1 & 2 \end{vmatrix} \overset{\{C_1 \leftrightarrow C_2\}}{=}$
$(-1)\begin{vmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 3 & 4 & 0 \\ 5 & 6 & 1 & 2 \end{vmatrix} = (-1)(2)(3)(4)(2) = -48.$

10. $\begin{vmatrix} 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 3 \\ 0 & 0 & 0 & 5 \\ 0 & 3 & 1 & 2 \end{vmatrix} = (-1) \begin{vmatrix} 1 & 2 & 1 & 3 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 \end{vmatrix} = -15.$

11. $\begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 2 & 6 & 3 \\ 2 & 4 & 1 & 5 \\ 0 & 0 & 0 & 4 \end{vmatrix} \overset{\{R_1 \leftrightarrow R_3\}}{=} (-1) \begin{vmatrix} 2 & 4 & 1 & 5 \\ 0 & 2 & 6 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{vmatrix} =$
$(-1)(2)(2)(1)(4) = -16.$

12. $\begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 3 \\ 2 & 1 & 0 & 6 \\ 3 & 2 & 2 & 4 \end{vmatrix} = (-1) \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 6 & 2 & 0 \\ 2 & 4 & 3 & 2 \end{vmatrix} = -12.$

13. $\det(B) = 3\det(A) = 6.$

14. $\det(B) = \det(A) = 2.$

15. $\det(B) = -\det(A) = -2.$

16. $\det(B) = \det(A) = 2.$

17. $\det(B) = -2\det(A) = -4.$

18. $\det(B) = \det(A) = 2.$

19. $\begin{vmatrix} 2 & 4 & 2 & 6 \\ 1 & 3 & 2 & 1 \\ 2 & 1 & 2 & 3 \\ 1 & 2 & 1 & 1 \end{vmatrix} \overset{\{R_1 - 2R_4\}}{=} \begin{vmatrix} 0 & 0 & 0 & 4 \\ 1 & 3 & 2 & 1 \\ 2 & 1 & 2 & 3 \\ 1 & 2 & 1 & 1 \end{vmatrix}$
$= (-4) \begin{vmatrix} 1 & 3 & 2 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix} \overset{\left\{ \begin{matrix} R_2 - 2R_1 \\ R_3 - R_1 \end{matrix} \right\}}{=} (-4) \begin{vmatrix} 1 & 3 & 2 \\ 0 & -5 & -2 \\ 0 & -1 & -1 \end{vmatrix} =$
$(-4) \begin{vmatrix} -5 & -2 \\ -1 & -1 \end{vmatrix} = -12.$

20. $-5.$

21. $\begin{vmatrix} 0 & 4 & 1 & 3 \\ 0 & 2 & 2 & 1 \\ 1 & 3 & 1 & 2 \\ 2 & 2 & 1 & 4 \end{vmatrix} \overset{\{R_4 - 2R_3\}}{=} \begin{vmatrix} 0 & 4 & 1 & 3 \\ 0 & 2 & 2 & 1 \\ 1 & 3 & 1 & 2 \\ 0 & -4 & -1 & 0 \end{vmatrix} =$

$$\begin{vmatrix} 4 & 1 & 3 \\ 2 & 2 & 1 \\ -4 & -1 & 0 \end{vmatrix} \overset{\{R_1 + R_3\}}{=} \begin{vmatrix} 0 & 0 & 3 \\ 2 & 2 & 1 \\ -4 & -1 & 0 \end{vmatrix} = (3) \begin{vmatrix} 2 & 2 \\ -4 & -1 \end{vmatrix} = 18.$$

22. 4.

23. $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \overset{\left\{ \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix} \right\}}{=} \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$

$= (b-a)(c-a) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix} \overset{\{R_3 - R_2\}}{=}$

$(b-a)(c-a) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{vmatrix}$

$= (b-a)(c-a)(c-b) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & 1 \end{vmatrix} = (b-a)(c-a)(c-b).$

24. $(b-a)(c-a)(c-b)(d-a)(d-b)(d-c)$.

25. Write $A = \begin{bmatrix} \mathbf{A_1} \\ \mathbf{A_2} \\ \vdots \\ \mathbf{A_n} \end{bmatrix}$ where $\mathbf{A_i} = [a_{i1}, a_{i2}, \ldots, a_{in}]$ is the i^{th} row of A. Then $cA = \begin{bmatrix} c\mathbf{A_1} \\ c\mathbf{A_2} \\ \vdots \\ c\mathbf{A_n} \end{bmatrix}$ so, by Theorem 7, $\det(cA) = c^n \det(A)$.

26. Suppose the i^{th} and j^{th} rows of A are identical and let B denote the matrix obtained by interchanging these two rows. By Theorem 6 $\det(B) = -\det(A)$. But $B = A$ so $\det(A) = \det(B)$. It follows that $\det(\mathrm{A}) = -\det(\mathrm{A})$, so $\det(A) = 0$.

27. $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is one possibility.

28. By Theorem 5, $\det(A) = \det(A^{\mathrm{T}})$. But $A^{\mathrm{T}} = -A$ so by Exercise 25, $\det(A^{\mathrm{T}}) = (-1)^n \det(A)$. Therefore $\det(A) = (-1)^n \det(A)$. In particular if n is odd then $\det(A) = -\det(A)$. It follows that $\det(A) = 0$ and hence A is singular.

4.4 Eigenvalues and the Characteristic Polynomial

1. $p(t) = (1-t)(3-t)$. The eigenvalues are $\lambda = 1$ and $\lambda = 3$, each with algebraic multiplicity 1.

2. $p(t) = (2-t)(-1-t)$. The eigenvalues are $\lambda = 2$ and $\lambda = -1$ each with algebraic multiplicity 1.

3. $p(t) = \begin{vmatrix} 2-t & -1 \\ -1 & 2-t \end{vmatrix} = (2-t)(2-t) - 1 = t^2 - 4t + 3 = (t-1)(t-3)$. The eigenvalues are $\lambda = 1$ and $\lambda = 3$, each with algebraic multiplicity 1.

4. $p(t) = (t-1)^2$. The only eigenvalue is $\lambda = 1$ and it has algebraic multiplicity 2.

5. $p(t) = \begin{vmatrix} 1-t & -1 \\ 1 & 3-t \end{vmatrix} = (1-t)(3-t) + 1 = t^2 - 4t + 4 = (t-2)^2$. The only eigenvalue is $\lambda = 2$ and it has algebraic multiplicity 2.

6. $p(t) = t(t-5)$. The eigenvalues are $\lambda = 0$ and $\lambda = 5$, each with algebraic multiplicity 1.

7. $p(t) = \begin{vmatrix} -6-t & -1 & 2 \\ 3 & 2-t & 0 \\ -14 & -2 & 5-t \end{vmatrix} = -t^3 + t^2 + t - 1 = -(t-1)^2(t+1)$. The eigenvalues are $\lambda = 1$ with algebraic multiplicity 2 and $\lambda = -1$ with algebraic multiplicity 1.

8. $p(t) = -t(t+1)^2$. The eigenvalues are $\lambda = 0$ with algebraic multiplicity 1, and $\lambda = -1$ with algebraic multiplicity 2.

9. $p(t) = \begin{vmatrix} 3-t & -1 & -1 \\ -12 & -t & 5 \\ 4 & -2 & -1-t \end{vmatrix} = -t^3 + 2t^2 + t - 2 =$

   $-(t-2)(t-1)(t+1)$. The eigenvalues are $\lambda = 2, \lambda = 1$, and $\lambda = -1$ each with algebraic multiplicity 1.

10. $p(t) = -(t-1)^3$.The only eigenvalue is $\lambda = 1$ and it has algebraic multiplicity 3.

11. $p(t) = \begin{vmatrix} 2-t & 4 & 4 \\ 0 & 1-t & -1 \\ 0 & 1 & 3-t \end{vmatrix} = (2-t)\begin{vmatrix} 1-t & -1 \\ 1 & 3-t \end{vmatrix} =$

    $(2-t)(t^2-4t+4) = -(t-2)^3$. The only eigenvalue is $\lambda = 2$ and it has algebraic multiplicity 3.

12. $p(t) = (t-5)^2(t+1)(t-15)$. The eigenvalues are $\lambda = 5$ with algebraic multiplicity 2, $\lambda = -1$ with algebraic multiplicity 1, and $\lambda = 15$ with algebraic multiplicity 1.

13. $p(t) = \begin{vmatrix} 5-t & 4 & 1 & 1 \\ 4 & 5-t & 1 & 1 \\ 1 & 1 & 4-t & 2 \\ 1 & 1 & 2 & 4-t \end{vmatrix} = t^4 - 18t^3 + 97t^2 - 180t + 100 = (t-1)(t-2)(t-5)(t-10)$. The eigenvalues are $\lambda = 1, \lambda = 2, \lambda = 5, \lambda = 10$, each with algebraic multiplicity 1.

14. $p(t) = (t-2)^3(t+2)$. The eigenvalues are $\lambda = 2$ with algebraic multiplicity 3, and $\lambda = -2$ with algebraic multiplicity 1.

15. Let $\mathbf{x}$ be an eigenvector corresponding to λ. Thus $\mathbf{x} \neq \theta$ and $A\mathbf{x} = \lambda\mathbf{x}$. Multiplication by A^{-1} yields $\mathbf{x} = A^{-1}(\lambda\mathbf{x}) = \lambda A^{-1}\mathbf{x}$. Since A is nonsingular $\lambda \neq 0$ (cf. Theorem 13). Thus multiplication by λ^{-1} gives $A^{-1}\mathbf{x} = \lambda^{-1}\mathbf{x}$.

16. If $A\mathbf{x} = \lambda\mathbf{x}$ then $(A + \alpha I)\mathbf{x} = A\mathbf{x} + \alpha I\mathbf{x} = \lambda\mathbf{x} + \alpha\mathbf{x} = (\lambda + \alpha)\mathbf{x}$.

17. Let $\mathbf{x}$ be an eigenvector corresponding to λ and suppose $A^k\mathbf{x} = \lambda^k\mathbf{x}$ for some integer $k \geq 2$. Then $A^{k+1}\mathbf{x} = A(A^k\mathbf{x}) =$
$A(\lambda^k\mathbf{x}) = \lambda^k(A\mathbf{x}) = \lambda^k(\lambda\mathbf{x}) = \lambda^{k+1}\mathbf{x}$. It follows by the principle of induction that $A^n\mathbf{x} = \lambda^n\mathbf{x}$ for each positive integer $n, n \geq 2$.

18. (a) $q(H)\mathbf{x} = (H^3 - 2H^2 - H + 2I)\mathbf{x} = H^3\mathbf{x} - 2H^2\mathbf{x} - H\mathbf{x} + 2I\mathbf{x} =$
$\lambda^3\mathbf{x} - 2\lambda^2\mathbf{x} - \lambda\mathbf{x} + 2\mathbf{x} = (\lambda^3 - 2\lambda^2 - \lambda + 2)\mathbf{x} = q(\lambda)\mathbf{x}$.

    (b) The eigenvalues for $q(A)$ are $q(1) = 0$ and $q(-1) = 0$. The eigenvalues for $q(B)$ are $q(0) = 2$ and $q(-1) = 0$.

19. $q(C) = C^3 - 2C^2 - C + 2I = \begin{bmatrix} 35 & -3 & -15 \\ -44 & 2 & 19 \\ 68 & -6 & -29 \end{bmatrix} -$
$$2\begin{bmatrix} 17 & -1 & -7 \\ -16 & 2 & 7 \\ 32 & -2 & -13 \end{bmatrix} - \begin{bmatrix} 3 & -1 & -1 \\ -12 & 0 & 5 \\ 4 & -2 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \mathcal{O}.$$

20. $p(t) = t^2 - 4t + 3$. $p(A) = A^2 - 4A + 3I = \mathcal{O}$.

21. $p(t) = t^2 - 2t + 1$. $p(A) = A^2 - 2A + I = \mathcal{O}$.

22. $p(t) = -t^3 + 2t^2 + t - 2$. $p(A) = -A^3 + 2A^2 + A - 2I = \mathcal{O}$.

23. $p(t) = t^4 - 18t^3 + 97t^2 - 180t + 100$. $p(A) = A^4 - 18A^3 + 97A^2 - 180A + 100I = \mathcal{O}$.

24. (a) Suppose $B = [\mathbf{B_1}, \mathbf{B_2}, \mathbf{B_3}]$. Then $\theta = B\mathbf{e_1} = \mathbf{B_1}, \theta =$
$B\mathbf{e_2} = \mathbf{B_2}$, and $\theta = B\mathbf{e_3} = \mathbf{B_3}$. Thus $B = \mathcal{O}$.

    (b) Since $A\mathbf{u_i} = \lambda_\mathbf{i}\mathbf{u_i}$ for $i = 1, 2, 3$, it follows, as in Exercise 18a, that $p(A)\mathbf{u_i} = p(\lambda_i)\mathbf{u_i} = (0)\mathbf{u_i} = \theta$. By property 3 of Theorem 9 in Section 2.5, $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ is a basis for

R^3. Therefore every vector $\mathbf{x}$ in R^3 can be expressed in the form $\mathbf{x} = a_1\mathbf{u_1} + a_2\mathbf{u_2} + a_3\mathbf{u_3}$. It follows that $p(A)\mathbf{x} = a_1 p(A)\mathbf{u_1} + a_2 p(A)\mathbf{u_2} + a_3 p(A)\mathbf{u_3} = \theta$. By part (a), $p(A) = \mathcal{O}$.

25. $p(A) = A^2 - (a+d)A + (ad-bc)I = \begin{bmatrix} a^2+bc & ab+bd \\ ca+dc & cb+d^2 \end{bmatrix}$
$- (a+d)\begin{bmatrix} a & b \\ c & d \end{bmatrix} + (ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

26. Expansion yields $p(t) = -[t^3 - (a+b+c)t^2 + (ab+ac+bc)t - abc]$. Similarly the properties of matrix multiplication imply that $-(A-aI)(A-bI)(A-cI) = -[A^3 - (a+b+c)A^2 + (ab+ac+bc)A - abcI] = p(A)$. Therefore
$p(A) = -(A-aI)(A-bI)(A-cI) =$
$$\begin{bmatrix} 0 & d & f \\ 0 & b-a & e \\ 0 & 0 & c-a \end{bmatrix}\begin{bmatrix} a-b & d & f \\ 0 & 0 & e \\ 0 & 0 & c-b \end{bmatrix}\begin{bmatrix} a-c & d & f \\ 0 & b-c & e \\ 0 & 0 & 0 \end{bmatrix}$$
$= \mathcal{O}$.

27. (a) For n = 2, $\det(A - tI) = \begin{vmatrix} -a_1 - t & -a_0 \\ 1 & -t \end{vmatrix} = t^2 + a_1 t + a_0 =$ $q(t)$. For n = 3, $\det(A-tI) = \begin{vmatrix} -a_2 - t & -a_1 & -a_0 \\ 1 & -t & 0 \\ 0 & 1 & -t \end{vmatrix}$. Expanding along the third column and applying the case n = 2 yields $\det(A-tI) = -a_0\begin{vmatrix} 1 & -t \\ 0 & 1 \end{vmatrix} - t\begin{vmatrix} -a_2 - t & -a_1 \\ 1 & -t \end{vmatrix} =$ $-t(t^2 + a_2 t + a_1) - a_0 = -q(t)$.

(b) $A = \begin{bmatrix} -3 & 1 & -2 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. The characteristic polynomial for A is $\det(A - tI) =$
$$\begin{vmatrix} -3-t & 1 & -2 & 2 \\ 1 & -t & 0 & 0 \\ 0 & 1 & -t & 0 \\ 0 & 0 & 1 & -t \end{vmatrix} = q(t).$$

(c) For some integer $k \geq 2$, assume that if

$$A = \begin{bmatrix} -a_{k-1} & -a_{k-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

then $\det(A - tI) = (-1)^k(t^k + a_{k-1}t^{k-1} + \cdots + a_1 t + a_0)$. If A is the $[(k+1)$ x $(k+1)]$ companion matrix then $\det(A - tI) = \begin{vmatrix} -a_k - t & -a_{k-1} & \cdots & -a_1 & -a_0 \\ 1 & -t & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 1 & -t \end{vmatrix}$.

Expanding along the $(k+1)^{st}$ column and using the case n = k yields $\det(A - tI) =$

$(-1)^{k+1}a_0 \begin{vmatrix} 1 & -t & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} - t \begin{vmatrix} -a_k - t & -a_{k-1} & \cdots & -a_1 \\ 1 & -t & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & -t \end{vmatrix} =$

$(-1)^{k+1}a_0 - t(-1)^k(t^k + a_k t^{k-1} + \cdots + a_2 t + a_1) = (-1)^{k+1}q(t)$. By mathematical induction $\det(A - tI) = (-1)^n q(t)$ for all $n, n \geq 2$.

28. $\mathbf{x_0} = [1,1,1]^{\mathrm{T}}, \mathbf{x_1} = [1,-7,1]^{\mathrm{T}}, \mathbf{x_2} = [9,-7,17]^{\mathrm{T}}, \mathbf{x_3} = [17,-23,33]^{\mathrm{T}}$, $\mathbf{x_4} = [41,-39,81]^{\mathrm{T}}, \mathbf{x_5} = [81,-87,161]^{\mathrm{T}}.\beta_0 = -5/3 \approx -1.667, \beta_1 = 75/51 \approx 1.471, \beta_2 = 875/419 \approx 2.088, \beta_3 = 4267/1907 \approx 2.238, \beta_4 = 19755/9763 \approx 2.023$.

29. Note that $\mathbf{x_j} = A^j\mathbf{x_0}$ so by property (a) of Theorem 11, $\mathbf{x_j} = c_1\lambda_1^j\mathbf{u_1} + c_2\lambda_2^j\mathbf{u_2} + \cdots + c_n\lambda_n^j\mathbf{u_n}$. Set $a_j = \lambda_j/\lambda_1, 1 \leq j \leq n$. Then $a_1 = 1$ whereas $|a_j| < 1$ for $2 \leq j \leq n$. In particular $\lim_{k\to\infty} a_j^k = 0$ if $j \neq 1$. It follows that $\mathbf{x_k}^{\mathrm{T}}\mathbf{x_{k+1}} = \sum_{1 \leq i,j \leq n} c_i c_j \lambda_i^k \lambda_j^{k+1}\mathbf{u_i}^{\mathrm{T}}\mathbf{u_j} =$

$\lambda_1^{2k+1}(c_1^2\mathbf{u_1}^{\mathrm{T}}\mathbf{u_1} + r_k)$, where $r_k = \sum_{\substack{1 \leq i,j \leq n \\ (i,j) \neq (1,1).}} c_i c_j a_i^k a_j^{k+1}\mathbf{u_i}^{\mathrm{T}}\mathbf{u_j}$ In particular $\lim_{k\to\infty} r_k = 0$. Similarly $\mathbf{x_k}^{\mathrm{T}}\mathbf{x_k} = \lambda_1^{2k}(c_1\mathbf{u_1}^{\mathrm{T}}\mathbf{u_1} + t_k)$ where $\lim_{k\to\infty} t_k = 0$. Therefore $\lim_{k\to\infty} \beta_k = \lambda_1$.

30. Let A and A^{T} have characteristic polynomials $p(t)$ and $q(t)$, respectively. Note that $(A - tI)^{\mathrm{T}} = A^{\mathrm{T}} - tI$. It follows that $p(t) = \det(A - tI) = \det(A - tI)^{\mathrm{T}} = \det(A^{\mathrm{T}} - tI) = q(t)$.

31. Suppose $p(t) = t^2 + a_1 t + a_0$. Then $p(0) = a_0 = \det(A - 0I) = 4$ and $p(1) = 1 + a_1 + a_0 = \det(A - I) = 1$. Thus we have

$$\begin{array}{lcl} a_0 & = & 4 \\ a_0 + a_1 & = & 0 \end{array}.$$

Solving yields $a_0 = 4, a_1 = -4$. Therefore $p(t) = t^2 - 4t + 4$.

32. $p(0) = a_0 = \det(A - 0I) = 0$ and $p(1) = 1 + a_1 + a_0 = \det(A - I) = -4$. Therefore $a_0 = 0, a_1 = -4$, and $p(t) = t^2 - 5t$.

33. Suppose $p(t) = -t^3 + a_2t^2 + a_1t + a_0$. Then $p(-1) = 1 + a_2 - a_1 + a_0 = \det(A + I) = 0, p(0) = a_0 = \det(A - 0I) = -1$, and $p(1) = -1 + a_2 + a_1 + a_0 = \det(A - I) = 0$. It follows that $a_0 = -1, a_1 = 1$, and $a_2 = 1$. Therefore $p(t) = -t^3 + t^2 + t - 1$.

34. $p(-1) = 1 + a_2 - a_1 + a_0 = \det(A + I) = 0, p(0) = a_0 = \det(A - 0I) = 0$, and $p(1) = -1 + a_2 + a_1 + a_0 = \det(A - I) = -4$. Therefore $a_0 = 0, a_1 = -1, a_2 = -2$, and $p(t) = -t^3 - 2t^2 - t$.

4.5 Eigenvalues and Eigenvectors

1. $(A - 3I)\mathbf{x} = \theta$ is the system

$$\begin{aligned} -x_1 - x_2 &= 0 \\ -x_1 - x_2 &= 0 \end{aligned} .$$

The solution is $x_1 = -x_2, x_2$ arbitrary, so E_λ consists of the vectors of the form $x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Thus $\{[-1, 1]^{\mathrm{T}}\}$ is a basis for E_λ. The eigenvalue $\lambda = 3$ has algebraic and geometric multiplicity 1.

2. The system $(A - I)\mathbf{x} = \theta$ has solution $x_1 = x_2, x_2$ arbitrary. $\{[1, 1]^{\mathrm{T}}\}$ is a basis for E_λ. The eigenvalue $\lambda = 1$ has algebraic and geometric multiplicity 1.

3. $(B - 2I)\mathbf{x} = \theta$ is the system

$$\begin{aligned} -x_1 - x_2 &= 0 \\ x_1 + x_2 &= 0 \end{aligned} .$$

The solution is $x_1 = -x_2, x_2$ arbitrary, so E_λ consists of the vectors of the form $x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Thus $\{[-1, 1]^{\mathrm{T}}\}$ is a basis for E_λ. The eigenvalue $\lambda = 2$ has algebraic multiplicity 2 and geometric multiplicity 1.

4. The system $(C - I)\mathbf{x} = \theta$ has solution $x_1 = (1/2)x_3, x_2 = (-3/2)x_3$, x_3 arbitrary. The set $\{[1, -3, 2]^{\mathrm{T}}\}$ is a basis for E_λ. The eigenvalue $\lambda = 1$ has algebraic multiplicity 2 and geometric multiplicity 1.

5. $(C + I)\mathbf{x} = \theta$ is the system

$$\begin{array}{rcrcrcl} -5x_1 & - & x_2 & + & 2x_3 & = & 0 \\ 3x_1 & + & 3x_2 & & & = & 0 \\ -14x_1 & - & 2x_2 & + & 6x_3 & = & 0 \end{array} .$$

The solution is $x_1 = (1/2)x_3, x_2 = (-1/2)x_3, x_3$ arbitrary, so E_λ consists of vectors of the form $a\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ where a is an arbitrary scalar. Thus $\{[1, -1, 2]^{\mathrm{T}}\}$ is a basis for E_λ. The eigenvalue $\lambda = -1$ has algebraic and geometric multiplicity 1.

6. The system $(D - I)\mathbf{x} = \theta$ has solution $x_1 = (1/2)x_2 - (3/8)x_3, x_2$ and x_3 arbitrary. The set $\{[1, 2, 0]^{\mathrm{T}}, [-3, 0, 8]^{\mathrm{T}}\}$ is a basis for E_λ. The eigenvalue $\lambda = 1$ has algebraic multiplicity 3 and geometric multiplicity 2.

7. $(E + I)\mathbf{x} = \theta$ is the system

$$\begin{array}{rcrcrcrcl} 7x_1 & + & 4x_2 & + & 4x_3 & + & x_4 & = & 0 \\ 4x_1 & + & 7x_2 & + & x_3 & + & 4x_4 & = & 0 \\ 4x_1 & + & x_2 & + & 7x_3 & + & 4x_4 & = & 0 \\ x_1 & + & 4x_2 & + & 4x_3 & + & 7x_4 & = & 0 \end{array}.$$

The solution is $x_1 = x_4, x_2 = -x_4, x_3 = -x_4, x_4$ arbitary, so E_λ consists of vectors of the form $x_4\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$. Thus $\{[1, -1, -1, 1]^{\mathrm{T}}\}$ is a basis for E_λ. The eigenvalue $\lambda = -1$ has geometric and algebraic multiplicity 1.

8. The system $(E - 5I)\mathbf{x} = \theta$ has solution $x_1 = -x_4, x_2 = -x_3, x_3$ and x_4 arbitrary. The set $\{[-1, 0, 0, 1]^{\mathrm{T}}, [0, -1, 1, 0]^{\mathrm{T}}\}$ is a basis for E_λ.
The eigenvalue $\lambda = 5$ has algebraic and geometric multiplicity 2.

9. $(E - 15I)\mathbf{x} = \theta$ is the system

$$\begin{array}{rcrcrcrcl} -9x_1 & + & 4x_2 & + & 4x_3 & + & x_4 & = & 0 \\ 4x_1 & - & 9x_2 & + & x_3 & + & 4x_4 & = & 0 \\ 4x_1 & + & x_2 & - & 9x_3 & + & 4x_4 & = & 0 \\ x_1 & + & 4x_2 & + & 4x_3 & - & 9x_4 & = & 0 \end{array}.$$

The solution is $x_1 = x_2 = x_3 = x_4, x_4$ arbitrary so E_λ consists of vectors of the form $x_4\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Thus $\{[1, 1, 1, 1]^{\mathrm{T}}\}$ is a basis for E_λ. The eigenvalue $\lambda = 15$ has algebraic and geometric multiplicity 1.

10. The system $(F+2I)\mathbf{x}=\theta$ has solution $x_1 = x_2 = x_3 = x_4, x_4$ arbitrary. The set $\{[1,1,1,1]^{\mathrm{T}}\}$ is a basis for E_λ. The eigenvalue $\lambda = -2$ has algebraic and geometric multiplicity 1.

11. $(F-2I)\mathbf{x}=\theta$ is the system

$$\begin{array}{rcrcrcrcl} -x_1 & - & x_2 & - & x_3 & - & x_4 & = & 0 \\ -x_1 & - & x_2 & - & x_3 & - & x_4 & = & 0 \\ -x_1 & - & x_2 & - & x_3 & - & x_4 & = & 0 \\ -x_1 & - & x_2 & - & x_3 & - & x_4 & = & 0 \end{array}.$$

The solution is $x_1 = -x_2 - x_3 - x_4, x_2, x_3, x_4$ arbitrary so E_λ consists of vectors of the form $x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. Thus $\{[-1,1,0,0]^{\mathrm{T}}, [-1,0,1,0]^{\mathrm{T}}, [-1,0,0,1]^{\mathrm{T}}\}$ is a basis for E_λ. The eigenvalue $\lambda = 2$ has algebraic and geometric multiplicity 3.

12. The characteristic equation is $p(t) = -(t-1)^2(t-2) = 0$ so the eigenvalues are $\lambda = 1$ and $\lambda = 2$. The eigenvectors for $\lambda = 1$ are the nonzero vectors of the form $\begin{bmatrix} x_1 \\ x_3 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. The eigenvectors for $\lambda = 2$ are the nonzero vectors of the form $\begin{bmatrix} x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. The matrix is not defective.

13. The characteristic equation for the given matrix A is $0 =$ $\det(A - tI) = \begin{vmatrix} 2-t & 1 & 2 \\ 0 & 3-t & 2 \\ 0 & 0 & 2-t \end{vmatrix} = -(t-2)^2(t-3)$.
The eigenvalues are $\lambda = 2$ and $\lambda = 3$. The system $(A-2I)\mathbf{x}=\theta$ is given by

$$\begin{array}{l} x_2 + 2x_3 = 0 \\ x_2 + 2x_3 = 0 \end{array}.$$

In the solution x_1 is arbitrary, $x_2 = -2x_3$, and x_3 is arbitrary. The eigenvectors for $\lambda = 2$ are the nonzero vectors of the form $\begin{bmatrix} x_1 \\ -2x_3 \\ x_3 \end{bmatrix}$

$= x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$. Therefore $\lambda = 2$ has algebraic and geometric multiplicity 2.
The system $(A - 3I)\mathbf{x} = \theta$ is given by

$$\begin{aligned} -x_1 + x_2 + 2x_3 &= 0 \\ 2x_3 &= 0\ . \\ -x_3 &= 0 \end{aligned}$$

The solution is $x_1 = x_2, x_2$ arbitrary, $x_3 = 0$. The eigenvectors for $\lambda = 3$ are the nonzero vectors of the form $\begin{bmatrix} x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.
Therefore $\lambda = 3$ has algebraic and geometric multiplicity 1. The matrix is not defective.

14. The characteristic polynomial is $p(t) = -(t-1)^3$ so $\lambda = 1$ is the only eigenvalue. The eigenvectors for $\lambda = 1$ are the nonzero vectors of the form $\begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Thus $\lambda = 1$ has algebraic multiplicity 3 and geometric multiplicity 1. The matrix is defective.

15. The given matrix A has characteristic equation $0 = \det(A - tI) = -(t-2)^2(t-1)$ so the eigenvalues are $\lambda = 1$ and $\lambda = 2$. The system of equations $(A - I)\mathbf{x} = \theta$ has solution $x_1 = -3x_3, x_2 = -x_3, x_3$ arbitrary so the eigenvectors for $\lambda = 1$ are the nonzero vectors of the form $\mathbf{x} = [-3x_3, -x_3, x_3]^{\mathrm{T}}$. For the system $(A - 2I)\mathbf{x} = \theta$, x_1 and x_2 are arbitrary and $x_3 = 0$. The eigenvectors for $\lambda = 2$ are the nonzero vectors of the form $\mathbf{x}$ $= x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. The matrix is not defective.

16. The characteristic polynomial is $p(t) = -(t-3)^2(t+4)$ so the eigenvalues are $\lambda = 3$ and $\lambda = -4$. The eigenvectors for $\lambda = 3$ are the nonzero vectors of the form $\mathbf{x} = [5x_3, 3x_3, x_3]^{\mathrm{T}}$. The eigenvectors for $\lambda = -4$ are the nonzero vectors of the form $\mathbf{x} = [-2x_3, (2/3)x_3, x_3]^{\mathrm{T}}$. Since $\lambda = 3$ has algebraic multiplicity 2 and geometric multiplicity 1, the matrix is defective.

17. The given matrix A has characteristic polynomial $p(t) =$ $-(t+1)(t-1)(t-2)$ so the eigenvalues for A are $\lambda = -1, \lambda = 1, \lambda = 2$. The system of equations $(A + I)\mathbf{x} = \theta$ has solution $x_1 = (1/2)x_3, x_2 = x_3, x_3$ arbitrary so the eigenvectors for $\lambda = -1$ are the nonzero vectors of the form $\mathbf{x} = [(1/2)x_3, x_3, x_3]^{\mathrm{T}}$. The system of equations $(A - I)\mathbf{x} = \theta$ has solution $x_1 = -3x_2, x_2$ arbitrary, $x_3 = -7x_2$ so the eigenvectors for $\lambda = 1$ are the nonzero vectors of the form $\mathbf{x} = [-3x_2, x_2, -7x_2]^{\mathrm{T}}$. The system of equations $(A - 2I)\mathbf{x} = \theta$ has solution $x_1 = (1/2)x_3, x_2 = (-1/2)x_3$ so the eigenvectors for $\lambda = 2$ are the nonzero vectors of the form $\mathbf{x} = [(1/2)x_3, (-1/2)x_3, x_3]^{\mathrm{T}}$. The matrix is not defective.

18. $\mathbf{u_1} = [1,1]^{\mathrm{T}}$ is an eigenvector for the eigenvalue $\lambda = 2$ and $\mathbf{u_2} = [2,5]^{\mathrm{T}}$ is an eigenvector for the eigenvalue $\lambda = -1$. Moreover $\mathbf{x} = -6\mathbf{u_1} + 3\mathbf{u_2}$ so $A^{10}\mathbf{x} = -6(2)^{10}\mathbf{u_1} + 3(-1)^{10}\mathbf{u_2}$ $= [-6138, -6129]^{\mathrm{T}}$.

19. The characteristic polynomial for A is $p(t) = -(t-1)^2(t-2)$ so the eigenvalues for A are $\lambda = 1$ and $\lambda = 2$. The vectors $\mathbf{u_1} = [1,0,0]^{\mathrm{T}}$ and $\mathbf{u_2} = [0,1,2]^{\mathrm{T}}$ are the eigenvectors for $\lambda = 1$ and $\mathbf{u_3} = [1,2,3]^{\mathrm{T}}$ is an eigenvector for $\lambda = 2$. Moreover $\mathbf{x} = \mathbf{u_1} + 2\mathbf{u_2} + \mathbf{u_3}$ so $A^{10}\mathbf{x} = (1)^{10}\mathbf{u_1} + 2(1)^{10}\mathbf{u_2} + (2)^{10}\mathbf{u_3} = [1025, 2050, 3076]^{\mathrm{T}}$.

20. Since λ is an eigenvalue for H, nullity $(H - \lambda I) \geq 1$. It follows that rank $(H - \lambda I) \leq 3$. But a, b and c° are nonzero so the first three columns of $H - \lambda I$ are linearly independent. Therefore
rank $(H - \lambda I) \geq 3$. Thus rank $(H - \lambda I) = 3$ and, hence,
nullity $(H - \lambda I) = 1$. This proves that λ has geometric multiplicity 1.

21. $P = P^{-1}P^2 = P^{-1}P = I$.

22. Suppose $P\mathbf{x} = \lambda\mathbf{x}$, $\mathbf{x} \neq \theta$. Then $\lambda^2\mathbf{x} = P^2\mathbf{x} = P\mathbf{x} = \lambda\mathbf{x}$ so $(\lambda^2 - \lambda)\mathbf{x} = \theta$. Since $\mathbf{x} \neq \theta$, $0 = \lambda^2 - \lambda = \lambda(\lambda - 1)$. Therefore either $\lambda = 0$ or $\lambda = 1$.

23. $P^2 = (\mathbf{u}\mathbf{u}^{\mathrm{T}})(\mathbf{u}\mathbf{u}^{\mathrm{T}}) = \mathbf{u}(\mathbf{u}^{\mathrm{T}}\mathbf{u})\mathbf{u}^{\mathrm{T}} = \mathbf{u}\mathbf{u}^{\mathrm{T}} = P$.

24. $(I - Q)^2 = I^2 - IQ - QI + Q^2 = I - Q$. Also $(I - 2Q)^2 = I^2 - 2IQ - 2QI + 4Q^2 = I$ so $(I - 2Q)^{-1} = I - 2Q$.

25. $P^2 = (\mathbf{u}\mathbf{u}^{\mathrm{T}} + \mathbf{v}\mathbf{v}^{\mathrm{T}})(\mathbf{u}\mathbf{u}^{\mathrm{T}} + \mathbf{v}\mathbf{v}^{\mathrm{T}}) = \mathbf{u}\,(\mathbf{u}^{\mathrm{T}}\mathbf{u})\mathbf{u}^{\mathrm{T}} + \mathbf{u}\,(\mathbf{u}^{\mathrm{T}}\mathbf{v})\mathbf{v}^{\mathrm{T}} +$
$\mathbf{v}\,(\mathbf{v}^{\mathrm{T}}\mathbf{u})\mathbf{u}^{\mathrm{T}} + \mathbf{v}\,(\mathbf{v}^{\mathrm{T}}\mathbf{v})\mathbf{v}^{\mathrm{T}} = \mathbf{u}\mathbf{u}^{\mathrm{T}} + \mathbf{v}\mathbf{v}^{\mathrm{T}} = P$.

26. $P(a\mathbf{u} + b\mathbf{v}) = aP\mathbf{u} + bP\mathbf{v} = a(\mathbf{u}\mathbf{u}^{\mathrm{T}} + \mathbf{v}\mathbf{v}^{\mathrm{T}})\mathbf{u} + b(\mathbf{u}\mathbf{u}^{\mathrm{T}} + \mathbf{v}\mathbf{v}^{\mathrm{T}})\mathbf{v} =$
$a\mathbf{u}(\mathbf{u}^{\mathrm{T}}\mathbf{u}) + a\mathbf{v}\,(\mathbf{v}^{\mathrm{T}}\mathbf{u}) + b\mathbf{u}(\mathbf{u}^{\mathrm{T}}\mathbf{v}) + b\mathbf{v}(\mathbf{v}^{\mathrm{T}}\mathbf{v}) = a\mathbf{u} + b\mathbf{v}$.

27. (a) A has eigenvalues $\lambda = 1$ and $\lambda = 3$ with corresponding eigenvectors $\mathbf{u_1} = [1/\sqrt{2}, 1/\sqrt{2}]^{\mathrm{T}}$ and $\mathbf{u_2} = [-1/\sqrt{2}, 1/\sqrt{2}]^{\mathrm{T}}$, respectively, where $\|\mathbf{u_1}\| = \|\mathbf{u_2}\| = 1$. It is easily checked that $\mathbf{u_1}\mathbf{u_1}^{\mathrm{T}} + 3\mathbf{u_2}\mathbf{u_2}^{\mathrm{T}} = A$.
(b) A has eigenvalues $\lambda = -1$ and $\lambda = 3$ with corresponding eigenvectors $\mathbf{u_1} = [-1/\sqrt{2}, 1/\sqrt{2}]^{\mathrm{T}}$ and $\mathbf{u_2} = [1/\sqrt{2}, 1/\sqrt{2}]^{\mathrm{T}}$, respectively, where $\|\mathbf{u_1}\| = \|\mathbf{u_2}\| = 1$. It is easily checked that $-\mathbf{u_1}\mathbf{u_1}^{\mathrm{T}} + 3\mathbf{u_2}\mathbf{u_2}^{\mathrm{T}} = A$.
(c) A has eigenvalues $\lambda = 4$ and $\lambda = -1$ with corresponding eigenvectors $\mathbf{u_1} = [2/\sqrt{5}, 1/\sqrt{5}]^{\mathrm{T}}$ and $\mathbf{u_2} = [1/\sqrt{5}, -2/\sqrt{5}]^{\mathrm{T}}$, respectively, where $\|\mathbf{u_1}\| = \|\mathbf{u_2}\| = 1$. It is easily checked that $4\mathbf{u_1}\mathbf{u_1}^{\mathrm{T}} - \mathbf{u_2}\mathbf{u_2}^{\mathrm{T}} = A$.

28. $\beta(\mathbf{u}^{\mathrm{T}}\mathbf{v}) = \mathbf{u}^{\mathrm{T}}(\beta\mathbf{v}) = \mathbf{u}^{\mathrm{T}}(A\mathbf{v}) = (A\mathbf{v})^{\mathrm{T}}\mathbf{u} = \mathbf{v}^{\mathrm{T}}A^{\mathrm{T}}\mathbf{u} = \mathbf{v}^{\mathrm{T}}(A\mathbf{u}) =$
$\mathbf{v}^{\mathrm{T}}(\lambda\mathbf{u}) = \lambda(\mathbf{v}^{\mathrm{T}}\mathbf{u}) = \lambda(\mathbf{u}^{\mathrm{T}}\mathbf{v})$. Since $\beta \neq \lambda$ it follows that $\mathbf{u}^{\mathrm{T}}\mathbf{v} = 0$.

29. Set $B = A - C$. For each $i, 1 \leq i \leq n, B\mathbf{u_i} = A\mathbf{u_i} - C\mathbf{u_i} = A\mathbf{u_i} - \lambda_1 \mathbf{u_1}(\mathbf{u_1}^{\mathrm{T}}\mathbf{u_i}) - \cdots - \lambda_i \mathbf{u_i}(\mathbf{u_i}^{\mathrm{T}}\mathbf{u_i}) - \cdots - \lambda_n \mathbf{u_n}(\mathbf{u_n}^{\mathrm{T}}\mathbf{u_i}) = \lambda_i \mathbf{u_i} - \lambda_i \mathbf{u_i} = \theta$. If $\mathbf{x}$ is in R^n then $\mathbf{x}$ may be written in the form $\mathbf{x} = a_1\mathbf{u_1} + \cdots + a_n\mathbf{u_n}$, so $B\mathbf{x} = a_1 B\mathbf{u_1} + \cdots + a_n B\mathbf{u_n} = \theta$. In particular $B\mathbf{e_j} = \theta$ for $1 \leq j \leq n$. If $B = [\mathbf{B_1}, \ldots, \mathbf{B_n}]$ then $B\mathbf{e_j} = \mathbf{B_j}$, so $\mathbf{B_j} = \theta$for $1 \leq j \leq n$. This shows that $A - C = \mathcal{O}$ so $A = C$.

4.6 Complex Eigenvalues and Eigenvectors

1. $\overline{u} = 3 + 2i$.

2. $\overline{z} = 1 - i$.

3. $u + \overline{v} = 7 - 3i$.

4. $\overline{z} + w = 3 - 2i$.

5. $u + \overline{u} = 6$.

6. $s - \overline{s} = 4i$.

7. $v\overline{v} = 17$.

8. $u\overline{v} = 10 - 11i$.

9. $s^2 - w = -5 + 5i$.

10. $z^2 w = 2 + 4i$.

11. $\overline{u}w^2 = 17 - 6i$.

12. $s(u^2 + v) = 31 + 7i$.

13. $u/v = (u\overline{v})/(v\overline{v}) = 10/17 - (11/17)i$.

14. $v/u^2 = 8/169 + (53/169)i$.

15. $s/z = (s\overline{z})/(z\overline{z}) = 3/2 + (1/2)i$.

16. $(w + \overline{v})/u = 22/13 + (6/13)i$.

17. $w + iz = 1$.

18. $s - iw = 0$.

19. For the given matrix A the characteristic polynomial is $p(t) = t^2 - 8t + 20$ and the eigenvalues are $\lambda = 4 + 2i$ and $\overline{\lambda} = 4 - 2i$. The system of equations $(A - (4+2i)I)\mathbf{x} = \theta$ is given by

$$\begin{array}{rcrcl} (2-2i)x_1 & + & 8x_2 & = & 0 \\ -x_1 & + & (-2-2i)x_2 & = & 0 \end{array}.$$

The solution is $x_1 = (-2-2i)x_2, x_2$ arbitrary. Thus the eigenvectors for $\lambda = 4 + 2i$ are the nonzero vectors of the form
$\mathbf{x} = [(-2-2i)x_2, x_2]^{\mathrm{T}}$. By Theorem 16, $\overline{\mathbf{x}}$ is an eigenvector corresponding to $\overline{\lambda}$.

20. The characteristic polynomial is $p(t) = t^2 + 4$ so the eigenvalues are $\lambda = 2i$ and $\overline{\lambda} = -2i$. The eigenvectors for $\lambda = 2i$ are the nonzero vectors of the form $\mathbf{x} = [(-1-i)x_2, x_2]^{\mathrm{T}}$. By Theorem 16, $\overline{\mathbf{x}}$ is an eigenvector for $\overline{\lambda} = -2i$.

21. The given matrix A has characteristic polynomial $p(t) = t^2 + 1$ so the eigenvalues are $\lambda = i$ and $\overline{\lambda} = -i$. The system $(A - iI)\mathbf{x} = \theta$ is given by

$$\begin{array}{rcrcl} (-2-i)x_1 & - & x_2 & = & 0 \\ 5x_1 & + & (2-i)x_2 & = & 0 \end{array}.$$

The solution is $x_1 = ((-2+i)/5)x_2$, so the eigenvectors for

$\lambda = i$ are the nonzero vectors of the form $\mathbf{x} = [((-2+i)/5)x_2, x_2]^{\mathrm{T}}$.

By Theorem 16, $\overline{\mathbf{x}}$ is an eigenvector for $\overline{\lambda} = -i$.

22. The characteristic polynomial is $p(t) = -(t-2)(t^2 - 4t + 5)$. The eigenvalues are $\lambda_1 = 2, \lambda_2 = 2 + i$, and $\overline{\lambda_2} = 2 - i$. The eigenvectors for $\lambda_1 = 2$ have the form $\mathbf{x} = [0, -x_3, x_3]^{\mathrm{T}}, x_3 \neq 0$. The eigenvectors for $\lambda_2 = 2 + i$ have the form $\mathbf{x} = [x_3, (-2/5 - (1/5)i)x_3, x_3]^{\mathrm{T}}$,
$x_3 \neq 0$ and $\overline{\mathbf{x}}$ is an eigenvector for $\overline{\lambda_2} = 2 - i$.

23. The given matrix A has characteristic polynomial $p(t) =$
$-(t-2)(t^2 - 4t + 13)$ so the eigenvalues for A are $\lambda_1 = 2, \lambda_2 =$
$2 + 3i$, and $\overline{\lambda}_2 = 2 - 3i$. The system $(A - 2I)\mathbf{x} = \theta$ is given by

$$\begin{array}{rcrcrcl} -x_1 & - & 4x_2 & - & x_3 & = & 0 \\ 3x_1 & & & + & 3x_3 & = & 0 \\ x_1 & + & x_2 & + & x_3 & = & 0 \end{array}.$$

The solution is $x_1 = -x_3, x_2 = 0, x_3$ arbitrary so the eigenvectors corresponding to $\lambda_1 = 2$ are of the form $\mathbf{x} = [-x_3, 0, x_3]^{\mathrm{T}}$. The system $[A - (2+3i)I]\mathbf{x} = \theta$ is given by

$$\begin{array}{rcrcrcl} (-1-3i)x_1 & - & 4x_2 & - & x_3 & = & 0 \\ 3x_1 & - & 3ix_2 & + & 3x_3 & = & 0 \\ x_1 & + & x_2 & + & (1-3i)x_3 & = & 0 \end{array}.$$

The solution is $x_1 = (-5/2 + (3/2)i)x_3$ and $x_2 = (3/2 + (3/2)i)x_3$

so the eigenvectors for $\lambda_2 = 2 + 3i$ are the nonzero vectors of the form $\mathbf{x} = [(-5/2 + (3/2)i)x_3, (3/2 + (3/2)i)x_3, x_3]^{\mathrm{T}}$. By Theorem 16, $\overline{\mathbf{x}}$ is an eigenvector for $\overline{\lambda}_2 = 2 - 3i$.

24. The characteristic polynomial is $p(t) = (t^2 - 2t + 26)(t^2 - 2t + 5)$ so the eigenvalues are $\lambda_1 = 1 + 5i, \overline{\lambda}_1 = 1 - 5i, \lambda_2 = 1 + 2i, \overline{\lambda}_2 = 1 - 2i$. Eigenvectors for λ_1 are the nonzero vectors of the form $\mathbf{x} = [ix_2, x_2, 0, 0]^{\mathrm{T}}$ and $\overline{\mathbf{x}}$ is an eigenvector for $\overline{\lambda}_1$. Eigenvectors for λ_2 are the nonzero vectors of the form $\mathbf{x} = [0, 0, ix_4, x_4]^{\mathrm{T}}$ and $\overline{\mathbf{x}}$ is an eigenvector for $\overline{\lambda}_2$.

25. $x = 2 - i, y = 3 - 2i$.

26. $x = i, y = 2$.

27. $\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{x} = (1 - i)(1 + i) + 2(2) = 6$ so $\|\mathbf{x}\| = \sqrt{6}$.

28. $\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{x} = (3 - i)(3 + i) + (2 + i)(2 - i) = 10 + 5 = 15$ so $\|\mathbf{x}\| = \sqrt{15}$.

29. $\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{x} = (1 + 2i)(1 - 2i) + (-i)(i) + (3 - i)(3 + i) = 5 + 1 + 10 = 16$. Thus $\|\mathbf{x}\| = \sqrt{16} = 4$.

30. $\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{x} = (-2i)(2i) + (1 + i)(1 - i) + 3(3) = 4 + 2 + 9 = 15$ so $\|\mathbf{x}\| = \sqrt{15}$.

31. $\lambda_1 = -1.4937 + 1.2616i$, $\mathbf{x}_1 = \begin{bmatrix} 0.5835 - 0.1460i \\ 0.1650 - 0.4762i \\ -0.4369 + 0.4397i \end{bmatrix}$; $\lambda_2 = -1.4937 - 1.2616i$, $\mathbf{x}_2 = \begin{bmatrix} 0.5835 + 0.1460i \\ 0.1650 + 0.4762i \\ -0.4369 - 0.4397i \end{bmatrix}$; $\lambda_3 = 10.9873$, $\mathbf{x}_3 = \begin{bmatrix} -0.4486 \\ -0.7312 \\ -0.5139 \end{bmatrix}$. In each case, the eigenvectors are chosen to have length 1.

32. $\lambda_1 = -3.6884 + 2.8416i$, $\mathbf{x}_1 = \begin{bmatrix} -0.0558 - 0.6977i \\ -0.4571 + 0.1436i \\ 0.3948 + 0.3532i \end{bmatrix}$; $\lambda_2 = -3.6884 - 2.8416i$, $\mathbf{x}_2 = \begin{bmatrix} -0.0558 + 0.6977i \\ -0.4571 - 0.1436i \\ 0.3948 - 0.3532i \end{bmatrix}$; $\lambda_3 = 13.3769$, $\mathbf{x}_3 = \begin{bmatrix} -0.4184 \\ -0.7889 \\ -0.4501 \end{bmatrix}$. For $i = 1, 2, 3$, $\|\mathbf{x_i}\| = 1$.

33. $\lambda_1 = 1.1857 + 2.6885i$, $\mathbf{x}_1 = \begin{bmatrix} -0.0781 - 0.6033i \\ -0.3495 + 0.5754i \\ 0.1199 - 0.1125i \\ 0.1963 + 0.3334i \end{bmatrix}$;

$\lambda_2 = 1.1857 - 2.6885i$, $\mathbf{x}_2 = \begin{bmatrix} -0.0781 + 0.6033i \\ -0.3495 - 0.5754i \\ 0.1199 + 0.1125i \\ 0.1963 - 0.3334i \end{bmatrix}$;

$\lambda_3 = 16.8037$, $\mathbf{x}_3 = \begin{bmatrix} -0.5484 \\ -0.0550 \\ -0.1746 \\ -0.8160 \end{bmatrix}$;

$\lambda_4 = 4.8249$, $\mathbf{x}_4 = \begin{bmatrix} 0.7046 \\ -0.6728 \\ -0.2027 \\ -0.0995 \end{bmatrix}$. For $i = 1, 2, 3, 4$, $\| \mathbf{x_i} \| = 1$.

34. $\lambda_1 = 0.2617 + 2.0076i$, $\mathbf{x}_1 = \begin{bmatrix} -0.2742 + 0.1318i \\ -0.6251 - 0.1139i \\ 0.1594 + 0.2495i \\ 0.5871 - 0.2672i \end{bmatrix}$;

$\lambda_2 = 0.2617 - 2.0076i$, $\mathbf{x}_2 = \begin{bmatrix} -0.2742 - 0.1318i \\ -0.6251 + 0.1139i \\ 0.1594 - 0.2495i \\ 0.5871 + 0.2672i \end{bmatrix}$;

$\lambda_3 = 16.6911$, $\mathbf{x}_3 = \begin{bmatrix} 0.5848 \\ 0.5644 \\ 0.1660 \\ 0.5585 \end{bmatrix}$;

$\lambda_4 = 3.7856$, $\mathbf{x}_4 = \begin{bmatrix} -0.4955 \\ 0.6192 \\ 0.2253 \\ -0.5659 \end{bmatrix}$

35. Let $z = a + bi$ and $w = c + di$.

(a) $z + w = (a + c) + (b + d)i$ so $\overline{z + w} = (a + c) - (b + d)i = \overline{z} + \overline{w}$.

(b) $zw = (ac - bd) + (ad + bc)i$ so $\overline{zw} = (ac - bd) - (ad + bc)i$. Therefore $\overline{z}\,\overline{w} = (a - bi)(c - di) = (ac - bd) - (ad + bc)i = \overline{zw}$.

(c) $z + \overline{z} = (a + bi) + (a - bi) = 2a$.

(d) $z - \overline{z} = (a + bi) - (a - bi) = 2bi$.

(e) $z\overline{z} = (a + bi)(a - bi) = a^2 - b^2 i^2 = a^2 + b^2$.

36. If $A = [a_{ij}]$ and $B = [b_{ij}]$ then AB is the (n x p) matrix $[c_{ij}]$ where $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$. Likewise $\overline{A}\ \overline{B}$ is the (n x p) matrix $[d_{ij}]$ where $d_{ij} = \sum_{k=1}^{n} \overline{a_{ik}}\,\overline{b_{kj}} = \sum_{k=1}^{n} \overline{a_{ik} b_{kj}} = \overline{\sum_{k=1}^{n} a_{ik} b_{kj}} = \overline{c_{ij}}$. Thus $\overline{A}\ \overline{B} = \overline{AB}$. If A is a real matrix and $\mathbf{x}$ is an (n x 1) vector then $\overline{A\mathbf{x}} = \overline{A}\overline{\mathbf{x}} = A\overline{\mathbf{x}}$.

37. $(AB)^* = \overline{(AB)}^{\mathrm{T}} = (\overline{A}\,\overline{B})^{\mathrm{T}} = \overline{B}^{\mathrm{T}} \overline{A}^{\mathrm{T}} = B^* A^*$.

38. (a) First note that for vectors $\mathbf{x}$ and $\mathbf{y}, \mathbf{y}^*\mathbf{x} = \mathbf{y}^*\mathbf{x}^{**} = (\mathbf{x}^*\mathbf{y})^* = \overline{\mathbf{x}^*\mathbf{y}}$. Now suppose that $A\mathbf{x} = \lambda\mathbf{x}$ where $\mathbf{x} \neq \theta$. Then $\lambda \parallel \mathbf{x} \parallel^2 = \lambda\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{x} = \lambda\mathbf{x}^*\mathbf{x} = \mathbf{x}^*(\lambda\mathbf{x}) = \mathbf{x}^*(A\mathbf{x}) = \overline{(A\mathbf{x})^*\mathbf{x}} = \overline{(\mathbf{x}^*A^*)\mathbf{x}} = \overline{\mathbf{x}^*(A\mathbf{x})} = \overline{\mathbf{x}^*(\lambda\mathbf{x})} = \overline{\lambda}\overline{\mathbf{x} * \mathbf{x}} = \overline{\lambda}\overline{\parallel \mathbf{x} \parallel^{\mathbf{2}}} = \overline{\lambda} \parallel \mathbf{x} \parallel^{\mathbf{2}}$. Since $\parallel \mathbf{x} \parallel^{\mathbf{2}} \neq \mathbf{0}, \lambda = \overline{\lambda}$ and λ is a real number.

(b) A^* is the (n x n) matrix $[b_{ij}]$ where $b_{ij} = \overline{a_{ji}}$. Since $A = A^*, a_{ij} = \overline{a_{ji}}$ for $1 \leq i, j \leq n$. In particular, $a_{ii} = \overline{a_{ii}}$ for $1 \leq i \leq n$, so a_{ii} is a real number.

39. (a) Since $p(r) = 0$ we have $0 = \overline{0} = \overline{p(r)} = \overline{a_0 + a_1 r + \cdots + a_n r^n} = \overline{a_0} + \overline{a_1}\,\overline{r} + \cdots + \overline{a_n}\overline{r^n} = a_0 + a_1\overline{r} + \cdots + a_n\overline{r}^n = p(\overline{r})$.

(b) Write $p(t) = c(t - r_1)(t - r_2)(t - r_3)$ where c is a real number and r_1, r_2, r_3 are the (not necessarily distinct) roots of $p(t)$. If all three roots are real numbers then there is nothing to prove, so assume that $r_1 = a + bi, b \neq 0$. By (a) we may also assume that $r_2 = \overline{r_1} = a - bi$. Thus $p(t) = c(t - r_1)(t - r_2)(t - r_3) = c[t^2 - 2at + (a^2 + b^2)](t - r_3)$. Since the coefficients of $p(t)$ are real numbers, it follows that r_3 is a real number.

(c) The characteristic polynomial $p(t) = \det(A - tI)$ has degree three and real coefficients. By (b) $p(t)$ has a real root so A has at least one real eigenvalue.

40. For any vector $\mathbf{x}, \parallel A\mathbf{x} \parallel^{\mathbf{2}} = \overline{(A\mathbf{x})}^{\mathrm{T}} A\mathbf{x} = (A\overline{\mathbf{x}})^{\mathrm{T}}(A\mathbf{x}) = \overline{\mathbf{x}}^{\mathrm{T}} A^{\mathrm{T}} A\mathbf{x} = \overline{\mathbf{x}}^{\mathrm{T}}\mathbf{x} = \parallel \mathbf{x} \parallel^2$. In particular if $A\mathbf{x} = \lambda\mathbf{x}, \mathbf{x} \neq \theta$, then $\parallel \mathbf{x} \parallel^2 = \parallel \lambda\mathbf{x} \parallel^{\mathbf{2}} = \overline{\lambda\mathbf{x}}^{\mathrm{T}}(\lambda\mathbf{x}) = \overline{\lambda}\lambda(\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{x}) = \overline{\lambda}\lambda \parallel \mathbf{x} \parallel^2$. Thus $\overline{\lambda}\lambda = 1$.

41. Let λ be an eigenvalue for A (note that λ is real by Theorem 17) and suppose $\mathbf{x}$ in R^n is a corresponding eigenvector. Then $\mathbf{x} \neq \theta$ and $0 < \mathbf{x}^{\mathrm{T}} A\mathbf{x} = \mathbf{x}^{\mathrm{T}}(\lambda\mathbf{x}) = \lambda\mathbf{x}^{\mathrm{T}}\mathbf{x} = \lambda \parallel \mathbf{x} \parallel^2$. Since $\parallel \mathbf{x} \parallel^2 > 0$ it follows that $\lambda > 0$.

42. cf. Exercise 40.

4.7 Similarity Transformations and Diagonalization

1. A has eigenvalues $\lambda = 1$ and $\lambda = 3$ with corresponding eigenvectors $\mathbf{u_1} = [1, 1]^{\mathrm{T}}$ and $\mathbf{u_2} = [-1, 1]^{\mathrm{T}}$, respectively. If $S = [\mathbf{u_1}, \mathbf{u_2}]$ then $S^{-1}AS = D$ where $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$. Now $D^5 = \begin{bmatrix} 1 & 0 \\ 0 & 243 \end{bmatrix}$ so $A^5 = SD^5S^{-1} = \begin{bmatrix} 122 & -121 \\ -121 & 122 \end{bmatrix}$.

2. For $S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, S^{-1}AS = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}. A^5 = S \begin{bmatrix} 0 & 0 \\ 0 & 2^5 \end{bmatrix} S^{-1} = \begin{bmatrix} 16 & -16 \\ -16 & 16 \end{bmatrix}$.

3. A has only one eigenvalue, $\lambda = -1$. The corresponding eigenvectors are the nonzero vectors of the form $\mathbf{x} = [x_2, x_2]^{\mathrm{T}}$. Since we cannot find a linearly independent set $\{\mathbf{u_1}, \mathbf{u_2}\}$ of eigenvectors for A, A is not diagonalizable.

4. A has eigenvalue $\lambda = 1$ with associated eigenvectors of the form $\mathbf{x} = [x_1, 0]^{\mathrm{T}}, x_1 \neq 0.$ A is not diagonalizable,

5. A has eigenvalues $\lambda = 1$ and $\lambda = 2$ with corresponding eigenvectors $\mathbf{u_1} = [1, -10]^{\mathrm{T}}$ and $\mathbf{u_2} = [0, 1]^{\mathrm{T}}$, respectively. If $S = [\mathbf{u_1}, \mathbf{u_2}]$

   then $S^{-1}AS = D$ where $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Thus $A^5 = SD^5S^{-1} =$

   $$S\begin{bmatrix} 1 & 0 \\ 0 & 32 \end{bmatrix}S^{-1} = \begin{bmatrix} 1 & 0 \\ 310 & 32 \end{bmatrix}.$$

6. For $S = \begin{bmatrix} 1 & 7 \\ 0 & 2 \end{bmatrix}, S^{-1}AS = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Thus $A^5 =$

   $$S\begin{bmatrix} (-1)^5 & 0 \\ 0 & (1)^5 \end{bmatrix}S^{-1} = \begin{bmatrix} -1 & 7 \\ 0 & 1 \end{bmatrix} = A.$$

7. A has eigenvalue $\lambda = 1$ with algebraic multiplicity 3. The eigenvectors for $\lambda = 1$ have the form $\mathbf{x} = [x_2 + 2x_3, x_2, x_3]^{\mathrm{T}}$. In particular we cannot obtain 3 linearly independent eigenvectors so A is not diagonalizable.

8. A has eigenvalue $\lambda = 1$ with algebraic multiplicity 3 and geometric multiplicity 1, so A is not diagonalizable.

9. A has eigenvalues $1, 2,$ and -1 with corresponding eigenvectors $\mathbf{u_1} = [-3, 1, -7]^{\mathrm{T}}, \mathbf{u_2} = [-1, 1, -2]^{\mathrm{T}}, \mathbf{u_3} = [1, 2, 2]^{\mathrm{T}}$, respectively. If $S = [\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}]$ then

   $S^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ 16/3 & 1/3 & 7/3 \\ 5/3 & 1/3 & -2/3 \end{bmatrix}$ and $S^{-1}AS = D$, where $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

   $$A^5 = SD^5S^{-1} = \begin{bmatrix} 163 & -11 & -71 \\ -172 & 10 & 75 \\ 324 & -22 & -141 \end{bmatrix}.$$

10. A has eigenvalues $\lambda = 1$ and $\lambda = 2$. For $\lambda = 1$, $\mathbf{u_1} = [1, 0, 0]^{\mathrm{T}}$ and $\mathbf{u_2} = [0, 1, 1]^{\mathrm{T}}$ are linearly independent eigenvectors. $\lambda = 2$ has corresponding eigenvector $\mathbf{u_3} = [1, 1, 0]^{\mathrm{T}}$. If

    $S = [\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}]$ then $S^{-1}AS = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = D$. Moreover $A^5 = SD^5S^{-1} =$

    $$\begin{bmatrix} 1 & 31 & -31 \\ 0 & 32 & -31 \\ 0 & 0 & 1 \end{bmatrix}.$$

11. A has eigenvalue $\lambda = 1$ with algebraic multiplicity 2 and geometric multiplicity 1 so A is not diagonalizable.

12. If $S = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 4 \\ 0 & 1 & 0 \end{bmatrix}$ then $S^{-1}AS = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. $A^5 =$
$S \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5^5 \end{bmatrix} S^{-1} = \begin{bmatrix} 1 & 2343 & 2343 \\ 0 & 3125 & 3124 \\ 0 & 0 & 1 \end{bmatrix}$.

13. $\mathbf{q_1}^{\mathrm{T}}\mathbf{q_2} = 0$ and $\mathbf{q_1}^{\mathrm{T}}\mathbf{q_1} = \mathbf{q_2}^{\mathrm{T}}\mathbf{q_2} = 1$ so Q is orthogonal.

14. Q is orthogonal.

15. $\mathbf{q_1}^{\mathrm{T}}\mathbf{q_1} = 5$ so Q is not orthogonal.

16. $\mathbf{q_1}^{\mathrm{T}}\mathbf{q_1} = 13$ so Q is not orthogonal.

17. $0 = \mathbf{q_1}^{\mathrm{T}}\mathbf{q_2} = \mathbf{q_1}^{\mathrm{T}}\mathbf{q_3} = \mathbf{q_2}^{\mathrm{T}}\mathbf{q_3}$ and $1 = \mathbf{q_1}^{\mathrm{T}}\mathbf{q_1} = \mathbf{q_2}^{\mathrm{T}}\mathbf{q_2} = \mathbf{q_3}^{\mathrm{T}}\mathbf{q_3}$ so Q is orthogonal.

18. $\mathbf{q_1}^{\mathrm{T}}\mathbf{q_1} = 6$ so Q is not orthogonal.

19. If Q is orthogonal then $2\alpha^2 = 1, 6\beta^2 = 1, a^2+b^2+c^2 = 1, \alpha a+\alpha c = 0,$ and $\beta a+2\beta b-\beta c = 0$. This implies that $\alpha = 1/\sqrt{2}, \beta = 1/\sqrt{6}, a = -c$ and $b = c$, where $c = \pm 1/\sqrt{3}$. Thus we one choice for Q is

$$Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}.$$

20. If Q is orthogonal then $\alpha = 1/\sqrt{3}$, $\beta = 1/\sqrt{14}$, $a = (-5/4)c$, and $b = (1/4)c$ where $c = \pm 4/\sqrt{42}$.

21. $Q = \begin{bmatrix} -0.8807 & 0.4332 & 0.1918 \\ 0.1849 & 0.6870 & -.7028 \\ 0.4361 & 0.5835 & 0.6851 \end{bmatrix}$ $T = \begin{bmatrix} 0.5048 & 2.9498 & -1.4966 \\ 0.0 & 8.3443 & 0.2429 \\ 0.0 & 0.0 & -2.8491 \end{bmatrix}$

22. $Q = \begin{bmatrix} 0.5892 & -0.7155 & -0.3754 \\ 0.7516 & 0.6559 & -0.0703 \\ -0.2965 & 0.2407 & -0.9242 \end{bmatrix}$ $T = \begin{bmatrix} -0.5223 & -7.6134 & -8.2480 \\ 0.0 & -5.6251 & 0.0735 \\ 0.0 & 0.0 & 7.1474 \end{bmatrix}$

23. $Q = \begin{bmatrix} -0.5276 & -0.5463 & -0.6406 & 0.1130 \\ -0.4235 & -0.4440 & 0.6477 & -0.4516 \\ -0.3315 & 0.0292 & 0.3989 & 0.8545 \\ -0.6576 & 0.7096 & -0.1044 & -0.2306 \end{bmatrix}$

$T = \begin{bmatrix} 19.2422 & 1.4541 & -3.5019 & 0.0983 \\ 0.0 & -3.8383 & -3.4646 & -0.5055 \\ 0.0 & 0.0 & 4.8409 & 3.5148 \\ 0.0 & 0.0 & 0.0 & 0.7552 \end{bmatrix}$

24. $Q = \begin{bmatrix} 0.5118 & -0.5287 & 0.6725 & -0.0794 \\ 0.6815 & -0.2030 & -0.6454 & 0.2788 \\ 0.3517 & 0.3331 & -0.1083 & -0.8681 \\ 0.3871 & 0.7539 & 0.3457 & 0.4029 \end{bmatrix}$

$T = \begin{bmatrix} 19.1640 & 5.4524 & 1.6686 & -1.9529 \\ 0.0 & 2.9467 & -2.5265 & -2.3201 \\ 0.0 & 0.0 & -3.8881 & 0.9528 \\ 0.0 & 0.0 & 0.0 & -2.2226 \end{bmatrix}$

25. (a) $(S^{-1}AS)^2 = (S^{-1}AS)(S^{-1}AS) = S^{-1}A(SS^{-1})AS = S^{-1}A^2S$.
$(S^{-1}AS)^3 = (S^{-1}AS)^2(S^{-1}AS) = (S^{-1}A^2S)(S^{-1}AS) = S^{-1}A^2(SS^{-1})AS = S^{-1}A^3S$.

(b) Suppose $(S^{-1}AS)^n = S^{-1}A^nS$ for some integer $n \geq 1$. Then $(S^{-1}AS)^{n+1} = (S^{-1}AS)^n(S^{-1}AS) = (S^{-1}A^nS)(S^{-1}AS) = S^{-1}A^n(SS^{-1})AS = S^{-1}A^{n+1}S$. By induction $(S^{-1}AS)^k = S^{-1}A^kS$ for any positive integer k.

26. Suppose $S^{-1}AS = D$, where D is a diagonal matrix, and suppose $W^{-1}AW = B$. If $T = W^{-1}S$ then T is invertible and $T^{-1}BT = S^{-1}WBW^{-1}S = S^{-1}AS = D$. Therefore B is diagonalizable.

27. Suppose that $S^{-1}AS = B$.

(a) $S^{-1}(A + \alpha I)S = S^{-1}AS + S^{-1}(\alpha I)S = B + \alpha S^{-1}IS = B + \alpha I$.

(b) Set $Q = (S^{-1})^{\mathrm{T}} = (S^{\mathrm{T}})^{-1}$. Then $Q^{-1}A^{\mathrm{T}}Q = S^{\mathrm{T}}A^{\mathrm{T}}(S^{-1})^{\mathrm{T}} = (S^{-1}AS)^{\mathrm{T}} = B^{\mathrm{T}}$.

(c) A product of nonsingular matrices is nonsingular so if A is nonsingular then so is B. Moreover $B^{-1} = (S^{-1}AS)^{-1} =$
$S^{-1}A^{-1}(S^{-1})^{-1} = S^{-1}A^{-1}S$. Therefore B^{-1} is similar to A^{-1}.

28. For (b) if $\mathbf{x}, \mathbf{y}$ are in R^n then $(Q\mathbf{x})^{\mathrm{T}}(Q\mathbf{y}) = \mathbf{x}^{\mathrm{T}}Q^{\mathrm{T}}Q\mathbf{y} = \mathbf{x}^{\mathrm{T}}I\mathbf{y} = \mathbf{x}^{\mathrm{T}}\mathbf{y}$.

For (c) let $t = \det(Q)$. Recall that from Theorem 5 of Section 3.3 that $\det(Q^{\mathrm{T}}) = \det(Q)$. Since $Q^{\mathrm{T}}Q = I$, Theorem 2 of Section 3.2 gives $1 = \det(I) = \det(Q^{\mathrm{T}}Q) = \det(Q^{\mathrm{T}})\det(Q) = t^2$. But $t^2 = 1$ implies that $t = \pm 1$.

29. First note that $Q^{\mathrm{T}} = (I - 2\mathbf{u}\mathbf{u}^{\mathrm{T}})^{\mathrm{T}} = I^{\mathrm{T}} - 2\mathbf{u}^{\mathrm{TT}}\mathbf{u}^{\mathrm{T}} =$
$I - 2\mathbf{u}\,\mathbf{u}^{\mathrm{T}} = Q$, so Q is symmetric. Thus $Q^{\mathrm{T}}Q = QQ =$
$(I - 2\mathbf{u}\mathbf{u}^{\mathrm{T}})(I - 2\mathbf{u}\mathbf{u}^{\mathrm{T}}) = I^2 - 2\mathbf{u}\,\mathbf{u}^{\mathrm{T}}I - 2I\mathbf{u}\mathbf{u}^{\mathrm{T}} + 4\mathbf{u}\,(\mathbf{u}^{\mathrm{T}}\mathbf{u})\mathbf{u}^{\mathrm{T}} =$
I. Thus Q is orthogonal. Moreover $Q\mathbf{u} = (I - 2\mathbf{u}\mathbf{u}^{\mathrm{T}})(\mathbf{u}) = I\mathbf{u} - 2\mathbf{u}(\mathbf{u}^{\mathrm{T}}\mathbf{u}) = \mathbf{u} - 2\mathbf{u} = -\mathbf{u}$, so $\mathbf{u}$ s an eigenvector corresponding to the eigenvalue $\lambda = -1$.

30. $(AB)^{\mathrm{T}}(AB) = B^{\mathrm{T}}(A^{\mathrm{T}}A)B = B^{\mathrm{T}}B = I$.

31. $\mathbf{y} = [b/\sqrt{a^2+b^2}, -a/\sqrt{a^2+b^2}]^{\mathrm{T}}$ is one choice. $-\mathbf{y}$ is another choice.

32. $Q^{\mathrm{T}}Q = \begin{bmatrix} \mathbf{u}^{\mathrm{T}}\mathbf{u} & \mathbf{u}^{\mathrm{T}}\mathbf{v} \\ \mathbf{v}^{\mathrm{T}}\mathbf{u} & \mathbf{v}^{\mathrm{T}}\mathbf{v} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ so Q is orthogonal. Now
$Q^{\mathrm{T}}AQ$ is the product $\begin{bmatrix} \mathbf{u}^{\mathrm{T}} \\ \mathbf{v}^{\mathrm{T}} \end{bmatrix} A[\mathbf{u}\,,\mathbf{v}\,] = \begin{bmatrix} \mathbf{u}^{\mathrm{T}} \\ \mathbf{v}^{\mathrm{T}} \end{bmatrix} [A\mathbf{u}\,, A\mathbf{v}\,] =$
$\begin{bmatrix} \mathbf{u}^{\mathrm{T}} \\ \mathbf{v}^{\mathrm{T}} \end{bmatrix} [\lambda\mathbf{u}\,, A\mathbf{v}\,] = \begin{bmatrix} \lambda\mathbf{u}^{\mathrm{T}}\mathbf{u} & \mathbf{u}^{\mathrm{T}}A\mathbf{v} \\ \lambda\mathbf{v}^{\mathrm{T}}\mathbf{u} & \mathbf{v}^{\mathrm{T}}A\mathbf{v} \end{bmatrix} = \begin{bmatrix} \lambda & \mathbf{u}^{\mathrm{T}}A\mathbf{v} \\ 0 & \mathbf{v}^{\mathrm{T}}A\mathbf{v} \end{bmatrix}$.

33. If $\mathbf{u} = [1/\sqrt{2}, -1/\sqrt{2}]^{\mathrm{T}}$ then $A\mathbf{u} = 2\mathbf{u}$ and $\mathbf{u}^{\mathrm{T}}\mathbf{u} = 1$. If $\mathbf{v} =$
$[1/\sqrt{2}, 1/\sqrt{2}]^{\mathrm{T}}$ then $\mathbf{u}^{\mathrm{T}}\mathbf{v} = 0$ and $\mathbf{v}^{\mathrm{T}}\mathbf{v} = 1$. If $Q = [\mathbf{u}\,,\mathbf{v}\,]$ then $Q^{\mathrm{T}}AQ =$
$\begin{bmatrix} 2 & -2 \\ 0 & 2 \end{bmatrix}$.

34. If $\mathbf{u} = [1/\sqrt{5}, 2/\sqrt{5}]^{\mathrm{T}}$ then $A\mathbf{u} = \mathbf{u}$ and $\mathbf{u}^{\mathrm{T}}\mathbf{u} = 1$. If $\mathbf{v} =$
$[2/\sqrt{5}, -1/\sqrt{5}]^{\mathrm{T}}$ then $\mathbf{u}^{\mathrm{T}}\mathbf{v} = 0$ and $\mathbf{v}^{\mathrm{T}}\mathbf{v} = 1$. If $Q = [\mathbf{u}\,,\mathbf{v}\,]$ then $Q^{\mathrm{T}}AQ = \begin{bmatrix} 1 & 8 \\ 0 & 2 \end{bmatrix}$.
Similarly if $\mathbf{u} = [2/\sqrt{13}, 3/\sqrt{13}]^{\mathrm{T}}$ and $\mathbf{v} =$
$[3/\sqrt{13}, -2/\sqrt{13}]^{\mathrm{T}}$ then $A\mathbf{u} = 2\mathbf{u}\,, \mathbf{u}^{\mathrm{T}}\mathbf{u} = \mathbf{v}^{\mathrm{T}}\mathbf{v} = 1$, and $\mathbf{u}^{\mathrm{T}}\mathbf{v} = 0$. If $Q = [\mathbf{u}\,,\mathbf{v}\,]$ then $Q^{\mathrm{T}}AQ = \begin{bmatrix} 2 & 8 \\ 0 & 1 \end{bmatrix}$.

35. If $\mathbf{u} = [1/\sqrt{2}, 1/\sqrt{2}]^{\mathrm{T}}$ then $A\mathbf{u} = \mathbf{u}$ and $\mathbf{u}^{\mathrm{T}}\mathbf{u} = 1$. If $\mathbf{v} =$
$[1/\sqrt{2}, -1/\sqrt{2}]^{\mathrm{T}}$ then $\mathbf{u}^{\mathrm{T}}\mathbf{v} = 0$ and $\mathbf{v}^{\mathrm{T}}\mathbf{v} = 1$. If $Q = [\mathbf{u}\,,\mathbf{v}\,]$ then
$Q^{\mathrm{T}}AQ = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$.

36. If $Q = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ then $Q^{\mathrm{T}}AQ = \begin{bmatrix} 0 & 1 \\ 0 & 5 \end{bmatrix}$. Also if $Q = \begin{bmatrix} 2/\sqrt{13} & 3/\sqrt{13} \\ 3/\sqrt{13} & -2/\sqrt{13} \end{bmatrix}$
then $Q^{\mathrm{T}}AQ = \begin{bmatrix} 5 & 1 \\ 0 & 0 \end{bmatrix}$.

37. Note that $A\mathbf{R_j}$ is the j^{th} column of AR and $\mathbf{R_i}^{\mathrm{T}}$ is the i^{th} row of R^{T}. Thus $\mathbf{R_i}^{\mathrm{T}} A\mathbf{R_j}$ is the ij^{th} entry of $R^{\mathrm{T}}AR$.

38. It is an consequence of Exercise 37 that

$$Q^{\mathrm{T}}AQ = \begin{bmatrix} \mathbf{u}^{\mathrm{T}}A\mathbf{u} & \mathbf{u}^{\mathrm{T}}A\mathbf{v} & \mathbf{u}^{\mathrm{T}}A\mathbf{w} \\ \mathbf{v}^{\mathrm{T}}A\mathbf{u} & \mathbf{v}^{\mathrm{T}}A\mathbf{v} & \mathbf{v}^{\mathrm{T}}A\mathbf{w} \\ \mathbf{w}^{\mathrm{T}}A\mathbf{u} & \mathbf{w}^{\mathrm{T}}A\mathbf{v} & \mathbf{w}^{\mathrm{T}}A\mathbf{w} \end{bmatrix}.$$

But $A\mathbf{u} = \lambda\mathbf{u}$, $\mathbf{u}^{\mathrm{T}}\mathbf{u} = 1, \mathbf{v}^{\mathrm{T}}\mathbf{u} = 0 = \mathbf{w}^{\mathrm{T}}\mathbf{u}$, so $Q^{\mathrm{T}}AQ$ has the desired form.

39. The matrix B has characteristic polynomial $p(t) = \det(B - tI)$.

Expansion along the first column gives $p(t) = (\lambda - t)\det(A_1 - tI) =$

$(\lambda - t)q(t)$ where $q(t)$ is the characteristic polynomial for A_1. Since every root of $q(t)$ is also a root of $p(t)$, each eigenvalue for A_1 is also an eigenvalue for B. Since Q is an orthogonal matrix $Q^{\mathrm{T}} = Q^{-1}$ and B is similar to A. Thus A and B have the same eigenvalues. In particular B has only real eigenvalues. It follows that A_1 has only real eigenvalues.

40. (a) It is straightforward to show that

$$R^{\mathrm{T}} = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{S^{T}}} \\ 0 & & \end{array}\right]$$

and that

$$R^{\mathrm{T}}R = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$S^{\mathrm{T}}S$}} \\ 0 & & \end{array}\right] = I.$$

(b) $R^{\mathrm{T}}Q^{\mathrm{T}}AQR = R^{\mathrm{T}}BR$ and $R^{\mathrm{T}}BR$ has the form

$$\left[\begin{array}{c|cc} \lambda & a & b \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$S^{\mathrm{T}}A_1S$}} \\ 0 & & \end{array}\right] =$$

$$\left[\begin{array}{c|cc} \lambda & a & b \\ \hline 0 & \multicolumn{2}{c}{T_1} \\ 0 & & \end{array}\right].$$

Since T_1 is upper triangular so is $R^{\mathrm{T}}BR$.

41. Assume that Theorem 22 is true for any [(k-1) x (k-1)] matrix with only real eigenvalues. Now let A be a (k x k) matrix with only real eigenvalues and suppose $A\mathbf{u} = \lambda\mathbf{u}$ where $\mathbf{u}^{\mathrm{T}}\mathbf{u} = 1$. By the Gram-Schmidt process there is an orthonormal basis $\{\mathbf{u_1}, \mathbf{u_2},$ $\ldots, \mathbf{u_k}\}$ for R^k such that $\mathbf{u_1} = \mathbf{u}$. The matrix $Q =$ $[\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_k}]$ is orthogonnal and

$$Q^{\mathrm{T}}AQ = \left[\begin{array}{c|ccc} \lambda & \mathbf{u_1}^{\mathrm{T}}A\mathbf{u_2} & \cdots & \mathbf{u_1}^{\mathrm{T}}A\mathbf{u_k} \\ \hline 0 & & & \\ \vdots & & A_1 & \\ 0 & & & \end{array}\right],$$

where A_1 is a [(k-1) x (k-1)] matrix. If $B = Q^{\mathrm{T}}AQ$ then B has characteristic polynomial $p(t) = \det(B - tI)$. Expanding along the first column yields $p(t) = (\lambda - t)\det(A_1 - tI) = (\lambda - t)q(t)$ where $q(t)$ is the characteristic polynomial for A_1. Thus every eigenvalue for A_1 is also an eigenvalue for B. But B is similar to A (since $Q^{\mathrm{T}} = Q^{-1}$) so B has only real eigenvalues and, therefore A_1 has only real eigenvalues. By assumption there exists a [(k-1) x (k-1)] orthogonal matrix S such that $S^{\mathrm{T}}A_1S = T_1$ where T_1 is upper triangular. If R is the (k x k) matrix

$$R = \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & S & \\ 0 & & & \end{array}\right]$$

then R is orthogonal and $P = QR$ is orthogonal. Furthermore
$P^{\mathrm{T}}AP = R^{\mathrm{T}}Q^{\mathrm{T}}AQR = R^{\mathrm{T}}BR$ and $R^{\mathrm{T}}BR$ has the form

$$\left[\begin{array}{c|ccc} \lambda & c_2 & \cdots & c_k \\ \hline 0 & & & \\ \vdots & & S^{\mathrm{T}}A_1S & \\ 0 & & & \end{array}\right] =$$

$$\left[\begin{array}{c|ccc} \lambda & c_2 & \cdots & c_k \\ \hline 0 & & & \\ \vdots & & T_1 & \\ 0 & & & \end{array}\right].$$

Since T_1 is upper triangular so is $R^{\mathrm{T}}BR$. The theorem now follows by induction.

42. $A\mathbf{u_1} = (2-n)\mathbf{u_1}$ while $A(\mathbf{e_1} - \mathbf{e_i}) = A\mathbf{e_1} - A\mathbf{e_i} = \mathbf{A_1} - \mathbf{A_i} = [2, 0, \ldots, 0, -2, 0, \ldots, 0]^{\mathrm{T}} = 2(\mathbf{e_1} - \mathbf{e_i})$, for $2 \le i \le n$.

43. Since A is symmetric we have $\beta(\mathbf{u}^{\mathrm{T}}\mathbf{v}) = \mathbf{u}^{\mathrm{T}}(\beta\mathbf{v}) = \mathbf{u}^{\mathrm{T}}A\mathbf{v} =$
$\mathbf{u}^{\mathrm{T}}A^{\mathrm{T}}\mathbf{v} = (A\mathbf{u})^{\mathrm{T}}\mathbf{v} = (\lambda\mathbf{u})^{\mathrm{T}}\mathbf{v} = \lambda\mathbf{u}^{\mathrm{T}}\mathbf{v}$. Since $\beta \ne \lambda, \mathbf{u}^{\mathrm{T}}\mathbf{v} = 0$.

4.8 Applications

1. $\mathbf{x_1} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \mathbf{x_2} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \mathbf{x_3} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \mathbf{x_4} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$

2. $\mathbf{x_1} = \mathbf{x_2} = \mathbf{x_3} = \mathbf{x_4} = \begin{bmatrix} 12 \\ 12 \end{bmatrix}.$

3. $\mathbf{x_1} = \begin{bmatrix} 80 \\ 112 \end{bmatrix}, \mathbf{x_2} = \begin{bmatrix} 68 \\ 124 \end{bmatrix}, \mathbf{x_3} = \begin{bmatrix} 65 \\ 127 \end{bmatrix}, \mathbf{x_4} = \begin{bmatrix} 64.25 \\ 127.75 \end{bmatrix}.$

4. $\mathbf{x_1} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}, \mathbf{x_2} = \begin{bmatrix} 11 \\ -7 \end{bmatrix}, \mathbf{x_3} = \begin{bmatrix} 29 \\ -25 \end{bmatrix}, \mathbf{x_4} = \begin{bmatrix} 83 \\ -79 \end{bmatrix}.$

5. $\mathbf{x_1} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}, \mathbf{x_2} = \begin{bmatrix} 11 \\ 8 \end{bmatrix}, \mathbf{x_3} = \begin{bmatrix} 43 \\ 19 \end{bmatrix}, \mathbf{x_4} = \begin{bmatrix} 119 \\ 62 \end{bmatrix}.$

6. $\mathbf{x_1} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}, \mathbf{x_2} = \begin{bmatrix} 26 \\ 48 \end{bmatrix}, \mathbf{x_3} = \begin{bmatrix} 126 \\ 248 \end{bmatrix}, \mathbf{x_4} = \begin{bmatrix} 626 \\ 1248 \end{bmatrix}.$

7. A has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$ with corresponding eigenvectors $\mathbf{u_1} = [1, 1]^{\mathrm{T}}$ and $\mathbf{u_2} = [-1, 1]^{\mathrm{T}}$, respectively. $\mathbf{x_0} = 3\mathbf{u_1} + \mathbf{u_2}$ so $\mathbf{x_k} = 3(1)^k\mathbf{u_1} + (-1)^k\mathbf{u_2} = [3 + (-1)^{k+1}, 3 + (-1)^k]^{\mathrm{T}}$. In particular $\mathbf{x_4} = [2, 4]^{\mathrm{T}} = \mathbf{x_{10}}$. The sequence $\{\mathbf{x_k}\}$ has no limit but $\| \mathbf{x_k} \| = \sqrt{20}$ for all k.

8. A has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 0$ with corresponding eigenvectors $\mathbf{u_1} = [1, 1]^{\mathrm{T}}$ and $\mathbf{u_2} = [-1, 1]^{\mathrm{T}}$, respectively. $\mathbf{x_0} = 12\mathbf{u_1} - 4\mathbf{u_2}$ so $\mathbf{x_k} = 12\mathbf{u_1}$. In particular $\mathbf{x_4} = \mathbf{x_{10}} = [12, 12]^{\mathrm{T}}$. The limit of the
sequence $\{\mathbf{x_k}\}$ is $\mathbf{x}^* = [12, 12]^{\mathrm{T}}$.

9. A has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 0.25$ with corresponding eigenvectors $\mathbf{u_1} = [1, 2]^{\mathrm{T}}$ and $\mathbf{u_2} = [-1, 1]^{\mathrm{T}}$, respectively. $\mathbf{x_0} = 64\mathbf{u_1} - 64\mathbf{u_2}$ so $\mathbf{x_k} = 64(1)^k\mathbf{u_1} - 64(0.25)^k\mathbf{u_2} = [64 + 64(0.25)^k, 128 - 64(0.25)^k]^{\mathrm{T}}$. In particular $\mathbf{x_4} = [64.25, 127.75]^{\mathrm{T}}$ and $\mathbf{x_{10}} =$
$[64.00006, 127.99994]^{\mathrm{T}}$. The sequence $\{\mathbf{x_k}\}$ converges to $\mathbf{x}^* =$
$[64, 128]^{\mathrm{T}}$.

10. A has eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 1$ with corresponding eigenvectors $\mathbf{u_1} = [-1, 1]^{\mathrm{T}}$ and $\mathbf{u_2} = [1, 1]^{\mathrm{T}}$, respectively. $\mathbf{x_0} = -\mathbf{u_1} + 2\mathbf{u_2}$, so $\mathbf{x_k} = -3^k \mathbf{u_1} + 2\mathbf{u_2} = [2 + 3^k, 2 - 3^k]^{\mathrm{T}}$. In particular $\mathbf{x_4} = [83, -79]^{\mathrm{T}}$

    and $\mathbf{x_{10}} = [59051, -59047]^{\mathrm{T}}$. The sequence $\{\mathbf{x_k}\}$ has no limit and $\lim_{k\to\infty} \|\mathbf{x_k}\| = \infty$.

11. A has eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -1$ with corresponding eigenvectors $\mathbf{u_1} = [2, 1]^{\mathrm{T}}$ and $\mathbf{u_2} = [-2, 1]^{\mathrm{T}}$, respectively. $\mathbf{x_0} =$

    $(3/4)\mathbf{u_1} + (5/4)\mathbf{u_2}$ so $\mathbf{x_k} = (3/4)(3)^k \mathbf{u_1} + (5/4)(-1)^k \mathbf{u_2} =$

    $(1/4)[2(3)^{k+1} - 10(-1)^k, 3^{k+1} + 5(-1)^k]^{\mathrm{T}}$. In particular $\mathbf{x_4} =$

    $[119, 62]^{\mathrm{T}}$ and $\mathbf{x_{10}} = [88571, 44288]^{\mathrm{T}}$. The sequence $\{\mathbf{x_k}\}$ has no limit and $\lim_{k\to\infty} \|\mathbf{x_k}\|$ $= \infty$.

12. A has eigenvalues $\lambda_1 = 5$ and $\lambda_2 = 1$ with corresponding eigenvectors $\mathbf{u_1} = [1, 2]^{\mathrm{T}}$ and $\mathbf{u_2} = [1, -2]^{\mathrm{T}}$, respectively. $\mathbf{x_0} = \mathbf{u_1} + \mathbf{u_2}$ so $\mathbf{x_k} = 5^k \mathbf{u_1} + \mathbf{u_2} = [5^k + 1, 2(5)^k - 2]^{\mathrm{T}}$. In particular, $\mathbf{x_4} = [626, 1248]^{\mathrm{T}}$ and $\mathbf{x_{10}} = [390626, 781248]^{\mathrm{T}}$. The sequence $\{\mathbf{x_k}\}$ has no limit and

    $\lim_{k\to\infty} \|\mathbf{x_k}\| = \infty$.

13. A has eigenvalues $\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = -1$ with corresponding eigenvectors $\mathbf{u_1} = [1, -1, 2]^{\mathrm{T}}, \mathbf{u_2} = [3, -1, 7]^{\mathrm{T}}$, and $\mathbf{u_3} = [1, 2, 2]^{\mathrm{T}}$, re-

    spectively. $\mathbf{x_0} = 2\mathbf{u_1} + 2\mathbf{u_2} - 5\mathbf{u_3}$, so

    $\mathbf{x_k} = 2^{k+1}\mathbf{u_1} + 2\mathbf{u_2} + 5(-1)^{k+1}\mathbf{u_3}$;

    $$\text{thus : } \mathbf{x_k} = \begin{bmatrix} 2^{k+1} + 6 + 5(-1)^{k+1} \\ -2^{k+1} - 2 + 10(-1)^{k+1} \\ 2^{k+2} + 14 + 10(-1)^{k+1} \end{bmatrix}.$$

    In particular $\mathbf{x_4} = [33, -44, 68]^{\mathrm{T}}$ and $\mathbf{x_{10}} = [2049, -2060, 4100]^{\mathrm{T}}$.

    The sequence $\{\mathbf{x_k}\}$ has no limit and $\lim_{k\to\infty} \|\mathbf{x_k}\| = \infty$.

14. A has eigenvalues $\lambda_1 = 4, \lambda_2 = 1, \lambda_3 = -1$ with corresponding eigenvectors $\mathbf{u_1} = [14, 11, 43]^{\mathrm{T}}, \mathbf{u_2} = [1, 1, 2]^{\mathrm{T}}$, and $\mathbf{u_3} =$

    $[1, -1, 2]^{\mathrm{T}}$, respectively. $\mathbf{x_0} = -0.2\mathbf{u_1} + 3.5\mathbf{u_2} + 0.3\mathbf{u_3}$ so $\mathbf{x_k} =$

    $-0.2(4)^k\mathbf{u_1} + 3.5\mathbf{u_2} + 0.3(-1)^k\mathbf{u_3}$; that is

    $$\mathbf{x_k} = \begin{bmatrix} (-2.8)4^k + 3.5 + 0.3(-1)^k \\ (-2.2)4^k + 3.5 - 0.3(-1)^k \\ (-8.6)4^k + 7 + 0.6(-1)^k \end{bmatrix}.$$

    In particular $\mathbf{x_4} = [-713, -560, -2195.2]^{\mathrm{T}}$ and $\mathbf{x_{10}} =$

$[-2936009, -2306864, -9017746]^{\mathrm{T}}$. The sequence $\{\mathbf{x_k}\}$ has no limit and $\lim_{k\to\infty} \|\mathbf{x_k}\| = \infty$.

15. If $\mathbf{x}(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}$ then $\mathbf{x}'(t) = A\mathbf{x}(t)$ for $A = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$. Eigenvalues and corresponding eigenvectors for A are $\lambda_1 = 2$, $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\lambda_2 = -1$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Setting $\mathbf{x}(t) = ae^{2t}\mathbf{u}_1 + be^{-t}\mathbf{u}_2$ and $\mathbf{x}(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ yields $a = 3$ and $b = -2$. Thus, $\mathbf{x}(t) = 3e^{2t}\mathbf{u}_1 - 2e^{-t}\mathbf{u}_2$. Equivalently, $u(t) = 6e^{2t} - 2e^{-t}$ and $v(t) = 3e^{2t} - 2e^{-t}$.

16. If $\mathbf{x}(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}$ then $\mathbf{x}'(t) = A\mathbf{x}(t)$ for $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Eigenvalues and corresponding eigenvectors for A are $\lambda_1 = 3$, $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\lambda_2 = -1$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Setting $\mathbf{x}(t) = ae^{3t}\mathbf{u}_1 + be^{-t}\mathbf{u}_2$ and $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ yields $a = 3$ and $b = 2$. Thus, $\mathbf{x}(t) = 3e^{3t}\mathbf{u}_1 + 2e^{-t}\mathbf{u}_2$. Equivalently, $u(t) = 3e^{3t} - 2e^{-t}$ and $v(t) = 3e^{3t} + 2e^{-t}$.

17. The matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ -2 & 1 & 1 \end{bmatrix}$ has eigenvalues $\lambda_1 = 0$, $\lambda_2 = 2$, $\lambda_3 = 3$. Corresponding eigenvectors are $\mathbf{u}_1 = [0, -1, 1]^T$, $\mathbf{u}_2 = [-2, -3, 1]^T$, $\mathbf{u}_3 = [1, 2, 0]^T$, respectively. The solution is $\mathbf{x}(t) = 2\mathbf{u}_1 - e^{2t}\mathbf{u}_2 + e^{3t}\mathbf{u}_3$. Equivalently, $u(t) = 2e^{2t} + e^{3t}$, $v(t) = -2 + 3e^{2t} + 2e^{3t}$, and $w(t) = 2 - e^{2t}$.

18. The matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ has eigenvalues $\lambda_1 = 5$ and $\lambda_2 = -3$, where λ_2 has algebraic multiplicity 2. An eigenvector for λ_1 is $\mathbf{u}_1 = [-1, -2, 1]^T$. Further, $\mathbf{u}_2 = [-2, 1, 0]^T$ and $\mathbf{u}_3 = [3, 0, 1]^T$ are linearly independent eigenvectors for λ_2. The solution is $\mathbf{x}(t) = e^{5t}\mathbf{u}_1 + e^{-3t}\mathbf{u}_2 + 2e^{-3t}\mathbf{u}_3$. Equivalently, $u(t) = -e^{5t} + 4e^{-3t}$, $v(t) = -2e^{5t} + e^{-3t}$, $w(t) = e^{5t} + 2e^{-3t}$.

19. (a) The eigenvectors corresponding to $\lambda = 1$ have the form $\mathbf{x} = [a, 0], a \neq 0$. In particular $\lambda = 1$ has algebraic multiplicity 2 and geometric multiplicity 1. Thus A is defective.

    (b) For $k = 0$ we have $\mathbf{x_0} = [1, 1]^{\mathrm{T}} = [2(0) + 1, 1]^{\mathrm{T}}$. Suppose $\mathbf{x_m} = [2m + 1, 1]^{\mathrm{T}}$ for some integer $m \geq 0$. Then $\mathbf{x_{m+1}} = A\mathbf{x_m} = [2m + 3, 1]^{\mathrm{T}} = [2(m + 1) + 1, 1]^{\mathrm{T}}$. It follows that $\mathbf{x_k} = [2k + 1, 1]^{\mathrm{T}}$ for each $k \geq 0$.

20. $\alpha = -0.5$. For each $k \geq 0, \mathbf{x_k} = \mathbf{x_0}$ so $\lim_{k\to\infty} \mathbf{x_k} = [1, 1]^{\mathrm{T}}$.

21. For $\alpha = -0.18$ A has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -0.18$ with corresponding eigenvectors $\mathbf{u_1} = [3, 10]^{\mathrm{T}}$ and $\mathbf{u_2} = [10, -6]^{\mathrm{T}}$, respectively. Moreover $\mathbf{x_0} = (16/118)\mathbf{u_1} + (7/118)\mathbf{u_2}$. Therefore $\mathbf{x_k} = (16/118)\mathbf{u_1} + (7/118)(-0.18)^k\mathbf{u_2}$. It follows that $\lim_{x\to\infty} \mathbf{x_k} = (16/118)\mathbf{u_1}$.

22. Note that $\mathbf{u_1} = A\mathbf{u_0} = A\mathbf{v_0} = \mathbf{v_1}$. Suppose we have shown that $\mathbf{u_m} = \mathbf{v_m}$ for some $m \geq 1$. Then $\mathbf{u_{m+1}} = A\mathbf{u_m} = A\mathbf{v_m} = \mathbf{v_{m+1}}$. It follows by induction that $\mathbf{u_k} = \mathbf{v_k}$ for all k.

23. $B\mathbf{w}$ is the vector $[c_1, c_2, \ldots, c_n]^{\mathrm{T}}$ where $c_i = b_{i1} + b_{i2} + \cdots +$

$b_{in} = 1$. Therefore $B\mathbf{w} = \mathbf{w}$ and $\lambda = 1$ is an eigenvalue for B with corresponding eigenvector $\mathbf{w}$.

24. Suppose $\mathbf{u} = [u_1, u_2, \ldots, u_n]^{\mathrm{T}}$ and choose i so that $|\, u_i \,| =$

$\max_{1\leq i\leq n}\{|\, u_1 \,|, \ldots, |\, u_n \,|\}$. Set $\alpha = 1/u_i$ and $\mathbf{v} = \alpha\mathbf{u}$. Then $\mathbf{v} = [v_1, v_2, \ldots, v_n]^{\mathrm{T}}$ where $v_i = 1$ and $|\, v_j \,| \leq 1$ for $1 \leq j \leq n$. Note also that $B\mathbf{v} = \mathbf{v}$. Equating the i^{th} component yields $b_{i1}v_1 + \cdots + b_{in}v_n = v_i = 1$. But $b_{i1} + \cdots + b_{in} = 1$ so it follows that $v_1 = \cdots = v_n = 1$. Thus $\mathbf{v} = \mathbf{w} = \alpha\mathbf{u}$, so $\mathbf{u} = \alpha^{-1}\mathbf{w}$.

25. Suppose $B\mathbf{u} = \lambda\mathbf{u}, \mathbf{u} \neq \theta$. As in Exercise 24, define $\mathbf{v} = [v_1, \ldots, v_n]^{\mathrm{T}}$ such that $B\mathbf{v} = \lambda\mathbf{v}$, $v_i = 1$ for some i, and $|\, v_j \,| \leq 1$ for $1 \leq j \leq n$. The i^{th} component of $\lambda\mathbf{v}$ is λ whereas the i^{th} component of $B\mathbf{v}$ is $b_{i1}v_1 + \cdots + b_{in}v_n$. Therefore $|\, \lambda \,| = |\, b_{i1}v_1 + \cdots + b_{in}v_n \,| \leq$ $b_{i1}\,|\, v_1 \,| + \cdots + b_{in}\,|\, v_n \,| \leq b_{i1} + \cdots + b_{in} = 1$.

26. The matrix A^{T} is a stochastic matrix. Moreover A and A^{T} have the same eigenvalues so we may apply Exercises 23 and 25.

27. If $a\mathbf{u} + b\mathbf{v} = \theta$ then $\theta = A\theta = A(a\mathbf{u} + b\mathbf{v}) = aA\mathbf{u} + bA\mathbf{v} = a\lambda\mathbf{u} +$

$b(\lambda\mathbf{v} + \mathbf{u}) = \lambda(a\mathbf{u} + b\mathbf{v}) + b\mathbf{u} = b\mathbf{u}$. Since $\mathbf{u} \neq \theta$ it follows that $b = 0$. Thus $a\mathbf{u} = \theta$ and $a = 0$. This proves that $\{\mathbf{u}, \mathbf{v}\}$ is a linearly independent set.

28. The formula holds for $k = 1$ by Exercise 27. Suppose $A^m\mathbf{v} = \lambda^m\mathbf{v} + m\lambda^{m-1}\mathbf{u}$ for some $m \geq 1$. Then $A^{m+1}\mathbf{v} = A(A^m\mathbf{v}) = A(\lambda^m\mathbf{v} + m\lambda^{m-1}\mathbf{u}) = \lambda^m A\mathbf{v} + m\lambda^{m-1}A\mathbf{u} = \lambda^m(\lambda\mathbf{v} + \mathbf{u}) + m\lambda^{m-1}\lambda\mathbf{u} = \lambda^{m+1}\mathbf{v} + (m+1)\lambda^m\mathbf{u}$. By induction the given formula holds for every integer $k \geq 1$.

29. (a) The vector $\mathbf{u} = [1, 0]^{\mathrm{T}}$ is an eigenvector for $\lambda = 1$ and $\mathbf{v} = [0, 1/2]^{\mathrm{T}}$ satisfies the equation $(A - I)\mathbf{v} = \mathbf{u}$.

(b) $\mathbf{x_0} = \mathbf{u} + 2\mathbf{v}$.

(c) $A^k\mathbf{x_0} = A^k(\mathbf{u} + 2\mathbf{v}) = A^k\mathbf{u} + 2A^k\mathbf{v} = \mathbf{u} + 2(\mathbf{v} + k\mathbf{u}) =$
$(2k + 1)\mathbf{u} + 2\mathbf{v}$.

(d) It follows immediately from (c) that $A^k\mathbf{x_0} = [2k + 1, 1]^{\mathrm{T}}$.

4.9 Supplementary Exercises

1. Det $(A) = x^2 - 9$ so A is singular when $x = \pm 3$.

2. $x \geq -1/4$.

3. $[1, 1]^T$ is an eigenvector for the eigenvalue $\lambda = 2$.

4. (a) $\det(A^{-1}B^2) = (\det B)^2 / \det A = 81/2$.
   (b) $\det(3A) = 3^3 \det(A) = 54$.
   (c) $\det(AB^2A^{-1}) = (\det B)^2 = 81$.

5. $x \neq 0$.

6. $A^2 = -3A + I$ so $A^2\mathbf{u} = -3A\mathbf{u} + \mathbf{u} = [-5, 0]^T$. $A^3 = -3A^2 + A$ so $A^3\mathbf{u} = -3A^2\mathbf{u} + A\mathbf{u} = [17, 1]^T$.

7. Suppose $A\mathbf{x} = \boldsymbol{\theta}$. Then $A^2\mathbf{x} = -3A\mathbf{x} + \mathbf{x}$ (cf. Exercise 6), so $\boldsymbol{\theta} = \mathbf{x}$.

8. $I = A^2 + 3A = A(A + 3I)$ so $A^{-1} = A + 3I$. Thus $A^{-1}\mathbf{u} = A\mathbf{u} + 3\mathbf{u} = [5, 10]^T$.

9. $A^2 = -3A + I$; $A^3 = 10A - 3I$; $A^4 = -33A + 10I$; $A^5 = 109A - 33I$.

10. $\lambda_1 = -1$, $\lambda_2 = -2$.

11. $\lambda_1 = 2$, $\lambda_2 = 3$.

12. $x = 8$, $\lambda = -6$.

13. $x = 0$, $y = -1$.

14. $x = 2$, $y = 3$.

4.10 Conceptual Exercises

1. False. $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

2. True. If $A\mathbf{x} = \lambda\mathbf{x}$ then $A^{-1}\mathbf{x} = (1/\lambda)\mathbf{x}$.

3. True. Det $(A^4) = (\det A)^4$.

4. False. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

5. True. $\| \mathbf{Ax} \|^2 = (A\mathbf{x})^T(A\mathbf{x}) = \mathbf{x}^T A^T A\mathbf{x} = \mathbf{x}^T\mathbf{x} = \| \mathbf{x} \|^2$.

6. True. Det $(S^{-1}AS - tI) = \det[S^{-1}(A - tI)S] = \det(A - tI)$.

7. False. $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.

8. False. $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

9. (a) $Q\mathbf{x}$ (b) $Q^T\mathbf{u}$.

10. If $A\mathbf{x} = \lambda\mathbf{x}$ then $A^3\mathbf{x} = \lambda^3\mathbf{x}$. But $A^3 = \mathcal{O}$.

11. If $P^{-1}AP = I$ then $A = PP^{-1} = I$.

12. $A^{-1}(AB)A = BA$.

13. Suppose $S^{-1}AS = B$. Then $S^{-1}A^2S = (S^{-1}AS)(S^{-1}AS) = B^2$. Similarly, $S^{-1}A^3S = (S^{-1}A^2S)(S^{-1}AS) = B^2B = B^3$ and $S^{-1}A^4S = (S^{-1}A^3S)(S^{-1}AS) = B^3B = B^4$.

14. (a) Yes. $A^T = (I - 2\mathbf{u}\mathbf{u}^T)^T = I^T - 2\mathbf{u}\mathbf{u}^T = I - 2\mathbf{u}\mathbf{u}^T = A$.
    (b) Yes. $AA^T = A^2 = (I - 2\mathbf{u}\mathbf{u}^T)^2 = I - 4\mathbf{u}\mathbf{u}^T + 4\mathbf{u}(\mathbf{u}^T\mathbf{u})\mathbf{u}^T = I$.
    (c) $A\mathbf{u} = (I - 2\mathbf{u}\mathbf{u}^T)\mathbf{u} = \mathbf{u} - 2\mathbf{u}(\mathbf{u}^T\mathbf{u}) = -\mathbf{u}$.
    (d) $A\mathbf{w} = (I - 2\mathbf{u}\mathbf{u}^T)\mathbf{w} = \mathbf{w} - 2\mathbf{u}(\mathbf{u}^T\mathbf{w}) = \mathbf{w}$.
    (e) $\lambda = 1$ has geometric multiplicity $n - 1$ and $\lambda = -1$ has geometric multiplicity 1.

Chapter 5

Vector Spaces and Linear Transformations

5.1 Introduction (No exercises)

5.2 Vector Spaces

1. $\mathbf{u} - 2\mathbf{v} = \begin{bmatrix} 0 & -7 & 5 \\ -11 & -3 & -12 \end{bmatrix}$; $\mathbf{u} - (2\mathbf{v} - 3\mathbf{w}) = \begin{bmatrix} 12 & -22 & 38 \\ -50 & -6 & -15 \end{bmatrix}$;
   $-2\mathbf{u} - \mathbf{v} + 3\mathbf{w} = \begin{bmatrix} 7 & -21 & 28 \\ -42 & -7 & -14 \end{bmatrix}$.

2. $\mathbf{u} - 2\mathbf{v} = -x^2 - 4x$; $\mathbf{u} - (2\mathbf{v} - 3\mathbf{w}) = -x^2 + 2x + 3$;
   $-2\mathbf{u} - \mathbf{v} + 3\mathbf{w} = -3x^2 + 4x + 8$.

3. $\mathbf{u} - 2\mathbf{v} = e^x - 2\sin x$; $\mathbf{u} - (2\mathbf{v} - 3\mathbf{w}) = e^x - 2\sin x + 3\sqrt{x^2+1}$; $-2\mathbf{u} - \mathbf{v} + 3\mathbf{w} = -2e^x - \sin x + 3\sqrt{x^2+1}$;

4. For $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ in Exercise 2 we may take $c_1 = c_3, c_2 = -c_3, c_3$ arbitrary. For example, $c_1 = 1, c_2 = -1, c_3 = 1$ is one choice. For $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ in Exercise 1, $c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = \theta$ if and only if $c_1 = c_2 = c_3 = 0$.

5. Note that $c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = (c_1 + c_2)x^2 + (2c_2 + 2c_3)x + (-2c_1 - c_2 + c_3)$. Thus $c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = x^2 + 6x + 1$ if and only if

$$\begin{array}{rcrcrcl} c_1 & + & c_2 & & & = & 1 \\ & & 2c_2 & + & 2c_3 & = & 0 \\ -2c_1 & - & c_2 & + & c_3 & = & 0. \end{array}$$

   Solving yields $c_1 = -2 + c_3, c_2 = 3 - c_3, c_3$ arbitrary. One choice is

$c_1 = -2, c_2 = 3, c_3 = 0$ and a direct calculation shows that $-2\mathbf{u}+$
$3\mathbf{v} = x^2 + 6x + 1$. Similarly $c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = x^2$ if and only if

$$\begin{array}{rcrcrcl} c_1 & + & c_2 & & & = & 1 \\ & & 2c_2 & + & 2c_3 & = & 6 \\ -2c_1 & - & c_2 & + & c_3 & = & 1. \end{array}$$

The system is easily seen to be inconsistent.

6. S is a vector space.

7. S is not a vector space. None of properties (c1), (c2), (a3), and (a4) of Definition 1 is satisfied. For example $\mathbf{v} = [1, 0, 0, 0]^{\mathrm{T}}$ and
$\mathbf{w} = [0, 0, 0, 1]^{\mathrm{T}}$ are in S but $\mathbf{u} + \mathbf{w}$ is not in S.

8. P is a vector space.

9. P is not a vector space. Properties (c1), (c2), and (a3) of Definition 1 fail to hold in P. For example $p(x) = 1 + 2x^2$ and $q(x) = x - 2x^2$ are in P but $p(x) + q(x)$ is not in P.

10. P is a vector space.

11. P is not a vector space (cf. Exercise 9).

12. S is a vector space.

13. S is a vector space.

14. S is not a vector space. For example the (3 x 4) zero matrix is not in S.

15. S is not a vector space. Properties (c1), (c2), and (a3) of Definition 1 fail to hold in S.

16. S is not a vector space. For example if $A = [a_{ij}]$ is a nonzero matrix in S then $\sqrt{2}\,A$ is not in S. Thus property (c2) of Definition 1
fails to hold. Note that S satisfies the remaining properties of Definition 1.

17. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and let $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then A and B are in Q but $A + B$ is not in Q. Also $0A$, the (2 x 2) zero matrix, is not in Q.

18. Q is not a vector space. For example, if $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ then A and B are in Q but $A + B$ is nonsingular.

19. The set of all (2 x 2) matrices is a vector space so axioms (a1), (a2), (m1), (m2), (m3), and (m4) are satisfied by any subset. Now let A and B be (2 x 2) symmetric matrices; that is $A = A^{\mathrm{T}}$ and $B = B^{\mathrm{T}}$. Therefore $(A+B)^{\mathrm{T}} = A^{\mathrm{T}} + B^{\mathrm{T}} = A + B$, so $A+B$ is symmetric. This verifies that property (c1) holds. For any scalar c, $(cA)^{\mathrm{T}} = cA^{\mathrm{T}} = cA$, so cA is symmetric and (c2) is satisfied. Clearly the (2 x 2) zero matrix is symmetric so (a3) holds. Finally if A is symmetric then so is $-A$, so (a4) holds. Therefore Q is a vector space.

20. Suppose **u, v,** and **w** are vectors in a vector space V such that **u+ v=u+w.** By property (a4) of Definition 1, V contains a vector **-u** such that **u+(-u)=** θ**.** By property (a1),**(-u)+u=** θ**.** Applying properties (a1), (a3), and (a2) yields **v=v+** θ**=** θ**+v=**

[(-u)+u]+ v= (-u)+(u+ v)= (-bfu)+(u+w)= [(-u)+u]+w=

θ **+w= w+** θ **=w.** Similarly, **v+u=w+u** implies

that **v=w.**

21. Let **u** and **w** be inverses for **v** . Thus $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}=\theta$ and $\mathbf{w}+\mathbf{v}=\mathbf{v}+\mathbf{w}=\theta$. Therefore $\mathbf{u}=\mathbf{u}+\theta=\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}=\theta+\mathbf{w}=\mathbf{w}$.

22. Note that $0\mathbf{v}+\theta=0\mathbf{v}=(0+0)\mathbf{v}=0\mathbf{v}+0\mathbf{v}$. By the cancellation laws, $\theta=0\mathbf{v}$.

23. If $a=0$ then we are done, so suppose that $a\neq 0$. Then $\mathbf{v}=1\mathbf{v}=$

$(a^{-1}a)\mathbf{v}=a^{-1}(av)=a^{-1}\theta=\theta$.

24. We show as illustrations that properties (a2) and (m1) hold.

Thus $\theta+(\theta+\theta)=\theta+\theta=\theta=\theta+\theta=(\theta+\theta)+\theta$, so (a2) holds. If a and b are scalars then $a(b\theta)=a\theta=\theta=(ab)\theta$ so (m1) is satisfied.

25. F is a vector space. Since F is a subset of $C[-1,1]$ and $C[-1,1]$ is a vector space, properties (a1), (a2), (m1), (m2), (m3), and (m4) hold in F. Now let $g(x), h(x)$ be in F; that is $g(x)$ and $h(x)$ are continuous, $g(-1)=g(1)$ and $h(-1)=h(1)$. It follows that $(g+h)(x)=g(x)+h(x)$ is continuous and $(g+h)(-1)=g(-1)+h(-1)=g(1)+h(1)=(g+h)(1)$. Therefore $(g+h)(x)$ is in F and property (c1) holds. If a is a scalar then $(ag)(x)=ag(x)$ is continuous and $(ag)(-1)=ag(-1)=ag(1)=(ag)(1)$, so $(ag)(x)$ is in F. This verifies that (c2) holds. The zero vector in $C[-1,1]$ is the function θ defined by $\theta(x)=0$ for all x, $-1\leq x\leq 1$. In particular $\theta(-1)=0=\theta(1)$ so $\theta(x)$ is in F. Thus $\theta(x)$ is also the zero vector for F and (a3) is satisfied. Property (a4) is an immediate consequence of (c2) since $-g(x)=(-1)g(x)$ for $g(x)$ in $C[-1,1]$. Since F satisfies the properties of Definition 1, F is a vector space.

26. F is a vector space.

27. F is not a vector space. For example set $f(x)=2x-1$ and $g(x)=2x^2-1$. Then $f(x)$ and $g(x)$ are in F whereas $f(x)+g(x)=2x^2+2x-2$ is not.

28. F is a vector space.

29. F is a vector space. As in Exercise 25, it suffices to check that properties (c1), (c2), (a3), and (a4) of Definition 1 are satisfied. To check (c1) for example, let $f(x)$ and $g(x)$ be in F. Then $\int_{-1}^{1}[f(x)+g(x)]dx = \int_{-1}^{1} f(x)dx + \int_{-1}^{1} g(x)dx = 0+0=0$, so $f(x)+g(x)$ is in F.

30. If $f(x)$ and $g(x)$ are in $C^2[a,b]$ then $(f+g)(x) = f(x)+g(x)$ is continuous on $[a,b]$, $(f+g)'(x) = f'(x)+g'(x)$ is continuous on $[a,b]$, and $(f+g)''(x) = f''(x)+g''(x)$ is continuous on $[a,b]$. Similarly if c is a scalar then the functions $(cf)(x) = cf(x), (cf)'(x) = cf'(x)$, and $(cf)''(x) = cf''(x)$ are all continuous on $[a,b]$. It follows that $C^2[a,b]$ is a vector space.

31. (a) F is a vector space. Since F is a subset of $C^2[-1,1]$ and $C^2[-1,1]$ is a vector space by Exercise 30, properties (a1), (a2), (m1), (m2), m(3), and (m4) hold in F. Now let $g(x), h(x)$ be in F. Thus $g''(x)+g(x)=0$ and $h''(x)+h(x)=0$. It follows that $(g+h)''(x)+(g+h)(x) = [g''(x)+g(x)]+[h''(x)+h(x)] = 0+0=0$ for $-1 \le x \le 1$. Therefore $(g+h)(x)$ is in F and property (c1) is satisfied. If a is any scalar then $(ag)''(x)+(ag)(x) = a[g''(x)+g(x)] = a0 = 0$ for $-1 \le x \le 1$. Thus $(ag)(x)$ is in F and property (c2) is satisfied. The zero vector in $C^2[-1,1]$ is the function θ defined by $\theta(x)=0, -1 \le x \le 1$. In particular $\theta''(x)+\theta(x) = 0+0=0$ for $-1 \le x \le 1$ so $\theta(x)$ is in F. Therefore $\theta(x)$ is the zero vector in F and (a3) is satisfied. Property (a4) follows from (c2) since $-f(x) = (-1)f(x)$ for every $f(x)$ in $C^2[-1,1]$.

    (b) F is not a vector space. For example suppose that $g(x)$ and $h(x)$ are in F; that is, assume that $g''(x)+g(x)=x^2$ and $h''(x)+h(x)=x^2$. Then $(g+h)''(x)+(g+h)(x) = [g''(x)+g(x)]+[h''(x)+h(x)] = 2x^2$. Therefore $(g+h)(x)$ is not in F.

32. Let $p(x)$ and $q(x)$ be in $\mathcal{P}$. Then we may write $p(x) = a_0+a_1x+\cdots+a_nx^n$ and $q(x) = b_0+b_1x+\cdots+b_nx^n$ where for $0 \le i \le n, a_i$ and b_i are real numbers. (By using zero coefficients as necessary we may list the same terms for both $p(x)$ and $q(x)$.) Thus $p(x)+q(x) = (a_0+b_0)+(a_1+b_1)x+\cdots+(a_n+b_n)x^n$ and for any scalar $c, cp(x) = ca_0+ca_1x+\cdots+ca_nx^n$. It is now straightforward to verify that $\mathcal{P}$ is a vector space.

33. The proof that $\mathcal{F}(R)$ is a vector space requires checking all ten properties of Definition 1. We illustrate by verifying properties (a2), (a3) and (m1). If $f(x), g(x), h(x)$ are in $\mathcal{F}(R)$ then $[f+(g+h)](x) = f(x)+[g+h](x) = f(x)+[g(x)+h(x)] = [f(x)+g(x)]+h(x) = [f+g](x)+h(x) = [(f+g))+h](x)$. Thus property (a2) holds. Define $\theta : R \to R$ by $\theta(x)=0$ for all x in R. If $f(x)$ is in $\mathcal{F}(R)$ then $(f+\theta)(x) = f(x)+\theta(x) = f(x)$; $f+\theta = f$ so θ is the zero of $\mathcal{F}(R)$. Therefore property (a3) is satisfied. To check (m1) let a and b be scalars and suppose $f(x)$ is in $\mathcal{F}(R)$. Then $[a(bf)](x) = a[(bf)(x)] = a[b(f(x))] = ab(f(x)) = [(ab)f](x)$.

34. (a) We will show that the given operations satisfy (a3) and (a4) of Definition 1. Note that $\mathbf{z} = [-1, 1]^{\mathrm{T}}$ is in V and $\mathbf{u} + \mathbf{z} = \mathbf{u}$ for every element $\mathbf{u}$ in V. Thus $\mathbf{z}$ is the zero of V. If $\mathbf{u} = [u_1, u_2]^{\mathrm{T}}$ then $\mathbf{w} = [-u_1 - 2, -u_2 + 2]$ is in V and $\mathbf{u} + \mathbf{w} = \mathbf{z}$. Thus $\mathbf{w}$ is an additive inverse for $\mathbf{u}$.

(b) Note that $2(\mathbf{e_1} + \mathbf{e_2}) = [4, 0]^{\mathrm{T}}$ whereas $2\mathbf{e_1} + 2\mathbf{e_2} = [3, 1]^{\mathrm{T}}$. Similarly, $\mathbf{e_1} + \mathbf{e_1} = [3, -1]^{\mathrm{T}}$ whereas $(1 + 1)\mathbf{e_1} = 2\mathbf{e_1} = [2, 0]^{\mathrm{T}}$.

35. To check (m2) as an illustration, note that $a(\mathbf{u} + \mathbf{v}) = \theta = \theta + \theta = a\mathbf{u} + a\mathbf{v}$ for any $\mathbf{u}, \mathbf{v}$ in V and scalar a. If $\mathbf{u} \neq \theta$ then $1\mathbf{u} = \theta$ and (m4) fails.

36. The zero of V is the vector $\mathbf{z} = [0, 1]^{\mathrm{T}}$. If $\mathbf{u} = [u_1, u_2]$ is in V then $\mathbf{w} = [-u_1, 1/u_2]$ is in V and $\mathbf{u} + \mathbf{w} = \mathbf{z}$. Thus $\mathbf{w}$ is the inverse of $\mathbf{u}$. To show that (m2) holds let $\mathbf{u} = [u_1, u_2]^{\mathrm{T}}$ and $\mathbf{v} = [v_1, v_2]^{\mathrm{T}}$ be in V and let a be a scalar. Then

$$a(\mathbf{u} + \mathbf{v}) = a \begin{bmatrix} u_1 + v_1 \\ u_2 v_2 \end{bmatrix} = \begin{bmatrix} a(u_1 + v_1) \\ (u_2 v_2)^a \end{bmatrix} = \begin{bmatrix} au_1 + av_1 \\ u_2^a v_2^a \end{bmatrix} =$$

$$\begin{bmatrix} au_1 \\ u_2^a \end{bmatrix} + \begin{bmatrix} av_1 \\ v_1^a \end{bmatrix} = a\mathbf{u} + a\mathbf{v}.$$

5.3 Subspaces

1. W is not a subspace of V. None of the properties (s1), (s2), and (s3) of Theorem 2 is satisfied. For example, if $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ then A and B are in W but $A + B$ is not in W.

2. W is a subspace of V.

3. W is a subspace of V. Clearly the (2 x 3) zero matrix is in W. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be in W. Therefore $a_{11} - a_{12} = 0, a_{12} + a_{13} = 0, a_{23} = 0, b_{11} - b_{12} = 0, b_{12} + b_{13} = 0, b_{23} = 0$. Now $A + B$ is the (2 x 3) matrix $A + B = [c_{ij}]$, where $c_{ij} = a_{ij} + b_{ij}$. Thus $c_{11} - c_{12} = (a_{11} + b_{11}) - (a_{12} + b_{12}) = (a_{11} - a_{12}) + (b_{11} - b_{12}) = 0$.Similarly, $c_{12} + c_{13} = 0$ and $c_{23} = 0$. This shows that $A + B$ is in W. If k is a scalar then kA is the (2 x 3) matrix $kA = [d_{ij}]$ where $d_{ij} = ka_{ij}$. Consequently $d_{11} - d_{12} = ka_{11} - ka_{12} = k(a_{11} - a_{12}) = k0 = 0$. Likewise $d_{12} + d_{13} = 0$ and $d_{23} = 0$. Therefore kA is in W. It follows from Theorem 2 that W is a subspace of V.

4. W is not a subspace of V. For Example if $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ then A and B are in W but $A + B$ is not in W. Note that properties (s1) and (s3) of Theorem 2 are satisfied.

5. W is a subspace of $\mathcal{P}_2$. If $\theta(x) = 0 + 0x + 0x^2$ is the zero polynomial then clearly $\theta(0)+\theta(2) = 0$, so $\theta(x)$ is in W. Suppose $g(x)$ and $h(x)$ are in W; that is $g(0)+g(2) = 0$ and $h(0) + h(2) = 0$. Then $(g + h)(0) = +(g + h)(2) = [g(0) + h(0)] + [g(2) + h(2)] = [g(0) + g(2)] + [h(0) + h(2)] = 0 + 0 = 0$ and it follows that $g(x) + h(x)$ is in W. If c is a scalar then $(cg)(0) + (cg)(2) = c[g(0) + g(2)] = 0$ and hence $(cg)(x)$ is in W. By Theorem 2, W is a subspace of $\mathcal{P}_2$.

6. W is a subspace of $\mathcal{P}_2$.

7. W is not a subspace of $\mathcal{P}_2$. For example if $p(x) = x^2 + x - 2$ and $q(x) = x^2 - 9$ then $p(x)$ and $q(x)$ are in W but $p(x)+q(x) = 2x^2+x-11$ is not in W. Note that properties (s1) and s(3) of Theorem 2 are satisfied.

8. W is a subspace of $\mathcal{P}_2$.

9. F is a subspace of $C[-1, 1]$. First recall that the zero of $C[-1, 1]$ is the function θ defined by $\theta(x) = 0$ for $-1 \le x \le 1$. Since $\theta(-1) = 0 = -\theta(1), \theta(x)$ is in F. Now assume that $g(x)$ and $h(x)$ are in F. Thus $g(-1) = -g(1)$ and $h(-1) = -h(1)$. It follows that

   $(g + h)(-1) = g(-1) + h(-1) = -g(1) - h(1) = -(g + h)(1).$

   Therefore $(g + h)(x)$ is in F. If c is a scalar then $(cg)(-1) =$

   $c(g(-1)) = c(-g(1)) = -(cg)(1)$ so $(cg)(x)$ is in F. by Theorem 2, F is a subspace of $C[-1, 1]$.

10. F is not a subspace of $C[-1, 1]$. If $f(x)$ is a nonzero function in F and $c < 0$ then $cf(x)$ is not in F . Note that properties (s1) and (s2) of Theorem 2 are satisfied.

11. F is not a subspace of $C[-1, 1]$. None of the properties (s1), s(2), (s3) of Theorem 2 is satisfied. For example if $g(x)$ and $h(x)$ are in F then $(g + h)(-1) = g(-1) + h(-1) = -2 + (-2) = -4$ so $(g + h)(x)$ is not in F .

12. F is a subspace of $C[-1, 1]$.

13. F is a subspace of $C^2[-1, 1]$. If $\theta(x)$ is the zero function then $\theta''(x) = \theta(x) = 0$ for $-1 \le x \le 1$. In particular $\theta''(0) = 0$ so $\theta(x)$ is in F. Let $g(x)$ and $h(x)$ be in F; that is $g''(0) = 0 = h''(0)$. Therefore $(g + h)''(0) = g''(0) + h''(0) = 0$, so $(g + h)(x)$ is in F . If c is a scalar then $(cg)''(0) = cg''(0) = 0$ and $(cg)(x)$ is in F . By Theorem 2, F is a subspace of $C^2[-1, 1]$.

14. F is a subspace of $C^2[-1, -]$.

15. F is not a subspace of $C^2[-1, 1]$. None of the properties (s1), (s2), s(3) of Theorem 2 is satisfied. For example suppose $g(x)$ and $h(x)$ are in F . Then $g''(x) + g(x) = \sin x$ and $h''(x)+h(x) = \sin x$ for $-1 \le x \le 1$. But $(g+h)''(x)+(g+h)(x) = 2\sin x$ for $-1 \le x \le 1$. Therefore $(g + h)(x)$ is not in F .

16. F is a subspace of $C^2[-1,1]$.

17. Note that $c_1p_1(x)+c_2p_2(x)+c_3p_3(x) = (c_1+2c_2+3c_3)+(2c_1+5c_2+8c_3)x+(c_1-2c_3)x^2$. Therefore $c_1p_1(x)+c_2p_2(x)+c_3p_3(x) = -1-3x+3x^2$ requires that $c_1+2c_2+3c_3 = -1, 2c_1+5c_2+8c_3 = -3,$ and $c_1-2c_3 = 3$. Solving we obtain $c_1 = -1, c_2 = 3, c_3 = -2$ and it is easily verified that $p(x) = -p_1(x)+3p_2(x)-2p_3(x)$.

18. $p(x) = \sum_{i=1}^{4} c_ip_i(x)$ if and only if $c_1 = -1-2c_3, c_2 = 2+3c_3, c_3$ is arbitrary, and $c_4 = -3$. For example $p(x) = -p_1(x)+2p_2(x)-3p_4(x)$.

19. From the matrix equation $A = \sum_{i=1}^{4} c_iB_i$ we obtain the system of equations

$$\begin{array}{rcrcrcrcr} c_1 & + & c_2 & - & c_3 & + & c_4 & = & -2 \\ & & c_2 & - & 3c_3 & + & 2c_4 & = & -4 \\ 2c_1 & + & & & 4c_3 & - & c_4 & = & 1 \\ c_1 & + & 2c_2 & - & 4c_3 & + & c_4 & = & 0 \end{array}.$$

The solution is $c_1 = -1-2c_3, c_2 = 2+3c_3, c_3$ arbitrary, and $c_4 = -3$. Taking $c_3 = 0$ we see that $A = -B_1+2B_2-3B_4$.

20. $e^x = \sinh x + \cosh x$.

21. $\cos 2x = (-1)\sin^2 x + (1)\cos^2 x$.

22. $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$.

23. Let $p(x) = a_0+a_1x+a_2x^2+a_3x^3$ be in W. The constraints $p(1) = p(-1)$ and $p(2) = p(-2)$ imply that $a_0+a_1+a_2+a_3 = a_0-a_1+a_2-a_3$ and $a_0+2a_1+4a_2+8a_3 = a_0-2a_1+4a_2-8a_3$. This forces $a_1 = a_3 = 0$ while a_0 and a_2 are arbitrary. Thus $\{1, x^2\}$ is a spanning set for W.

24. $\{-3+2x+x^2, 2-3x+x^3\}$.

25. For Exercise 2, a matrix $A = [a_{ij}]$ is in W if and only if A has the form $A = \begin{bmatrix} a_{12}-2a_{13} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$, where $a_{12}, a_{13}, a_{21}, a_{22}, a_{23}$ are arbitrary. Therefore

$$W = \mathrm{Sp}\left\{ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -2 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

For Exercise 3, $A = [a_{ij}]$ is in W if and only if $A =$
$\begin{bmatrix} a_{11} & a_{11} & -a_{11} \\ a_{21} & a_{22} & 0 \end{bmatrix}$, where a_{11}, a_{21}, and a_{22} are arbitrary.
Therefore

$$W = \mathrm{Sp}\left\{ \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right\}.$$

For Exercise 5 let $p(x) = a_0 + a_1x + a_2x^2$. The condition $p(0) + p(2) = 0$ implies that $2a_0 + 2a_1 + 4a_2 = 0$. Therefore $p(x) = (-a_1 - 2a_2) + a_1x + a_2x^2$ where a_1 and a_2 are arbitrary. It follows that $W = \mathrm{Sp}\{-1 + \mathrm{x}, -2 + \mathrm{x}^2\}$.

For Exercise 6, $W = \mathrm{Sp}\{\ 1, -4x + x^2\}$.

For Exercise 8, $W = \mathrm{Sp}\{\ x, 1 - x^2\}$.

26. That W is a subspace follows from Theorem 2 and from the properties of the transpose given in Theorem 10 of Section 1.6.

$$W = \mathrm{Sp}\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right.,$$
$$\left. \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\}.$$

27. It is straightforward to show that $tr(A + B) = tr(A) + tr(B)$ and $tr(cA) = ctr(A)$. It then follows easily from Theorem 2 that W is a subspace of V. If A is in W then A has the form $A =$
$\begin{bmatrix} -a_{22} - a_{33} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. It follows that $W = \mathrm{Sp}\{\mathrm{B}_1, \mathrm{B}_2, \mathrm{E}_{12}, \mathrm{E}_{13}, \mathrm{E}_{21}, \mathrm{E}_{23}, \mathrm{E}_{31}, \mathrm{E}_{32}\}$
where $B_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $B_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

28. $B^{\mathrm{T}} = [(1/2)(A + A^{\mathrm{T}})]^{\mathrm{T}} = (1/2)(A^{\mathrm{T}} + A^{\mathrm{TT}}) = (1/2)(A + A^{\mathrm{T}}) = B$, so B is symmetric. Similarly $C^{\mathrm{T}} = [(1/2)(A - A^{\mathrm{T}})]^{\mathrm{T}} =$

$(1/2)(A^{\mathrm{T}} - A) = -C$ and C is skew-symmetric.

29. For any (n x n) matrix $A, A = B + C$ where B and C are the matrices given in Exercise 28.

30. The vector space of all (3 x 3) matrices is spanned by the set $\{E_{11}, E_{22}, E_{23}, A_1, A_2, A_3, B_1, B_2, B_3\}$ where $A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

$A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

$B_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}, B_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$. If $A = [a_{ij}]$ is a (3 x 3) matrix then $A = a_{11}E_{11} + a_{22}E_{22} + a_{33}E_{33} + (1/2)(a_{12} + a_{21})A_1 + (1/2)(a_{13} + a_{31})A_2 + (1/2)(a_{23} + a_{32})A_3 + (1/2)(a_{12} - a_{21})B_1 + (1/2)(a_{13} - a_{31})B_2 + (1/2)(a_{23} - a_{32})B_3$.

31. (a) A is in W if and only if A has the form $A = \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{bmatrix}$, where a_{12}, a_{13}, a_{23} are arbitrary. Thus $W = \text{Sp}\{\text{E}_{12}, \text{E}_{13}, \text{E}_{23}\}$.

(b) A is in W if and only if A has form $A = \begin{bmatrix} -a_{22} - a_{33} & -a_{23} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$. Therefore

$$W = \text{Sp}\left\{ \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}.$$

(c) A is in W if and only if A has the form $A = \begin{bmatrix} a_{11} & a_{11} & a_{13} \\ 0 & a_{22} & a_{13} \\ 0 & 0 & a_{22} \end{bmatrix}$.. Therefore

$$W = \text{Sp}\left\{ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

(d) A is in W if and only if A has the form $A = \begin{bmatrix} a_{11} & -a_{23} & a_{13} \\ 0 & a_{11} & a_{23} \\ 0 & 0 & a_{11} \end{bmatrix}$. Therefore $W = \text{Sp}\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \right.$

$$\left.\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}.$$

32. $p(x) = (a_0 - a_1 + a_2)(1) + (a_1 - 2a_2)(x+1) + a_2(x+1)^2$. In particular $q(x) = 4(1) - 5(x+1) + 2(x+1)^2$ and $r(x) = 6(1) - 5(x+1) + (x+1)^2$.

33. The equation $A = \sum_{i=1}^{4} x_i B_i$ implies that

$$\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & - & x_3 & + & x_4 & = & a \\ & & x_2 & + & 3x_3 & + & x_4 & = & b \\ x_1 & + & x_2 & - & 3x_3 & - & 2x_4 & = & c \\ -2x_1 & - & 2x_2 & + & 6x_3 & + & 5x_4 & = & d \end{array}.$$

Solving we obtain $x_1 = -6a+5b+37c+15d, x_2 = 3a-2b-17c-7d, x_3 = -a+b+5c+2d, x_4 = 2c+d$. Therefore $C = -12B_1 + 6B_2 - B_3 - B_4$ and $D = 8B_1 - 3B_2 + B_3 + B_4$.

5.4 Linear Independence, Bases, and Coordinates

1. If A is in W then $A = \begin{bmatrix} -b-c-d & b \\ c & d \end{bmatrix} = b\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. The set $\left\{ \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for W.

2. The set $\left\{ \begin{bmatrix} -1 & 2 \\ -3 & 1 \end{bmatrix} \right\}$ is a basis for W.

3. The set $\{E_{12}, E_{21}, E_{22}\}$ is a basis for W.

4. The set $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \right\}$ is a basis for W.

5. For $p(x)$ in $W, p(x) = a_0 + a_1x + (a_0 - 2a_1)x^2 = a_0(1+x^2) + a_1(x - 2x^2)$. The set $\{1+x^2, x-2x^2\}$ is a basis for W.

6. The set $\{3 - x + x^2\}$ is a basis for W.

7. The set $\{x, x^2\}$ is a basis for W.

8. The set $\{1 - 2x + x^2\}$ is a basis for W.

9. Let $p(x) = \sum_{i=0}^{4} a_i x^i$ be in V. The given constraints are as follows:

$$\begin{array}{lr} p(0) = 0 : & a_0 = 0 \\ p^{'}(1) = 0 : & a_1 + 2a_2 + 3a_3 + 4a_4 = 0 \\ p^{''}(-1) = 0 : & 2a_2 - 6a_3 + 12a_4 = 0 \end{array} .$$

Solving yields $a_0 = 0, a_1 = -9a_3 + 8a_4, a_2 = 3a_3 - 6a_4, a_3$ and a_4 arbitrary. Thus $\{-9x + 3x^2 + x^3, 8x - 6x^2 + x^4\}$ is a basis for V.

10. The set $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ is a basis for the subspace of (2 x 2) symmetric matrices.

11. If $A = [a_{ij}]$ is a (2 x 2) matrix then $A = a_{11}E_{11} + a_{12}E_{12} + a_{21}E_{21} + a_{22}E_{22}$ so B spans V. It is easy to see that B is a linearly independent set, so B is a basis for W.

12. (a) $[1, -1, 1]^{\mathrm{T}}$, (b) $[-1, 4, 1]^{\mathrm{T}}$ (c) $[5, 2, 0]^{\mathrm{T}}$.

13. (a) $[2, -1, 3, 2]^{\mathrm{T}}$ (b) $[1, 0, -1, 1]^{\mathrm{T}}$ (c) $[2, 3, 0, 0]^{\mathrm{T}}$.

14. Set $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ and assume $p(x) = \theta(x)$. Then $p^{(n)}(x) = \theta(x)$ so $n!a_n = 0$. It follows that $a_n = 0$. Suppose we have seen that $a_{m+1} = \cdots = a_n = 0$ where $0 \leq m < n$. Then $p^{(m)}(x) = \theta(x)$ so $m!a_m = 0$. Thus $a_m = 0$. By continuing the process we see that $a_i = 0, 0 \leq i \leq n$. Therefore the set $\{1, x, \ldots, x^n\}$ is linearly independent.

15. The given matrices have coordinate vectors $\mathbf{u_1} = [2, 1, 2, 1]^{\mathrm{T}}$,

$\mathbf{u_2} = [3, 0, 0, 2]^{\mathrm{T}}, \mathbf{u_3} = [1, 1, 2, 1]^{\mathrm{T}}$, respectively. The equation $x_1\mathbf{u_1} + x_2\mathbf{u_2} + x_3\mathbf{u_3} = \theta$ has only the trivial solution so $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ is a linearly independent subset of R^4. By property (2) of Theorem 5, the set $\{A_1, A_2, A_3\}$ is a linearly independent subset of the vector space of (2 x 2) matrices.

16. The set is linearly dependent. For example $-2A_1 - A_2 + A_3 = \mathcal{O}$.

17. The given matrices have coordinate vectors $\mathbf{u_1} = [2, 2, 1, 3]^{\mathrm{T}}$,

$\mathbf{u_2} = [1, 4, 0, 5]^{\mathrm{T}}, \mathbf{u_3} = [4, 10, 1, 13]^{\mathrm{T}}$, respectively. The set

$\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ is linearly dependent in R^4. For example $-\mathbf{u_1} - 2\mathbf{u_2} + \mathbf{u_3} = \theta$. It follows that $\{A_1, A_2, A_3\}$ is linearly dependent; indeed $-A_1 - 2A_2 + A_3 = \mathcal{O}$.

18. The set is linearly independent.

19. The polynomials $p_1(x), p_2(x), p_3(x)$, have coordinate vectors

$\mathbf{u_1} = [-1, 2, 1, 0]^{\mathrm{T}}, \mathbf{u_2} = [2, -5, 1, 0]^{\mathrm{T}}, \mathbf{u_3} = [0, -1, 3, 0]^{\mathrm{T}}$, respectively. The set $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ is linearly dependent; for example $-2\mathbf{u_1} - \mathbf{u_2} + \mathbf{u_3} = \theta$. It follows that the set $\{p_1(x), p_2(x), p_3(x)\}$ is linearly dependent; indeed $-2p_1(x) - p_2(x) + p_3(x) = 0$.

20. The set is linearly dependent; for example $-p_1(x) - p_2(x) - p_3(x) + p_4(x) = 0$.

21. The given polynomials have coordinate vectors $\mathbf{u_1} = [1, 0, 0, 1]^{\mathrm{T}}$, $\mathbf{u_2} = [1, 0, 1, 0]^{\mathrm{T}}, \mathbf{u_3} = [1, 1, 0, 0]^{\mathrm{T}}, \mathbf{u_4} = [1, 0, 0, 0]^{\mathrm{T}}$, respectively.
Since $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}\}$ is a linearly independent subset of R^4, the given set of polynomials is a linearly independent subset of $\mathcal{P}_3$.

22. A basis for Sp*(S)* is $\{1 + 2x + x^2, x - 2x^2\}$.

23. The given polynomials have coordinate vectors $\mathbf{u_1} = [1, 2, 1]^{\mathrm{T}}$, $\mathbf{u_2} = [2, 5, 0]^{\mathrm{T}}, \mathbf{u_3} = [3, 7, 1]^{\mathrm{T}}, \mathbf{u_4} = [1, 1, 3]^{\mathrm{T}}$, respectively. In the equation $x_1\mathbf{u_1} + x_2\mathbf{u_2} + x_3\mathbf{u_3} + x_4\mathbf{u_4} = \theta$, x_3 and x_4 are arbitrary. It follows that $\{\mathbf{u_1}, \mathbf{u_2}\}$ is a basis for Sp$\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}\}$. Therefore $\{p_1(x), p_2(x)\}$ is a basis for Sp(S).

24. The set $\left\{ \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for Sp(S).

25. The given matrices have coordinate vectors $\mathbf{u_1} = [1, 2, -1, 3]^{\mathrm{T}}$, $\mathbf{u_2} = [-2, 1, 2, -1]^{\mathrm{T}}, \mathbf{u_3} = [-1, -1, 1, -3]^{\mathrm{T}}, \mathbf{u_4} = [-2, 2, 2, 0]^{\mathrm{T}}$, respectively. In the equation $x_1\mathbf{u_1} + x_2\mathbf{u_2} + x_3\mathbf{u_3} + x_4\mathbf{u_4} = \theta$, x_4 is arbitrary so $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ is a basis for Sp$\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}\}$. It follows that $\{A_1, A_2, A_3\}$ is a basis for Sp(S).

26. The coordinate vectors $\mathbf{u_1} = [-1, 1, 2]^{\mathrm{T}}, \mathbf{u_2} = [0, 1, 3]^{\mathrm{T}}, \mathbf{u_3} = [1, 2, 8]^{\mathrm{T}}$ form a linearly independent set in R^3. Since $\dim(R^3) = 3$ the set $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ is a basis for R^3. By the Corollary to Theorem 5, Q is a basis for $\mathcal{P}_2$.

27. $p(x) = -4p_1(x) + 11p_2(x) - 3p_3(x)$ so $[p(x)]_Q = [-4, 11, -3]^{\mathrm{T}}$.

28. $[p(x)]_Q = [-2a_0 - 3a_1 + a_2, 4a_0 + 10a_1 - 3a_2, -a_0 - 3a_1 + a_2]^{\mathrm{T}}$.

29. The coordinate vectors for the given matrices are $\mathbf{u_1} = [1, 0, 0, 0]^{\mathrm{T}}$, $\mathbf{u_2} = [1, -1, 0, 0]^{\mathrm{T}}, \mathbf{u_3} = [0, 2, 1, 0]^{\mathrm{T}}, \mathbf{u_4} = [-3, 0, 2, 1]^{\mathrm{T}}$. It is easily verified that $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}\}$ is a basis for R^4. It follows from the Corollary to Theorem 5 that Q is a basis for V.

30. $[A]_Q = [9, -5, -1, -1]^{\mathrm{T}}$.

31. $A = (a + b - 2c + 7d)A_1 + (-b + 2c - 4d)A_2 + (c - 2d)A_3 + dA_4$ so $[A]_Q = [a + b - 2c + 7d, -b + 2c - 4d, c - 2d, d]^{\mathrm{T}}$.

32. The suggested constraints yield the following system of equations:

$$\begin{array}{rcll} p(-1) & = & 0: & a_0 - a_1 + a_2 = 0 \\ p(0) & = & 0: & a_0 \qquad\qquad\;\; = 0 \\ p(1) & = & 0: & a_0 + a_1 + a_2 = 0 \end{array}$$

Solving we obtain $a_0 = a_1 = a_2 = 0$ as the unique solution.

33. Let $f(x) = c_1 \sin x + c_2 \cos x$ and suppose $f(x) = \theta(x)$. Then $f(0) = c_2 = 0$ and $f(\pi/2) = c_1 = 0$. It follows that $\{\sin x, \cos x\}$ is a linearly independent set.

34. The conditions $h(0) = h'(0) = h''(0) = h'''(0) = 0$ yield the system of equations

$$\begin{array}{rcrcrcrcl} c_1 & + & c_2 & + & c_3 & + & c_4 & = & 0 \\ c_1 & + & 2c_2 & + & 3c_3 & + & 4c_4 & = & 0 \\ c_1 & + & 4c_2 & + & 9c_3 & + & 16c_4 & = & 0 \\ c_1 & + & 8c_2 & + & 27c_3 & + & 64c_4 & = & 0 \end{array}$$

The system has only the trivial solution so B is a linearly independent set. Since $V = \mathrm{Sp}(B)$ by definition, B is a basis for V.

35. Note that $[g_1(x)]_B = [1, 0, 0, -1]^{\mathrm{T}} = \mathbf{u_1}$, $[g_2(x)]_B = [0, 1, 1, 0]^{\mathrm{T}} = \mathbf{u_2}$, and $[g_3(x)]_B = [-1, 0, 1, 1]^{\mathrm{T}} = \mathbf{u_3}$. It is easy to verify that $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ is a linearly independent subset of R^4. By Theorem 5 the set $\{g_1(x), g_2(x), g_3(x)\}$ is linearly independent in V.

36. It follows from the note that $\mathbf{w} \neq \theta$. Suppose that $a_1\mathbf{v_1} + \cdots + a_m\mathbf{v_m} + b\mathbf{w} = \theta$. If $b \neq 0$ then we can solve for $\mathbf{w}$, contradicting the assumption that $\mathbf{w}$ is not in Sp(Q). Thus $b = 0$. This leaves $a_1\mathbf{v_1} + \cdots + a_m\mathbf{v_m} = \theta$. But the set Q is linearly independent so $a_1 = \cdots = a_m = 0$. This proves that $Q \cup \{\mathbf{w}\}$ is linearly independent.

37. Suppose $a_1\mathbf{v_1} + \cdots + a_n\mathbf{v_n} = \theta$, where $a_i \neq 0$. Then $v_i = b_1\mathbf{v_1} + \cdots + b_{i-1}\mathbf{v_{i-1}} + b_{i+1}\mathbf{v_{i+1}} + \cdots + b_n\mathbf{v_n}$, where $b_j = -a_j/a_i$, for $1 \leq j \leq n, j \neq i$.

38. A set of two vectors is linearly dependent if and only if one of the vectors is a scalar multiple of the other. The sets given in (a) and (b) are linearly independent whereas the sets given in (c), (d), and (e) are linearly dependent.

5.5 Dimension

1. (a) We show that V_1 is a subspace. The proof that V_2 is a subspace is similar. Clearly the (3 x 3) zero matrix is lower-triangular so it is in V_1. Now let A and B be in V_1,

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & 0 & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{32} & b_{33} \end{bmatrix}.$$

$$\text{Then } A + B = \begin{bmatrix} a_{11} + b_{11} & 0 & 0 \\ a_{21} + b_{21} & a_{22} + b_{22} & 0 \\ a_{31} + b_{31} & a_{32} + b_{32} & a_{33} + b_{33} \end{bmatrix},$$

so $A + B$ is in V_1. If c is any scalar then

$$cA = \begin{bmatrix} ca_{11} & 0 & 0 \\ ca_{21} & ca_{22} & 0 \\ ca_{31} & ca_{32} & ca_{33} \end{bmatrix},$$

so cA is in V_1. By Theorem 2 of Section 4.3, V is a subspace of V_1.

(b) The set $\{E_{11}, E_{21}, E_{22}, E_{31}, E_{32}, E_{33}\}$ is a basis for V_1 and $\{E_{11}, E_{12}, E_{13}, E_{22}, E_{23}, E_{33}\}$ is a basis for V_2.

(c) $\dim(V) = 9; \dim(V_1) = 6; \dim(V_2) = 6$.

2. Since V_1 and V_2 are subspaces of V, θ is in $V_1 \cap V_2$. Suppose $\mathbf{u}$ and $\mathbf{v}$ are vectors in $V_1 \cap V_2$. Then $\mathbf{u}$ and $\mathbf{v}$ are in V_1 and V_1 is a vector space. Therefore $\mathbf{u}+\mathbf{v}$ is in V_1 and for any scalar a, $a\mathbf{u}$ is in V_1. Similarly,

$\mathbf{u}+\mathbf{v}$ and $a\mathbf{u}$ are in V_2. This shows that $\mathbf{u}+\mathbf{v}$ and $a\mathbf{u}$ are in $V_1 \cap V_2$ so by Theorem 2 of Section 4.3, $V_1 \cap V_2$ is a subspace of V.

Set $V_1 = \{[a, 0]^{\mathrm{T}} : a$ any real number $\}$ and let $V_2 = \{[0, b]^{\mathrm{T}} : b$ any real number $\}$. Then V_1 and V_2 are subspaces of R^2 but $V_1 \cup V_2$ is not a subspace. For example $\mathbf{u} = [1, 0]^{\mathrm{T}}$ and $\mathbf{v} = [0, 1]^{\mathrm{T}}$ are in $V_1 \cup V_2$ but $\mathbf{u}+\mathbf{v}$ is not in $V_1 \cup V_2$.

3. Let $A = [a_{ij}]$ be in $V_1 \cap V_2$. Since A is in $V_1, a_{ij} = 0$ for $i < j$. Likewise A is in V_2 so $a_{ij} = 0$ for $i > j$. Thus $a_{ij} = 0$ if $i \neq j$ and $V_1 \cap V_2$ is the set of (3 x 3) diagonal matrices. $\dim(V_1 \cap V_2) = 3$.

4. $\dim(W) = 6$.

5. The set $\left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\}$ is a basis for W so $\dim(W) = 3$.

6. $\dim(W) = 2$.

7. $p(x) = \sum_{i=0}^{4} a_i x^i$ is in W if and only if $a_0 = 4a_4, a_2 = -5a_4$,

a_1, a_3, a_4 arbitrary. A basis for W is $\{x, x^3, 4 - 5x^2 + x^4\}$ so

$\dim(W) = 3$.

8. The set S does not span V by property (1) of Theorem 9.

9. S contains 4 elements and $\dim(\mathcal{P}_2) = 3$. By property (1) of Theorem 8, S is linearly dependent.

10. The set S is a basis for V by property (2) of Theorem 8.

11. S contains only two vectors and $\dim(V) = 4$. By property (1) of Theorem 9, S does not span V.

12. The set S is a basis for V by property (2) of Theorem 8.

13. The set S contains 5 elements whereas $\dim(V) = 4$. By property (1) of Theorem 8, S is a linearly dependent set.

14. $\dim(W) = 2$.

15. (a) First note that since V is not a subset of an already familiar vector space, we must check all the properties of Definition 1 given in Section 4.2. The closure properties (c1) and (c2) are evident. As illustrations we will check (a2), (a3), and (m2).
Let $\mathbf{x} = \{x_i\}_{i=1}^{\infty}, \mathbf{y} = \{y_i\}_{i=1}^{\infty}$, and $\mathbf{z} = \{z_i\}_{i=1}^{\infty}$ be in V. Then $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = \{x_i + (y_i + z_i)\}_{i=1}^{\infty} = \{(x_i + y_i) + z_i\}_{i=1}^{\infty} = (\mathbf{x} + \mathbf{y}) +$
$\mathbf{z}$. Therefore (a2) is satisfied. If $\theta = \{\theta_i\}_{i=1}^{\infty}$, where $\theta_i = 0$ for each i then θ is the zero for V and property (a3) holds. To check (m2) let a be a scalar. Then $a(\mathbf{x} + \mathbf{y}) = \{a(x_i + y_i)\}_{i=1}^{\infty} = \{ax_i + ay_i\}_{i=1}^{\infty} = \{ax_i\}_{i=1}^{\infty} + \{ay_i\}_{i=1}^{\infty} = a\mathbf{x} + a\mathbf{y}$.

    (b) If $\mathbf{x} = a_1\mathbf{s_1} + \cdots + a_n\mathbf{s_n}$ then $\mathbf{x}$ is the sequence $\{x_i\}_{i=1}^{\infty}$ where $x_i = a_i, 1 \le i \le n$, and $x_i = 0$ for $i > n$. In particular if $\theta = a_1\mathbf{s_1} + \cdots + a_n\mathbf{s_n}$ (where θ is the zero sequence described in (a)) then $a_1 = \cdots = a_n = 0$. It follows that $\{\mathbf{s_1}, \mathbf{s_2}, \ldots, \mathbf{s_n}\}$ is a linearly independent subset of V. Since n is arbitrary, it follows from property (1) of Theorem 8 that V has infinite dimension.

16. Apply property (1) of Theorem 8.

17. Suppose $\dim(V) = n$ and let $\mathbf{w_1}$ be in W, $\mathbf{w_1} \neq \theta$. If $\{\mathbf{w_1}\}$ spans W then it is a basis. If not then by Exercise 36, Section 4.4, there is a vector $\mathbf{w_2}$ in W such that $\{\mathbf{w_1}, \mathbf{w_2}\}$ is a linearly independent set. In general suppose we have constructed a linearly independent subset $S_k = \{\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_k}\}$ of W. If $W = \mathrm{Sp}(S_k)$ then S_k is a basis and we are done. If S_k does not span W then, by Exercise 36, Section 4.4, there is a vector $\mathbf{w_{k+1}}$ in W such that $S_{k+1} = \{\mathbf{w_1}, \ldots, \mathbf{w_k},$

    $\mathbf{w_{k+1}}\}$ is linearly independent. This process must stop since, by property (1) of Theorem 8, any set of n+1 vectors in V is linearly dependent. Therefore there exists an integer m, $1 \le m \le n$, and a linearly independent subset $S_m = \{\mathbf{w_1}, \ldots, \mathbf{w_m}\}$ of W such that $W = \mathrm{Sp}(S_m)$. Thus S_m is a basis for W and $\dim(W) = m \le n$.

18. The set $T = \{[\mathbf{u_1}]_B, \ldots, [\mathbf{u_k}]_B\}$ contains k vectors in R^p, where $k \ge p + 1$. Thus T is a linearly dependent subset of R^p. By Theorem 5, $\{\mathbf{u_1}, \ldots \mathbf{u_k}\}$ is a linearly dependent subset of V.

19. Since B is a linearly independent subset of V containing p vectors and Q is a basis of V containing m vectors, Theorem 6 implies that $p \le m$. Reversing the roles of B and Q gives $m \le p$. Therefore

$m = p$.

20. Property (1) of Theorem 8 is a direct consequence of Theorem 6. To prove property (2) let B be a basis for V and let $S = \{\mathbf{u_1}, \ldots, \mathbf{u_p}\}$ be a linearly independent subset of V. By property (2) of Theorem 5 the set $T = \{[\mathbf{u_1}]_B, \ldots, [\mathbf{u_p}]_B\}$ is a linearly independent subset of R^p. Thus T is a basis for R^p. If $\mathbf{v}$ is in V then $[\mathbf{v}]_B$ is in Sp(T). By property 1 of Theorem 5, $\mathbf{v}$ is in Sp(S). It follows that S spans V, so S is a basis for V.

21. Note that $[\mathbf{w}]_B = [d_1, \ldots, d_n]^{\mathrm{T}}$ and $[\mathbf{w}]_C = [c_1, \ldots, c_n]^{\mathrm{T}}$. Thus $A[\mathbf{w}]_C = c_1[\mathbf{u_1}]_B + \cdots + c_n[\mathbf{u_n}]_B = [c_1\mathbf{u_1} + \cdots + c_n\mathbf{u_n}]_B = [\mathbf{w}]_B$.

(a) $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) $[p(x)]_C = [8, 4, 1]^{\mathrm{T}}$ and $[p(x)]_B = A[p(x)]_C = [5, 2, 1]^{\mathrm{T}}$.

22. (a) $p(x) = -4+(x+1)+(x+1)^2$. (b) $p(x) = 15-9(x+1)+2(x+1)^2$. (c) $p(x) = 4-(x+1)^2$. (d) $p(x) = -10 + (x + 1)$.

23. $A^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) $p(x) = 6 + 11x + 7x^2$. (b) $p(x) = 4 + 2x - x^2$. (c) $p(x) = 5 + x$.

(d) $p(x) = 8 - 2x - x^2$.

24. $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(a) $p(x) = -9 + 4x + x(x - 1) + x(x - 1)(x - 2)$.

(b) $p(x) = -2 + 8x + x(x - 1)$.

(c) $p(x) = 1 + x + 3x(x - 1) + x(x - 1)(x - 2)$.

(d) $p(x) = 3 + 5x + 5x(x - 1) + x(x - 1)(x - 2)$.

5.6 Inner-products

1. (1) $<\mathbf{x}, \mathbf{x}> = 4x_1^2 + x_2^2 \geq 0$ and $<\mathbf{x}, \mathbf{x}> = 0$ if and only if $x_1 = x_2 = 0$.

(2) $<\mathbf{x}, \mathbf{y}> = 4x_1y_1 + x_2y_2 = 4y_1x_1 + y_2x_2 = <\mathbf{y}, \mathbf{x}>$.

(3) $< a\mathbf{x}, \mathbf{y}> = 4ax_1y_1 + ax_2y_2 = a(4x_1y_1 + x_2y_2) = a <\mathbf{x}, \mathbf{y}>$.

(4) Let $\mathbf{z} = [z_1, z_2]^{\mathrm{T}}$. Then $<\mathbf{x}, \mathbf{y} + \mathbf{z}> = 4x_1(y_1 + z_1) + x_2(y_2 + z_2) = (4x_1y_1 + x_2y_2) + (4x_1z_1 + x_2z_2) = <\mathbf{x}, \mathbf{y}> + <\mathbf{x}, \mathbf{z}>$.

2. (i) $<\mathbf{x},\mathbf{x}>=\sum_{i=1}^{n} a_i x_i^2 \geq 0$ with equality if and only if $x_i = 0$ for each i.

   (2) $<\mathbf{x},\mathbf{y}>=\sum_{i=1}^{n} a_i x_i y_i = \sum_{i=1}^{n} a_i y_i x_i =<\mathbf{y},\mathbf{x}>$.

   (3) $<a\mathbf{x},\mathbf{y}>=\sum_{i=1}^{n} a_i a x_i y_i = a\sum_{i=1}^{n} a_i x_i y_i = a <\mathbf{x},\mathbf{y}>$.

   (4) $<\mathbf{x},\mathbf{y}+\mathbf{z}>=\sum_{i=1}^{n} a_i x_i(y_i+z_i) = \sum_{i=1}^{n} a_i x_i y_i + \sum_{i=1}^{n} a_i x_i z_i =$ $<\mathbf{x},\mathbf{y}>+<\mathbf{x},\mathbf{z}>$.

3. (1) is immediate since A is positive definite.

   (2) $<\mathbf{x},\mathbf{y}>=\mathbf{x}^{\mathrm{T}}A\mathbf{y} = (\mathbf{x}^{\mathrm{T}}A\mathbf{y})^{\mathrm{T}} = \mathbf{y}^{\mathrm{T}}A^{\mathrm{T}}\mathbf{x}^{\mathrm{TT}} =$

   $\mathbf{y}^{\mathrm{T}}A\mathbf{x} =<\mathbf{y},\mathbf{x}>$.

   (3) $<a\mathbf{x},\mathbf{y}>=(a\mathbf{x})^{\mathrm{T}}A\mathbf{y} = a[\mathbf{x}^{\mathrm{T}}A\mathbf{y}] = a <\mathbf{x},\mathbf{y}>$.

   (4) $<\mathbf{x},\mathbf{y}+\mathbf{z}>=\mathbf{x}^{\mathrm{T}}A(\mathbf{y}+\mathbf{z}) = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{y} + \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{z} =<\mathbf{x},\mathbf{y}>+$ $<\mathbf{x},\mathbf{z}>$.

4. $\mathbf{x}^{\mathrm{T}}A\mathbf{x} = x_1^2 + 2x_1x_2 + 2x_2^2 = (x_1+x_2)^2 + x_2^2$.

5. (1) $<p,p>= a_0^2 + a_1^2 + a_2^2 \geq 0$ with equality if and only if $a_i = 0$ for $0 \leq i \leq 2$.

   (2) $<p,q>= a_0b_0 + a_1b_1 + a_2b_2 = b_0a_0 + b_1a_1 + b_2a_2 =<q,p>$.

   (3) $<ap,q>= aa_0b_0 + aa_1b_1 + aa_2b_2 = a(a_0b_0 + a_1b_1 + a_2b_2) =$ $a <p,q>$.

   (4) Let $r(x) = c_0 + c_1x + c_2x^2$. Then $<p,q+r>= a_0(b_0+c_0) + a_1(b_1+c_1) + a_2(b_2+c_2) =$ $(a_0b_0 + a_1b_1 + a_2b_2) + (a_0c_0 + a_1c_1 + a_2c_2) =$

   $<p,q> + <p,r>$.

6. Clearly $< p,p >= p(0)^2 + p(1)^2 + p(2)^2 \geq 0$. Suppose $0 =< p,p >= a_0^2 + (a_0+a_1+a_2)^2 + (a_0+2a_1+4a_2)^2$. Then $0 = a_0 = a_0+a_1+a_2 = a_0+2a_1+4a_2$. It follows that $a_0 = a_1 = a_2 = 0$. The remaining properties of Definition 7 are straightforward to verify.

7. (1) $<A,A>= a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2 \geq 0$ with equality if and only if $A = \mathcal{O}$.

   (2) $<A,B>= a_{11}b_{11}+a_{12}b_{12}+a_{21}b_{21}+a_{22}b_{22} = b_{11}a_{11}+b_{12}a_{12}+b_{21}a_{21}+b_{22}a_{22} =<B,A>$.

   (3) $<aA,B>= aa_{11}b_{11}+aa_{12}b_{12}+aa_{21}b_{21}+aa_{22}b_{22} = a(a_{11}b_{11}+a_{12}b_{12}+a_{21}b_{21}+a_{22}b_{22}) =$ $a <A,B>$.

   (4) Let $C = [c_{ij}]$. Then $< A,B+C >= a_{11}(b_{11}+c_{11}) + a_{12}(b_{12}+c_{12}) + a_{21}(b_{21}+c_{21}) + a_{22}(b_{22}+c_{22}) = (a_{11}b_{11}+a_{12}b_{12}+a_{21}b_{21}+a_{22}b_{22}) + (a_{11}c_{11}+a_{12}c_{12}+a_{21}c_{21}+a_{22}c_{22}) =<$ $A,B> + <A,C>$.

8. $<\mathbf{x},\mathbf{y}>= -2$; $\|\mathbf{x}\| = 2\sqrt{2}$; $\|\mathbf{y}\| = 1$; $\|\mathbf{x}-\mathbf{y}\| = \sqrt{13}$.

9. $<\mathbf{x},\mathbf{y}>= [1,-2]\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -3; \|\mathbf{x}\|^2 =<\mathbf{x},\mathbf{x}>=$
$\mathbf{x}^{\mathrm{T}}A\mathbf{x}= 5$ so $\|\mathbf{x}\| = \sqrt{5}; \|\mathbf{y}\| = \sqrt{2}; \mathbf{x}-\mathbf{y} = [1,-3]^{\mathrm{T}}$ and
$\|\mathbf{x}-\mathbf{y}\|^2 = (\mathbf{x}-\mathbf{y})^{\mathrm{T}}A(\mathbf{x}-\mathbf{y}) = 13$. Thus $\|\mathbf{x}-\mathbf{y}\| = \sqrt{13}$.

10. $<p,q>= -1; \|\mathrm{p}\| = \sqrt{6}; \|\mathrm{q}\| = \sqrt{6}; \|\mathrm{p}-\mathrm{q}\| = \sqrt{14}$.

11. $<p,q>= (-1)1 + (2)2 + 7(7) = 52; \|\mathrm{p}\|^2 =<p,p>= (-1)^2 + 2^2 + 7^2 = 54$ so $\|p\|= 3\sqrt{6}; \|q\|^2=<q,q>= 1^2 + 2^2 + 7^2 = 54$ so $\|q\|= 3\sqrt{6}; \|p-q\|^2=<p-q,p-q>= 2^2$ so $\|p-q\|= 2$.

12. With the inner product defined in Exercise 5, $<1,x>=<1,x^2>=$
$<x,x^2>= 0$ so $\{1,x,x^2\}$ is an orthogonal set. With the inner product defined in Exercise 6, $<1,x>= 3$ so $\{1,x,x^2\}$ is not an orthogonal set.

13. For $<\mathbf{x},\mathbf{y}>= \mathbf{x}^{\mathrm{T}}\mathbf{y}$ the graph of S is the circle with equation $x^2 + y^2 = 1$. For $<\mathbf{x},\mathbf{y}> = 4x_1y_1 + x_2y_2$ the graph of S is the ellipse with equation $4x^2 + y^2 = 1$.

14. $\mathbf{u_1} = [1,0]^T, \mathbf{u_2} = [-1,1]^T$.

15. $\mathbf{v} = a_1\mathbf{u_1} + a_2\mathbf{u_2}$ where $a_1 =<\mathbf{u_1},\mathbf{v}> / <\mathbf{u_1},\mathbf{u_1}>= 7$ and
$a_2 =<\mathbf{u_2},\mathbf{v}> / <\mathbf{u_2},\mathbf{u_2}>= 4$.

16. $p_0(x) = 1, p_1(x) = x-1, p_2(x) = x^2 - 2x + 1/3$.

17. $q(x) = a_0p_0(x) + a_1p_1(x) + a_2p_2(x)$ where $a_0 =<p_0,q> / <p_0,p_0>= -5/3, a_1 =<p_1,q> / <p_1,p_1>= -5$, and
$a_2 =<p_2,q> / <p_2,p_2>= -4$.

18. For every scalar c, if $p(x) = 2cx - 3cx^2 + cx^3$ then $<p,p>= 0$.

19. $p_0(x) = 1, p_1(x) = x - a_0p_0$ where $a_0 =<x,p_0>/<p_0,p_0>= 0$.
Thus $p_1(x) = x$. $p_2(x) = x^2 - b_0p_0 - b_1p_1$ where $b_0 =$
$<x^2,p_0>/<p_0,p_0>= 2$ and $b_1 =<x^2,p_1>/<p_1,p_1>= 0$. Thus $p_2(x) = x^2 - 2$. $p_3(x) = x^3 - c_0p_0 - c_1p_1 - c_2p_2$ where $c_0 =$
$<x^3,p_0>/<p_0,p_0>= 0, c_1 =<x^3,p_1>/<p_1,p_1>= 17/5$, and $c_2 =<x^3,p_2>/<p_2,p_2>= 0$. Therefore $p_3(x) = x^3 - (17/5)x$.
$p_4(x) = x^4 - d_0p_0 - d_1p_1 - d_2p_2 - d_3p_3$ where $d_0 =$
$<x^4,p_0> / <p_0,p_0>= 34/5, d_1 =<x^4,p_1> / <p_1,p_1>= 0, d_2 =$
$<x^4,p_2> / <p_2,p_2>= 31/7, d_3 =<x^4,p_3> / <p_3,p_3>= 0$. Therefore $p_4(x) = x^4 - (31/7)x^2 + 72/35$.

20. $<\mathbf{v},\theta>=<\mathbf{v},\theta+\theta>=<\mathbf{v},\theta>+<\mathbf{v},\theta>$. Therefore $<\mathbf{v},\theta>=0$.

21. By assumption $\|\mathbf{u}\|^2=<\mathbf{u},\mathbf{u}>=0$. Therefore $\mathbf{u}=\theta$.

22. $\|a\mathbf{v}\|=\sqrt{<a\mathbf{v},a\mathbf{v}>}=\sqrt{a^2<\mathbf{v},\mathbf{v}>}=|a|\,\|\mathbf{v}\|$.

23. Suppose $a_1\mathbf{v_1}+a_2\mathbf{v_2}+\cdots+a_k\mathbf{v_k}=\theta$. For each $i, 1\le i\le k$, $0=$
$<\mathbf{v_i},\theta>=<\mathbf{v_i},\sum_{j=1}^k a_j\mathbf{v_j}>=\sum_{j=1}^k a_j<\mathbf{v_i},\mathbf{v_j}>=$
$a_i<\mathbf{v_i},\mathbf{v_i}>=a_i\|\mathbf{v_i}\|^2$. Thus $a_i=0$ and the set is linearly independent.

24. Suppose $\mathbf{u}=\sum_{j=1}^n a_j\mathbf{v_j}$. Then $<\mathbf{v_i},\mathbf{u}>=<\mathbf{v_i},\sum_{j=1}^n a_j\mathbf{v_j}>=$
$\sum_{j=1}^n a_j<\mathbf{v_i},\mathbf{v_j}>=a_i<\mathbf{v_i},\mathbf{v_i}>$. Therefore $a_i=$
$<\mathbf{v_i},\mathbf{u}>/<\mathbf{v_i},\mathbf{v_i}>$.

25. From Examples 4 and 5,$p_0(x)=1, p_1(x)=x-(1/2), p_2(x)=x^2-x+(1/6), <p_0,p_0>=1, <p_1,p_1>=1/12$, and $<p_2,p_2>=1/180$. Moreover $<x^3,p_0>=1/4, <x^3,p_1>=3/40$, and $<x^3,p_2>=1/120$. By Theorem 13,$p^*(x)=(1/4)p_0(x)+$
$(9/10)p_1(x)+(3/2)p_2(x)=(3/2)x^2-(3/5)x+(1/20)$.

26. The required constants are $<p_0,x^4>=1/5$, $<p_1,x^4>=1/15$, $<p_2,x^4>=1/105$, and $<p_3,x^4>=1/1400$. The remaining constants have already been calculated in Examples 4, 5, and 7. If follows that $p_4(x)=x^4-2x^3+(9/7)x^2-(2/7)x+1/70$.

27. With $p_3(x)$ as determined in Example 7 and with the calculations done in Example 6 we obtain $p^*(x)\simeq 0.841471p_0(x)-$
$0.467544p_1(x)-0.430920p_2(x)+0.07882p_3(x)$.

28. $T_0(x)=1, T_1(x)=x, T_2(x)=2x^2-1, T_3(x)=4x^3-3x$.

29. (a) Clearly $T_0(\cos\theta)=1=\cos(0\theta)$ and $T_1(cos\theta)=\cos\theta=\cos(1\theta)$. Suppose we have seen that $T_k(cos\theta)=cos(k\theta)$ for $0\le k\le n$, where $n\ge 1$. Then $T_{n+1}(\cos\theta)=2\cos\theta\, T_n(\cos\theta)-T_{n-1}(\cos\theta)=2\cos\theta\cos(n\theta)-\cos(n-1)\theta=\cos(n+1)\theta$ [since $\cos(\alpha+\beta)=2\cos\alpha\cos\beta-\cos(\alpha-\beta)$].

(b) $<T_i,T_j>=(2/\pi)\int_{-1}^1[T_i(x)T_j(x)/\sqrt{1-x^2}]dx=$
$-(2/\pi)\int_\pi^0\cos(i\theta)\cos(j\theta)d\theta=0$ if $i\ne j$.

(c) $T_0(x)=1$ has degree zero and $T_1(x)=x$ has degree one. Suppose $T_k(x)$ has degree k for $0\le k\le n$, where $n\ge 1$. Thus $T_n(x)=a_nx^n+\cdots+a_1x+a_0$ and $T_{n-1}(x)=b_{n-1}x^{n-1}+\cdots+b_1x+b_0$, where $a_n\ne 0$. Using (R) we obtain $T_{n+1}(x)=2a_nx^{n+1}+\cdots+(2a_0-b_1)x-b_0$. In particular, $T_{n+1}(x)$ has degree $n+1$. It follows by induction that $T_k(x)$ is a polynomial of degree k for each integer $k\ge 0$.

(d) $T_2(x) = 2x^2 - 1, T_3(x) = 4x^3 - 3x, T_4(x) = 8x^4 - 8x^2 + 1, T_5(x) = 16x^6 - 20x^3 + 5x.$

30. It follows from Exercise 29 that $\{T_0(x), T_1(x), \ldots, T_n(x)\}$ is an orthogonal basis for $\mathcal{P}_n$. Moreover $< T_0, T_0 >= 2$ whereas $< T_j, T_j >= 1$ for $j \geq 1$. The given formula is now an immediate consequence of Theorem 13.

32. By property (2) of Definition 7, $< \mathbf{e}_i, \mathbf{e}_j >=< \mathbf{e}_j, \mathbf{e}_i >$ for unit vectors, $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ in R^2. But $< \mathbf{e}_i, \mathbf{e}_j >= \mathbf{e}_i^T A \mathbf{e}_j = a_{ij}$, whereas $< \mathbf{e}_j, \mathbf{e}_i >= \mathbf{e}_j^T A \mathbf{e}_i = a_{ji}$. It follows that $a_{ij} = a_{ji}$, so A is symmetric. If $\mathbf{x}$ is a nonzero vector in R^n then $\mathbf{x}^T A \mathbf{x} =< \mathbf{x}, \mathbf{x} >> 0$ by property (1) of Definition 7. Therefore, A is positive definite.

5.7 Linear Transformations

1. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then $T(A+B) =$
$det(A+B) = det(I) = 1$ whereas $T(A) + T(B) = det(A) + det(B) = 0$. Therefore T is not a linear transformation.

2. T is a linear transformation.

3. T is a linear transformation. If A and B are (2 x 2) matrices it is straightforward to see that $tr(A+B) = tr(A) + tr(B)$; thus
$T(A+B) = T(A) + T(B)$. Likewise if c is a scalar, $tr(cA) =$
$c\, tr(A)$ so $T(cA) = cT(A)$.

4. T is not a linear transformation. For example if $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ then
$T(A) = T(B) = 0$ whereas $T(A+B) = 1$.

5. Let f and g be in $C[-1,1]$ and let c be a scalar. Then $T(f+g) = (f+g)'(0) = f'(0) + g'(0) = T(f) + T(g)$, and $T(cf) = (cf)'(0) =$
$cf'(0) = cT(f)$. Therefore T is a linear transformation.

6. T is a linear transformation.

7. T is not a linear transformation. For example $T(1 + \theta(x)) = T(1) = 2 + x + x^2$ whereas $T(1) + T(\theta(x)) = (2 + x + x^2) + (1 + x + x^2) = 3 + 2x + 2x^2$.

8. T is a linear transformation.

9. (a) $T(p) = 3T(1) - 2T(x) + 4T(x^2) = 3(1 + x^2) - 2(x^2 - x^3) +$
$4(2 + x^3) = 11 + x^2 + 6x^3$.

(b) $T(a_0+a_1x+a_2x^2) = a_0T(1)+a_1T(x)+a_2T(x^2) = a_0(1+x^2)+a_1(x^2-x^3)+a_2(2+x^3) = (a_0+2a_2)+(a_0+a_1)x^2+(-a_1+a_2)x^3$.

10. $p(x) = (-5)1 + 3(x+1) + (x^2+2x+1)$ so $T(p(x)) = -5x^4 + 3(x^3-2x) + x = -5x^4 + 3x^3 - 5x$. Similarly, $q(x) = (-3)1 + 7(x+1) + (x^2+2x+1)$ so $T(q(x)) = -3x^4 + 7x^3 - 13x$.

11. (a) $T(A) = -2T(E_{11}) + 2T(E_{12}) + 3T(E_{21}) + 4T(E_{22}) = 8 + 14x - 9x^2$.

(b) $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = aT(E_{11}) + bT(E_{12}) + cT(E_{21}) + dT(E_{22}) = (a+b+2d) + (-a+b+2c+d)x + (b-c-2d)x^2$.

12. (a) Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be (2 x 2) matrices. Then $T(A+B) = T([a_{ij}+b_{ij}]) = \begin{bmatrix} (a_{11}+b_{11}) + 2(a_{22}+b_{22}) \\ (a_{12}+b_{12}) - (a_{21}+b_{21}) \end{bmatrix} = \begin{bmatrix} a_{11}+2a_{22} \\ a_{12}-a_{21} \end{bmatrix} + \begin{bmatrix} b_{11}+2b_{22} \\ b_{12}-b_{21} \end{bmatrix} = T(A) + T(B)$.

If c is a scalar then $T(cA) = T([ca_{ij}]) = \begin{bmatrix} ca_{11}+ca_{22} \\ ca_{12}-ca_{21} \end{bmatrix} = c\begin{bmatrix} a_{11}+a_{22} \\ a_{12}-a_{21} \end{bmatrix} = cT(A)$.
Therefore T is a linear transformation.

(b) $\mathcal{N}(T) = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a = -2d, b = c, c \text{ and d arbitrary}\right\}$.

(c) $\left\{\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}$ is a basis for $\mathcal{N}(T)$.

(d) nullity $(T) = 2$ and rank $(T) = 2$.

(e) is a subspace of R^2 and $2 = \dim(R^2) = \text{rank}\,(T) = \dim(\mathcal{R}(T))$. Therefore $\mathcal{R}(T) = R^2$.

(f) $T(A) = \mathbf{v}$ where A is any matrix of the form $A = \begin{bmatrix} x-2d & y+c \\ c & d \end{bmatrix}$, where c and d are arbitrary. For example, $A = \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}$ is one choice for A.

13. (a) By property 1 of Theorem 15, $\mathcal{R}(T) = \text{Sp}\,\{T(1), T(x), T(x^2), T(x^3), T(x^4)\} = \text{Sp}\{0, 0, 2, 6x, 12x^2\} = \text{Sp}\{2, 6x, 12x^2\}$. It follows that rank $(T) = 3$. Since $\mathcal{R}(T) \subseteq \mathcal{P}_2$ and $\dim(\mathcal{P}_2) = 3$, we have $\mathcal{R}(T) = \mathcal{P}_2$.

(b) By property 3 of Theorem 15, nullity $(T) = \dim(\mathcal{P}_4) - \text{rank}\,(T) = 5 - 3 = 2$. Since nullity $(T) > 0, T$ is not one to one.

(c) We wish to determine $q(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4$ in $\mathcal{P}_4$ such that $a_0 + a_1x + a_2x^2 = T(q) = 2b_2 + 6b_3x + 12b_4x^2$.
Equating coefficients gives $b_2 = a_0/2, b_3 = a_1/6, b_4 = a_2/12, b_0$ and b_1 arbitrary. In particular,$q(x) = (a_0/2)x^2 + (a_1/6)x^3 + (a_2/12)x^4$ is one choice.

14. $\mathcal{R}(T) = \mathrm{Sp}\{1 - \mathrm{x} + 2\mathrm{x}^2 + 3\mathrm{x}^3, -1 + 3\mathrm{x} - 3\mathrm{x}^2 - \mathrm{x}^3, 2 - 2\mathrm{x} + 5\mathrm{x}^2 + 7\mathrm{x}^3, -1 + 3\mathrm{x} - \mathrm{x}^2 + 2\mathrm{x}^3, 1 - \mathrm{x} + \mathrm{x}^2 + 2\mathrm{x}^3\}$. Utilizing the spanning set we obtain the basis $\{1 - x + 2x^2 + 3x^3, 2x - x^2 + 2x^3, x^2 + x^3, x^3\}$ for $\mathcal{R}(T)$. In particular, $\mathrm{rank}\,(T) = 4$. Thus $\mathrm{nullity}\,(T) = 1$ and T is not one to one.

15. $\mathcal{N}(T) = \{p(x) = a_0 + a_1x + a_2x^2 : a_0 + 2a_1 + 4a_2 = 0\}$. It follows that $\mathrm{nullity}\,(T) = 2$. Consequently $\mathrm{rank}\,(T) = 1$ and $\mathcal{R}(T) = R^1$.

16. $\mathcal{N}(T) = \{f$ in $C[0,1] : \int_0^1 f(t)\,dt = 0\}$. For any a in $R^1, T(2ax) = \int_0^1 2atdt = a,$ so $\mathcal{R}(T) = R^1$.

17. (a) Let $\mathbf{u}, \mathbf{v}$ be vectors in V and let c be a scalar. Then $I(\mathbf{u}+\mathbf{v}) = \mathbf{u}+\mathbf{v} = I(\mathbf{u}) + I(\mathbf{v})$ and $I(c\mathbf{u}) = c\mathbf{u} = cI(\mathbf{u})$.
Therefore I is a linear transformation.

(b) The vector $\mathbf{v}$ is in $\mathcal{N}(I)$ if and only if $\theta = I(\mathbf{v}) = \mathbf{v}$. Thus $\mathcal{N}(I) = \{\theta\}$. For each $\mathbf{v}$ in $V, I(\mathbf{v}) = \mathbf{v}$ so $\mathcal{R}(I) = V$.

18. (a) Let $\mathbf{u_1}, \mathbf{u_2}$ be in U and let a be a scalar. Then $T(\mathbf{u_1} + \mathbf{u_2}) = \theta_V = \theta_V + \theta_V = T(\mathbf{u_1}) + \mathbf{T}(\mathbf{u_2})$ and $T(a\mathbf{u_1}) = \theta_V = a\,\theta_V = aT(\mathbf{u_1})$. This proves that T is a linear transformation.

(b) $\mathcal{N}(T) = U$ and $\mathcal{R}(T) = \{\theta_V\}$.

19. Recall that $5 = \dim(\mathcal{P}_4) = \mathrm{rank}\,(T) + \mathrm{nullity}\,(T)$. Moreover $\mathcal{R}(T) \subseteq \mathcal{P}_2$ so $\mathrm{rank}\,(T) \le 3$. The possibilities are:

rank (T)	3	2	1	0
nullity (T)	2	3	4	5.

Since $\mathrm{nullity}\,(T) \ge 2, T$ cannot be one to one.

20. By property 3 of Theorem 15, $\dim(U) = \mathrm{rank}\,(T) + \mathrm{nullity}\,(T)$. But $\mathcal{R}(T) \subseteq V$ so $\mathrm{rank}\,(T) \le \dim(V) < \dim(U)$. It follows that $\mathrm{nullity}\,(T) > 0$ and, hence, T cannot be one to one.

21. Recall that $3 = \dim(R^3) = \text{rank}\,(T) + \text{nullity}\,(T)$. Moreover $\mathcal{N}(T) \subseteq R^3$ so nullity $(T) \leq 3$. The possibilities are:

rank (T)	3	2	1	0
nullity (T)	0	1	2	3.

Since $\dim(\mathcal{P}_3) = 4$ and rank $(T) < 4, \mathcal{R}(T) = \mathcal{P}_3$ is not a possibility.

22. By property 3 of Theorem 15, $\dim(U) = \text{rank}\,(T) + \text{nullity}\,(T)$. In particular, rank $(T) \leq \dim(U) < \dim(V)$ so $\mathcal{R}(T) = V$ is not possible.

23. It follows from property 1 of Theorem 14 that θ_V is in $\mathcal{R}(T)$. Suppose that $\mathbf{v_1}$ and $\mathbf{v_2}$ are in $\mathcal{R}(T)$; thus there exist vectors $\mathbf{u_1}$ and $\mathbf{u_2}$ in U such that $T(\mathbf{u_1}) = \mathbf{v_1}$ and $T(\mathbf{u_2}) = \mathbf{v_2}$. Therefore $\mathbf{v_1} + \mathbf{v_2} = T(\mathbf{u_1}) + T(\mathbf{u_2}) = T(\mathbf{u_1} + \mathbf{u_2})$ so $\mathbf{v_1} + \mathbf{v_2}$ is in $\mathcal{R}(T)$. If a is a scalar then $a\mathbf{v_1} = aT(\mathbf{u_1}) = T(a\mathbf{u_1})$, and $a\mathbf{v_1}$ is in $\mathcal{R}(T)$. This proves that $\mathcal{R}(T)$ is a subspace of V.

24. Suppose T is one to one and let $\mathbf{u}$ be in $\mathcal{N}(T)$. It follows from property 1 of Theorem 14 that $T(\mathbf{u}) = \theta_V = T(\theta_U)$. Since T is one to one, $\mathbf{u} = \theta_U$ so $\mathcal{N}(T) = \{\theta_U\}$.

25. (a) If rank $(T) = p$ then, in the notation of Theorem 15, $C = \{T(\mathbf{u_1}), \ldots, T(\mathbf{u_p})\}$ is a basis for $\mathcal{R}(T)$ [cf. property 2 of Theorem 9 in Section 4.5]. In particular the set C is linearly independent. By property 2 of Theorem 15, T is one to one so, by property 4 of Theorem 14, nullity $(T) = 0$.

    (b) If rank $(T) = 0$ then $\mathcal{R}(T) = \{\theta_V\}$ and T is the zero linear transformation defined by $T(\mathbf{u}) = \theta_V$ for all $\mathbf{u}$ in U. Thus $\mathcal{N}(T) = U$ and nullity $(T) = \dim(U) = p$.

26. T is one to one if and only if $\mathcal{N}(T) = \{\theta\}$. Thus T is one to one if and only if $A\mathbf{x} = \theta$ has only the trivial solution, that is, if and only if A is nonsingular.

27. (a) Let A and B be (2 x 2) matrices. Then $T(A + B) = (A + B)^{\mathrm{T}} = A^{\mathrm{T}} + B^{\mathrm{T}} = T(A) + T(B)$. If c is a scalar then $T(cA) = (cA)^{\mathrm{T}} = cA^{\mathrm{T}} = cT(A)$. This proves that T is a linear transformation.

    (b) If A is in $\mathcal{N}(T)$ then $T(A) = A^{\mathrm{T}} = \mathcal{O}$. It follows that $A = \mathcal{O}$ and that nullity $(T) = 0$. Therefore rank $(T) = 4$. Consequently T is one to one and $\mathcal{R}(T) = V$.

    (c) Let B be in V and set $C = B^{\mathrm{T}}$. Then $T(C) = C^{\mathrm{T}} = B^{\mathrm{TT}} = B$. Therefore $\mathcal{R}(T) = V$.

5.8 Operations with Linear Transformations

1. $(S+T)(p) = S(p)+T(p) = p'(0)+(x+2)p(x)$. In particular, $(S+T)(x) = 1+(x+2)x = x^2+2x+1$ and $(S+T)(x^2) = 0+(x+2)x^2 = x^3+2x^2$.

2. $(2T)(p) = 2[T(p)] = 2(x+2)p(x) = (2x+4)p(x)$. Therefore $(2T)(x) = 2x^2+4x$.

3. $(H \circ T)(p) = H(T(p)) = H((x+2)p(x)) = [(x+2)p(x)]'+2p(0) = (x+2)p'(x)+p(x)+2p(0)$. The domain for $H \circ T$ is $\mathcal{P}_3$ and $(H \circ T)(x) = 2x+2$.

4. $(T \circ H)(p) = (x+2)[p'(x)+p(0)]; T \circ H$ has domain $\mathcal{P}_4$; $(T \circ H)(x) = x+2$.

5. (a) If $p(x) = \sum_{i=0}^{3} a_i x^i$ then $T(p) = 2a_0+(a_0+2a_1)x+(a_1+2a_2)x^2+(a_2+2a_3)x^3+a_3x^4$. In particular $T(p) = \theta(x)$ if and only if $p(x) = \theta(x)$. Therefore T is one to one. Now $\text{rank}\,(T) = \dim(\mathcal{P}_3) - \text{nullity}\,(T) = 4$. Since $\mathcal{R}(T) \subseteq \mathcal{P}_4$ and $\dim(\mathcal{P}_4) = 5, \mathcal{R}(T) \neq \mathcal{P}_4$; that is, T is not onto.

   (b) It is easy to verify that $T(p) = x$ is impossible. Therefore $T^{-1}(x)$ is not defined.

6. (a) $\mathcal{R}(H) = \text{Sp}\ \{H(1), H(x), H(x^2), H(x^3), H(x^4)\} = \text{Sp}\{\ 1, 1, 2x, 3x^2, 4x^3\} = \text{Sp}\{\ 1, 2x, 3x^2, 4x^3\}$. It follows that $\text{rank}\,(H) = 4$ and $\text{nullity}\,(H) = 1$. Therefore H is onto but not one to one.

   (b) Note that $H((1/2)x^2) = x = H((1/2)x^2+x-1)$. Therefore $H^{-1}(x)$ is not uniquely determined.

7. Let $p(x) = ae^x + be^{2x} + ce^{3x}$ be in V. Then $T(p(x)) = p'(x) = ae^x + 2be^{2x} + 3ce^{3x}$. Since $B = \{e^x, e^{2x}, e^{3x}\}$ is a linearly independent set, it follows that $T(p(x)) = \theta(x)$ if and only if $a = b = c = 0$. Thus $\mathcal{N}(T) = \{\theta(x)\}$ and T is one to one. The set B is a basis for V so $\dim(V) = 3$. Thus $\text{rank}\,(T) = \dim(V) - \text{nullity}\,(T) = 3$. It follows that T is onto. Therefore T is invertible. Moreover $T^{-1}(e^x) = e^x, T^{-1}(e^{2x}) = (1/2)e^{2x}$, and $T^{-1}(e^{3x}) = (1/3)e^{3x}$. This implies that $T^{-1}(ae^x + be^{2x} + ce^{3x}) = ae^x + (b/2)e^{2x} + (c/3)e^{3x}$.

8. $T^{-1}(\sin x) = -\cos x, T^{-1}(\cos x) = \sin x$, and $T^{-1}(e^{-x}) = -e^{-x}$.

   Therefore $T^{-1}(a\sin x + b\cos x + ce^{-x}) = -a\cos x + b\sin x - ce^{-x}$.

9. If A is in $\mathcal{N}(T)$ then $T(A) = A^T = \mathcal{O}$. Therefore $A = \mathcal{O}$, so $\mathcal{N}(T) = \{\mathcal{O}\}$. It follows that T is one to one. Further $\text{rank}\,(T) = \dim(V) - \text{nullity}\,(T) = 4$ so T is onto. Therefore T is invertible. In fact $T^{-1} = T$ since $(A^T)^T = A$.

10. $T^{-1}(A) = QAQ^{-1}$.

11. (a) Since $\dim(V) = 4, V$ is isomorphic to R^4 by Theorem 17.

(b) Since $\dim(\mathcal{P}_3) = 4 = \dim(V)$, V and $\mathcal{P}_3$ are isomorphic by the corollary to Theorem 17.

(c) It is easily shown that $T : V \to \mathcal{P}_3$ defined by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + bx + cx^2 + dx^3$ is an isomorphism.

12. (a) Note that $\dim(U) = 3$.

(b) $\dim(U) = 3 = \dim(\mathcal{P}_2)$.

(c) Define $T : U \to \mathcal{P}_2$ by $T\left(\begin{bmatrix} a & b \\ b & c \end{bmatrix}\right) = a + bx + cx^2$.

13. If $\mathbf{u}$ and $\mathbf{w}$ are in U then $S(\mathbf{u}+\mathbf{w}) = T_1(\mathbf{u}+\mathbf{w}) + T_2(\mathbf{u}+\mathbf{w}) =$
$T_1(\mathbf{u}) + T_1(\mathbf{w}) + T_2(\mathbf{u}) + T_2(\mathbf{w}) = T_1(\mathbf{u}) + T_2(\mathbf{u}) + T_1(\mathbf{w}) +$
$T_2(\mathbf{w}) = \mathbf{S}(\mathbf{u}) + \mathbf{S}(\mathbf{w})$. If c is a scalar then $S(c\mathbf{u}) = T_1(c\mathbf{u}) +$
$T_2(c\mathbf{u}) = cT_1(\mathbf{u}) + cT_2(\mathbf{u}) = c[T_1(\mathbf{u}) + T_2(\mathbf{u})] = cS(\mathbf{u})$.
This proves that S is a linear transformation.

14. Let $\mathbf{u}$ and $\mathbf{w}$ be in U. $[aT](\mathbf{u}+\mathbf{w}) = a[T(\mathbf{u}+\mathbf{w})] =$
$a[T(\mathbf{u}) + T(\mathbf{w})] = a[T(\mathbf{u})] + a[T(\mathbf{w})] = [aT](\mathbf{u}) + [aT](\mathbf{w})$. If c is a scalar then
$[aT](c\mathbf{u}) = a[T(c\mathbf{u})] = a[cT(\mathbf{u})] = c[aT(\mathbf{u})] = c[aT](\mathbf{u})$. Therefore aT is a linear transformation.

15. Suppose that $T^{-1}(\mathbf{v}) = \mathbf{u}$. Then $T(\mathbf{u}) = \mathbf{v}$ and $T(c\mathbf{u}) =$
$cT(\mathbf{u}) = c\mathbf{v}$. Therefore $T^{-1}(c\mathbf{v}) = c\mathbf{u} = cT^{-1}(\mathbf{v})$.

16. By formula 1, $(T^{-1})^{-1}(\mathbf{u}) = \mathbf{v}$ where $\mathbf{v}$ is chosen so that $T^{-1}(\mathbf{v}) =$
$\mathbf{u}$. But $T^{-1}(\mathbf{v}) = \mathbf{u}$ precisely when $T(\mathbf{u}) = \mathbf{v}$. Therefore $(T^{-1})^{-1}$
$= T$.

17. Let $\mathbf{u}$ be in U and set $T(\mathbf{u}) = \mathbf{v}$. Then $T^{-1}(\mathbf{v}) = \mathbf{u}$ so
$(T^{-1} \circ T)(\mathbf{u}) = T^{-1}(T(\mathbf{u})) = T^{-1}(\mathbf{v}) = \mathbf{u}$. It follows that
$T^{-1} \circ T = I_U$. Likewise $(T \circ T^{-1})(\mathbf{v}) = \mathbf{T}(\mathbf{T^{-1}}(\mathbf{v})) = T(\mathbf{u}) = \mathbf{v}$, so
$T \circ T^{-1} = I_V$.

18. (a) Suppose S and T are both one to one and let $\mathbf{u}$ be in $\mathcal{N}(T \circ S)$.
Thus $\theta_W = (T \circ S)(\mathbf{u}) = T(S(\mathbf{u}))$. Since T is one to one it follows that $S(\mathbf{u}) = \theta_V$. But S is also one to one so $\mathbf{u} = \theta_U$.

(b) Suppose that S and T are onto and let $\mathbf{w}$ be in W. By assumption there exists $\mathbf{v}$ in V such that $T(\mathbf{v}) = \mathbf{w}$. Likewise, S is onto so there exists $\mathbf{u}$ in U such that $S(\mathbf{u}) = \mathbf{v}$. Therefore $(T \circ S)(\mathbf{u}) = T(S(\mathbf{u})) = T(\mathbf{v}) = \mathbf{w}$. It follows that $T \circ S$ is onto.

(c) S and T are both one to one and both onto so by (a) and (b) $T \circ S$ is one to one and onto. Therefore $T \circ S$ is invertible. To see that $(T \circ S)^{-1} = S^{-1} \circ T^{-1}$, let $\mathbf{w}$ be in W. Since $T \circ S$ is onto there exists $\mathbf{u}$ in U such that $(T \circ S)(\mathbf{u}) = \mathbf{w}$. Therefore $(T \circ S)^{-1}(\mathbf{w}) = \mathbf{u}$. Now set $\mathbf{v} = S(\mathbf{u})$. Then $T(\mathbf{v}) = \mathbf{w}$ so $T^{-1}(\mathbf{w}) = \mathbf{v}$ and $S^{-1}(\mathbf{v}) = \mathbf{u}$. Therefore $(S^{-1} \circ T^{-1})(\mathbf{w}) = S^{-1}(T^{-1}(\mathbf{w})) = S^{-1}(\mathbf{v}) = \mathbf{u} = (T \circ S)^{-1}(\mathbf{w})$.

19. Let $S : U \to V$ be an isomorphism and let $T : V \to W$ be an isomorphism. By Exercise 18, $T \circ S : U \to W$ is an isomorphism.

20. (a) Since T is one to one, nullity $(T) = 0$. Therefore rank $(T) = n -$ nullity $(T) = n$ so $\mathcal{R}(T) = V$. Since T is onto, T is invertible.

(b) By assumption rank $(T) = n$. Thus nullity $(T) = n -$ rank $(T) = 0$ and T is one to one. Therefore T is invertible.

21. It is easy to show that $T(p(x)) = \theta(x)$ if and only if $p(x) = \theta(x)$.

Thus $\mathcal{N}(T) = \{\theta(x)\}$ and T is one to one. Clearly there exists no polynomial $p(x)$ in $\mathcal{P}$ such that $T(p(x)) = 1$. Therefore T is not onto. This does not contradict Exercise 20(a) since $\mathcal{P}$ has infinite dimension.

22. Let $q(x) = b_0 + b_1 x + \cdots + b_n x^n$ be in $\mathcal{P}$ and set $p(x) = b_0 x + (b_1/2)x^2 + \cdots + [b_n/(n+1)]x^{n+1}$. Then $S(p) = p'(x) = q(x)$ so S is onto. Note that $\mathcal{N}(T)$ is the set of all constant polynomials. In particular, $\mathcal{N}(T) \neq \{\theta(x)\}$ so T is not one to one. This does not contradict Exercise 20(b) since $\mathcal{P}$ has infinite dimension.

23. If $\mathbf{u}$ is in $\mathcal{N}(S)$ then $S(\mathbf{u}) = \theta_V$. Therefore $(T \circ S)(\mathbf{u}) = T(S(\mathbf{u})) = T(\theta_V) = \theta_W$ and $\mathbf{u}$ is in $\mathcal{N}(T \circ S)$. If $T \circ S$ is one to one then $\mathcal{N}(T \circ S) = \{\theta_U\}$. Therefore $\mathcal{N}(S) = \{\theta_U\}$ and S is one to one.

24. If $\mathbf{w}$ is in $\mathcal{R}(T \circ S)$ then there exists $\mathbf{u}$ in U such that $\mathbf{w} = (T \circ S)(\mathbf{u})$. Set $\mathbf{v} = S(\mathbf{u})$. Then $\mathbf{v}$ is in V and $T(\mathbf{v}) = T(S(\mathbf{u})) = (T \circ S)(\mathbf{u}) = \mathbf{w}$. This shows that $\mathbf{w}$ is in $\mathcal{R}(T)$. If $T \circ S$ is onto we have $\mathcal{R}(T \circ S) = V$ and $\mathcal{R}(T \circ S) \subseteq \mathcal{R}(T) \subseteq V$. It follows that $\mathcal{R}(T) = V$ and T is onto.

25. Assume that $T \circ S$ is invertible. Then $T \circ S$ is one to one so, by Exercise 23, S is one to one. Exercise 20(a) now implies that S is invertible. Since $T \circ S$ is also onto, Exercise 24 implies that T is onto. Exercise 20(b) now implies that T is invertible.

26. Define $S : R^n \to R^p$ by $S(\mathbf{x}) = B\mathbf{x}$ and define $T : R^p \to R^m$ by $T(\mathbf{y}) = A\mathbf{y}$. Then $T \circ S : R^n \to R^m$ is defined by $(T \circ S)(\mathbf{x}) = AB\mathbf{x}$. Therefore nullity $(B) =$ nullity (S), nullity $(AB) =$

nullity $(T \circ S)$, rank $(A) =$ rank (T), and rank $(AB) =$ rank $(T \circ S)$.

Now apply Exercises 23 and 24.

27 It follows from Exercise 20(a) that T is invertible if and only if T is one to one. Now apply Exercise 26 of Section 4.7.

28. Define $S : R^n \to R^n$ by $S(\mathbf{x}) = B(\mathbf{x})$ and define $T : R^n \to R^n$ by $T(\mathbf{x}) = A\mathbf{x}$. Then $T \circ S : R^n \to R^n$ is defined by $(T \circ S)(\mathbf{x}) = AB\mathbf{x}$. If AB is nonsingular then $T \circ S$ is invertible by Exercise 27. By Exercise 25, both T and S are invertible. Applying Exercise 27 again, we see that A and B are nonsingular.

29. To prove that $L(U, V)$ is a vector space requires checking all ten properties of Definition 1 in Section 4.2. We shall verify only properties (c1), (c2), (a2), (a3), (a4), and (m2).

If S and T are in $L(U, V)$ and c is a scalar then $S + T$ and cT are in $L(U, V)$ by Exercises 13 and 14. Thus properties (c1) and (c2) hold. Now let R be in $L(U, V)$. To show that $R+(S+T) = (R+S)+T$ we must show that each of the transformations has the same action on any vector $\mathbf{u}$ in U. But addition is associative in V so $[R + (S + T)](\mathbf{u}) = R(\mathbf{u}) + (S + T)(\mathbf{u}) = R(\mathbf{u})+(S(\mathbf{u})+T(\mathbf{u})) = (R(\mathbf{u})+S(\mathbf{u}))+T(\mathbf{u}) = (R+S)(\mathbf{u})+T(\mathbf{u}) = [(R+S)+T](\mathbf{u})$. Recall that the zero linear transformation $T_0 : U \to V$ is defined by $T_0(\mathbf{u}) = \theta_V$ for every $\mathbf{u}$ in U. Thus $(T + T_0)(\mathbf{u}) = T(\mathbf{u}) + T_0(\mathbf{u}) = T(\mathbf{u}) + \theta_V = T(\mathbf{u})$. It follows that $T + T_0 = T$, so T_0 is the zero of $L(U, V)$. For T in $L(U, V)$ it is easily seen that $T + (-1)T = T_0$, so $(-1)T = -T$ and property (a4) of Definition 1 is satisfied,

To check (m2) let a be a scalar and let S and T be in $L(U, V)$. For any $\mathbf{u}$ in U $[a(S+T)](\mathbf{u}) = a[(S + T)(\mathbf{u})] = a[S(\mathbf{u}) + T(\mathbf{u})] = aS(\mathbf{u}) + aT(\mathbf{u}) = [aS](\mathbf{u}) + [aT](\mathbf{u}) = [aS + aT](\mathbf{u})$. It follows that $a(S + T) = aS + aT$.

5.9 Matrix Representations for Linear Transformations

1. $S(1) = 0, S(x) = 1, S(x^2) = 0,$ and $S(x^3) = 0$. Thus $[S(1)]_C = [S(x^2)]_C = [S(x^3)]_C = [0, 0, 0, 0, 0]^{\mathrm{T}}$, while $[S(x)]_C = [1, 0, 0, 0, 0]^{\mathrm{T}}$.

The matrix for S is $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

2. The matrix for T is $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

3. (a) $(S+T)(1) = 2+x, (S+T)(x) = 1+2x+x^2, (S+T)(x^2) = 2x^2+x^3, (S+T)(x^3) = 2x^3+x^4$. Therefore $[(S+T)(1)]_C = [2,1,0,0,0]^{\mathrm{T}}, [(S+T)(x)]_C = [1,2,1,0,0]^{\mathrm{T}}, [(S+T)](x^2)]_C = [0,0,2,1,0]^{\mathrm{T}}$, and $[(S+T)(x^3)]_C = [0,0,0,2,1]^{\mathrm{T}}$.

   The matrix for $S+T$ is the matrix $\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

   (b) By Theorem 19 the matrix for $S+T$ is the sum of matrices for S and T. This is easily verified.

4. The matrix for $2T$ is $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

5. $H(1) = 1, H(x) = 1, H(x^2) = 2x, H(x^3) = 3x^2$, and $H(x^4) = 4x^3$. Therefore $[H(1)]_B = [H(x)]_B = [1,0,0,0]^{\mathrm{T}}, [H(x^2)]_B$
$= [0,2,0,0]^{\mathrm{T}}, [H(x^3)]_B = [0,0,3,0]^{\mathrm{T}}$, and $[H(x^4)]_B = [0,0,0,4]^{\mathrm{T}}$.

   The matrix for H is the matrix $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$.

6. (a) The matrix for $H \circ T$ is the matrix $\begin{bmatrix} 3 & 2 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 4 \end{bmatrix}$.

   (b) Denote by D, E, and F the matrices in Exercises 5,2, and 6(a), respectively. By Theorem 20, $F = DE$ and it is easily verified that this is the case.

7. (a) $(T \circ H)(1) = 2+x, (T \circ H)(x) = 2+x, (T \circ H)(x^2) = 4x+2x^2, (T \circ H)(x^3) = 6x^2+3x^3, (T \circ H)(x^4) = 8x^3+4x^4$. Therefore
$[(T \circ H)(1)]_C = [(T \circ H)(x)]_C = [2,1,0,0,0]^{\mathrm{T}}, [(T \circ H)(x^2)]_C$
$= [0,4,2,0,0]^{\mathrm{T}}, [(T \circ H)(x^3)]_C = [0,0,6,3,0]^{\mathrm{T}}$, and

$[(T\circ H)(x^4)]_C = [0,0,0,8,4]^{\mathrm{T}}$. Thus the matrix for $T\circ H$ is the matrix

$$\begin{bmatrix} 2 & 2 & 0 & 0 & 0 \\ 1 & 1 & 4 & 0 & 0 \\ 0 & 0 & 2 & 6 & 0 \\ 0 & 0 & 0 & 3 & 8 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}.$$

(b) Let D, E, and F denote the matrices for T, H, and $T\circ H$, respectively (cf. Exercises 2, 5, and 7(a)). By Theorem 20, $F = DE$ and it is easily verified that this the case.

8. (a) $[p]_B = [a_0, a_1, a_2, a_3]^{\mathrm{T}}$, S(p)=$a_1$ so $[\mathrm{S(p)}]_{\mathrm{C}} = [a_1, 0, 0, 0, 0]^{\mathrm{T}}$.

9. (a) $[p]_B = [a_0, a_1, a_2, a_3]^{\mathrm{T}}; T(p) = 2a_0 + (a_0 + 2a_1)x + (a_1 + 2a_2)x^2 + (a_2 + 2a_3)x^3 + a_3x^4$ so $[T(p)]_C = [2a_0, a_0 + 2a_1, a_1 + 2a_2, a_2 + 2a_3, a_3]^{\mathrm{T}}$.

10. $[q]_C = [a_0, a_1, a_3, a_4]^{\mathrm{T}}; H(q) = (a_0 + a_1) + 2a_2x + 3a_3x^2 + 4a_4x^3$ so $[H(q)]_B = [a_0 + a_1, 2a_2, 3a_3, 4a_4]^{\mathrm{T}}$. It is easily seen that $N[q]_C = [H(q)]_B$.

11. (a) $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

(b) $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$.

(c) Clearly $P = Q^{-1}$.

12. (a) $Q = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

(b) $P = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

(c) Clearly $P = Q^{-1}$.

13. (a) $T(E_{11}) = E_{11}, T(E_{12}) = E_{21}, T(E_{21}) = E_{12}$, and $T(E_{22}) = E_{22}$. Therefore $Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(b) If $A = [a_{ij}]$ is a (2 x 2) matrix then $[A]_B = [a_{11}, a_{12}, a_{21}, a_{22}]^{\mathrm{T}}$ whereas $[A^{\mathrm{T}}]_B = [a_{11}, a_{21}, a_{12}, a_{22}]^{\mathrm{T}}$. Clearly $Q[A]_B = [A^{\mathrm{T}}]_B$.

14. $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$.

15. $S(x+1) = 3+3x-x^2, S(x+2) = 6+3x-x^2,$ and $S(x^2) = 3x^2$. Therefore $[S(x+1)]_C = [3,3,-1,0]^{\mathrm{T}}, [S(x+2)]_C = [6,3,-1,0]^{\mathrm{T}},$

and $[S(x^2)]_C = [0,0,3,0]^{\mathrm{T}}$ and the matrix representation for S

is $\begin{bmatrix} 3 & 6 & 0 \\ 3 & 3 & 0 \\ -1 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$.

16. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

17. $T(1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, T(x) = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$, and $T(x^2) = \begin{bmatrix} 0 \\ 6 \\ 4 \end{bmatrix}$, so the matrix for T is given by $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 6 \\ 0 & 1 & 4 \end{bmatrix}$.

18. $\begin{bmatrix} 0 & -2 & -2 \\ -1 & 1 & 4 \\ 1 & 2 & 2 \end{bmatrix}$.

19. $T(\mathbf{v_1}) = 0\mathbf{v_1}+1\mathbf{v_2}+0\mathbf{v_3}+0\mathbf{v_4}, T(\mathbf{v_2}) = 0\mathbf{v_1}+0\mathbf{v_2}+1\mathbf{v_3}+0\mathbf{v_4}, T(\mathbf{v_3}) = 1\mathbf{v_1}+1\mathbf{v_2}+0\mathbf{v_3}+0\mathbf{v_4},$

and $T(\mathbf{v_4}) = 1\mathbf{v_1}+0\mathbf{v_2}+0\mathbf{v_3}+3\mathbf{v_4}$, so the matrix of T is $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$.

20. $T(\mathbf{e_1}) = A\mathbf{e_1} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, T(\mathbf{e_2}) = A\mathbf{e_2} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $T(\mathbf{e_3}) = A\mathbf{e_3} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$. Therefore the matrix for T is the given matrix A.

21. $T(1) = -4+3x-x^2, T(x) = -2+3x+2x^2,$ and $T(x^2) = 3x^2$ so the matrix for T is $\begin{bmatrix} -4 & -2 & 0 \\ 3 & 3 & 0 \\ -1 & 2 & 3 \end{bmatrix}$.

22. $[p]_B = [a_0, a_1, a_2]^{\mathrm{T}}$ and $[T(p)]_B = [-4a_0 - 2a_1, 3a_0 + 3a_1, -a_0 + 2a_1 + 3a_2]^{\mathrm{T}}$. It is easily verified that $Q[p]_B = [T(p)]_B$.

23. $T(1 - 3x + 7x^2) = 2(1 - 3x + 7x^2), T(6 - 3x + 2x^2) = -3(6 - 3x + 2x^2)$, and $T(x^2) = 3x^2$. Therefore the matrix of T is $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

24. Let $P = [\mathbf{P_1}, \mathbf{P_2}, \ldots, \mathbf{P_k}]$ be a matrix such that $P[\mathbf{u}]_B = [\mathbf{T}(\mathbf{u})]_C$ for every vector $\mathbf{u}$ in U. Since $[\mathbf{u}]_B$ is in R^n and $[T(\mathbf{u})]_C$ is in R^m, P is necessarily an $(m \times n)$ matrix. Therefore $k = n$. Suppose $Q = [\mathbf{Q_1}, \mathbf{Q_2}, \ldots, \mathbf{Q_n}]$ and assume that $B = \{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}\}$. Then for $1 \le j \le n, \mathbf{P_j} = P\mathbf{e_j}$ $= P[\mathbf{u_j}]_B = [T(\mathbf{u_j})]_C = Q[\mathbf{u_j}]_B = Q\mathbf{e_j} = \mathbf{Q_j}$. It follows that $P = Q$.

25. Let $B = \{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}\}$ and $C = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}\}$. Suppose

$$\begin{array}{rcl} T_1(\mathbf{u_1}) & = & a_{11}\mathbf{v_1} + a_{21}\mathbf{v_2} + \cdots + a_{m1}\mathbf{v_m} \\ T_1(\mathbf{u_2}) & = & a_{12}\mathbf{v_1} + a_{22}\mathbf{v_2} + \cdots + a_{m2}\mathbf{v_m} \\ \vdots & & \vdots \\ T_1(\mathbf{u_n}) & = & a_{1n}\mathbf{v_1} + a_{2n}\mathbf{v_2} + \cdots + a_{mn}\mathbf{v_m} \end{array}.$$

Also assume that

$$\begin{array}{rcl} T_2(\mathbf{u_1}) & = & b_{11}\mathbf{v_1} + b_{21}\mathbf{v_2} + \cdots + b_{m1}\mathbf{v_m} \\ T_2(\mathbf{u_2}) & = & b_{12}\mathbf{v_1} + b_{22}\mathbf{v_2} + \cdots + b_{m2}\mathbf{v_m} \\ \vdots & & \vdots \\ T_2(\mathbf{u_n}) & = & b_{1n}\mathbf{v_1} + b_{2n}\mathbf{v_2} + \cdots + b_{mn}\mathbf{v_m} \end{array}.$$

Then T_1 and T_2 are represented by the $(m \times n)$ matrices $Q_1 = [a_{ij}]$ and $Q_2 = [b_{ij}]$, respectively. To obtain the matrix for $T_1 + T_2$ note that

$(T_1 + T_2)(\mathbf{u_1}) = T_1(\mathbf{u_1}) + T_2(\mathbf{u_1}) =$

$$(a_{11} + b_{11})\mathbf{v_1} + (a_{21} + b_{21})\mathbf{v_2} + \cdots + (a_{m1} + b_{m1})\mathbf{v_m}$$

$(T_1 + T_2)(\mathbf{u_2}) = T_1(\mathbf{u_2}) + T_2(\mathbf{u_2}) =$

$$(a_{12} + b_{12})\mathbf{v_1} + (a_{22} + b_{22})\mathbf{v_2} + \cdots + (a_{m2} + b_{m2})\mathbf{v_m}$$

$$\vdots \qquad\qquad \vdots$$

$(T_1 + T_2)(\mathbf{u_n}) = T_1(\mathbf{u_n}) + T_2(\mathbf{u_n}) =$

$$(a_{1n} + b_{1n})\mathbf{v_1} + (a_{2n} + b_{2n})\mathbf{v_2} + \cdots + (a_{mn} + b_{mn})\mathbf{v_m}$$

Therefore the matrix for $T_1 + T_2$ is the $(m \times n)$ matrix $[a_{ij} + b_{ij}] = Q_1 + Q_2$.

26. Let $B = \{\mathbf{u_1}, \ldots, \mathbf{u_n}\}$ and $C = \{\mathbf{v_1}, \ldots, \mathbf{v_m}\}$. If $T(\mathbf{u_j}) = \sum_{i=1}^{m} q_{ij}\mathbf{v_i}$ for $1 \le j \le n$, then the matrix for T is the $(m \times n)$ matrix $Q = [q_{ij}]$. Moreover $[aT](\mathbf{u_j}) = a\sum_{i=1}^{m} q_{ij}\mathbf{v_i} = \sum_{i=1}^{m}(aq_{ij})\mathbf{v_i}$ for $1 \le j \le n$. Therefore the matrix for aT is the $(m \times n)$ matrix $[aq_{ij}] = aQ$.

27. By assumption $Q[\mathbf{u}]_B = [T(\mathbf{u})]_C$ for every vector $\mathbf{u}$ in U. If P is the matrix for aT then P is the unique matrix such that $P[\mathbf{u}]_B = [(aT)(\mathbf{u})]_C$ for every vector $\mathbf{u}$ in U. But $(aQ)[\mathbf{u}]_B = a(Q[\mathbf{u}]_B) = a[T(\mathbf{u})]_C = [aT(\mathbf{u})]_C = [(aT)(\mathbf{u})]_C$. It follows that $P = aQ$.

28. If $B = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is a basis for V then $I_V(\mathbf{v_j}) = \mathbf{v_j}$ for $1 \le j \le n$. Therefore $[I_V(\mathbf{v_j})]_B = \mathbf{e_j}$ and the matrix representation for I_V is $[\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}] = I$.

29. If $B = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is a basis for V then $T_0(\mathbf{v_j}) = \theta_V = 0\mathbf{v_1} + 0\mathbf{v_2} + \cdots + 0\mathbf{v_n}$ for $1 \le j \le n$. Thus $[T_0(\mathbf{v_j})]_B = \theta$ (the zero vector in R^n) and the matrix for T_0 is the $(n \times n)$ zero matrix.

30. It is an immediate consequence of Theorem 20 and Exercise 28 that $PQ = I$ and $QP = I$. Therefore $P = Q^{-1}$.

31. If Q is the matrix for T then $T(1) = 1 - x^2, T(x) = x + x^2, T(x^2) = 2,$ and $T(x^3) = x - x^2$. Therefore $T(a_0 + a_1x + a_2x^2 + a_3x^3) = a_0T(1) + a_1(T(x) + a_2T(x^2) + a_3T(x^3) = (a_0 + 2a_2) + (a_1 + a_3)x + (-a_0 + a_1 - a_3)x^2$.

32. $S(a_0 + a_1x + a_2x^2) = \begin{bmatrix} a_0 - a_2 & a_1 + a_2 \\ 2a_0 & a_1 - a_2 \end{bmatrix}$.

33. To see that ψ is one to one let $T : U \to V$ be a linear transformation and assume that T is in $\mathcal{N}(\psi)$; that is $\psi(T)$ is the $(m \times n)$ zero matrix.

    Let $B = \{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}\}$ and $C = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}\}$ be the given bases for U and V, respectively. By assumption, $T(\mathbf{u_j}) = \sum_{\mathrm{i=1}}^{\mathrm{m}} 0\mathbf{v_i} = \theta_V$ for each vector $\mathbf{u_j}$ in $B, 1 \le j \le n$. It follows that $T(\mathbf{u}) = \theta_V$ for each $\mathbf{u}$ in U. Therefore $T = T_0$, where T_0 is the zero transformation from U to V (cf. Exercise 29). But T_0 is the zero vector in the vector space $L(U, V)$, and $\mathcal{N}(\psi) = \{T_0\}$. This proves that ψ is one to one.

5.10 Change of Basis and Diagonalization

1. $T(\mathbf{u_1}) = \mathbf{u_1}$ and $T(\mathbf{u_2}) = 3\mathbf{u_2}$. Therefore $\mathbf{u_1}$ and $\mathbf{u_2}$ are eigenvectors for T corresponding to the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 3$, respectively. The matrix of T with respect to C is $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$.

2. The matrix for T with respect to C is $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

3. $T(A_1) = 2A_1, T(A_2) = -2A_2, T(A_3) = 3A_3,$ and $T(A_4) = -3A_4$.

Therefore A_1, A_2, A_3 and A_4 are eigenvectors for T corresponding to the eigenvalues $\lambda_1 = 2, \lambda_2 = -2, \lambda_3 = 3,$ and $\lambda_4 = -3,$ respectively. The matrix for T with respect to C is given by $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$.

4. The transition matrix is the matrix $P = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$. Note that $P\mathbf{a} = [-1, 3]^{\mathrm{T}} = [\mathbf{a}]_{\mathbf{C}}$. Therefore $\mathbf{a} = -\mathbf{u_1} + 3\mathbf{u_2}$. Similarly $\mathbf{b} = \mathbf{u_1} - \mathbf{u_2}$, $\mathbf{c} = -2\mathbf{u_1} + 7\mathbf{u_2}$, and $\mathbf{d} = (-a/2 + b/2)\mathbf{u_1}$ $+ (a/2 + b/2)\mathbf{u_2}$.

5. The transition matrix is the matrix $P = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$.

Now $[p(x)]_B = [2, 1, 0]^{\mathrm{T}}$ and $P[p(x)]_B = [1, 1, 0]^{\mathrm{T}} = [p(x)]_C$. Denote the polynomials in C by $g_1(x), g_2(x), g_3(x),$ respectively. It follows that $p(x) = g_1(x) + g_2(x)$. Similarly $s(x) = -2g_1(x) - g_2(x) + 2g_3(x), q(x) = -5g_1(x) - 3g_2(x) + 7g_3(x),$ and $r(x) =$

$(a_0 - a_1 - a_2)g_1(x) + (a_0 - a_1)g_2(x) + (-a_0 + 2a_1 + a_2)g_3(x)$.

6. The transition matrix is $P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$. $A = 4A_1 + 3A_2 - A_3 - 3A_4; B = 3A_1 + A_3 - 4A_4; C = dA_1 + cA_2 + (b - c)A_3 + (a - d)A_4$.

7. Since $\mathbf{u_1} = (1/3)\mathbf{w_1} + (1/3)\mathbf{w_2}$ and $\mathbf{u_2} = (5/3)\mathbf{w_1} - (1/3)\mathbf{w_2}$, the transition matrix is $P = \begin{bmatrix} 1/3 & 5/3 \\ 1/3 & -1/3 \end{bmatrix}$.

8. $P = \begin{bmatrix} 5/3 & 2/3 \\ -4/3 & -1/3 \end{bmatrix}$.

9. The transition matrix is $P = \begin{bmatrix} -1 & 1 & 2 & 3 \\ 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Since $[p(x)]_B = [2, -7, 1, 0]^{\mathrm{T}}, P[p(x)]_B = [-7, 2, 1, 0]^{\mathrm{T}} = [p(x)]_C$. Let the polynomials in C be denoted by $g_1(x), g_2(x), g_3(x),$ and $g_4(x),$ respectively. It follows that $p(x) = -7g_1(x) + 2g_2(x) + g_3(x)$. Similarly $q(x) = 13g_1(x) - 4g_2(x) + g_4(x)$ and $r(x) = -7g_1(x) + 3g_2(x) - 2g_3(x) + g_4(x)$.

10. The transition matrix is $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. $p(x) = -3 + 6x + x(x-1); q(x) = 8 - 4x + 2x(x-1); r(x) = -5 + x + x(x-1)$.

11. Note that $T(\mathbf{e_1}) = [2,1]^{\mathrm{T}} = 2\mathbf{e_1} + \mathbf{e_2}$ and $T(\mathbf{e_2}) = [1,2]^{\mathrm{T}} = \mathbf{e_1} + 2\mathbf{e_2}$. Therefore the matrix of T with respect to B is the matrix $Q_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. The transition matrix from B to C is the matrix $P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$. By Theorem 24 the matrix of T with respect to C is the matrix Q_2 given by $Q_2 = P^{-1}Q_1P = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$.

12. The matrix for T with respect to B is $Q_1 = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ -1 & 1 & 2 \end{bmatrix}$. The transition matrix from C to B is $P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$. By Theorem 24 the matrix of T with respect to C is the matrix $Q_2 = P^{-1}Q_1P = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

13. The matrix of T with respect to B is $Q_1 = \begin{bmatrix} -3 & 0 & 0 & 5 \\ 0 & 3 & -5 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$. The transition matrix from C to B is $P = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$. By Theorem 24 the matrix of T with respect to C is the matrix $Q_2 = P^{-1}Q_1P = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$.

14. (a) $Q = \begin{bmatrix} 4 & 3 \\ -2 & -3 \end{bmatrix}$.

(b) If $S = \begin{bmatrix} -3 & 1 \\ 1 & -2 \end{bmatrix}$ then $S^{-1}QS = R$ where $R = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$.

(c) $C = \{-3 + x, 1 - 2x\}$.

(d) $P = \begin{bmatrix} -2/5 & -1/5 \\ -1/5 & -3/5 \end{bmatrix}$.

(e) $[\mathbf{w_1}]_B = [2,3]^{\mathrm{T}}$ so $[\mathbf{w_1}]_C = P[\mathbf{w_1}]_B = [-7/5, -11/5]^{\mathrm{T}}$. Therefore $[T(\mathbf{w_1})]_C = R[\mathbf{w_1}]_C = [-21/5, 22/5]^{\mathrm{T}}$. It follows that $T(\mathbf{w_1}) = (-21/5)[-3+x] + (22/5)[1-2x] = 17 - 13x$. Similarly $[T(\mathbf{w_2})]_C = [3/5, 4/5]^{\mathrm{T}}$ so $T(\mathbf{w_2}) = (3/5)[-3+x] + (4/5)[1-2x] = -1 - x$. Finally, $[T(\mathbf{w_3})]_C = [-3/5, 6/5]^{\mathrm{T}}$ so $T(\mathbf{w_3}) = (-3/5)[-3+x] + (6/5)[1-2x] = 3 - 3x$.

15. (a) $T(1) = 1, T(x) = 1 + 2x,$ and $T(x^2) = 4x + 3x^2$.

Therefore $Q = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}$.

(b) Q has characteristic polynomial $p(t) = -(t-1)(t-2)(t-3)$.

Therefore Q has eigenvalues $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$. The corresponding eigenvectors are $\mathbf{u_1} = [1,0,0]^{\mathrm{T}}, \mathbf{u_2} = [1,1,0]^{\mathrm{T}}$ and $\mathbf{u_3} = [2,4,1]^{\mathrm{T}}$, respectively. If $S = [\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}]$ then $S^{-1}QS = R$

where $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

(c) $C = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ where $[\mathbf{v_i}]_B = \mathbf{u_i}$, the i^{th} column of S. Thus $[\mathbf{v_1}]_B = [1,0,0]^{\mathrm{T}}$ so $\mathbf{v_1} = 1$; $[\mathbf{v_2}]_B = [1,1,0]^{\mathrm{T}}$ so $\mathbf{v_2} = 1 + x$; $[\mathbf{v_3}]_B = [2,4,1]^{\mathrm{T}}$ so $\mathbf{v_3} = 2 + 4x + x^2$.

(d) $1 = \mathbf{v_1}$, $x = -\mathbf{v_1} + \mathbf{v_2}$, and $x^2 = 2\mathbf{v_1} - 4\mathbf{v_2} + \mathbf{v_3}$ so the transition matrix is $P = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$.

(e) $[\mathbf{w_1}]_B = [-8,7,1]^{\mathrm{T}}$ and $[\mathbf{w_1}]_C = P[\mathbf{w_1}]_B = [-13,3,1]^{\mathrm{T}}$. Therefore $[T(\mathbf{w_1})]_C = R[\mathbf{w_1}]_C = [-13,6,3]^{\mathrm{T}}$. It follows that $T(\mathbf{w_1}) = -13\mathbf{v_1} + 6\mathbf{v_2} + 3\mathbf{v_3} = -1 + 18x + 3x^2$. Similarly $[T(\mathbf{w_2})]_C = [7,-8,3]^{\mathrm{T}}$ so $T(\mathbf{w_2}) = 7\mathbf{v_1} - 8\mathbf{v_2} + 3\mathbf{v_3} = 5 + 4x + 3x^2$.

Finally, $[T(\mathbf{w_3})]_{\mathbf{C}} = [11,-22,6]^{\mathrm{T}}$ so $T(\mathbf{w_3}) = 11\mathbf{v_1} - 22\mathbf{v_2} + 6\mathbf{v_3} = 1 + 2x + 6x^2$.

16. (a) $Q = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 0 & 10 \end{bmatrix}$.

(b) If $S = \begin{bmatrix} 1 & -1 & 1 & -3 \\ 0 & 1 & -4 & 27 \\ 0 & 0 & 6 & -108 \\ 0 & 0 & 0 & 180 \end{bmatrix}$ then $S^{-1}QS = R$ where $R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix}$.

(c) $C = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -4 \\ 6 & 0 \end{bmatrix}, \right.$
$\left. \begin{bmatrix} -3 & 27 \\ -108 & 180 \end{bmatrix} \right\}.$

(d) $P = \begin{bmatrix} 1 & 1 & 1/2 & 1/6 \\ 0 & 1 & 2/3 & 1/4 \\ 0 & 0 & 1/6 & 1/10 \\ 0 & 0 & 0 & 1/180 \end{bmatrix}.$

(e) $T(\mathbf{w_1}) = \begin{bmatrix} -3 & 6 \\ -3 & 10 \end{bmatrix}$; $T(\mathbf{w_2}) = \begin{bmatrix} 5 & -8 \\ 5 & 0 \end{bmatrix}$;
$T(\mathbf{w_3}) = \begin{bmatrix} 15 & -14 \\ -6 & 20 \end{bmatrix}.$

17. Let Q be the matrix of I_V with respect to C and B. Since P is the matrix of I_V with respect to B and C it follows from Theorem 20 in Section 4.9 that PQ is the matrix of $I_V \circ I_V = I_V$ with respect to C. Thus $PQ = I$. Similarly, QP is the matix of I_V with respect to B, so $QP = I$. Therefore $Q = P^{-1}$ and P is nonsingular.

18. (a) By assumption $T(\mathbf{v}) = \lambda\mathbf{v}$. Therefore $Q[\mathbf{v}]_B = [T(\mathbf{v})]_B = [\lambda\mathbf{v}]_B = \lambda[\mathbf{v}]_B$.

(b) By assumption $Q\mathbf{x} = \lambda\mathbf{x}$ and $\mathbf{x} = [\mathbf{v}]_B$. Thus $Q\mathbf{x} = Q[\mathbf{v}]_B = [T(\mathbf{v})]_B$. It follows that $[T(\mathbf{v})]_B = \lambda\mathbf{x} = \lambda[\mathbf{v}]_B = [\lambda\mathbf{v}]_B$.
Therefore $T(\mathbf{v}) = \lambda\mathbf{v}$.

19. Let $\mathbf{v}$ be an eigenvector for T corresponding to λ. Then $T^2(\mathbf{v}) = T(T(\mathbf{v})) = T(\lambda(\mathbf{v}) = \lambda T(\mathbf{v}) = \lambda(\lambda\mathbf{v}) = \lambda^2\mathbf{v}$.

20. Suppose T is one to one and let $\mathbf{v}$ be a vector in V such that $T(\mathbf{v}) = 0\mathbf{v}$. Then $T(\mathbf{v}) = \boldsymbol{\theta}$ so $\mathbf{v}$ is in $\mathcal{N}(T)$. But $\mathcal{N}(T) = \{\boldsymbol{\theta}\}$ so $\mathbf{v} = \boldsymbol{\theta}$. Therefore 0 is not an eigenvalue for T (since eigenvectors must be nonzero). Next assume that 0 is not an eigenvalue for T and let $\mathbf{u}$ be in $\mathcal{N}(T)$. Then $T(\mathbf{u}) = \boldsymbol{\theta} = 0\mathbf{u}$. Since 0 is not an eigenvalue, it must be the case that $\mathbf{u} = \boldsymbol{\theta}$. Therefore $\mathcal{N}(T) = \{\boldsymbol{\theta}\}$ and T is one to one.

21. Suppose $\mathbf{v}$ is an eigenvector for T corresponding to λ. Thus $T(\mathbf{v}) = \lambda\mathbf{v}$ and $T^{-1}(\lambda\mathbf{v}) = \mathbf{v}$. But $T^{-1}(\lambda\mathbf{v}) = \lambda T^{-1}(\mathbf{v})$ so it follows that $T^{-1}(\mathbf{v}) = \lambda^{-1}\mathbf{v}$.

5.11 Supplementary Exercises

1. V is not a vector space. For example, if $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ then $1A \neq A$.

3. (a) For arbitrary c, $-2cA_1 - 3cA_2 + cA_3 = \mathcal{O}$. In particular, with $c = 1$, $A_3 = 2A_1 + 3A_2$.
   (b) As in (a), $p_3(x) = 2p_1(x) + 3p_2(x)$
   (c) $\mathbf{v}_3 = 2\mathbf{v}_1 + 3\mathbf{v}_2$.

4. (a) $B = \left\{ \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -3 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is one basis for W.
   (b) Set $A = 2\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -3 & 0 \\ 1 & 0 \end{bmatrix} - 2\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & -2 \end{bmatrix}$.

5. (a) $\text{Sp}\,(S) = \{a + bx + cx^2 : 7a - 3b - 5c = 0\}$.
   (b) $q_1(x)$, $q_3(x)$, $q_4(x)$ are in $\text{Sp}\,(S)$.
   (c) A polynomial $p(x) = a + bx + cx^2$ is in $\text{Sp}\,(S)$ if and only if $c = (7/5)a - (3/5)b$. Thus, $p(x) = a[1 + (7/5)x^2] + b[x - (3/5)x^2]$. The set $B = \{1 + (7/5)x^2, x - (3/5)x^2\}$ is one choice of a basis for $\text{Sp}\,(S)$. Moreover, for this choice of B, $[p(x)]_B = [a,\ b]^T$.
   (d) $[q_1(x)]_B = [5,\ 5]^T$; $[q_3(x)]_B = [0,\ -5]^T$; $[q_4(x)]_B = [5,\ 0]^T$

6. (a) The dependence relation $x_1A_1 + x_2A_2 + x_3A_3 + x_4A_4 + x_5A_5 = \mathcal{O}$ has solution $x_1 = -x_3 - 2x_5$, $x_2 = x_3 - 3x_5$, $x_4 = -4x_5$, x_3 and x_5 arbitrary. $\{A_1, A_2, A_4\}$ is a basis for $\text{Sp}\,(S)$. Setting $x_3 = 1$ and $x_5 = 0$ yields $-A_1 + A_2 + A_3 = \mathcal{O}$, so $A_3 = A_1 - A_2$. Setting, $x_3 = 0$ and $x_5 = 1$ gives $-2A_1 - 3A_2 - 4A_4 + A_5 = \mathcal{O}$, so $A_5 = 2A_1 + 3A_2 + 4A_4$.
   (b) Using the same calculations as in (a), $\{p_1(x), p_2(x), p_4(x)\}$ is a basis for $\text{Sp}\,(S)$, $p_3(x) = p_1(x) - p_2(x)$, and $p_5(x) = 2p_1(x) + 3p_2(x) + 4p_4(x)$.
   (c) $\{f_1(x), f_2(x), f_3(x)\}$ is a basis for $\text{Sp}\,(S)$, $f_3(x) = f_1(x) - f_2(x)$ and $f_5(x) = 2f_1(x) + 3f_2(x) + 4f_4(x)$.

7. $\left\{ \begin{bmatrix} 1 & 0 & 2 \\ 3 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \right\}$

8. $\{p_1(x), p_2(x), p_5(x)\}$

9. A polynomial $p(x) = a + bx + cx^2 + dx^3$ is in $\text{Sp}\,(S)$ if and only if $a - 3b - c + d = 0$ and in this case $q(x) = (4a - 3b - 2c)p_1(x) + (-3a + 3b + c)p_2(x) + (-2a + 2b + c)p_5(x)$. Therefore, $q(x)$ is in $\text{Sp}\,(S)$ and $q(x) = 2p_2(x) + p_5(x)$.

10. $\text{Sp}\,(S) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a - 3b - c + d = 0 \right\}$. $\left\{ \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \right\}$, is a basis for $\text{Sp}\,(S)$.

11. (a) The matrix of T is A.

(b) $\operatorname{rank}(T) = 3$ and $\operatorname{nullity}(T) = 3$.

(c) $\mathcal{R}(T) = \{p(x) = a + bx + cx^2 + dx^3 : a - 3b - c + d = 0\}$.
The set $\{3 + x, 1 + x^2, -1 + x^3\}$ is a bais for $\mathcal{R}(T)$. (cf Exercise 10).

(d) If $B = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ then $T(B) = q(x)$ (cf. Exercise 9).

(e) $T\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\right) = \boldsymbol{\theta}(x)$ if and only if $a_{11} = -2a_{13} - 3a_{21} + a_{23}$, $a_{12} = a_{13} - 2a_{21} - 3a_{23}$, $a_{22} = -2a_{23}$, a_{13}, a_{21}, a_{23} arbitrary.
Therefore, $\left\{\begin{bmatrix} -2 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -3 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \end{bmatrix}\right\}$ is a basis for $\mathcal{N}(T)$.

12. $\begin{bmatrix} a \\ b \end{bmatrix} = (b + a)\begin{bmatrix} 0 \\ 1 \end{bmatrix} - a\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, so $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = (b + a)T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) - aT\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = (b + a)(1 + 2x + x^2) - a(2 - x) = (b - a) + (3a + 2b)x + (a + b)x^2$.

13. If $T(1) = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ then $T(a + bx + cx^2) = aT(1) + bT(x) + cT(x^2) = \begin{bmatrix} au_1 + b \\ bu_2 + c \end{bmatrix}$. In particular, $T(a + bx + cx^2) = \begin{bmatrix} b \\ c \end{bmatrix}$ is one such linear transformation.

14. (a) $\begin{bmatrix} 1 & -1 & 1 & -4 \\ 0 & 1 & 1 & 3 \\ 1 & 0 & 2 & -1 \end{bmatrix}$

(b) $\mathcal{R}(T) = \{u + vx + wx^2 : u + v - w = 0\}$. Moreover, if $q(x) = u + vx + wx^2$ is in $\mathcal{R}(T)$ then $T\left(\begin{bmatrix} u + v - 2c + d & v - c - 3d \\ c & d \end{bmatrix}\right) = q(x)$ for arbitrary c and d. If $q_1(x) = 1 + x^2$ and $q_2(x) = x + x^2$, then $S = \{q_1(x), q_2(x)\}$ is a basis for $\mathcal{R}(T)$.

(c) If $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ then $T(A_1) = q_1(x)$ and $T(A_2) = q_2(x)$.

(d) $\mathcal{N}(T) = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a = -2c + d \text{ and } b = -c - 3d, c \text{ and } d \text{ arbitrary}\right\}$. Therefore, if $A_3 = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix}$ and $A_4 = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$ then $B_2 = \{A_3, A_4\}$ is a basis for $\mathcal{N}(T)$.

5.12 Conceptual Exercises

1. True. $\mathbf{u} = a^{-1}(a\mathbf{u}) = a^{-1}(a\mathbf{v}) = \mathbf{v}$.

2. True. $(a-b)\mathbf{v} = \boldsymbol{\theta}$ and $\mathbf{v} \neq \boldsymbol{\theta}$, so $a - b = 0$.

3. False. Each vector $\mathbf{v}$ in V has a unique inverse $-\mathbf{v}$ in V, but as $\mathbf{v}$ varies, so does $-\mathbf{v}$.

4. False. If $n = 1$ then $p(x) = 1 - x$ and $q(x) = 1 + x$ are in V but $p(x) + q(x)$ is not in V.

5. True. Every basis for W is also a basis for V.

6. True. If $\dim(W) = k$ then a basis B for W is a linearly independent susbset of V containing k vectors. Therefore, $k \leq n$.

7. True. $a\boldsymbol{\theta} = \boldsymbol{\theta}$ for every nonzero scalar a.

8. True.

9. False. In R^2 let $S_1 = \{[1,0]^T, [0,1]^T\}$ and $S_2\{[1,0]^T, [0,1]^T, [1,1]^T\}$.

10. True. Since $V = \mathrm{Sp}\,(S_1)$, $\dim(V) \leq k$. Since S_2 is a linearly independent subset of V, $l \leq \dim(V)$.

11. $\mathbf{u} = (1/2)(\mathbf{u}+\mathbf{v}) + (1/2)(\mathbf{u}-\mathbf{v})$ and $\mathbf{v} = (1/2)(\mathbf{u}+\mathbf{v}) - (1/2)(\mathbf{u}-\mathbf{v})$

Chapter 6

Determinants

6.1 Introduction (No exercises)

6.2 Cofactor Expansion of Determinants

1. $\det(A) = 1(1) - 3(2) = -5.$

2. $\det(A) = -31.$

3. $\det(A) = 2(8) - 4(4) = 0; \mathbf{x} = a \begin{bmatrix} -2 \\ 1 \end{bmatrix}, a \neq 0.$

4. $\det(A) = 2.$

5. $\det(A) = 4(7) - 3(1) = 25.$

6. $\det(A) = 3.$

7. $\det(A) = 4(1) - 1(-2) = 6.$

8. $\det(A) = 0; \mathbf{x} = a \begin{bmatrix} -3 \\ 1 \end{bmatrix}, a \neq 0.$

9. $A_{11} = (-1)^2 \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} = -2; A_{12} = (-1)^3 \begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix} = 6;$

   $A_{13} = (-1)^4 \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} = -2; A_{33} = (-1)^6 \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1.$

10. $A_{11} = -2; A_{12} = 4; A_{13} = 1, A_{33} = -4.$

11. $A_{11} = \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} = -2; A_{12} = -\begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} = 7;$

$A_{13} = \begin{vmatrix} -1 & 2 \\ 3 & 2 \end{vmatrix} = -8; A_{33} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3.$

12. $A_{11} = -1; A_{12} = 3; A_{13} = -1; A_{33} = 0.$

13. $A_{11} = \begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} = 3; A_{12} = -\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = -6;$

$A_{13} = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2; A_{33} = \begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} = -3.$

14. $A_{11} = 6; A_{12} = -8; A_{13} = 0; A_{33} = 4.$

15. $\det(A) = A_{11} + 2A_{12} + A_{13} = -2 + 2(6) + (-2) = 8.$

16. $\det(A) = 14.$

17. $\det(A) = 2A_{11} - A_{12} + 3A_{13} = 2(-2) - 7 + 3(-8) = -35.$

18. $\det(A) = 1.$

19. $\det(A) = -A_{11} + A_{12} - A_{13} = -3 - 6 - 2 = -11.$

20. $\det(A) = 8.$

21. $\det(A) = 2\begin{vmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 0 & 1 \\ 2 & 2 & 0 \\ 3 & 1 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 1 & 2 \end{vmatrix} +$

$2(-1)\begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & 2 \\ 3 & 1 & 1 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} + (-1)\left[3\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} \right] +$

$(-1)\left[3\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} \right] + (-2)(3)\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -9.$

22. $\det(A) = 2.$

23. $\det(A) = 2\begin{vmatrix} 3 & 1 & 2 \\ 1 & 2 & 1 \\ 3 & 1 & 4 \end{vmatrix} + 2\begin{vmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 3 & 4 \end{vmatrix} =$

$2\left[3\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \right] +$

$2\left[\begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix} - 3\begin{vmatrix} 0 & 1 \\ 0 & 4 \end{vmatrix} + 2\begin{vmatrix} 0 & 1 \\ 0 & 3 \end{vmatrix} \right] = 22.$

24. $\det(A) = 4.$

25. $\det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = (1)(10) + (3)(5) + (2)(-10) = 5$; $a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} = (-1)(-5) + (4)(-1) + (1)(4) = 5$; $a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} = (2)(-5) + (2)(-3) + (3)(7) = 5$.

26. $\det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = (2)(-7) + (4)(0) + (1)(7) = -7$; $a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} = (3)(-5) + (1)(2) + (3)(2) = -7$; $a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} = (2)(11) + (3)(-3) + (2)(-10) = -7$.

27. $a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = (1)(-5) + (3)(-1) + (2)(4) = 0$; $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = (1)(-5) + (3)(-1) + (2)(7) = 0$.

28. $a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = (2)(-5) + (4)(2) + (1)(2) = 0$; $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = (2)(11) + (4)(-3) + (1)(-10) = 0$.

29. $C = \begin{bmatrix} 10 & 5 & -10 \\ -5 & -1 & 4 \\ -5 & -3 & 7 \end{bmatrix}$, $C^TA = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = [\det(A)]I$.
So $A^{-1} = (1/5)C^T = [1/\det(A)]C^T$.

30. $C = \begin{bmatrix} -7 & 0 & 7 \\ -5 & 2 & 2 \\ 11 & -3 & -10 \end{bmatrix}$, $C^TA = \begin{bmatrix} -7 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -7 \end{bmatrix} = [\det(A)]I$. So $A^{-1} = -(1/7)C^T = [1/\det(A)]C^T$.

31. $\det(A) = -a_{12}\begin{vmatrix} 0 & a_{23} \\ 0 & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} 0 & a_{22} \\ 0 & a_{32} \end{vmatrix} = 0.$

32. $\det(U) = u_{11}\begin{vmatrix} u_{22} & u_{23} & u_{24} \\ 0 & u_{33} & a_{34} \\ 0 & 0 & u_{44} \end{vmatrix} - u_{12}\begin{vmatrix} 0 & u_{23} & u_{24} \\ 0 & u_{33} & u_{34} \\ 0 & 0 & u_{44} \end{vmatrix} +$

$u_{13}\begin{vmatrix} 0 & u_{22} & u_{24} \\ 0 & 0 & u_{34} \\ 0 & 0 & u_{44} \end{vmatrix} - u_{14}\begin{vmatrix} 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \\ 0 & 0 & 0 \end{vmatrix} =$

$u_{11}\begin{vmatrix} u_{22} & u_{23} & u_{24} \\ 0 & u_{33} & u_{34} \\ 0 & 0 & u_{44} \end{vmatrix} = u_{11}u_{22}\begin{vmatrix} u_{33} & u_{34} \\ 0 & u_{44} \end{vmatrix} = u_{11}u_{22}u_{33}u_{44}.$

33. $A^{\mathrm{T}} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$ so $\det(A^{\mathrm{T}}) = a_{11}a_{22} - a_{21}a_{12} = \det(A)$.

34. (a) If A is positive definite then $0 < \mathbf{e_1}^{\mathrm{T}}A\mathbf{e_1} = a_{11}$. If $\mathbf{x} = [u, v]^{\mathrm{T}}$ and $\mathbf{x} \neq \theta$ then $0 < \mathbf{x}^{\mathrm{T}}\mathbf{Ax} = a_{11}u^2 + 2a_{12}uv + a_{22}v^2$ (since A is symmetric $a_{12} = a_{21}$). In particular if $u = a_{12}$ and $v = -a_{11}$ Then $0 < a_{11}a_{12}^2 - 2a_{12}^2a_{11} + a_{22}a_{11}^2 = a_{11}(a_{11}a_{22} - a_{12}^2) = a_{11}\det(A)$. It follows that $\det(A) > 0$.

(b) Suppose $a_{11} > 0$ and $\det(A) > 0$. For $\mathbf{x} = [u, v]^{\mathrm{T}}$ we have $a_{11}(\mathbf{x}^{\mathrm{T}}A\mathbf{x}) = a_{11}^2u^2 + 2a_{11}a_{12}uv + a_{11}a_{22}v^2 = (a_{11}u + a_{12}v)^2 + v^2(a_{11}a_{22} - a_{12}^2) = (a_{11}u + a_{12}v)^2 + v^2\det(A)$. For $\mathbf{x} \neq \theta$ it follows that $\mathbf{x}^{\mathrm{T}}A\mathbf{x} > \mathbf{0}$.

35. (a) For $n = 3$ and $n = 4, H(n) = n!/2$. For some integer $k \geq 4$ suppose we have seen that $H(k) = k!/2$. If A is a $((k+1)$ x $(k+1))$ matrix then $\det(A)$ can be obtained by evaluating $k+1$ $(k$ x $k)$ determinants. Thus the number of (2 x 2) determinants in the expansion of $\det(A)$ is $(k+1)H(k) = (k+1)!/2$. It follows by induction that $H(n) = n!/2$ for every positive integer $n, n \geq 2$.

(b) Note that evaluating a single (2 x 2) determinant requires 3 operations, two multiplications and one subtraction.

n	$H(n)$	Time required
2	1	3 seconds
5	60	3 minutes
10	1,814,400	1512 hours.

6.3 Elementary Operations and Determinants

1. $$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 1 \end{vmatrix} \begin{matrix} C_2 - 2C_1 \\ C_3 - C_1 \\ = \end{matrix} \begin{vmatrix} 1 & 0 & 0 \\ 2 & -4 & -1 \\ 1 & -3 & 0 \end{vmatrix} = \begin{vmatrix} -4 & -1 \\ -3 & 0 \end{vmatrix} = -3.$$

2. $$\begin{vmatrix} 2 & 4 & -2 \\ 0 & 2 & 3 \\ 1 & 1 & 2 \end{vmatrix} \begin{matrix} C_2 - 2C_1 \\ C_3 + C_1 \\ = \end{matrix} \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 3 \\ 1 & -1 & 3 \end{vmatrix} = 2\begin{vmatrix} 2 & 3 \\ -1 & 3 \end{vmatrix} = 18.$$

3. $$\begin{vmatrix} 0 & 1 & 2 \\ 3 & 1 & 2 \\ 2 & 0 & 3 \end{vmatrix} \begin{matrix} C_1 \leftrightarrow C_2 \\ = \end{matrix} -\begin{vmatrix} 1 & 0 & 2 \\ 1 & 3 & 2 \\ 0 & 2 & 3 \end{vmatrix} \begin{matrix} C_3 - 2C_1 \\ = \end{matrix} -\begin{vmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 2 & 3 \end{vmatrix} =$$
$$-\begin{vmatrix} 3 & 0 \\ 2 & 3 \end{vmatrix} = -9.$$

4. $$\begin{vmatrix} 2 & 2 & 4 \\ 1 & 0 & 1 \\ 2 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 1 & -1 & -1 \\ 2 & -1 & -2 \end{vmatrix} = 2.$$

5. $$\begin{vmatrix} 0 & 1 & 3 \\ 2 & 1 & 2 \\ 1 & 1 & 2 \end{vmatrix} \begin{matrix} C_1 \leftrightarrow C_2 \\ = \end{matrix} -\begin{vmatrix} 1 & 0 & 3 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{vmatrix} \begin{matrix} C_3 - 3C_1 \\ = \end{matrix}$$

$$-\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 1 & -1 \end{vmatrix} = -\begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} = 1.$$

6. $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 0 & 2 \end{vmatrix} \begin{matrix} C_2 - C_1 \\ C_3 - C_1 \\ = \end{matrix} \begin{vmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 3 & -3 & -1 \end{vmatrix} = 1.$

7. $\det(B) = -2\det(A) = -6.$

8. $\det(B) = -6\det(A) = -18.$

9. $\det(B) = \det(A) = 3.$

10. $\det(B) = 2\det(A) = 6.$

11. $\det(B) = \det(A) = 3.$

12. $\det(B) = 4\det(A) = 12.$

13. $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 3 \\ 1 & 1 & 0 & 1 \\ 1 & 4 & 2 & 2 \end{vmatrix} \begin{matrix} C_2 \leftrightarrow C_4 \\ = \end{matrix} -\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 2 & 2 & 4 \end{vmatrix} \begin{matrix} C_3 \leftrightarrow C_4 \\ = \end{matrix}$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 4 & 2 \end{vmatrix} = 6.$$

14. $\begin{vmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 4 & 1 & 3 \\ 2 & 1 & 5 & 6 \end{vmatrix} = -\begin{vmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 3 & 4 & 0 \\ 5 & 6 & 1 & 2 \end{vmatrix} = -48.$

15. $\begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 3 \\ 2 & 1 & 0 & 6 \\ 3 & 2 & 2 & 4 \end{vmatrix} \begin{matrix} C_1 \leftrightarrow C_2 \\ C_3 \leftrightarrow C_4 \\ = \end{matrix} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 1 & 2 & 6 & 0 \\ 2 & 3 & 4 & 2 \end{vmatrix} \begin{matrix} C_2 \leftrightarrow C_3 \\ = \end{matrix}$

$$-\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 6 & 2 & 0 \\ 2 & 4 & 3 & 2 \end{vmatrix} = -12.$$

16. $$\begin{vmatrix} 1 & 2 & 0 & 3 \\ 2 & 5 & 1 & 1 \\ 2 & 0 & 4 & 3 \\ 0 & 1 & 6 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & -4 & 8 & -23 \\ 0 & 1 & 5 & 7 \end{vmatrix} = \begin{vmatrix} 8 & -23 \\ 5 & 7 \end{vmatrix} = 171.$$

17. $$\begin{vmatrix} 2 & 4 & -2 & -2 \\ 1 & 3 & 1 & 2 \\ 1 & 3 & 1 & 3 \\ -1 & 2 & 1 & 2 \end{vmatrix} \begin{matrix} C_2 - 2C_1 \\ C_3 + C_2 \\ C_4 + C_1 \\ = \end{matrix} \begin{vmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 4 \\ -1 & 4 & 0 & 1 \end{vmatrix} \begin{matrix} C_3 - 2C_2 \\ C_4 - 3C_2 \\ = \end{matrix}$$

$$\begin{vmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ -1 & 4 & -8 & -11 \end{vmatrix} = 2\begin{vmatrix} 0 & 1 \\ -8 & -11 \end{vmatrix} = 16.$$

18. $$\begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 4 & 1 \\ 2 & 1 & 3 & 0 \\ 2 & 2 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & -1 & 3 & -1 \\ 2 & 0 & -3 & 0 \end{vmatrix} = -3.$$

19. $$\begin{vmatrix} 1 & 2 & 0 & 3 \\ 2 & 5 & 1 & 1 \\ 2 & 0 & 4 & 3 \\ 0 & 1 & 6 & 2 \end{vmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - 2R_1 \\ = \end{matrix} \begin{vmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & -5 \\ 0 & -4 & 4 & -3 \\ 0 & 1 & 6 & 2 \end{vmatrix} \begin{matrix} R_3 + 4R_4 \\ R_4 - R_2 \\ = \end{matrix}$$

$$\begin{vmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & 8 & -23 \\ 0 & 0 & 5 & 7 \end{vmatrix} = \begin{vmatrix} 8 & -23 \\ 5 & 7 \end{vmatrix} = 171.$$

20. $$\begin{vmatrix} 2 & 4 & -2 & -2 \\ 1 & 3 & 1 & 2 \\ 1 & 3 & 1 & 3 \\ -1 & 2 & 1 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & -1 & -1 \\ 1 & 3 & 1 & 2 \\ 1 & 3 & 1 & 3 \\ -1 & 2 & 1 & 2 \end{vmatrix} =$$

$$2\begin{vmatrix} 1 & 2 & -1 & -1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -8 & -11 \end{vmatrix} = 16.$$

21. $$\begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 4 & 1 \\ 2 & 1 & 3 & 0 \\ 2 & 2 & 1 & 2 \end{vmatrix} \begin{matrix} R_3 - 2R_1 \\ R_4 - 2R_1 \\ = \end{matrix} \begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 4 & 1 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & -3 & 0 \end{vmatrix} \begin{matrix} R_3 + R_2 \\ = \end{matrix}$$

$$\begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & -3 & 0 \end{vmatrix} = \begin{vmatrix} 3 & -1 \\ -3 & 0 \end{vmatrix} = -3.$$

22. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ then $\det(A) = \det(B) = 0$ whereas $\det(A+B) = 1$. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ then $\det(A) = \det(B) = 0$ and $\det(A+B) = 0 = \det(A) + \det(B)$.

23. $$\begin{vmatrix} a+1 & a+4 & a+7 \\ a+2 & a+5 & a+8 \\ a+3 & a+6 & a+9 \end{vmatrix} \left\{ \begin{array}{c} C_3 - C_2 \\ = \\ C_2 - C_1 \end{array} \right\} \begin{vmatrix} a+1 & 3 & 3 \\ a+2 & 3 & 3 \\ a+3 & 3 & 3 \end{vmatrix} = 0;$$
$$\begin{vmatrix} a & 4a & 7a \\ 2a & 5a & 8a \\ 3a & 6a & 9a \end{vmatrix} \left\{ \begin{array}{c} C_3 - C_2 \\ = \\ C_2 - C_1 \end{array} \right\} \begin{vmatrix} a & 3a & 3a \\ 2a & 3a & 3a \\ 3a & 3a & 3a \end{vmatrix} = 0;$$
$$\begin{vmatrix} a & a^4 & a^7 \\ a^2 & a^5 & a^8 \\ a^3 & a^6 & a^9 \end{vmatrix} = (a)(a^4)(a^7) \begin{vmatrix} 1 & 1 & 1 \\ a & a & a \\ a^2 & a^2 & a^2 \end{vmatrix} = 0.$$

24. (a) Set $B = [\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3]$. Then $AB = [A\mathbf{B}_1, A\mathbf{B}_2, A\mathbf{B}_3]$ where $A\mathbf{B}_1 = A \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = 2\mathbf{A}_1 + 3\mathbf{A}_2 + \mathbf{A}_3$, $A\mathbf{B}_2 = A \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} = -\mathbf{A}_2 + 3\mathbf{A}_3$, $A\mathbf{B}_3 = A \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} = 4\mathbf{A}_3$.

(b)

$$\begin{aligned} \det(AB) &= \det[2\mathbf{A}_1 + 3\mathbf{A}_2 + \mathbf{A}_3, -\mathbf{A}_2 + 3\mathbf{A}_3, 4\mathbf{A}_3] \\ (C_3/4): &= (4)\det[2\mathbf{A}_1 + 3\mathbf{A}_2 + \mathbf{A}_3, -\mathbf{A}_2 + 3\mathbf{A}_3, \mathbf{A}_3] \\ (C_2 - 3C_3): &= (4)\det[2\mathbf{A}_1 + 3\mathbf{A}_2 + \mathbf{A}_3, -\mathbf{A}_2, \mathbf{A}_3] \\ (-C_2): &= (-4)\det[2\mathbf{A}_1 + 3\mathbf{A}_2 + \mathbf{A}_3, \mathbf{A}_2, \mathbf{A}_3] \\ (C_1 - C_3): &= (-4)\det[2\mathbf{A}_1 + 3\mathbf{A}_2, \mathbf{A}_2, \mathbf{A}_3] \\ (C_1 - 3C_2): &= (-4)\det[2\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3] \\ (C_1/2): &= (-8)\det[\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3] = (-8)\det(A). \end{aligned}$$

(c) $\det(B) = -8$, so $\det(AB) = \det(A)\det(B)$.

25. It follows from Theorem 3 (with $c = 0$) that if A has a zero column then $\det(A) = 0$. Since the first column of U_{1j} contains only zeros for $2 \leq j \leq n, \det(U_{1j}) = 0$.

26. If U is a (2 x 2) upper triangular matrix then $U = \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$ and $\det(U) = u_{11}u_{22}$. Suppose we have seen that $\det(U) = u_{11}u_{22}\cdots u_{kk}$ for a $(k \,\mathrm{x}\, \mathrm{k})$ upper triangular matrix U. If U is a $((k+1)\,\mathrm{x}\,(\mathrm{k}+1))$ upper triangular matrix then $\det(U) = u_{11}U_{11} + u_{12}U_{12} + \cdots +$ $u_{1,k+1}U_{1,k+1} = u_{11}U_{11}$. But $U_{11} = \begin{vmatrix} u_{22} & u_{23} & \cdots & u_{2,k+1} \\ 0 & u_{33} & \cdots & u_{3,k+1} \\ \vdots & & & \\ 0 & 0 & \cdots & u_{k+1,k+1} \end{vmatrix} =$ $u_{22}u_{33}\cdots u_{k+1,k+1}$ by assumption. Thus $\det(U) =$ $u_{11}u_{22}\cdots u_{k+1,k+1}$. It follows by induction that if $U = (u_{ij})$ is an $(n \,\mathrm{x}\, \mathrm{n})$ upper triangular matrix, $n \geq 2$, then $\det(U) = u_{11}u_{22}\cdots u_{nn}$.

27. First note that $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$ is a linear equation in x and y. It follows from Theorem 5 that $x = x_1, y = y_1$ and $x = x_2, y = y_2$ are solutions. Consequently the equation describes the line through the points (x_1, y_1) and (x_2, y_2).

28. Consider the case represented by the figure below.

Clearly area$(ABC) =$ area$(ADEC) +$ area$(CEFB) -$ area$(ADFB)$. Therefore area$(ABC) = (1/2)(x_3 - x_1)(y_1 + y_3) + (1/2)(x_2 - x_3)(y_2 + y_3) -$ $(1/2)(x_2 - x_1)(y_1 + y_2) =$

$$(1/2)[x_1y_2 - x_1y_3 - x_2y_1 + x_2y_3 + x_3y_1 - x_3y_2] = (1/2)\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

29. Let $\mathbf{x} = [x_1, x_2, x_3]^{\mathrm{T}}$ and $\mathbf{y} = [y_1, y_2, y_3]^{\mathrm{T}}$ and let $B = \mathbf{x}\mathbf{y}^{\mathrm{T}} =$ $[\mathbf{B_1}, \mathbf{B_2}, \mathbf{B_3}]$ where $\mathbf{B_j} = [x_jy_1, x_jy_2, x_jy_3]^{\mathrm{T}}$. Then $A =$ $[\mathbf{B_1} + \mathbf{e_1}, \mathbf{B_2} + \mathbf{e_2}, \mathbf{B_3} + \mathbf{e_3}]^{\mathrm{T}}$. Repeated applications of Theorem 4 yield; $\det(A) =$ $\det[\mathbf{B_1}, \mathbf{B_2}, \mathbf{B_3}] + \det[\mathbf{B_1}, \mathbf{B_2}, \mathbf{e_3}] +$

$\det[\mathbf{B_1}, \mathbf{e_2}, \mathbf{B_3}] + \det[\mathbf{e_1}, \mathbf{B_2}, \mathbf{B_3}] + \det[\mathbf{B_1}, \mathbf{e_2}, \mathbf{e_3}] +$
$\det[\mathbf{e_1}, \mathbf{B_2}, \mathbf{e_3}] + \det[\mathbf{e_1}, \mathbf{e_2}, \mathbf{B_3}] + \det[\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}]$.

Since each $\mathbf{B_j}$, $1 \le j \le 3$, is a scalar multiple of $\mathbf{y}$, Theorem 5 implies that

$\det(A) = \det[\mathbf{B_1}, \mathbf{e_2}, \mathbf{e_3}] + \det[\mathbf{e_1}, \mathbf{B_2}, \mathbf{e_3}] + \det[\mathbf{e_1}, \mathbf{e_2}, \mathbf{B_3}] +$

$\det[\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}] = x_1y_1 + x_2y_2 + x_3y_3 + 1 = 1 + \mathbf{y}^{\mathrm{T}}\mathbf{x}$.

30. $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \begin{matrix} C_3 - aC_2 \\ C_2 - aC_1 \\ = \end{matrix} \begin{vmatrix} 1 & 0 & 0 \\ 1 & b-a & b(b-a) \\ 1 & c-a & c(c-a) \end{vmatrix} =$

$(b-a)(c-a)\begin{vmatrix} 1 & b \\ 1 & c \end{vmatrix} = (b-a)(c-a)(c-b).$

31. $\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} \begin{matrix} C_4 - aC_3 \\ C_3 - aC_2 \\ C_2 - aC_1 \\ = \end{matrix} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & b-a & b(b-a) & b^2(b-a) \\ 1 & c-a & c(c-a) & c^2(c-a) \\ 1 & d-a & d(d-a) & d^2(d-a) \end{vmatrix} =$

$(b-a)(c-a)(d-a)\begin{vmatrix} 1 & b & b^2 \\ 1 & c & c^2 \\ 1 & d & d^2 \end{vmatrix} =$

$(b-a)(c-a)(d-a)(c-b)(d-b)(d-c).$

32. Write $A = [\mathbf{A_1}, \mathbf{A_2}, \ldots, \mathbf{A_n}]$. Then $cA = [c\mathbf{A_1}, c\mathbf{A_2}, \ldots, c\mathbf{A_n}]$.

By Theorem 3, $\det(cA) = c\ \det[\mathbf{A_1}, c\mathbf{A_2}, \ldots, c\mathbf{A_n}] =$

$c^2\ \det[\mathbf{A_1}, \mathbf{A_2}, \ldots, c\mathbf{A_n}] = \cdots = c^n\ \det[\mathbf{A_1}, \mathbf{A_2}, \ldots, \mathbf{A_n}] =$

$c^n\ \det(A)$.

6.4 Cramer's Rule

1. $\begin{vmatrix} 0 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 4 & 1 \end{vmatrix} \begin{matrix} \{C_1 \leftrightarrow C_2\} \\ = \end{matrix} -\begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \\ 4 & 3 & 1 \end{vmatrix} \begin{matrix} \{C_3 - 3C_1\} \\ = \end{matrix}$

$-\begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & -5 \\ 4 & 3 & -11 \end{vmatrix} \begin{matrix} \{C_3 + 5C_2\} \\ = \end{matrix} -\begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 3 & 4 \end{vmatrix} = -4.$

2. $\begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 2 & 1 & 3 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & -3 \end{vmatrix} = 3.$

3. $\begin{vmatrix} 2 & 2 & 4 \\ 1 & 3 & 4 \\ -1 & 2 & 1 \end{vmatrix} \overset{\left\{\begin{array}{c} C_2 - C_1 \\ C_3 - 2C_1 \end{array}\right\}}{=} \begin{vmatrix} 2 & 0 & 0 \\ 1 & 2 & 2 \\ -1 & 3 & 3 \end{vmatrix} \overset{\{C_3 - C_2\}}{=}$

$\begin{vmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ -1 & 3 & 0 \end{vmatrix} = 0.$

4. $\begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 2 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2.$

5. $\begin{vmatrix} 1 & 0 & -2 \\ 3 & 1 & 3 \\ 0 & 1 & 2 \end{vmatrix} \overset{\{C_3 + 2C_1\}}{=} \begin{vmatrix} 1 & 0 & 0 \\ 3 & 1 & 9 \\ 0 & 1 & 2 \end{vmatrix} \overset{\{C_3 - 9C_2\}}{=} \begin{vmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 1 & -7 \end{vmatrix}$

$\overset{\{(-1/7)C_3\}}{=} -7 \begin{vmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} \overset{\{C_2 - C_3\}}{=}$

$-7 \begin{vmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \overset{\{C_1 - 3C_2\}}{=} -7 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -7.$

6. A is singular.

7. (a) $\det(AB) = \det(A)\det(B) = (2)(3) = 6$ (b) $\det(AB^2) = \det(A)[\det(B)]^2 = (2)(9) = 18$ (c) $\det(A^{-1}B) = \det(B)/\det(A) = 3/2$ (d) $\det(2A^{-1}) = 8\det(A^{-1}) = 8/\det(A) = 4$ (e) $\det{(2A)}^{-1} = \det((1/2)A^{-1}) = (1/8)\det(A^{-1}) = 1/[8\det(A)] = 1/16$.

8. Both matrices have determinant $\sin^2\theta + \cos^2\theta = 1$, so the matrices are nonsingular for all values of θ.

9. $\det(B(\lambda)) = 2\lambda - \lambda^2 = \lambda(2-\lambda); B(\lambda)$ is singular provided $\lambda = 0$ or $\lambda = 2$.

10. $\det(B(\lambda)) = \lambda^2 - 1; B(\lambda)$ is singular for $\lambda = \pm 1$.

11. $\det(B(\lambda)) = 4 - \lambda^2; B(\lambda)$ is singular for $\lambda = \pm 2$.

12. $\det(B(\lambda)) = 2(1-\lambda)(3-\lambda); B(\lambda)$ is singular provided $\lambda = 1$ or $\lambda = 3$.

13. $\det(B(\lambda)) = (\lambda - 1)^2(\lambda + 2); B(\lambda)$ is singular provided $\lambda = 1$ or $\lambda = -2$.

14. $\det(B(\lambda)) = \lambda(\lambda - 3)(\lambda + 1); B(\lambda)$ is singular provided $\lambda = 0, \lambda = 3,$ or $\lambda = -1$.

15. $\det(A) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2; \det(B_1) = \begin{vmatrix} 3 & 1 \\ -1 & -1 \end{vmatrix} = -2;$

$\det(B_2) = \begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} = -4.$

$x_1 = \det(B_1)/\det(A) = 1; x_2 = \det(B_2)/\det(A) = 2.$

16. $x_1 = x_2 = 1.$

17. $\det(A) = \begin{vmatrix} 1 & -2 & 1 \\ 1 & 0 & 1 \\ 1 & -2 & 0 \end{vmatrix} = -2; \det(B_1) = \begin{vmatrix} -1 & -2 & 1 \\ 3 & 0 & 1 \\ 0 & -2 & 0 \end{vmatrix} = -8;$

$\det(B_2) = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 3 & 1 \\ 1 & 0 & 0 \end{vmatrix} = -4; \det(B_3) = \begin{vmatrix} 1 & -2 & -1 \\ 1 & 0 & 3 \\ 1 & -2 & 0 \end{vmatrix} = 2.$

$x_1 = \det(B_1)/\det(A) = 4; x_2 = \det(B_2)/\det(A) = 2;$

$x_3 = \det(B_3)/\det(A) = -1.$

18. $x_1 = -1, x_2 = 0, x_3 = 3.$

19. $\det(A) = \begin{vmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 2 \end{vmatrix} = 3; \det(B_1) = \begin{vmatrix} 2 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 3 & 0 & 1 & 2 \end{vmatrix} = 3;$

$\det(B_2) \begin{vmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 3 & 1 & 2 \end{vmatrix} = 3; \det(B_3) = \begin{vmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 3 & 2 \end{vmatrix} = 3;$

$\det(B_4) = \begin{vmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 \end{vmatrix} = 3.$

$x_1 = \det(B_1)/\det(A) = 1; x_2 = \det(B_2)/\det(A) = 1;$

$x_3 = \det(B_3)/\det(A) = 1; x_4 = \det(B_4)/\det(A) = 1.$

20. $x_1 = 2, x_2 = 1, x_3 = 0.$

21. $\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1; \det(B_1) = \begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ c & 0 & 1 \end{vmatrix} = a - b;$

$\det(B_2) = \begin{vmatrix} 1 & a & 1 \\ 0 & b & 1 \\ 0 & c & 1 \end{vmatrix} = b - c; \det(B_3) = \begin{vmatrix} 1 & 1 & a \\ 0 & 1 & b \\ 0 & 0 & c \end{vmatrix} = c.$

$x_1 = a - b, x_2 = b - c, x_3 = c.$

22. $\det(A)^2 = \det(A^2) = \det(I) = 1$ so $\det(A) = \pm 1$.

23. Suppose $\hat{B}$ is produced by interchanging the i^{th} and the j^{th} columns of B. Thus $A\hat{B} = A[\mathbf{B_1}, \ldots, \mathbf{B_j}, \ldots, \mathbf{B_i}, \ldots, \mathbf{B_n}] =$

$[A\mathbf{B_1}, \ldots, A\mathbf{B_j}, \ldots, A\mathbf{B_i} \ldots, A\mathbf{B_n}] = \hat{C}$ where $\hat{C}$ is obtained by interchanging the i^{th} and j^{th} columns of $C = AB$.

Suppose $\hat{B}$ is produced by replacing $\mathbf{B_i}$ with $\mathbf{B_i} + a\mathbf{B_j}$. Then

$A\hat{B} = [A\mathbf{B_1}, \ldots, A\mathbf{B_i} + aA\mathbf{B_j}, \ldots, A\mathbf{B_n}] = \hat{C}$ where $C = AB$ and $\hat{C}$ is obtained by adding a times the j^{th} column of C to the i^{th} column of C.

Finally suppose $\hat{B}$ is produced by replacing $\mathbf{B_i}$ with $a\mathbf{B_i}$. Then $A\hat{B} = [A\mathbf{B_1}, \ldots, aA\mathbf{B_i}, \ldots, A\mathbf{B_n}] = \hat{C}$ where $C = AB$ and $\hat{C}$ is obtained by multiplying the i^{th} column of C by a..

24. $\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$.

25. $\det(B) = \det(SAS^{-1}) = \det(S)\det(A)\det(S^{-1}) =$

$\det(S)\det(S)^{-1}\det(A) = \det(A)$.

26. Either $\det(A) = 0$ or $\det(A) = 1$.

27. $\det(A^5) = \det(A)^5 = 3^5 = 243$.

28. Set $x = \det(A)$. Then $x^{-1} = \det(A^{-1})$. Moreover, since A and A^{-1} both have only integer entries, both $\det(A)$ and $\det(A^{-1})$ are integers. It follows that $x = \pm 1$.

29. $\det(Q) = -33 = (-3)(11) = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \begin{vmatrix} 1 & 2 & 2 \\ 3 & 5 & 1 \\ 1 & 4 & 1 \end{vmatrix}$.

30. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 5 & 1 \\ 1 & 4 & 1 \end{bmatrix}$ then $\det(A)\det(B) = (-3)(11) = -33 = \det(Q)$.

6.5 Applications of Determinants

1. $$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ -1 & 4 & 1 \end{vmatrix} \begin{Bmatrix} R_2 - 2R_1 \\ R_3 + R_1 \end{Bmatrix} \overset{}{=} \begin{vmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 6 & 2 \end{vmatrix} \{R_3 + 6R_2\} =$$

$$\begin{vmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{vmatrix} = -2.$$

2. $\begin{vmatrix} 0 & 3 & 1 \\ 1 & 2 & 1 \\ 2 & -2 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{vmatrix} = -6.$

3. $\begin{vmatrix} 0 & 1 & 3 \\ 1 & 2 & 2 \\ 3 & 1 & 0 \end{vmatrix} \overset{\{R_1 \leftrightarrow R_2\}}{=} -\begin{vmatrix} 1 & 2 & 2 \\ 0 & 1 & 3 \\ 3 & 1 & 0 \end{vmatrix} \overset{\{R_3 - 3R_1\}}{=}$

$-\begin{vmatrix} 1 & 2 & 2 \\ 0 & 1 & 3 \\ 0 & -5 & -6 \end{vmatrix} \overset{\{R_3 + 5R_2\}}{=} \begin{vmatrix} 1 & 2 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 9 \end{vmatrix} = -9.$

4. $\begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & 4 \\ 3 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & -6 \end{vmatrix} = -12.$

5. $\operatorname{adj}(A) = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}; \det(A) = -2; A^{-1} = (-1/2)\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}.$

6. $\operatorname{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}; \det(A) = ad - bc;$

$A^{-1} = [1/(ad-bc)]\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$

7. $\operatorname{adj}(A) = \begin{bmatrix} 0 & 1 & -1 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}; \det(A) = 1; A^{-1} = \operatorname{adj}(A).$

8. $\operatorname{adj}(A) = \begin{bmatrix} -1 & -1 & 1 \\ -3 & 2 & -2 \\ 3 & -2 & -3 \end{bmatrix}; \det(A) = -5;$

$A^{-1} = (-1/5)\begin{bmatrix} -1 & -1 & 1 \\ -3 & 2 & -2 \\ 3 & -2 & -3 \end{bmatrix}.$

9. $\operatorname{adj}(A) = \begin{bmatrix} -4 & 2 & 0 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}; \det(A) = -2;$

$A^{-1} = (-1/2)\begin{bmatrix} -4 & 2 & 0 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}.$

10. $\text{adj}\,(A) = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$; $\det(A) = 1$; $A^{-1} = \text{adj}\,(A)$.

11. $W(x) = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2$. Since $W(0) = 2$ the given set of functions is linearly independent.

12. $W(x) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 2e^{6x}$. Since $W(0) = 6 \neq 0$, the set of functions is linearly independent.

13. $W(x) = \begin{vmatrix} 1 & \cos^2 x & \sin^2 x \\ 0 & -2\cos x \sin x & 2\cos x \sin x \\ 0 & 2\sin^2 x - 2\cos^2 x & 2\cos^2 x - 2\sin^2 x \end{vmatrix} =$
$4\cos x \sin x \cos 2x \begin{vmatrix} 1 & \cos^2 x & \sin^2 x \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{vmatrix} = 0$. The Wronskian gives no information, but $1 - \cos^2 x - \sin^2 x = 0$ so the set is linearly dependent.

14. $W(x) = 4\sin x \cos 2x - 2\sin 2x \cos x . W(\pi/4) \neq 0$ so the set of functions is linearly independent.

15. Note that $x \mid x \mid = x^2$ for $x \geq 0$ and $x \mid x \mid = -x^2$ for $x < 0$. Therefore $W(x) = \begin{vmatrix} x^2 & x^2 \\ 2x & 2x \end{vmatrix} = 0$ if $x \geq 0$ and $W(x) = \begin{vmatrix} x^2 & -x^2 \\ 2x & -2x \end{vmatrix}$ when $x < 0$. Since $W(x) = 0$ for all $x, -1 \leq x \leq 1$, the Wronskian test is inconclusive. Thus suppose that $c_1x^2 + c_2x \mid x \mid = 0$ for all $x, -1 \leq x \leq 1$. Then for $x = 1$ we have $c_1 + c_2 = 0$ and for $x = -1$ we obtain $c_1 - c_2 = 0$. It follows that $c_1 = c_2 = 0$ and the set $\{x^2, x \mid x \mid\}$ is linearly independent.

16. $W(x) = 0$ for all $x, -1 \leq x \leq 1$. The set is linearly dependent since $3x^2 - 2(1 + x^2) + (2 - x^2) = 0$.

17. The column operations $C_1 \leftrightarrow C_2, C_3 - 3C_1, C_3 + 2C_2$ reduce A to the matrix $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & -1 \end{bmatrix}$. Therefore $Q = E_1E_2E_3$ where $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

Multiplication yields $Q = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -3 \\ 0 & 0 & 1 \end{bmatrix}$. It is easily seen that

$\det(Q) = \det(Q^{\mathrm{T}}) = -1$.

18. $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}; E_2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix};$

$Q = E_1E_2E_3 = \begin{bmatrix} 0 & 1 & -5 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}; AQ = L = \begin{bmatrix} -1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix};$

$\det(Q) = \det(Q^{\mathrm{T}}) = -1$.

19. The column operations $C_2 - 2C_1, C_3 + C_1, C_3 + 4C_4$ transform A to $L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & -1 & 0 \\ 4 & -8 & -26 \end{bmatrix}$.

If $E_1 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$

$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$, and $Q = E_1E_2E_3 = \begin{bmatrix} 1 & -2 & -7 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

then $AQ = L$ and $\det(Q) = \det(Q^{\mathrm{T}}) = 1$.

20. $E_1 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; E_2 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix};$

$Q = E_1E_2E_3 = \begin{bmatrix} 1 & -2 & -5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}; AQ = L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ 3 & -4 & -6 \end{bmatrix};$

$\det(Q) = \det(Q^{\mathrm{T}}) = 1$.

21. $\det(A(x)) = x^2 + 1 > 0$ for all real x. $\operatorname{adj}(A(x)) = \begin{bmatrix} x & -1 \\ 1 & x \end{bmatrix}$ so $A^{-1} = [1/(x^2 + 1)] \begin{bmatrix} x & -1 \\ 1 & x \end{bmatrix}$.

22. $\det(A(x)) = x^2 + 2 > 0$ for all real x. $A^{-1} = [1/(x^2 + 2)] \begin{bmatrix} 2 & -x \\ x & 1 \end{bmatrix}$.

23. $\det(A(x)) = 4x^2 + 8 > 0$ for all real x. $\text{adj}\,(A) =$
$\begin{bmatrix} x^2+4 & -2x & x^2 \\ 2x & 4 & -2x \\ x^2 & 2x & x^2+4 \end{bmatrix}$ so $A^{-1} = [1/(4x^2+8)]\,\text{adj}\,(A)$.

24. $\det(A(x)) = 1$ for all $x . A^{-1} = \text{adj}\,(A) = \begin{bmatrix} \sin x & 0 & -\cos x \\ 0 & 1 & 0 \\ \cos x & 0 & \sin x \end{bmatrix}$.

25. $\det(L) = 1$ so $L^{-1} = \text{adj}\,(L) = \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ac-b & -c & 1 \end{bmatrix}$. $\det(U) = 1$ so $U^{-1} = \text{adj}\,(U) =$
$\begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$.

26. Let $L = [l_{ij}]$ be a (4 x 4) nonsingular, lower-triangular matrix. Direct calculations show that $\text{adj}\,(L)$ is also a lower-triangular matrix. By Theorem 14, L^{-1} is lower-triangular.

27. Clearly each cofactor, A_{ij} of A is an integer. Therefore $A^{-1} =$
$\text{adj}\,(A)$ contains only integer entries.

28. (a) Let $A = [\mathbf{A_1}, \ldots, \mathbf{A_n}]$. By assumption
$E = [\mathbf{e_1}, \ldots, \mathbf{e_j}, \ldots, \mathbf{e_i}, \ldots, \mathbf{e_n}]$. Therefore
$AE = [A\mathbf{e_1} \ldots, A\mathbf{e_j}, \ldots, A\mathbf{e_i}, \ldots, A\mathbf{e_n}] =$
$[\mathbf{A_1}, \ldots, \mathbf{A_j}, \ldots, \mathbf{A_i}, \ldots, \mathbf{A_n}]$.
    (b) Let $E = [e_{rs}]$. If suffices to note that $e_{rs} = e_{sr}$ when $s \neq r$. If $(r, s) \neq (i, j)$ and $(r, s) \neq (j, i)$ then $e_{rs} = e_{sr} = 0$. But $e_{ij} = e_{ji} = 1$ so E is symmetric.

29. If A is an $(n \,\mathrm{x}\, \mathrm{n})$ skew symmetric matrix then $\det(A^{\mathrm{T}}) = \det(-A) = (-1)^n \det(A)$. But $\det(A) = \det(A^{\mathrm{T}})$ so $\det(A) - \det(A^{\mathrm{T}}) = \det(A) - (-1)^n \det(A) =$. For n odd this implies that $2\det(A) = 0$. Therefore $\det(A) = 0$ and A is singular.

30. Let $x = \det(A)$. Then $x = \det(A^T) = \det(A^{-1}) = 1/x$. Thus, $x^2 = 1$ and it follows that $x = \pm 1$.

31. Set $c = \det(A)$. Since A is nonsingular, $c \neq 0$. Moreover, $\text{Adj}\,(A) = \det(A)A^{-1} = cA^{-1}$, so $\det[\text{Adj}\,(A)] = \det(cA^{-1}) = c^n \det(A^{-1}) = c^n/\det(A) = c^n/c = c^{n-1}$.

32. (a) $I = AA^{-1} = A[\frac{1}{\det(A)}\,\text{Adj}\,(A)] = [\frac{1}{\det(A)}A]\,\text{Adj}\,(A)$.
Therefore, $[\text{Adj}\,(A)]^{-1} = [1/\det(A)]A$.
    (b) $A = (A^{-1})^{-1} = \left[1/\det(A^{-1})\right] \text{Adj}\,(A^{-1}) = \det(A)\,\text{Adj}\,(A^{-1})$. Therefore, $\text{Adj}\,(A^{-1}) = [1/\det(A)]A$.

6.6 Supplementary Exercises

1. $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}$.

2. Note that if $[\mathbf{A}_{n-2}, \ldots, \mathbf{A}_1]$ can be obtained from $[\mathbf{A}_1, \ldots, \mathbf{A}_{n-2}]$ with m column interchanges, then $[\mathbf{A}_1, \mathbf{A}_{n-1}, \ldots, \mathbf{A}_2, \mathbf{A}_n]$ can be obtained from $[\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n]$ with m column interchanges. Thus, $m+1$ column interchanges yields $[\mathbf{A}_n, \mathbf{A}_{n-1}, \ldots, \mathbf{A}_1]$. Therefore, if $n = 4k$ or $n = 4k+1$ then an even number of column interchanges is required and $\det(B) = \det(A)$. If $n = 2k$ or $n = 2k+1$, where k is odd, then an odd number of column interchanges is required and $\det(B) = -\det(A)$.

3. If $x = \det(A)$ then $x^3 = x$ so $x = 0$, $x = 1$, or $x = -1$.

4. Let $x = \det(A)$. Then $x \neq 0$, $x = \det(A^T)$, and if A is a (2×2) matrix, $\det(cA) = c^2x$. From $x = c^2x$ it follows that $c = \pm 1$. Similarly, if A is a (3×3) matrix, $c^3 = 1$ so $c = 1$.

5. $AB = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$

6. $\operatorname{Det}(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = -1$ and $A^{-1} = (1/\det(A))\operatorname{Adj}(A) = -C^T$. Therefore, $A = (-C^T)^{-1} = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 4 \\ 2 & 2 & 1 \end{bmatrix}$.

7. $\operatorname{Det}(B+I) = \det[\mathbf{b}+\mathbf{e}_1, \mathbf{b}+\mathbf{e}_2, \ldots, \mathbf{b}+\mathbf{e}_n] = \sum_{i=1}^{n} \det(A_i) + \det(I) = b_1 + \cdots + b_n + 1$.

8. $(x^5 + x^3)e^x$.

6.7 Conceptual Exercises

1. True. A is nonsingular so $B = A^{-1}(AB) = A^{-1}(AC) = C$.

2. True. $\operatorname{Det}(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(AB)$.

3. False. If $A = I_n$ then $\det(cI_n - A) = (c-1)^n$.

4. False. $\operatorname{Det}(cA) = c^n \det(A)$.

5. True. $0 = \det(A^k) = (\det(A))^k$, so $\det(A) = 0$.

6. True. $0 \neq \det(B) = \det(A_1)\cdots\det(A_m)$, so $\det(A_i) \neq 0$ for $1 \leq i \leq m$.

7. True. If $C = [A_{ij}]$ is the cofactor matrix for A then C is symmetric.

8. True. $A^{-1} = \text{Adj}\,(A)$.

9. If $x = \det(A)$ and $A^2 = -I$ then $x^2 = -1$, which is not possible.

10. A is nonsingular (cf. Exercise 6) so A^{-1} exists. Thus $I = A^{-1}IA = A^{-1}(AB)A = BA$.

11. $A^T - cI = (A - cI)^T$.

12. (a) $\text{Det}\,(B^{-1}AB - cI) = \det[B^{-1}(A - cI)B] = \det(B^{-1})\det(A - cI)\det(B) = \det(A - cI)$.
    (b) $\text{Det}\,(AB - cI) = \det(B^{-1}(BA)B - cI) = \det(BA - cI)$.

13. $A[\text{Adj}\,(A)] = (\det(A))I$ so, by Exercise 6, $\text{Adj}\,(A)$ is nonsingular.

14. (a) If A is nonsingular then $B = IB = A^{-1}(AB) = A^{-1}\mathcal{O} = \mathcal{O}$. Similarly, if B is nonsingular, then $A = \mathcal{O}$. If follows that A and B are both singular.
    (b) If A is singular then $\det(A) = 0$ so $A[\text{Adj}\,(A)] = \mathcal{O}$. By (a), $\text{Adj}\,(A)$ is singular.

15. $A^T = A^{-1} = (1/\det(A))\,\text{Adj}\,(A)$ so $\text{Adj}\,(A)^T = (\det(A))A$.

Chapter 7

Eigenvalues and Applications

7.1 Quadratic Forms

1. $A = \begin{bmatrix} 2 & 2 \\ 2 & -3 \end{bmatrix}$.

2. $A = \begin{bmatrix} -1 & 3 \\ 3 & 1 \end{bmatrix}$.

3. $A = \begin{bmatrix} 1 & 1 & -3 \\ 1 & -4 & 4 \\ -3 & 4 & 3 \end{bmatrix}$.

4. $A = \begin{bmatrix} 1 & 1 & 5 & -2 \\ 1 & 0 & 2 & -1 \\ 5 & 2 & 4 & 3 \\ -2 & -1 & 3 & -1 \end{bmatrix}$.

5. $A = \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}$.

6. $A = \begin{bmatrix} 1 & 4 & 2 \\ 4 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$.

7. $q(\mathbf{x}) = \mathbf{x}^{\mathrm{T}} A\mathbf{x}$ where $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$. A has eigenvalues $\lambda_1 = 5, \lambda_2 = -1$ with corresponding eigenvectors $a \begin{bmatrix} 1 \\ 1 \end{bmatrix}, b \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, respectively, where $a \neq 0$ and $b \neq 0$. In particular $Q = (1/\sqrt{2}) \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. The form is indefinite.

8. $q(\mathbf{x}) = \mathbf{x}^{\mathrm{T}} A\mathbf{x}$ for $A = \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}$. A has eigenvalues $\lambda_1 = 7$ and $\lambda_2 = 3$. We may take $Q = (1/\sqrt{2}) \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$. The form is positive definite.

9. $q(\mathbf{x}) = \mathbf{x}^{\mathrm{T}} A\mathbf{x}$ for $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$. The eigenvalues for A are $\lambda_1 = 5$ and $\lambda_2 = -1$ (algebraic multiplicity 2). An eigenvector for $\lambda_1 = 5$ is $\mathbf{u_1} = [1, 1, 1]^{\mathrm{T}}$. The vectors $\mathbf{w_2} = [-1, 1, 0]^{\mathrm{T}}$ and $\mathbf{w_3} = [-1, 0, 1]^{\mathrm{T}}$

are eigenvectors for $\lambda_2 = -1$. The Gram-Schmidt process yields orthogonal eigenvectors $\mathbf{u_2} = \mathbf{w_2} = [-1, 1, 0]^{\mathrm{T}}$ and $\mathbf{u_3} = [-1, -1, 2]^{\mathrm{T}}$.

We form Q by normalizing the set $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ of eigenvectors;

$$Q = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}.$$

The form is indefinite.

10. $q(\mathbf{x}) = \mathbf{x}^{\mathrm{T}} A\mathbf{x}$ for $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. A has eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 0$ (algebraic multiplicity 2). One choice for Q is

$$Q = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}.$$

The form is positive semidefinite.

11. $q(\mathbf{x}) = \mathbf{x}^{\mathrm{T}} A\mathbf{x}$ where $A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$. A has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 4$ with corresponding eigenvectors $\mathbf{u_1} = [1, 1]^{\mathrm{T}}$ and $\mathbf{u_2} =$

$[-1, 1]^{\mathrm{T}}$, respectively. The set $\{\mathbf{u_1}, \mathbf{u_2}\}$ is orthogonal. We normalize $\mathbf{u_1}$ and $\mathbf{u_2}$ to obtain $Q; Q = (1/\sqrt{2}) \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. The form is positive definite.

12. $q(\mathbf{x}) = \mathbf{x}^{\mathrm{T}} A\mathbf{x}$ where $A = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$. A has eigenvalues $\lambda_1 = 2$ (with algebraic

multiplicity 3) and $\lambda_2=-2$. We may take $Q=\begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & -1/2\sqrt{3} & 1/2 \\ 1/\sqrt{2} & -1/\sqrt{6} & -1/2\sqrt{3} & 1/2 \\ 0 & 2/\sqrt{6} & -1/2\sqrt{3} & 1/2 \\ 0 & 0 & 3/2\sqrt{3} & 1/2 \end{bmatrix}$. The form is indefinite.

13. Set $q(\mathbf{x})=2x^2+\sqrt{3}\,xy+y^2$. Then $q(\mathbf{x})=\mathbf{x}^{\mathrm{T}}A\mathbf{x}$ for $A=$ $\begin{bmatrix} 2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1 \end{bmatrix}$. A has eigenvalues $\lambda_1=1/2, \lambda_2=5/2$ with corresponding eigenvectors $\mathbf{u_1}=[-1,\sqrt{3}]^{\mathrm{T}}$ and $\mathbf{u_2}=[\sqrt{3},1]^{\mathrm{T}}$, respect- ively. Since $\{\mathbf{u_1}, \mathbf{u_2}\}$ is an orthogonal set we may normalize and obtain$Q=\begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$. The substitution $\mathbf{x}=Q\mathbf{y}$ yields $q(\mathbf{x})=(1/2)u^2+(5/2)v^2=10$. The graph corresponds to the ellipse $u^2/20+v^2/4=1$.

14. $Q=(1/\sqrt{2})\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and the graph corresponds to the ellipse $u^2/2+v^2/4=1$.

15. Set $q(\mathbf{x})=x^2+6xy-7y^2$. Then $q(\mathbf{x})=\mathbf{x}^{\mathrm{T}}A\mathbf{x}$ where $A=\begin{bmatrix} 1 & 3 \\ 3 & -7 \end{bmatrix}$.

A has eigenvalues $\lambda_1=-8$ and $\lambda_2=2$ with corresponding eigenvectors $\mathbf{u_1}=[-1,3]^{\mathrm{T}}$ and $\mathbf{u_2}=[3,1]^{\mathrm{T}}$, respectively. Since $\{\mathbf{u_1}, \mathbf{u_2}\}$ is an orthogonal set we may normalize to obtain $Q=(1/\sqrt{10})\begin{bmatrix} -1 & 3 \\ 3 & 1 \end{bmatrix}$.

The substitution $\mathbf{x}=Q\mathbf{y}$ yields $q(\mathbf{x})=-8u^2+2v^2=8$. The graph corresponds to the hyperbola $v^2/4-u^2=1$.

16. $Q=(1/\sqrt{2})\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and the graph corresponds to the ellipse $u^2+v^2/5=1$.

17. If $q(\mathbf{x})=xy$ then $q(\mathbf{x})=\mathbf{x}^{\mathrm{T}}A\mathbf{y}$ where $A=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The mat- rix A is diagonalized by $Q=(1/\sqrt{2})\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $Q^{\mathrm{T}}AQ=D=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. The substitution $\mathbf{x}=Q\mathbf{y}$ yields $q(\mathbf{x})=u^2-v^2=4$. The graph corresponds to the hyperbola $u^2/4-v^2/4=1$.

18. $Q=\begin{bmatrix} \sqrt{3}/2 & 1/2 \\ 1/2 & -\sqrt{3}/2 \end{bmatrix}$. The transformed equation is the parabola $v=-2u^2-\sqrt{3}\,u+2$.

19. If $q(\mathbf{x}) = 3x^2 - 2xy + 3y^2$ then $q(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}A\mathbf{x}$ where $A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$.

A is diagonalized by $Q = (1/\sqrt{2})\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ and $Q^{\mathrm{T}}AQ = D$ where $D = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$. The transformed equation is the ellipse $4u^2 + 2v^2 = 16$, or $u^2/4 + v^2/8 = 1$.

20. $Q = (1/\sqrt{2})\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and the transformed equation is $2u^2 = -1$.

21. Note that $a_{ii} = \mathbf{e_i}^{\mathrm{T}}A\mathbf{e_i} = \mathbf{e_i}^{\mathrm{T}}C\mathbf{e_i} = c_{ii}$ for $1 \le i \le n$. For $r \ne s$ set $\mathbf{x} = \mathbf{e_r} + \mathbf{e_s}$. Then $\mathbf{x}^{\mathrm{T}}A\mathbf{x} = a_{rr} + a_{rs} + a_{sr} + a_{ss} = a_{rr} + 2a_{rs} + a_{ss}$, since A is symmetric. Similarly $\mathbf{x}^{\mathrm{T}}C\mathbf{x}$ $= c_{rr} + 2c_{rs} + c_{ss}$ it follows that $a_{rs} = c_{rs}$.

22. (a) Suppose that $A\mathbf{x} = \lambda_i\mathbf{x}$ where $\mathbf{x} \ne \theta$. Then $0 < \mathbf{x}^{\mathrm{T}}A\mathbf{x} = \lambda_i\mathbf{x}^{\mathrm{T}}\mathbf{x} =$ $\lambda_i \, \|\mathbf{x}\|^2$. It follows that $\lambda_i > 0$.

(b) By Equation (3) $q(\mathbf{x}) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$ where $\mathbf{y} = [y_1, \ldots, y_n]^{\mathrm{T}}$ and $\mathbf{x}$ $= Q\mathbf{y}$. If $\mathbf{x} \ne \theta$ then $\mathbf{y} \ne \theta$ so $q(\mathbf{x}) > 0$.

23. See Exercise 22.

24. See Exercise 22.

25. If $q(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}A\mathbf{x}$ is indefinite then it follows from properties (a) - (d) of Theorem 2 that A has positive and negative eigenvalues. Conversely assume that A has eigenvalues $\lambda_1 > 0$ and $\lambda_2 < 0$ and let $\mathbf{x_1}$ and $\mathbf{x_2}$ be corresponding eigenvectors, respectively. Then $\mathbf{x_1}^{\mathrm{T}}A\mathbf{x_1} =$ $\lambda_1 \, \|x\|_1^2 > 0$ and $\mathbf{x_2}^{\mathrm{T}}A\mathbf{x_2} = \lambda_2 \, \|x\|_2^2 < 0$. This shows that $q(\mathbf{x})$ is indefinite.

26. Following the hint we obtain $R(\mathbf{x}) = (\sum_{i=1}^n a_1^2\lambda_i)/(\sum_{i=1}^n a_i^2)$. Therefore $\lambda_1 = (\lambda_1 \sum_{i=1}^n a_i^2)/(\sum_{i=1}^n a_i^2) \le (\sum_{i=1}^n a_i^2\lambda_i)/(\sum_{i=1}^n a_i^2) =$ $R(\mathbf{x}) \le (\lambda_n \sum_{i=1}^n a_i^2)/(\sum_{i=1}^n a_i^2) = \lambda_n$.

27. If $\|\mathbf{x}\| = 1$ then $R(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}A\mathbf{x} = q(\mathbf{x})$ [cf. Exercise 26]. By Exercise 26, $\lambda_1 \le R(\mathbf{x}) \le \lambda_n$. Let $\mathbf{u_1}$ and $\mathbf{u_n}$ be eigenvectors corresponding to λ_1 and λ_n, respectively, where $\|\mathbf{u_1}\|$ $= \|\mathbf{u_n}\| = 1$. Then $R(\mathbf{u_1}) = \mathbf{u_1}^{\mathrm{T}}A\mathbf{u_1} = \lambda_1$ and, similarly, $R(\mathbf{u_n}) = \boldsymbol{\lambda_n}$.

28. (a) $B^{\mathrm{T}} = (S^{\mathrm{T}}AS)^{\mathrm{T}} = S^{\mathrm{T}}A^{\mathrm{T}}S^{\mathrm{TT}} = S^{\mathrm{T}}AS = B$.

(b) Suppose that q_1 is positive definite and assume $\mathbf{x} \ne \theta$. Since S is nonsingular $\mathbf{y}$ $= S\mathbf{x} \ne \theta$. Therefore $q_2(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}B\mathbf{x} = \mathbf{x}^{\mathrm{T}}S^{\mathrm{T}}AS\mathbf{x} = (S\mathbf{x})^{\mathrm{T}}A(S\mathbf{x}) = \mathbf{y}^{\mathrm{T}}A\mathbf{y} = q_1(\mathbf{y}) > \mathbf{0}$. This shows that q_2 is positive definite. The reverse argument is similar.

7.2 Systems of Differential Equations

1. The given system has matrix equation $\mathbf{x}'(t) = A\mathbf{x}(t)$ where $\mathbf{x}(t) = [u(t), v(t)]^{\mathrm{T}}$ and $A = \begin{bmatrix} 5 & -2 \\ 6 & -2 \end{bmatrix}$. The eigenvalues for A are $\lambda_1 = 1$ and $\lambda_2 = 2$ and the corresponding eigenvectors are $\mathbf{u_1} = [1,2]^{\mathrm{T}}, \mathbf{u_2} = [2,3]^{\mathrm{T}}$. Thus $\mathbf{x_1}(t) = e^t\mathbf{u_1}$ and $\mathbf{x_2}(t) = e^{2t}\mathbf{u_2}$ are solutions. The general solution is given by $\mathbf{x}(t) = b_1\mathbf{x_1}(t) +$ $b_2\mathbf{x_2}(t)$; that is $\mathbf{x}(t) = b_1e^t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + b_2e^t \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. It is easily seen that $\mathbf{x_0} = \mathbf{u_1} + 2\mathbf{u_2}$ so the solution $\mathbf{x}(t) = e^t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2e^{2t} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} e^t + 4e^{2t} \\ 2e^t + 6e^{2t} \end{bmatrix}$ satisfies the initial condition.

2. $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. The general solution is $\mathbf{x}(t) = b_1e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b_2e^{3t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. The solution that satisfies the initial condition is $\mathbf{x}(t) = (1/2)e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} - (3/2)e^{3t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

3. The system is $\mathbf{x}'(t) = A\mathbf{x}(t)$ where $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$. A has eigen- values $\lambda_1 = 0$ and $\lambda_2 = 3$ with corresponding eigenvectors $\mathbf{u_1} =$ $[-1,1]^{\mathrm{T}}$ and $\mathbf{u_2} = [1,2]^{\mathrm{T}}$. The general solution is given by $\mathbf{x}(t) =$ $b_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + b_2e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The particular solution that satisfies the initial condition is $\mathbf{x}(t) = -3 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 2e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 + 2e^{3t} \\ -3 + 4e^{3t} \end{bmatrix}$.

4. $A = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$. The general solution is $\mathbf{x}(t) = b_1e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} +$ $b_2e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The solution that satisfies the initial condition is $\mathbf{x}(t) = e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2e^{2t} + e^{-t} \\ e^{2t} + e^{-t} \end{bmatrix}$.

5. The system is $\mathbf{x}'(t) = A\mathbf{x}(t)$ where $A = \begin{bmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix}$. A has eigenvalues $\lambda_1 = 0.5 + 0.5i$ and $\lambda_2 = 0.5 - 0.5i$ with corresponding eigenvectors $\mathbf{u_1} = [-i, 1]^{\mathrm{T}}$ and $\mathbf{u_2} = [i, 1]^{\mathrm{T}}$. The general solution is $\mathbf{x}(t) = b_1e^{\lambda_1 t}\mathbf{u_1} + b_2e^{\lambda_2 t}\mathbf{u_2}$ where $e^{\lambda_1 t} = e^{(0.5+0.5i)t} =$ $e^{t/2}[\cos(t/2) + i\sin(t/2)]$ and $e^{\lambda_2 t} = e^{(0.5-0.5i)t} = e^{t/2}[\cos(t/2) - i\sin(t/2)]$. The equation

$\mathbf{x_0} = b_1\mathbf{u_1} + b_2\mathbf{u_2}$ has solution $b_1 = 2+2i$ and $b_2 = 2-2i$ so the particular solution that satisfies the initial condition is $\mathbf{x}(t) = 4e^{(t/2)}\begin{bmatrix} \cos(t/2)+\sin(t/2) \\ \cos(t/2)-\sin(t/2) \end{bmatrix}$.

6. $A = \begin{bmatrix} 6 & 8 \\ -1 & 2 \end{bmatrix}$. The general solution is
$\mathbf{x}(t) = b_1e^{(4+2i)t}\begin{bmatrix} -2-2i \\ 1 \end{bmatrix} + b_2e^{(4-2i)t}\begin{bmatrix} -2+2i \\ 1 \end{bmatrix}$. The solution that satisfies the initial condition is $\mathbf{x}(t) = 2ie^{(4+2i)t}\begin{bmatrix} -2-2i \\ 1 \end{bmatrix} -$
$2ie^{(4-2i)t}\begin{bmatrix} -2+2i \\ 1 \end{bmatrix} = 4e^{4t}\begin{bmatrix} 2\cos 2t + 4\sin 2t \\ -\sin 2t \end{bmatrix}$.

7. The system is $\mathbf{x}'(t) = A\mathbf{x}(t)$ where $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$. A has eigenvalues $\lambda_1 = 1, \lambda_2 = 2$, and $\lambda_3 = 3$ with corresponding eigenvectors $\mathbf{u_1} = [0,1,0]^{\mathrm{T}}, \mathbf{u_2} = [1,-2,-2]^{\mathrm{T}}$, and $\mathbf{u_3} = [-1,1,1]^{\mathrm{T}}$, respectively. Therefore the general solution is $\mathbf{x}(t) =$
$b_1e^t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + b_2e^{2t}\begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix} + b_3e^{3t}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$. Since $\mathbf{x_0} = \mathbf{u_1} + \mathbf{u_2} + 2\mathbf{u_3}$ the solution $\mathbf{x}$
$(t) = e^t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + e^{2t}\begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix} + 2e^{3t}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ satisfies the initial condition.

8. $A = \begin{bmatrix} 3 & 1 & -2 \\ -1 & 2 & 1 \\ 4 & 1 & -3 \end{bmatrix}$. The general solution is
$\mathbf{x}(t) = b_1e^t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b_2e^{-t}\begin{bmatrix} 7 \\ -2 \\ 13 \end{bmatrix} + b_3e^{2t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. The solution $\mathbf{x}(t) = 3e^t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} -$
$e^{-t}\begin{bmatrix} 7 \\ -2 \\ 13 \end{bmatrix} + 2e^{2t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ satisfies the initial condition.

9. (a) The system is $\mathbf{x}'(t) = A\mathbf{x}(t)$ for $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$. A has only one eigenvalue, $\lambda = 2$, with corresponding eigenvector $u = [1,-1]^{\mathrm{T}}$.

Therefore $\mathbf{x_1}(t) = e^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is a solution for the system.

(b) Set $\mathbf{x_2}(t) = te^{\lambda t}\mathbf{u} + e^{\lambda t}\mathbf{y_0}$. Then $\mathbf{x_2}'(t) = e^{\lambda t}(t\lambda\mathbf{u} + \mathbf{u} + \lambda\mathbf{y_0})$ whereas $A\mathbf{x_2}(t) = e^{\lambda t}(t\lambda\mathbf{u} + A\mathbf{y_0})$. Therefore we require that $A\mathbf{y_0} = \mathbf{u} + \lambda\mathbf{y_0}$; that is $(A - \lambda I)\mathbf{y_0} = \mathbf{u}$. One choice is $\mathbf{y_0} = [-2, 1]^{\mathrm{T}}$. Thus $\mathbf{x_2}(t) = te^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{2t}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is a solution.

(c) If $\mathbf{y}(t) = c_1\mathbf{x_1}(t) + c_2\mathbf{x_2}(t)$ note that $\mathbf{y}(0) = c_1\mathbf{u} + c_2\mathbf{y_0}$. Since $\{\mathbf{u}, \mathbf{y_0}\}$ is a linearly independent set for every $\mathbf{x_0}$ in R^2 we may find c_1 and c_2 such that $\mathbf{x_0} = c_1\mathbf{u} + c_2\mathbf{y_0}$.

10. (a) $A = \begin{bmatrix} 2 & -1 \\ 4 & 6 \end{bmatrix}$ and $\mathbf{x_1}(t) = e^{4t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

(b) One choice for $\mathbf{y_0}$ is $\mathbf{y_0} = [0, -1]^{\mathrm{T}}$. In this case $\mathbf{x_2}(t) = te^{4t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + e^{4t}\begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

(c) The solution $\mathbf{x}(t) = \mathbf{x_1}(t) - 3\mathbf{x_2}(t) = \begin{bmatrix} e^{4t}(1 - 3t) \\ e^{4t}(1 + 6t) \end{bmatrix}$ satisfies the initial condition.

7.3 Transformation to Hessenberg Form

1. The desired elementary row operation is $R_3 - 4R_2$. Performing this operation on the (3 x 3) identity matrix yields $Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix}$.

$Q_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$ and $Q_1AQ_1^{-1} = H = \begin{bmatrix} -7 & 16 & 3 \\ 8 & 9 & 3 \\ 0 & 1 & 1 \end{bmatrix}$.

2. $Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$; $H = Q_1AQ_1^{-1}\begin{bmatrix} -6 & 31 & -14 \\ -1 & 6 & -2 \\ 0 & 2 & 1 \end{bmatrix}$.

3. Let Q_1 denote the permutation matrix $Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Then $Q_1 = Q_1^{-1}$ and Q_1A interchanges the second and third rows of A. Further $(Q_1A)Q_1$ interchanges the second and third columns of Q_1A. Therefore $H = Q_1AQ_1^{-1} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 1 \\ 0 & 4 & 2 \end{bmatrix}$.

4. $Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$ and $H = Q_1AQ_1^{-1} = \begin{bmatrix} 1 & 4 & -1 \\ 3 & 0 & 1 \\ 0 & -5 & 5 \end{bmatrix}$.

5. The desired elementary row operation is $R_3 + 3R_2$. Performing this operation on the (3 x 3) identity matrix yields $Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$. $Q_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$ and $H = Q_1 A Q_1^{-1} = \begin{bmatrix} 3 & 2 & -1 \\ 4 & 5 & -2 \\ 0 & 20 & -6 \end{bmatrix}$.

6. $Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $H = Q_1 A Q_1^{-1} = \begin{bmatrix} 4 & 3 & 0 \\ 3 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$.

7. Performing the elementary row operations $R_3 - R_2$ and $R_4 - R_2$ on the (4 x 4) identity matrix yields $Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$. $Q_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ and $H = Q_1 A Q_1^{-1} = \begin{bmatrix} 1 & -3 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

8. Let $Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & -4 & 0 & 1 \end{bmatrix}$ and $Q_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$. Then $H = Q_2 Q_1 A Q_1^{-1} Q_2^{-1} = \begin{bmatrix} 6 & 33 & 8 & 4 \\ 1 & 38 & 8 & 4 \\ 0 & -120 & -25 & -15 \\ 0 & 0 & 0 & 5 \end{bmatrix}$.

9. If $Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ then $Q_1 = Q_1^{-1}$ and $Q_1 A$ interchanges the second and fourth rows of A whereas $(Q_1 A) Q_1$ interchanges the second and fourth columns 0f $Q_1 A$. Therefore

$$Q_1 A Q_1 = \begin{bmatrix} 1 & 3 & 1 & 2 \\ 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 1 & 1 \end{bmatrix}.$$

Now the desired elementary row operation is $R_4 - 2R_3$ so set $Q_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix}$.

Then $Q_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$ and

$$H = Q_2 Q_1 A Q_1^{-1} Q_2^{-1} = \begin{bmatrix} 1 & 3 & 5 & 2 \\ 1 & 2 & 4 & 2 \\ 0 & 1 & 7 & 3 \\ 0 & 0 & -11 & -5 \end{bmatrix}.$$

10. If $Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ and $Q_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -5/3 & 1 \end{bmatrix}$ Then

$$H = Q_2 Q_1 A Q_1^{-1} Q_2^{-1} = \begin{bmatrix} 2 & -1 & -5/3 & -1 \\ -1 & 2 & -1/3 & 1 \\ 0 & 0 & 7 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

11. Since λ is an eigenvalue for H, the matrix $H-\lambda I$ is singular; that is nullity $(H-\lambda I) \geq 1$. It follows that rank $(H - \lambda I) \leq 3$. But $H - \lambda I = \begin{bmatrix} a_1-\lambda & b_1 & c_1 & d_1 \\ a_2 & b_2-\lambda & c_2 & d_2 \\ 0 & b_3 & c_3-\lambda & d_3 \\ 0 & 0 & c_4 & d_4-\lambda \end{bmatrix}$ and clearly the first three columns of $H - \lambda I$ are linearly independent. Therefore rank $(H - \lambda I) \geq 3$. Thus rank $(H - \lambda I) = 3$ and so, nullity $(H - \lambda I) = 1$. Hence, λ has geometric multiplicity equal to 1.

12. Since H is similar to a symmetric matrix, H is diagonizable. Therefore the algebraic multiplicity for λ equals the geometric multiplicity. Now apply Exercise 11.

13. $p(t) = (t-2)^3(t+2)$ is the characteristic polynomial for H. Since H and A are similar, $p(t)$ is also the characteristic polynomial for A. Therefore A has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -2$ and $\lambda_1 = 2$ has algebraic (and hence geometric) multiplicity 3 (cf. Exercise 12).

14. $p(t) = (t-5)^2(t-15)(t+1)$ so A has eigenvalues $\lambda_1 = 5, \lambda_2 = 15,$ and $\lambda_3 = -1$.

15. $[\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}], [\mathbf{e_1}, \mathbf{e_3}, \mathbf{e_2}], [\mathbf{e_2}, \mathbf{e_1}, \mathbf{e_3}], [\mathbf{e_2}, \mathbf{e_3}, \mathbf{e_1}], [\mathbf{e_3}, \mathbf{e_1}, \mathbf{e_2}],$

$[\mathbf{e_3}, \mathbf{e_2}, \mathbf{e_1}]$.

16. $[\mathbf{e_1},\mathbf{e_2},\mathbf{e_3},\mathbf{e_4}], [\mathbf{e_1},\mathbf{e_2},\mathbf{e_4},\mathbf{e_3}], [\mathbf{e_1},\mathbf{e_3},\mathbf{e_2},\mathbf{e_4}], [\mathbf{e_1},\mathbf{e_3},\mathbf{e_4},\mathbf{e_2}],$
$[\mathbf{e_1},\mathbf{e_4},\mathbf{e_2},\mathbf{e_3}], [\mathbf{e_1},\mathbf{e_4},\mathbf{e_3},\mathbf{e_2}], [\mathbf{e_2},\mathbf{e_1},\mathbf{e_3},\mathbf{e_4}], [\mathbf{e_2},\mathbf{e_1},\mathbf{e_4},\mathbf{e_3}],$
$[\mathbf{e_2},\mathbf{e_3},\mathbf{e_1},\mathbf{e_4}], [\mathbf{e_2},\mathbf{e_3},\mathbf{e_4},\mathbf{e_1}], [\mathbf{e_2},\mathbf{e_4},\mathbf{e_1},\mathbf{e_3}], [\mathbf{e_2},\mathbf{e_4},\mathbf{e_3},\mathbf{e_1}],$
$[\mathbf{e_3},\mathbf{e_1},\mathbf{e_2},\mathbf{e_4}], [\mathbf{e_3},\mathbf{e_1},\mathbf{e_4},\mathbf{e_2}], [\mathbf{e_3},\mathbf{e_2},\mathbf{e_1},\mathbf{e_4}], [\mathbf{e_3},\mathbf{e_2},\mathbf{e_4},\mathbf{e_1}],$
$[\mathbf{e_3},\mathbf{e_4},\mathbf{e_1},\mathbf{e_2}], [\mathbf{e_3},\mathbf{e_4},\mathbf{e_2},\mathbf{e_1}], [\mathbf{e_4},\mathbf{e_1},\mathbf{e_2},\mathbf{e_3}], [\mathbf{e_4},\mathbf{e_1},\mathbf{e_3},\mathbf{e_2}],$
$[\mathbf{e_4},\mathbf{e_2},\mathbf{e_1},\mathbf{e_3}], [\mathbf{e_4},\mathbf{e_2},\mathbf{e_3},\mathbf{e_1}], [\mathbf{e_4},\mathbf{e_3},\mathbf{e_1},\mathbf{e_2}], [\mathbf{e_4},\mathbf{e_3},\mathbf{e_2},\mathbf{e_1}].$

17. There are $n!$ $(n \times n)$ permutation matrices.

18. Since the columns of P are some ordering of $\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}$, they form an orthonormal set.

19. $AP = A[\mathbf{e_i}, \mathbf{e_j}, \mathbf{e_k}, \ldots, \mathbf{e_r}] = [\mathbf{A}\mathbf{e_i}, A\mathbf{e_j}, A\mathbf{e_k}, \ldots, A\mathbf{e_r}] =$
$[\mathbf{A_i}, \mathbf{A_j}, \mathbf{A_k}, \ldots, \mathbf{A_r}]$.

20. Let $A = \begin{bmatrix} \mathbf{a_1} \\ \mathbf{a_2} \\ \vdots \\ \mathbf{a_n} \end{bmatrix}$ where $\mathbf{a_j}$ is the j^{th} row of A.

By Exercise 19, $A^{\mathrm{T}}P = [\mathbf{a_i}^{\mathrm{T}}, \mathbf{a_j}^{\mathrm{T}}, \mathbf{a_k}^{\mathrm{T}}, \ldots, \mathbf{a_r}^{\mathrm{T}}]$. Therefore $P^{\mathrm{T}}A = (A^{\mathrm{T}}P)^{\mathrm{T}} = \begin{bmatrix} \mathbf{a_i} \\ \mathbf{a_j} \\ \mathbf{a_k} \\ \vdots \\ \mathbf{a_r} \end{bmatrix}$.

21. Apply Exercise 19.

22. By Exercise 21 each of the matrices $P^1, P^2, P^3, \ldots$ is a permutation matrix. By Exercise 17 there are $n!$ distinct $(n \times n)$ permutation matrices. Therefore there exists integers r and s such that $r > s$ and $P^r = P^s$. Since P is nonsingular this implies that $P^{r-s} = I$.

7.4 Eigenvalues of Hessenberg Matrices

1. Note that the given matrix H is in unreduced Hessenberg form. We have $\mathbf{w_0} = \mathbf{e_1} = [1,0]^{\mathrm{T}}, \mathbf{w_1} = H\mathbf{w_0} = [2,1]^{\mathrm{T}}$, and $\mathbf{w_2} = H\mathbf{w_1} = [4,3]^{\mathrm{T}}$. The vector equation $a_0\mathbf{w_0} + a_1\mathbf{w_1} = -\mathbf{w_2}$ is equivalent to the system

$$\begin{array}{rcrcr} a_0 & + & 2a_1 & = & -4 \\ & & a_1 & = & -3 \end{array}.$$

The system has solution $a_0 = 2, a_1 = -3$ so $p(t) = 2 - 3t + t^2$.

2. $\mathbf{w_0} = \mathbf{e_1} = [1,0]^{\mathrm{T}}; \mathbf{w_1} = [0,3]^{\mathrm{T}}; \mathbf{w_2} = [0,0]^{\mathrm{T}}$, The vector equation $a_0\mathbf{w_0} + a_1\mathbf{w_1} = -\mathbf{w_2}$ has solution $a_0 = a_1 = 0$, so $p(t) = t^2$.

3. Note that the given matrix H is in unreduced Hessenberg form. We have $\mathbf{w_0} = \mathbf{e_1} = [1,0,0]^{\mathrm{T}}, \mathbf{w_1} = H\mathbf{w_0} = [1,2,0]^{\mathrm{T}}, \mathbf{w_2} = H\mathbf{w_1} = [1,4,2]^{\mathrm{T}}$, and $\mathbf{w_3} = H\mathbf{w_2} = [3,6,8]^{\mathrm{T}}$. The vector equation

   $a_0\mathbf{w_0} + a_1\mathbf{w_1} + a_2\mathbf{w_2} = -\mathbf{w_3}$ is equivalent to the system of equations

$$\begin{array}{rcrcrcr} a_0 & + & a_1 & + & a_2 & = & -3 \\ & & 2a_1 & + & 4a_2 & = & -6 \\ & & & & 2a_2 & = & -8 \end{array}.$$

   The system has unique solution $a_0 = -4, a_1 = 5, a_2 = -4$ so $p(t) = -4 + 5t - 4t^2 + t^3$.

4. $\mathbf{w_0} = \mathbf{e_1} = [1,0,0]^{\mathrm{T}}; \mathbf{w_1} = [1,1,0]^{\mathrm{T}}; \mathbf{w_2} = [3,4,1]^{\mathrm{T}}; \mathbf{w_3} = [12,14,6]^{\mathrm{T}}$.

   The vector equation $a_0\mathbf{w_0} + a_1\mathbf{w_1} + a_2\mathbf{w_2} = -\mathbf{w_3}$ has solution

   $a_0 = -4, a_1 = 10, a_2 = -6$, so $p(t) = -4 + 10t - 6t^2 + t^3$.

5. Note that the given matrix H is in unreduced Hessenberg form. We have $\mathbf{w_0} = \mathbf{e_1} = [1,0,0]^{\mathrm{T}}, \mathbf{w_1} = H\mathbf{w_0} = [2,1,0]^{\mathrm{T}}, \mathbf{w_2} = H\mathbf{w_1} = [8,3,1]^{\mathrm{T}}$, and $\mathbf{w_3} = H\mathbf{w_2} = [29,14,8]^{\mathrm{T}}$. The vector equation $a_0\mathbf{w_0} + a_1\mathbf{w_1} + a_2\mathbf{w_2} = -\mathbf{w_3}$ is equivalent to the system of equations

$$\begin{array}{rcrcrcr} a_0 & + & 2a_1 & + & 8a_2 & = & -29 \\ & & a_1 & + & 3a_2 & = & -14 \\ & & & & a_2 & = & -8 \end{array}.$$

   The system has unique solution $a_0 = 15, a_1 = 10, a_2 = -8$ so $p(t) = 15 + 10t - 8t^2 + t^3$.

6. $\mathbf{w_0} = \mathbf{e_1}; \mathbf{w_1} = \mathbf{e_2}; \mathbf{w_2} = \mathbf{e_3}, \mathbf{w_3} = \mathbf{e_1}$. The vector equation $a_0\mathbf{w_0} + +a_1\mathbf{w_1} + a_2\mathbf{w_2} = -\mathbf{w_3}$ has solution $a_0 = -1, a_1 = a_2 = 0$. Therefore $p(t) = -1 + t^3$.

7. Note that the given matrix H is in unreduced Hessenberg form. We have $\mathbf{w_0} = \mathbf{e_1} = [1,0,0,0]^{\mathrm{T}}, \mathbf{w_1} = H\mathbf{w_0} = [0,1,0,0]^{\mathrm{T}}$,
   $\mathbf{w_2} = H\mathbf{w_1} = [1,2,1,0]^{\mathrm{T}}, \mathbf{w_3} = H\mathbf{w_2} = [2,6,2,2]^{\mathrm{T}}$, and $\mathbf{w_4} =$

   $H\mathbf{w_3} = [8,18,8,6]^{\mathrm{T}}$. The vector equation $a_0\mathbf{w_0} + a_1\mathbf{w_1} + a_2\mathbf{w_2} + a_3\mathbf{w_3} = -\mathbf{w_4}$ is equivalent to the system of equations

$$\begin{array}{rcrcrcrcr} a_1 & & & + & a_2 & + & 2a_3 & = & -8 \\ & & a_1 & + & 2a_2 & + & 6a_3 & = & -18 \\ & & & & a_2 & + & 2a_3 & = & -8 \\ & & & & & & 2a_3 & = & -6 \end{array}.$$

   The system has unique solution $a_0 = 0, a_1 = 4, a_2 = -2, a_3 = -3$, so $p(t) = 4t - 2t^2 - 3t^3 + t^4$.

8. $\mathbf{w_0} = \mathbf{e_1} = [1,0,0,0]^{\mathrm{T}}, \mathbf{w_1} = [0,1,0,0]^{\mathrm{T}}, \mathbf{w_2} = [2,0,2,0]^{\mathrm{T}}, \mathbf{w_3} = [2,4,0,2]^{\mathrm{T}}$, and $\mathbf{w_4} = [12,0,12,2]^{\mathrm{T}}$. The vector equation $a_0\mathbf{w_0} +$

   $a_1\mathbf{w_1} + a_2\mathbf{w_2} + a_3\mathbf{w_3} = -\mathbf{w_4}$ has solution $a_0 = 2, a_1 = 4, a_2 = -6,$

   $a_3 = -1$. Therefore $p(t) = 2 + 4t - 6t^2 - t^3 + t^4$.

9. $H = \begin{bmatrix} B_{11} & B_{12} \\ \mathcal{O} & B_{22} \end{bmatrix}$ where $B_{11} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}, B_{12} = \begin{bmatrix} 1 & 4 \\ -2 & 1 \end{bmatrix}$ and $B_{22} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. B_{11} has eigenvalue $\lambda_1 = 2$ (with algebraic multiplicity 2) with corresponding eigenvector $\mathbf{u_1} = [-1,1]^{\mathrm{T}}$.$B_{22}$ has eigenvalues $\lambda_2 = 1, \lambda_3 = 3$ with corresponding eigenvectors $\mathbf{v_2} = [1,1]^{\mathrm{T}}$ and $\mathbf{v_3} = [-1,1]^{\mathrm{T}}$, respectively. Thus H has eigenvalues $\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = 3$. The vector $\mathbf{x_1} = \begin{bmatrix} \mathbf{u_1} \\ \theta \end{bmatrix} = [-1,1,0,0]^{\mathrm{T}}$ is an eigenvector for H corresponding to $\lambda_1 = 2$. The system of equations $(B_{11} - I)\mathbf{u} = -B_{12}\mathbf{v_2}$ has solution $\mathbf{u_2} = [-9,5]^{\mathrm{T}}$, so $\mathbf{x_2} = \begin{bmatrix} \mathbf{u_2} \\ \mathbf{v_2} \end{bmatrix} = [-9,5,1,1]^{\mathrm{T}}$ is an eigenvector of H corresponding to $\lambda_2 = 1$. Similarly $(B_{11} - 3I)\mathbf{u} = -B_{12}\mathbf{v_3}$ has solution $\mathbf{u_3} = [-3,9]^{\mathrm{T}}$ so $\mathbf{x_3} = \begin{bmatrix} \mathbf{u_3} \\ \mathbf{v_3} \end{bmatrix} = [-3,9,-1,1]^{\mathrm{T}}$ is an eigenvector of H corresponding to $\lambda_3 = 3$.

10. $B_{11} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B_{22} = \begin{bmatrix} 3 & 0 \\ 1 & 4 \end{bmatrix}$. B_{11} has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 2$ and B_{22} has eigenvalues $\lambda_3 = 3, \lambda_4 = 4$. The corresponding eigenvectors are $\mathbf{x_1} = [-1,1,0,0]^{\mathrm{T}}, \mathbf{x_2} = [1,1,0,0]^{\mathrm{T}}, \mathbf{x_3} = [0,1,-1,1]^{\mathrm{T}}$ and $\mathbf{x_4} = [3/4,5/4,0,1]^{\mathrm{T}}$.

11. $H = \begin{bmatrix} B_{11} & B_{12} \\ \mathcal{O} & B_{22} \end{bmatrix}$ where $B_{11} = \begin{bmatrix} -2 & 0 & -2 \\ -1 & 1 & -2 \\ 0 & 1 & -1 \end{bmatrix}, B_{12} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$,

    and $B_{22} = [2]$. B_{11} has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -1$ (algebraic multiplicity 2) with corresponding eigenvectors $\mathbf{u_1} = [-1,1,1]^{\mathrm{T}}$ and $\mathbf{u_2} = [-2,0,1]^{\mathrm{T}}$.$B_{22}$ has eigenvalue $\lambda_3 = 2$ with corresponding eigenvector $\mathbf{v_3} = [1]$. Thus H has eigenvalues $\lambda_1 = 0, \lambda_2 = -1$, and $\lambda_3 = 2$. The vectors $\mathbf{x_1} = \begin{bmatrix} \mathbf{u_1} \\ \theta \end{bmatrix} = [-1,1,1,0]^{\mathrm{T}}$ and $\mathbf{x_2} = \begin{bmatrix} \mathbf{u_2} \\ \theta \end{bmatrix} = [-2,0,1,0]^{\mathrm{T}}$ are eigenvectors for H corresponding to $\lambda_1 = 0$ and $\lambda_2 = -1$, respectively. The system of equations $(B_{11} - 2I)\mathbf{u_1} = -B_{12}\mathbf{v_3}$ has solution $\mathbf{u_3} = [1/6,15/6,1/6]^{\mathrm{T}}$ so $\mathbf{x_3} = \begin{bmatrix} \mathbf{u_3} \\ \mathbf{v_3} \end{bmatrix} =$

    $[1/6,15/6,1/6,1]^{\mathrm{T}}$ is an eigenvector for H corresponding to $\lambda_3 = 2$.

12. $B_{11} = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$ and $B_{22} = \begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}$. B_{11} has eigenvalues $\lambda_1 = 5$ and $\lambda_2 = -1$ and B_{22} has eigenvalue $\lambda_3 = 3$. The corresponding eigenvectors for H are $\mathbf{x_1} = [1,1,0,0]^{\mathrm{T}}, \mathbf{x_2} = [-1,1,0,0]^{\mathrm{T}}$, and $\mathbf{x_3} = [-7/8,-13/8,0,1]^{\mathrm{T}}$.

13. $\det(B) = afwz - afyx - ebwz + ebyx = (af - eb)(wz - yx) = \det(B_{11})\det(B_{22})$.

14. $\det(H) = \begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix} \text{ x } \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = (4)(3) = 12.$

15. $P = [\mathbf{e_2}, \mathbf{e_3}, \mathbf{e_1}]$.

16. $P = [\mathbf{e_2}, \mathbf{e_3}, \mathbf{e_4}, \mathbf{e_1}]$.

17. $P = [\mathbf{e_2}, \mathbf{e_3}, \ldots, \mathbf{e_n}, \mathbf{e_1}]$.

18. Write $P = [\mathbf{P_1}, \mathbf{P_2}, \ldots, \mathbf{P_n}]$ where, as shown in Exercise 17, $\mathbf{P_1} = \mathbf{e_2}, \mathbf{P_2} = \mathbf{e_3}, \ldots, \mathbf{P_{n-1}} = \mathbf{e_n}$, and $\mathbf{P_n} = \mathbf{e_1}$. Thus $\mathbf{w_0} = \mathbf{e_1}, \mathbf{w_1} = P\mathbf{w_0} = P\mathbf{e_1} = \mathbf{P_1} = \mathbf{e_2}, \mathbf{w_2} = P\mathbf{w_1} = P\mathbf{e_2} = \mathbf{P_2} = \mathbf{e_3}, \ldots, \mathbf{w_{n-1}} = P\mathbf{w_{n-2}} = P\mathbf{e_{n-1}} = \mathbf{P_{n-1}} = \mathbf{e_n}$, and $\mathbf{w_n} = P\mathbf{w_{n-1}} = P\mathbf{e_n} = \mathbf{P_n} = \mathbf{e_1}$. Obviously the vector equation $a_0\mathbf{w_0} + a_1\mathbf{w_1} + \cdots + a_{n-1}\mathbf{w_{n-1}} = -\mathbf{w_n}$ has solution $a_0 = -1, a_1 = \cdots = a_{n-1} = 0$. Therefore $p(t) = t^n - 1$.

19. Let $H = [h_{ij}]$ and let λ be an eigenvalue for H, Then

$$H - \lambda I = \begin{bmatrix} h_{11}-\lambda & h_{12} & \cdots & h_{1,n-1} & h_{1n} \\ h_{21} & h_{22}-\lambda & & h_{2,n-1} & h_{2n} \\ 0 & h_{32} & & h_{3,n-1} & h_{3n} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & h_{n-1,n-1}-\lambda & h_{n-1,n} \\ 0 & 0 & & h_{n,n-1} & h_{nn}-\lambda \end{bmatrix}.$$

Since $h_{21}, h_{32}, \ldots, h_{n,n-1}$ are nonzero, the first $n-1$ columns of $H - \lambda I$ form a linearly independent set. Therefore rank $(H - \lambda I) \geq n - 1$. It follows that nullity $(H - \lambda I) \leq 1$. Since λ is an eigenvalue for H it follows that nullity $(H - \lambda I) = 1$; that is, λ has geometric multiplicity 1.

20. Since H is symmetric, it is diagonalizable. Therefore the algebraic multiplicity of λ equals the geometric multiplicity. It follows from Exercise 19 that λ has algebraic multiplicity 1. Therefore H neces-
sarily has n distinct eigenvalues.

21. If H is unreduced then $b \neq 0$. Thus $p(t) = t^2 - (a+c)t - b^2$. The eigenvalues for H are $\lambda = [(a+c) \pm \sqrt{(a+c)^2 + 4b^2}]/2$. Since $(a+c)^2 + 4b^2 > 0, H$ has two distinct eigenvalues.

22. Set $\mathbf{u} = [u_1, u_2, \ldots, u_n]^{\mathrm{T}}$ and assume that $u_n = 0$. Set $H = [\mathbf{h_1}, \mathbf{h_2}, \ldots, \mathbf{h_n}]$. Thus $\lambda\mathbf{u} = H\mathbf{u} = u_1\mathbf{h_1} + u_2\mathbf{h_2} + \cdots + u_{n-1}\mathbf{h_{n-1}}$. There-

fore the n^{th} component of $H\mathbf{u}$ is $u_{n-1}h_{n,n-1}$. Since $h_{n,n-1} \neq 0$ it follows that $u_{n-1} = 0$. Repetition of this argument yields $u_1 = u_2 = \cdots = u_n = 0$, so $\mathbf{u} = \theta$. Therefore if $\mathbf{u} \neq \theta$ then $u_n \neq 0$.

23. Let k be an integer, $1 \leq k \leq n$, and suppose we have shown that $\mathbf{w_{k-1}}$ has the form $\mathbf{w_{k-1}} = [a_1 \ldots, a_k, 0, \ldots, 0]^{\mathrm{T}}$, where $a_k \neq 0$. If $H = [\mathbf{h_1}, \ldots, \mathbf{h_n}]$ then $\mathbf{w_k} = H\mathbf{w_{k-1}} = a_0\mathbf{h_1} + \cdots + a_k\mathbf{h_k}$. But H is in Hessenberg form so $h_{ij} = 0$ when $i > j+1$. Therefore the $k+1$ component of $\mathbf{w_k}$ is $a_k h_{k+1,k}$ and is nonzero since H is unreduced. Thus $\mathbf{w_k}$ has the form $\mathbf{w_k} = [b_1, \ldots, b_{k+1}, 0, \ldots, 0]^{\mathrm{T}}$, where $b_{k+1} \neq 0$.

7.5 Householder Transformations

1. $Q\mathbf{x} = \mathbf{x} - \gamma\mathbf{u}$ where $\gamma = 2\mathbf{u}^{\mathrm{T}}\mathbf{x}/\mathbf{u}^{\mathrm{T}}\mathbf{u} = (-2)(2)/4 = -1$. Thus $Q\mathbf{x} = [4, 1, 6, 7]^{\mathrm{T}}$.

2. $Q\mathbf{x} = [4, -3, 5, 4[^{\mathrm{T}}$.

3. Set $\gamma_1 = 2\mathbf{u}^{\mathrm{T}}\mathbf{A_1}/\mathbf{u}^{\mathrm{T}}\mathbf{u} = -1$ and $\gamma_2 = 2\mathbf{u}^{\mathrm{T}}\mathbf{A_2}/\mathbf{u}^{\mathrm{T}}\mathbf{u} = -2$. Then $Q\mathbf{A_1} = \mathbf{A_1} - \gamma_1\mathbf{u} = [3, 5, 5, 1]^{\mathrm{T}}$ and $Q\mathbf{A_2} = \mathbf{A_2} - \gamma_2\mathbf{u} = [3, 1, 4, 2]^{\mathrm{T}}$.

   Therefore $QA = \begin{bmatrix} 3 & 3 \\ 5 & 1 \\ 5 & 4 \\ 1 & 2 \end{bmatrix}$.

4. $QA = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 0 & 2 \\ 3 & 6 & 2 \\ 1 & 5 & 3 \end{bmatrix}$.

5. Set $\gamma = 2\mathbf{u}^{\mathrm{T}}\mathbf{x}/\mathbf{u}^{\mathrm{T}}\mathbf{u} = -1$. Then $Q\mathbf{x} = \mathbf{x} - \gamma\mathbf{u} = \mathbf{x} + \mathbf{u} = [4, 1, 3, 4]^{\mathrm{T}}$.
   Thus $\mathbf{x}^{\mathrm{T}}Q = (Q\mathbf{x})^{\mathrm{T}} = [4, 1, 3, 4]$.

6. $\mathbf{x}^{\mathrm{T}}Q = [2, 2, 3, 1]$.

7. Set $\mathbf{x} = [2, 1, 2, 1]^{\mathrm{T}}$ and $\mathbf{y} = [1, 0, 1, 4]^{\mathrm{T}}$. Then $QA^{\mathrm{T}} = Q[\mathbf{x}, \mathbf{y}] =$

   $[Q\mathbf{x}, Q\mathbf{y}] = \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 1 & 2 \\ 2 & 3 \end{bmatrix}$. Therefore $AQ = (QA^{\mathrm{T}})^{\mathrm{T}} =$

   $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & -1 & 2 & 3 \end{bmatrix}$.

8. $BQ = \begin{bmatrix} 1/2 & 5/2 & 7/2 & 5/2 \\ 6 & 0 & -1 & -1 \\ -7/2 & 13/2 & 11/2 & 5/2 \\ 3 & -7 & 1 & 5 \end{bmatrix}$.

9. Set $u_1 = 0$. If $a = -\sqrt{4+4+1} = -3$ then $u_2 = v_2 - a = 2+3 = 5$. Finally take $u_3 = v_3 = 2$ and $u_4 = v_4 = 1$. Thus $\mathbf{u} = [0, 5, 2, 1]^{\mathrm{T}}$.

10. $a = -2, \mathbf{u} = [3, 1, 1, 1]^{\mathrm{T}}$.

11. $a = -\sqrt{4^2 + 3^2} = -5; u_1 = u_2 = 0; u_3 = v_3 - a = 4 + 5 = 9; u_4 = v_4 = 3$. Therefore $\mathbf{u} = [0, 0, 9, 3]^{\mathrm{T}}$.

12. $a = 3; \mathbf{u} = [0, 0, -5, 2, 1]^{\mathrm{T}}$.

13. $a = \sqrt{(-3)^2 + 4^2} = 5; u_1 = u_2 = u_3 = 0; u_4 = v_4 - a = -8; u_5 = v_5 = 4$. Therefore $\mathbf{u} = [0, 0, 0, -8, 4]^{\mathrm{T}}$.

14. $a = -4; \mathbf{u} = [0, 0, 8, 0, 0]^{\mathrm{T}}$.

15. We want $QA_1 = [1, a, 0]^{\mathrm{T}}$. Therefore $a = -\sqrt{3^2 + 4^2} = -5$ and
$\mathbf{u} = [u_1, u_2, u_3]^{\mathrm{T}}$ where $u_1 = 0, u_2 = 3 - (-5) = 8,$ and $u_3 = 4$.
Then $\mathbf{u} = [0, 8, 4]^{\mathrm{T}}$.

16. $\mathbf{u} = [0, -5, 5]^{\mathrm{T}}$.

17. We want $Q\mathbf{A_1} = \begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix}$. Therefore $a = \sqrt{(-4)^2 + 3^2} = 5$ and
$\mathbf{u} = [u_1, u_2, u_3]^{\mathrm{T}}$ where $u_1 = 0, u_2 = -4 - 5 = -9,$ and $u_3 = 3$.
Thus $\mathbf{u} = [0, -9, 3]^{\mathrm{T}}$.

18. $\mathbf{u} = [0, 0, 8, 4]^{\mathrm{T}}$.

19. We want $Q\mathbf{A_2} = \begin{bmatrix} 1 \\ 4 \\ a \\ 0 \end{bmatrix}$ so $a = \sqrt{(-3)^2 + 4^2} = 5$. $\mathbf{u} =$
$[u_1, u_2, u_3, u_4]^{\mathrm{T}}$ where $u_1 = u_2 = 0, u_3 = -3 - 5 = -8,$ and $u_4 = 4$.
Thus $\mathbf{u} = [0, 0, -8, 4]^{\mathrm{T}}$.

20. $\mathbf{u} = [0, 0, -1, 1]^{\mathrm{T}}$.

21. $Q^{\mathrm{T}} = (I - b\mathbf{u}\mathbf{u}^{\mathrm{T}})^{\mathrm{T}} = I^{\mathrm{T}} - (b\mathbf{u}\mathbf{u}^{\mathrm{T}})^{\mathrm{T}} = I - b\mathbf{u}^{\mathrm{TT}}\mathbf{u}^{\mathrm{T}} = I - b\mathbf{u}\mathbf{u}^{\mathrm{T}} = Q$.

22. Set $b = 2/\mathbf{u}^{\mathrm{T}}\mathbf{u}$. Then $Q\mathbf{u} = (I - b\mathbf{u}\mathbf{u}^{\mathrm{T}})\mathbf{u} = I\mathbf{u} - (b\mathbf{u}\mathbf{u}^{\mathrm{T}})\mathbf{u} =$
$\mathbf{u} - b\mathbf{u}(\mathbf{u}^{\mathrm{T}}\mathbf{u}) = \mathbf{u} - 2\mathbf{u} = -\mathbf{u}$. If $\mathbf{u}^{\mathrm{T}}\mathbf{v} = 0$ then $Q\mathbf{v} = (I - b\mathbf{u}\mathbf{u}^{\mathrm{T}})\mathbf{v} =$
$I\mathbf{v} - b\mathbf{u}(\mathbf{u}^{\mathrm{T}}\mathbf{v}) = \mathbf{v}$.

23. Let $\{\mathbf{u}, \mathbf{w_2}, \ldots, \mathbf{w_n}\}$ be as given in the hint. By Exercise 22, $Q\mathbf{u} = -\mathbf{u}$ and $Q\mathbf{w_i} = \mathbf{w_i}$ for $2 \le i \le n$. Thus R^n has a basis consisting of eigenvectors for Q; that is Q is diagonalizable. Moreover Q is similar to the $(n \,\mathrm{x}\, \mathrm{n})$ diagonal matrix D with diagonal entries $d_{11} = 1, d_{22} = \cdots = d_{nn} = -1$. Since Q and D have the same eigenvalues, Q has eigenvalues 1 and -1.

24. To prove (a) note that $Q^{-1} = (Q_{n-2} \cdots Q_2 Q_1)^{-1} =$
$Q_1^{-1} Q_2^{-1} \cdots Q_{n-2}^{-1} = Q_1^{\mathrm{T}} Q_2^{\mathrm{T}} \cdots Q_{n-2}^{\mathrm{T}} = (Q_{n-2} \cdots Q_2 Q_1)^{\mathrm{T}} = Q^{\mathrm{T}}$. To prove (b) assume that A is symmetric. Thus $H^{\mathrm{T}} = (QAQ^{\mathrm{T}})^{\mathrm{T}} =$
$Q^{\mathrm{TT}} A^{\mathrm{T}} Q^{\mathrm{T}} = QAQ^{\mathrm{T}} = H$ and H is symmetric.

25. (a) Set $B^{\mathrm{T}} = [\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}]$. Then $QB^{\mathrm{T}} = [Q\mathbf{v_1}, Q\mathbf{v_2}, Q\mathbf{v_3}, Q\mathbf{v_4}]$, and for $1 \le j \le 4, Q\mathbf{v_j} = \mathbf{v_j} - \gamma_{\mathbf{j}}\mathbf{u}$, where γ_j is a constant. Since $\mathbf{u} = [0, a, b, c]^{\mathrm{T}}$ it follows that $Q\mathbf{v_j}$ and $\mathbf{v_j}$ have the same first coordinate. Thus B^{T} and QB^{T} have the same first row. It follows that B and $BQ = (QB^{\mathrm{T}})^{\mathrm{T}}$ have the same first column.
(b) It follows from (a) that $\mathbf{x} = [b_{11}, b_{12}, 0, 0]^{\mathrm{T}}$ is the first column of BQ. Thus $Q\mathbf{x}$ is the first column of QBQ. But $Q\mathbf{x} = \mathbf{x} - \gamma\mathbf{u}$ where $\gamma = 2\mathbf{u}^{\mathrm{T}}\mathbf{x}/\mathbf{u}^{\mathrm{T}}\mathbf{u} = 0$; that is $Q\mathbf{x} = \mathbf{x}$.

7.6 QR Factorization & Least-Squares

1. $\mathbf{x}^*$ is the unique solution to $R\mathbf{x} = \mathbf{c}$, where $R = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and
$\mathbf{c} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Thus $\mathbf{x}^* = [1, 1]^{\mathrm{T}}$.

2. $\mathbf{x}^* = [2, -1]^{\mathrm{T}}$.

3. $\mathbf{x}^*$ is the unique solution to $R\mathbf{x} = \mathbf{c}$ where $R = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$ and
$\mathbf{c} = \begin{bmatrix} 6 \\ 7 \\ 4 \end{bmatrix}$. Thus $\mathbf{x}^* = [2, 1, 2]^{\mathrm{T}}$.

4. $\mathbf{x}^* = [1, 2, 1]^{\mathrm{T}}$.

5. We require that $S\mathbf{A_1} = [a, 0]^{\mathrm{T}}$. Therefore $a = \pm\sqrt{a_{11}^2 + a_{21}^2} = -5, u_1 = a_{11} - a = 8$, and $u_2 = a_{21} = 4$. Consequently $\mathbf{u} = [8, 4]^{\mathrm{T}}$ and $S\mathbf{A_1} = \mathbf{A_1} - \mathbf{u} = [-5, 0]^{\mathrm{T}}$. $S\mathbf{A_2} = \mathbf{A_2} - 2\mathbf{u} = [-11, 2]^{\mathrm{T}}$, so $SA = R = \begin{bmatrix} -5 & -11 \\ 0 & 2 \end{bmatrix}$.

6. $\mathbf{u} = [1, 1]^{\mathrm{T}}; R = \begin{bmatrix} -1 & -5 \\ 0 & -3 \end{bmatrix}$.

7. We require that $S\mathbf{A_1} = \begin{bmatrix} a \\ 0 \end{bmatrix}$ so take $a = -\sqrt{a_{11}^2 + a_{21}^2} = -4$, $u_1 = a_{11} - a = 4$, and $u_2 = a_{21} = 4$. Thus $\mathbf{u} = [4, 4]^{\mathrm{T}}$ and $S\mathbf{A_1} = \mathbf{A_1} - \mathbf{u} = [-4, 0]^{\mathrm{T}}$. Also $S\mathbf{A_2} = \mathbf{A_2} - 2\mathbf{u} = [-6, -2]^{\mathrm{T}}$. Therefore $SA = R = \begin{bmatrix} -4 & -6 \\ 0 & -2 \end{bmatrix}$.

8. $\mathbf{u} = [-22, 4]^{\mathrm{T}}; R = \begin{bmatrix} 5 & -22 \\ 0 & 4 \end{bmatrix}$.

9. We require that $S\mathbf{A_2} = [2, a, 0]^{\mathrm{T}}$ so set $a = -\sqrt{a_{22}^2 + a_{32}^2} = -1; u_1 = 0, u_2 = a_{22} - a = 1$, and $u_3 = a_{32} = 1$. Therefore $\mathbf{u} = [0, 1, 1]^{\mathrm{T}}, S\mathbf{A_1} = \mathbf{A_1}, S\mathbf{A_2} = \mathbf{A_2} - \mathbf{u} = [2, -1, 0]^{\mathrm{T}}$, and $S\mathbf{A_3} = \mathbf{A_3} - 14\mathbf{u} = [1, -8, -6]^{\mathrm{T}}$. Consequently $SA = R = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -8 \\ 0 & 0 & -6 \end{bmatrix}$.

10. $\mathbf{u} = [0, 8, 4]^{\mathrm{T}}; R = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -5 & -11 \\ 0 & 0 & 2 \end{bmatrix}$.

11. We first require Q_1 such that $Q_1\mathbf{A_1} = [a, 0, 0, 0]^{\mathrm{T}}$ so take $\mathbf{u_1} = [6, 2, 2, 4]^{\mathrm{T}}$. Then $Q_1\mathbf{A_1} = \mathbf{A_1} - \mathbf{u_1} = [-5, 0, 0, 0]^{\mathrm{T}}$ and $Q_1\mathbf{A_2} = \mathbf{A_2} - 3\mathbf{u} = [-59/3, 6, 2, 3]^{\mathrm{T}}$. We now require Q_2 such that $Q_2(Q_1\mathbf{A_2}) = [-59/3, a, 0, 0]^{\mathrm{T}}$. Thus set $\mathbf{u_2} = [0, 13, 2, 3]^{\mathrm{T}}$. Then $Q_2(Q_1\mathbf{A_1}) = Q_1\mathbf{A_1}$ and $Q_2(Q_1\mathbf{A_2}) = Q_1\mathbf{A_2} - \mathbf{u_2} = [-59/3, -7, 0, 0]^{\mathrm{T}}$. Therefore $Q_2Q_1A = \begin{bmatrix} -5 & -59/3 \\ 0 & -7 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

12. $\mathbf{u_1} = [3, 1, 1, 1]^{\mathrm{T}}$ and $Q_1A = \begin{bmatrix} -2 & -7 \\ 0 & 0 \\ 0 & 0 \\ 0 & 3 \end{bmatrix}$. $\mathbf{u_2} = [0, 3, 0, 3]^{\mathrm{T}}$ and

$$Q_2Q_1A = \begin{bmatrix} -2 & -7 \\ 0 & -3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

13. We require a matrix Q such that $Q\mathbf{A_2} = [4, a, 0, 0]^{\mathrm{T}}$. Thus $\mathbf{u} = [0, 8, 0, 4]^{\mathrm{T}}$ and $QA = \begin{bmatrix} 2 & 4 \\ 0 & -5 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

14. $\mathbf{u} = [0, 5, 1, 2]^{\mathrm{T}}$ and $QA = \begin{bmatrix} 3 & 5 \\ 0 & -3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

15. Let $Q_1, \mathbf{u_1}, Q_2, \mathbf{u_2}$ be as in Exercise 11. Then $Q_1\mathbf{b} = \mathbf{b} - \mathbf{u_1} = [-5, 8, -2, -3]^{\mathrm{T}}$ and $Q_2(Q_1\mathbf{b}) = Q_1\mathbf{b} - \mathbf{u_2} = [-5, -5, -4, -6]^{\mathrm{T}}$.

The least-squares solution is the unique solution $\mathbf{x}^*$ to $R\mathbf{x} = \mathbf{c}$ where $R = \begin{bmatrix} -5 & -59/3 \\ 0 & -7 \end{bmatrix}$ and $\mathbf{c} = [-5, -5]^{\mathrm{T}}$. Thus $\mathbf{x}^* =$ $[-38/21, 15/21]^{\mathrm{T}}$.

16. $Q_2Q_1\mathbf{b} = [-4, 2, -1, 3]$. $\mathbf{x}^*$ is the unique solution to $R\mathbf{x} = \mathbf{c}$ where $R = \begin{bmatrix} -2 & -7 \\ 0 & -3 \end{bmatrix}$ and $\mathbf{c} = [-4, 2]^{\mathrm{T}}$. Thus $\mathbf{x}^* = [13/3, -2/3]^{\mathrm{T}}$.

17. With Q and $\mathbf{u}$ as in Exercise 13, $Q\mathbf{b} = \mathbf{b} - (12/5)\mathbf{u} =$ $[2, -56/5, 16, -8/5]^{\mathrm{T}}$. Therefore $\mathbf{x}^*$ is the unique solution to $R\mathbf{x} = \mathbf{c}$ where $R = \begin{bmatrix} 2 & 4 \\ 0 & -5 \end{bmatrix}$ and $\mathbf{c} = [2, -56/5]^{\mathrm{T}}$. Solving yields $\mathbf{x}^* = [-87/25, 56/25]^{\mathrm{T}}$.

18. $Q\mathbf{b} = [5, -10/3, -19/3, -8/15]^{\mathrm{T}}$. $\mathbf{x}^*$ is the unique solution to $R\mathbf{x} = \mathbf{c}$ where $R = \begin{bmatrix} 3 & 5 \\ 0 & -3 \end{bmatrix}$ and $\mathbf{c} = [5, -10/3]^{\mathrm{T}}$. Solving yields $\mathbf{x}^* = [-5/27, 10/9]^{\mathrm{T}}$.

19. Write $[\mathbf{A_1}, \mathbf{A_2}, \ldots, \mathbf{A_n}]$, where $\{\mathbf{A_1}, \mathbf{A_2}, \ldots, \mathbf{A_n}\}$ is a linearly independent subset of R^n. Now $SA = [S\mathbf{A_1}, S\mathbf{A_2}, \ldots, S\mathbf{A_n}]$. Suppose $c_1, c_2, \ldots, c_n$ are scalars such that $\theta = c_1 S\mathbf{A_1} + c_2 S\mathbf{A_2} + \cdots + c_n S\mathbf{A_n}$.

Then $\theta = S(c_1\mathbf{A_1} + c_2\mathbf{A_2} + \cdots + c_n\mathbf{A_n})$ and S is nonsingular. Therefore $\theta = c_1\mathbf{A_1} + c_2\mathbf{A_2} + \cdots + c_n\mathbf{A_n}$. It follows that $c_1 = c_2 = \cdots = c_n = 0$ and hence, the set $\{S\mathbf{A_1}, S\mathbf{A_2}, \ldots, S\mathbf{A_n}\}$ is linearly independent. For each $j, 1 \le j \le n, S\mathbf{A_j} = \begin{bmatrix} \mathbf{R_j} \\ \theta \end{bmatrix}$ where $\mathbf{R_j}$ is the j^{th} column of R and θ is in R^{m-n}. Therefore the set $\{\mathbf{R_1}, \mathbf{R_2}, \ldots, \mathbf{R_n}\}$ is linearly independent in R^n and the matrix R is nonsingular.

7.7 Matrix Polynomials & The Cayley-Hamilton Theorem

1. $q(A) = A^2 - 4A + 3I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$; $q(B) = B^2 - 4B + 3I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; $q(C) = C^2 - 4C + 3I = \begin{bmatrix} 15 & -2 & 14 \\ 5 & -2 & 10 \\ -1 & -4 & 6 \end{bmatrix}$.

2. (a) $p(A) = (A - I)^3 = \mathcal{O}$; $p(B) = (B - I)^3 = \mathcal{O}$;
   $p(C) = (C - I)^3 = \mathcal{O}$; $p(I) = (I - I)^3 = \mathcal{O}^3 = \mathcal{O}$

   (b) Set $q(t) = (t - 1)^2 = t^2 - 2t + 1$.

3. (a) $q(t) = s(t)p(t) + r(t)$ where $s(t) = t^3 + t - 1$ and $r(t) = t + 2$.

   (b) $q(B) = s(B)p(B)+r(B) = r(B)$ since $p(B) = \mathcal{O}$. Thus $q(B) = B+2I = \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}$.

4. $H_{11} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}, H_{22} = \begin{bmatrix} 4 & 1 & 3 \\ 6 & 1 & 2 \\ 0 & 4 & 1 \end{bmatrix}$ and $H_{33} = \begin{bmatrix} 6 & 5 \\ 7 & 3 \end{bmatrix}$. Using Algorthim 1 we obtain $p_1(t) = t^2 - 9t - 1, p_2(t) = t^3 - 6t^2 - 5t - 38,$ and $p_3(t) = t^2 - 9t - 17$.

5. Note that $H^2 = (SAS^{-1})(SAS^{-1}) = (SA^2S^{-1})$. For some positive integer $k \ge 2$ suppose we have shown that $H^k = SA^kS^{-1}$. Then $H^{k+1} = H^kH = (SA^kS^{-1})(SAS^{-1}) = SA^{k+1}S^{-1}$. It follows by induction that $H^n = SA^nS^{-1}$ for each positive integer n. Now let $q(t) = a_nt^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0$. Then $q(H) = a_nH^n + a_{n-1}H^{n-1} + \cdots + a_1H + a_0I = a_nSA^nS^{-1} + a_{n-1}SA^{n-1}S^{-1} + \cdots + a_1SAS^{-1} + a_0SIS^{-1} = S(a_nA^n + a_{n-1}A^{n-1} + \cdots + a_1A + a_0I)S^{-1} = Sq(A)S^{-1}$.

6. Since $\mathbf{u_1}^{\mathrm{T}}A\mathbf{u_2}$ is a (1 x 1) matrix, $\mathbf{u_1}^{\mathrm{T}}A\mathbf{u_2} = (\mathbf{u_1}^{\mathrm{T}}A\mathbf{u_2})^{\mathrm{T}} = \mathbf{u_2}^{\mathrm{T}}A^{\mathrm{T}}\mathbf{u_1}^{\mathrm{TT}} = \mathbf{u_2}^{\mathrm{T}}A\mathbf{u_1}$. Now $\mathbf{u_1}^{\mathrm{T}}A\mathbf{u_2} = \lambda_2\mathbf{u_1}^{\mathrm{T}}\mathbf{u_2}$ whereas $\mathbf{u_2}^{\mathrm{T}}A\mathbf{u_1} = \lambda_1\mathbf{u_2}^{\mathrm{T}}\mathbf{u_1} = \lambda_1\mathbf{u_1}^{\mathrm{T}}\mathbf{u_2}$. Therefore $\lambda_1\mathbf{u_1}^{\mathrm{T}}\mathbf{u_2} = \lambda_2\mathbf{u_1}^{\mathrm{T}}\mathbf{u_2}$.
   Since $\lambda_1 \ne \lambda_2$ it follows that $\mathbf{u_1}^{\mathrm{T}}\mathbf{u_2} = \mathbf{0}$.

7. (a) By assumption, $A\mathbf{x_0}$ is in W. For some positive integer k, suppose we have shown that $A^k\mathbf{x_0}$ is in W. Then $A^{k+1}\mathbf{x_0} = A(A^k\mathbf{x_0})$ is in W by the assumed property of A. By induction, $A^n\mathbf{x_0}$ is in W for each positive integer n.

   (b) $\theta = m(A)\mathbf{x_0} = (A - rI)s(A)\mathbf{x_0}$. Since $s(t)$ has degree $k-1$,

   $S(A)\mathbf{x_0} \neq \theta$. Thus if $\mathbf{u} = S(A)\mathbf{x_0}$ then $(A - rI)\mathbf{u} = \theta$. It follows that $\mathbf{u}$ is an eigenvector of A corresponding to the eigenvalue r. Now if $s(t) = b_{k-1}t^{k-1} + \cdots + b_1 t + b_0$ then $S(A)\mathbf{x_0} = b_{k-1}A^{k-1}\mathbf{x_0} + \cdots + b_1 A\mathbf{x_0} + b_0\mathbf{x_0}$. Since r is an eigenvalue for A, r is real. It follows that $b_0, \ldots, b_{k-1}$ are real. Thus $S(A)\mathbf{x}_0$ is a linear combination of the vectors $A^{k-1}\mathbf{x}_0, \ldots, A\mathbf{x}_0, \mathbf{x}_0$ in W.

   Hence $S(A)\mathbf{x_0}$ is in W.

8. (a) Let $\mathbf{x}$ and $\mathbf{y}$ be in W; that is, $\mathbf{x}\mathbf{u_i}^{\mathrm{T}} = 0$ and $\mathbf{y}\mathbf{u_i}^{\mathrm{T}} = \mathbf{0}$ for $1 \leq i \leq k$. Therefore $(\mathbf{x}+\mathbf{y})\mathbf{u_i}^{\mathrm{T}} = \mathbf{x}\mathbf{u_i}^{\mathrm{T}} + \mathbf{y}\mathbf{u_i}^{\mathrm{T}} = 0 + 0 = 0$ for $1 \leq i \leq k$. Consequently, $\mathbf{x}+\mathbf{y}$ is in W. Likewise if c is a scalar then $(c\mathbf{x})\mathbf{u_i}^{\mathrm{T}} = c(\mathbf{x}\mathbf{u_i}^{\mathrm{T}}) = c0 = 0$, for $1 \leq i \leq k$, so $c\mathbf{x}$ is in W. Certainly θ is in W, so W is a subspace of R^n.

   (b) Suppose that $A\mathbf{u_i} = \lambda_i\mathbf{u_i}$, $1 \leq i \leq k$, and let $\mathbf{x}$ be in W. Thus $\mathbf{x}^{\mathrm{T}}\mathbf{u_i} = 0$ for $1 \leq i \leq k$. Now $(A\mathbf{x})^{\mathrm{T}}\mathbf{u_i} = \mathbf{x}^{\mathrm{T}}A^{\mathrm{T}}\mathbf{u_i} = \mathbf{x}^{\mathrm{T}}A\mathbf{u_i} = \mathbf{x}^{\mathrm{T}}(\lambda_{\mathbf{i}}\mathbf{u_i}) = \lambda_{\mathbf{i}}(\mathbf{x}^{\mathrm{T}}\mathbf{u_i}) = 0$ for $1 \leq i \leq k$. Therefore $A\mathbf{x}$ is in W. It now follows from Exercise 7 that A has an eigenvector $\mathbf{u_{k+1}}$ in W. By definition of W, $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_k}, \mathbf{u_{k+1}}\}$ is an orthogonal set of eigenvectors for A.

7.8 Generalized Eigenvectors & Differential Equations

1. (a) The given matrix H has characteristic polynomial $p(t) =$

   $(t-2)^2$, so $\lambda = 2$ is the only eigenvalue and it has algebraic multiplicity 2. The vector $\mathbf{v_1} = [1, -1]^{\mathrm{T}}$ is an eigenvector corresponding to $\lambda = 2$. If we solve the system of equations $(H - 2I)\mathbf{x} = \mathbf{v_1}$ we see that $\mathbf{x} = [-1-a, a]^{\mathrm{T}}$, where a is arbitrary. Taking $a = 0$ we obtain a generalized eigenvector $\mathbf{v_2} = [-1, 0]^{\mathrm{T}}$.

   (b) The given matrix H has characteristic polynomial $p(t) =$

   $t(t+1)^2$. The eigenvalue $\lambda = -1$ has corresponding eigenvector $\mathbf{v_1} = [-2, 0, 1]^{\mathrm{T}}$. Solving the system $(H - (-1)I)\mathbf{x} = \mathbf{v_1}$ yields $\mathbf{x} = [2-2a, 1, a]^{\mathrm{T}}$ where a is arbitrary. Thus $\mathbf{v_2} = [0, 1, 1]^{\mathrm{T}}$ is a generalized eigenvector for $\lambda = -1$. The eigenvalue $\lambda = 0$ has corresponding eigenvector $\mathbf{w_1} = [-1, 1, 1]^{\mathrm{T}}$.

   (c) The given matrix H has characteristic polynomial $p(t) =$

   $(t-1)^2(t+1)$. The eigenvalue $\lambda = 1$ has corresponding eigenvector $\mathbf{v_1} = [-2, 0, 1]^{\mathrm{T}}$. Solving $(H - I)\mathbf{x} = \mathbf{v_1}$ yields $\mathbf{x} = [(5/2) - 2a, 1/2, a]^{\mathrm{T}}$, where a is arbitrary. Thus $\mathbf{v_2} = [5/2, 1/2, 0]^{\mathrm{T}}$ is a generalized eigenvector of $\lambda = 1$. The eigenvalue $\lambda = -1$ has corresponding eigenvector $\mathbf{w_1} = [-9, -1, 1]^{\mathrm{T}}$.

2. For $A, \lambda = 1$ is the only eigenvalue. Corresponding generalized eigenvectors are $\mathbf{v_1} = [0,0,0,1]^{\mathrm{T}}, \mathbf{v_2} = [0,0,1,0]^{\mathrm{T}}, \mathbf{v_3} = [0,1,0,0]^{\mathrm{T}}$,
and $\mathbf{v_4} = [1,0,0,0]^{\mathrm{T}}$.
For B generalized eigenvectors are $\mathbf{v_1} = [-3,-5,-1,2]^{\mathrm{T}}, \mathbf{v_2} = [0,0,0,1]^{\mathrm{T}}, \mathbf{v_3} = [0,1/2,0,1/2]^{\mathrm{T}}$, and $\mathbf{v_4} = [1/4,1/4,0,1/4]^{\mathrm{T}}$.

3. (a) If $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$ then $Q^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$ and $H =$
$QAQ^{-1} = \begin{bmatrix} 8 & -69 & 21 \\ 1 & -10 & 3 \\ 0 & -4 & 1 \end{bmatrix}$ is in unreduced Hessenberg form. H has characteristic polynomial $p(t) = (t+1)^2(t-1)$. The eigenvalue $\lambda = -1$ has corresponding eigenvector $\mathbf{v_1} = [3,1,2]^{\mathrm{T}}$.
Solving the system $(H-(-1)I)\mathbf{u} = \mathbf{v_1}$ yields $\mathbf{u} = [-(7/2)+(3/2)a, (-1/2)+(1/2)a, a]^{\mathrm{T}}$, where a is arbitrary.
Therefore $\mathbf{v_2} = [-2,0,1]^{\mathrm{T}}$ is a generalized eigenvector for $\lambda = -1$. The eigenvalue $\lambda = 1$ has corresponding eigenvector $\mathbf{w_1} = [-3,0,1]^{\mathrm{T}}$. Set $\mathbf{y}(t) = Q\mathbf{x}(t)$ and $\mathbf{y_0} = Q\mathbf{x_0} = [-1,-1,-2]^{\mathrm{T}}$.
The system $\mathbf{y}' = H\mathbf{y}$ has general solution $\mathbf{y}(t) = c_1 e^{-t}\mathbf{v_1} + c_2 e^{-t}(\mathbf{v_2} + t\mathbf{v_1}) + c_3 e^{t}\mathbf{w_1}$ and $\mathbf{y_0} = \mathbf{y}(0) = c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{w_1}$. Solving we obtain $c_1 = -1, c_2 = 2, c_3 = -2$, so $\mathbf{y}(t) =$
$\begin{bmatrix} e^{-t}(6t-7) & + & 6e^t \\ e^{-t}(2t-1) & & \\ e^{-t}(4t) & - & 2e^t \end{bmatrix}$. Therefore

$$\mathbf{x}(t) = Q^{-1}\mathbf{y}(t) = \begin{bmatrix} e^{-t}(6t-7)+6e^t \\ e^{-t}(2t-1) \\ e^{-t}(-2t+3)-2e^t \end{bmatrix}.$$

(b) If $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$ then $Q^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$ and
$H = QAQ^{-1} = \begin{bmatrix} 2 & 4 & -1 \\ -3 & -4 & 1 \\ 0 & 3 & -1 \end{bmatrix}$ is in unreduced Hessenberg form. H has characteristic polynomial $p(t) = (t+1)^3$. The eigenvalue $\lambda = -1$ has corresponding eigenvector $\mathbf{v_1} = [1,0,3]^{\mathrm{T}}$. The system $(H-(-1)I)\mathbf{u} = \mathbf{v_1}$ has solution $\mathbf{u} = [-1+(1/3)a, 1, a]^{\mathrm{T}}$, where a is arbitrary. Therefore $\mathbf{v_2} = [0,1,3]^{\mathrm{T}}$ is a generalized eigenvector of order 2 for $\lambda = -1$. The system $(H-(-1)I)\mathbf{u} = \mathbf{v_2}$ has solution $\mathbf{u} = [(-4/3)+(1/3)a, 1, a]^{\mathrm{T}}$ so $\mathbf{v_3} = [0,1,4]^{\mathrm{T}}$ is a generalized eigenvector of order 3 for $\lambda = -1$.

Set $\mathbf{y}\,(t) = Q\mathbf{x}\,(t)$ and $\mathbf{y_0} = Q\mathbf{x_0} = [-1,-1,-2]^{\mathrm{T}}$. The system $\mathbf{y}' = H\mathbf{y}$ has general solution $\mathbf{y}(t) = c_1 e^{-t}\mathbf{v_1} + c_2 e^{-t}(\mathbf{v_1} + t\mathbf{v_1}) + c_3 e^{-t}(\mathbf{v_3} + t\mathbf{v_2} + (t^2/2)\mathbf{v_1})$ and $\mathbf{y_0} = \mathbf{y}(0) = c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3}$. Solving we obtain $c_1 = -1, c_2 = -5, c_3 = 4$, so $\mathbf{y}$ $(t) = \begin{bmatrix} e^{-t}(2t^2-5t-1) \\ e^{-t}(4t-1) \\ e^{-t}(6t^2-3t-2) \end{bmatrix}$. Therefore $\mathbf{x}\,(t) = Q^{-1}\mathbf{y}\,(t) = \begin{bmatrix} e^{-t}(2t^2-5t-1) \\ e^{-t}(4t-1) \\ e^{-t}(6t^2-15t+1) \end{bmatrix}$.

(c) If $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$ then $Q^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$ and $H =$ $QAQ^{-1} = \begin{bmatrix} 1 & 4 & -1 \\ -3 & -5 & 1 \\ 0 & 3 & -2 \end{bmatrix}$ is in unreduced Hessenberg form. H has characteristic polynomial $p(t) = (t+2)^3$. The eigenvalue $\lambda = -2$ has eigenvector $\mathbf{v_1} = [1,0,3]^{\mathrm{T}}$, generalized eigenvector $\mathbf{v_2} = [0,1,3]^{\mathrm{T}}$ of order two, and generalized eigenvector $\mathbf{v_3} = [0,1,4]^{\mathrm{T}}$ of order 3.

Set $\mathbf{y}(t) = Q\mathbf{x}(t)$ and $\mathbf{y_0} = Q\mathbf{x_0} = [-1,-1,-2]^{\mathrm{T}}$. The system $\mathbf{y}' = H\mathbf{y}$ has general solution $\mathbf{y}(\mathbf{t}) = e^{-2t}[c_1\mathbf{v_1} + c_2(\mathbf{v_2} + t\mathbf{v_1}) + c_3(\mathbf{v_3} + t\mathbf{v_2} + (t^2/2)\mathbf{v_1})]$ and $\mathbf{y_0} = \mathbf{y}(\mathbf{0}) = c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3}$. Solving yields $c_1 = -1, c_2 = -5$, and $c_3 = 4$, so $\mathbf{y}(t) = \begin{bmatrix} e^{-2t}(2t^2-5t-1) \\ e^{-2t}(4t-1) \\ e^{-2t}(6t^2-3t-2) \end{bmatrix}$. Therefore

$$\mathbf{x}\,(t) = Q^{-1}\mathbf{y}\,(t) = \begin{bmatrix} e^{-2t}(2t^2-5t-1) \\ e^{-2t}(4t-1) \\ e^{-2t}(6t^2-15t+1) \end{bmatrix}.$$

4. $\mathbf{x}\,(t) = \begin{bmatrix} e^t c_4 \\ e^t(c_4t + c_3) \\ e^t(c_4t^2/2 + c_3t + c_2) \\ e^t(c_4t^3/6 + c_3t^2/2 + c_2t + c_1) \end{bmatrix}$.

5. We see that from part(c) of Exercise 1 that $\mathbf{x}(t) =$ $c_1e^t\mathbf{v_1} + c_2e^t(\mathbf{v_2} + t\mathbf{v_1}) + c_3e^{-t}\mathbf{w_1} =$

$$c_1e^t\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + c_2e^t\left(\begin{bmatrix} 5/2 \\ 1/2 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\right) + c_3e^{-t}\begin{bmatrix} -9 \\ -1 \\ 1 \end{bmatrix}.$$

6. Note that $(H-\lambda I)\mathbf{v_2} \neq \theta$ since $\mathbf{v_1} \neq \theta$. But $(H-\lambda I)^2\mathbf{v_2} =$ $(H-\lambda I)\mathbf{v_1} = \theta$, so $\mathbf{v_2}$ is a generalized eigenvector of order 2. Suppose we have seen that $\mathbf{v_j}$ is a generalized eigenvector of order j for $1 \le j \le k$ where $1 \le k < m$. Then $(H-\lambda I)^k\mathbf{v_{k+1}} = (H-\lambda I)^{k-1}\mathbf{v_k} \neq \theta$ whereas $(H-\lambda I)^{k+1}\mathbf{v_{k+1}} = (H-\lambda I)^k\mathbf{v_k} = \theta$.

Therefore $\mathbf{v_{k+1}}$ is a generalized eigenvector of order $k+1$. It follows by induction that $\mathbf{v_r}$ is a generalized eigenvector of order r for $1 \leq r \leq m$.

Clearly the set $\{\mathbf{v_1}\}$ is linearly independent since $\mathbf{v_1} \neq \theta$. Suppose we have seen that the set $\{\mathbf{v_1}, \ldots, \mathbf{v_k}\}$ is linearly independent for some $k, 1 \leq k < m$. Now assume that $c_1\mathbf{v_1} + \cdots + c_k\mathbf{v_k} +$

$c_{k+1}\mathbf{v_{k+1}} = \theta$. Note that $(H - \lambda I)^k\mathbf{v_j} = \theta$ for $1 \leq j \leq k$ whereas

$(H - \lambda I)^k\mathbf{v_{k+1}} = \mathbf{v_1}$. It follows that $\theta =$

$(H - \lambda I)^k\theta = (H - \lambda I)^k(c_1\mathbf{v_1} + \cdots + c_k\mathbf{v_k} + c_{k+1}\mathbf{v_{k+1}}) = c_{k+1}\mathbf{v_1}$. Therefore $c_{k+1} = 0$. Since the set $\{\mathbf{v_1}, \ldots, \mathbf{v_k}\}$ is linearly independent, $c_1 = \cdots = c_k = 0$. This proves that $\{\mathbf{v_1}, \ldots, \mathbf{v_k}, \mathbf{v_{k+1}}\}$ is a linearly independent set. It follows by induction that $\{\mathbf{v_1}, \ldots, \mathbf{v_m}\}$ is linearly independent.

7. Note that $H\mathbf{v_j} = \lambda\mathbf{v_j} + \mathbf{v_{j-1}}$ for $2 \leq j \leq r$ whereas $H\mathbf{v_1} = \lambda\mathbf{v_1}$. It is straightforward to see that $\mathbf{x_r}'(t) = H\mathbf{x_r}(\mathbf{t})$.

8. First note that $q(H)(H - \lambda_1 I)^{m_1-1} = (H - \lambda_1 I)^{m_1-1}q(H)$. It follows from the equations in (5) that if Eq.(9) is multiplied by

$(H - \lambda_1 I)^{m_1-1}$ then we obtain $a_{m_1}q(H)\mathbf{v_1} = \theta$. Since $\mathbf{v_1}$ is an eigenvector corresponding to $\lambda_1, q(H)\mathbf{v_1} = q(\lambda_1)\mathbf{v_1} \neq \theta$ since $\mathbf{v_1} \neq \theta$ and $q(\lambda_1) \neq 0$. Therefore $a_{m_1} = 0$. By a similar argument, multiplication of (9) by $(H - \lambda_1 I)^{m_1-2}$ shows that $a_{m_1-1} = 0$. We may continue the process to show that $a_j = 0$ for each $j, 1 \leq j \leq m_1$.

7.9 Supplementary Exercises

1. $A = \begin{bmatrix} 1 & a \\ 3-a & 1 \end{bmatrix}$, a arbitrary.

2. $a = 0$ or $a = -6$.

3. (a) If $\mathbf{x} = [1,\ -1]^T$ then $q(\mathbf{x}) = -2$.

   (b) The matrix B is not symmetric.

5. (a) $L = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$ (b) $L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}$

7.10 Conceptual Exercises

1. Let A have characterstic polynomial $p(t) = t^3 + ut^2 + vt + w$. Since A is nonsingular, $\lambda = 0$ is not an eigenvalue for A. Therefore, $w \neq 0$. Since $A^3 + uA^2 + vA + wI = \mathcal{O}$, it follows that $[(-1/w)A^2 - (u/w)A - (v/w)I]A = I$.

2. If $B = P^{-1}AP$ then $B^n = (P^{-1}AP)^n = P^{-1}A^nP$. It follows that $p(B) = p(P^{-1}AP) = P^{-1}p(A)P$.

3. For $1 \leq i \leq n$, $a_{ii} = \mathbf{e}_i^T A\mathbf{e}_i > 0$.