Chapter 6

6.1

1. Fails to be invertible; since $\det\begin{bmatrix}1 & 2\\ 3 & 6\end{bmatrix} = 6 - 6 = 0$.

2. Invertible; since $\det\begin{bmatrix}2 & 3\\ 4 & 5\end{bmatrix} = 10 - 12 = -2$.

3. Invertible; since $\det\begin{bmatrix}3 & 5\\ 7 & 11\end{bmatrix} = 33 - 35 = -2$.

4. Fails to be invertible; since $\det\begin{bmatrix}1 & 4\\ 2 & 8\end{bmatrix} = 8 - 8 = 0$.

5. Invertible; since $\det\begin{bmatrix}2 & 5 & 7\\ 0 & 11 & 7\\ 0 & 0 & 5\end{bmatrix} = 2\cdot 11\cdot 5 + 0 + 0 - 0 - 0 - 0 = 110$.

6. Invertible; since $\det\begin{bmatrix}6 & 0 & 0\\ 5 & 4 & 0\\ 3 & 2 & 1\end{bmatrix} = 6\cdot 4\cdot 1 + 0 + 0 - 0 - 0 - 0 = 24$.

7. This matrix is clearly not invertible, so the determinant must be zero.

8. This matrix fails to be invertible, since the $\det(A) = 0$.

9. Invertible; since $\det\begin{bmatrix}0 & 1 & 2\\ 7 & 8 & 3\\ 6 & 5 & 4\end{bmatrix} = 0 + 3\cdot 6 + 2\cdot 7\cdot 5 - 7\cdot 4 - 2\cdot 8\cdot 6 = -36$.

10. Invertible; since $\det\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 3 & 6\end{bmatrix} = 1\cdot 2\cdot 6 + 1\cdot 3\cdot 1 + 1\cdot 1\cdot 3 - 3\cdot 3\cdot 1 - 2\cdot 1\cdot 1 - 6\cdot 1\cdot 1 = 1$.

11. $\det\begin{bmatrix}k & 2\\ 3 & 4\end{bmatrix} \neq 0$ when $4k \neq 6$, or $k \neq \frac{3}{2}$.

12. $\det\begin{bmatrix}1 & k\\ k & 4\end{bmatrix} \neq 0$ when $k^2 \neq 4$, or $k \neq 2, -2$.

13. $\det\begin{bmatrix}k & 3 & 5\\ 0 & 2 & 6\\ 0 & 0 & 4\end{bmatrix} = 8k$, so $k \neq 0$ will ensure that this matrix is invertible.

14. $\det\begin{bmatrix}4&0&0\\3&k&0\\2&1&0\end{bmatrix}=0$, so the matrix will never be invertible, no matter which k is chosen.

15. $\det\begin{bmatrix}0&k&1\\2&3&4\\5&6&7\end{bmatrix}=6k-3$. This matrix is invertible when $k\neq\frac{1}{2}$.

16. $\det\begin{bmatrix}1&2&3\\4&k&5\\6&7&8\end{bmatrix}=60+84+8k-18k-35-64=45-10k$. So this matrix is invertible when $k\neq 4.5$.

17. $\det\begin{bmatrix}1&1&1\\1&k&-1\\1&k^2&1\end{bmatrix}=2k^2-2=2(k^2-1)=2(k-1)(k+1)$. So k cannot be 1 or -1.

18. $\det\begin{bmatrix}0&1&k\\3&2k&5\\9&7&5\end{bmatrix}=30+21k-18k^2=-3(k-2)(6k+5)$. So k cannot be 2 or $-\frac{5}{6}$.

19. $\det\begin{bmatrix}1&1&k\\1&k&k\\k&k&k\end{bmatrix}=-k^3+2k^2-k=-k(k-1)^2$. So k cannot be 0 or 1.

20. $\det\begin{bmatrix}1&k&1\\1&k+1&k+2\\1&k+2&2k+4\end{bmatrix}=(k+1)(2k+4)+k(k+2)+(k+2)-(k+1)-k(2k+4)-(k+2)(k+2)=(k+1)(3k+6)-(3k^2+9k+5)=1$. Thus, A will always be invertible, no matter the value of k, meaning that k can have any value.

21. $\det\begin{bmatrix}k&1&1\\1&k&1\\1&1&k\end{bmatrix}=k^3-3k+2=(k-1)^2(k+2)$. So k cannot be -2 or 1.

22. $\det\begin{bmatrix}\cos k&1&-\sin k\\0&2&0\\\sin k&0&\cos k\end{bmatrix}=2\cos^2 k+2\sin^2 k=2$. So k can have any value.

23. $\det(A-\lambda I_2)=\det\begin{bmatrix}1-\lambda&2\\0&4-\lambda\end{bmatrix}=(1-\lambda)(4-\lambda)=0$ if λ is 1 or 4.

24. $\det(A-\lambda I_2)=\det\begin{bmatrix}2-\lambda&0\\1&0-\lambda\end{bmatrix}=(2-\lambda)(-\lambda)=0$ if λ is 2 or 0.

25. $\det(A-\lambda I_2) = \det\begin{bmatrix} 4-\lambda & 2 \\ 4 & 6-\lambda \end{bmatrix} = (4-\lambda)(6-\lambda) - 8 = (\lambda-8)(\lambda-2) = 0$ if λ is 2 or 8.

26. $\det(A-\lambda I_2) = \det\begin{bmatrix} 4-\lambda & 2 \\ 2 & 7-\lambda \end{bmatrix} = (4-\lambda)(7-\lambda) - 4 = (\lambda-8)(\lambda-3) = 0$ if λ is 3 or 8.

27. $A-\lambda I_3$ is a lower triangular matrix with the diagonal entries $(2-\lambda), (3-\lambda)$ and $(4-\lambda)$. Now, $\det(A-\lambda I_3) = (2-\lambda)(3-\lambda)(4-\lambda) = 0$ if λ is 2, 3 or 4.

28. $A-\lambda I_3$ is an upper triangular matrix with the diagonal entries $(2-\lambda), (3-\lambda)$ and $(5-\lambda)$. Now, $\det(A-\lambda I_3) = (2-\lambda)(3-\lambda)(5-\lambda) = 0$ if λ is 2, 3 or 5.

29. $\det(A-\lambda I_3) = \det\begin{bmatrix} 3-\lambda & 5 & 6 \\ 0 & 4-\lambda & 2 \\ 0 & 2 & 7-\lambda \end{bmatrix} = (3-\lambda)(\lambda-8)(\lambda-3) = 0$ if λ is 3 or 8.

30. $\det(A-\lambda I_3) = \det\begin{bmatrix} 4-\lambda & 2 & 0 \\ 4 & 6-\lambda & 0 \\ 5 & 2 & 3-\lambda \end{bmatrix} = (4-\lambda)(6-\lambda)(3-\lambda) - 8(3-\lambda)$

    $= (3-\lambda)(8-\lambda)(2-\lambda) = 0$ if λ is 3, 8 or 2.

31. This matrix is upper triangular, so the determinant is the product of the diagonal entries, which is 24.

32. This matrix is upper triangular, so the determinant is the product of the diagonal entries, which is 210.

33. By Fact 6.1.8, the determinant is equal to $\det\begin{bmatrix} 1 & 2 \\ 8 & 7 \end{bmatrix} \det\begin{bmatrix} 2 & 3 \\ 7 & 5 \end{bmatrix} = (7-16)(10-21) = 99$.

34. By Fact 6.1.8, the determinant is equal to $\det\begin{bmatrix} 4 & 5 \\ 3 & 6 \end{bmatrix} \det\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = (24-15)(3-8) = -45$.

35. We will use Fact 6.1.5 and expand down the third column. Our determinant equals:

    $$-2\det\begin{bmatrix} 2 & 3 & 2 \\ 6 & 0 & 3 \\ 7 & 0 & 4 \end{bmatrix}.$$

    Then we expand down the second column, so we have $-2(-3)\det\begin{bmatrix} 6 & 3 \\ 7 & 4 \end{bmatrix} = 6(24-21) = 18$.

36. We use Fact 6.1.5, first expanding down the second column. Our determinant equals:

$$-2\det\begin{bmatrix}2&2&2\\1&2&2\\1&1&2\end{bmatrix}=-2(8+4+2-4-4-4)=-4.$$

37. Here we use Fact 6.1.8 to find $\det\begin{bmatrix}5&4\\6&7\end{bmatrix}\det\begin{bmatrix}5&6&7\\0&1&2\\0&0&1\end{bmatrix}=(35-24)(5)=55.$

38. By Fact 6.1.8, the determinant is equal to $\det\begin{bmatrix}1&2&3\\3&0&4\\2&1&2\end{bmatrix}\det\begin{bmatrix}6&5\\5&6\end{bmatrix}$

    $=(9+16-12-4)(36-25)=99.$

39. We repeatedly expand down the first column, finding the determinant to be

    $5\cdot 4(-2)\det\begin{bmatrix}0&1\\3&0\end{bmatrix}=120.$

40. We repeatedly expand down the first column, finding the determinant to be

    $5(4)(3)\det\begin{bmatrix}0&2\\1&0\end{bmatrix}=-120.$

41. Here we expand down the fourth column: $2\det\begin{bmatrix}0&0&1&2\\1&3&5&7\\2&0&4&6\\0&0&3&4\end{bmatrix}$, then down the second column: $2(3)\det\begin{bmatrix}0&1&2\\2&4&6\\0&3&4\end{bmatrix}=2(3)(-2)\det\begin{bmatrix}1&2\\3&4\end{bmatrix}=-12(4-6)=24.$

42. We first expand across the fourth row to obtain

    $3\det\begin{bmatrix}1&1&0&1\\0&2&0&0\\2&3&4&0\\3&4&5&6\end{bmatrix}$, then across the second row to obtain

    $3(2)\det\begin{bmatrix}1&0&1\\2&4&0\\3&5&6\end{bmatrix}=6(24+10-12)=132.$

43. For a 2×2 matrix $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$, we have $\det(-A)=\det\begin{bmatrix}-a&-b\\-c&-d\end{bmatrix}=ad-bc=\det(A)$. Use Sarrus' rule to see that $\det(-A)=-\det(A)$ for a 3×3 matrix. We may conjecture

that $\det(-A) = \det(A)$ for an $n \times n$ matrix with even n, and $\det(-A) = -\det(A)$ when n is odd; in both cases we can write $\det(-A) = (-1)^n \det(A)$. A proof of this conjecture can be based upon the following rule: If a square matrix B is obtained from matrix A by multiplying all the entries in the i^{th} row of A by a scalar k, then $\det(B) = k\det(A)$. We can show this by expansion along the i^{th} row:

$$\det(B) = \sum_{j=1}^{n}(-1)^{i+j} b_{ij} \det(B_{ij}) = \sum_{j=1}^{n}(-1)^{i+j} k a_{ij} \det(A_{ij})$$

$$= k\sum_{j=1}^{n}(-1)^{i+j} a_{ij} \det(A_{ij}) = k\det(A).$$

We can obtain $-A$ by multiplying all the entries in the first row of A by $k = -1$, then all the entries in the second row, and so forth down to the n^{th} row; each time the determinant gets multiplied by $k = -1$. Thus, $\det(-A) = k^n \det(A) = (-1)^n \det(A)$, as claimed.

44. $\det(kA) = k^n \det(A)$

The argument is analogous to the one in Exercise 43.

45. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det(A^T) = \det \begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - cb = \det(A)$. It turns out that $\det(A^T) = \det(A)$.

46. Let $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$. If $a_1a_4 - a_2a_3 \neq 0$, then $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} a_4 & -a_2 \\ -a_3 & a_1 \end{bmatrix}$.

By Exercise 44, $\det(A^{-1}) = \left(\frac{1}{\det(A)}\right)^2 (a_1a_4 - a_2a_3) = \left(\frac{1}{\det(A)}\right)^2 \cdot \det(A)$ so $\det(A^{-1}) = \frac{1}{\det(A)}$.

47. We have $\det(A) = (ah - cf)k + bef + cdg - aeg - bdh$. Thus matrix A is invertible for all k if (and only if) the coefficient $(ah - cf)$ of k is 0, while the sum $bef + cdg - aeg - bdh$ is nonzero. A numerical example is $a = c = d = f = h = g = 1$ and $b = e = 2$, but there are infinitely many other solutions as well.

48. Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ so $\det(A) = \det(B) = \det(C) = \det(D) = 0$ hence $\det(A)\det(D) - \det(B)\det(C) = 0$ but $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = -1$.

49. The kernel of T consists of all vectors $\vec{x}$ such that the matrix $[\vec{x}\ \vec{v}\ \vec{w}]$ fails to be invertible. This is the case if $\vec{x}$ is a linear combination of $\vec{v}$ and $\vec{w}$, as discussed on Pages 247 and 248. Thus $\ker(T) = \text{span}(\vec{v}, \vec{w})$. The image of T isn't $\{0\}$, since $T(\vec{v} \times \vec{w}) \neq 0$, for example. Being a subspace of $\mathbb{R}$, the image must be all of $\mathbb{R}$.

50. Fact 6.1.1 tells us that $\det[\vec{u}\ \vec{v}\ \vec{w}] = \vec{u}\cdot(\vec{v}\times\vec{w}) = \|\vec{u}\|\cos(\theta)\|\vec{v}\times\vec{w}\| = \|\vec{u}\|\cos(\theta)\|\vec{v}\|\sin(\alpha)\|\vec{w}\| = \cos(\theta)\sin(\alpha)$, where θ is the angle enclosed by vectors $\vec{u}$ and $\vec{v}\times\vec{w}$, and α is the angle between $\vec{v}$ and $\vec{w}$. Thus $\det[\vec{u}\ \vec{v}\ \vec{w}]$ can be any number on the closed interval $[-1, 1]$.

51. We expand down the first column:

$$\det(M_n) = 1\det\begin{bmatrix} 2 & 1 & 0 & \cdots & 0 \\ 1 & 2 & 1 & \cdots & 0 \\ 0 & 1 & 2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 2 \end{bmatrix} - 1\det\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 1 & \cdots & 0 \\ 0 & 1 & 2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 2 \end{bmatrix}$$

$$= \det(A_{n-1}) - \det\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & & & & \\ 0 & & & & \\ \vdots & & A_{n-2} & & \\ 0 & & & & \\ 0 & & & & \end{bmatrix} = \det(A_{n-1}) - 1\det(A_{n-2}).$$ Using Example 9, this equals $n - (n-1) = 1$.

52. $$\det(D_n) = \det\begin{bmatrix} 2 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & & & & \\ 0 & 0 & 0 & & & & & \\ \vdots & \vdots & \vdots & & & A_{n-3} & & \\ 0 & 0 & 0 & & & & & \\ 0 & 0 & 0 & & & & & \end{bmatrix}$$

Then we expand down the first column:

$$= 2\det\begin{bmatrix} 2 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & & & & \\ 0 & 0 & & & & & \\ \vdots & \vdots & & & A_{n-3} & & \\ 0 & 0 & & & & & \\ 0 & 0 & & & & & \end{bmatrix} + \det\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 2 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & & & & \\ 0 & 0 & & & & & \\ \vdots & \vdots & & & A_{n-3} & & \\ 0 & 0 & & & & & \\ 0 & 0 & & & & & \end{bmatrix}$$

$= 2\det(A_{n-1}) - 2\det(A_{n-3}) = 2(n) - 2(n-2) = 4$. $\det(D_n)$ will never be 0.

53. $\det(E_n) = \det \begin{bmatrix} 2 & 0 & 0 & 1 & \cdots & 0 \\ 0 & & & & & \\ 0 & & & & & \\ 1 & & & A_{n-1} & & \\ \vdots & & & & & \\ 0 & & & & & \end{bmatrix}$

$$= 2\det(A_{n-1}) - 1\det \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 1 & 2 \end{bmatrix}$$

$$= 2\det(A_{n-1}) - 1\det \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & & & & \\ 0 & 0 & 0 & & & & & \\ \vdots & \ddots & \ddots & & & A_{n-4} & & \\ 0 & 0 & 0 & & & & & \\ 0 & 0 & 0 & & & & & \end{bmatrix}$$

$= 2\det(A_{n-1}) - \det \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \det(A_{n-4})$ (by Fact 6.1.8) $= 2\det(A_{n-1}) - 3\det(A_{n-4})$

$= 2n - 3(n-3) = 2n - 3n + 9 = 9 - n$. So, $\det(E_9) = 0$.

54. a. We will expand down the first column:

$$d_n = \det(M_n) = 5\det(M_{n-1}) - 1\det \begin{bmatrix} 6 & 0 & 0 & \cdots & 0 & 0 \\ 1 & & & & & \\ 0 & & & & & \\ \vdots & & & M_{n-2} & & \\ 0 & & & & & \\ 0 & & & & & \end{bmatrix}$$

$$= 5\det(M_{n-1}) - 1(6)\det(M_{n-2}).$$

b. $d_1 = \det[2] = 2, d_2 = \det\begin{bmatrix}5 & 6\\1 & 2\end{bmatrix} = 4,$

$d_3 = 5(4) - 6(2) = 8, d_4 = 5(8) - 6(4) = 16.$

c. $d_n = 2^n$. The base case has already been shown. If we assume that $d_{n-1} = 2^{n-1}$ and $d_{n-2} = 2^{n-2}$, then $d_n = 5d_{n-1} - 6d_{n-2} = 5(2^{n-1}) - 6(2^{n-2}) = 5(2^{n-1}) - 3(2^{n-1}) = 2(2^{n-1}) = 2^n$.

55. a. $d_n = \det\begin{bmatrix}1 & 1 & 0 & 0 & 0 & \cdots & 0\\1 & 1 & 1 & 0 & 0 & \cdots & 0\\0 & 1 & 1 & 1 & 0 & \cdots & 0\\0 & 0 & 1 & 1 & 1 & \ddots & 0\\0 & 0 & 0 & 1 & 1 & \ddots & 0\\\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots\\0 & 0 & 0 & 0 & 0 & \cdots & 1\end{bmatrix}$

$$= 1\det\begin{bmatrix}1 & 1 & 0 & 0 & \cdots & 0\\1 & 1 & 1 & 0 & \cdots & 0\\0 & 1 & 1 & 1 & \cdots & 0\\0 & 0 & 1 & 1 & \ddots & 0\\\vdots & \vdots & \vdots & \ddots & \ddots & \vdots\\0 & 0 & 0 & 0 & \cdots & 1\end{bmatrix} - 1\det\begin{bmatrix}1 & 0 & 0 & 0 & \cdots & 0\\1 & 1 & 1 & 0 & \cdots & 0\\0 & 1 & 1 & 1 & \cdots & 0\\0 & 0 & 1 & 1 & \ddots & 0\\\vdots & \vdots & \vdots & \ddots & \ddots & \vdots\\0 & 0 & 0 & 0 & \cdots & 1\end{bmatrix}$$

$$= d_{n-1} - 1\det\begin{bmatrix}1 & 0 \quad 0 \quad 0 \quad \cdots \quad 0\\1 & \\0 & \\0 & M_{n-2}\\\vdots & \\0 & \end{bmatrix} = d_{n-1} - d_{n-2}$$

b. $d_1 = \det[1] = 1$. $d_2 = \det\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} = 0$. $d_3 = 0 - 1 = -1$. $d_4 = -1 - 0 = -1$. $d_5 = -1 - (-1) = 0$. $d_6 = 0 - (-1) = 1$. $d_7 = 1 - 0 = 1$. $d_8 = 1 - 1 = 0$.....

Note that $d_7 = d_1 = 1$ and $d_8 = d_2 = 0$, so that the values are repeated, with a period of 6, meaning that $d_k = d_{k+6}$ for all k.

c. Following part b, we have $d_{100} = d_{100-16(6)} = d_4 = -1$.

56. a. $d_n = \det \begin{bmatrix} 0 & & \\ \vdots & M_{n-1} & \\ 0 & & \\ 1 & & \end{bmatrix}$.

    Now, if we expand down the first column, we find that $d_n = \det(M_{n-1})$ if n is odd, or $d_n = -\det(M_{n-1})$ if n is even. We can model this by simply saying: $d_n = (-1)^{n-1}d_{n-1}$.

    b. $d_1 = 1, d_2 = -1, d_3 = -1, d_4 = 1, d_5 = 1, d_6 = -1, d_7 = -1, d_8 = 1$. We notice the pattern that it keeps switching between -1 and 1 with every other increase. We see that $d_{n+4} = d_n$.

    c. By the periodicity in part b, we see that d_{100} will be equal to $d_{100-24(4)} = d_4 = 1$.

57. Repeatedly expanding down the first column, we see that the determinant will be 1 or -1, since it is the product of n terms that are all 1 or -1.

58. a. If a, b, c, d are distinct prime numbers, then $ad \neq bc$, since the prime factorization of a positive integer is unique. Thus $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0$: No matrix of the required form exists.

    b. We are looking for a noninvertible matrix $A = [\vec{u}\ \vec{v}\ \vec{w}]$ whose entries are nine distinct prime numbers. The last column vector, $\vec{w}$, must be redundant; to keep things simple, we will make $\vec{w} = \vec{u} + 2\vec{v}$. Now we have to pick six distinct prime entries for the first two columns, $\vec{u}$ and $\vec{v}$, such that the entries of $\vec{w} = \vec{u} + 2\vec{v}$ are prime as well. This can be done in many different ways; one solution is $A = \begin{bmatrix} 7 & 2 & 11 \\ 17 & 3 & 23 \\ 19 & 5 & 29 \end{bmatrix}$.

59. Let M_n be the number of multiplications required to compute the determinant of an $n \times n$ matrix by Laplace expansion. We will use induction on n to prove that $M_n > n!$, for $n \geq 3$.

    In the lowest applicable case, $n = 3$, we can check that $M_3 = 9$ and $3! = 6$.

    Now let's do the induction step. If A is an $n \times n$ matrix, then $\det(A) = a_{11}\det(A_{11}) + \cdots + (-1)^{n+1}a_{n1}\det(A_{n1})$, by Definition 6.1.4. We need to compute n determinants of $(n-1) \times (n-1)$ matrices, and then do n extra multiplications $a_{i1}\det(A_{i1})$, so that $M_n = nM_{n-1} + n$. If $n > 3$, then $M_{n-1} > (n-1)!$, by induction hypothesis, so that $M_n > n(n-1)! + n > n!$, as claimed.

60. To compute $\det(A)$ for an $n \times n$ matrix A by Laplace expansion, $\det(A) = a_{11}\det(A_{11}) - a_{21}\det(A_{21}) + \cdots + (-1)^{n+1}a_{n1}\det(A_{n1})$, we first need to compute the n minors, which requires nL_{n-1} operations; then we compute the n products $a_{i1}\det(A_{i1})$; and finally we have to do $n-1$ additions. Altogether,

$L_n = nL_{n-1} + n + (n-1) = nL_{n-1} + 2n - 1.$

Now we can prove the formula $\frac{L_n}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!} - \frac{1}{n!}$ by induction on n.

For $n = 2$, the formula gives $\frac{L_2}{2} = 1 + 1 - \frac{1}{2}$, or $L_2 = 3$, which is correct: We have to perform 2 multiplications and 1 addition to compute the determinant of a 2×2 matrix.

For an $n \times n$ matrix A we can use the recursive formula derived above to see that $\frac{L_n}{n!} = \frac{nL_{n-1}+2n-1}{n!} = \frac{L_{n-1}}{(n-1)!} + \frac{2}{(n-1)!} - \frac{1}{n!}$. Applying the induction hypothesis to the first summand, we find that $\frac{L_n}{n!} = \frac{L_{n-1}}{(n-1)!} + \frac{2}{(n-1)!} - \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-2)!} - \frac{1}{(n-1)!} + \frac{2}{(n-1)!} - \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!} - \frac{1}{n!}$, as claimed.

Now recall from the theory of Taylor series in calculus that $e = e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!} + \frac{1}{n!} + \cdots$. Thus $L_n = (1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!} - \frac{1}{n!})n! < en!$, as claimed.

6.2

1. $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 2 & 2 & 5 \end{bmatrix} \begin{array}{l} \\ -I \\ -2I \end{array} \rightarrow B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$. Now $\det(A) = \det(B) = 6$, by Algorithm 6.2.3b.

2. $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 6 & 8 \\ -2 & -4 & 0 \end{bmatrix} \begin{array}{l} \\ -I \\ +2I \end{array} \rightarrow B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$. Now $\det(A) = \det(B) = 24$, by Algorithm 6.2.3b.

3. $A = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 1 & 6 & 4 & 8 \\ 1 & 3 & 0 & 0 \\ 2 & 6 & 4 & 12 \end{bmatrix} \begin{array}{l} \\ -I \\ -I \\ -2I \end{array} \rightarrow B = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & 3 & 2 & 4 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & 0 & 4 \end{bmatrix}$. Now $\det(A) = \det(B) = -24$, by Algorithm 6.2.3b.

4. $A = \begin{bmatrix} 1 & -1 & 2 & -2 \\ -1 & 2 & 1 & 6 \\ 2 & 1 & 14 & 10 \\ -2 & 6 & 10 & 33 \end{bmatrix} \begin{array}{l} \\ +I \\ -2I \\ +2I \end{array} \rightarrow$

$$\begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & 3 & 4 \\ 0 & 3 & 10 & 14 \\ 0 & 4 & 14 & 29 \end{bmatrix} \begin{array}{l} \\ \\ -3II \\ -4II \end{array} \rightarrow$$

$\begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 13 \end{bmatrix} \begin{matrix} \\ \\ \\ -2III \end{matrix} \to B = \begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 9 \end{bmatrix}$. Now $\det(A) = \det(B) = 9$, by Algorithm 6.2.3b.

5. After three row swaps, we end up with $B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}$. Now, by Algorithm 6.2.3b, $\det(A) = (-1)^3 \det(B) = -24$.

6. $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 4 & 4 \\ 1 & -1 & 2 & -2 \\ 1 & -1 & 8 & -8 \end{bmatrix} \begin{matrix} \\ -I \\ -I \\ -I \end{matrix} \to$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & -2 & 1 & -3 \\ 0 & -2 & 7 & -9 \end{bmatrix} \begin{matrix} \text{swap:} \\ II \leftrightarrow III \end{matrix} \to$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 3 & 3 \\ 0 & -2 & 7 & -9 \end{bmatrix} \begin{matrix} \\ \div -2 \\ \\ \\ \end{matrix} \to$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 3 & 3 \\ 0 & -2 & 7 & -9 \end{bmatrix} \begin{matrix} \\ \\ \\ +2II \end{matrix} \to$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & -6 \end{bmatrix} \begin{matrix} \\ \\ \\ -2III \end{matrix} \to \quad B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & -12 \end{bmatrix}$$

$\det(A) = -\frac{1}{2}(-1)\det(B) = -72$, by Algorithm 6.2.3b.

7. After two row swaps, we end up with an upper triangular matrix B with all 1's along the diagonal. Now $\det(A) = (-1)^2 \det(B) = 1$, by Algorithm 6.2.3b.

8. After four row swaps, we end up with an upper triangular matrix B with all 1's along the diagonal, except for a 2 in the bottom right corner. Now $\det(A) = (-1)^4 \det(B) = 2$, by Algorithm 6.2.3b.

9. If we subtract the first row from every other row, then we have an upper triangular matrix B, with diagonal entries 1, 1, 2, 3 and 4. Then $\det(A) = \det(B) = 24$ by Algorithm 6.2.3b.

10. $$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{bmatrix} \begin{matrix} \\ -I \\ -I \rightarrow \\ -I \\ -I \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 5 & 9 & 14 \\ 0 & 3 & 9 & 19 & 34 \\ 0 & 4 & 14 & 34 & 69 \end{bmatrix} \begin{matrix} \\ \\ -2II \rightarrow \\ -3II \\ -4II \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 3 & 10 & 22 \\ 0 & 0 & 6 & 22 & 53 \end{bmatrix} \begin{matrix} \\ \\ \rightarrow \\ -3III \\ -6III \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 4 & 17 \end{bmatrix} \begin{matrix} \\ \\ \\ \\ -4IV \end{matrix} \quad \rightarrow B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now $\det(A) = \det(B) = 1$ by Algorithm 6.2.3b.

11. By Fact 6.2.1a, the desired determinant is $(-9)(8) = -72$.

12. By Fact 6.2.1b, the desired determinant is -8.

13. By Fact 6.2.1b, applied twice, since there are two row swaps, the desired determinant is $(-1)(-1)(8) = 8$.

14. By Fact 6.2.1c, the desired determinant is 8.

15. By Fact 6.2.1c, the desired determinant is 8.

16. This determinant is 0, since the first row is twice the last.

17. The standard matrix of T is $A = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 2 & 6 \\ 0 & 0 & 2 \end{bmatrix}$, so that $\det(T) = \det(A) = 8$.

18. The standard matrix of T is $A = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 3 & -12 \\ 0 & 0 & 9 \end{bmatrix}$, so that $\det(T) = \det(A) = 27$.

19. The standard matrix of T is $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so that $\det(T) = \det(A) = -1$.

20. The standard matrix of L is $M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, so that $\det(L) = \det(M) = -1$.

21. The standard matrix of T is $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$, so that $\det(T) = \det(A) = 1$.

22. Using Exercises 19 and 21 as a guide, we observe that the standard matrix A of T is diagonal, of size $(n+1) \times (n+1)$, with diagonal entries $(-1)^0, (-1)^1, (-1)^2, \ldots, (-1)^n$. Thus $\det(T) = \det(A) = (-1)^{1+2+\cdots+n} = (-1)^{n(n+1)/2}$.

23. Consider the matrix M of T with respect to a basis consisting of $n(n+1)/2$ symmetric matrices and $n(n-1)/2$ skew-symmetric matrices (see Exercises 54 and 55 or Section 5.3). Matrix M will be diagonal, with $n(n+1)/2$ entries 1 and $n(n-1)/2$ entries -1 on the diagonal. Thus, $\det(T) = \det(M) = (-1)^{n(n-1)/2}$.

24. The standard matrix of T is $A = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$, so that $\det(T) = \det(A) = 13$.

25. The standard matrix of T is $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$, so that $\det(T) = \det(A) = 16$.

26. The matrix of T with respect to the basis $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is $A = \begin{bmatrix} 2 & 4 & 0 \\ 2 & 4 & 2 \\ 0 & 4 & 6 \end{bmatrix}$, so that $\det(T) = \det(A) = -16$.

27. The matrix of T with respect to the basis $\cos(x), \sin(x)$ is $A = \begin{bmatrix} -b & a \\ -a & -b \end{bmatrix}$, so that $\det(T) = \det(A) = a^2 + b^2$.

28. The matrix of T with respect to the basis $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ is $A = \begin{bmatrix} -6 & -10 \\ 5 & 6 \end{bmatrix}$, so that $\det(T) = \det(A) = 14$.

29. Expand down the first column, realizing that all but the first contribution are zero, since $a_{21} = 0$ and A_{i1} has two equal rows for all $i > 2$. Therefore, $\det(P_n) = \det(P_{n-1})$.

    Since $\det(P_1) = 1$, we can conclude that $\det(P_n) = 1$, for all n.

30. a. $f(t) = \det \begin{bmatrix} 1 & 1 & 1 \\ a & b & t \\ a^2 & b^2 & t^2 \end{bmatrix} = (ab^2 - a^2b) + (a^2 - b^2)t + (b - a)t^2$ so $f(t)$ is a quadratic function of t. The coefficient of t^2 is $(b - a)$.

    b. In the cases $t = a$ and $t = b$ the matrix has two identical columns. It follows that $f(t) = k(t - a)(t - b)$ with $k =$ coefficient of $t^2 = (b - a)$.

    c. The matrix is invertible for the values of t for which $f(t) \neq 0$, i.e., for $t \neq a, t \neq b$.

31. a. If $n = 1$, then $A = \begin{bmatrix} 1 & 1 \\ a_0 & a_1 \end{bmatrix}$, so $\det(A) = a_1 - a_0$ (and the product formula holds).

    b. Expanding the given determinant down the right-most column, we see that the coefficient k of t^n is the $n - 1$ Vandermonde determinant which we assume is

    $$\prod_{n-1 \geq i > j} (a_i - a_j).$$

    Now $f(a_0) = f(a_1) = \cdots = f(a_{n-1}) = 0$, since in each case the given matrix has two identical columns, hence its determinant equals zero. Therefore

    $$f(t) = \left(\prod_{n-1 \geq i > j} (a_i - a_j) \right) (t - a_0)(t - a_1) \cdots (t - a_{n-1})$$

    and

    $$\det(A) = f(a_n) = \prod_{n \geq i > j} (a_i - a_j),$$

    as required.

32. By Exercise 31, we need to compute $\prod_{i>j} (a_i - a_j)$ where $a_0 = 1, a_1 = 2, a_2 = 3, a_3 = 4, a_4 = 5$ so

    $\prod_{i>j} (a_i - a_j) = (2-1)(3-1)(3-2)(4-1)(4-2)(4-3)(5-1)(5-2)(5-3)(5-4) = 288.$

33. Think of the i^{th} column of the given matrix as $a_i \begin{bmatrix} 1 \\ a_i \\ a_i^2 \\ \vdots \\ a_i^{n-1} \end{bmatrix}$, so by Fact 6.2.1a, the determinant can be written as $(a_1 a_2 \cdots a_n) \det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{bmatrix}$. The new determinant is a Vandermonde determinant (see Exercise 31), and we get

$$\prod_{i=1}^{n} a_i \prod_{i>j} (a_i - a_j).$$

34. a. The hint pretty much gives it away. Since the columns of matrix $\begin{bmatrix} B \\ -I_n \end{bmatrix}$ are in the kernel of $[\, I_n \quad M \,]$, we have $[\, I_n \quad M \,] \begin{bmatrix} B \\ -I_n \end{bmatrix} = B - M = 0$, and $M = B$, as claimed.

b. If $B = A^{-1}$ we get rref$[A \vdots I_n] = [I_n \vdots A^{-1}]$ which tells us how to compute A^{-1} (see Fact 2.3.5).

35. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ must satisfy $\det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & a_1 & b_1 \\ x_2 & a_2 & b_2 \end{bmatrix} = 0$, i.e., must satisfy the linear equation

$(a_1 b_2 - a_2 b_1) - x_1(b_2 - a_2) + x_2(b_1 - a_1) = 0.$

We can see that $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ satisfy this equation, since the matrix has two identical columns in these cases.

36. Expanding down the first column we see that the equation has the form $A - Bx_1 + Cx_2 - D(x_1^2 + x_2^2) = 0$. If $D \neq 0$ this equation defines a circle; otherwise it is a line. From Exercise 35 we know that $D = 0$ if and only if the three given points $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ are collinear. Note that the circle or line runs through the three given points.

37. Applying Fact 6.2.4 to the equation $AA^{-1} = I_n$ we see that $\det(A)\det(A^{-1}) = 1$. The only way the product of the two integers $\det(A)$ and $\det(A^{-1})$ can be 1 is that they are both 1 or both -1. Therefore, $\det(A) = 1$ or $\det(A) = -1$.

38. $$\det(A^TA) \underset{\substack{\uparrow \\ \text{Fact 6.2.4}}}{=} \det(A^T)\det(A) \underset{\substack{\uparrow \\ \text{Fact 6.2.7}}}{=} [\det(A)]^2 = 9$$

39. $$\det(A^TA) \underset{\substack{\uparrow \\ \text{Fact 6.2.4}}}{=} \det(A^T)\det(A) \underset{\substack{\uparrow \\ \text{Fact 6.2.7}}}{=} [\det(A)]^2 > 0$$

40. By Exercise 38, $\det(A^TA) = [\det(A)]^2$. Since A is orthogonal, $A^TA = I_n$ so that $1 = \det(I_n) = \det\ (A^TA) = [\det(A)]^2$ and $\det(A) = \pm 1$.

41. $\det(A) = \det(A^T) = \det(-A) = (-1)^n(\det A) = -\det(A)$, so that $\det(A) = 0$. We have used Facts 6.2.7 and 6.2.1a.

42. $$\det(A^TA) \underset{\substack{\uparrow \\ \text{Definition of } A}}{=} \det((QR)^TQR) = \det(R^TQ^TQR) \underset{\substack{\uparrow \\ \text{Since columns of } Q \text{ are orthonormal}}}{=} \det(R^TI_mR) = \det(R^TR) \underset{\substack{\uparrow \\ \text{Fact 6.2.4}}}{=} \det(R^T)\det(R)$$

$$\underset{\substack{\uparrow \\ \text{Fact} \\ \text{6.2.7}}}{=} [\det(R)]^2 \underset{\substack{\uparrow \\ \text{Since } R \\ \text{is triangular.}}}{=} \left(\prod_{i=1}^{m} r_{ii}\right)^2 > 0$$

43. $$\det(A^TA) = \det\left(\begin{bmatrix} \vec{v}^T \\ \vec{w}^T \end{bmatrix} [\vec{v}\ \vec{w}]\right) = \det\begin{bmatrix} \vec{v}\cdot\vec{v} & \vec{v}\cdot\vec{w} \\ \vec{v}\cdot\vec{w} & \vec{w}\cdot\vec{w} \end{bmatrix} = \det\begin{bmatrix} \|\vec{v}\|^2 & \vec{v}\cdot\vec{w} \\ \vec{v}\cdot\vec{w} & \|\vec{w}\|^2 \end{bmatrix}$$

$= \|\vec{v}\|^2\|\vec{w}\|^2 - (\vec{v}\cdot\vec{w})^2 \geq 0$ by the Cauchy-Schwarz inequality (Fact 5.1.11).

44. a. We claim that $\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n \neq \vec{0}$ if and only if the vectors $\vec{v}_2, \ldots, \vec{v}_n$ are linearly independent. If the vectors $\vec{v}_2, \ldots, \vec{v}_n$ are linearly independent, then we can find a basis $\vec{x}, \vec{v}_2, \ldots, \vec{v}_n$ of $\mathbb{R}^n$ (any vector $\vec{x}$ that is not in span $(\vec{v}_2, \ldots, \vec{v}_n)$ will do). Then $\vec{x} \cdot (\vec{v}_2 \times \cdots \times \vec{v}_n) = \det[\vec{x}\,\vec{v}_2 \cdots \vec{v}_n] \neq 0$, so that $\vec{v}_2 \times \cdots \times \vec{v}_n \neq \vec{0}$. Conversely, suppose that $\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n \neq 0$; say the ith component of this vector is nonzero. Then $0 \neq \vec{e}_i \cdot (\vec{v}_2 \times \cdots \times \vec{v}_n) = \det[\vec{e}_i\,\vec{v}_2 \cdots \vec{v}_n]$, so that the vectors $\vec{v}_2, \ldots, \vec{v}_n$ are linearly independent (being columns of an invertible matrix).

b. ith component of $\vec{e}_2 \times \vec{e}_3 \times \cdots \times \vec{e}_n = \det \begin{bmatrix} 1 & 1 & & 1 \\ \vec{e}_i & \vec{e}_2 & \cdots & \vec{e}_n \\ 1 & 1 & & 1 \end{bmatrix} = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{if } i > 1 \end{cases}$

so $\vec{e}_2 \times \vec{e}_3 \times \cdots \times \vec{e}_n = \vec{e}_1$.

c. $\vec{v}_i \cdot (\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n) = \det \begin{bmatrix} 1 & 1 & 1 & & 1 \\ \vec{v}_i & \vec{v}_2 & \vec{v}_3 & \cdots & \vec{v}_n \\ 1 & 1 & 1 & & 1 \end{bmatrix} = 0$

for any $2 \leq i \leq n$ since the above matrix has two identical columns.

d. Compare the ith components of the two vectors:

$$\det \begin{bmatrix} 1 & 1 & 1 & & 1 \\ \vec{e}_i & \vec{v}_2 & \vec{v}_3 & \cdots & \vec{v}_n \\ 1 & 1 & 1 & & 1 \end{bmatrix} \text{ and } \det \begin{bmatrix} 1 & 1 & 1 & & 1 \\ \vec{e}_i & \vec{v}_3 & \vec{v}_2 & \cdots & \vec{v}_n \\ 1 & 1 & 1 & & 1 \end{bmatrix}.$$

The two determinants differ by a factor of -1 by Fact 6.2.1b, so that
$\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n = -\vec{v}_3 \times \vec{v}_2 \times \cdots \times \vec{v}_n$.

e. $\det[\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n\ \vec{v}_2\ \vec{v}_3 \cdots \vec{v}_n] = (\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n) \cdot (\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n) = \|\vec{v}_2 \times \cdots \times \vec{v}_n\|^2$

f. In Definition 6.1.1 we saw that the "old" cross product satisfies the defining equation of the "new" cross product: $\vec{x} \cdot (\vec{v}_2 \times \vec{v}_3) = \det[\,\vec{x}\ \vec{v}_2\ \vec{v}_3\,]$.

45. $f(x)$ is a linear function, so $f'(x)$ is the coefficient of x (the slope). Expanding down the first column, we see that the coefficient of x is $-\det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix} = -24$, so $f'(x) = -24$.

46\. a. $\det\begin{bmatrix} a & 3 & d \\ b & 3 & e \\ c & 3 & f \end{bmatrix} = 3\det\begin{bmatrix} a & 1 & d \\ b & 1 & e \\ c & 1 & f \end{bmatrix} = 21$

↑

Facts 6.2.1

b. $\det\begin{bmatrix} a & 3 & d \\ b & 5 & e \\ c & 7 & f \end{bmatrix} = \det\begin{bmatrix} a & 2(1)+1 & d \\ b & 2(2)+1 & e \\ c & 2(3)+1 & f \end{bmatrix} = \det\begin{bmatrix} a & 2(1) & d \\ b & 2(2) & e \\ c & 2(3) & f \end{bmatrix} + \det\begin{bmatrix} a & 1 & d \\ b & 1 & e \\ c & 1 & f \end{bmatrix}$

↑

Fact 6.2.8

$= 2\det\begin{bmatrix} a & 1 & d \\ b & 2 & e \\ c & 3 & f \end{bmatrix} + \det\begin{bmatrix} a & 1 & d \\ b & 1 & e \\ c & 1 & f \end{bmatrix} = 2\cdot 11 + 7 = 29$

↑

Fact 6.2.1a

47\. Yes! For example, $T\begin{bmatrix} x & b \\ y & d \end{bmatrix} = dx + by$ is given by the matrix $[d\ b]$, so that T is linear in the first column.

48\. Since $\vec{v}_2, \ldots, \vec{v}_n$ are linearly independent, $T(\vec{x}) = 0$ only if $\vec{x}$ is a linear combination of the $\vec{v}_i$ ‘s, (otherwise the matrix $[\vec{x}\,\vec{v}_2 \cdots \vec{v}_n]$ is invertible, and $T(\vec{x}) \neq 0$). Hence, the kernel of T is the span of $\vec{v}_2, \ldots, \vec{v}_n$, an $(n-1)$-dimensional subspace of $\mathbb{R}^n$. The image of T is the real line $\mathbb{R}$ (since it must be 1-dimensional).

49\. For example, we can start with an upper triangular matrix B with $\det(B) = 13$, such as $B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 13 \end{bmatrix}$. Adding the first row of B to both the second and the third to make all entries nonzero, we end up with $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 14 \end{bmatrix}$. Note that $\det(A) = \det(B) = 13$.

50\. There are many ways to do this problem; here is one possible approach:

Subtracting the second to last row from the last, we can make the last row into

$[0 \quad 0 \quad \cdots \quad 0 \quad 1]$.

Now expanding along the last row we see that $\det(M_n) = \det(M_{n-1})$.

Since $\det(M_1) = 1$ we can conclude that $\det(M_n) = 1$ for all n.

51. Notice that it takes n row swaps (swap row i with $n+i$ for each i between 1 and n) to turn A into I_{2n}. So, $\det(A) = (-1)^n \det(I_{2n}) = (-1)^n$.

52. a. We build B column-by-column:

$$\begin{bmatrix} T\begin{bmatrix} 1 \\ 0 \end{bmatrix} & T\begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} d \\ -c \end{bmatrix} & \begin{bmatrix} -b \\ a \end{bmatrix} \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

b. $\det(A) = ad - bc = \det(B)$. The two determinants are equal.

c. $BA = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} da - bc & 0 \\ 0 & -cb + ad \end{bmatrix} = (ad - bc)I_2$.

$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & -cb + da \end{bmatrix} = (ad - bc)I_2$ also.

d. Any vector $\vec{u}$ in the image of A will be of the form $c_1\begin{bmatrix} a \\ c \end{bmatrix} + c_2\begin{bmatrix} b \\ d \end{bmatrix}$. We note that $B\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} ad - bc \\ -ca + ac \end{bmatrix} = \vec{0}$. The same is true of $B\begin{bmatrix} b \\ d \end{bmatrix}$. Thus, anything in the image of A will be in the kernel of B. Since both matrices have a rank of 1, the dimensions of the kernel and image of each will be exactly 1. So, it must be that $\text{im}(A) = \ker(B)$.

Also, any vector $\vec{u}$ in the image of B will be of the form $c_1\begin{bmatrix} d \\ -c \end{bmatrix} + c_2\begin{bmatrix} -b \\ a \end{bmatrix}$. However, we see that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -b \\ a \end{bmatrix} = \begin{bmatrix} -ab + ba \\ -bc + ad \end{bmatrix} = \vec{0}$. The same is true for $A\begin{bmatrix} d \\ -c \end{bmatrix}$. Thus, by the same reasoning as above, the image of B will equal the kernel of A.

e. $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}B$.

53. a. See Exercise 37.

b. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ has integer entries.

54. $f(t) = (\det(A+tB))^2 - 1 = \left(\det \begin{bmatrix} a_1 + tb_1 & a_2 + tb_2 \\ a_3 + tb_3 & a_4 + tb_4 \end{bmatrix}\right)^2 - 1$ assuming $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$, $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$.

    Then the determinant above is a polynomial of degree ≤ 2 so its square is a polynomial of degree ≤ 4. Hence $f(t)$ is a polynomial of degree ≤ 4.

    Since $A, A+B, A+2B, A+3B, A+4B$ are invertible and their inverses have integer entries, by Exercise 53a, it follows that their determinants are ± 1. Hence $f(0) = f(1) = f(2) = f(3) = f(4) = 0$.

    Since f is a polynomial of degree ≤ 4 with at least 5 roots, it follows that $f(t) = 0$ for all t, in particular for $t = 5$, so $\det(A+5B) = \pm 1$. Hence $A + 5B$ is an invertible 2×2 matrix whose inverse has integer entries by Exercise 53b.

55. We start out with a preliminary remark: If a square matrix A has two equal rows, then $D(A) = 0$. Indeed, if we swap the two equal rows and call the resulting matrix B, then $B = A$, so that $D(A) = D(B) = -D(A)$, by property b, and $D(A) = 0$ as claimed.

    Next we need to understand how the elementary row operations affect D. Properties a and b tell us about how row multiplications and row swaps, but we still need to think about row additions.

    We will show that if B is obtained from A by adding k times the i^{th} row to the j^{th}, then $D(B) = D(A)$. Let's label the row vectors of A by $\vec{v}_1, \ldots \vec{v}_n$. By linearity of D in the j^{th} row (property c) we have

$$D(B) = D\left(\begin{bmatrix} & \vdots & \\ — & \vec{v}_i & — \\ & \vdots & \\ — & \vec{v}_j + k\vec{v}_i & — \\ & \vdots & \end{bmatrix}\right) = D(A) + kD\left(\begin{bmatrix} & \vdots & \\ — & \vec{v}_i & — \\ & \vdots & \\ — & \vec{v}_i & — \\ & \vdots & \end{bmatrix}\right) = D(A).$$

    Note that in the last step we have used the preliminary remark. Now, using the terminology introduced on Page 262, we can write $D(A) = (-1)^s k_1 k_2 \cdots k_r D(\text{rref } A)$.

    Next we observe that $D(\text{rref } A) = \det(\text{rref } A)$ for all square matrices A. Indeed, if A is invertible, then $\text{rref}(A) = I_n$, and $D(I_n) = 1 = \det(I_n)$ by property c of function D. If A fails to be invertible, then $D(\text{rref } A) = 0 = \det(\text{rref } A)$ by linearity in the last row.

    It follows that $D(A) = (-1)^s k_1 k_2 \cdots k_r D(\text{rref } A) = (-1)^s k_1 k_2 \cdots k_r \det(\text{rref } A) = \det(A)$ for all square matrices, as claimed.

56. a. We show first that D is linear in the ith row.

$$\underset{\displaystyle A}{\underset{\uparrow}{D\begin{bmatrix}\vec{v}_1\\ \vdots\\ \vec{x}\\ \vdots\\ \vec{v}_n\end{bmatrix}}} = \frac{1}{\det M}\det \underset{\displaystyle AM}{\underset{\uparrow}{\begin{bmatrix}\vec{v}_1M\\ \vdots\\ \vec{x}M\\ \vdots\\ \vec{v}_nM\end{bmatrix}}}$$

The entries in the ith row of AM are linear combinations of the components x_i of the vector $\vec{x}$, while the other entries of AM are constants. Therefore, $\det(AM)$ is a linear combination of the x_i (expand along the i^{th} row). Since $\frac{1}{\det M}$ is a constant, we have $D\begin{bmatrix}\vec{v}_1\\ \vdots\\ \vec{x}\\ \vdots\\ \vec{v}_n\end{bmatrix} = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ for some constants c_i, as claimed.

b. Secondly, we need to show that $D(B) = -D(A)$ if B is obtained from A by a row swap:

$$A = \begin{bmatrix}\vec{v}_1\\ \vdots\\ \vec{v}_i\\ \vdots\\ \vec{v}_j\\ \vdots\\ \vec{v}_n\end{bmatrix} \longrightarrow B = \begin{bmatrix}\vec{v}_1\\ \vdots\\ \vec{v}_j\\ \vdots\\ \vec{v}_i\\ \vdots\\ \vec{v}_n\end{bmatrix}$$

$$D(B) = \frac{1}{\det M}\det\begin{bmatrix}\vec{v}_1M\\ \vdots\\ \vec{v}_jM\\ \vdots\\ \vec{v}_iM\\ \vdots\\ \vec{v}_nM\end{bmatrix} = -\frac{1}{\det M}\det\begin{bmatrix}\vec{v}_1M\\ \vdots\\ \vec{v}_iM\\ \vdots\\ \vec{v}_jM\\ \vdots\\ \vec{v}_nM\end{bmatrix} = -D(A).$$

c. The property $D(I_n) = 1$ is obvious.

It now follows from Exercise 41 that $\det(A) = D(A) = \frac{\det(AM)}{\det(M)}$ and therefore $\det(AM) = \det(A)\det(M)$.

57. Note that matrix A_1 is invertible, since $\det(A_1) \neq 0$. Now

$T\begin{bmatrix}\vec{y}\\ \vec{x}\end{bmatrix} = [A_1\ A_2]\begin{bmatrix}\vec{y}\\ \vec{x}\end{bmatrix} = A_1\vec{y} + A_2\vec{x} = \vec{0}$ when $A_1\vec{y} = -A_2\vec{x}$, or,

$\vec{y} = -A_1^{-1}A_2\vec{x}$. This shows that for every $\vec{x}$ there is a unique $\vec{y}$ (that is, $\vec{y}$ is a function of $\vec{x}$); furthermore, this function is linear, with matrix $M = -A_1^{-1}A_2$.

58. Using the approach of Exercise 57, we have $A_1 = \begin{bmatrix}1 & 2\\ 3 & 7\end{bmatrix}$, $A_2 = \begin{bmatrix}1 & 2\\ 4 & 3\end{bmatrix}$,

and $M = -A_1^{-1}A_2 = \begin{bmatrix}1 & -8\\ -1 & 3\end{bmatrix}$. The function is $\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}1 & -8\\ -1 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$.

Alternatively, we can solve the linear system

$$\begin{aligned} y_1 + 2y_2 + x_1 + 2x_2 &= 0\\ 3y_1 + 7y_2 + 4x_1 + 3x_2 &= 0 \end{aligned}$$

Gaussian Elimination gives

$$\begin{aligned} y_1 - x_1 + 8x_2 &= 0 \\ y_2 + x_1 - 3x_2 &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} y_1 &= x_1 - 8x_2 \\ y_2 &= -x_1 + 3x_2 \end{aligned}$$

59. $\det\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} = 0 = \det\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$, but these matrices fail to be similar.

60. We argue using induction on n. The base case ($n = 2$) is discussed on page 261. Now we assume that B is obtained from the $n \times n$ matrix A by adding k times the p^{th} row to the q^{th} row.

We will evaluate the determinant of B by expanding across the i^{th} row (where i is neither p nor q).

$$\det(B) = \Sigma_{j=1}^{n}(-1)^{i+j}b_{ij}\det(B_{ij})$$

$$= \Sigma_{j=1}^{n}(-1)^{i+j}a_{ij}\det(B_{ij}) = \Sigma_{j=1}^{n}(-1)^{i+j}a_{ij}\det(A_{ij}) = \det(A)$$

Note that the $(n-1)\times(n-1)$ matrix B_{ij} is obtained from A_{ij} by adding k times some row to another row. Now, $\det(B_{ij}) = \det(A_{ij})$ by induction hypothesis.

61. We follow the hint: $\begin{bmatrix} I_n & 0 \\ -C & A \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ -CA+AC & -CB+AD \end{bmatrix}$

$= \begin{bmatrix} A & B \\ 0 & AD-CB \end{bmatrix}$. So, $\det\left(\begin{bmatrix} I_n & 0 \\ -C & A \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \det(A)\det(AD-CB)$.

Thus, $\det(I_n)\det(A)\det\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \det(A)\det(AD-CB)$, which leads to

$\det\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \det(AD-CB)$, since $\det(A) \neq 0$.

62. a. We compute $\begin{bmatrix} I_n & 0 \\ -CA^{-1} & I_n \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & -CA^{-1}B+D \end{bmatrix}$. Since the matrix $\begin{bmatrix} I_n & 0 \\ -CA^{-1} & I_n \end{bmatrix}$ is invertible (its determinant is 1),the product $\begin{bmatrix} A & B \\ 0 & -CA^{-1}B+D \end{bmatrix}$ will have the same rank as $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, namely, n. With A being invertible, this implies that $-CA^{-1}B+D=0$, or $CA^{-1}B=D$, as claimed.

b. Take determinants on both sides of the equation $D = CA^{-1}B$ from part (a) to find that $\det(D) = \det(C)(\det A)^{-1}\det(B), \textit{or} \det(A)\det(D) - \det(B)\det(C) = 0$, proving the claim.

6.3

1. By Fact 6.3.3, the area equals $|\det\begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix}| = |-50| = 50$.

2. By Fact 6.3.3 Area $= \frac{1}{2}\left|\det\begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix}\right| = \frac{1}{2}|-50| = 25$

3. Area of triangle $= \frac{1}{2}|\det\begin{bmatrix} 6 & 1 \\ -2 & 4 \end{bmatrix}| = 13$ (See Figure 6.1.)

4. Note that area of triangle $= \frac{1}{2}\left|\det\begin{bmatrix} b_1-a_1 & c_1-a_1 \\ b_2-a_2 & c_2-a_2 \end{bmatrix}\right|$. (See Figure 6.2.)

On the other hand, $\det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{bmatrix} = \det\begin{bmatrix} a_1 & b_1-a_1 & c_1-a_1 \\ a_2 & b_2-a_2 & c_2-a_2 \\ 1 & 0 & 0 \end{bmatrix}$, by subtracting the first column from the second and third.

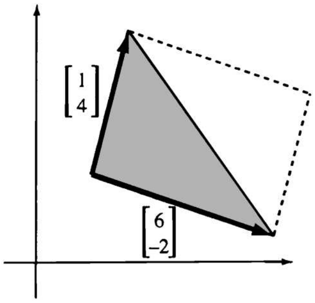

Figure 6.1: for Problem 6.3.3.

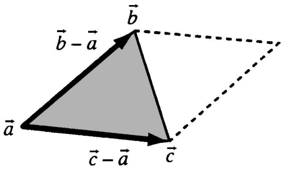

Figure 6.2: for Problem 6.3.4.

This, in turn, equals $\det\begin{bmatrix} b_1-a_1 & c_1-a_1 \\ b_2-a_2 & c_2-a_2 \end{bmatrix}$, by expanding across the bottom row.

Therefore, area of triangle $= \frac{1}{2}\left|\det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{bmatrix}\right|$.

5. The volume of the tetrahedron T_0 defined by $\vec{e}_1, \vec{e}_2, \vec{e}_3$ is $\frac{1}{3}$(base)(height) $= \frac{1}{6}$.

   Here we are using the formula for the volume of a pyramid. (See Figure 6.3.)

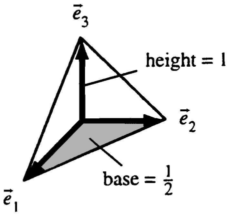

Figure 6.3: for Problem 6.3.5.

The tetrahedron T defined by $\vec{v}_1, \vec{v}_2, \vec{v}_3$ can be obtained by applying the linear transformation with matrix $[\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$ to T_0.

Now we have

$$\text{vol}(T) \underset{\substack{\uparrow \\ \text{Fact 6.3.8 and Page 282}}}{=} |\det[\vec{v}_1\ \vec{v}_2\ \vec{v}_3]|\text{vol}(T_0) = \tfrac{1}{6}|\det[\vec{v}_1\ \vec{v}_2\ \vec{v}_3]| \underset{\substack{\uparrow \\ \text{Fact 6.3.5}}}{=} \tfrac{1}{6}V(\vec{v}_1, \vec{v}_2, \vec{v}_3).$$

6. From Exercise 5 we know that volume of tetrahedron $= \frac{1}{6}\left|\det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{bmatrix}\right|$, and Exercise 4 tells us that area of triangle $= \frac{1}{2}\left|\det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{bmatrix}\right|$, so that area of tetrahedron $= \frac{1}{3}$ (area of triangle).

   We can see this result more directly if we think of the tetrahedron as an inverted pyramid whose base is the triangle and whose height is 1. (See Figure 6.4.)

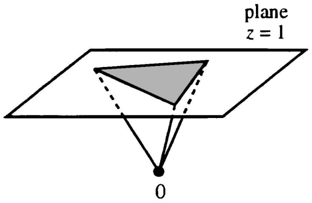

Figure 6.4: for Problem 6.3.6.

   The three vertices of the shaded triangle are $\begin{bmatrix} a_1 \\ a_2 \\ 1 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \\ 1 \end{bmatrix}, \begin{bmatrix} c_1 \\ c_2 \\ 1 \end{bmatrix}$.

7. Area $= \frac{1}{2}\left|\det\begin{bmatrix} 10 & -2 \\ 11 & 13 \end{bmatrix}\right| + \frac{1}{2}\left|\det\begin{bmatrix} 8 & 10 \\ 2 & 11 \end{bmatrix}\right| = 110$. (See Figure 6.5.)

8. We need to show that both sides of the equation in Fact 6.3.4 give zero.

   $|\det(A)| = 0$ since A is not invertible. On the other hand, since A is not invertible, the $\vec{v}_i$ will be linearly dependent, i.e., one of the $\vec{v}_i$ will be redundant. This implies that $\vec{v}_i^{\parallel} = \vec{v}_i$ and $\vec{v}_i^{\perp} = \vec{0}$, so that the right-hand side of the equation is 0, as claimed.

9. By Fact 6.3.3, $|\det[\vec{v}_1\ \vec{v}_2]|$ = area of the parallelogram defined by $\vec{v}_1$ and $\vec{v}_2$. But $\|\vec{v}_1\|$ is the base of that parallelogram and $\|\vec{v}_2\|\sin\theta$ is its height, hence $|\det[\vec{v}_1\ \vec{v}_2]| = \|\vec{v}_1\|\|\vec{v}_2\|\sin\theta$.

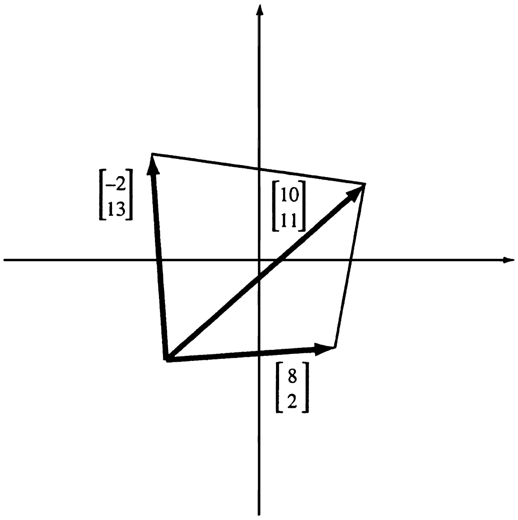

Figure 6.5: for Problem 6.3.7.

10. $|\det(A)| \leq \|\vec{v}_1\|\|\vec{v}_2\|\cdots\|\vec{v}_n\|$ since $|\det(A)| = \|\vec{v}_1\|\|\vec{v}_2^{\perp}\|\cdots\|\vec{v}_n^{\perp}\|$ and $\|\vec{v}_i\| \geq \|\vec{v}_i^{\perp}\|$. The equality holds if $\|\vec{v}_i\| = \|\vec{v}_i^{\perp}\|$ for all i, that is, if the $\vec{v}_i$'s are mutually perpendicular.

11. The matrix of the transformation T with respect to the basis $\vec{v}_1, \vec{v}_2$ is $B = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$, so that $\det(A) = \det(B) = 12$, by Fact 6.2.5.

12. Denote the columns by $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$. From Fact 6.3.4 and Exercise 8 we know that $|\det(A)| \leq \|\vec{v}_1\|\|\vec{v}_2\|\ \|\vec{v}_3\|\|\vec{v}_4\|$; equality holds if the columns are orthogonal. Since the entries of the $\vec{v}_i$ are 0, 1, and -1, we have $\|\vec{v}_i\| \leq \sqrt{1+1+1+1} = 2$. Therefore, $|\det A| \leq 16$.

    To build an example where $\det(A) = 16$ we want all 1's and -1's as entries, and the columns need to be orthogonal. A little experimentation produces $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$ (there are other solutions). Note that we need to *check* that $\det(A) = 16$ (and not -16).

13. By Fact 6.3.7, the desired 2-volume is

$$\sqrt{\det\left(\begin{bmatrix}1&1&1&1\\1&2&3&4\end{bmatrix}\begin{bmatrix}1&1\\1&2\\1&3\\1&4\end{bmatrix}\right)}=\sqrt{\det\begin{bmatrix}4&10\\10&30\end{bmatrix}}=\sqrt{20}.$$

14. By Fact 6.3.7, the desired 3-volume is

$$\sqrt{\det\left(\begin{bmatrix}1&0&0&0\\1&1&1&1\\1&2&3&4\end{bmatrix}\begin{bmatrix}1&1&1\\0&1&2\\0&1&3\\0&1&4\end{bmatrix}\right)}=\sqrt{\det\begin{bmatrix}1&1&1\\1&4&10\\1&10&30\end{bmatrix}}=\sqrt{6}.$$

15. If $\vec{v}_1,\vec{v}_2,\dots,\vec{v}_m$ are linearly dependent and if $A=[\vec{v}_1\cdots\vec{v}_m]$, then $\det(A^TA)=0$ since A^TA and A have equal and nonzero kernels (by Fact 5.4.2), hence A^TA fails to be invertible.

    On the other hand, since the $\vec{v}_i$ are linearly dependent, at least one of them will be redundant. For such a redundant $\vec{v}_i$, we will have $\vec{v}_i=\vec{v}_i^{\|}$ and $\vec{v}_i^{\perp}=\vec{0}$, so that $V(\vec{v}_1,\dots,\vec{v}_m)=0$, by Definition 6.3.6. This discussion shows that $V(\vec{v}_1,\dots,\vec{v}_m)=0=\sqrt{\det(A^TA)}$ if the vectors $\vec{v}_1,\dots,\vec{v}_m$ are linearly dependent.

16. False

    If T is given by $A=2I_3$ then $|\det(A)|=8$. But if Ω is the square defined by $\vec{e}_1,\vec{e}_2$ in $\mathbb{R}^3$ (of area 1), then $T(\Omega)$ is the square defined by $2\vec{e}_1,2\vec{e}_2$ and the area of $T(\Omega)$ is 4.

17. a. Let $\vec{w}=\vec{v}_1\times\vec{v}_2\times\vec{v}_3$. Note that $\vec{w}$ is orthogonal to $\vec{v}_1,\vec{v}_2$ and $\vec{v}_3$, by Exercise 6.2.44c. Then $V(\vec{v}_1,\vec{v}_2,\vec{v}_3,\vec{w})\underset{\substack{\uparrow\\ \text{Definition 6.3.6}}}{=}V(\vec{v}_1,\vec{v}_2,\vec{v}_3)\|\vec{w}^{\perp}\|=V(\vec{v}_1,\vec{v}_2,\vec{v}_3)\|\vec{w}\|$.

    b. By Exercise 6.2.44e,

    $$\begin{aligned}V(\vec{v}_1,\vec{v}_2,\vec{v}_3,\vec{v}_1\times\vec{v}_2\times\vec{v}_3)&=|\det[\,\vec{v}_1\quad\vec{v}_2\quad\vec{v}_3\quad\vec{v}_1\times\vec{v}_2\times\vec{v}_3\,]|\\&=|\det[\,\vec{v}_1\times\vec{v}_2\times\vec{v}_3\quad\vec{v}_1\quad\vec{v}_2\quad\vec{v}_3\,]|=\|\vec{v}_1\times\vec{v}_2\times\vec{v}_3\|^2.\end{aligned}$$

    c. By parts a and b, $V(\vec{v}_1,\vec{v}_2,\vec{v}_3)=\|\vec{v}_1\times\vec{v}_2\times\vec{v}_3\|$. If the vectors $\vec{v}_1,\vec{v}_2,\vec{v}_3$ are linearly dependent, then both sides of the equation are 0, by Exercise 15 and Exercise 6.2.44a.

18. a. (See Figure 6.6.) $\begin{bmatrix}p&0\\0&q\end{bmatrix}\begin{bmatrix}\cos(t)\\\sin(t)\end{bmatrix}=\begin{bmatrix}p\cdot\cos(t)\\q\cdot\sin(t)\end{bmatrix}$, the ellipse with semi-axes $\pm\begin{bmatrix}p\\0\end{bmatrix}$ and $\pm\begin{bmatrix}0\\q\end{bmatrix}$.

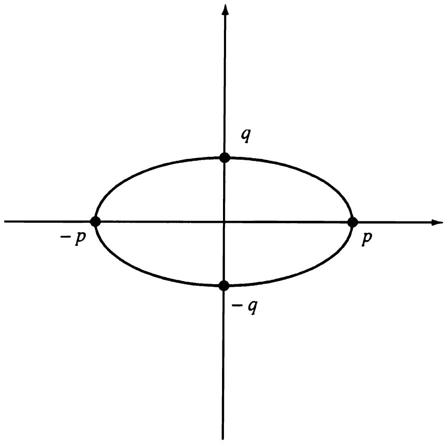

Figure 6.6: for Problem 6.3.18a.

(area of the ellipse) $= |\det(A)|$(area of the unit circle) $= pq\pi$

b. By Fact 6.3.8, $|\det(A)| = \dfrac{\text{area of the ellipse}}{\text{area of the unit circle}} = \frac{ab\pi}{\pi} = ab$ so $|\det(A)| = ab$.

c. The unit circle consists of all vectors of the form $\vec{x} = \cos(t)\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} + \sin(t)\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$; its image is the ellipse consisting of all vectors

$$T(\vec{x}) = \cos(t)\underbrace{2\sqrt{2}\begin{bmatrix}1\\1\end{bmatrix}}_{\text{semi-major axis}} + \sin(t)\underbrace{\sqrt{2}\begin{bmatrix}1\\-1\end{bmatrix}}_{\text{semi-minor axis}}$$. (See Figure 6.7.)

19. $\det[\vec{v}_1\ \vec{v}_2\ \vec{v}_3] = \vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3) = \|\vec{v}_1\|\|\vec{v}_2 \times \vec{v}_3\| \cos\theta$ where θ is the angle between $\vec{v}_1$ and $\vec{v}_2 \times \vec{v}_3$ so $\det[\vec{v}_1\ \vec{v}_2\ \vec{v}_3] > 0$ if and only if $\cos\theta > 0$, i.e., if and only if θ is acute $(0 \le \theta \le \frac{\pi}{2})$. (See Figure 6.8.)

20. By Exercise 19, $\vec{v}_1, \vec{v}_2, \vec{v}_3$ constitute a positively oriented basis if and only if $\det[\vec{v}_1\, \vec{v}_2\, \vec{v}_3] > 0$. Assume that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is such a basis. We want to show that $A\vec{v}_1, A\vec{v}_2, A\vec{v}_3$ is positively oriented if and only if $\det(A) > 0$. We have $\det[A\vec{v}_1\, A\vec{v}_2\, A\vec{v}_3] = \det(A[\vec{v}_1\, \vec{v}_2\, \vec{v}_3]) = \det(A)\det[\vec{v}_1\, \vec{v}_2\, \vec{v}_3]$ so since $\det[\vec{v}_1\, \vec{v}_2\, \vec{v}_3] > 0$ by assumption, $\det[A\vec{v}_1\, A\vec{v}_2\, A\vec{v}_3] > 0$ if and only if $\det(A) > 0$. Hence A is orientation preserving if and only if $\det(A) > 0$.

21. a. Reverses

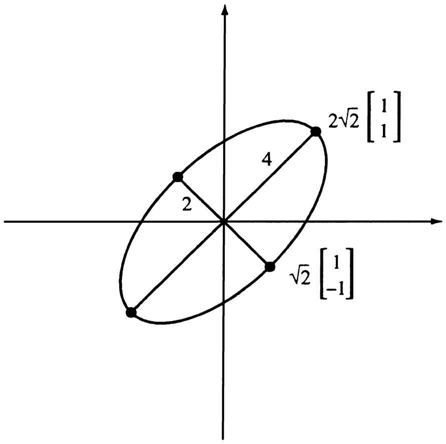

Figure 6.7: for Problem 6.3.18c.

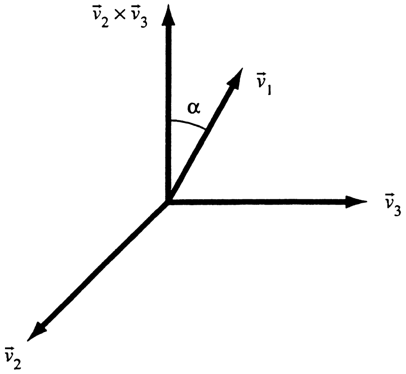

Figure 6.8: for Problem 6.3.19.

Consider $\vec{v}_2$ and $\vec{v}_3$ in the plane (not parallel), and let $\vec{v}_1 = \vec{v}_2 \times \vec{v}_3$; then $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is a positively oriented basis, but $T(\vec{v}_1) = -\vec{v}_1, T(\vec{v}_2) = \vec{v}_2, T(\vec{v}_3) = \vec{v}_3$ is negatively oriented.

b. Preserves

Consider $\vec{v}_2$ and $\vec{v}_3$ orthogonal to the line (not parallel), and let $\vec{v}_1 = \vec{v}_2 \times \vec{v}_3$; then $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is a positively oriented basis, and $T(\vec{v}_1) = \vec{v}_1, T(\vec{v}_2) = -\vec{v}_2, T(\vec{v}_3) = -\vec{v}_3$ is positively oriented as well.

c. Reverses

The standard basis $\vec{e}_1, \vec{e}_2, \vec{e}_3$ is positively oriented, but $T(\vec{e}_1) = -\vec{e}_1, T(\vec{e}_2) = -\vec{e}_2, T(\vec{e}_3) = -\vec{e}_3$ is negatively oriented.

22. Here $A = \begin{bmatrix} 3 & 7 \\ 4 & 11 \end{bmatrix}$, $\det(A) = 5$, $\vec{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, so by Fact 6.3.9

$$x = \frac{\det \begin{bmatrix} 1 & 7 \\ 3 & 11 \end{bmatrix}}{5} = -2,\ y = \frac{\det \begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}}{5} = 1.$$

23. Here $A = \begin{bmatrix} 5 & -3 \\ -6 & 7 \end{bmatrix}$, $\det(A) = 17, \vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so by Fact 6.3.9

$$x_1 = \frac{\det \begin{bmatrix} 1 & -3 \\ 0 & 7 \end{bmatrix}}{17} = \frac{7}{17}, x_2 = \frac{\det \begin{bmatrix} 5 & 1 \\ -6 & 0 \end{bmatrix}}{17} = \frac{6}{17}.$$

24. Here $A = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 4 & 5 \\ 6 & 0 & 7 \end{bmatrix}$, $\det(A) = 146$, $\vec{b} = \begin{bmatrix} 8 \\ 3 \\ -1 \end{bmatrix}$, so by Fact 6.3.9,

$$x = \frac{\det \begin{bmatrix} 8 & 3 & 0 \\ 3 & 4 & 5 \\ -1 & 0 & 7 \end{bmatrix}}{146} = 1,\ y = \frac{\det \begin{bmatrix} 2 & 8 & 0 \\ 0 & 3 & 5 \\ 6 & -1 & 7 \end{bmatrix}}{146} = 2,\ z = \frac{\det \begin{bmatrix} 2 & 3 & 8 \\ 0 & 4 & 3 \\ 6 & 0 & -1 \end{bmatrix}}{146} = -1.$$

25. By Fact 6.3.10, the ij^{th} entry of $\text{adj}(A)$ is given by $(-1)^{i+j}\det(A_{ji})$, so since

$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ for $i = 1, j = 1$, we get $(-1)^2 \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$, and for $i = 1, j = 2$ we get $(-1)^3 \det \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = 0$, and so forth.

Completing this process gives $\text{adj}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$, hence by Fact 6.3.10,

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A) = \frac{1}{-1}\begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}.$$

26. By Fact 6.3.10, $A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$, so if $\det(A) = 1, A^{-1} = \text{adj}(A)$. If A has integer entries then $(-1)^{i+j}\det(A_{ji})$ will be an integer for all $1 \le i, j \le n$, hence $\text{adj}(A)$ will have integer entries. Therefore, A^{-1} will also have integer entries.

27. By Fact 6.3.9, using $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, $\det(A) = a^2 + b^2, \vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we get

$$x = \det \begin{bmatrix} 1 & -b \\ 0 & a \end{bmatrix} \left(\frac{1}{a^2+b^2}\right) = \frac{a}{a^2+b^2}, y = \det \begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix} \left(\frac{1}{a^2+b^2}\right) = \frac{-b}{a^2+b^2},$$

so x is positive, y is negative (since $a, b > 0$), and x decreases as b increases.

28. Here $A = \begin{bmatrix} s & a \\ m & -h \end{bmatrix}$, $\det(A) = -sh - ma, \vec{b} = \begin{bmatrix} I^\circ + G \\ M_s + M^\circ \end{bmatrix}$ so, by Fact 6.3.9

$$Y = \frac{\det \begin{bmatrix} I^\circ + G & a \\ M_s - M^\circ & -h \end{bmatrix}}{-sh-ma} = \frac{-h(I^\circ+G)-a(M_s-M^\circ)}{-sh-ma} = \frac{h(I^\circ+G)+a(M_s-M^\circ)}{sh+ma},$$

$$r = \frac{\det \begin{bmatrix} s & I^\circ + G \\ m & M_s - M^\circ \end{bmatrix}}{-sh-ma} = \frac{s(M_s-M^\circ)-m(I^\circ+G)}{-sh-ma} = \frac{m(I^\circ+G)-s(M_s-M^\circ)}{sh+ma}.$$

29. By Fact 6.3.9,

$$dx_1 = \frac{\det \begin{bmatrix} 0 & R_1 & -(1-\alpha) \\ 0 & 1-\alpha & -(1-\alpha)^2 \\ -R_2de_2 & -R_2 & -\frac{(1-\alpha)^2}{\alpha} \end{bmatrix}}{D} = \frac{R_1R_2(1-\alpha)^2de_2 - R_2(1-\alpha)^2de_2}{D}$$

$$dy_1 = \frac{\det \begin{bmatrix} -R_1 & 0 & -(1-\alpha) \\ \alpha & 0 & -(1-\alpha)^2 \\ R_2 & -R_2de_2 & -\frac{(1-\alpha)^2}{\alpha} \end{bmatrix}}{D} = \frac{R_2de_2(R_1(1-\alpha)^2+\alpha(1-\alpha))}{D} > 0$$

$$dp = \frac{\det \begin{bmatrix} -R_1 & R_1 & 0 \\ \alpha & 1-\alpha & 0 \\ R_2 & -R_2 & -R_2de_2 \end{bmatrix}}{D} = \frac{R_1R_2de_2}{D} > 0.$$

30. Using the procedure outlined in Exercise 25, we find $\text{adj}(A) = \begin{bmatrix} 18 & 0 & 0 \\ -12 & 6 & 0 \\ -2 & -5 & 3 \end{bmatrix}$.

31. Using the procedure outlined in Exercise 25, we find $\text{adj}(A) = \begin{bmatrix} -6 & 0 & 1 \\ -3 & 5 & -2 \\ 4 & -5 & 1 \end{bmatrix}$.

32. Using the procedure outlined in Exercise 25, we find that $\text{adj}(A) = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$.

33. Using the procedure outlined in Exercise 25, we find that $\operatorname{adj}(A) = \begin{bmatrix} 24 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$.

    Note that the matrix $\operatorname{adj}(A)$ is diagonal, and the i^{th} diagonal entry of $\operatorname{adj}(A)$ is the product of all a_{jj} where $j \neq i$.

34. For an invertible $n \times n$ matrix A, $A\operatorname{adj}(A) = A(\det(A)A^{-1}) = \det(A)AA^{-1} = \det(A)I_n$. The same is true for $\operatorname{adj}(A)A$.

35. $\det(\operatorname{adj}(A)) = \det(\det(A)A^{-1})$. Taking the product $\det(A)A^{-1}$ amounts to multiplying each row of A^{-1} by $\det(A)$, so that $\det(\operatorname{adj}(A)) = (\det A)^n \det(A^{-1}) = (\det A)^n \frac{1}{\det(A)} = (\det A)^{n-1}$.

36. $\operatorname{adj}(\operatorname{adj}A) = \operatorname{adj}(\det(A)A^{-1})$

    $= \det(\det(A)A^{-1})(\det(A)A^{-1})^{-1} = (\det A)^n \det(A^{-1})(\det(A)A^{-1})^{-1}$

    $= (\det A)^{n-1}(\det(A)A^{-1})^{-1} = (\det A)^{n-1}\frac{1}{\det(A)}(A^{-1})^{-1}$

    $= (\det A)^{n-2}A$.

37. $\operatorname{adj}(A^{-1}) = \det(A^{-1})(A^{-1})^{-1} = (\det A)^{-1}(A^{-1})^{-1} = (\operatorname{adj}A)^{-1}$.

38. $\operatorname{adj}(AB) = \det(AB)(AB)^{-1}$

    $= \det(A)(\det(B)B^{-1})A^{-1}$

    $= \det(B)B^{-1}(\det(A)A^{-1})$

    $= \operatorname{adj}(B)\operatorname{adj}(A)$.

39. Yes, let S be an invertible matrix such that $AS = SB$, or $SB^{-1} = A^{-1}S$. Multiplying both sides by $\det(A) = \det(B)$, we find that $S(\det(B)B^{-1}) = (\det(A)A^{-1})S$, or, $S(\operatorname{adj}B) = (\operatorname{adj}A)S$, as claimed.

40. The ij^{th} entry of the matrix B of T is (i^{th} component of $T(\vec{e}_j)) = \det(A(\vec{e}_{j,i}))$, which is the ij^{th} entry of $\operatorname{adj}(A)$ (see the first paragraph on Page 286 and Fact 6.3.10). Thus $B = \operatorname{adj}(A)$.

41. If A has a nonzero minor $\det(A_{ij})$, then the $n-1$ columns of the invertible matrix A_{ij} will be independent, so that the $n-1$ columns of A, minus the j^{th}, will be independent as well. Thus, the rank of A (the dimension of the image) is at least $n-1$.

    Conversely, if $\operatorname{rank}(A) \geq n-1$, then we can find $n-1$ independent columns of A. The $n \times (n-1)$ matrix consisting of those $n-1$ columns will have rank $n-1$, so that there will be exactly one redundant row (compare with Exercises 3.3.49 through 51). Omitting this redundant row produces an invertible $(n-1) \times (n-1)$ submatrix of A, giving us a nonzero minor of A.

42. By Definition 6.3.10, $\operatorname{adj}(A) = 0$ if (and only if) all the minors A_{ji} of A are zero. By Exercise 41, this is the case if (and only if) $\operatorname{rank}(A) \leq n - 2$.

43. A direct computation shows that $A(\operatorname{adj}A) = (\operatorname{adj}A)A = (\det A)(I_n)$ for all square matrices. Thus we have $A(\operatorname{adj}A) = (\operatorname{adj}A)A = 0$ for noninvertible matrices, as claimed.

    Let's write $B = \operatorname{adj}(A)$, and let's verify the equation $AB = (\det A)(I_n)$ for the diagonal entries; the verification for the off-diagonal entries is analogous. The i^{th} diagonal entry of AB is

$$[i^{th} \text{ row of } A]\begin{bmatrix} i^{th} \\ \text{column} \\ \text{of } B \end{bmatrix} = a_{i1}b_{1i} + \cdots + a_{in}b_{ni} = \sum_{j=1}^{n} a_{ij}b_{ji}.$$

    Since B is the adjunct of A, $b_{ji} = (-1)^{j+i}\det(A_{ij})$.

    So, our summation equals

$$\sum_{j=1}^{n} a_{ij}(-1)^{i+j}\det(A_{ij})$$

    which is our formula for Laplace expansion across the i^{th} row, and equals $\det(A)$, proving our claim for the diagonal entries.

44. The equation $A(\operatorname{adj}A) = 0$ from Exercise 43 means that $\operatorname{im}(\operatorname{adj}A)$ is a subspace of $\ker(A)$. Thus $\operatorname{rank}(\operatorname{adj}A) = \dim(\operatorname{im}(\operatorname{adj}A)) \leq \dim(\ker A) = n - \operatorname{rank}(A) = n - (n-1) = 1$, implying that $\operatorname{rank}(\operatorname{adj}A) \leq 1$. Since $\operatorname{adj}(A) \neq 0$, by Exercise 42, we can conclude that $\operatorname{rank}(\operatorname{adj}A) = 1$.

45. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We want $A^T = \operatorname{adj}(A)$, or $\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. So, $a = d$ and $b = -c$. Thus, the equation $A^T = \operatorname{adj}(A)$ holds for all matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$.

46. In the simple case when $f(x,y) = 1$ we have $\int_{\Omega_2} f(x,y)dA = \int_{\Omega_2} dA = \text{area of } \Omega_2 = |\det M|$ and

$$\int_{\Omega_1} g(u,v)dA = \int_{\Omega_1} dA = \text{area of } \Omega_1 = 1, \text{ so that } \int_{\Omega_2} f(x,y)dA = |\det M| \cdot \int_{\Omega_1} g(u,v)dA.$$

    This formula holds, in fact, for any continuous function $f(x,y)$; see an introductory text in multivariable calculus for a justification.

47. Note that $\frac{1}{2}\det\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$ is the area of the triangle OP_1P_2, where O denotes the origin. This is likewise true for one-half the second matrix. However, because of the reversal

in orientation, $\frac{1}{2}\det\begin{bmatrix} x_3 & x_4 \\ y_3 & y_4 \end{bmatrix}$ is negative the area of the triangle OP_3P_4; likewise for the last matrix. (See the discussion on Page 277.) Finally, note that the area of the quadrilateral $P_1P_2P_3P_4$ is equal to:

$$\text{the area of triangle } OP_1P_2 + \text{ area of triangle } OP_2P_3$$

$$- \text{ area of triangle } OP_3P_4 - \text{ area of triangle } OP_4P_1.$$

48. In what follows, we will freely use the fact that an invertible linear transformation L from $\mathbb{R}^2$ to $\mathbb{R}^2$ maps an ellipse into an ellipse (see Exercise 2.2.52).

    Now consider a linear transformation L that transforms our 3-4-5 right triangle R into an equilateral triangle T. If we place the vertices of the right triangle R at the points $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$, and the vertices of the equilateral triangle T at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$, then the transformation L has the matrix $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{\sqrt{3}} \end{bmatrix}$, with $\det(A) = \frac{1}{2\sqrt{3}}$.

    According to the hint, L will map the largest ellipse E inscribed into R into the circle C inscribed into T. The Figure 6.9 illustrates that the radius of C is $\tan(\pi/6) = 1/\sqrt{3}$, so that the area of C is $\pi/3$. Using the interpretation of the determinant as an expansion fact, we find that (area of C) = (det A)(area of E), or (area of E) $= \frac{\text{area of } C}{\det(A)} = \frac{2\pi}{\sqrt{3}} \approx 3.6$

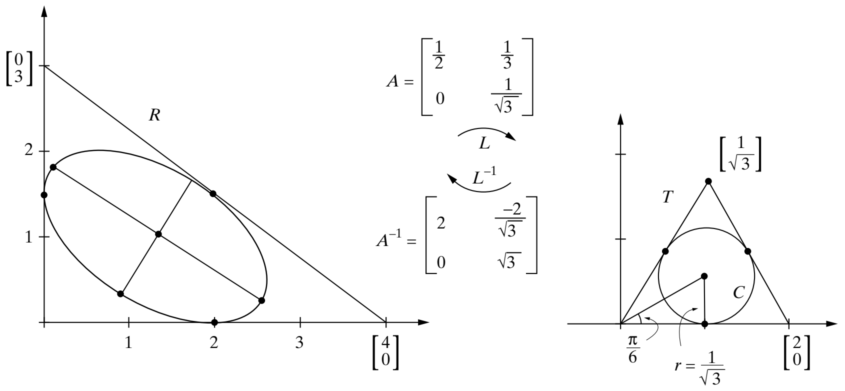

Figure 6.9: for Problem 6.3.48 and Problem 49.

49. We will use the terminology introduced in the solution of Exercise 48 throughout. Note that the transformation L^{-1}, with matrix $A^{-1} = \begin{bmatrix} 2 & -2/\sqrt{3} \\ 0 & \sqrt{3} \end{bmatrix}$, maps the circle C (with

radius $1/\sqrt{3}$) into the ellipse E. Now consider a radial vector $\vec{v} = \frac{1}{\sqrt{3}}\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix}$ of C, and find the maximal value M and the minimal value m of $\|A^{-1}\vec{v}\|^2 = \frac{4}{3} + \frac{1}{9}\sin^2\theta - \frac{8}{3\sqrt{3}}(\sin\theta)(\cos\theta) = \frac{25}{18} - \frac{1}{18}(\cos 2\theta) - \frac{4}{3\sqrt{3}}(\sin 2\theta)$ (we are taking the square to facilitate the computations). Then $\sqrt{M}$ and $\sqrt{m}$ will be the lengths of the semi-axes of E. The function above is sinusoidal with average value $\frac{25}{18}$ and amplitude $\sqrt{\frac{1}{18^2} + \frac{16}{27}} = \frac{\sqrt{193}}{18}$. Thus $M = \frac{25+\sqrt{193}}{18}$ and $m = \frac{25-\sqrt{193}}{18}$, so that the length of the semi-major axis of E is

$\sqrt{M} = \sqrt{\frac{25+\sqrt{193}}{18}} \approx 1.47$, and for the semi-minor axis we get

$\sqrt{m} = \sqrt{\frac{25-\sqrt{193}}{18}} \approx 0.79$.

True or False

1. T, by Fact 6.1.6 (a diagonal matrix is triangular as well)
2. T, by Fact 6.2.1b.
3. T, by Definition 6.1.1
4. F; We have $\det(4A) = 4^4 \det(A)$, by Fact 6.2.1a.
5. F; Let $A = B = I_5$, for example
6. T; We have $\det(-A) = (-1)^6 \det(A) = \det(A)$, by Fact 6.2.1a.
7. F; In fact, $\det(A) = 0$, since A fails to be invertible
8. F; The matrix A fails to be invertible if $\det(A) = 0$, by Fact 6.2.2.
9. T, by Fact 6.2.1a, applied to the columns.
10. T, by Fact 6.2.4
11. T, by Fact 6.2.5.
12. F, by Fact 6.3.1. The determinant can be -1.
13. T, by Fact 6.2.4.
14. F; The second and the fourth column are linearly dependent.

15. T; The determinant is 0 for $k = -1$ or $k = -2$, so that the matrix is invertible for all *positive* k.

16. F; Expand down the first column to see that $\det(A) = -\det(I_3) = -1$.

17. T; It would be tedious to compute the exact value of the determinant, but we can show that the determinant is nonzero without finding its exact value.

    One method is to show that the determinant is an odd integer. Expanding down the third column, we see that $\det(A) = 3\det\begin{bmatrix} 5 & 4 & 8 \\ 100 & 9 & 7 \\ 6 & 5 & 100 \end{bmatrix} + \text{ even terms}$

    $= 3(-7\det\begin{bmatrix} 5 & 4 \\ 6 & 5 \end{bmatrix} + \text{ even terms }) + \text{ even terms}$

    $= -21 + \text{ even terms } = \text{ odd, as claimed.}$

    Alternatively, we can use the permutation formula for the determinant (Fact 6.2.10), and argue that one of the 24 terms is very large (namely $100^4 = 10^8$) compared to the other terms [those are less than $(100^2)(10^2) = 10^6$ each in absolute value], so that the determinant turns out to be positive. (In fact, this determinant is 97,763,383.)

18. F; The correct formula is $\det(A^{-1}) = \frac{1}{\det(A^T)}$, by Facts 6.2.6 and 6.2.7.

19. T; Matrix A is invertible.

20. T; Any nonzero noninvertible matrix A will do.

21. T, by Fact 6.3.4, since $\|\vec{v}_i^{\perp}\| \le \|\vec{v}_i\| = 1$ for all column vectors $\vec{v}_i$.

22. T; We have $\det(A) = \det(\text{rref } A) = 0$.

23. F; Let $A = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$, for example.

24. F; Let $A = 2I_2$, for example

25. T; Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$. The column vectors of A are orthogonal and they all have length 2.

26. F; Let $A = \begin{bmatrix} 8 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, for example.

27. F; In fact, $\det(A) = \det[\vec{u}\ \vec{v}\ \vec{w}] = -\det[\vec{v}\ \vec{u}\ \vec{w}] = -\vec{v} \cdot (\vec{u} \times \vec{w})$. We have used Fact 6.2.1b and Definition 6.1.1.

28. T; Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, for example.

29. F; Note that $\det \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = 2$.

30. T, by Fact 6.3.10

31. F; Let $A = 2I_2$, for example.

32. F; Let $A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, for example.

33. F; Let $A = I_2$ and $B = -I_2$, for example.

34. T; Note that $\det(B) = -\det(A) < \det(A)$, so that $\det(A) > 0$.

35. T; Let's do Laplace expansion along the first row, for example (see Fact 6.1.5).

    Then $\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j}) \neq 0$. Thus $\det(A_{1j}) \neq 0$ for at least one j, so that A_{1j} is invertible.

36. T; Note that $\det(A)$ and $\det(A^{-1})$ are both integers, and $(\det A)(\det A^{-1}) = 1$. This leaves only the possibilities $\det(A) = \det(A^{-1}) = 1$ and $\det(A) = \det(A^{-1}) = -1$.

37. T, since $\operatorname{adj}(A) = (\det A)(A^{-1})$, by Fact 6.3.10.

38. F; Note that $\det(A^2) = (\det A)^2$ cannot be negative, but $\det(-I_3) = -1$.

39. F; Note that $\det(S^{-1}AS) = \det(A)$ but $\det(2A) = 2^3(\det A) = 8(\det A)$.

40. F; Note that $\det(S^T AS) = (\det S)^2(\det A)$ and $\det(-A) = -(\det A)$ have opposite signs.

41. T; We can use induction on n to show that $\det(A)$ is an odd integer. If we expand $\det(A)$ down the first column, then the first term, $a_{11}\det(A_{11})$, will be odd, since a_{11} is odd, and $\det(A_{11})$ is odd by the induction hypothesis. However, all the other terms $a_{i1}\det(A_{i1})$, where $i > 1$, will be even, since a_{i1} is even in this case. Thus, $\det(A)$ is odd as claimed, and A is invertible.

42. F; Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 5 & 2 \end{bmatrix}$, for example

43. T; Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $a \neq 0$, let $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$; if $b \neq 0$, let $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$; if $c \neq 0$, let $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and if $d \neq 0$, let $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

44. T; Use Gaussian elimination for the first column only to transform A into a matrix of the form

$$B = \begin{bmatrix} 1 & \pm 1 & \pm 1 & \pm 1 \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix}$$

Note that $\det(B) = \det(A)$ or $\det(B) = -(\det A)$. The stars in matrix B all represent numbers $(\pm 1) \pm (\pm 1)$, so that they are 2, 0, or -2. Thus the determinant of the 3×3 matrix M containing the stars is divisible by 8, since each of the 6 terms in Sarrus' rule is 8, 0 or -8. Now perform Laplace expansion down the first column of B to see that $\det(M) = \det(B) = +/-\det(A)$.

45. T; $A(\text{adj}A) = A(\det(A)A^{-1}) = \det(A)I_n = \det(A)A^{-1}A = \text{adj}(A)A$.

46. T; Laplace expansion along the second row gives $\det(A) = -k \det \begin{bmatrix} 1 & 2 & 4 \\ 8 & 9 & 7 \\ 0 & 0 & 5 \end{bmatrix} + C = 35k + C$, for some constant C (we need not compute that $C = -259$). Thus A is invertible except for $k = \frac{-C}{35}$ (which turns out to be $\frac{259}{35} = \frac{37}{5} = 7.4$).

47. F; $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are both orthogonal and $\det(A) = \det(B) = 1$. However, $AB \neq BA$.